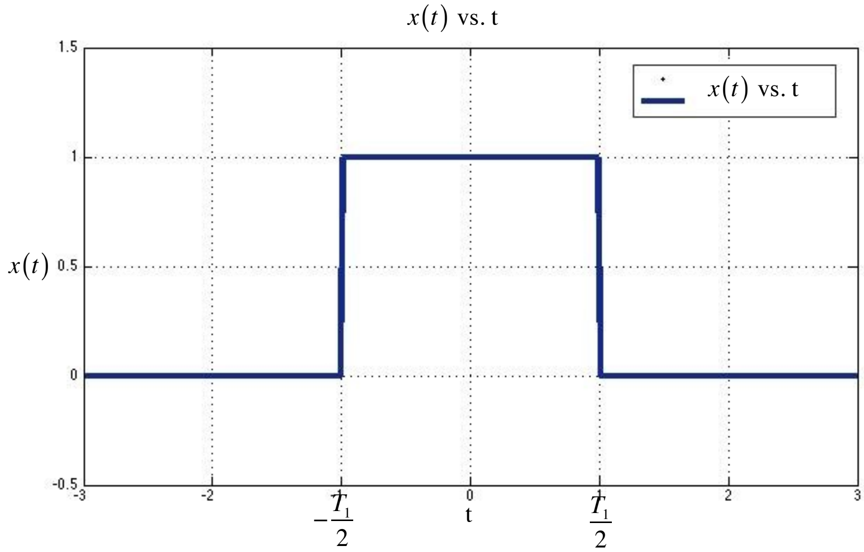
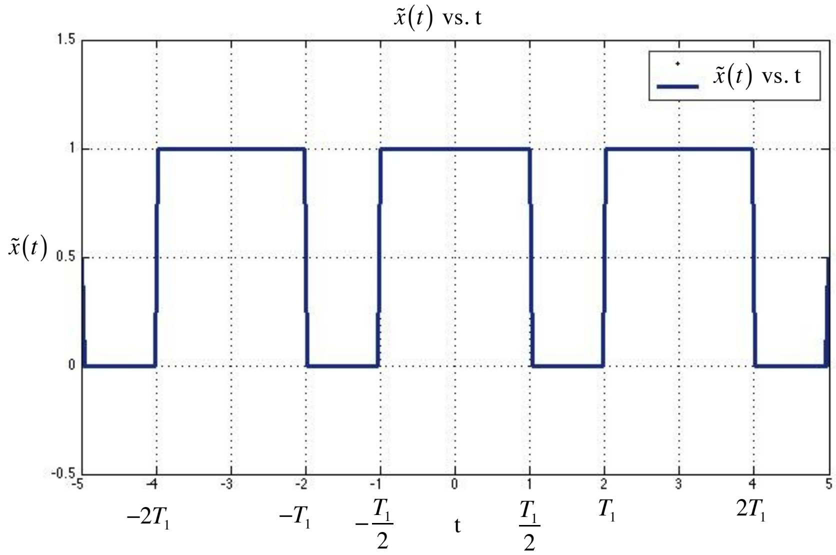
HW Assignment 4

Solution

Question 1

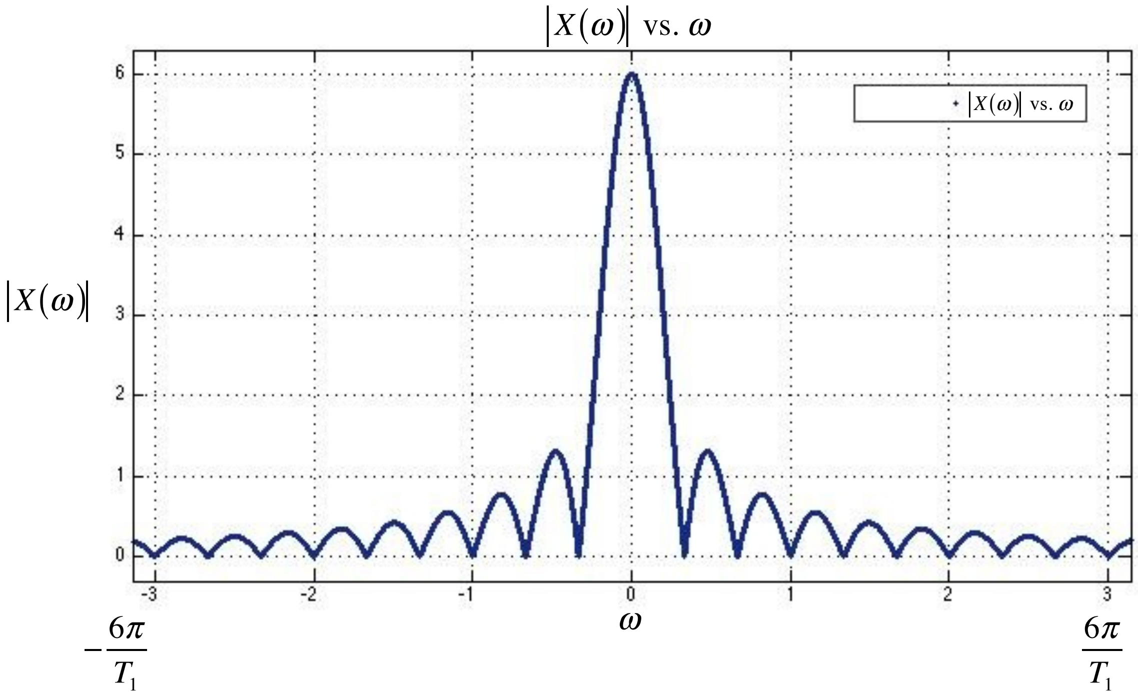
1.





2. $$X(\omega)=\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} e^{-i\omega t}\,dt=\frac{-1}{i\omega}\left[e^{-i\omega t}\right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}=\frac{1}{i\omega}\left(e^{i\frac{T_1}{2}\omega}-e^{-i\frac{T_1}{2}\omega}\right)=\frac{2}{\omega}\cdot\frac{1}{2i}\left(e^{i\frac{T_1}{2}\omega}-e^{-i\frac{T_1}{2}\omega}\right)=.$$

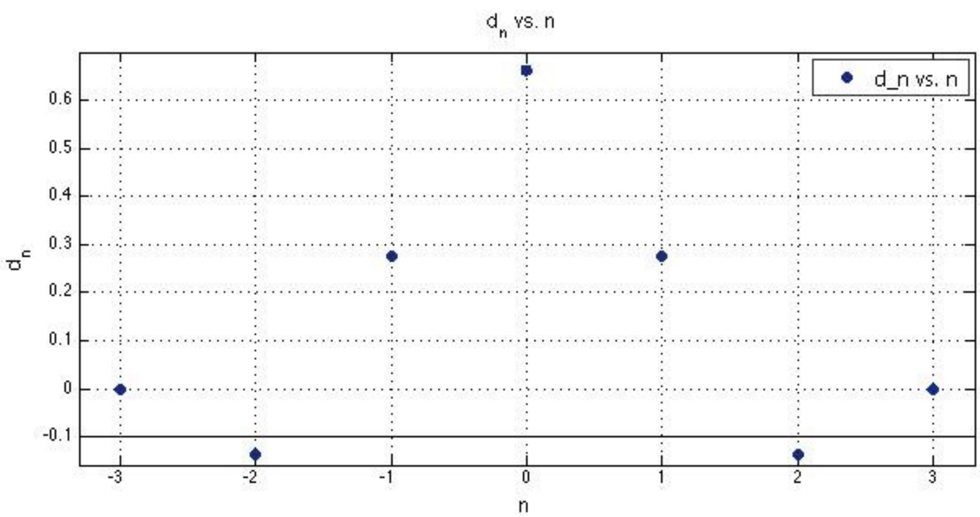
$$=\frac{2}{\omega}\sin\left(\frac{T_1}{2}\omega\right)=\frac{T_1}{\frac{T_1}{2}\omega}\sin\left(\frac{T_1}{2}\omega\right)=\boxed{T_1\sin c\left(\frac{T_1}{2}\omega\right)}$$



3. $d_n = \frac{2}{3T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} e^{-i\frac{4\pi n}{3T_1}t}\,dt = \frac{1}{-i2\pi n}\left[e^{-i\frac{4\pi n}{3T_1}t}\right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}} = \frac{1}{\pi n}\cdot\frac{1}{2i}\left(e^{i\frac{2\pi n}{3}} - e^{-i\frac{2\pi n}{3}}\right) = .$

$$= \frac{1}{\pi n}\sin\left(\frac{2\pi n}{3}\right) = \boxed{\frac{2}{3}\operatorname{sin}c\left(\frac{2\pi n}{3}\right)}$$

$$\boxed{x(t) = \underbrace{\frac{2}{3}}_{n=0} + \sum_{m=-\infty}^{\infty}\frac{\sqrt{3}}{2\pi(3m+1)}e^{i\frac{4\pi(3m+1)}{3T_1}t} - \sum_{m=-\infty}^{\infty}\frac{\sqrt{3}}{2\pi(3m+2)}e^{i\frac{4\pi(3m+2)}{3T_1}t}}$$ (don't have to calculate the series) $\boxed{d_3 = d_{-3} = 0\ ,\ d_2 = d_{-2} = -\frac{\sqrt{3}}{4\pi}\ ,\ d_1 = d_{-1} = \frac{\sqrt{3}}{2\pi}\ ,\ d_0 = \frac{2}{3}}$



4. $\boxed{d_n = \frac{1}{T_0} X\left(\omega = \frac{2\pi n}{T_0}\right) = \sin\frac{\left(2\pi n/3\right)}{\pi n}}$.

Question 2

For the following signals, calculate the Fourier transform and sketch the amplitude and phase:

1. $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t-5) e^{-i\omega t}\, dt = \boxed{e^{-i5\omega}}$.

   Amplitude: $\left|e^{-i5\omega}\right| = 1$.

   Phase: $\phi = \tan^{-1}\left(\frac{\mathrm{Im}\left(e^{-i5\omega}\right)}{\mathrm{Re}\left(e^{-i5\omega}\right)}\right) = \tan^{-1}\left(\frac{-\sin(5\omega)}{\cos(5\omega)}\right) = \tan^{-1}\left(-\tan(5\omega)\right) = -5\omega$ (π cycles)

2. $X(\omega) == \int_{0}^{\infty} e^{-i\omega t} e^{(-1+2i)t}\, dt = \int_{0}^{\infty} e^{(-1+(2-\omega)i)t}\, dt = \frac{1}{(-1+(2-\omega)i)}\left[e^{(-1+(2-\omega)i)t}\right]_0^{\infty} = \boxed{\frac{1}{1+(\omega-2)i}}$.

   Amplitude: $|X(\omega)| = \sqrt{\left(\frac{1}{\left(1+(\omega-2)^2\right)^2}\right) + \left(\frac{(\omega-2)^2}{\left(1+(\omega-2)^2\right)^2}\right)} = \frac{1}{\sqrt{1+(\omega-2)^2}}$.

   Phase: $\phi = \tan^{-1}\left(\frac{\mathrm{Im}(X(\omega))}{\mathrm{Re}(X(\omega))}\right) = \tan^{-1}\left(\frac{2-\omega}{1}\right) = -\tan^{-1}(\omega-2)$

3.
   - $X_1(\omega) = \int_{-1/2}^{1/2} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-1/2}^{1/2} = \frac{1}{j\omega}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) = \boxed{\mathrm{sinc}\left(\frac{\omega}{2}\right)}$.
   - $X_2(\omega) = \int_{-\pi}^{\pi} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-\pi}^{\pi} = \frac{1}{j\omega}\left(e^{j\pi\omega} - e^{-j\pi\omega}\right) = \frac{2}{\omega}\sin(\pi\omega) = \boxed{2\pi\mathrm{sinc}(\pi\omega)}$
   - $x_1(t)$ width is smaller than $x_2(t)$ width (in time) $\rightarrow X_1(\omega)$ band width is larger than $X_2(\omega)$ band width (time-frequency uncertainty).

Question 3

1. $fourier\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = fourier\left\{x(t) * H(t)\right\} = fourier\left\{x(t)\right\} \cdot fourier\left\{H(t)\right\}$.

$$\left\{\begin{aligned} & fourier\{H(t)\} = \lim_{a\to 0} fourier\left\{e^{-at} H(t)\right\} = \lim_{a\to 0}\left(\frac{1}{a+j\omega}\right) = \lim_{a\to 0}\left(\frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}\right) = \\ & = \lim_{a\to 0}\left(\frac{a}{a^2+\omega^2}\right) + \lim_{a\to 0}\left(-\frac{j\omega}{a^2+\omega^2}\right) \rightarrow \pi\delta(\omega) + \frac{1}{j\omega} \end{aligned}\right\}$$

$$\Rightarrow fourier\{x(t)\}\cdot fourier\{H(t)\}=X(\omega)\cdot\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]=\frac{1}{j\omega}X(\omega)+\pi X(\omega)\delta(\omega)$$

since $\delta(\omega)=\begin{cases}\infty & \omega=0\\ 0 & otherwise\end{cases}$ we can say that $\pi X(\omega)\delta(\omega)=\pi X(0)\delta(\omega)$ and therefore:

$$\boxed{fourier\left\{\int_{-\infty}^{t}x(\tau)d\tau\right\}=\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)}$$

2. $$Fourier\{f(at)\}=\int_{-\infty}^{\infty}f(at)e^{-j\omega t}\,dt \underbrace{=}_{\substack{t'=at\\ dt=\frac{1}{a}dt'}} \begin{cases}\frac{1}{|a|}\int_{-\infty}^{\infty}f(t')e^{-j\omega\frac{t'}{a}}dt' & a>0\Rightarrow\begin{cases}a=|a|\\ sign\{t'\}=sign\{t\}\end{cases}\\ -\frac{1}{|a|}\int_{\infty}^{-\infty}f(t')e^{-j\omega\frac{t'}{a}}dt' & a<0\Rightarrow\begin{cases}a=-|a|\\ sign\{t'\}=-sign\{t\}\end{cases}\end{cases}=$$

$$=\begin{cases}\frac{1}{|a|}\int_{-\infty}^{\infty}f(t')e^{-j\omega\frac{t'}{a}}dt'=\frac{1}{|a|}F\left(\frac{\omega}{a}\right) & a>0\\ \frac{1}{|a|}\int_{-\infty}^{\infty}f(t')e^{-j\omega\frac{t'}{a}}dt'=\frac{1}{|a|}F\left(\frac{\omega}{a}\right) & a<0\end{cases}=\boxed{\frac{1}{|a|}F\left(\frac{\omega}{a}\right)}$$

3. $$Fourier\{f(t)\}=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\left(\mathrm{Re}\{f(t)\}+i\,\mathrm{Im}\{f(t)\}\right)\left(\cos(\omega t)-i\sin(\omega t)\right)dt=$$

$$\underbrace{\int_{-\infty}^{\infty}\left(\mathrm{Re}\{f(t)\}\cos(\omega t)+\mathrm{Im}\{f(t)\sin(\omega t)\}\right)dt}_{\substack{\triangleq\mathrm{Re}\{F(\omega)\}\in\mathbb{R}\\ (\text{an integral over } Real-valued\ function)}}+i\underbrace{\int_{-\infty}^{\infty}\left(\mathrm{Im}\{f(t)\}\cos(\omega t)-\mathrm{Re}\{f(t)\}\sin(\omega t)\right)dt}_{\substack{\triangleq\mathrm{Im}\{F(\omega)\}\in\mathbb{R}\\ (\text{an integral over } Real-valued\ function)}}=$$

$$\mathrm{Re}\{F(\omega)\}+i\,\mathrm{Im}\{F(\omega)\}.$$

$$Fourier\left\{f(t)^*\right\}=\int_{-\infty}^{\infty}f(t)^*e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\left(\mathrm{Re}\{f(t)\}-i\,\mathrm{Im}\{f(t)\}\right)\left(\cos(\omega t)-i\sin(\omega t)\right)dt=$$

$$=\int_{-\infty}^{\infty}\left(\mathrm{Re}\{f(t)\}\cos(\omega t)-\mathrm{Im}\{f(t)\sin(\omega t)\}\right)dt-i\int_{-\infty}^{\infty}\left(\mathrm{Im}\{f(t)\}\cos(\omega t)+\mathrm{Re}\{f(t)\}\sin(\omega t)\right)dt\underset{\substack{-\sin(\omega t)=\sin(-\omega t)\\ \cos(\omega t)=\cos(-\omega t)}}{=}$$

$$\underbrace{\int_{-\infty}^{\infty}\left(\mathrm{Re}\{f(t)\}\cos(-\omega t)+\mathrm{Im}\{f(t)\sin(-\omega t)\}\right)dt}_{\triangleq\mathrm{Re}\{F(-\omega)\}}-i\underbrace{\int_{-\infty}^{\infty}\left(\mathrm{Im}\{f(t)\}\cos(-\omega t)-\mathrm{Re}\{f(t)\}\sin(-\omega t)\right)dt}_{\triangleq\mathrm{Im}\{F(-\omega)\}\in\mathbb{R}}=$$

$$\boxed{=\mathrm{Re}\{F(-\omega)\}-i\,\mathrm{Im}\{F(-\omega)\}=F(-\omega)^*}$$

Question 4

The given function can be written as: $X(\omega)=H(\omega+4)+2H(\omega+1)-2H(\omega-1)-H(\omega-4)$.

Using the dual solution for question 1a and the frequency-shift property:

- $u(t)\xrightarrow{Fourier}\pi\delta(\omega)+\frac{1}{j\omega}$.

$$u(\omega)\xrightarrow{Fourier^{-1}}\frac{1}{2\pi}u(-t)=\frac{1}{2}\delta(-t)-\frac{1}{j2\pi t}=\frac{1}{2}\delta(t)-\frac{1}{j2\pi t}$$

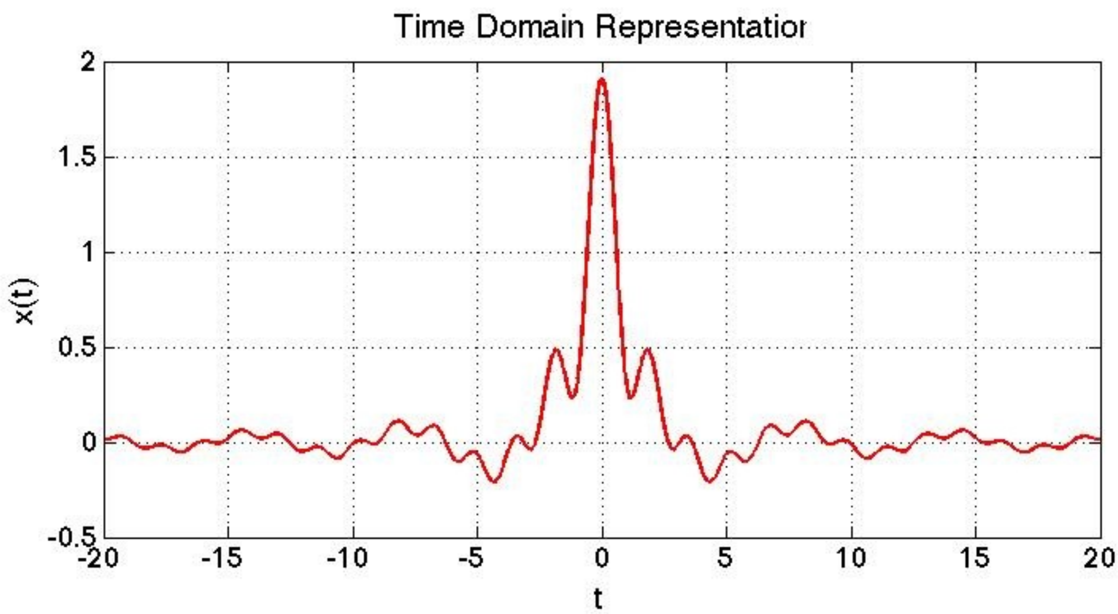
- $F(\omega-\omega_0)\xrightarrow{Fourier^{-1}} e^{j\omega_0 t}f(t)$.

We get:

$$x(t)=\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)e^{-i4t}+2\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)e^{-it}-2\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)e^{it}-\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)e^{i4t}=$$

$$=-\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)\left(e^{i4t}-e^{-i4t}\right)-2\left(\frac{1}{2}\delta(t)-\frac{1}{i2\pi t}\right)\left(e^{it}-e^{-it}\right)=\left[-i\delta(t)+\frac{1}{\pi t}\right]\cdot\left[\sin(4t)+2\sin(t)\right]\underset{\underbrace{\delta(0)\sin(0)}_{0}=0}{=}$$

$$\boxed{=\frac{1}{\pi t}\left[\sin(4t)+2\sin(t)\right]}$$



Question 5

1. $$X(\omega)=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\left(1+\cos(at)\right)e^{-i\omega t}\,dt=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{1}{2}\left(2e^{-i\omega t}+e^{-i(\omega+a)t}+e^{-i(\omega-a)t}\right)dt=$$

$$=\frac{1}{2}\left[-\frac{2}{i\omega}e^{-i\omega t}-\frac{1}{i(\omega+a)}e^{-i(\omega+a)t}-\frac{1}{i(\omega-a)}e^{-i(\omega-a)t}\right]_{-\frac{\pi}{a}}^{\frac{\pi}{a}}=$$

$$=\frac{1}{2}\left[\frac{2}{i\omega}\left(e^{i\frac{\pi}{a}t}-e^{-i\frac{\pi}{a}t}\right)+\frac{1}{i(\omega+a)}\left(e^{i(\omega+a)\frac{\pi}{a}}-e^{-i(\omega+a)\frac{\pi}{a}}\right)+\frac{1}{i(\omega-a)}\left(e^{i(\omega-a)\frac{\pi}{a}}-e^{-i(\omega-a)\frac{\pi}{a}}\right)\right]=$$

$$=\frac{2}{\omega}\sin\left(\frac{\pi}{a}\omega\right)+\frac{1}{\omega+a}\sin\left(\frac{\pi}{a}\omega+\pi\right)+\frac{1}{\omega-a}\sin\left(\frac{\pi}{a}\omega-\pi\right)=$$

$$\boxed{\frac{2\pi}{a}\sin c\left(\frac{\pi}{a}\omega\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega+\pi\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega-\pi\right)}.$$

2. Using $(-jt)f(t)\overset{fourier}{\rightarrow}\frac{d}{d\omega}F(\omega) \Rightarrow t\cdot f(t)\overset{fourier}{\rightarrow} j\frac{d}{d\omega}F(\omega)$ we get:

$$fourier\{tx(t)\}=j\frac{d}{d\omega}\left[\frac{2\pi}{a}\sin c\left(\frac{\pi}{a}\omega\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega+\pi\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega-\pi\right)\right]=.$$

$$\boxed{=j\left[\frac{2\pi}{a\omega}\cos\left(\frac{\pi}{a}\omega\right)-\frac{2}{\omega^2}\sin\left(\frac{\pi}{a}\omega\right)+\frac{1}{(a+\omega)^2}\sin\left(\frac{\pi}{a}\omega\right)-\frac{\pi}{a(a+\omega)}\cos\left(\frac{\pi}{a}\omega\right)+\frac{1}{(a-\omega)^2}\sin\left(\frac{\pi}{a}\omega\right)-\frac{\pi}{a(a-\omega)}\cos\left(\frac{\pi}{a}\omega\right)\right]}$$

3. $$\boxed{\frac{d}{dx}x(t)=\begin{cases}-a\sin(at) & |t|<\pi/a\\ 0 & otherwise\end{cases}}.$$

using $\frac{d}{dt}x(t)\overset{Fourier}{\rightarrow} j\omega F(\omega)$ we get:

$$\boxed{fourier\left\{\frac{d}{dt}x(t)\right\}=j\omega\left[\frac{2\pi}{a}\sin c\left(\frac{\pi}{a}\omega\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega+\pi\right)+\frac{\pi}{a}\sin c\left(\frac{\pi}{a}\omega-\pi\right)\right]}$$

Question 6

1. $Fourier^{-1}\{f(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{j\omega t}\,dt=\frac{1}{2\pi}F(-\omega)$.
2. $Fourier^{3}\{f(t)\}=2\pi\int_{-\infty}^{\infty}f(-t)e^{-j\omega t}\,dt=-2\pi\int_{\infty}^{-\infty}f(t)e^{j\omega t}\,dt=2\pi\int_{-\infty}^{\infty}f(t)e^{j\omega t}\,dt=2\pi F(-\omega)$.
3. $Fourier^{4}\{f(t)\}=2\pi\int_{-\infty}^{\infty}F(-\omega)e^{-j\omega t}\,d\omega=-2\pi\int_{\infty}^{-\infty}F(\omega)e^{j\omega t}\,d\omega=2\pi\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}\,d\omega=4\pi^2 f(t)$