Observer-based output-feedback control for linear systems

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Useful Lemmas

Lemma 1 (Congruent transform). For any nonsingular (i.e., invertible) matrix M, the following two conditions are equivalent:

- $\Omega < 0$ (or $\Omega \leq 0$) (1)
- $M^T \Omega M < 0$ (or $M^T \Omega M \leq 0$). (2)

Lemma 2. For a positive scalar ϵ and any matrices $M \in \mathbb{R}^{r\times s}$ and $N \in \mathbb{R}^{s\times r}$, the following inequality holds:

$$\mathbf{He}\{MN\} \leq \epsilon^{-1}MM^T + \epsilon N^T N.$$

Lemma 3. For a symmetric matrix $\Omega < 0$ and any matrices M and N, the following inequality holds:

$$M^T\Omega M \leq -\mathbf{He}\{M^T N\} - N^T\Omega^{-1}N.$$

Lemma 4 (Schur complement). For any matrices $S = S^T$, $R = R^T$, and N, if the following condition is satisfied:

$$\left[\begin{array}{c:c} S & N \\ \hdashline N^T & R \end{array}\right] < 0 \left(\text{or} \left[\begin{array}{c:c} R & N^T \\ \hdashline N & S \end{array}\right] < 0 \right) \tag{3}$$

then it holds that

$$S - NR^{-1}N^T < 0,\ S < 0,\ R < 0. \tag{4}$$

1 State observer design

• To begin with, let us consider the following linear state-space model:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ represents the control input, and $y(t) \in \mathbb{R}^p$ represents the measurable output.

• In most cases, a so-called Luenberger observer is utilized for estimating the state:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - \tilde{y}(t)\big) \\ \tilde{y}(t) = C\tilde{x}(t). \end{cases} \tag{6}$$

where $\tilde{x}(t)$ denotes the estimated state.

• In (6), the matrix $L \in \mathbb{R}^{n\times p}$ is called the observer gain to be designed later.

- Now, let us define the estimation error as

$$e(t) = x(t) - \tilde{x}(t).$$

- Then, our aim is to design the observer gain L that achieve

$$\lim_{t \to \infty} e(t) \to 0. \tag{7}$$

- To accomplish this aim, we need to derive the error dynamic system model from (5) and (6):

$$\begin{aligned} \dot{e}(t) &= Ae(t) + L\Big(y(t) - \tilde{y}(t)\Big) \\ &= \Big(A + LC\Big)e(t). \end{aligned} \tag{8}$$

The following theorem provides LMI-based observer design conditions obtained in the sense of Lyapunov stability.

Theorem 1. Convergence condition (7) holds if there exist matrices $P = P^T \in \mathbb{R}^{n\times n}$ and $\bar{L} \in \mathbb{R}^{n\times p}$, such that

$$\begin{aligned} &P > 0 \\ &\mathbf{He}\Big\{PA + \bar{L}C\Big\} < 0. \end{aligned}$$

Furthermore, the observer gain can be reconstructed in this manner:

$$L = P^{-1}\bar{L}.$$

Example 1.1. Design an observer gain for

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0.5 & 0 \end{bmatrix}.$$

And see if the estimated state approaches to the real state, where the initial condition is given as $x(0) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$ and $\tilde{x}(0) = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$.

2 Observer-based control design via separation principle

• Let us consider a continuous-time linear system of the following general form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{9}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ represents the control input, and $y(t) \in \mathbb{R}^p$ represents the measurable output.

- In observer-based control systems,

(i) the observer is used to estimate the internal state of the system based on the available input and output measurements.

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - C\tilde{x}(t)\big). \tag{10}$$

(ii) This estimated state $\tilde{x}(t)$ is then used by the controller to generate control signals to regulate the behavior of system.

$$u(t) = K\tilde{x}(t). \tag{11}$$

• To sum up, our main goal is to design the control gain K and the observer gain L such that

$$\textbf{(i)} \quad \lim_{t\to\infty} x(t) \to 0$$
$$\textbf{(ii)} \quad \lim_{t\to\infty} e(t) \to 0$$

where $x(t)$ is the state and $e(t) = x(t) - \tilde{x}(t)$ is the estimation error.

• To accomplish this goal, we first have to obtain a dynamic system model with respect to $x(t)$ and $e(t)$:

$$\dot{x}(t) = ...$$
$$\dot{e}(t) = ...$$

- Based on $\tilde{x}(t) = x(t) - e(t)$, Eqs. (9) and (11) provide

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BK\tilde{x}(t) \\ &= (A + BK)x(t) - BKe(t).\end{aligned}$$

- From (9) and (10), it follows that

$$\dot{e}(t) = (A + LC)e(t).$$

- As a result, the closed-loop control system is described as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \left[\begin{array}{c|c} A + BK & -BK \\ \hline 0 & A + LC \end{array}\right] \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \qquad (12)$$

Separation principle: The separation principle refers to the ability to design the observer and the controller independently.

In other words, the separation principle states that the design of the observer can be carried out independently of the controller design.

- Controller design: Theorem 1 in "State-feedback control"
- Observer design: Theorem 1 in Section 1.

Proof: The stability of closed-loop system (12) is ensured if the following condition holds:

$$\text{real number of } \lambda_i \left(\left[\begin{array}{c:c} A+BK & -BK \\ \hdashline 0 & A+LC \end{array} \right] \right) < 0, \ \forall i = 1, 2, \cdots, 2n.$$

This condition is equivalent to

- real number of $\lambda_i \left(A + BK\right) < 0, \ \forall i = 1, 2, \cdots, n$
- real number of $\lambda_i \left(A + LC\right) < 0, \ \forall i = 1, 2, \cdots, n.$

Consequently, it can be said that both control and observer gains can be designed separately.

Example 2.1. By using the separation principle, obtain both control and observer gains for

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \ C = \begin{bmatrix} 0.1 & 0 \end{bmatrix}.$$

And see if the state trajectory on the x_1-x_2 plane converges to the origin, where $x(0) = \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}$ and $\tilde{x}(0) = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$.

Remark 1. The separation principle offers convenience when designing observer-based control. However, it also presents the following disadvantages:

- When the system contains uncertainties, it is impossible to utilize the separation principle because these uncertainties cannot be incorporated into the observer.
- In performance-based control designs, the separation principle fails to yield an optimal solution that includes both control and observer gains.

3 Observer-based control design via Lyapunov stability approach

• Let us consider a continuous-time linear system of the following general form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{13}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ represents the control input, and $y(t) \in \mathbb{R}^p$ represents the output.

- Based on (10) and (11), the observer-based control is described as

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - C\tilde{x}(t)\big) \\ u(t) = K\tilde{x}(t) \end{cases} \tag{14}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ represents the estimated state; $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ are the control gain and the observer gain, respectively, to be designed later.

- As a result, based on

$$\zeta(t) = \left[\begin{array}{c} x(t) \\ \hdashline e(t) \end{array}\right] \in \mathbb{R}^{2n \times 2n}$$

the closed-loop system with (13) and (14) is described as follows:

$$\dot{\zeta}(t) = \bar{A}\zeta(t) \tag{15}$$

where

$$\zeta(t) = \left[\begin{array}{c} x(t) \\ \hdashline e(t) \end{array}\right] \in \mathbb{R}^{2n \times 2n},\ \bar{A} = \left[\begin{array}{c:c} A + BK & -BK \\ \hdashline 0 & A + LC \end{array}\right].$$

Theorem 2. For a prescribed value μ, close-loop control system (15) is stable at origin if there exist matrices $\bar{P}_1 = \bar{P}_1^T \in \mathbb{R}^{n\times n}$, $P_2 = P_2^T \in \mathbb{R}^{n\times n}$, $\bar{K} \in \mathbb{R}^{m\times n}$, and $\bar{L} \in \mathbb{R}^{n\times p}$ such that

$$\bar{P}_1 > 0 \tag{16}$$

$$P_2 > 0 \tag{17}$$

$$\left[\begin{array}{cc|c} \mathbf{He}\{A\bar{P}_1 + B\bar{K}\} & -B\bar{K} & 0 \\ \star & -\mathbf{He}\{\mu\bar{P}_1\} & \mu I \\ \hline \star & \star & \mathbf{He}\{P_2A + \bar{L}C\} \end{array}\right] < 0. \tag{18}$$

Furthermore, the control and observer gains are constructed as follows:

$$K = \bar{K}\bar{P}_1^{-1}, \quad L = P_2^{-1}\bar{L}.$$

Example 3.1. By using Theorem 2 for $\mu = 100$, obtain both control and observer gains for

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \; C = \begin{bmatrix} 0.1 & 0 \end{bmatrix}.$$

And see if the state trajectory on the x_1-x_2 plane converges to the origin, where $x(0) = \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}$ and $\tilde{x}(0) = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$. Finally, compare it with the state trajectory of Example 2.1.

4 Robust observer-based control

• Let us consider a continuous-time uncertain system of the following general form:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \\ y(t) = Cx(t) \end{cases} \tag{25}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times m}$ denote the parameter uncertainties expressed as

$$\Delta A = E\Upsilon H_1,\ \Delta B = E\Upsilon H_2. \tag{26}$$

- As in (14), the observer-based control is configured by using the nominal matrices A, B, and C:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - C\tilde{x}(t)\big) \\ u(t) = K\tilde{x}(t) \end{cases}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ represents the estimated state; $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ are the control gain and the observer gain, respectively, to be designed later.

- Hence, based on

$$\zeta(t) = \left[\begin{array}{c} x(t) \\ \hdashline e(t) \end{array}\right] \in \mathbb{R}^{2n\times 2n}$$

the closed-loop system with (25) and (14) is described as follows:

$$\dot{\zeta}(t) = (\bar{A} + \Delta\bar{A})\zeta \tag{27}$$

where

$$\bar{A} = \left[\begin{array}{c:c} A+BK & -BK \\ \hdashline 0 & A+LC \end{array}\right], \ \Delta\bar{A} = \left[\begin{array}{cc} \Delta A + \Delta BK & -\Delta BK \\ \Delta A + \Delta BK & -\Delta BK \end{array}\right].$$

- Furthermore, the closed-loop system can be transformed into

$$\dot{\zeta}(t) = (\bar{A} + \bar{E}\Upsilon\bar{H})\zeta(t)$$

where

$$\bar{A} = \left[\begin{array}{c:c} A + BK & -BK \\ \hdashline 0 & A + LC \end{array}\right], \quad \bar{E} = \left[\begin{array}{c} E \\ \hdashline E \end{array}\right]$$

$$\bar{H} = \left[\begin{array}{c:c} H_1 + H_2 K & -H_2 K \end{array}\right].$$