State-feedback control systems

University of Ulsan

Prof. KIM

6 Sliding mode control

• Sliding mode control (SMC) is a nonlinear control method to regulate the behavior of dynamic systems subject to disturbances or uncertainties.

• The core idea behind SMC is to create a sliding surface in the state space of the system, such that

"Once the system reaches this sliding surface, it ideally remains there regardless of disturbances or uncertainties."

- In other words, this control law is designed such that it drives the system toward the sliding surface and then maintains it there.
- One of prominent features is that this control involves the use of a discontinuous control signal which switches based on the system's position relative to the sliding surface.
- Sliding mode control has applications in various fields including aerospace, automotive, robotics, and power systems.

Let us consider the following linear systems with uncertainties and external disturbances:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + N_1 w(t). \\ z(t) = Cx(t) + N_2 w(t) \end{cases} \tag{44}$$

Assumption: In general, the matrix B is of full column rank.

In this study, we employ the following switching function $s(t)$:

$$s(t) = Gx(t) - \int_0^t G(A + BK)x(\tau)d\tau \tag{45}$$

where $G \in \mathbb{R}^{m \times n}$ is chosen so that $GB \in \mathbb{R}^{m \times m}$ is invertible and K denote the linear control gain to be designed later.

From (45), it follows that

$$\dot{s}(t) = G\dot{x}(t) - G(A + BK)x(t). \tag{46}$$

Additionally, according to (44), it is also available that

$$\dot{s}(t) = GBu(t) + G\Delta Ax(t) + GN_1w(t) - GBKx(t). \tag{47}$$

Theorem 5. The trajectories of system (44) can be driven onto the surface $s(t) = 0$ if the control input is given as

$$u(t) = Kx(t) + (GB)^{-1}\psi(t) \tag{48}$$

where

$$\psi(t) = -\mathrm{sgn}(s(t)) \cdot (\alpha + \epsilon(x(t))) \tag{49}$$
$$\epsilon(x(t)) = \epsilon_A \cdot ||G|| \cdot ||x(t)|| + \epsilon_w \cdot ||GN_1||.$$

Remark 2. The decision variables K and G will be designed from the stability condition of closed-loop system given that $\dot{s}(t) = 0$.

Proof: Consider the Lyapunov function:

$$V_s(t) = \frac{1}{2} s^T(t) s(t).$$

Then, we have

$$\begin{aligned}
\dot{V}_s(t) &= s^T(t)\dot{s}(t) \\
&= s^T(t)\Big(GBu(t) + G\Delta Ax(t) + GN_1 w(t) - GBKx(t)\Big).
\end{aligned}$$

Using (48), it is given that

$$\begin{aligned}
\dot{V}_s(t) &= s^T(t)\Big(\psi(t) + G\Delta Ax(t) + GN_1 w(t)\Big) \\
&\leq s^T(t)\psi(t) + ||s(t)|| \cdot ||G\Delta Ax(t)|| + ||s(t)|| \cdot ||GN_1 w(t)||.
\end{aligned}$$

Noting that

$$\begin{aligned} & ||G\Delta A x(t)|| + ||GN_1 w(t)|| \\ & \leq ||G|| \cdot ||\Delta A|| \cdot ||x(t)|| + ||GN_1|| \cdot ||w(t)|| \\ & \leq \underbrace{\epsilon_A \cdot ||G|| \cdot ||x(t)|| + \epsilon_w \cdot ||GN_1||}_{=:\epsilon(x(t))} \end{aligned}$$

we have

$$\dot{V}_s(t) \leq s^T(t)\psi(t) + ||s(t)||\epsilon(x(t)).$$

Finally, using (49) implies

$$\dot{V}_s(t) \leq -\alpha ||s(t)|| < 0, \ \forall s(t) \neq 0$$

which means that

$$\lim_{t\to\infty} s(t) \to 0.$$

■

According to sliding mode control theory, when the sliding motion takes place, it follows that $s(t) = 0$ and $\dot{s}(t) = 0$.

That is, from (47), the equivalent control be given as

$$u_{\mathrm{eq}}(t) = Kx(t) - (GB)^{-1}G\big(\Delta A x(t) + N_1 w(t)\big).$$

Thus, the sliding motion is described as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu_{\mathrm{eq}}(t) + \Delta A x(t) + N_1 w(t) \\
&= \Big(A + BK\Big)x(t) + \Big(I - B(GB)^{-1}G\Big)\Big(\Delta A x(t) + N_1 w(t)\Big) \\
&= \Big(\bar{A} + \bar{G}\Delta A\Big)x(t) + \bar{G}N_1 w(t) \qquad (50)
\end{aligned}$$

where

$$\bar{A} = A + BK, \;\; \bar{G} = I - B(GB)^{-1}G.$$

Theorem 6. System (50) is stable under the condition of $s(t) = 0$ if there exits matrices $\bar{P} = \bar{P}^T \in \mathbb{R}^{n\times n}$, $\bar{K} \in \mathbb{R}^{m\times n}$ and scalars $\epsilon > 0$, $\gamma > 0$, such that

$$\bar{P} > 0$$

$$\left[\begin{array}{c|ccc|c} -I & C\bar{P} & & N_2 & 0 \\ \hline (*) & \mathrm{He}\{A\bar{P} + B\bar{K}\} + \epsilon\bar{G}EE^T\bar{G}^T & & \bar{G}N_1 & \bar{P}H^T \\ (*) & (*) & & -\gamma^2 I & 0 \\ \hline 0 & (*) & & 0 & -\epsilon I \end{array}\right] < 0.$$

Furthermore, the control gain can be reconstructed as follows:

$$K = \bar{K}\bar{P}^{-1}.$$

Proof: Let us choose

$$V(t) = x^T(t)Px(t).$$

Then we have

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \Psi \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

where

$$\Psi = \left[\begin{array}{c:c} \mathrm{He}\{P\bar{A} + P\bar{G}\Delta A\} & P\bar{G}N_1 \\ \hdashline (*) & -\gamma^2 I \end{array}\right] + \begin{bmatrix} C^T \\ N_2^T \end{bmatrix} \begin{bmatrix} C & N_2 \end{bmatrix}.$$

By the Schur complement, the robust $\mathcal{H}_\infty$ stability condition $\Psi < 0$ is transformed into

$$\left[\begin{array}{c|cc} -I & C & N_2 \\ \hline (*) & \mathrm{He}\{P\bar{A} + P\bar{G}\Delta A\} & P\bar{G}N_1 \\ (*) & (*) & -\gamma^2 I \end{array}\right] < 0. \tag{51}$$

Furthermore, pre- and post-multiplying (51) by $\mathrm{diag}(I, \bar{P} := P^{-1}, I)$ and its transpose yields

$$\left[\begin{array}{c|cc} -I & C\bar{P} & N_2 \\ \hline (*) & \mathrm{He}\{\bar{A}\bar{P} + \bar{G}\Delta A\bar{P}\} & \bar{G}N_1 \\ (*) & (*) & -\gamma^2 I \end{array}\right] < 0. \tag{52}$$

Noting that

$$\mathrm{He}\{\bar{G}\Delta A\bar{P}\} = \mathrm{He}\{\bar{G}E\Upsilon H\bar{P}\} \leq \epsilon\bar{G}EE^T\bar{G}^T + \epsilon^{-1}\bar{P}H^T H\bar{P}$$

and by apply the Schur complement, we can see that the following condition implies (52):

$$\left[\begin{array}{c:ccc:c} -I & C\bar{P} & & N_2 & 0 \\ \hdashline (*) & \mathrm{He}\{A\bar{P} + B\bar{K}\} + \epsilon\bar{G}EE^T\bar{G}^T & & \bar{G}N_1 & \bar{P}H^T \\ (*) & (*) & & -\gamma^2 I & 0 \\ \hdashline 0 & (*) & & 0 & -\epsilon I \end{array}\right] < 0$$

where $\bar{K} = K\bar{P}$.