Midterm Examination

April 15, 2015

1. (10pts) If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$, prove

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

2. (30pts) Let c be a constant, and X and Y be random variables. Show that

   a. (10pt) $Var(cX) = c^2 Var(X)$

   b. (10pt) $Var(c + X) = Var(X)$

   c. (10pts) $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$.

3. (20pts) Let X have the Poisson distribution with pmf

$$P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}.$$

   a. (5pt) Find the moment generating function of X.

   b. (5pt) Find the expectation of X, $E[X]$.

   c. (5pt) Find the second moment of X, $E[X^2]$.

   d. (5pt) Find the variance of X.

4. (40pts) Let $N(t)$ be a Poisson process with rate λ, which means that the probability that n events occurs during time period t is

$$P\{N(t + s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

   for $t \geq 0$.

   a. (10pt) Let us denote the time of the first event by T_1. For $n > 1$, let T_n denote the elapsed time between the (n-1)st and the nth event. Find the distribution of the T_n.

   b. (10pt) Find the expected time until 4 event occur.

c. (10pt) Find the expected time until 4 event occur for given that two events occurred at time 2.

d. (10pt) Find the expected number of events between times 2 and 4 for given that three events occurred at time 1.

5. (10pts) Independent trials, each of which is a success with probability p, are performed until there are 2 consecutive successes. What is the mean number of necessary trials?

6. (20pts) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. Let the process be in state 0 when it rains and state 1 when it does not rain.

   a. (10pt) Find a transition probabilities of two-state Markov chain.

   b. (10pt) Find the probability that it will rain the day after tomorrow when it rains today.