# Design a Linear Quadratic Gaussian Controller for Piezo-Positioning Stage



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***Abstract*—The piezo-positioning stage driven by piezoelectric (PZT) actuators has been widely used in the precision engineering field. Because that in an actual piezo-positioning control system, it is usually difficult to obtain an accurate mathematical model of the controlled object, and there is also the problem of some disturbances in the system, a Linear Quadratic Gaussian (LQG) control system is given. In this control system, an optimal controller for a deterministic linear quadratic regulator problem is obtained by solving a state feedback law equation. A Kalman filter is designed to obtain the best state estimation value of the control system, so as to solve the problem that the state must be measurable and usable for feedback. According to simulation results, the dynamic response and robustness performance of designed LQG control system is given, it obviously seen that the designed control system has no-overshoot, good dynamic characteristics and stability, it can effectively suppress the step disturbance at the output. These indicate that the designed controller is effective and feasible.**

***Keywords—piezo-positioning stage, linear quadratic Gaussian control, Kalman filter***

## I. Introduction

Positioning stages driven by PZT actuators have the characteristic of small displacement resolution, they are widely used in the precision engineering field. In an actual piezo-positioning control system, it is usually difficult to obtain an accurate mathematical model of the controlled object, and there is also the problem of unknown noise disturbance in the system. Traditional control system design algorithms are generally aimed at deterministic systems and environments, it can not give good robustness for system parameter changes and external disturbance signals. Therefore, it is necessary to design a robust control system to solve these problems.

The traditional Proportion Integral Differential (PID) controllers have been used widely for piezo-positioning stages, because the controller is ease of design and simplicity in implementation[1-4]. However, the PID controller does not offer flexibility in design, good robustness and disturbance suppression capability [5-6]. In order to strengthen the robustness of the system and improve the dynamic performance, feedforward-feedback control algorithm is designed for piezo-positioning stages [7]. Chen Hui uses feedforward-feedback control technology to achieve decoupling and control of piezo-positioning stage [8]. Ziqiang Chi et al. used the inverse model of the Preisach model in the feedforward control, and designed a fuzzy PID feedback control [9]. Yung-Tien Liu et al. designed the feedforward control based on the Bouc-Wen model, and then proposed an adaptive feedback control method [10]. However, in the feedforward-feedback controller, it needs to know the more accurate mathematical model of the controlled object to design the feedforward compensation controller. The parameter setting is more complicated than the simple feedback controller, especially the adjustment of dynamic feedforward [11]. Aiming at these problem, an adaptive controller is designed, it can automatically modify its own characteristics to adapt to the changing characteristics of the controlled object. Shome SK et al. designed an adaptive feedforward controller for piezo-positioning stage [12]. Zhang Y. et al. developed a robust adaptive feedback control method that can simultaneously handle various model uncertainties and disturbances to achieve high-precision control [13]. Although adaptive control has its advantages, it is difficult to design a large number of preselected parameters that are closely related to the characteristics of the controlled object.

In this paper, in order to make the control system of a piezo-positioning stage have good dynamic response performances and robustness for external disturbance signals, a LQG control system with robustness was designed. First, an optimal state feedback controller for a deterministic linear quadratic regulator problem is given, it can be obtained by solving a state feedback law equation. Second, a Kalman filter is designed to get the best state estimation value of the control system, so as to solve the problem that the state must be measurable and usable for feedback. Finally, simulation results give dynamic response and robustness performances, it confirms the designed controller is effective and feasible.

## II. Mathematical Model of Piezo-Positioning Stage

A piezo-positioning stage can be simplified as a mass-spring-damping system that is a second-order linear system, as shown in figure 1.

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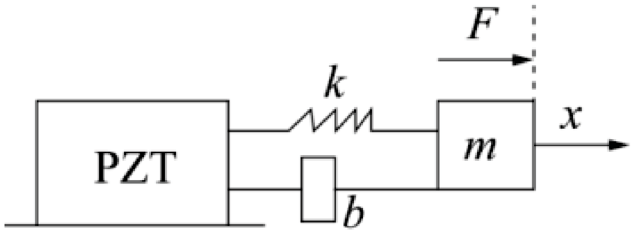


Fig. 1. Equivalent damping system of the piezo-positioning stage.

According to figure 1, the mathematical model is represented as,

$$m\ddot{x}+b\dot{x}+kx=F \quad (1)$$

Where, $m$ is the equivalent mass of the moving stage, $b$ is the equivalent damping coefficient, $k$ is the equivalent spring coefficient, $F$ is a force, $x$ is the displacement of the stage.

Assuming the initial state is zero and then Laplace transforming equation (1), $(ms^2+bs+k)X(s)=F(s)$ is obtained. Its transfer function is $G_1(s)=\dfrac{X(s)}{F(s)}=\dfrac{1}{ms^2+bs+k}$, $F=kau$, where $a$ is the effective piezoelectric coefficient, $u$ is the input voltage. It can be expressed as $G_2=F(s)/U(s)=ka$. And then the system model is $G(s)=\dfrac{X(s)}{U(s)}=G_1(s)G_2(s)=\dfrac{ka}{ms^2+bs+k}$. It is equivalent to $G(s)=\dfrac{k^*\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, where $k^*$ is amplification factor, $k^*=ka/m\omega_n^2$, $\xi$ is damping ratio, $\xi=b/2m\omega_n$, $\omega_n$ is natural oscillation frequency, $\omega_n=\sqrt{k/m}$

## III. LQG Controller Design

The LQG problem and its solution can be decomposed into two different parts, as shown in the figure 2. The equations needed to solve the optimal state feedback matrix *Kr* and Kalman filter will be given below.

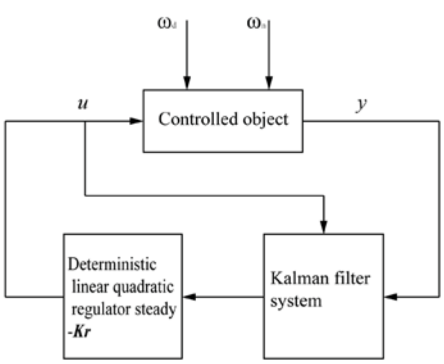


Fig. 2. Separation theorem.

### A. Optimal State Feedback

In this part, an optimal controller for a deterministic linear quadratic regulator problem is obtained by solving a state feedback law equation. Given a system with a non-zero initial state $x(0)$,

$$\dot{x}=Ax+Bu \quad (2)$$

Find the input signal $u(t)$ that can make the system reach zero state ($x$=0) in the best way, That is to minimize the deterministic cost function,

$$J_r=\int_0^\infty (x(t)^{\mathrm{T}}Qx(t)+u(t)^{\mathrm{T}}Ru(t))dt \quad (3)$$

The optimal solution (for any initial state) is

$$u(t)=-K_r x(t) \quad (4)$$

where,

$$K_r=R^{-1}B^{\mathrm{T}}X \quad (5)$$

$X=X^{\mathrm{T}}\geq 0$ is the only positive semi-definite solution of the following algebraic Riccati equation.

$$A^TX+XA-XBR^{-1}B^TX+Q=0 \quad (6)$$

### B. Kalman Filter

A Kalman filter is designed to obtain the best state estimation value, so as to solve the problem that the state must be measurable and usable for feedback. As shown in the figure 3, the Kalman filter has the same structure as the ordinary state estimator or observer, where,

$$\dot{\hat{x}}=A\hat{x}+Bu+K_f(y-C\hat{x}) \quad (7)$$

The optimal $K_f$ should minimize $E\{[x-\hat{x}]^{\mathrm{T}}[x-\hat{x}]\}$, which can be given by the following formula,

$$K_f=YC^{\mathrm{T}}V^{-1} \quad (8)$$

where, $Y=Y^{\mathrm{T}}\geq 0$ is the only positive semi-definite solution of the following algebraic Riccati equation.

$$YA^{\mathrm{T}}+AY-YC^{\mathrm{T}}V^{-1}CY+W=0 \quad (9)$$

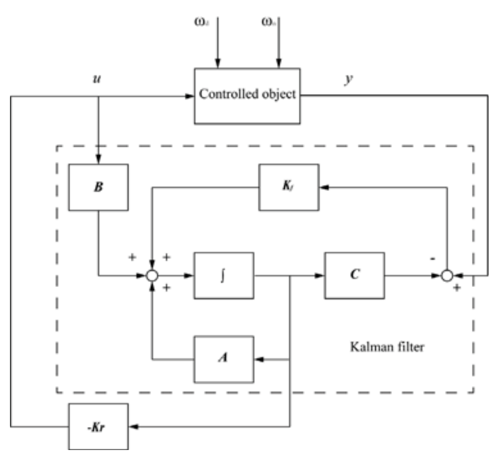


Fig. 3. LQG controller and objects with noise.

*C. LQG Controller Design with Integral Function*

Combining the above optimal state estimation and optimal state feedback, the purpose of the LQG control problem is to minimize $J$ .

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T [x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u] dt \right\} \tag{10}$$

The figure 4 shows the structure of the LQG controller, and the transfer function from $y$ to $u$ can be obtained as

$$\begin{aligned} K_{LQG}(s) &= \begin{bmatrix} A - BK_r - K_f C & K_f \\ -K_r & 0 \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}B^{\mathrm{T}}X - YC^{\mathrm{T}}V^{-1}C & YC^{\mathrm{T}}V^{-1} \\ -R^{-1}B^{\mathrm{T}}X & 0 \end{bmatrix} \end{aligned} \tag{11}$$

It has the same order (number of poles) as the object.

Since the standard LQG design method cannot give an integral controller, consider using the structure shown in the figure 4 to add an integrator to the object $G(s)$ before designing the state feedback regulator.

For the objective function $J = \int (x^T Q x + u^T R u) dt$ , choose $Q$ so that only the integrated state $y$-$r$ is weighted, and select the input weight function $R$=1. Only the ratio of $Q$ to $R$ is meaningful, reducing $R$ will make the response faster. The setting of the Kalman filter allows us not to estimate the integration state. Regarding the noise weight function, choose $W = \omega I$ , this process noise directly affects the state, and take $\omega = 1$ , $V = 1$ . For measurement noise, only the ratio $\omega$ to $V$ is meaningful, reducing $V$ will make the response faster.

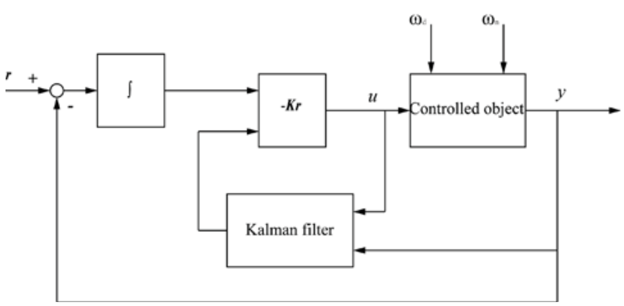

Fig. 4. LQG controller with integral action and reference input.

## IV. Simulation and Analysis Results

In the piezo-positioning system, the motion stage mass is 7g, natural oscillation frequency is 3.1kHz, the stiffness is 100N/um. According to model formulas in the section1, the positioning system transfer function is established as follows,

$$G(s) = \frac{128861028.16}{s^2 + 38936s + 379003024}$$

The open-loop step response with 1 amplitude of this piezo-positioning system is shown in the figure 5, It can be seen that the overshoot of the step response is too large, and there is strong oscillation, which easily leads to system instability. The steady-state error of the step response is relatively large and cannot reach the set value.

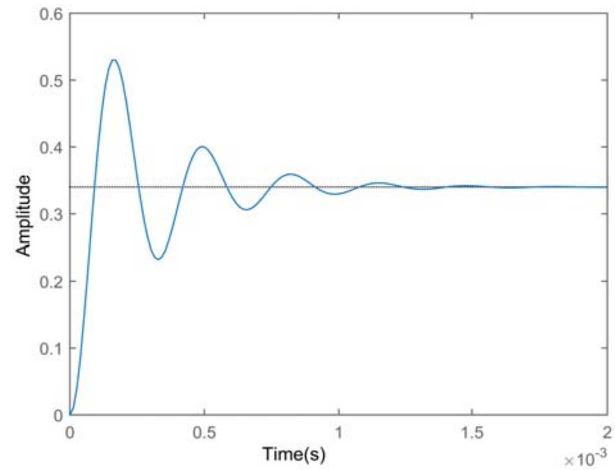

Fig. 5. Open-loop step response.

Apply the designed control system to this piezo-positioning system, and give the performance analysis of the control system as follows.

*A. Control System Dynamic Response Performances*

The step response with 1 amplitude of the LQG control system is shown in the figure 6.

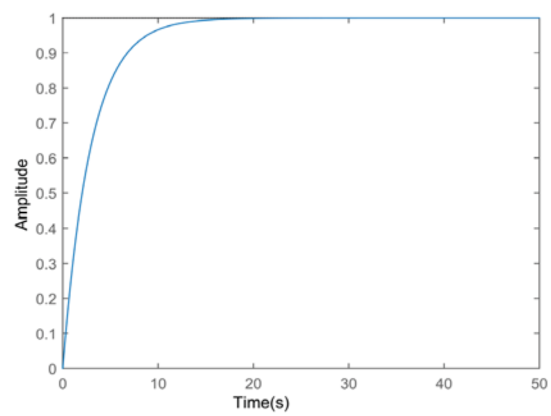

Fig. 6. LQG control system step response.

It can be seen that the step response of LQG control system has no large overshoot and strong oscillation, and the final steady-state value can reach the set value, achieving fast and low overshoot and accurate tracking of the step signal, these confirm the effectiveness of the control system.

*B. Control System Robustness Performances*

Before carrying out the robustness verification of the control system, the stability margin analysis of the control system is given first. In order to ensure the reliability and stability of the system, sufficient stability margin must be left for the system. Figure 7 shows the gain margin of the control system. It can be seen that the shape of the Bode diagram of the control system has a relatively gentle slope near the crossover frequency, which corresponds to a good margin. The gain margin is 86.3db at frequency 1.95e+04 rad/s. It proves that the designed control system is stable and has a good stability margin.

In order to verify the robustness of the control system to external disturbances, a step disturbance with amplitude 1 is added to the output of piezo-positioning stage. According to the transfer function of the disturbance signal to the output of the system, the control system's suppression result of the disturbance signal is obtained. Observing the effect of the disturbance signal from the output end is shown in figure 8. It can be seen that when a sudden disturbance is applied, the maximum disturbance output amplitude reaches about 1 at the very beginning, but because of the control system, this disturbance output is effectively suppressed.

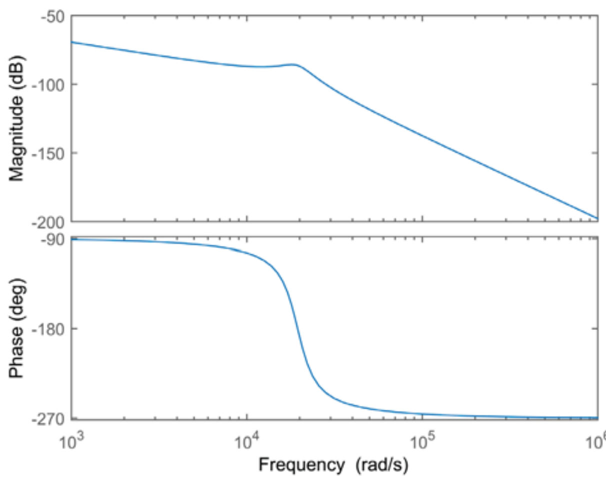


Fig. 7. Bode diagram of the control system.

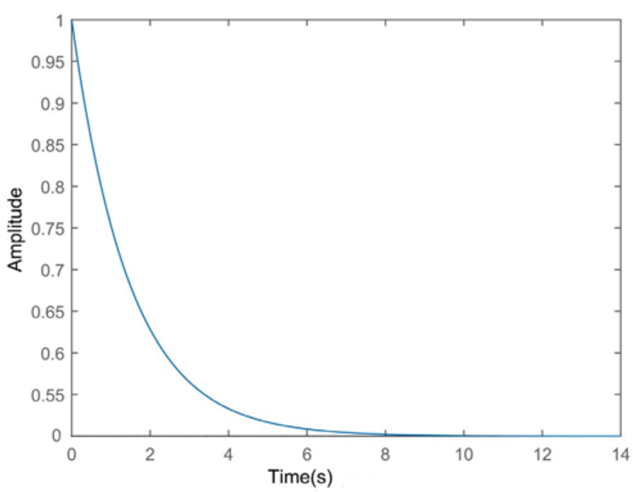


Fig. 8. Control system step disturbance suppression curve.

## V. Conclusions

This paper designs a LQG controller for piezo-positioning stage. The controller uses an optimal state feedback controller and a Kalman filter to improve performance of controlling the piezo-positioning stage. According to simulation results, the designed controller makes the system has good dynamic response characteristics and robustness. The performances of the designed controller are confirmed.

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