HW Assignment 10

Solution

Question 1

1.
$$\bar{X}(s)=\Phi(s)\left[\bar{X}(0)+\bar{\bar{B}}\bar{F}(s)\right].$$

$$\Phi(s)=(sI-A)^{-1}=\begin{pmatrix} s+5 & 6 \\ -1 & s \end{pmatrix}^{-1}=\frac{1}{\underbrace{s(s+5)+6}_{s^2+5s+6=(s+3)(s+2)}}\begin{pmatrix} s & -6 \\ 1 & s+5 \end{pmatrix}=\begin{pmatrix} \frac{s}{(s+3)(s+2)} & -\frac{6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{pmatrix}$$

$$\Rightarrow X(s)=\begin{pmatrix} \frac{s}{(s+3)(s+2)} & -\frac{6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{pmatrix}\left[\begin{pmatrix} 5 \\ 4 \end{pmatrix}+\begin{pmatrix} Laplace\{\sin(100t)\} \\ 0 \end{pmatrix}\right]=$$

$$=\begin{pmatrix} \frac{-34.01}{s+2}+\frac{39.03}{s+3}-\frac{10^{-2}s}{s^2+10^4} \\ \frac{17.01}{s+2}-\frac{13.01}{s+3}-\frac{0}{s^2+10^4} \end{pmatrix}$$

$$\boxed{x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=Laplace^{-1}\{X(s)\}=\begin{pmatrix} -34.02e^{-2t}+39.03e^{-3t}-0.01\cos 100t \\ 17.01e^{-2t}-13.01e^{-3t} \end{pmatrix}u(t)}$$

2.
$$\bar{X}(s)=\Phi(s)\left[\bar{X}(0)+\bar{\bar{B}}\bar{F}(s)\right].$$

$$\Phi(s)=(sI-A)^{-1}=\begin{pmatrix} s+1 & -1 \\ 0 & s+2 \end{pmatrix}^{-1}=\frac{1}{(s+1)(s+2)}\begin{pmatrix} s+2 & 1 \\ 0 & s+1 \end{pmatrix}=\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}$$

$$\Rightarrow X(s)=\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \underbrace{Laplace\{u(t)\}}_{1/s} \\ \underbrace{Laplace\{\delta(t)\}}_{1} \end{pmatrix}\right]=$$

$$=\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\begin{pmatrix} \frac{s+1}{s} \\ 1 \end{pmatrix}\right]=\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}\begin{pmatrix} \frac{2s+1}{s} \\ 3 \end{pmatrix}=$$

$$= \begin{pmatrix} \frac{(2s+1)(s+2)+3s}{s(s+1)(s+2)} \\ \frac{3}{s+2} \end{pmatrix} = \begin{pmatrix} \frac{1}{s}+\frac{4}{s+1}-\frac{3}{s+2} \\ \frac{3}{s+2} \end{pmatrix}$$

$$\boxed{x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = Laplace^{-1}\{X(s)\} = \begin{pmatrix} 1+4e^{-4}-3e^{-2t} \\ 3e^{-2t} \end{pmatrix} u(t)}$$

Question 2

1. $$\overline{x}(t)=\underbrace{\overline{\overline{e^{at}}}\,\overline{x}(0)}_{ZIR}+\underbrace{\overline{\overline{e^{at}}} * \overline{\overline{B}}\overline{f}(t)}_{ZSR}$$ where $e^{At} = Laplace^{-1}\{\Phi(s)\}$.

$$e^{At} = Laplace^{-1}\left\{\begin{pmatrix} \frac{s}{(s+3)(s+2)} & -\frac{6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{pmatrix}\right\} = Laplace^{-1}\left\{\begin{pmatrix} \frac{-2}{s+2}+\frac{3}{s+3} & \frac{-6}{s+2}+\frac{6}{s+3} \\ \frac{1}{s+2}-\frac{1}{s+3} & \frac{3}{s+2}-\frac{2}{s+3} \end{pmatrix}\right\} =$$

$$= \begin{pmatrix} -2e^{-2t}+3e^{-3t} & -6e^{-2t}+6e^{-3t} \\ e^{-2t}-e^{-3t} & 3e^{-2t}-2e^{-3t} \end{pmatrix} u(t)$$

$$e^{At}x(0) = u(t)\begin{pmatrix} -2e^{-2t}+3e^{-3t} & -6e^{-2t}+6e^{-3t} \\ e^{-2t}-e^{-3t} & 3e^{-2t}-2e^{-3t} \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -34e^{-2t}+39e^{-3t} \\ 17e^{-2t}-13e^{-3t} \end{pmatrix}$$

$$\overline{\overline{e^{at}}} * \overline{\overline{B}}\overline{f}(t) = \begin{pmatrix} -2e^{-2t}+3e^{-3t} & -6e^{-2t}+6e^{-3t} \\ e^{-2t}-e^{-3t} & 3e^{-2t}-2e^{-3t} \end{pmatrix} u(t) * \begin{pmatrix} \sin(100t) \\ 0 \end{pmatrix} = \begin{pmatrix} -2e^{-2t}u(t)*\sin(100t)+3e^{-3t}u(t)*\sin(100t) \\ e^{-2t}u(t)*\sin(100t)-e^{-3t}u(t)*\sin(100t) \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{2e^{-2t}}{100}+\frac{2\cos(100t)}{100}+\frac{3e^{-3t}}{100}-\frac{3\cos(100t)}{100} \\ \frac{e^{-2t}}{100}-\frac{\cos(100t)}{100}-\frac{e^{-3t}}{100}+\frac{\cos(100t)}{100} \end{pmatrix} = \begin{pmatrix} -0.02e^{-2t}+0.03e^{-3t}-0.01\cos(100t) \\ 0.01e^{-2t}-0.01e^{-3t} \end{pmatrix}$$

$$\boxed{\Rightarrow x(t) = \begin{pmatrix} -34e^{-2t}+39e^{-3t} \\ 17e^{-2t}-13e^{-3t} \end{pmatrix} + \begin{pmatrix} -0.02e^{-2t}+0.03e^{-3t}-0.01\cos(100t) \\ 0.01e^{-2t}-0.01e^{-3t} \end{pmatrix} = \begin{pmatrix} -34.02e^{-2t}+39.03e^{-3t}+0.01\cos(100t) \\ 17.01e^{-2t}-13.01e^{-3t} \end{pmatrix}}$$

This is the same result as in question 1.1

2. $$\overline{x}(t)=\underbrace{\overline{\overline{e^{at}}}\,\overline{x}(0)}_{ZIR}+\underbrace{\overline{\overline{e^{at}}} * \overline{\overline{B}}\overline{f}(t)}_{ZSR}$$ where $e^{At} = Laplace^{-1}\{\Phi(s)\}$.

$$e^{At} = Laplace^{-1}\left\{\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{pmatrix}\right\} = Laplace^{-1}\left\{\begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+1}-\frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{pmatrix}\right\} =$$

$$= \begin{pmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{pmatrix} u(t)$$

$$e^{At}x(0) = u(t)\begin{pmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3e^{-t}-2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\overline{\overline{e^{at}}} * \overline{\overline{B}}\overline{f}(t) = \begin{pmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{pmatrix} u(t) * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u(t) \\ \delta(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{pmatrix} u(t) * \begin{pmatrix} u(t)+\delta(t) \\ \delta(t) \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-t}u(t)*u(t)+e^{-t}u(t)*\delta(t)+e^{-t}u(t)*\delta(t)-e^{-2t}u(t)*\delta(t) \\ e^{-2t}u(t)*\delta(t) \end{pmatrix} =$$

$$= \begin{pmatrix} (1-e^{-t})+e^{-t}+e^{-t}-e^{-2t} \\ e^{-2t} \end{pmatrix} u(t) = \begin{pmatrix} 1+e^{-t}-e^{-2t} \\ e^{-2t} \end{pmatrix} u(t)$$

$$\boxed{\Rightarrow x(t) = \begin{pmatrix} 3e^{-t}-2e^{-2t} \\ 2e^{-2t} \end{pmatrix} u(t) + \begin{pmatrix} 1+e^{-t}-e^{-2t} \\ e^{-2t} \end{pmatrix} u(t) = \begin{pmatrix} 1+4e^{-t}-3e^{-2t} \\ 3e^{-2t} \end{pmatrix} u(t)}$$

This is the same result as in question 1.1

Question 3

$$\bar{Y}(s) = C\Phi(s)X(0) + [C\Phi(s)B + D]V(s) = C\{\Phi(s)[X(0)+BF(s)]\} + DF(s)$$

$$\Phi(s) = (sI-A)^{-1} = \begin{pmatrix} s+1 & -1 \\ 1 & s+1 \end{pmatrix}^{-1} = \frac{1}{\underbrace{(s+1)^2+1}_{s^2+2s+2}}\begin{pmatrix} s+1 & 1 \\ -1 & s+1 \end{pmatrix} = \begin{pmatrix} \frac{s+1}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ \frac{-1}{s^2+2s+2} & \frac{s+1}{s^2+2s+2} \end{pmatrix}$$

$$X(0)+BF(s) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/s \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{s+1}{s} \end{pmatrix}$$

$$\Rightarrow \Phi(s)[X(0)+BF(s)] = \begin{pmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{pmatrix}\begin{pmatrix} 2 \\ \frac{s+1}{s} \end{pmatrix} = \begin{pmatrix} \frac{2(s+1)}{(s+1)^2+1} + \frac{s+1}{s((s+1)^2+1)} \\ \frac{-2}{(s+1)^2+1} + \frac{(s+1)^2}{s((s+1)^2+1)} \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{2s^2+3s+1}{s\left((s+1)^2+1\right)} \\ \dfrac{s^2+1}{s\left((s+1)^2+1\right)} \end{pmatrix}$$

$$\Rightarrow C\left\{\Phi(s)\left[X(0)+BF(s)\right]\right\} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{2s^2+3s+1}{s\left((s+1)^2+1\right)} \\ \dfrac{s^2+1}{s\left((s+1)^2+1\right)} \end{pmatrix} = \frac{3s^2+3s+2}{s\left((s+1)^2+1\right)}$$

$$\Rightarrow Y(s) = C\left\{\Phi(s)\left[X(0)+BF(s)\right]\right\} + DF(s) = \frac{3s^2+3s+2}{s\left((s+1)^2+1\right)} + \frac{1}{s} = \frac{4s^2+5s+4}{s\left((s+1)^2+1\right)}$$

Using partial fractions and clearing fractions we get:

$$Y(s) = \frac{2}{s} + \frac{2s+1}{(s+1)^2+1^2} = \frac{2}{s} + 2\frac{s+1}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$$

and so:

$$y(t) = Laplace^{-1}\left\{Y(s)\right\} = \left(2+e^{-t}\cos t - e^{-t}\sin t\right)u(t)$$

Question 4

$$\bar{x}(t) = \underbrace{\bar{\bar{e^{at}}}\,\bar{x}(0)}_{ZIR} + \underbrace{\bar{\bar{e^{at}}} * \bar{\bar{B}}\bar{f}(t)}_{ZSR} \text{ where } e^{At} = Laplace^{-1}\left\{\Phi(s)\right\}$$

$$e^{At} = Laplace^{-1}\left\{\begin{pmatrix} \dfrac{s+1}{(s+1)^2+1^2} & \dfrac{1}{(s+1)^2+1^2} \\ \dfrac{-1}{(s+1)^2+1^2} & \dfrac{s+1}{(s+1)^2+1^2} \end{pmatrix}\right\} = \begin{pmatrix} e^{-t}\cos(t)u(t) & e^{-t}\sin(t)u(t) \\ -e^{-t}\sin(t)u(t) & e^{-t}\cos(t)u(t) \end{pmatrix}$$

$$e^{At}X(0) = \begin{pmatrix} e^{-t}\cos(t)u(t) & e^{-t}\sin(t)u(t) \\ -e^{-t}\sin(t)u(t) & e^{-t}\cos(t)u(t) \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-t}\cos(t)u(t) + e^{-t}\sin(t)u(t) \\ e^{-t}\cos(t)u(t) - 2e^{-t}\sin(t)u(t) \end{pmatrix}$$

$$e^{At} * BF(s) = \begin{pmatrix} e^{-t}\cos(t)u(t) & e^{-t}\sin(t)u(t) \\ -e^{-t}\sin(t)u(t) & e^{-t}\cos(t)u(t) \end{pmatrix} * \begin{pmatrix} 0 \\ u(t) \end{pmatrix} = \begin{pmatrix} e^{-t}\sin(t)u(t)*u(t) \\ e^{-t}\cos(t)u(t)*u(t) \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{\cos\left(\frac{\pi}{2}-\phi\right)}{\sqrt{2}} - \dfrac{e^{-t}}{\sqrt{2}}\cos\left(t-\frac{\pi}{2}-\phi\right) \\ \dfrac{\cos(-\phi)}{\sqrt{2}} - \dfrac{e^{-t}}{\sqrt{2}}\cos(t-\phi) \end{pmatrix} \text{ where } \phi = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$\Rightarrow x(t) = e^{at}x(0) + e^{at} * Bf(t) = \begin{pmatrix} \frac{1}{2}+\frac{3}{2}e^{-t}\cos(t)+\frac{1}{2}e^{-t}\sin(t) \\ \frac{1}{2}+\frac{1}{2}e^{-t}\cos(t)-\frac{3}{2}e^{-t}\sin(t) \end{pmatrix}$$

$$\boxed{y(t) = Cx(t) + Df(t) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}+\frac{3}{2}e^{-t}\cos(t)+\frac{1}{2}e^{-t}\sin(t) \\ \frac{1}{2}+\frac{1}{2}e^{-t}\cos(t)-\frac{3}{2}e^{-t}\sin(t) \end{pmatrix} + u(t) = \left(2+2e^{-t}\cos(t)-e^{-t}\sin t\right)u(t)}$$