# Design of State-feedback Controller by Pole Placement for a Coupled Set of Inverted Pendulums

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***Abstract*** *–This paper presents controller design based on state-space pole-placement method for a non-linear dynamic system described by a double-parallel inverted pendulum i.e. two identical inverted pendulums on carts coupled by a spring. The state space model of this multivariable, dynamic, unstable and strongly coupled system developed elsewhere is taken and controller design task using pole placement method is carried out. The designed controller is tested on a real time control system. The experimental results verify the significance and performance of the designed controller both in terms of simulations based experiments and on a real laboratory scale system.*

***Keywords*** *–state space; pole-placement; double-parallel inverted pendulum*

## I. INTRODUCTION

The inverted pendulum has been used as a classical control example for nearly half a century because of its nonlinear, unstable, and non-minimum-phase characteristics [1]. Control methods and strategies of inverted pendulum have been widely used in the study of pitch dynamics of a rocket, robot system, bipedal dynamic walking in biomechanics, wheeled motion, and balancing mechanism [2]. In addition, the inverted pendulum has always been adopted as a classical control example to test the advantages and disadvantages of various control algorithms such as PID control, state feedback control, fuzzy control, neural network control, adaptive control and genetic algorithms [3].

Concerning the control of inverted pendulum, there are two main control problems: (i) swing-up control of the pendulum from the downward position to the upward unstable equilibrium position by considering the traveling position of the cart; (ii) stabilizing control of the pendulum at the upward unstable equilibrium position including cart travel control [4].

In the study of inverted pendulums, new pendulum is hinged on the original system one after another to increase the series and complexity of the system, such as double inverted pendulum, triple inverted pendulum, etc. Another kind of novel double inverted pendulums with a small cart and two different pendulums also emerges whose control objective is to keep both inverted pendulums stable in vertical plane. However, most of these systems are SISO systems and their bonding forces are not strong enough.

Meanwhile, many algorithm studies for the inverted pendulum systems are limited to the theoretical modeling and simulation duo to the high price of inverted pendulum devices. Inadequate consideration to the practical constrained problems also makes it difficult for the algorithm to be achieved in actual application.

In this paper, the model for a coupled set of inverted pendulums is proposed which is different from a single-stage inverted pendulum in modeling, because there is an interaction between two pendulums. Due to this coupling force, some control algorithms in classical theory based on SISO systems and relatively accurate models are difficult to be applied in this double-parallel inverted pendulum. In view of this coupled model, the state-space pole assignment algorithm in modern control theory is put forward, which is based on modern mathematical tools and extends the concepts in classical control theory to MIMO systems.

The paper is organized as follows. The next section provides the double-parallel inverted pendulum model. Section 3 introduces the pole-placement algorithm and presents the designed controller. The simulation and experimental results are given in Section 4 and Section 5concludes the paper.

## II. MATHEMATICAL MODEL OF DOUBLE-PARALLEL INVERTED PENDULUM

As shown in Fig.1 [3], the double-parallel inverted pendulum system considered here consists of two straight line rails, two carts moving on the rails respectively and a spring connecting two pendulums. In the same vertical plane with the rails, the two pendulums can rotate freely around their own pivots.

Here, the parameter $\theta_1$, $\theta_2$ (rad) is separately the angle between the vertical line and the pendulum, whose counterclockwise direction is positive and clockwise direction is negative. A, B is the point that spring fixed point $a$, $b$ projects on line rail respectively. $w$ is the distance between two projection points in rail direction. $h$ is the distance between spring fixed point and pendulum pivot. $L_S$ is the length of the spring after stretching. $d$ is the vertical distance between two rails. The position of two carts from the rail origin is denoted as $X_1$, $X_2$ respectively, and is positive when the cart locates on the right side of the rail origin. The driving force applied horizontally to the carts is denoted as $F_1$, $F_2$ (N) separately. Also, note that friction is neglected in this model.

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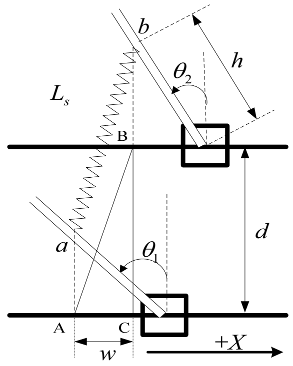


Fig.1 double-parallel inverted pendulum

Table.1 detailed value

| Notations | Physical Meaning | Value/Unit |
|---|---|---|
| $h$ | the distance between spring fixed point and pendulum pivot | 0.40m |
| $L$ | original length of spring | 0.181 |
| $d$ | the vertical distance between two rails | 0.187 |
| $l_1$ , $l_2$ | the length from the pivot to the center of mass of the pendulum | 0.25m |
| $M_1$ , $M_2$ | The mass of the cart | 1.096Kg |
| $m_1$ , $m_2$ | the mass of the pendulum | 0.109Kg |
| $J_1$ , $J_2$ | the moment of inertia of the pendulum | 0.009083Kg·m2 |
| $k$ | The stiffness of spring | 24.5/m |
| g | The acceleration of gravity | 9.8 $m/s^2$ |

According to Fixed Axis Rotation Law, the following dynamic equation of such a double-parallel inverted pendulum system can be obtained [3]:

$$w = (X_2 - h\sin\theta_2) - (X_1 - h\sin\theta_1) \quad (1)$$

$$F_X = k(L_s - L)\frac{w}{L_s} = kw\left(1 - \frac{L}{\sqrt{d^2 + w^2}}\right) \quad (2)$$

Where $F_X$ is the spring force decomposed in rail direction.

$$J_1\ddot{\theta}_1 = l_1 m_1 \ddot{X}_1 \cos\theta_1 + l_1 m_1 g \sin\theta_1 - hF_X \cos\theta_1 \quad (3)$$

$$J_2\ddot{\theta}_2 = l_2 m_2 \ddot{X}_2 \cos\theta_2 + l_2 m_2 g \sin\theta_2 + hF_X \cos\theta_2 \quad (4)$$

From the mathematical perspective, it is obviously a strongly coupled nonlinear system with multiple inputs and variables. The cart driving force is applied to maintain the upright position of two pendulums in the vertical direction.

Non-linear systems occur extensively in the field of dynamic analysis and the control of such systems presents the designer with a difficult challenge. The central idea of feedback linearization is to algebraically transform the non-linear system into a linear form, in order to cancel the non-linear dynamics in the closed-loop [5]. Therefore, the model can be linearized near the equilibrium point (in the range of $|\theta_1| < 0.2$ , $|\theta_2| < 0.2$ ). Suppose $\theta_1 \approx 0$ , $\theta_2 \approx 0$ , and $\cos\theta_1 = 1$ , $\cos\theta_2 = 1$ , $\sin\theta_1 = \theta_1$ , $\sin\theta_2 = \theta_2$ can be obtained by approximate treatment.

Then, the controller of double-parallel inverted pendulum can be designed in line with the linearized model [6]. The simplified dynamic equations are supposed as follows:

$$F_X \approx kw\left(1 - \frac{L}{d}\right) \approx k\left(1 - \frac{L}{d}\right)[(X_2 - X_1) + h(\theta_1 - \theta_2)] \quad (5)$$

$$J_1\ddot{\theta}_1 \approx l_1 m_1 u_{X1} + l_1 m_1 g\theta_1 - hF_X \quad (6)$$

$$J_2\ddot{\theta}_2 \approx l_2 m_2 u_{X2} + l_2 m_2 g\theta_2 + hF_X \quad (7)$$

Where $u_{X1} = \ddot{X}_1$ , $u_{X2} = \ddot{X}_2$ is the acceleration command input of the system.

Finally, the state space model can be obtained as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0_{4\times1} \\ 0_{4\times1} & b_2 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{X2} \end{bmatrix} \quad (8)$$

Where $x_i = [X_i \ \theta_i \ \dot{X}_i \ \dot{\theta}_i]^T$ $\quad x = [x_1 \quad x_2]^T$

$u_X = [\ddot{X}_1 \quad \ddot{X}_2]^T$ $\qquad \beta = hk\left(1 - \frac{L}{d}\right)$

$$A_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \frac{\beta}{J_1} & \frac{l_1 m_1 g - h\beta}{J_1} & 0 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-\beta}{J_1} & \frac{h\beta}{J_1} & 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-\beta}{J_2} & \frac{h\beta}{J_2} & 0 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \frac{\beta}{J_2} & \frac{l_2 m_2 g - h\beta}{J_2} & 0 & 0 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{l_1 m_1}{J_1} \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{l_2 m_2}{J_2} \end{bmatrix}$$

## III. POLE-PLACEMENT STATE-FEEDBACK CONTROLLER DESIGN

The stability and various control performance indicators of a closed-loop linear time-invariant system largely depend on pole locations. Therefore, during control system design, the poles of the closed-loop system should be located in those positions which have rational and expected performance.

When the system is analyzed in a comprehensive way, state feedback can provide more control information. Therefore, state feedback has been widely applied in deriving optimal control law, inhibiting or eliminating the influence of disturbance and realizing decoupling control.

The pole-placement method is used to place the poles of closed- loop system in the desired positions by state feedback or output feedback. As the system performance is closely related to the pole positions, the pole-placement in system design is very important. There are two main steps to carry out. The first step is the placement or assignment of poles and the second step is the identification of the feedback gain matrix [7].

The sufficient and necessary condition of arbitrary closed-loop pole-placement by state feedback is that the system must be controllable [8].

*A. System Property Analysis*

After substituting the model parameters into equation (8), the state space model is as follows:

$$\begin{bmatrix}\ddot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}b_1 & 0_{4\times1}\\ 0_{4\times1} & b_2\end{bmatrix}\begin{bmatrix}u_{X1}\\ u_{X2}\end{bmatrix} \tag{9}$$

$$x_i=[X_i\ \theta_i\ \dot{X}_i\ \dot{\theta}_i]^T \qquad x=[x_1\quad x_2]^T \qquad u_X=[\ddot{X}_1\quad \ddot{X}_2]^T$$

$$A_{11}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 34.62 & 15.55 & 0 & 0\end{bmatrix} \quad A_{12}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -34.62 & 13.85 & 0 & 0\end{bmatrix}$$

$$A_{21}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -34.62 & 13.85 & 0 & 0\end{bmatrix} \quad A_{22}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 34.62 & 15.55 & 0 & 0\end{bmatrix}$$

$$b_1=\begin{bmatrix}0\\ 0\\ 1\\ 3\end{bmatrix} \qquad b_2=\begin{bmatrix}0\\ 0\\ 1\\ 3\end{bmatrix}$$

The system open loop poles are given below:

$S_1=-5.4222$ , $S_2=5.4222$ , $S_3=-1.3038$ , $S_4=1.3038$, $S_5=S_6=S_7=S_8=0$.

The system is unstable because there are two poles in the right-half S plane. According to linear control theory, if the rank of matrix $M=[B\ AB\ AB^2\ AB^3\ AB^4\ AB^5\ AB^6\ AB^7]$ is 8, then the system poles can be placed anywhere in the left-half complex plane [9]. From the system property, we can see that the system is both controllable and observable [10].

*B. Controller design and simulation*

The model (31) is used for the design of the linear controllers as follows. The control law is U=-KX, Where U is the control voltage, X is the state parameters and K is the state-feedback gain matrix [11]. Now the controller design task is accomplished considering the following two performance objectives [12].

Overshoot: $\sigma \le 0.05$ Regulation Time: $t_s \le 4.5\ s$

Poles are chosen according to the damping factor and natural frequency of the system. The error range is 2%, according to the following formula:

$$\sigma = e\frac{-\delta\pi}{\sqrt{1-\delta^2}} \tag{10}$$

$$t_s=\frac{4-In\sqrt{1-\delta^2}}{\delta w_n} \tag{11}$$

Calculated $\delta$ and $w_n$ should satisfy the following inequality: $\delta \ge 0.69$, $w_n \ge 1.39$, and the dominant poles $S_1$ and $S_2$ are obtained as $S_1=-1+j$, $S_2=-1-j$ by calculating the formula

$$S_1\ ,S_2=-w_n\delta \pm jw_n\sqrt{1-\delta^2} \tag{12}$$

Where $\delta=0.7$, $w_n=1.4$. Other six non-dominant poles are placed at p= $S_i$ (i = 3, 4, 5, 6, 7, 8), where $S_3$=-2+2j, $S_4$ = -2-2j, $S_5$ =-3+3j, $S_6$ = -3-3j, $S_7$ =-4+4j, and $S_8$ = -4-4j.

According to the above poles, the state-feedback gain matrix is given as below.

$$K=\begin{bmatrix}28.4769 & 12.1501 & 1.8701 & 2.7648 & -30.3567 & 8.7722 & -3.5845 & 0.3059\\ -21.5800 & 11.4979 & -3.5873 & 1.4572 & 19.2733 & 14.9451 & 0.3391 & 3.1655\end{bmatrix}$$

For the initial conditions

$x_0=[0.05\ \ 0.15\ \ 0\ \ 0\ \ -0.05\ \ -0.1\ \ 0\ \ 0]$, the simulation results are shown in Fig.2.

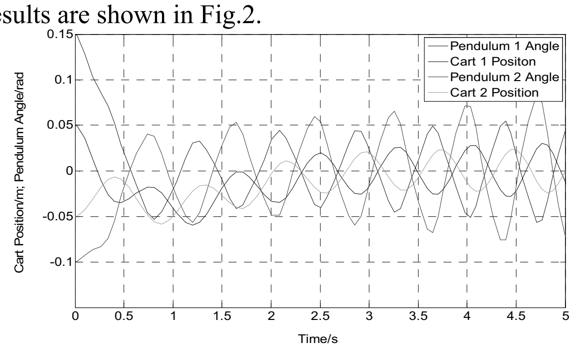


Fig.2 Simulation results with first placement of poles

From Fig.2 it can be seen the system is divergent. Generally speaking, the poles should be placed on left of the origin. The farther are the poles from the origin the quicker is the control action. But, it requires higher control force. So the poles should be replaced at some appropriate distance from the origin. The newly chosen poles are p= $S_i$ (i = 1, 2, 3, 4, 5, 6, 7, 8), where $S_1$=-1+2j, $S_2$=-1-2j, $S_3$ =-3+2j, $S_4$ = -3-2j, $S_5$ =-6+4j, $S_6$ = -6-4j, $S_7$ =-6+5j, and $S_8$ = -6-5j, so the corresponding state-feedback gain matrix is as follows

$$K=\begin{bmatrix}37.7311 & 29.6950 & -3.1682 & 6.4655 & -58.7869 & 27.4932 & -10.3414 & 3.4371\\ -38.0460 & 23.8797 & -7.5654 & 3.1190 & 21.9757 & 35.1021 & -5.9364 & 7.2360\end{bmatrix}$$

The simulation results under same initial conditions with replaced poles are shown in Fig.3.

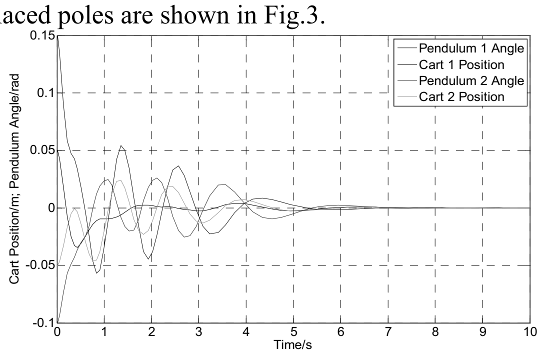


Fig.3 Simulation results with second placement of poles

Now from Fig.3 it can be seen that the system is stable, but the regulation time is a little longer. When the distances among desired non-dominant poles are too short, the error between desired non-dominant poles and actual non-dominant poles will be large because of the interaction among close non-dominant poles in calculation. On the other hand, if the distance among them is set too large, the error becomes small but it results in a high feed-back gain matrix which saturates the system. Therefore, in the design of such a higher-order system, many tests are needed to choose the best poles to meet the performance index completely. After many trials and experiments with different pole positions the system poles were finally placed as follows:

p= $S_i$ (i = 1,2,3,4,5,6,7,8), where $S_1$ =-1.5+4j, $S_2$ =-1.5-4j, $S_3$ =-3.5+2j, $S_4$ = -3.5-2j, $S_5$ =-6+4j, $S_6$ = -6-4j, $S_7$ =-6+4.5j, and $S_8$ = -6-4.5j.

The state-feedback gain matrix K of the system now is

$$K=\begin{bmatrix} 15.5575 & 44.7970 & -11.0454 & 9.3731 & -48.7105 & 29.2018 & -9.8515 & 3.9453 \\ -44.1769 & 26.3908 & -8.3075 & 3.4332 & 17.2278 & 42.2108 & -9.9158 & 8.9473 \end{bmatrix}.$$

The simulation results of the system now are shown in Fig.4.

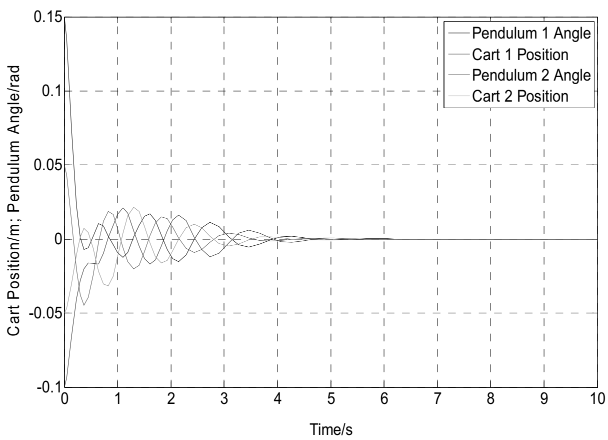


Fig.4 Simulation results with final placement of poles

From Fig.4, it can be seen that the state space pole-placement controller takes about 4.5 seconds to stabilize the system, and the overshoot is small. The above state-feedback can stabilize the system in given initial states. Although there exists some external interferences due to slight tilting of pendulums and the devastations of the carts from the baseline which will be considered in the real time control in the following section.

## IV. REAL-TIME CONTROL EXPERIMENTS

In the previous section simulation results based on Matlab were presented. This section presents the real time control results .The computer code (programme) for the real- time control task was written in Visual C++6.0.

The following state-feedback gain matrix as in Section3.2 was used to control the system.

$$K=\begin{bmatrix} 15.5575 & 44.7970 & -11.0454 & 9.3731 & -48.7105 & 29.2018 & -9.8515 & 3.9453 \\ -44.1769 & 26.3908 & -8.3075 & 3.4332 & 17.2278 & 42.2108 & -9.9158 & 8.9473 \end{bmatrix}$$

The real-time experimental control results with and without interference are shown in the following Fig.5, Fig.6 and Fig.7.

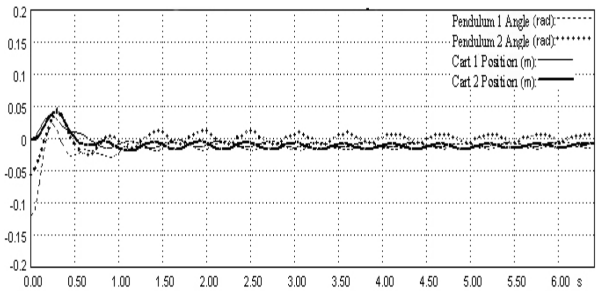


Fig.5 Real-time control results without interference

From Fig.5 it can be seen that the settling time for controller based on state space pole placement method is about one second i.e. the controller can stabilize both pendulums within one second and has a very small overshoot.

The following Fig.6 and Fig.7 show the control performance of the aforesaid method when a sinusoidal interference of magnitude 40 mm and of 2 Hz frequency after about 3.2 seconds as well as a square-wave interference of magnitude 50mm and of 3Hz frequency after about 2.7 seconds were introduced into the system respectively.

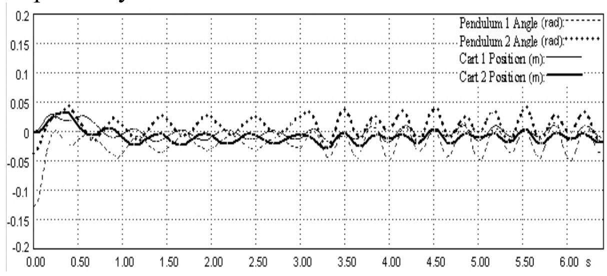


Fig.6 Real-time control results with sinusoidal interference

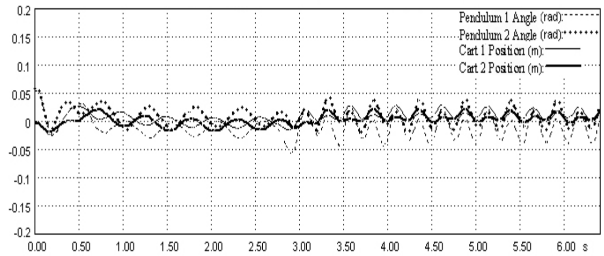


Fig.7 Real-time control results with square-wave interference

From Fig. 6 and Fig.7 it can be seen that the interference introduced oscillations in the system but the system remained stable in the presence of disturbance. Hence the designed pole placement state feed back controller exhibited good disturbance rejection characteristics.

## V. CONCLUSIONS

Although the control of coupled set of inverted pendulums is a very complex and challenging problem due to the nonlinear characteristics of the system and interactions. But the designed pole placement state feed back controller for the system has shown robustness not only in simulation based experiments but also in the real time experiments on a laboratory scale set up for a pair of inverted pendulums.

Therefore, based on simulation results and actual experiments, it can be concluded that the pole placement method is of significance and it can be useful to design controllers for coupled inverted pendulums with satisfactory performance. Also the proposed method is less complicated and can be implemented with lesser effort in comparison to other methods proposed in literature.

## ACKNOWLEDGMENT

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