Solving Differential Equations (DE)

(1) $\dot{x} + 3x = 0$, where $x(0) = 5$

integrating both sides

$\Rightarrow \frac{dx}{dt} = -3x \Rightarrow \frac{dx}{x} = -3dt \Rightarrow \ln|x| = -3t + C$

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$\Rightarrow x = e^{-3t+C} = e^{C}e^{-3t}$ (1)

We have $x(0) = 5$, (1) $\Rightarrow 5 = e^{C}e^{0} \Rightarrow e^{C} = 5$

$\therefore$ the solution to the DE is $x(t) = 5e^{-3t}$

(2) $\dot{x} + 3x = e^{-2t}$ (*), where $x(0) = 5$

Consider the form $\dot{x} + p(t)x = q(t)$ (1)

(1) can be solved using $\mu(t) = e^{\int p(t)dt}$

where $p(t) = 3 \Rightarrow \mu(t) = e^{3t}$

Multiplying both sides by e^{3t}, (*) $\Rightarrow e^{3t}\dot{x} + 3e^{3t}x = e^{3t}e^{-2t} = e^{t}$

$\Leftrightarrow \frac{d}{dt}(xe^{3t}) = e^{t}$

Integrate both sides : $\int \frac{d}{dt}(xe^{3t})dt = \int e^{t}dt$

$\Rightarrow xe^{3t} = e^{t} + C \Rightarrow x = e^{-3t}(e^{t} + C)$

$\Rightarrow x = e^{-2t} + Ce^{-3t}$

We have $x(0) = 5 \Rightarrow 5 = e^{0} + Ce^{0} \Rightarrow C = 4$

$\therefore$ the solution to the DE is $x(t) = e^{-2t} + 4e^{-3t}$

$\boxed{\text{Partial Fration Expansion}}$

Case 1: Distinct Poles

$$F(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

Assume: $F(s) = \frac{A}{s+1} + \frac{B}{s+2}$

Multiply both sides by $(s+1)(s+2)$

$$\Rightarrow \quad s+3 = A(s+2) + B(s+1)$$

- $s = -1 \Rightarrow 2 = A(1) + 0 \Rightarrow A = 2$
- $s = -2 \Rightarrow 1 = 0 + B(-1) \Rightarrow B = -1$

$$\therefore \quad F(s) = \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}$$

Case 2: Repeated Poles

$$F(s) = \frac{1}{s(s+1)^2}$$

The PFE takes the form $\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

Multiply both sides by $s(s+1)^2 \Rightarrow 1 = A(s+1)^2 + Bs(s+1) + Cs$

$$\Rightarrow 1 = (A+B)s^2 + (2A+B+C)s + A$$

We may have the system $\begin{cases} A+B=0 \\ 2A+B+C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \end{cases}$

$$\therefore \quad F(s) = \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Solve DEs using Laplace transform

$$\dot{x} + x = \sin 2t, \text{ when } x(0) = 6$$

Use Laplace transform:
$$\begin{cases} \mathcal{L}\{\dot{x}\} = sX(s) - x(0) \\ \mathcal{L}\{x\} = X(s) \\ \mathcal{L}\{\sin(2t)\} = \dfrac{2}{s^2+4} \end{cases}$$

$$\Rightarrow sX(s) - 6 + X(s) = \frac{2}{s^2+4} \Rightarrow (s+1)X(s) = \frac{2}{s^2+4} + 6$$

$$\Rightarrow (s+1)X(s) = \frac{6s^2+26}{s^2+4} \Rightarrow X(s) = \frac{6s^2+26}{(s+1)(s^2+4)}$$

$$X(s) = \frac{6s^2+26}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

Multiply both sides by $(s+1)(s^2+4) \Rightarrow 6s^2 + 26 = A(s^2+4) + (Bs+C)(s+1)$

$$\Rightarrow 6s^2 + 26 = (A+B)s^2 + (B+C)s + (4A+C)$$

$$\Rightarrow \begin{cases} A+B = 6 \\ B+C = 0 \\ 4A+C = 26 \end{cases} \Rightarrow \begin{cases} A = 32/5 \\ B = -2/5 \\ C = 2/5 \end{cases}$$

$$\Rightarrow X(s) = \frac{32/5}{s+1} + \frac{(-2s/5 + 2/5)}{s^2+4}$$

Take inverse Laplace transform
$$\mathcal{L}^{-1}\left[\frac{A}{s+a}\right] = Ae^{-at}$$
$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos(\omega t)$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{\sin(\omega t)}{\omega}$$

$$\Rightarrow x(t) = \frac{32}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{2}{10}\sin(2t)$$

$$\therefore x(t) = \frac{32}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)$$