Stability analysis of linear systems

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Contents

- Preliminaries
- Hurwitz stability (for linear systems)
- Lyapunov stability (for linear systems)
- Robust stability (for uncertain linear systems)
- $\mathcal{H}_\infty$ stability (for linear systems with disturbances)

1 Preliminaries

- The state-space model of linear time-invariant systems is given as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state vector and the control input vector, respectively.

- When $u(t) \equiv 0$, the state-space model reduces to

$$\dot{x}(t) = Ax(t) \tag{2}$$

which is called the **autonomous** system.

2 Stability analysis

Definition 1. **Stability** is the ability of an autonomous system to drive its state towards a desired equilibrium point (often denoted as the origin) over time, even if the system starts from non-zero initial conditions.

More formally, if for every initial state $x(0)$, the following condition holds when $u(t) \equiv 0$:

$$\lim_{t \to \infty} x(t) = 0 \tag{3}$$

then system (2) can be said to be **stable** at the origin.

2.1 Hurwitz stability criterion

• It is worth noting that the eigenvalue of matrix A corresponds to the pole of the transfer function.

Example 2.1. Consider

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = u(t).$$

• The autonomous system is stable if and only if matrix A has all its eigenvalues in the open left-half plane $\mathbb{C}^-$.

$$\text{“system is stable”} \;\Leftrightarrow\; \mathbf{Re}\Big[\lambda(A)\Big] < 0,\ \forall i = 1, 2, \cdots, n.$$

Example 2.2. Find the eigenvalue of A given by

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

and assess the stability of $\dot{x}(t) = Ax(t)$.

Example 2.3. Find the range of k such that $\dot{x}(t) = Ax(t)$ is stable, where

$$A = \begin{bmatrix} -1+k & -2 \\ 1 & -4 \end{bmatrix}.$$

2.2 Lyapunov stability

• The Lyapunov stability is a well-known method for evaluating the stability of

$$\dot{x}(t) = Ax(t).$$

• To employ the Lyapunov stability method effectively, our first step is to select a proper **Lyapunov function**, described as $V(t)$.

- In linear systems, the standard choice for a Lyapunov function is as follows:

$$V(t) = x^T(t)Px(t) \tag{4}$$

and $P \in \mathbb{R}^{n\times n}$ must be symmetric and positive definite.

- The Lyapunov function becomes a positive real-valued function:

$$V(t) = x^T(t)Px(t) > 0, \ \forall x(t) \neq 0.$$

Lemma 1 (Lyapunov stability criterion). If it holds that

$$\dot{V}(t) < 0 \tag{5}$$

system (2) can be said to be stable at the origin.

Proof: If (5) holds, then $V(t)$ is strictly monotonically decreasing. Accordingly, $V(t)$ converges to zero as time goes to infinity:

$$\lim_{t \to \infty} x^T(t) P x(t) = 0.$$

Since $P > 0$, this limitation is achieved when the state $x(t)$ converges to the origin, written as follows:

$$\lim_{t \to \infty} x(t) = 0.$$

As a result, according to Definition 1, we can claim that system (2) is stable at the origin.

- Now, the Lyapunov stability criterion in (5) will be transformed into a set of linear matrix inequalities (LMIs).

- This is because LMIs can be solved through numerical tools that use various optimization techniques.

With this aim, the following lemma provides a set of LMIs to assess the stability of (2).

Lemma 2. Suppose that there exists a symmetric matrix P that satisfies

$$P > 0 \tag{6}$$

$$A^T P + PA < 0. \tag{7}$$

Then, system (2) is stable at the origin.

Proof: From (4), it follows that

$$\begin{aligned}\dot{V}(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &= x^T(t)A^TPx(t) + x^T(t)PAx(t) \\ &= x^T(t)\Big(A^TP + PA\Big)x(t).\end{aligned}$$

Thus, if (7) holds, it is satisfied that $\dot{V}(t) < 0$.

Therefore, according to Lemma 1, we can claim that system (2) is stable at the origin. ∎

Example 2.4. By specifying $P = \mathbf{diag}(p_1, p_2) > 0$, show the stability of (2) with

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$

And when $p_1 = 1$, provide a possible value of $p_2 > 0$.

Example 2.5. Using the robust toolbox in MATLAB, analyze the stability of (2) with

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$

Example 2.6. The following inequality provides the exponential stability criterion:

$$\dot{V}(t) < -\alpha V(t), \ \alpha > 0.$$

Please formulate a set of linear matrix inequalities to check this stability criterion.

2.3 Robust stability

• **Robust stability** is the ability of an autonomous system to remain stable despite variations or uncertainties in its parameters.

• For this reason, robust stability analysis is addressed considering the following uncertain linear system:

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) \tag{8}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ denotes the parameter uncertainty.

Definition 2. If system (8) maintains stability even in the presence of uncertainty, it is said that the system is **robustly stable**.

- As studied in Chap. 1, the following decomposition is available:

$$\Delta A = E \Upsilon H \tag{9}$$

where $E \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{q\times n}$ are known real constant matrices; and $\Upsilon \in \mathbb{R}^{p\times q}$ is an unknown time-varying or time-invariant matrix satisfying

$$\Upsilon^T \Upsilon \leq I. \tag{10}$$

The following two lemmas will be used to obtain the robust stability condition in terms of LMIs.

Lemma 3. For a positive scalar ϵ and any matrices $M \in \mathbb{R}^{r \times s}$ and $N \in \mathbb{R}^{s \times r}$, the following inequality holds:

$$\mathbf{He}\{MN\} \leq \epsilon^{-1} M M^T + \epsilon N^T N.$$

Example 2.7. For

$$M = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ N = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

provide ϵ to minimize the maximum eigenvalue of

$$\epsilon^{-1} M M^T + \epsilon N^T N.$$

Lemma 4 (Schur complement). For any matrices $S = S^T$, $R = R^T$, and N, if the following condition is satisfied:

$$\left[\begin{array}{c|c} S & N \\ \hline N^T & R \end{array}\right] < 0 \quad \left(\text{or} \quad \left[\begin{array}{c|c} R & N^T \\ \hline N & S \end{array}\right] < 0 \right) \tag{11}$$

then it holds that

$$S - NR^{-1}N^T < 0,\ S < 0,\ R < 0. \tag{12}$$

Example 2.8. Without using the eigenvalue approach, provide the condition of k such that the following inequality holds:

$$\begin{bmatrix} -1+k & 2 \\ 2 & -5 \end{bmatrix} < 0.$$

Example 2.9. Find the condition of k such that the following inequality holds:

$$\begin{bmatrix} -1+k & 2 & 1 \\ 2 & -5 & 0 \\ 1 & 0 & -1 \end{bmatrix} < 0.$$

The following theorem provides a set of linear matrix inequalities (LMIs) for robust stability.

Theorem 1. System (8) is robustly stable at the origin if there exist a symmetric matrix P and a positive scalar ϵ satisfying

$$P > 0 \tag{13}$$

$$\left[\begin{array}{c|c} \mathbf{He}\{PA\} + \epsilon H^T H & PE \\ \hline E^T P & -\epsilon I \end{array}\right] < 0. \tag{14}$$

Example 2.10. Analyze the robust stability of (8) through the utilization of MATLAB:

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}, \; E = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Remark 1. Let us recall the following robust stability condition:

$$\mathbf{He}\{PA\} + \underbrace{\left(\epsilon^{-1}PEE^TP + \epsilon H^TH\right)}_{\geq 0} < 0.$$

If the maximum eigenvalues of matrices EE^T and H^TH are larger, then it becomes more difficult to identify a feasible matrix P.

For instance, from $\Delta A = \begin{bmatrix} -\delta R & -\delta \\ 0 & 0 \end{bmatrix}$ with $\delta \in [-0.1, 0.1]$, it follows that

$$\begin{aligned} \Delta A &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} (\delta) \begin{bmatrix} R & 1 \end{bmatrix} \;\mapsto\; \lambda_{\max}(EE^T) = 1 \\ &= \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} (10\delta) \begin{bmatrix} R & 1 \end{bmatrix} \;\mapsto\; \lambda_{\max}(EE^T) = 0.01. \end{aligned}$$

2.4 $\mathcal{H}_\infty$ stability

• This stability requires to simultaneously analyze **(i)** the stability and **(ii)** the $\mathcal{H}_\infty$ performance for the following linear systems with external disturbances:

$$\begin{cases} \dot{x}(t) = Ax(t) + Nw(t) \\ z(t) = Gx(t) \end{cases} \tag{18}$$

• As mentioned, $w(t) \in \mathbb{R}^d$ denotes the external disturbances, and $z(t) \in \mathbb{R}^q$ denotes the performance output.

• To consider the $\mathcal{H}_\infty$ performance, it is essential to assume that $w(t) \in \mathcal{L}_2$, that is,

$$\int_0^\infty w^T(\tau)w(\tau)d\tau < \infty.$$

Definition 3. If the following two conditions hold:

(i) for $x(0) \neq 0$ and $w(t) = 0$,

$$\dot{V}(t) < 0 \quad :\sim \text{stability criterion}$$

(ii) for $x(0) = 0$ and $w(t) \neq 0$,

$$\int_0^\infty z^T(\tau)z(\tau)d\tau \leq \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau \quad :\sim \mathcal{H}_\infty \text{ performance}$$

then it can be said that (18) is stable and has an $\mathcal{H}_\infty$ disturbance attenuation level γ.

The following lemma provides the $\mathcal{H}_\infty$ stability criterion.

Lemma 5. If it holds that

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \tag{19}$$

then system (18) is stable and has an $\mathcal{H}_\infty$ disturbance attenuation level γ.

The following theorem provides a set of linear matrix inequalities (LMIs) for (19).

Theorem 2. System (18) is stable and has an $\mathcal{H}_\infty$ disturbance attenuation level γ, if there exist a symmetric matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar γ satisfying

$$P > 0 \tag{22}$$

$$\begin{bmatrix} \mathbf{He}\{PA\} + G^T G & PN \\ N^T P & -\gamma^2 I \end{bmatrix} < 0. \tag{23}$$

Example 2.11. Analyze the $\mathcal{H}_\infty$ stability of (8) through the utilization of MATLAB:

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}, \; N = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; \gamma = 0.64.$$

What happens when γ becomes less than 0.64?

Example 2.12. Obtain a set of linear matrix inequalities (LMIs) that ensures the robust $\mathcal{H}_\infty$ stability of

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + Nw(t) \\ z(t) = Gx(t). \end{cases}$$

Simulation report: Recall Example 2.2 in Chap.1, which addresses the state-space model of a dc servomotor. The parameter values are configured as follows:

$$R_a = 5.385,\ L_a = 3.694 \times 10^{-3},\ K_T = K_\theta = 0.0583$$
$$J = 6.88627 \times 10^{-6},\ B = 3.1346 \times 10^{-5}\ \pm 20\%.$$

Note that B (:∼ viscous friction coefficient) has an uncertainty of $\pm 20\%$. Using the LMIs obtained in Example 2.12, for

$$N = \begin{bmatrix} 0.1 \\ 0.01 \\ 0 \end{bmatrix},\ G = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(1) **find** the minimum value of γ (i.e., $\mathcal{H}_\infty$ performance level) through the utilization of MATLAB, and (2) **report** the overall process.