5 Linear systems with external disturbances

5.1 Norms of signals

- The $\mathcal{L}_1$-norm of vector-valued signal $v(t) \in \mathbb{R}^w$ is given by

$$||v(t)||_1 := \sum_{i=1}^{w} \int_0^\infty |v_i(t)|dt.$$

If $||v(t)||_1 < \infty$, it can be said that $v(t)$ belongs to the $\mathcal{L}_1$-space defined as

$$\mathcal{L}_1 := \left\{ v(t) \in \mathbb{R}^w \;\middle|\; ||v(t)||_1 < \infty \right\}.$$

- The $\mathcal{L}_2$-norm of vector-valued signal $v(t) \in \mathbb{R}^w$ is given by

$$||v(t)||_2 := \left(\sum_{i=1}^{w} \int_0^\infty v_i^2(t)dt \right)^{\frac{1}{2}} = \left(\int_0^\infty v^T(t)v(t)dt \right)^{\frac{1}{2}}.$$

If $||v(t)||_2 < \infty$, it can be said that $v(t)$ belongs to the $\mathcal{L}_2$-space defined as

$$\mathcal{L}_2 := \left\{ v(t) \in \mathbb{R}^w \;\middle|\; ||v(t)||_2 < \infty \right\}.$$

- The $\mathcal{L}_\infty$-norm of vector-valued signal $v(t) \in \mathbb{R}^w$ is given by

$$||v(t)||_\infty := \max_{i=1,2,\ldots,w} \left(\sup_{t \geq 0} |v_i(t)| \right).$$

If $||v(t)||_\infty < \infty$, it can be said that $v(t)$ belongs to the $\mathcal{L}_\infty$-space defined as

$$\mathcal{L}_\infty := \left\{ v(t) \in \mathbb{R}^w \;\middle|\; ||v(t)||_\infty < \infty \right\}.$$

5.2 External disturbances

In general, the linear systems with external disturbances are described as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nw(t) \\ z(t) = Gx(t) + Hu(t) \end{cases} \tag{13}$$

where $w(t) \in \mathbb{R}^d$ denotes the external disturbances, and $z(t) \in \mathbb{R}^q$ denotes the performance output.

To handle the disturbance attenuation problem, the influence of $w(t)$ on $z(t)$ is expressed as a performance index. At that time, the disturbance $w(t)$ is characterized through some vector-valued signal spaces, such as $w(t) \in \mathcal{L}_2$ and $w(t) \in \mathcal{L}_\infty$.

As is well known, one representative performance index is the $\mathcal{H}_\infty$ performance.

Definition 1. For $w(t) \in \mathcal{L}_2$, system (13) has the $\mathcal{H}_\infty$ performance level γ if the response $z(t)$ satisfies

$$\frac{||z(t)||_2^2}{||w(t)||_2^2} \leq \gamma^2$$

that is,

$$\int_0^\infty z^T(\tau)z(\tau)d\tau \leq \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau.$$