The Exponential Distribution and the Poisson Process

Introduction

- In making a mathematical model for a real-world phenomenon it is always necessary to make certain simplifying assumptions so as to render the mathematics tractable.

- One simplifying assumption that is often made is to assume that certain random variables are exponentially distributed. The reason for this is that the exponential distribution is both relatively easy to work with and is often a good approximation to the actual distribution.

- Properties in Sect. 5.2

- Poisson process in Sect. 5.3

Definition of the Exponential Distribution

- A continuous random variable X is said to have an *exponential distribution* with parameter λ, $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, if\ x \geq 0 \\ 0, otherwise \end{cases}$$

$$F(x) = \int_0^x \lambda e^{-\lambda y} dy = \begin{cases} 1 - e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

- Statistics
  - Mean: $E[X] = \frac{1}{\lambda}$
  - Moment generating function: $\phi(t) = E[e^{Xt}] = \frac{\lambda}{\lambda - t}$
  - The second moment: $E[X^2] = \frac{2}{\lambda^2}$
  - Variance: $Var[X] = \frac{1}{\lambda^2}$

Properties of the Exponential Distribution

- A random variable X is said to be without memory, or *memoryless*, if $P\{X > s + t | X > t\} = P\{X > s\}$ for all s, t >0 (5.2)
  - In other words, if the instrument is alive at time t, then the distribution of the remaining amount of time that it survives is the same as the original lifetime distribution; that is, the instrument does not remember that it has already been in use for a time t.

- Eq. (5.2) equals to

$$\frac{P\{X>s+t, X>t\}}{P\{X>t\}} = P\{X > s\} \text{ or } P\{X > s + t\} = P\{X > s\}P\{X > t\} \quad (5.3)$$

- Since Equation (5.3) is satisfied when X is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$), it follows that exponentially distributed random variables are memoryless.

Examples

- **Example 5.2** Suppose that the amount of time one spends in a bank is exponentially distributed with mean ten minutes, that is, λ = 1/10 . What is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that she is still in the bank after ten minutes?
  - $P\{X > 15\} = e^{-15\lambda} = e^{-3/2} \approx 0.22$
  - $P\{X > 5\} = e^{-5\lambda} = e^{-1/2} \approx 0.62$

- **Example 5.3** Consider a post office that is run by two clerks. Suppose that when Mr. Smith enters the system he discovers that Mr. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Jones or Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with mean $1/\lambda$, what is the probability that, of the three customers, Mr. Smith is the last to leave the post office?
  - By the lack of memory of the exponential, the answer must equal 1/2. LoL

Examples

- **Example 5.4** The dollar amount of damage involved in an automobile accident is an exponential random variable with mean 1000. Of this, the insurance company only pays that amount exceeding (the deductible amount of) 400. Find the expected value and the standard deviation of the amount the insurance company pays per accident.
  - Let $Y = (X - 400)^+$ be the amount paid.
  - Let $I = \begin{cases} 1, \ if\ X \geq 400 \\ 0, if\ X < 400 \end{cases}$ be the condition whether X exceeds 400.
  - By the lack of memory property of the exponential, it follows that if a damage amount exceeds 400, then the amount by which it exceeds it is exponential with mean 1000. Therefore,
    - $\left.\begin{matrix} E[Y|I=0] = 1000 \\ E[Y|I=1] = 0 \end{matrix}\right\} \rightarrow E[Y|I] = 1000I$
    - $\left.\begin{matrix} Var[Y|I=0] = 1000^2 \\ Var[Y|I=1] = 0 \end{matrix}\right\} \rightarrow Var[Y|I] = 10^6 I$
    - $E[Y] = E\big[E[Y|I]\big] = 10^3 E[I] = 10^3 e^{-0.4} \approx 670.32$
    - $Var[Y] = E\big[Var[Y|I]\big] + Var\big[E[Y|I]\big] = 10^6 e^{-0.4} - 10^6 e^{-0.4}(1 - e^{-0.4}) \approx 994.09^2$

- It turns out that not only is the exponential distribution “memoryless,” but it is the unique distribution possessing this property.

Examples

- **Example 5.5** A store must decide how much of a certain commodity to order so as to meet next month's demand, where that demand is assumed to have an exponential distribution with rate λ. If the commodity costs the store *c* per pound, and can be sold at a price of *s* > *c* per pound, how much should be ordered so as to maximize the store's expected profit? Assume that any inventory left over at the end of the month is worthless and that there is no penalty if the store cannot meet all the demand.
  - Let X equal the demand. If the store orders the amount t, then the profit, call it P, is given by $P = s\min(X,t) - ct$, where $\min(X,t) = X - (X-t)^+$.
  - $E[(X-t)^+] = E[(X-t)^+|X > t]P(X > t) + E[(X-t)^+|X \le t)P(X \le t) = E[(X-t)^+|X > t]e^{-\lambda t} = \frac{1}{\lambda}e^{-\lambda t}$
  - $E[\min(X,t)] = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t}$
  - $E[P] = \frac{s}{\lambda} - \frac{s}{\lambda}e^{-\lambda t} - ct$
  - For maximal profit $t = \frac{1}{\lambda}\log(s/c)$

Another Interpretation of Memoryless

- The memoryless property is further illustrated by the failure rate function (also called the hazard rate function) of the exponential distribution. Consider a continuous positive random variable *X* having distribution function *F* and density *f* . The *failure* (or *hazard*) *rate* function *r(t)* is defined by
  $r(t) = \frac{f(t)}{1-F(t)}$ (5.4)

- To interpret *r(t)*, suppose that an item, having lifetime *X*, has survived for *t* hours, and we desire the probability that it does not survive for an additional time *dt*.
  $P\{X \in (t, t+dt) | X > t\} = \frac{P\{X \in (t,t+dt), X>t\}}{P\{X>t\}} = \frac{P\{X \in (t,t+dt)\}}{P\{X>t\}} \approx \frac{f(t)dt}{1-F(t)}$
  - That is, *r(t)* represents the conditional probability density that a *t*-year-old item will fail.

- Suppose now that the lifetime distribution is exponential. Then, by the memoryless property, it follows that the distribution of remaining life for a *t*-year-old item is the same as for a new item.
  - $r(t) = \frac{f(t)}{1-F(t)} = \frac{e^{-\lambda t}}{\lambda e^{-\lambda t}} = \frac{1}{\lambda}$

Examples

- **Example 5.6** Let $X_1, \dots, X_n$ be independent exponential random variables with respective rates $\lambda_1, \dots, \lambda_n$, where $\lambda_i \neq \lambda_j$ when $i \neq j$. Let T be independent of these random variables and suppose that

$$\sum_{j=1}^{n} P_j = 1$$

where $P_j = P\{T = j\}$. The random variable X_T is said to be a *hyperexponential* random variable. What is the failure rate?

  - $1 - F(t) = P\{X > t\} = \sum_{i=1}^{n} P\{X > t | T = i\} P\{T = i\} = \sum_{i=1}^{n} P_i \, e^{-\lambda_i t}$
  - $f(t) = \sum_{i=1}^{n} \lambda_i P_i \, e^{-\lambda_i t}$
  - From (5.4), $r(t) = \frac{\sum_{i=1}^{n} \lambda_i P_i e^{-\lambda_i t}}{\sum_{i=1}^{n} P_i e^{-\lambda_i t}}$

Further Properties of the Exponential Distribution

- Let $X_1, \ldots, X_n$ be independent and identically distributed exponential random variables having mean $1/\lambda$. $S_n = X_1 + \cdots + X_n$ has a gamma distribution with parameters n and λ.

$$f_{S_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

- [Example] Suppose that X_1 and X_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$; what is $P\{X_1 < X_2\}$?
  - $P\{X_1 < X_2\} = \int_0^\infty P\{X_1 < X_2 | X_1 = x\}\lambda_1 e^{-\lambda_1 t}dx = \int_0^\infty P\{x < X_2\}\lambda_1 e^{-\lambda_1 t}dx = \int_0^\infty e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}dx = \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)t}dx = \frac{\lambda_1}{\lambda_1+\lambda_2}$

- Suppose that $X_1, \ldots, X_n$ are independent exponential random variables, with X_i having rate λ_i, $i = 1, \ldots, n$.

$$P\{\min\{X_1, \ldots, X_n\} > x\} = P\{X_i > x \; for \; each \; i = 1, \ldots, n\} = \prod_{i=1}^{n} P\{X_i > x\} = \prod_{i=1}^{n} e^{-\lambda_i t}$$
$$= e^{-(\sum_{i=1}^{n}\lambda_i)t}$$

independency

Convolutions of Exponential Random Variables

- Let $X_1, \ldots, X_n$ be independent exponential random variables with respective rates $\lambda_1, \ldots, \lambda_n$, where $\lambda_i \neq \lambda_j$ when $i \neq j$. The random variable $\sum_{i=1}^{n} X_i$ is said to be a *hypoexponential* random variable. To compute its probability density function,
  - let us start with the case $n = 2$. Now, $f_{X_1+X_2}(t) = f_{X_1}(t) * f_{X_2}(t) = \int_{-\infty}^{\infty} f_{X_1}(s) f_{X_2}(t-s)\,ds = \frac{\lambda_1}{\lambda_1-\lambda_2}\lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1}\lambda_1 e^{-\lambda_1 t}$
  - In the general case,

$$f_{X_1+\cdots+X_n}(t) = f_{X_1}(t) * \cdots * f_{X_2}(t) = \sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i t}$$

$$C_{i,n} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

Counting Process

- A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that occur by time t.
  - Some examples of counting processes are the following:
    - If we let $N(t)$ equal the number of persons who enter a particular store at or prior to time t, then $\{N(t), t \geq 0\}$ is a counting process in which an event corresponds to a person entering the store.
    - If we say that an event occurs whenever a child is born, then $\{N(t), t \geq 0\}$ is a counting process when $N(t)$ equals the total number of people who were born by time t.
    - If $N(t)$ equals the number of goals that a given soccer player scores by time t, then $\{N(t), t \geq 0\}$ is a counting process.

- The properties of the counting process
  1. $N(t) \geq 0$.
  2. $N(t)$ is integer valued.
  3. If $s < t$, then $N(s) \leq N(t)$.
  4. For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

Poisson Process (1/2)

- One of the most important counting processes is the Poisson process

- **Definition 5.1** The counting process {*N(t)*, *t*≥0} is said to be a *Poisson process having rate* $\lambda, \lambda > 0$, if
  1. N(0) = 0
  2. The process has independent increments.
  3. The number of events in any interval of length *t* is Poisson distributed with mean λt. That is, for all s, t 0
     $P\{N(t+s) - N(s) = n\} = \ e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, *n* = 0, 1, . . .
     - Note that it follows from condition (iii) that a Poisson process has stationary increments and also that $[N(t)] = \lambda t,$ which explains why λ is called the rate of the process

Poisson Process (2/2)

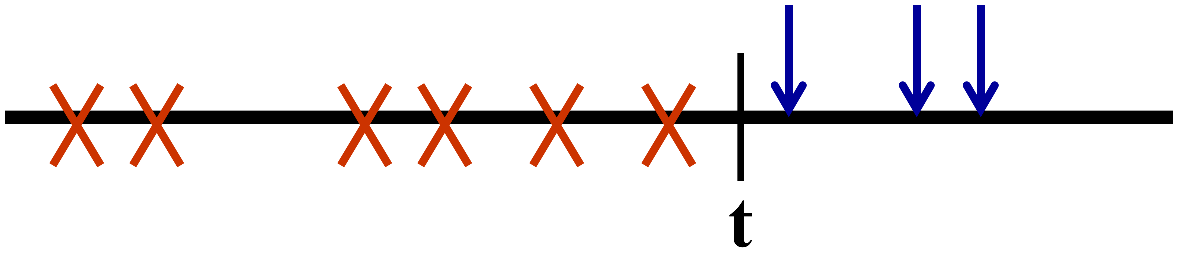
- To determine if an arbitrary counting process is actually a Poisson process, we must show that conditions (i), (ii), and (iii) are satisfied. We have an alternate definition of a Poisson process.

- **Definition 5.3** The counting process {*N(t)*, t 0} is said to be a Poisson process having rate λ, $\lambda > 0$, if
  1. N(0) = 0.
  2. The process has stationary and independent increments.
     1. For any t , s ≥0, the distribution of N(t+s) – N(t) is independent of t.
     2. For any t ≥0, s ≥0, N(t +s) – N(t) is independent of {N(u): u≤t}.
  3. P{N(h) = 1} = λh + o(h).
  4. P{N(h) 2} = o(h).

- **Theorem 5.1** Definitions 5.1 and 5.3 are equivalent.
- **Definition 5.2** The function *f (·)* is said to be *o(h)* if $\lim_{h \to 0} \frac{f(h)}{h} = 0$
  - $f(x) = x^2$ is o(h) while $f(x) = x$ is not o(h)

Independent Increments Property

- $N(t+s) - N(t)$ is independent of $\{N(t)\colon u \leq t\}$. Arrivals in the future (beyond time t) are independent of the entire past history up to time t (very strong property).



Stationary Increments Property

- Strong Property (but less so than Ind. Inc):
  - For any $t, t \geq 0$ the distribution of $N(t+s) - N(t)$ is independent of t.
  - $P\{N(t+s) - N(t) = n\}$
    $$= P\{N(t_1 + s) - N(t_1) = n\}$$
    $$= P\{N(s) = n\}$$

- Note $N(t+s) - N(t) \neq P(s)$

Numerical Example

- N(t) is a Poisson Process with rate λ = 8.
- We would like to compute:

$$P\{N(2.5) = 17, N(3.7) = 22, N(4.3) = 36\}$$

- $P = P\{N(2.5) = 17, N(3.7) = 22, N(4.3) = 36\}$

$$= P\{N(2.5) = 17, N(3.7) - N(2.5) = 5, N(4.3) - N(3.7) = 14\}$$

$$= P\{N(2.5 = 17\}P\{N(3.7) - N(2.5) = 5\}P\{N(4.3) - N(3.7) = 14\}$$

(From Independent Increments)

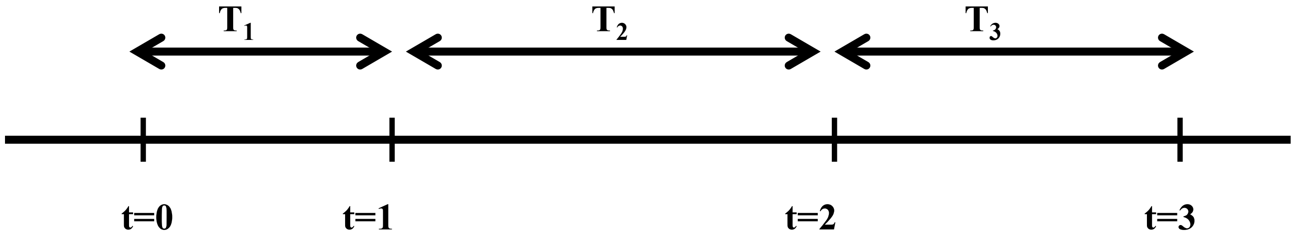
$$= P\{N(2.5) = 17\}P\{N(1.2) = 5\}P\{N(0.6) = 14\}$$

(From Stationary Increments)

$$= e^{-2.5\times 8}\frac{(2.5\times 8)^{17}}{17!}e^{-1.2\times 8}\frac{(1.2\times 8)^{5}}{5!}e^{-0.6\times 8}\frac{(0.6\times 8)^{14}}{14!}$$

Interarrival Time Distributions

- Consider a Poisson process, and let us denote the time of the first event by T_1. Further, for $n > 1$, let T_n denote the elapsed time between the $(n-1)$st and the nth event. The sequence $\{T_n, n = 1, 2, \dots\}$ is called the *sequence of interarrival times*.



Interarrival Time Distributions

- We shall now determine the distribution of the T_n.
- To do so, we first note that the event $\{T_1 > t\}$ takes place if and only if no events of the Poisson process occur in the interval $[0, t]$ and thus, $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$.
- Now, $P\{T_2 > t\} = E[P\{T_2 > t|T_1\}]$

$$\begin{aligned} &P\{T_2 > t|T_1 = s\} \\ &= P\{0\ eveints\ in\ (s, s+t]|T_1 = s\} \\ &= P\{0\ eveints\ in\ (s, s+t]\} \\ &= e^{-\lambda t} \end{aligned}$$

- **Proposition 5.1** $T_n, n = 1,2, \ldots,$ are independent identically distributed exponential random variables having mean $\frac{1}{\lambda}$.

Waiting Time Distributions (1/2)

- The arrival time of the *n*th event, also called the *waiting time* until the *n*th event. It is easily seen that $S_n = \sum_{i=1}^{n} T_i, n \geq 1$
- Hence from Proposition 5.1 and the results of Section 2.2 it follows that S_n has a gamma distribution with parameters *n* and *λ*.
  $f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, t \geq 0$ (5.13)
- Another way to find $f_{S_n}(t)$ using the Poisson process property
-

Waiting Time Distributions (2/2)

- Equation (5.13) may also be derived by noting that the *n*th event will occur prior to or at time t if and only if the number of events occurring by time t is at least n. That is,

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

- Hence, $F_{S_n}(t) = P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$.

- By differentiating both sides,
  - $f_{S_n}(t) = F_{S_n}(t)' = -\sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \sum_{j=n+1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

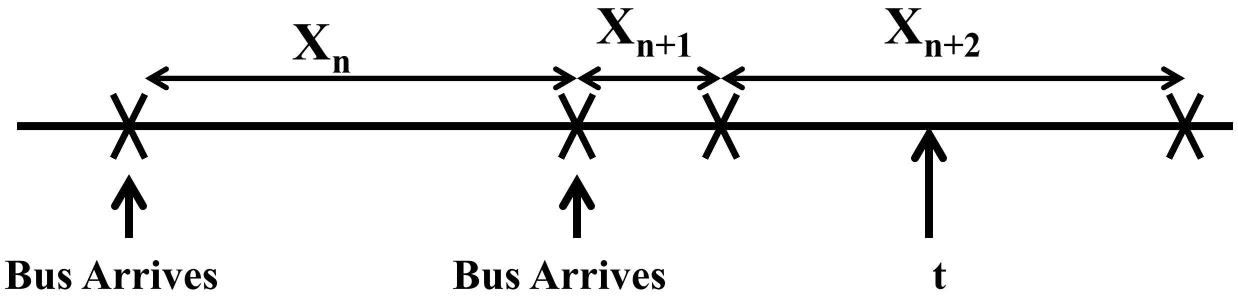
$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Paradox of Residual Life

- Suppose that buses arrive at a stop according to a Poisson process with rate.
- Assume that you arrive at that bus-stop at some arbitrary point in time.
- Question: How long would you expect to wait?
- Two “logical” answers.

Answer 1: Average wait = 1/2λ

- Note that average time between bus arrivals = $\frac{1}{\lambda}$.



- On average you will arrive exactly in the middle of an inter-arrival time.

- Your average wait for the next bus $= \frac{1}{2E[X_i]} = \frac{1}{2\lambda}$

Answer 2: Average wait = 1/λ

- Since buses arrive according to a Poisson Process with rate λ, the time you have to wait is independent of how long it has taken since the last bus, i.e. , the Avg. wait $= \frac{1}{\lambda}$.

- In fact by the second argument, if buses have been operational for a long time, then by the memory-less property, the expected time since the last bus arrival is also $\frac{1}{\lambda}$

- This implies that the expected time between the last bus and the next bus arrival in your interval $= \frac{2}{\lambda}$????

- So which answer is correct?

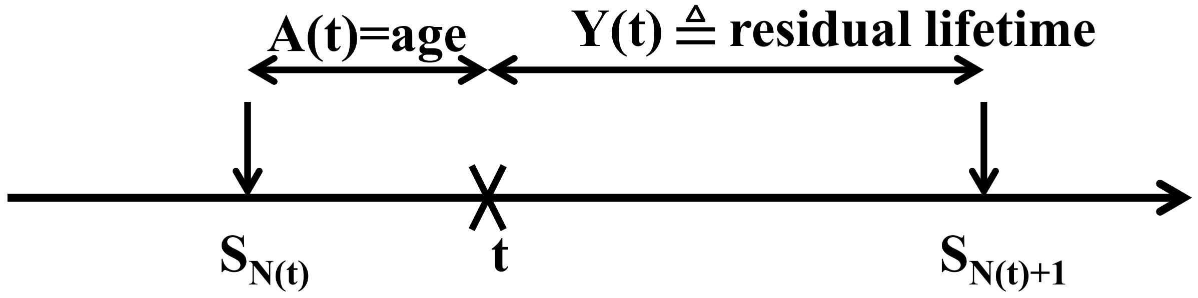
Paradox of Residual Life

- Answer 2 is in fact correct!
  - This is because the interval that you arrive in is not a typical interval, i.e. it is in fact more likely that you will arrive in a larger time interval.
  - In the case of a Poisson process if you are sufficiently far from the origin, this interval that you arrive in is in fact two times as long as the average interval.

Paradox of Residual Life

- Formally:
  - Let $\{N(t), t \geq 0\}$ be a Poisson Process with rate ⍰
  - Let S_1, S_2,… be successive arrival times, and $S_0 \triangleq 0$
  - The number of arrivals in $[0,t] \triangleq N(t)$
  - The time of the last arrival before $t \triangleq S_N(t)$
  - The time until the next arrival after $t = S_N(t)+1$

Paradox of Residual Life



- To Prove: $P\{Y(t) \leq u\} = 1 - e^{-\lambda u}, u \geq 0$

- In other words what we want to prove is that the time until the next arrival is exponentially distributed with mean $\frac{1}{\lambda}$.
  - Proof: For $u \geq 0 : P\{Y(t) > u\}$ $= P\{S_{N(t)} + 1 - t > u\}$
    $= P\{S_{N(t)} + 1 > u + t\}$
    $= P\{N(t+u) - N(t) = 0\}$
  - $P\{Y(t) > u\} = P\{N(t+u) - N(t) = 0\} = = P\{N(u) = 0\} = e^{-\lambda u}, u \geq 0$
  - Hence, $P\{Y(t) \leq u\} = 1 - e^{-\lambda u}, u \geq 0$

Paradox of Residual Life (2/2)

- $P\{Y(t) > u\}$
  $= P\{N(t+u) - N((t) = 0\}$
  $= P\{N(u) = 0\}$
  $= e^{-\lambda u}, u \geq 0$

- $P\{Y(t) \leq u\} = 1 - e^{-\lambda u}, u \geq 0$

- The Paradox of residual life is important in calculating the $E(W)$ in the queueing system.

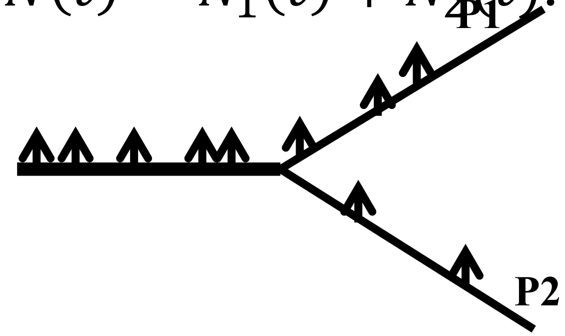
➢ **Homework:** Find the distribution of the age of the process. In other words find $P\{A(t) \geq u\}$?

$\Rightarrow E\left[S_{N(t)+1} - S_{N(t)}\right]$ =? as t→∞

$\Rightarrow E\left[S_{N(t)+1} - S_{N(t)}\right] = \frac{2}{\lambda}$ as t→∞

Further Properties of Poisson Processes

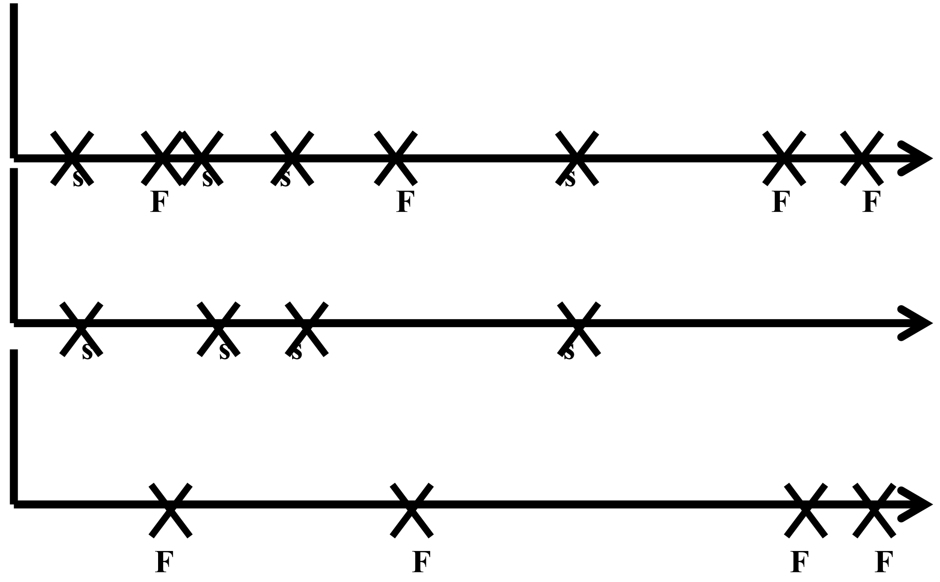
- Consider a Poisson process $\{N(t), t \geq 0\}$ having rate λ, and suppose that each time an event occurs it is classified as either a type I or a type II event. Suppose further that each event is classified as a type I event with probability p or a type II event with probability $(1-p)$, independently of all other events.
- Let $N_1(t)$ and $N_2(t)$ denote respectively the number of type I and type II events occurring in $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$.



- **Proposition 5.2** $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$. Furthermore, the two processes are independent.

Proof of Proposition 5.2 (Decomposition of a Poisson Process)

- We have a Poisson Process $\{N(t), t \geq 0\}$ with rate λ
- For each arrival/event, we toss a coin with probability "p" of success
- We construct two processes from it:
  - $U(t)$ = contains all "successful" events
  - $V(t)$ = contains all "failed" events



- Next page: Proof of $P\{U(t) = m, V(t) = n\} = e^{-\lambda tp} \frac{(\lambda tp)^m}{m!} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^n}{n!}$

Decomposition of a Poisson Process

- $P\{U(t)=m, V(t)=n\} = \sum_{k=0}^{\infty} P\{U(t)=m, V(t)=n|N(t)=k\}\, P\{N(t)=k\}$

$$= \underbrace{P\{U(t)=m, V(t)=n|N(t)=m+n\}}_{binomial\ probability}\, \underbrace{P\{N(t)=m+n\}}_{Poision\ process}$$

$$= \left\{\binom{m+n}{n} p^m (1-p)^n\right\} \frac{e^{-\lambda t}(\lambda t)^{m+n}}{(m+n)!}$$

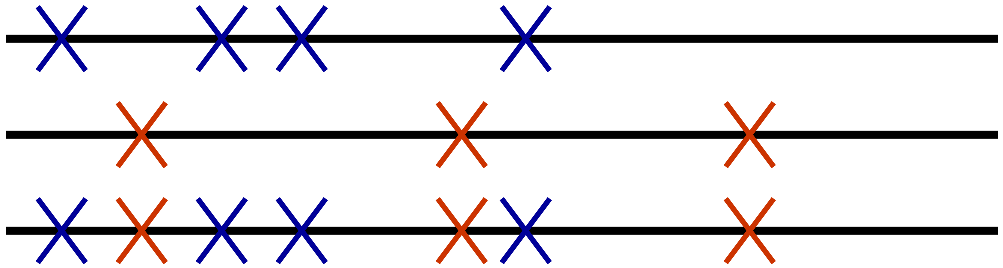
$$= \frac{(m+n)!}{n!\,m!} p^m (1-p)^n \frac{e^{-\lambda t}(\lambda t)^{m+n}}{(m+n)!}$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^m}{m!} \frac{e^{-\lambda(1-p)t}[\lambda(1-p)t]^n}{n!}$$

- It follows from Proposition 5.2 that if each of a Poisson number of individuals is independently classified into one of two possible groups with respective probabilities p and $1-p$, then the number of individuals in each of the two groups will be independent Poisson random variables. Because this result easily generalizes to the case where the classification is into any one of *r* possible groups.

Superposition of Poisson

- Let $\{L(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ be two independent Poisson Processes with rates λ and μ, respectively. For $t \geq 0$, let $N(t) = L(t) + M(t)$. Then the resulting process $\{N(t), t \geq 0\}$ is called the superposition of processes $\{L(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ and is a Poisson Process with rate $\lambda + \mu$.



➢Proof: To show $P\{N(t)=n\} = \dfrac{e^{-(\lambda+\mu)t}[(\lambda+\mu)t]^n}{n!}$

$$P\{N(t)=n\} = \sum_{k=0}^{n} P\{L(t)=k, M(t)=n-k\}$$

$$Note: [\{N(t)=n\} = \bigcup_{k=0}^{n} \{L(t)=k, M(t)=n-k\}]$$

$$= \sum_{k=0}^{n} P\{L(t)=k\}P\{M(t)=n-k\}$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda t}[\lambda t]^k}{k!}\frac{e^{-\mu t}[\mu t]^{n-k}}{(n-k)!}$$

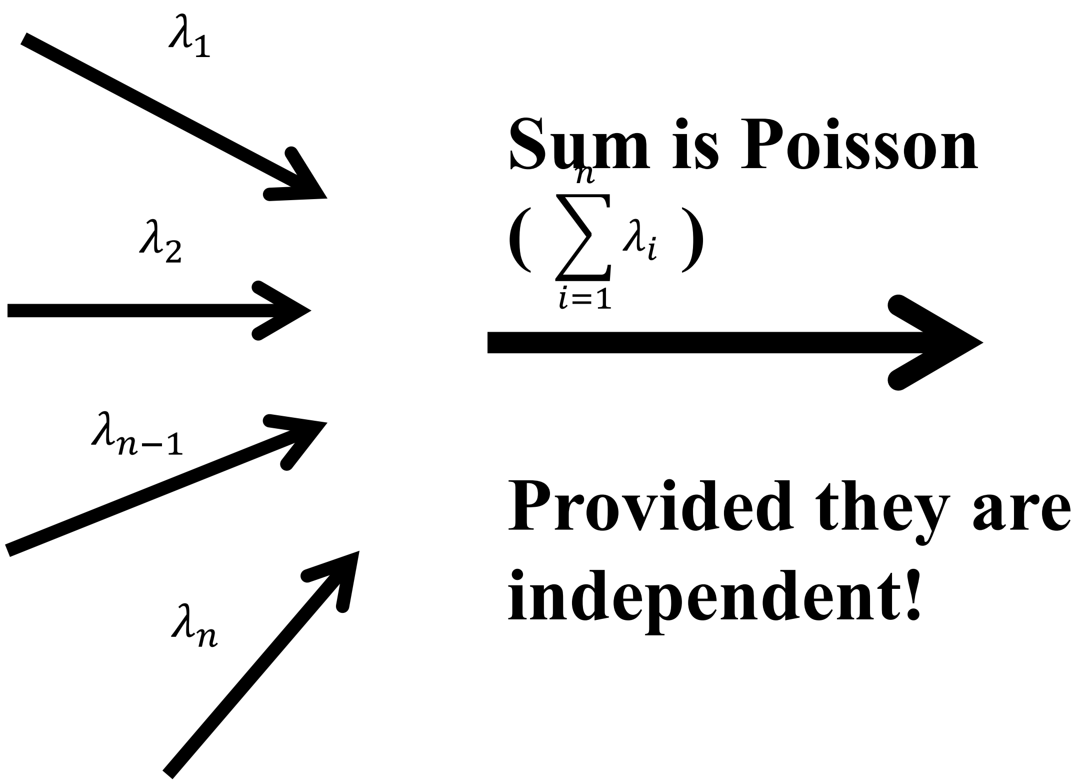
$$= \frac{e^{-(\lambda+\mu)t}[(\lambda+\mu)t]^n}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

$$= \frac{e^{-(\lambda+\mu)t}[(\lambda+\mu)t]^n}{n!}\left(\frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}\right)^n$$

Proof (Cont.)

- Finally: $= \dfrac{\mathrm{e}^{-(\lambda+\mu)\mathrm{t}}[(\lambda+\mu)t]^n}{n!}$

- Thus: $P\{N(t)=n\} = \dfrac{\mathrm{e}^{-(\lambda+\mu)\mathrm{t}}[(\lambda+\mu)t]^n}{n!}$



Further Properties of Poisson Processes

- We shall determine is the probability that *n* events occur in one Poisson process before *m* events have occurred in a second and independent Poisson process. More formally let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent Poisson processes having respective rates λ_1 and λ_2. Also, let $S^1{}_n$ denote the time of the *n*th event of the first process, and $S^2{}_m$ the time of the *m*th event of the second process. That is Pr{$S^2{}_m$ $S^2{}_m$}
  - Let us consider the special case *n* = *m* = 1
    - $\Pr\{S^1{}_1 \leq S^2{}_1\} = \frac{\lambda_1}{\lambda_1+\lambda_2}$
  - Let us consider the special case *n* = 2, *m* = 1
    - $\Pr\{S^1{}_2 \leq S^2{}_1\} = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$
    - *Each event that occurs is going to be an event of the $N_1(t)$ process with probability $\lambda_1/(\lambda_1 + \lambda_2)$ or an event of the $N_2(t)$ process with probability $\lambda_2/(\lambda_1 + \lambda_2)$, independent of all that has previously occurred.*
  - This event will occur if and only if the first *n* + *m* − 1 events result in *n* or more *N*1*(t)* events, we see that our desired probability is given by
    - $\Pr\{S^1{}_2 \leq S^2{}_1\} = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$

Conditional Distribution of the Arrival Times

- Suppose we are told that exactly one event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred. Now, since a Poisson process possesses stationary and independent increments it seems reasonable that each interval in $[0, t]$ of equal length should have the same probability of containing the event. In other words, the time of the event should be uniformly distributed over $[0, t]$. This is easily checked since, for $s \le t$,
  - $\Pr\{T_1 < s|N(t) = 1\} = \frac{\Pr\{T_1 < s, N(t)=1\}}{\Pr\{N(t)=1\}}$

$$= \frac{\Pr\{1\ event\ in\ [0, s), 0\ event\ \ in\ [s, t)\}}{\Pr\{N(t) = 1\}}$$

$$= \frac{\Pr\{1\ event\ in\ [0, s), \}\Pr\{0\ event\ \ in\ [s, t)\}}{\Pr\{N(t) = 1\}}$$

$$= \frac{s}{t}$$

Generalizations of the Poisson Process

- **Definition 5.3** The counting process $\{N(t), t \geq 0\}$ is said to be *nonhomogeneous Poisson process with intensity function $\lambda(t)$*, $t \geq 0$, if
  1. $N(0) = 0$.
  2. $\{N(t), t \geq 0\}$ has independent increments.
  3. $P\{N(h) = 1\} = \lambda(t)h + o(h)$.
  4. $P\{N(t+h) - N(h) \geq 2\} = o(h)$.