B. Examples

We now consider some specific cases.

EXAMPLE 11.1. We consider the *simple pendulum* discussed in Example 4.4 and described by the equation

$$\ddot{x} + k \sin x = 0, \tag{11.11}$$

where $k > 0$ is a constant. Letting $x_1 = x$ and $x_2 = \dot{x}$, (11.11) can be expressed as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k \sin x_1. \end{aligned} \tag{11.12}$$

It is easily verified that $\phi_1(t) \equiv 0$ and $\phi_2(t) \equiv 0$ is a solution of (11.12). Letting $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = -k \sin x_1$, the Jacobian of $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))^T$ evaluated at $(x_1, x_2)^T = (0, 0)^T$ is given by

$$J(0) \triangleq A = \begin{bmatrix} 0 & 1 \\ -k \cos x_1 & 0 \end{bmatrix}_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}.$$

The linearized equation of (11.12) about the solution $\phi_1(t) \equiv 0$, $\phi_2(t) \equiv 0$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

■

EXAMPLE 11.2. The system of equations

$$\begin{aligned} \dot{x}_1 &= a x_1 - b x_1 x_2 - c x_1^2 \\ \dot{x}_2 &= d x_2 - e x_1 x_2 - f x_2^2 \end{aligned} \tag{11.13}$$

describes the growth of two competing species (e.g., two species of small fish) that prey on each other (e.g., the adult members of one species prey on the young members of the other species, and vice versa). In (11.13) a, b, c, d, e, and f are positive parameters and it is assumed that $x_1 \geq 0$ and $x_2 \geq 0$. For (11.13), $\phi_1(t) = \phi_1(t, 0, 0) \equiv 0$ and $\phi_2(t) = \phi_2(t, 0, 0) \equiv 0$, $t \geq 0$, is a solution of (11.13). A simple computation yields

$$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

and thus the system of equations

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

constitutes the linearized equation of (11.13) about the solution $\phi_1(t) = 0$, $\phi_2(t) = 0$, $t \geq 0$. ■

EXAMPLE 11.3. Consider a unit mass subjected to an inverse square law force field, as depicted in Fig. 1.11. In this figure, r denotes radius and θ denotes angle, and it is assumed that the unit mass (representing, e.g., a satellite) can thrust in the radial and in the tangential directions with thrusts u_1 and u_2, respectively. The equations that govern this system are given by

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\ \ddot{\theta} &= \frac{-2\dot{\theta}\dot{r}}{r} + \frac{1}{r} u_2. \end{aligned} \tag{11.14}$$

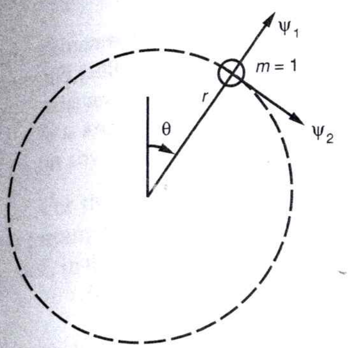

FIGURE 1.11
A unit mass subjected to an inverse square law force field

When $r(0) = r_0$, $\dot{r}(0) = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = \omega_0$ and $u_1(t) \equiv 0$, $u_2(t) \equiv 0$ for $t \geq 0$, it is easily verified that the system of equations (11.14) has as a solution the circular orbit given by

$$\begin{aligned} r(t) &\equiv r_0 = \text{constant} \\ \dot{\theta}(t) &= \omega_0 = \text{constant} \end{aligned} \tag{11.15}$$

for all $t \geq 0$, which implies that

$$\theta(t) = \omega_0 t + \theta_0, \tag{11.16}$$

where $\omega_0 = (k/r_0^3)^{1/2}$.

If we let $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$, the equations of motion (11.14) assume the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\frac{2x_2 x_4}{x_1} + \frac{u_2}{x_1}. \end{aligned} \tag{11.17}$$

The linearized equation of (11.17) about the solution (11.16) [with $u_1(t) \equiv 0$, $u_2(t) \equiv 0$] is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{r_0} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

■

EXAMPLE 11.4. In this example we consider systems described by equations of the form

$$\dot{x} + Af(x) + Bg(x) = u, \tag{11.18}$$

where $x \in R^n$, $A = [a_{ij}] \in R^{n \times n}$, $B = [b_{ij}] \in R^{n \times n}$ with $a_{ii} > 0$, $b_{ii} > 0$, $1 \leq i \leq n$, $f, g \in C^1(R^n, R^n)$, $u \in C(R^+, R^n)$, and $f(x) = 0$, $g(x) = 0$ if and only if $x = 0$.

Equation (11.18) can be used to model a great variety of physical systems. In particular, (11.18) has been used to model a large class of integrated circuits consisting