Deriving State Equations for a DC Servo Motor

A. System Model

A useful component in many real control systems is a permanent magnet DC servo motor. The input signal to the motor is the armature voltage $V_a(t)$, and the output signal is the angular position $\theta(t)$. A schematic diagram for the motor is shown in Fig. 1. The terms R_a and L_a are the resistance and inductance of the armature winding in the motor, respectively. The voltage V_b is the back EMF generated internally in the motor by the angular rotation. J is the inertia of the motor and load (assumed lumped together), and B is the damping in the motor and load relative to the fixed chassis.

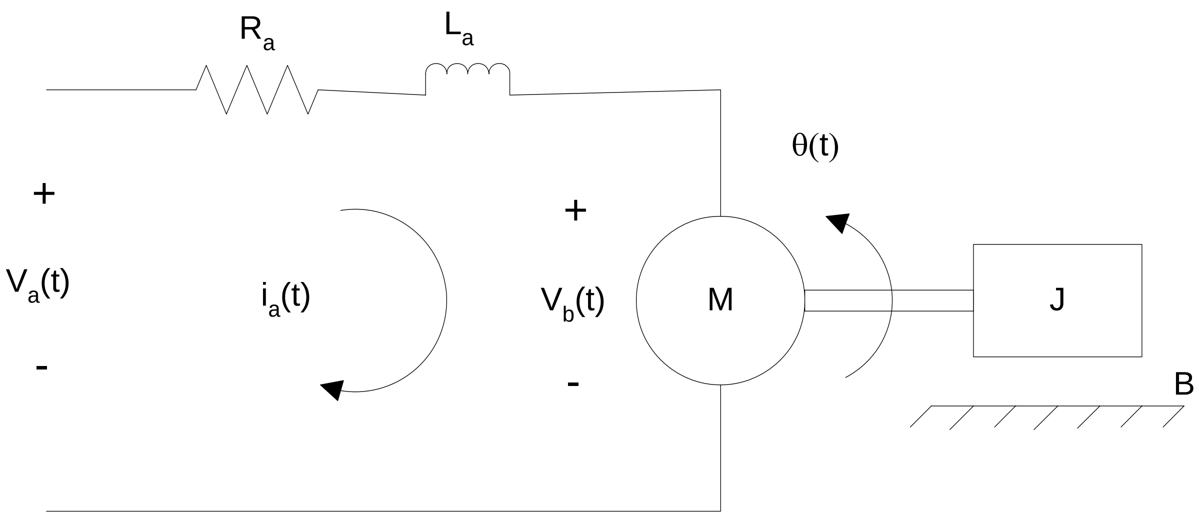

Fig. 1. Schematic diagram of a DC servo motor.

The equations for the electrical side of the system are

$$V_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + V_b(t) \qquad \text{with} \qquad V_b(t) = K_b \frac{d\theta(t)}{dt} \tag{1}$$

$$V_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b \frac{d\theta(t)}{dt} \tag{2}$$

where K_b is the motor's back EMF constant. The equations for the mechanical side of the system are

$$J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} = T_{app}(t) \qquad \text{with} \qquad T_{app(t)} = K_T i_a(t) \tag{3}$$

$$J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} = K_T i_a(t) \tag{4}$$

where T_{app} is the applied torque, and K_T is the torque constant that relates the torque to the armature current.

B. Developing The State Equations

There are three derivative terms in the system model for the DC servo motor—first derivative of $i_a(t)$ in Eqn. (2) and first and second derivatives of $\theta(t)$ in Eqn. (4). Therefore, there are a total of 3 state variables in the state space model. Although the first derivative of $\theta(t)$ appears in both of those equations, it is the same variable and so does not add an additional state variable.

A simulation diagram can be drawn for this system by solving Eqns. (2) and (4) for the highest derivative term in each. This yields the expressions in (5) and (6). The simulation diagram is shown in Fig. 2. The state variables will be defined as the outputs of the integrators, with x_1 being i_a, x_2 being θ, and x_3 being $d\theta/dt$.

$$\frac{di_a(t)}{dt} = -\left(\frac{R_a}{L_a}\right) i_a(t) - \left(\frac{K_b}{L_a}\right) \frac{d\theta(t)}{dt} + \left(\frac{1}{L_a}\right) V_a(t) \tag{5}$$

$$\frac{d^2\theta(t)}{dt^2} = -\left(\frac{B}{J}\right) \frac{d\theta(t)}{dt} + \left(\frac{K_T}{J}\right) i_a(t) \tag{6}$$

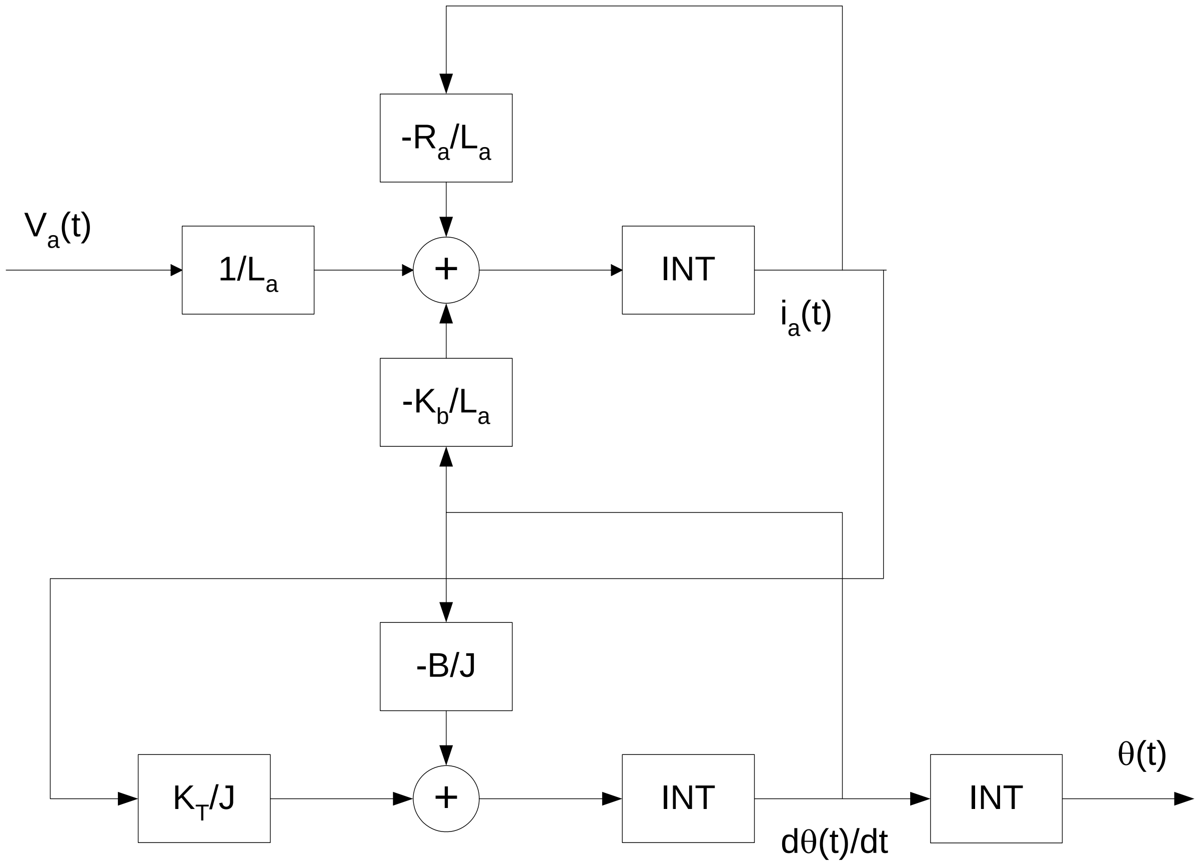

Fig. 2. Simulation diagram for the DC servo motor example.

With these definitions for the state variables, and defining $u(t) = V_a(t)$ and $y(t) = \theta(t)$, the state and output equations are:

$$\begin{aligned} \dot{x}_1(t) &= -\left(\frac{R_a}{L_a}\right) x_1(t) - \left(\frac{K_b}{L_a}\right) x_3(t) + \left(\frac{1}{L_a}\right) u(t) & (7) \\ \dot{x}_2(t) &= x_3(t) & (8) \\ \dot{x}_3(t) &= -\left(\frac{B}{J}\right) x_3(t) + \left(\frac{K_T}{J}\right) x_1(t) & (9) \end{aligned}$$

$$y(t) = x_2(t) \tag{10}$$

In vector-matrix form, the state and output equations are:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -R_a/L_a & 0 & -K_b/L_a \\ 0 & 0 & 1 \\ K_T/J & 0 & -B/J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} u(t) \tag{11}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{12}$$

In short-hand notation, the state and output equations are

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{13}$$

The state variables chosen for this example are all real physical variables representing energy stored in the system. The structure of the A, B, and C matrices are different than what was seen using the Direct Method or Nested Integration. When the state variable are physical variables in the system, there will not be a fixed structure to those matrices. The sizes of the matrices will always be $n \times n$ for A, $n \times r$ for B, $m \times n$ for C, and $m \times r$ for the D matrix if it is non-zero.