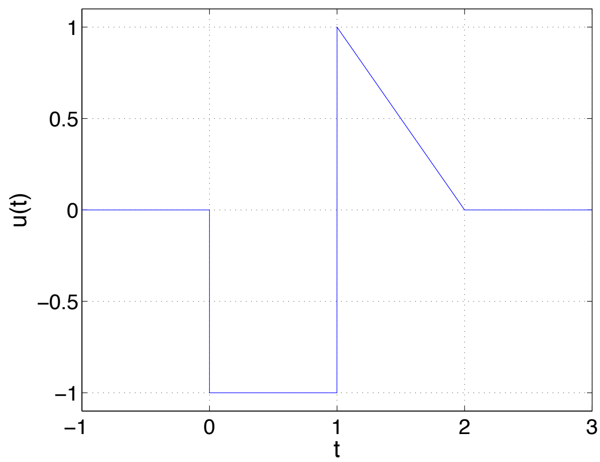
HW Assignment 1

Solution

Question 1

1.



2. $T[u](t) = \begin{cases} -1 & \left(\frac{3}{2}t+1\right) \in [0,1] \\ 2-\left(\frac{3}{2}t+1\right) & \left(\frac{3}{2}t+1\right) \in [1,2] \\ 0 & else \end{cases}$.

3. Linear: $T[\alpha_1 x_1 + \alpha_2 x_2](t) = (\alpha_1 x_1 + \alpha_2 x_2)\left(\frac{3}{2}t+1\right) = \alpha_1 x\left(\frac{3}{2}t+1\right) + \alpha_2 x\left(\frac{3}{2}t+1\right) =$

   $\alpha_1 T[x_1](t) + \alpha_2 T[x_2](t) \to linear$.

   Time-invariant: $\left.\begin{array}{l} s_\tau[T[x]](t) = s_\tau[x]\left(\frac{3}{2}t+1\right) = x\left(\frac{3}{2}t+1-\tau\right) \\ T[s_\tau[x]](t) = T[x](t-\tau) = x\left(\frac{3}{2}(t-\tau)+1\right) \end{array}\right\} \to$ not time-invariant

4. $R[T[u]](t) = (T[u](t))^2 = \begin{cases} 1 & t \in \left[-\frac{2}{3}, 0\right] \\ \left(1-\frac{3}{2}t\right)^2 & t \in \left[0, \frac{2}{3}\right] \\ 0 & else \end{cases}$

5. Linear: $R[\alpha_1 x_1 + \alpha_2 x_2](t) = (\alpha_1 x_1 + \alpha_2 x_2)^2 \neq \alpha_1 x_1^2 + \alpha_2 x_2^2 \to$ not linear .

   Time-invariant: $\left.\begin{array}{l} R[s_\tau[x]](t) = R[x](t-\tau) = x^2(t-\tau) \\ s_\tau[R[x]](t) = s_\tau[x^2](t) = x^2(t-\tau) \end{array}\right\} \to$ time-invariant

6. $R \circ T$ is neither linear not time-invariant.

Question 2

Full solution: in Lathi's book, question 1.1-3 (I changed the names if the signals from x and y, to u and v - to make it more obscure).

Question 3

1. $D\left[\alpha_1 x_1 + \alpha_2 x_2\right](t) = \frac{d}{dt}\left(\alpha_1 x_1(t) + \alpha_2 x_2(t)\right) = \alpha_1 \frac{d}{dt} x_1(t) + \alpha_2 \frac{d}{dt} x_2 = \alpha_1 D\left[x_1\right](t) + \alpha_2 D\left[x_2\right](t)$.

$$\begin{cases} s_\tau\left[\left[D[x]\right]\right](t) = s_\tau\left[\frac{dx}{dt}\right](t) = \frac{dx}{dt}(t-\tau) \\ D\left[s_\tau[x]\right](t) = \frac{d}{dt}\left[x(t-\tau)\right] = \frac{dx}{dt}(t-\tau) \end{cases}$$

2. The direct method is same as above (only with d^2 / dx^2). The indirect method is by stating that the second derivative is equal to applying the derivative twice, i.e. two LTI systems, which means that their product is also LTI.

$D^2\left[\alpha_1 x_1 + \alpha_2 x_2\right] = D\left[D\left[\alpha_1 x_1 + \alpha_2 x_2\right]\right] = D\left[\alpha_1 D\left[x_1\right]\right] \alpha_2 D\left[x_2\right] = \alpha_1 D\left[D\left[x_1\right]\right] + \alpha_2 D\left[D\left[x_2\right]\right]$.

$S_\tau\left[D^2[x]\right] = S_\tau\left[D\left[D[x]\right]\right] = D\left[S_\tau\left[D[x]\right]\right] = D\left[D\left[S_\tau[x]\right]\right] = D^2\left[S_\tau[x]\right]$

3. - Base case: n=1 you've shown that $D[x](t)$ is LTI.

- Assumption: $D^k[x]$ for $1 \le k \le n-1$ is LTI.

- Inductive step:

linearity: $D^n\left[\alpha_1 x_1 + \alpha_2 x_2\right](t) = D\left[D^{n-1}\left[\alpha_1 x_1 + \alpha_2 x_2\right]\right](t) \underset{assumption}{=} D\left[\alpha_1 D^{n-1}\left[x_1\right] + \alpha_2 D^{n-1}\left[x_2\right]\right] =$

$\underset{assumption}{=} \alpha_1 D\left[D^{n-1}\left[x_1\right]\right] + \alpha_2 D\left[D^{n-1}\left[x_2\right]\right](t) = \alpha_1 D^n\left[x_1\right](t) + \alpha_2 D^n\left[x_2\right](t)$.

time- invariant:

$s_\tau\left[D^n[x]\right](t) = s_\tau\left[D\left[D^{n-1}[x]\right]\right](t) D\left[s_\tau\left[D^{n-1}[x]\right]\right](t) \underset{assumption}{=} D\left[D^{n-1}\left[s_\tau[x]\right]\right](t) = D^n\left[s_\tau[x]\right](t)$

Question 4

1. Linear: $T_{sum}\left[\alpha_1 x_1 + \alpha_2 x_2\right](t) = T_a\left[\alpha_1 x_1 + \alpha_2 x_2\right](t) + T_b\left[\alpha_1 x_1 + \alpha_2 x_2\right](t) \underbrace{=}_{\text{linearity of } T_a, T_b}$

$= \alpha_1 T_a\left[x_1\right](t) + \alpha_2 T_a\left[x_2\right](t) + \alpha_1 T_b\left[x_1\right](t) + \alpha_2 T_b\left[x_2\right](t) =$

$= \alpha_1\left(T_a + T_b\right)\left[x_1\right](t) + \alpha_2\left(T_a + T_b\right)\left[x_2\right](t) = \alpha_1 T_{sum}\left[x_1\right](t) + \alpha_2 T_{sum}\left[x_2\right](t)$

Time-invariant: $T_{sum}\left[s_\tau[x]\right](t) = T_{sum}[x](t-\tau) = T_a[x](t-\tau) + T_b[x](t-\tau) =$

$= T_a\left[s_\tau[x]\right](t) + T_b\left[s_\tau[x]\right](t) \underbrace{=}_{T_1, T_2 \text{ are time-invatiant}} s_\tau\left[T_a[x]\right](t) + s_\tau\left[T_b[x]\right](t) =$

$\underbrace{=}_{s_\tau \text{ is linear}} s_\tau\left[T_a[x] + T_b[x]\right](t) + s_\tau\left[T_{sum}[x]\right](t)$

2. $s_\tau\left[T_{series}\left[x\right]\right](t) = s_\tau\left[T_b\left[T_a\left[x\right]\right]\right](t) = T_b\left[s_\tau\left[T_a\left[x\right]\right]\right](t) = T_b\left[T_a\left[s_\tau\left[x\right]\right]\right](t) = T_{series}\left[s_\tau\left[x\right]\right](t)$.

Question 5

Proof by induction. Same as previous question.