Final Examination

December 11, 2023

1. (20pts) Prove the following sentences.

   a. (10pts) If $f(x) = 0$ for $x < 0$, then, for any $\alpha > 0$, prove $P\{X \geq \alpha\} \leq \frac{E[X]}{\alpha}$.

   b. (10pts) If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$, prove

   $$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

2. (20pts) The random variable X has the following probability mass function: $P\{X = 0\} = \frac{1}{2}$, $P\{X = 1\} = \frac{1}{3}$, $P\{X = 3\} = \frac{1}{6}$. Answer the following questions.

   a. (5pts) Find $P\{X \leq 2\}$

   b. (5pts) Find $E[X]$

   c. (5pts) Find $E[X^2]$

   d. (5pts) Find the variance of X, $Var(X)$

3. (10pts) Let X be an exponential random variables with mean $\frac{1}{\lambda}$.

   a. (5pt) Find the probability density function of X

   b. (5pt) Find the moment generation function of random variable X.

4. (40pts) Let $N(t)$ be a Poisson process with rate λ, which means that the probability that n events occurs during time period t is

   $$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

   for $t \geq 0$.

   a. (10pt) Let us denote the time of the first event by T_1. For $n > 1$, let T_n denote the elapsed time between the (n-1)st and the nth event. Find the distribution of the T_n.

   b. (10pt) Find the expected time until 4 event occur.

c. (10pt) Find the expected time until 4 event occur for given that two events occurred at time 2.

d. (10pt) Find the expected number of events between times 2 and 4 for given that three events occurred at time 1.

5. (20pts) There are independent trials, each of which is a success with probability p. Answer the following questions

   a. (10pt) The trials are performed until ONE success happens. What is the mean number of necessary trials?

   b. (10pt) The trials are performed until there are THREE consecutive successes. What is the mean number of necessary trials?

6. (20pts) Let X and Y are independent random variables both uniformly distributed on $(0, 1)$. When a new random variable Z is defined as $Z = X + Y$. Answer the following questions.

   a. (10pt) Find the probability density function of X

   b. (10pt) Find the probability density function of Z