Tutorials on Quantum Mechanics II

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1. **Tensor Products and entangled states**

Consider a two dimensional Hilbert space $\mathcal{H}$ describing a spin-$\frac{1}{2}$ particle. Let $|+\rangle, |-\rangle$ be an ONB. Consider now the tensor product $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ describing two spin-$\frac{1}{2}$ particles. Show

A general state

$$|\psi\rangle = \alpha| + -\rangle + \beta| - +\rangle,$$

with $\alpha, \beta \neq 0$ cannot be written as $|v\rangle \otimes |w\rangle$ for two vectors $|v\rangle \in \mathcal{H}^{(1)}$ and $|w\rangle \in \mathcal{H}^{(2)}$.

2. **The C-NOT gate**

Consider a two dimensional Hilbert space $\mathcal{H}$ describing a spin-$\frac{1}{2}$ particle. Let $|0\rangle, |1\rangle$ be an ONB. Consider now the tensor product $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ describing two spin-$\frac{1}{2}$ particles. The C-NOT operator is given by

$$U_{C-NOT} = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x,$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a) Write down a matrix representation for this operator.

b) Find the eigenstates and eigenvalues of U_{C-NOT}. Are the eigenstates entangled?

c) Give an example of a state that is not entangled, but mapped to an entangled state by U_{C-NOT}.

d) Show that the action of U_{C-NOT} on a state $|x\rangle \otimes |y\rangle$ $x, y \in \{0, 1\}$ can be written as

$$U_{C-NOT} = (|x\rangle \otimes |y\rangle) = |x\rangle \otimes |x \oplus y\rangle,$$

where $\oplus$ denotes addition modulo 2.

3. **Time evolution**

Consider a system with a two dimensional Hilbert space. Denote the basis of the two-dimensional Hilbert space by $|0\rangle, |1\rangle$. In this basis, the Hamilton-operator is given by

$$\hat{H} = \frac{\omega}{2}\sigma_y,$$

where

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a) Compute the time evolved state $|\psi(t)\rangle$ from $|\psi(0)\rangle = |0\rangle$.

b) Determine the (time dependent) probabilities to measure $|0\rangle$ and to measure $|1\rangle$.

c) Suppose you measure in this basis at very small time intervals. What will happen with the probability that the system goes to the state $|1\rangle$?