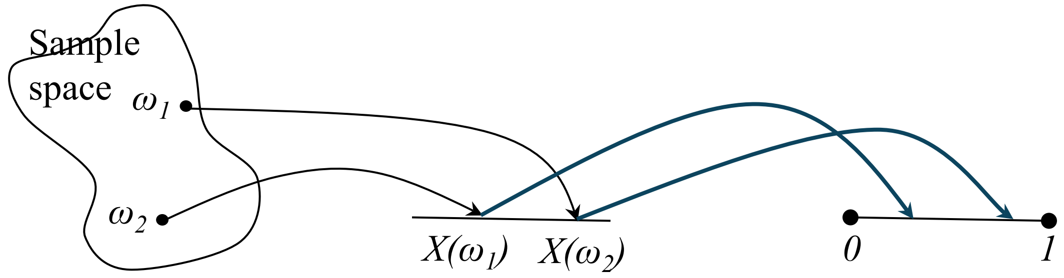
Random Variable

Random Variable Definition

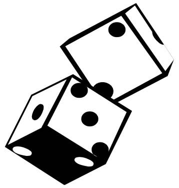
- It frequently occurs that in performing an experiment we are mainly interested in some functions of the outcome as opposed to the outcome itself
  - Rotery, casino, game, etc.
- These quantities of interest, or more formally, these real-valued functions defined on the sample space, are known as random variables (RVs).
- More mathematical expression of RVs
  - Given a probability space (S, F, P), a random variable is a measurable function (mapping) from S to the real line X: $S \rightarrow R$
    - $X(\{H\}) = 100, X(\{H\}) = -50$
    - $X(\{1,2\}) = 100, X(\{3,4,5,6\}) = -50$
- Since the value of a random variable is defined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable

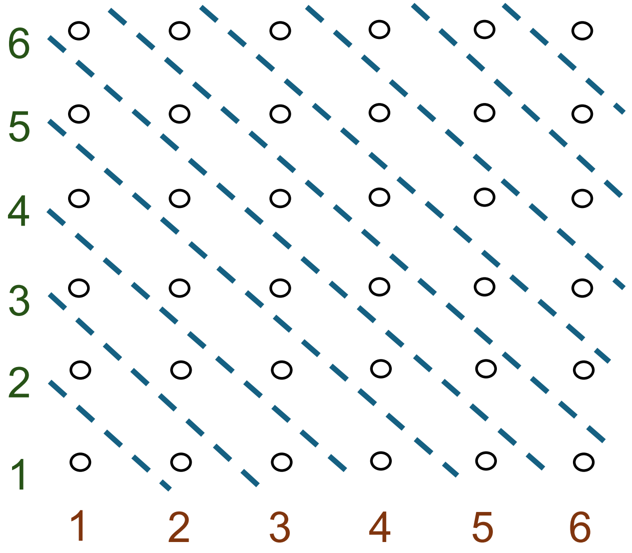


Random Variable: Examples

- Ex 2.1] Letting X denote the random variable that is defined as the sum of two fair dice
  - The outcomes of two dice = (ω_1, ω_2)
  - RV $X(\omega_1, \omega_2) \; = \; \omega_1 + \omega_2$
  - Possible values of $X \; = \; \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
  - Probabilities to the possible values of the random variable
    - $P(X = 2) = P\{(1,1)\} = 1/36$
    - $P(X = 3) = P\{(1,2), (2,1)\} = 2/36$

    ...
    - $P(X = 11) = P\{(5,6), (6,5)\} = 2/36$
    - $P(X = 12) = P\{(6,6)\} = 1/36$
    - $1 = P\{\cup_{n=2}^{12}\{X = n\}\} = \sum_{n=2}^{12} P(X = n)$





Random Variable: Examples

- Ex.2.3] Suppose that we toss a coin having a probability p of coming up heads, until the first head appears. Letting N denote the number of flips required, then assuming that the outcome of successive flips are independent, N is a random variable taking on one of the values 1, 2, 3, . . . , with respective probabilities

$$P\{N = 1\} = P\{H\} = p,$$
$$P\{N = 2\} = P\{(T,H)\} = (1 - p)p,$$
$$P\{N = 3\} = P\{(T,T,H)\} = (1 - p)2p,$$
$$\dots$$
$$P\{N = n\} = P\{(T,T,\dots,T,H)\} = (1 - p)^{n - 1}p, n \geq 1$$

- Indicator random variable

  $I_E(x) = 1$ if $x \in E$, 0 otherwise.

Cumulative Distribution Function (CDF)

- If a random variable takes on either a finite or a countable number of possible values, the RV is called discrete.

- If a continuum of possible values, continuous.

- The cumulative distribution function (CDF) $F(\ )$ of the random variable X is defined for any real number b, $-\infty < b < \infty$ by $F(b) = P\{X \leq b\}$.

- Properties of CDF F
  1. $F(b)$ is a nondecreasing function of b
  2. $lim_{b \to \infty} F(b) = F(\infty) = 1$
  3. $lim_{b \to -\infty} F(b) = F(-\infty) = 0$

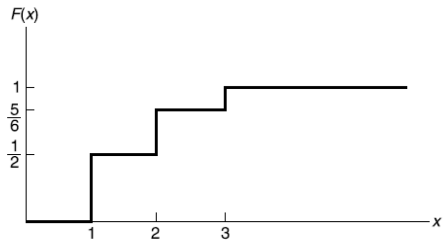
- Ex. $P\{a < X \leq b\} = F(b) - F(a)$ for all $a < b$

Discrete RVs

- A random variable that can take on at most countable number of possible values

- Probability Mass Function (pmf): for a discrete random variable X, pmf $p(a)$ of X is defined a $p(a) = P\{X = a\}$.

- Properties of pmf
  1. $p(x_i) > 0, i = 1,2, \ldots$
  2. $p(x) = 0$ all other values of x
  3. $\sum_{i = \{1, \ldots, \infty\}} p(x_i) = 1$
  4. cdf vs. pmf: $F(a) = \sum_{x_i \leq a} p(x_i)$



- Ex. $p(1) = 1/2, p(2) = 1/3, p(3) = 1/6$
  $F(a) = 0 \quad a < 1, F(a) = \frac{1}{2} \; 1 \leq a < 2, F(a) = \frac{5}{6} \; 2 \leq a < 3, F(a) = 1 \; 3 \leq a$

- Discrete random variables are often classified according to their pmf
  - Ex: Bernoulli RV, Binomial RV, Geometric RV, Poisson RV

Example of Discrete RVs: The Bernoulli RV

- A random variable X is said to be a Bernoulli RV for given sample space $S = \{A, A^c\}$ and some $p \in (0,1)$ if its pmf is given by
$$P\{A\} = p, P\{A^c\} = 1 - p$$

- For example, suppose that a trial, or an experiment, whose outcome can be classified as either a "success" or as a "failure" is performed. If we let X equal 1 if the outcome is a success and 0 if it is a failure, then the probability mass function of X is given by
$$p(0) = P\{X = 0\} = 1 - p,$$
$$p(1) = P\{X = 1\} = p \qquad (2.2)$$
where $p, 0 \leq p \leq 1$, is the probability that the trial is a "success."

Example of Discrete RVs: The Binomial RV

- Suppose that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability $1 - p$, are to be performed.

- If X represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p).

- The pmf of a binomial RV with (n,p)
  $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$ for $i = 0, 1, \ldots, n$ (2.3)
  where $\binom{n}{i} = \frac{n!}{(n-i)!i!}$.

- $\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + (1-p))^n = 1$

Example of Discrete RVs: The Binomial RV

- Ex. 2.7] It is known that any item produced by a certain machine will be defective with probability 0.1, independently of any other item. What is the probability that in a sample of three items, at most one will be defective?
  - If X is the number of defective items in the sample, then X is a binomial random variable with parameters $(3, 0.1)$. Hence, the desired probability is given by
  - $P\{X = 0\} + P\{X = 1\} \; = (3,0)(0.1)0(0.9)3 + (3,1)(0.1)1(0.9)2 = 0.972$

Example of Discrete RVs: The Geometric RV

- Suppose that independent trials, each having probability p of being a success, are performed until a success occurs.

- If we let X be the number of trials required until the first success, then X is said to be a geometric random variable with parameter p.

- Its probability mass function is given by
$p(n) = P\{X = n\} = (1 - p)^{n-1}p, n = 1, 2, \ldots$ (2.4)

- To check that p(n) is a probability mass function, we note that
$\sum_{n=1}^{\infty} p(n) = p\sum_{n=1}^{\infty}(1-p)^{n-1} = 1$

Example of Discrete RVs: The Poisson RV

- A random variable X, taking on one of the values $0, 1, 2, \dots,$ is said to be a Poisson random variable with parameter λ, if for some $\lambda > 0$,
  $p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, i = 0,1,2,3, \dots$ (2.5)

- To check that p(n) is a probability mass function, we note that
  $\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$

- Binomial vs. Poisson RVs
  - $n >> 1, p << 1$, let $\lambda = np$
  - $P\{X = 1\} = \binom{n}{i} p^i(1-p)^{n-i} = \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} = \frac{n(n-1)\dots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$
  - For large n and small p
    - $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}, \left(1-\frac{\lambda}{n}\right)^i \approx 1, \frac{n(n-1)\dots(n-i+1)}{n^i} \approx 1$
  - $P\{X = i\} \approx e^{-\lambda}\frac{\lambda^i}{i!}$

Example of Discrete RVs: The Poisson RV

- Ex. 2.10] Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on this page.
  - $P\{X \geq 1\} = 1 - P\{X = 0\} = 1 - e^{-1} = 0.633$

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$

Continuous RVs

- RV X is said to be a continuous RV if there exists a non-negative function $f(x)$, defined for all real $x \in \{-\infty, \infty\}$ having the property that for any set B of real numbers
$$P\{X \in B\} = \int_B f(x)dx = 1$$

- The function f(x) is called the probability density function (pdf) of the random variable X

- Properties of pdf of X
  - $P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx = 1$
  - $P\{a \leq X \leq b\} = \int_a^b f(x)dx$
  - $P\{X = a\} = \int_a^a f(x)dx = 0$
  - $F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$
  - $\frac{dF(a)}{da} = f(a)$

- Ex. Uniform Rv, Exponential Rv, Gamma RV, Normal RV

Example of Continuous RVs: Uniform RV

- A RV is said to be uniformly distributed over the interval (α,β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, \alpha < x < \beta \\ 0, otherwise \end{cases}$$

- Ex. 2.14] $X \sim U(0,10)$
  - $P\{X < 3\} = \int_{-\infty}^{3} f(x)dx = \int_{0}^{3} \frac{1}{10} dx = \frac{3}{10}$
  - $P\{X > 7\} = \int_{7}^{\infty} f(x)dx = \int_{7}^{10} \frac{1}{10} dx = \frac{3}{10}$
  - $P\{1 \leq X < 6\} = \int_{1}^{6} f(x)dx = \int_{1}^{6} \frac{1}{10} dx = \frac{5}{10} = \frac{1}{2}$

Example of Continuous RVs: Exponential RV

- A continuous RV is said to be an exponential RV with parameter λ if its probability density function is given, for $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, if\ x \geq 0 \\ 0, otherwise \end{cases}$$

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}$$

Example of Continuous RVs: Gamma RV

- A continuous RV is said to be a gamma RV with parameter α, λ, if its probability density function is given, for $\alpha > 0, \lambda > 0$, by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, if\ x \geq 0 \\ 0, otherwise \end{cases}$$

where

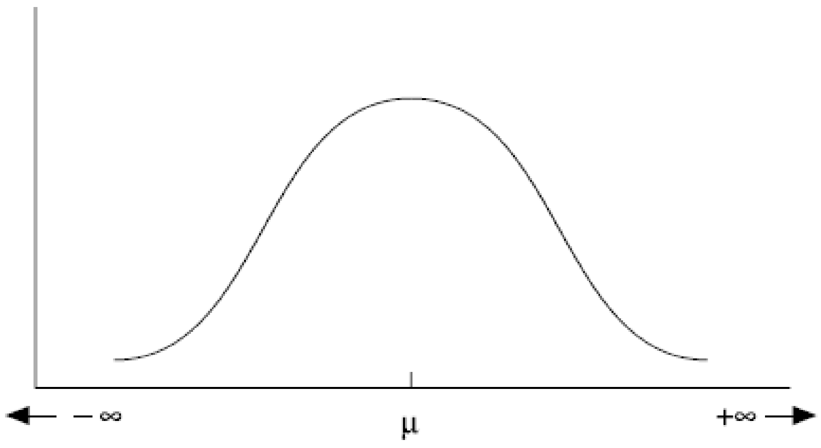
$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

Example of Continuous RVs: Normal RV

- A continuous RV is said to be a normal RV with parameter μ and σ^2, if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < \infty$.



Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\},$ $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \dfrac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\dfrac{\lambda}{\lambda - t}\right)^n$	$\dfrac{n}{\lambda}$	$\dfrac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Expectation of a function of a RV

- The expected value of X , $E[X]$, is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes that value.

- If RV X is a discrete RV having a pmf $p(x)$, then $E[X]$ is defined by
$$E[X] = \sum_{x:p(x)>0} xp(x)$$
  - Ex. 2.16~2.19] find the followings
    - Expectation of a Bernoulli Random Variable
    - Expectation of a Binomial Random Variable
    - Expectation of a Geometric Random Variable
    - Expectation of a Poisson Random Variable

- If RV X is a continuous RV having a pdf $f(x)$, then $E[X]$ is defined by
$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$
  - Ex. 2.20~2.22] find the followings
    - Expectation of a Uniform Random Variable
    - Expectation of a Exponential Random Variable
    - Expectation of a Normal Random Variable

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Expectation of a function of a RV

- Given a RV X and its probability distribution, what is the expectation of a function of X?
  1. Since $g(X)$ is itself a random variable, it must have a probability distribution, which should be computable from a knowledge of the distribution of X. Once we have obtained the distribution of $g(X)$, we can then compute $E[g(X)]$ by the definition of the expectation.
  2. Another way is to compute the expectation of a function of X from a knowledge of the distribution of X. See Proposition 2.1.

Expectation of a function of a RV

- Ex. 2.23] Suppose X has the following probability mass function: $p(0) = 0.2, p(1) = 0.5, p(2) = 0.3$. Calculate $E[X^2]$.
  - Letting $Y = X^2$, we have that Y is a random variable that can take on one of the values $0^2, 1^2, 2^2$ with respective probabilities
$$p_Y(0) = P\{Y = 0^2\} = 0.2,$$
$$p_Y(1) = P\{Y = 1^2\} = 0.5,$$
$$p_Y(4) = P\{Y = 2^2\} = 0.3$$
  - Hence, $E[X^2] = E[Y] = 0(0.2) + 1(0.5) + 4(0.3) = 1.7$
  - Note that $E[X^2] \neq E[X]^2$

Expectation of a function of a RV

- Proposition 2.1
  1. If *X* is a discrete random variable with probability mass function *p(x)*, then for any real-valued function g,

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

  2. If *X* is a continuous random variable with probability density function *f (x)*, then for any real-valued function *g*,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

- Ex 2.26] Let X be uniformly distributed over (0,1), E[X^3]?
  - $E[X^3] = \int_0^1 X^3\,dx = \frac{1}{4}$

- Corollary 2.2
  - If *a* and *b* are constants, then *E*[*aX* + *b*] = *aE*[*X*] + *b*

Expectation of a function of a RV

- The expected value of a RV X, $E[X]$, is also referred to as the mean or the first moment of X.

- The quantity $E[X^n], n \geq 1$, is called the n_{th} moment of X

- The variance of a RV X, denoted by $Var(X)$, is defined by
  $Var(X) = E[(X - E[X])^2]$,
  deviation of X from the mean.

- $Var(X) = E[(X - E[X])^2]$
  $= E[X^2 - 2E[X]X + E[X]^2]$
  $= E[X^2] - E[2E[X]X] + E[E[X]^2]$
  $= E[X^2] - 2E[X]^2 + E[X]^2$
  $= E[X^2] - E[X]^2$

Expectation of a function of a RV

- Ex 2.27] $Var(X)$ of the normal RV with μ and σ.
  - $Var(X) = E[(X-\mu)^2]$
    $= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$
    $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy$
    $= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$
    $= \sigma^2$

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \leq p \leq 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leq p \leq 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\dfrac{\lambda}{\lambda - t}\right)^n$	$\dfrac{n}{\lambda}$	$\dfrac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}$	μ	σ^2