3 Observer-based control design via Lyapunov stability approach

• Let us consider a continuous-time linear system of the following general form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{13}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ represents the control input, and $y(t) \in \mathbb{R}^p$ represents the output.

- Based on (10) and (11), the observer-based control is described as

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - C\tilde{x}(t)\big) \\ u(t) = K\tilde{x}(t) \end{cases} \tag{14}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ represents the estimated state; $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ are the control gain and the observer gain, respectively, to be designed later.

- As a result, based on

$$\zeta(t) = \left[\begin{array}{c} x(t) \\ \hdashline e(t) \end{array}\right] \in \mathbb{R}^{2n \times 2n}$$

the closed-loop system with (13) and (14) is described as follows:

$$\dot{\zeta}(t) = \bar{A}\zeta(t) \tag{15}$$

where

$$\bar{A} = \left[\begin{array}{c:c} A + BK & -BK \\ \hdashline 0 & A + LC \end{array}\right].$$

Theorem 2. For a prescribed value μ, close-loop control system (15) is stable at origin if there exist matrices $\bar{P}_1 = \bar{P}_1^T \in \mathbb{R}^{n\times n}$, $P_2 = P_2^T \in \mathbb{R}^{n\times n}$, $\bar{K} \in \mathbb{R}^{m\times n}$, and $\bar{L} \in \mathbb{R}^{n\times p}$ such that

$$\bar{P}_1 > 0 \tag{16}$$

$$P_2 > 0 \tag{17}$$

$$\left[\begin{array}{cc|c} \mathbf{He}\{A\bar{P}_1 + B\bar{K}\} & -B\bar{K} & 0 \\ \star & -\mathbf{He}\{\mu\bar{P}_1\} & \mu I \\ \hline \star & \star & \mathbf{He}\{P_2A + \bar{L}C\} \end{array}\right] < 0. \tag{18}$$

Furthermore, the control and observer gains are constructed as follows:

$$K = \bar{K}\bar{P}_1^{-1}, \quad L = P_2^{-1}\bar{L}.$$

Proof: Let us choose a Lyapunov function of the following form:

$$V(t) = x^T(t)P_1x(t) + e^T(t)P_2e(t) = \zeta^T(t)P\zeta(t)$$

where $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, and

$$P = P^T = \left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & P_2 \end{array}\right] > 0.$$

The time derivative of $V(t)$ along with (15) is given by

$$\dot{V}(t) = \zeta^T(t)\mathbf{He}\{P\bar{A}\}\zeta(t).$$

Thus, the observer-based control design condition becomes

$$\mathbf{He}\{P\bar{A}\} < 0 \tag{19}$$

that is,

$$\mathbf{He}\left\{\left[\begin{array}{c:c} P_1(A+BK) & -P_1BK \\ \hdashline 0 & P_2(A+LC) \end{array}\right]\right\} < 0. \tag{20}$$

Furthermore, using

$$M = \left[\begin{array}{c|c} P_1^{-1} & 0 \\ \hline 0 & P_1^{-1} \end{array}\right]$$

the congruent transformation of (20) is given from Lemma 1 as follows:

$$M^T \cdot \mathbf{He}\left\{\left[\begin{array}{c|c} P_1(A+BK) & -P_1BK \\ \hline 0 & P_2(A+LC) \end{array}\right]\right\} \cdot M < 0$$

that is,

$$\mathbf{He}\left\{\left[\begin{array}{c|c} (A+BK)P_1^{-1} & -BKP_1^{-1} \\ \hline 0 & P_1^{-1}(P_2A+P_2LC)P_1^{-1} \end{array}\right]\right\} < 0. \qquad (21)$$

By replacing

$$\bar{P}_1 = P_1^{-1},\ \bar{K} = KP_1^{-1},\ \bar{L} = P_2L$$

condition (21) can be expressed as

$$\left[\begin{array}{c|c} \mathbf{He}\{A\bar{P}_1 + B\bar{K}\} & -B\bar{K} \\ \hline \star & \bar{P}_1\mathbf{He}\{P_2A + \bar{L}C\}\bar{P}_1 \end{array}\right] < 0. \qquad (22)$$

Under the following condition:

$$\mathbf{He}\left\{P_2 A + \bar{L}C\right\} < 0 \tag{23}$$

Lemma 3 allows

$$\bar{P}_1 \mathbf{He}\left\{P_2 A + \bar{L}C\right\} \bar{P}_1 \leq -\mathbf{He}\left\{\mu \bar{P}_1\right\} - \mu^2 \left(\mathbf{He}\left\{P_2 A + \bar{L}C\right\}\right)^{-1}.$$

Thus, condition (22) is ensured by (23) and

$$\left[\begin{array}{c:c} \mathbf{He}\left\{A\bar{P}_1 + B\bar{K}\right\} & -B\bar{K} \\ \hdashline \star & -\mathbf{He}\left\{\mu \bar{P}_1\right\} - \mu^2 \left(\mathbf{He}\left\{P_2 A + \bar{L}C\right\}\right)^{-1} \end{array}\right] < 0. \tag{24}$$

Finally, by Lemma 4, condition (24) is converted into

$$0 > \left[\begin{array}{cc|c} \mathbf{He}\{A\bar{P}_1 + B\bar{K}\} & -B\bar{K} & 0 \\ \star & -\mathbf{He}\{\mu\bar{P}_1\} & \mu I \\ \hline \star & \star & \mathbf{He}\{P_2 A + \bar{L}C\} \end{array}\right]. \qquad (25)$$

Here, it is worth noting that condition (25) guarantees (23). Thus, there is no need to additionally include (23) in the observer-based control design condition. ∎

Example 3.1. By using Theorem 2 for $\mu = 100$, obtain both control and observer gains for

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \ C = \begin{bmatrix} 0.1 & 0 \end{bmatrix}.$$

And see if the state trajectory on the x_1-x_2 plane converges to the origin, where $x(0) = \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}$ and $\tilde{x}(0) = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$. Finally, compare it with the state trajectory of Example 2.1.

4 Robust observer-based control

• Let us consider a continuous-time uncertain system of the following general form:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \\ y(t) = Cx(t) \end{cases} \tag{26}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times m}$ denote the parameter uncertainties expressed as

$$\Delta A = E\Upsilon H_1, \ \Delta B = E\Upsilon H_2. \tag{27}$$

- As in (14), the observer-based control is configured by using the nominal matrices A, B, and C:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) - L\big(y(t) - C\tilde{x}(t)\big) \\ u(t) = K\tilde{x}(t) \end{cases}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ represents the estimated state; $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ are the control gain and the observer gain, respectively, to be designed later.

- Hence, based on

$$\zeta(t) = \left[\begin{array}{c} x(t) \\ \hdashline e(t) \end{array}\right] \in \mathbb{R}^{2n\times 2n}$$

the closed-loop system with (26) and (14) is described as follows:

$$\dot{\zeta}(t) = (\bar{A} + \Delta\bar{A})\zeta \tag{28}$$

where

$$\bar{A} = \left[\begin{array}{c:c} A + BK & -BK \\ \hdashline 0 & A + LC \end{array}\right], \ \Delta\bar{A} = \left[\begin{array}{c:c} \Delta A + \Delta BK & -\Delta BK \\ \hdashline \Delta A + \Delta BK & -\Delta BK \end{array}\right].$$

- Furthermore, the closed-loop system can be transformed into

$$\dot{\zeta}(t) = (\bar{A} + \bar{E}\bar{\Upsilon}\bar{H})\zeta(t)$$

where

$$\bar{A} = \left[\begin{array}{c:c} A+BK & -BK \\ \hdashline 0 & A+LC \end{array}\right], \quad \bar{E} = \left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & E \end{array}\right]$$

$$\bar{\Upsilon} = \left[\begin{array}{c:c} \Upsilon & 0 \\ \hdashline 0 & \Upsilon \end{array}\right], \quad \bar{H} = \left[\begin{array}{c:c} H_1 + H_2K & -H_2K \\ \hdashline H_1 + H_2K & -H_2K \end{array}\right].$$

Theorem 3. For prescribed values μ and $\epsilon > 0$, close-loop control system (15) is robustly stable at origin if there exist matrices $\bar{P}_1 = \bar{P}_1$, $P_2 = P_2^T$, $\bar{K}$, and $\bar{L}$ such that

$$\bar{P}_1 > 0 \tag{29}$$

$$P_2 > 0 \tag{30}$$

$$\left[\begin{array}{cc|cc|c|c} -\epsilon I & 0 & \Psi_c & -H_2\bar{K} & 0 & 0 \\ 0 & -\epsilon I & \Psi_c & -H_2\bar{K} & 0 & 0 \\ \hline \star & \star & \Omega_c + \epsilon E E^T & -B\bar{K} & 0 & 0 \\ \star & \star & \star & -\mathbf{He}\{\mu \bar{P}_1\} & \mu I & 0 \\ \hline \star & \star & \star & \star & \Omega_o & \epsilon P_2 E \\ \hline \star & \star & \star & \star & \star & -\epsilon I \end{array}\right] < 0 \tag{31}$$

where

$$\Omega_c = \mathbf{He}\big\{A\bar{P}_1 + B\bar{K}\big\},\ \Omega_o = \mathbf{He}\big\{P_2 A + \bar{L}C\big\}$$
$$\Psi_c = H_1\bar{P}_1 + H_2\bar{K}.$$

Moreover, the control and observer gains are constructed as follows:

$$K = \bar{K}\bar{P}_1^{-1}, \quad L = P_2^{-1}\bar{L}.$$

Proof: let us choose a Lyapunov function of the following form:

$$V(t) = \zeta^T(t) P \zeta(t)$$

where

$$P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0.$$

The time derivative of $V(t)$ is given by

$$\dot{V}(t) = \zeta^T(t) \mathbf{He}\{P(\bar{A} + \bar{E}\bar{\Upsilon}\bar{H})\}\zeta(t).$$

Thus, the observer-based control design condition becomes

$$\mathbf{He}\{P(\bar{A} + \bar{E}\bar{\Upsilon}\bar{H})\} < 0. \tag{32}$$

To be specific, condition (32) is represented as

$$\mathbf{He}\left\{\left[\begin{array}{c|c} P_1(A+BK) & -P_1BK \\ \hdashline 0 & P_2(A+LC) \end{array}\right] + \left[\begin{array}{c|c} P_1E & 0 \\ \hdashline 0 & P_2E \end{array}\right]\bar{\Upsilon}\left[\begin{array}{c|c} H_1+H_2K & -H_2K \\ \hdashline H_1+H_2K & -H_2K \end{array}\right]\right\} < 0. \quad (33)$$

Furthermore, using

$$M = \left[\begin{array}{c|c} P_1^{-1} & 0 \\ \hline 0 & P_1^{-1} \end{array}\right]$$

the congruent transformation of (33) is given from Lemma 1 as follows:

$$\mathbf{He}\left\{\left[\begin{array}{c|c} (A+BK)P_1^{-1} & -BKP_1^{-1} \\ \hline 0 & P_1^{-1}(P_2A+P_2LC)P_1^{-1} \end{array}\right] + \left[\begin{array}{c|c} E & 0 \\ \hline 0 & P_1^{-1}P_2E \end{array}\right] \bar{\Upsilon} \left[\begin{array}{c|c} (H_1+H_2K)P_1^{-1} & -H_2KP_1^{-1} \\ \hline (H_1+H_2K)P_1^{-1} & -H_2KP_1^{-1} \end{array}\right]\right\} < 0. \tag{34}$$

By replacing

$$\bar{P}_1 = P_1^{-1},\ \bar{K} = KP_1^{-1},\ \bar{L} = P_2L$$

condition (34) can be expressed as

$$\left[\begin{array}{c:c} \Omega_c & -B\bar{K} \\ \hdashline \star & \bar{P}_1\Omega_o\bar{P}_1 \end{array}\right] + \mathbf{He}\{\bar{\mathbf{E}}\bar{\Upsilon}\bar{\mathbf{H}}\} < 0 \tag{35}$$

where

$$\Omega_c = \mathbf{He}\{A\bar{P}_1 + B\bar{K}\},\ \Omega_o = \mathbf{He}\{P_2A + \bar{L}C\}$$

$$\bar{\mathbf{E}} = \left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & \bar{P}_1P_2E \end{array}\right],\ \bar{\mathbf{H}} = \left[\begin{array}{c:c} \Psi_c & -H_2\bar{K} \\ \hdashline \Psi_c & -H_2\bar{K} \end{array}\right],\ \Psi_c = H_1\bar{P}_1 + H_2\bar{K}.$$

From Lemma 2, it follows that

$$\mathbf{He}\left\{\bar{\mathbf{E}}\Upsilon\bar{\mathbf{H}}\right\} \leq \epsilon\bar{\mathbf{E}}\bar{\mathbf{E}}^T + \epsilon^{-1}\bar{\mathbf{H}}^T\Upsilon^T\Upsilon\bar{\mathbf{H}} \leq \epsilon\bar{\mathbf{E}}\bar{\mathbf{E}}^T + \epsilon^{-1}\bar{\mathbf{H}}^T\bar{\mathbf{H}}.$$

Thus, by the Schur complement, condition (35) holds if

$$\left[\begin{array}{c|c} -\epsilon I & \bar{\mathbf{H}} \\ \hline \star & \begin{bmatrix} \Omega_c & -B\bar{K} \\ \star & \bar{P}_1\Omega_o\bar{P}_1 \end{bmatrix} + \epsilon\bar{\mathbf{E}}\bar{\mathbf{E}}^T \end{array}\right] < 0$$

that is,

$$\left[\begin{array}{cc|cc} -\epsilon I & 0 & \Psi_c & -H_2\bar{K} \\ 0 & -\epsilon I & \Psi_c & -H_2\bar{K} \\ \hline \star & \star & \Omega_c + \epsilon EE^T & -B\bar{K} \\ \star & \star & \star & \bar{P}_1(\Omega_o + \epsilon P_2EE^TP_2)\bar{P}_1 \end{array}\right] < 0. \tag{36}$$

Under the following condition:

$$\Omega_o + \epsilon P_2 E E^T P_2 < 0 \tag{37}$$

Lemma 3 allows

$$\bar{P}_1 \Big(\Omega_o + \epsilon P_2 E E^T P_2\Big) \bar{P}_1 \leq -\mathbf{He}\{\mu \bar{P}_1\} - \mu^2 \Big(\Omega_o + \epsilon P_2 E E^T P_2\Big)^{-1}.$$

Accordingly, condition (36) is ensured by (37) and

$$\left[\begin{array}{cc|cc} -\epsilon I & 0 & \Psi_c & -H_2 \bar{K} \\ 0 & -\epsilon I & \Psi_c & -H_2 \bar{K} \\ \hline \star & \star & \Omega_c + \epsilon E E^T & -B\bar{K} \\ \star & \star & \star & \begin{pmatrix} -\mathbf{He}\{\mu \bar{P}_1\} \\ -\mu^2(\Omega_o + \epsilon P_2 E E^T P_2)^{-1} \end{pmatrix} \end{array}\right] < 0. \tag{38}$$

Using the Schur complement, condition (38) is transformed into

$$\left[\begin{array}{cc:cc:c} -\epsilon I & 0 & \Psi_c & -H_2\bar{K} & 0 \\ 0 & -\epsilon I & \Psi_c & -H_2\bar{K} & 0 \\ \hdashline \star & \star & \Omega_c + \epsilon E E^T & -B\bar{K} & 0 \\ \star & \star & \star & -\mathbf{He}\{\mu\bar{P}_1\} & \mu I \\ \hdashline \star & \star & \star & \star & \Omega_o + \epsilon P_2 E E^T P_2 \end{array}\right] < 0$$

and it is transformed once more into (31).

Finally, it is worth noting that (31) implies

$$\begin{bmatrix} \Omega_o & \epsilon P_2 E \\ \star & -\epsilon I \end{bmatrix} < 0$$

which guarantees (37) according to Schur complement. Thus, there is no need to additionally include (37) in the robust observer-based control design condition. ∎

5 $\mathcal{H}_\infty$ observer-based control

- Let us consider the following linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \\ y(t) = Cx(t) + Dw(t) \\ z(t) = Gx(t) + Hu(t) \end{cases} \tag{39}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^d$, $y(t) \in \mathbb{R}^p$, and $z(t) \in \mathbb{R}^q$.

- Thus, the closed-loop system is given by

$$\begin{cases} \dot{\zeta}(t) = \bar{A}\zeta(t) + \bar{E}w(t) \\ z(t) = \bar{G}\zeta(t) \end{cases} \tag{40}$$

where

$$\bar{A} = \left[\begin{array}{c:c} A + BK & -BK \\ \hdashline 0 & A + LC \end{array} \right], \quad \bar{E} = \left[\begin{array}{c} E \\ \hdashline E + LD \end{array} \right]$$

$$\bar{G} = \left[\begin{array}{c:c} G + HK & -HK \end{array} \right].$$

Theorem 4. For a prescribed value $\mu > 0$, close-loop control system (40) is asymptotically stable at origin and has the $\mathcal{H}_\infty$ performance γ, if there exist matrices $\bar{P}_1 = \bar{P}_1^T$, $P_2 = P_2^T$, $\bar{K}$, $\bar{L}$, and a scalar $\gamma > 0$, such that

$$\bar{P}_1 > 0 \tag{41}$$

$$P_2 > 0 \tag{42}$$

$$\left[\begin{array}{cc|cc:cc} -I & \Psi_c & -H\bar{K} & 0 & 0 & 0 \\ \star & \Omega_c & -B\bar{K} & E & 0 & 0 \\ \hline \star & \star & -2\mu\bar{P}_1 & -N_1 & \mu I & N_1 \\ \star & \star & \star & -\mathbf{He}\{N_2\} & 0 & N_2 \\ \hdashline \star & \star & \star & \star & \Omega_o & \Psi_o \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{array}\right] < 0 \tag{43}$$

where

$$\Omega_c = \mathbf{He}\big\{A\bar{P}_1 + B\bar{K}\big\},\ \Omega_o = \mathbf{He}\big\{P_2 A + \bar{L}C\big\}$$
$$\Psi_c = G\bar{P}_1 + H\bar{K},\ \Psi_o = P_2 E + \bar{L}D.$$

Moreover, the control and observer gains are constructed as follows:

$$K = \bar{K}\bar{P}_1^{-1}, \quad L = P_2^{-1}\bar{L}.$$

Proof: Let us choose a Lyapunov function of the following form:

$$V(t) = \zeta^T(t) P \zeta(t)$$

where

$$P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0.$$

Then it is obtained that

$$
\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&= \zeta^T(t)P\Big(\bar{A}\zeta(t) + \bar{E}w(t)\Big) + \Big(\bar{A}\zeta(t) + \bar{E}w(t)\Big)^T P\zeta(t) \\
&\quad + \zeta^T(t)\bar{G}^T\bar{G}\zeta(t) - \gamma^2 w^T(t)w(t) \\
&= \left[\begin{array}{c} \zeta(t) \\ \hline w(t) \end{array}\right]^T \left[\begin{array}{c|c} \mathbf{He}\{P\bar{A}\} + \bar{G}^T\bar{G} & P\bar{E} \\ \hline \star & -\gamma^2 I \end{array}\right] \left[\begin{array}{c} \zeta(t) \\ \hline w(t) \end{array}\right].
\end{aligned}
$$

Thus, the $\mathcal{H}_\infty$ control design condition is given as follows:

$$0 > \left[\begin{array}{c|c} \mathbf{He}\{P\bar{A}\} + \bar{G}^T\bar{G} & P\bar{E} \\ \hline \star & -\gamma^2 I \end{array}\right]. \tag{44}$$

Furthermore, by the Schur complement, condition (44) is transformed into

$$0 > \left[\begin{array}{c:c|c} -I & \bar{G} & 0 \\ \hdashline \star & \mathbf{He}\{P\bar{A}\} & P\bar{E} \\ \hline \star & \star & -\gamma^2 I \end{array}\right]. \tag{45}$$

Specifically, condition (45) becomes

$$0 > \left[\begin{array}{c:cc|c} -I & G + HK & -HK & 0 \\ \hdashline \star & \mathbf{He}\{P_1(A + BK)\} & -P_1 BK & P_1 E \\ \star & \star & \mathbf{He}\{P_2(A + LC)\} & P_2(E + LD) \\ \hline \star & \star & \star & -\gamma^2 I \end{array}\right]. \tag{46}$$

Furthermore, using

$$M = \mathbf{diag}(I, P_1^{-1}, P_1^{-1}, I)$$

the congruent transformation of (46) is given from Lemma 1 as follows:

$$0 > \left[\begin{array}{c:cc|c} -I & \Psi_c & -HKP_1^{-1} & 0 \\ \hdashline \star & \Omega_c & -BKP_1^{-1} & E \\ \star & \star & P_1^{-1}\Omega_o P_1^{-1} & P_1^{-1}\Psi_o \\ \hline \star & \star & \star & -\gamma^2 I \end{array}\right]. \tag{47}$$

where

$$\Omega_c = \mathbf{He}\{(A + BK)P_1^{-1}\},\ \Omega_o = \mathbf{He}\{P_2(A + LC)\}$$

$$\Psi_c = (G + HK)P_1^{-1},\ \Psi_o = P_2(E + LD).$$

Letting

$$\bar{P}_1 = P_1^{-1},\ \bar{F} = F\bar{P}_1,\ \bar{L} = P_2 L$$

condition (47) is represented as

$$0 > \left[\begin{array}{c:cc|c} -I & \Psi_c & -H\bar{K} & 0 \\ \hdashline \star & \Omega_c & -B\bar{K} & E \\ \star & \star & \bar{P}_1\Omega_o\bar{P}_1 & \bar{P}_1\Psi_o \\ \hline \star & \star & \star & -\gamma^2 I \end{array}\right] \tag{48}$$

where

$$\Omega_c = \mathbf{He}\Big\{A\bar{P}_1 + B\bar{K}\Big\},\ \Omega_o = \mathbf{He}\Big\{P_2 A + \bar{L}C\Big\}$$
$$\Psi_c = G\bar{P}_1 + H\bar{K},\ \Psi_o = P_2 E + \bar{L}D.$$

Under the following condition:

$$\begin{bmatrix} \Omega_o & \Psi_o \\ \star & -\gamma^2 I \end{bmatrix} < 0 \tag{49}$$

Lemma 3 allows

$$\begin{aligned}
\begin{bmatrix} \bar{P}_1 \Omega_o \bar{P}_1 & \bar{P}_1 \Psi_o \\ \star & -\gamma^2 I \end{bmatrix} &= \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega_o & \Psi_o \\ \star & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & I \end{bmatrix} \\
&\leq -\mathbf{He}\left\{ \begin{bmatrix} \mu I & N_1 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & I \end{bmatrix} \right\} \\
&\quad - \begin{bmatrix} \mu I & N_1 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \Omega_o & \Psi_o \\ \star & -\gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} \mu I & N_1 \\ 0 & N_2 \end{bmatrix}^T .
\end{aligned}$$

Thus, condition (48) holds by (49) and

$$0 > \left[\begin{array}{cc|cc} -I & \Psi_c & -H\bar{K} & 0 \\ \star & \Omega_c & -B\bar{K} & E \\ \hline \star & \star & -2\mu\bar{P}_1 & -N_1 \\ \star & \star & \star & -\mathbf{He}\{N_2\} \end{array}\right]$$

$$- \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \mu I & N_1 \\ 0 & N_2 \end{array}\right] \left[\begin{array}{cc} \Omega_o & \Psi_o \\ \star & -\gamma^2 I \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \mu I & N_1 \\ 0 & N_2 \end{array}\right]^T . \qquad (50)$$

Finally, by the Schur complement, condition (50) can be transformed into

$$0 > \left[\begin{array}{cc|cc:cc} -I & \Psi_c & -H\bar{K} & 0 & 0 & 0 \\ \star & \Omega_c & -B\bar{K} & E & 0 & 0 \\ \hline \star & \star & -2\mu\bar{P}_1 & -N_1 & \mu I & N_1 \\ \star & \star & \star & -\mathbf{He}\{N_2\} & 0 & N_2 \\ \hdashline \star & \star & \star & \star & \Omega_o & \Psi_o \\ \star & \star & \star & \star & \star & -\gamma^2 I \end{array}\right]$$

which ensures (49). Thus, there is no need to additionally include (49) in the $\mathcal{H}_\infty$ observer-based control design condition. ∎

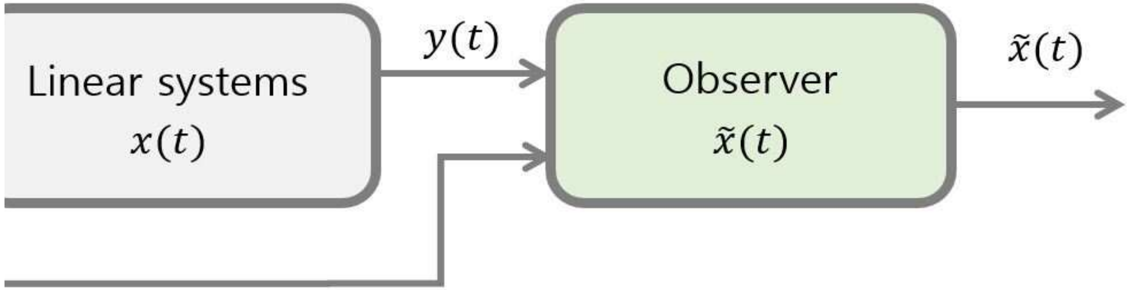

Figure 1: State estimation through an observer

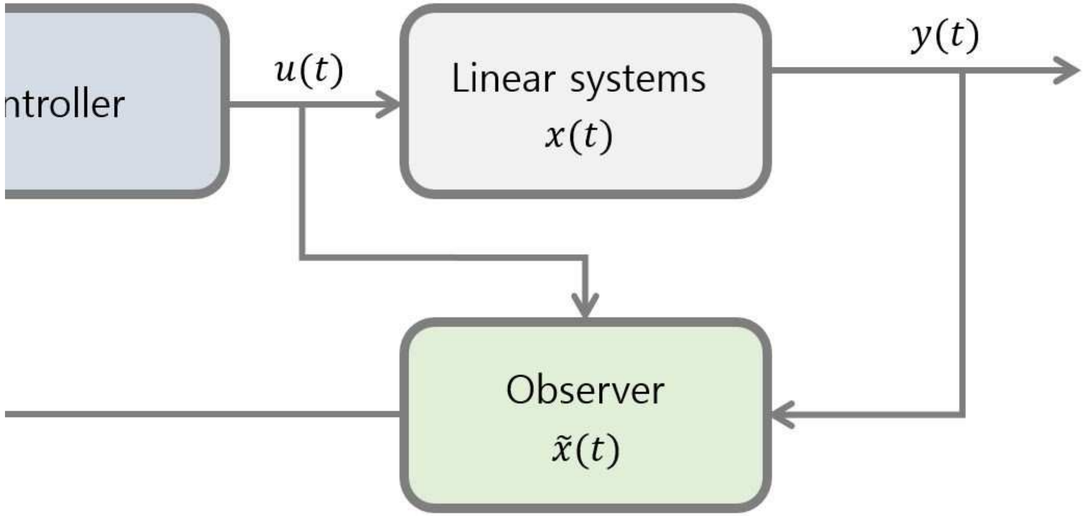

Figure 2: Observer-based output-feedback control