HW Assignment 3

Solution

Question 1

1. Writing the Cosine in it's complex exponents form we get:

$$x(t)=1+\cos\left(5\pi t+\frac{\pi}{3}\right)=1+\frac{1}{2}e^{i\frac{\pi}{3}}e^{i5\pi t}+\frac{1}{2}e^{-i\frac{\pi}{3}}e^{-i5\pi t}$$

We got an exponent series in the form of: $x(t)=\sum_{n\in\mathbb{Z}}d_n e^{in\omega_0 t}$. Define $\omega_0=5\pi$ and we get:

$$d_0=1\ ,\ d_1=\frac{1}{2}e^{i\frac{\pi}{3}}\ ,\ d_{-1}=\frac{1}{2}e^{-\frac{\pi}{3}}\ ,\ d_{n\neq 0,1,-1}=0$$

(one can also define $\omega_0=\pi$ and then: $d_0=1\ ,\ d_5=\frac{1}{2}e^{i\frac{\pi}{3}}\ ,\ d_{-5}=\frac{1}{2}e^{-\frac{\pi}{3}}\ ,\ d_{n\neq 0,5,-5}=0$)

2. Same way: $x(t)=\cos^2(\frac{\pi}{2}t)\sin(\pi t)=\left(\frac{1}{2}+\frac{1}{2}\cos(\pi t)\right)\sin(\pi t)=\frac{1}{2}\sin(\pi t)+\frac{1}{2}\cos(\pi t)\sin(\pi t)=$

$$=\frac{1}{2}\sin(\pi t)+\frac{1}{4}\sin(2\pi t)=\frac{1}{4i}e^{i\pi t}-\frac{1}{4i}e^{-i\pi t}+\frac{1}{8i}e^{i2\pi t}-\frac{1}{8i}e^{-i2\pi t}$$

Define $\omega_0=\pi$ and we get:

$$d_1=\frac{1}{4i}\ ,\ d_{-1}=-\frac{1}{4i}\ ,\ d_2=\frac{1}{8i}\ ,\ d_{-2}=-\frac{1}{8i}\ ,\ d_{n\neq\pm1,\pm2}=0$$

3. $x(t)=\left(1+\sin(2\pi t)\right)\cos\left(4\pi t+\frac{\pi}{3}\right)=\cos\left(4\pi t+\frac{\pi}{3}\right)+\sin(2\pi t)\cos\left(4\pi t+\frac{\pi}{3}\right)=$

$$=\frac{1}{2}e^{i\frac{\pi}{3}}e^{i4\pi t}+\frac{1}{2}e^{-i\frac{\pi}{3}}e^{-i4\pi t}+\frac{1}{4i}\left(e^{i2\pi t}-e^{-i2\pi t}\right)\left(e^{i\frac{\pi}{3}}e^{i4\pi t}+e^{-i\frac{\pi}{3}}e^{-i4\pi t}\right)=$$

$$=\frac{1}{4i}e^{-i\frac{\pi}{3}}e^{-i2\pi t}-\frac{1}{4i}e^{i\frac{\pi}{3}}e^{i2\pi t}+\frac{1}{2}e^{i\frac{\pi}{3}}e^{i4\pi t}+\frac{1}{2}e^{-i\frac{\pi}{3}}e^{-i4\pi t}+\frac{1}{4i}e^{i\frac{\pi}{3}}e^{i6\pi t}-\frac{1}{4i}e^{-i\frac{\pi}{3}}e^{-i6\pi t}=$$

define $\omega_0=2\pi$ and we get:

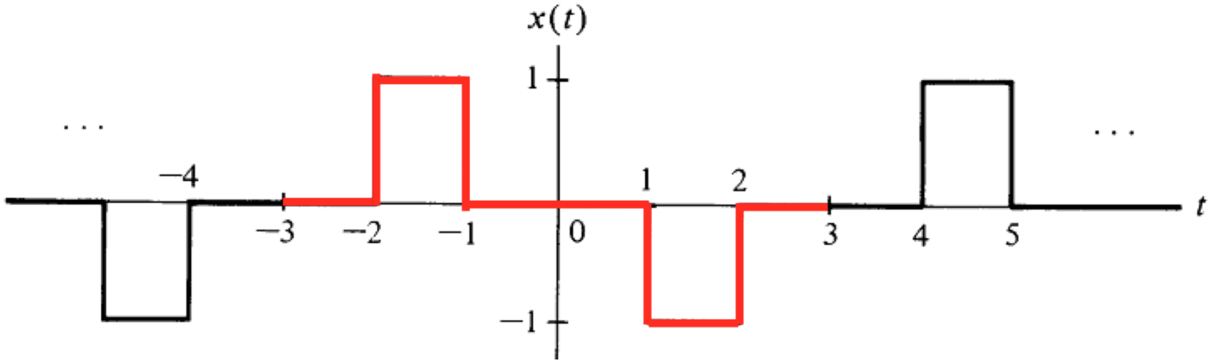
$$d_1=-\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_{-1}=\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_2=\frac{1}{2}e^{i\frac{\pi}{3}}\ ,\ d_{-2}=\frac{1}{2}e^{-i\frac{\pi}{3}}\ ,\ d_3=\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_{-3}=-\frac{1}{4i}e^{-i\frac{\pi}{3}}\ ,\ d_{n\neq\pm1,\pm2,\pm3}=0$$

We can also define $\omega_0=\pi$ and then:

$$d_2=-\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_{-2}=\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_4=\frac{1}{2}e^{i\frac{\pi}{3}}\ ,\ d_{-4}=\frac{1}{2}e^{-i\frac{\pi}{3}}\ ,\ d_6=\frac{1}{4i}e^{i\frac{\pi}{3}}\ ,\ d_{-6}=-\frac{1}{4i}e^{-i\frac{\pi}{3}}\ ,\ d_{n\neq\pm2,\pm4,\pm6}=0$$

Question 2

The period is $T_0 = 6$ so $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$.



We have to span 1 cycle over the exp basis. Define $\tilde{x}(t) = u(t+2) - u(t+1) - u(t-1) + u(t-2)$ (one cycle) and we get:

$$c_n = \frac{1}{6}\int_{-3}^{3} \tilde{x}(t) e^{-i\frac{\pi n}{3}t} dt = \frac{1}{6}\int_{-2}^{-1} e^{-i\frac{\pi n}{3}t} dt - \frac{1}{6}\int_{1}^{2} e^{-i\frac{\pi n}{3}t} dt = \frac{1}{6}\left[\frac{3i}{\pi n} e^{-i\frac{\pi n}{3}t}\right]_{-2}^{-1} - \frac{1}{6}\left[\frac{3i}{\pi n} e^{-i\frac{\pi n}{3}t}\right]_{1}^{2} =$$

$$= \frac{i}{2\pi n}\left(e^{i\frac{\pi n}{3}} - e^{i\frac{2\pi n}{3}}\right) - \frac{i}{2\pi n}\left(e^{-i\frac{2\pi n}{3}} - e^{-i\frac{\pi n}{3}}\right) = -\frac{i}{2\pi n}\left(e^{i\frac{2\pi n}{3}} + e^{-i\frac{2\pi n}{3}}\right) + \frac{i}{2\pi n}\left(e^{i\frac{\pi n}{3}} + e^{-i\frac{\pi n}{3}}\right) =$$

$$= \frac{i}{\pi n}\left[\cos\left(\frac{\pi n}{3}\right) - \cos\left(\frac{2\pi n}{3}\right)\right] = \begin{cases} \frac{i}{\pi n} & n = 6m+1 \\ -\frac{2i}{\pi n} & n = 6m+3 \\ \frac{i}{\pi n} & n = 6m+5 \\ 0 & otherwise \end{cases} \quad (n, m \in \mathbb{Z})$$

Which is zero when $n = 6m$ for all integer m.
and so:

$$\boxed{x(t) = \sum_{m\in\mathbb{Z}}\left(\frac{i}{\pi(6m+1)}\right) e^{i\frac{\pi(6m+1)}{3}t} - \sum_{m\in\mathbb{Z}} \frac{2i}{\pi(6m+3)} e^{i\frac{\pi(6m+3)}{3}t} + \sum_{m\in\mathbb{Z}} \frac{i}{\pi(6m+5)} e^{i\frac{\pi(6m+5)}{3}t}}$$

$d_n \propto \frac{1}{n}$ since the signal has a jump discontinuity $\rightarrow$ we need lots of frequencies to reconstruct the signal.

Question 3

The period is $T_0 = 2$ so $\omega_0 = \dfrac{2\pi}{T_0} = \pi$.

$$d_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \left.\frac{t^3}{6}\right|_{-1}^{1} = \frac{2}{6} = \frac{1}{3}$$

$$d_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-jn\pi t}\,dt = \frac{1}{2}\int_{-1}^{1} t^2 e^{-jn\pi t}\,dt$$

integration by parts: $\left.\begin{array}{l} u = t^2 \rightarrow u' = 2t \\ v' = e^{-jn\pi g} \rightarrow v = \dfrac{e^{-jn\pi t}}{-jn\pi} \end{array}\right\} \int uv'dt = uv - \int vu'dt$

$$\frac{1}{2}\int_{-1}^{1} t^2 e^{-jn\pi t}\,dt = \left.\frac{1}{-2jn\pi} t^2 e^{-jn\pi t}\right|_{-1}^{1} + \frac{1}{jn\pi}\int_{-1}^{1} te^{-jn\pi t}\,dt = \underbrace{\frac{1}{\pi n} t^2 \sin(\pi n)}_{=0} + \frac{1}{jn\pi}\int_{-1}^{1} te^{-jn\pi t}dt = \frac{1}{j\pi n}\int_{-1}^{1} te^{-jn\pi t}dt$$

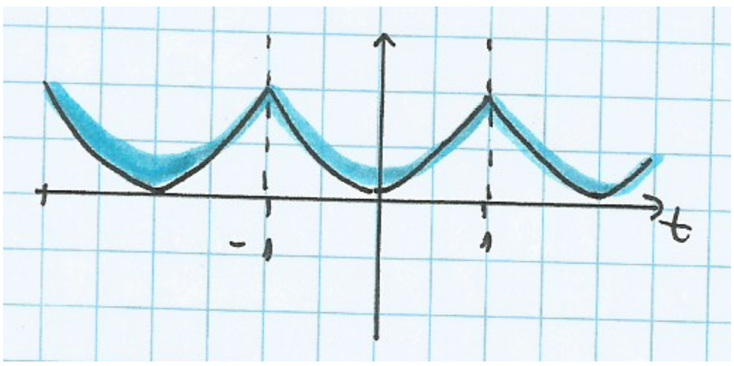
integration by parts: $\left.\begin{array}{l} u = t \rightarrow u' = 1 \\ v' = e^{-jn\pi g} \rightarrow v = \dfrac{e^{-jn\pi t}}{-jn\pi} \end{array}\right\} \int uv'dt = uv - \int vu'dt$

$$\frac{1}{j\pi n}\int_{-1}^{1} te^{-n\pi t}dt = \frac{1}{j\pi n}\left[\left.-\frac{1}{j\pi n} te^{-jn\pi t}\right|_{-1}^{1} + \frac{1}{j\pi n}\int_{-1}^{1} e^{-jn\pi t}\,dt\right] = \frac{1}{j\pi n}\left[-\frac{1}{j\pi n}\left(e^{-j\pi n} + e^{j\pi n}\right) + \left.\frac{1}{(j\pi n)^2} e^{-jn\pi t}\right|_{-1}^{1}\right] =$$

$$\frac{1}{j\pi n}\left[-\frac{2}{j\pi n}\cos(\pi n) + \underbrace{\frac{2}{j(\pi n)^2}\sin(\pi n)}_{=0}\right] = \frac{2}{\pi^2 n^2}\cos(\pi n) = \frac{2}{\pi^2 n^2}(-1)^n$$

and so we get: $\boxed{f(t) = \frac{1}{3} + \frac{2}{\pi^2}\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{(-1)^n}{n^2} e^{jn\pi t}}$

$d_n \propto \dfrac{1}{n^2}$ since the signal doesn't have discontinuities but the signal's Fourier isn't smooth $\rightarrow$ we need lots of frequencies to reconstruct the signal but less than before.



Question 4

- $$d_{-n}{}^{*} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{f(t)^{*}}_{=f(t)} e^{i(-n)\omega_0 t}\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-in\omega_0 t}\,dt = d_n$$

- $$d_n = \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)e^{-in\omega_0 t}\,dt = \frac{1}{T_0}\int_{-\infty}^{\infty} \underbrace{f(t)\cos(n\omega_0 t)}_{\substack{\textit{symmetric}\text{ over }\left[-T_0/2,\,T_0/2\right] \\ \to\int\ \neq 0}}\,dt + \frac{1}{T_0}i\int_{-\infty}^{\infty} \underbrace{f(t)\sin(n\omega_0 t)}_{\substack{\textit{anti-symmetric}\text{ over }\left[-T_0/2,\,T_0/2\right] \\ \to\int\ =0}}\,dt =$$

$$= \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)\cos(n\omega_0 t)\,dt \in \mathbb{R}$$

- $$d_n = \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)e^{-in\omega_0 t}\,dt = \frac{1}{T_0}\int_{-\infty}^{\infty} \underbrace{f(t)\cos(n\omega_0 t)}_{\substack{\textit{anti-symmetric}\text{ over }\left[-T_0/2,\,T_0/2\right] \\ \to\int\ =0}}\,dt - \frac{1}{T_0}i\int_{-\infty}^{\infty} \underbrace{f(t)\sin(n\omega_0 t)}_{\substack{\textit{symmetric}\text{ over }\left[-T_0/2,\,T_0/2\right] \\ \to\int\ \neq 0}}\,dt = .$$

$$= -\frac{1}{T_0}i\int_{-\infty}^{\infty} f(t)\sin(n\omega_0 t)\,dt \in j\mathbb{R}$$

- define a_n the Fourier series coefficients of $\alpha f(t) + \beta g(t)$. we get:

$$a_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left(\alpha f(t) + \beta g(t)\right) e^{-j\omega_0 nt}\,dt = \alpha\left(\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-j\omega_0 nt}\,dt\right) + \beta\left(\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)e^{-j\omega_0 nt}\,dt\right) = \alpha d_n + \beta c_n$$

Question 5

1. $x(t)$ is real and symmetric $\to$ $d_n \in \mathbb{R}$.
2. $x(t)$ is real and anti-symmetric $\to$ $d_n \in j\mathbb{R}$.
3. $x(t)$ is neither symmetric nor anti-symmetric $\to$ can be either.