Mathematical Description of Linear Systems with various constraints

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3 Linearization

3.1 Without control input

Let us consider the system of first-order nonlinear ordinary differential equations without control input, given by

$$\dot{x} = f(t,x) = \begin{bmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{bmatrix} \in \mathbb{R}^n. \tag{7}$$

Assume that f has $\mathcal{C}^1$ continuity and $\phi(t)$ is a given solution of (7). Then we can linearize the nonlinear system (7) about $\phi(t)$ in the following manner.

Define $z(t) = x(t) - \phi(t)$. Then

$$\dot{z}(t) = A(t)z(t)$$

where

$$A(t) = \frac{\partial f}{\partial x}(t,x)\bigg|_{x=\phi(t)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(t,x) & \cdots & \frac{\partial f_1}{\partial x_n}(t,x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(t,x) & \cdots & \frac{\partial f_n}{\partial x_n}(t,x) \end{bmatrix}\right|_{x=\phi(t)}.$$

Remark 1. $\dfrac{\partial f}{\partial x}(t,x)$ is called the Jacobian matrix of $f(t,x)$ with respect to x.

Remark 2. If the solution $\phi(t) = \mathbf{0} \in \mathbb{R}^n$, then due to $z(t) = x(t)$, it holds that

$$\dot{x}(t) = A(t)x(t).$$

3.2 With control input

Let us consider the system of first-order nonlinear ordinary differential equations with control input, given by

$$\dot{x} = f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix} \in \mathbb{R}^n . \tag{8}$$

Assume that f has $\mathcal{C}^1$ continuity and $\phi(t)$ is a given solution of (8) for the given specific function $\psi(t)$.
Then we can linearize the nonlinear system (8) about $\phi(t)$ in the following manner.

$$\dot{z}(t) = A(t)z(t) + B(t)v(t)$$

where $z(t) = x(t) - \phi(t)$, $v(t) = u(t) - \psi(t)$,

$$A(t) = \frac{\partial f}{\partial x}(t, x, u)\bigg|_{x=\phi(t), u=\psi(t)}$$

$$B(t) = \frac{\partial f}{\partial u}(t, x, u)\bigg|_{x=\phi(t), u=\psi(t)}.$$

Examples

Example 3.1. Let us consider a simple pendulum described by

$$\ddot{\theta}(t) + k \sin \theta(t) = 0$$

where $k > 0$ is a constant.

Example 3.2. Let us consider the following nonlinear system:

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$

$$\ddot{\theta}(t) = -2\frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t).$$

Example 3.3. Let us consider the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}$$

where $g = 10$ (m/sec^2) and $l = 1$ (m). Then, a linearized model of this system about the solution $\phi(t) = [0, 0]^T$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) For $x(0) = [x_0, 0]^T$ with $x_0 = \pi/18, \pi/12, \pi/6$, and $\pi/3$, plot the states for $t \geq 0$, for the nonlinear model, using MATLAB.

(b) Repeat (a) for the linear model.

(c) Compare the results in (a) and (b).

Assignment 3.1. Let us consider the following nonlinear system:

$$\ddot{\varphi} - \left(\frac{g}{L'}\right)\sin\varphi + \left(\frac{1}{L'}\right)\ddot{S}\cos\varphi = 0$$

$$M\ddot{S} + F\dot{S} = \mu(t).$$

Linearize this nonlinear system about $\phi(t) = 0$.