HW Assignment 9

Solution

Question 1

A is a 2x2 matrix $\rightarrow$ we want to find β_0, β_1 such that $e^{At} = \beta_1 A + \beta_0 I$.

Eigen values:

$$|A - \lambda I| = \begin{vmatrix} \sigma - \lambda & \omega \\ -\omega & \sigma - \lambda \end{vmatrix} = (\sigma - \lambda)^2 + \omega^2 = \lambda^2 - 2\sigma\lambda + \sigma^2 + \omega^2$$

$$\lambda_{1,2} = \frac{2\sigma \pm \sqrt{4\sigma^2 - 4\sigma^2 - 4\omega^2}}{2} = \sigma \pm j\omega$$

$$\Downarrow$$

$$\begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} e^{(\sigma + j\omega)t} \\ e^{(\sigma - j\omega)t} \end{pmatrix} = \begin{pmatrix} 1 & \sigma + j\omega \\ 1 & \sigma - j\omega \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & \sigma + j\omega \\ 1 & \sigma - j\omega \end{pmatrix}^{-1} = \frac{1}{2j\omega} \begin{pmatrix} -\sigma + j\omega & \sigma + j\omega \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{(\sigma + j\omega)t} \\ e^{(\sigma - j\omega)t} \end{pmatrix} =$$

$$= \frac{1}{2j\omega} \begin{pmatrix} (-\sigma + j\omega) e^{(\sigma + j\omega)t} + (\sigma + j\omega) e^{(\sigma - j\omega)t} \\ e^{(\sigma + j\omega)t} - e^{(\sigma - j\omega)t} \end{pmatrix} = \frac{1}{2j\omega} \begin{pmatrix} -\sigma e^{\sigma t} \left(e^{j\omega t} - e^{-j\omega t}\right) + j\omega e^{\sigma t} \left(e^{j\omega t} + e^{-j\omega t}\right) \\ e^{\sigma t} \left(e^{j\omega t} - e^{-j\omega t}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) + e^{\sigma t} \cos(\omega t) \\ \frac{1}{\omega} e^{\sigma t} \sin(\omega t) \end{pmatrix}$$

$$e^{At} = \beta_0 I + \beta_1 A = \left(-\frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) + e^{\sigma t} \cos(\omega t)\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\omega} e^{\sigma t} \sin(\omega t) \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) + e^{\sigma t} \cos(\omega t) + \frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) & e^{\sigma t} \sin(\omega t) \\ -e^{\sigma t} \sin(\omega t) & -\frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) + e^{\sigma t} \cos(\omega t) + \frac{\sigma}{\omega} e^{\sigma t} \sin(\omega t) \end{pmatrix} =$$

$$\boxed{= \begin{pmatrix} e^{\sigma t} \cos(\omega t) & e^{\sigma t} \sin(\omega t) \\ -e^{\sigma t} \sin(\omega t) & e^{\sigma t} \cos(\omega t) \end{pmatrix}}$$

Question 2

$$\exp\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}t\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}t\right)^n = I + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}t + \frac{1}{2}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^2 t^2 + \dots.$$

$$\underbrace{=}_{\substack{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} is \\ a\ Diagonal \\ matrix}} \begin{pmatrix} I + At + A^2t^2 + \dots & 0 \\ 0 & I + Bt + \frac{1}{2}B^2t^2 + \dots \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty}\frac{A^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{B^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{At} & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

Question 3

1. $$H(s) = \frac{Y(s)}{V(s)} = \frac{1/s^2}{1 + a/s + b/s^2} = \frac{1}{s^2 + as + b}$$ (See exercise 7)

2. $$\left(s^2 + as + b\right)Y(s) = V(s)$$
$$Laplace^{-1}\downarrow \quad .$$
$$\ddot{y}(t) + a\dot{y}(t) + by(t) = v(t)$$

3. $$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = v(t).$$

Define:

$$x_1(t) = y(t) \quad ; \quad x_2(t) = \dot{x}_1(t) = \dot{y}(t)$$
$$\overline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

The new diff. eq:

$$\dot{x}_2(t) + 5x_2(t) + 6x_1(t) = v(t)$$

$$\Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = v(t) - 5x_2(t) - 6x_1(t) \end{cases}$$ and so we get:

$$\boxed{\frac{d}{dt}\overline{x} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\overline{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}v(t)}$$

4. $$y(t) = x_1(t) = \begin{pmatrix} 1 & 0 \end{pmatrix}\overline{x}(t)$$