Stability analysis of linear systems

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Example 2.5. The following inequality provides the exponential stability criterion:

$$\dot{V}(t) < -\alpha V(t), \ \alpha > 0.$$

Please formulate a set of linear matrix inequalities to check this stability criterion.

2.3 Robust stability

• **Robust stability** is the ability of an autonomous system to remain stable despite variations or uncertainties in its parameters.

• For this reason, robust stability analysis is addressed considering the following uncertain linear system:

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) \tag{8}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ denotes the parameter uncertainty.

Definition 2. If system (8) maintain stability even in the presence of uncertainty, it is said that the system is **robustly stable**.

- As studied in Chap. 1, the following decomposition is available:

$$\Delta A = E \Upsilon H \tag{9}$$

where $E \in \mathbb{R}^{n \times p}$ and $H \in \mathbb{R}^{q \times n}$ are known real constant matrices; and $\Upsilon \in \mathbb{R}^{p \times q}$ is an unknown time-varying or time-invariant matrix satisfying

$$\Upsilon^T \Upsilon \leq I. \tag{10}$$

The following two lemmas will be used to obtain the robust stability condition in terms of LMIs.

Lemma 3. For a positive scalar ϵ and any matrices $M \in \mathbb{R}^{r\times s}$ and $N \in \mathbb{R}^{s\times r}$, the following inequality holds:

$$\mathbf{He}\{MN\} \leq \epsilon^{-1}MM^T + \epsilon N^T N.$$

Lemma 4 (Schur complement). For any matrices $S = S^T$, $R = R^T$, and N, if the following condition is satisfied:

$$\left[\begin{array}{c:c} S & N \\ \hdashline N^T & R \end{array}\right] < 0 \left(\text{ or } \left[\begin{array}{c:c} R & N^T \\ \hdashline N & S \end{array}\right] < 0 \right) \tag{11}$$

then it holds that

$$S - NR^{-1}N^T < 0, \; S < 0, \; R < 0. \tag{12}$$

The following theorem provides a set of linear matrix inequalities (LMIs) for robust stability.

Theorem 1. System (8) is robustly stable at the origin if there exist a symmetric matrix P and a positive scalar ϵ satisfying

$$P > 0 \tag{13}$$

$$\left[\begin{array}{c:c} \mathbf{He}\{PA\} + \epsilon H^T H & PE \\ \hdashline E^T P & -\epsilon I \end{array}\right] < 0. \tag{14}$$

Remark 1. Let us recall the following stability criteria:

- Stability: $\mathbf{He}\{PA\} < 0$
- Robust stability: $\mathbf{He}\{PA\} < -(\epsilon^{-1}PEE^TP + \epsilon H^TH)$.

If the maximum eigenvalues of matrices EE^T and H^TH are larger, then it becomes more difficult to identify a feasible matrix P that guarantees the stability condition.

For instance,

$$\begin{aligned}\Delta A &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \delta \begin{bmatrix} R & 1 \end{bmatrix}, \ \delta \in [-0.1, 0.1] \ \mapsto \ \lambda_{\max}(EE^T) = 1 \\ &= \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} (10\delta) \begin{bmatrix} R & 1 \end{bmatrix}, \ 10\delta \in [-1, 1] \ \mapsto \ \lambda_{\max}(EE^T) = 0.01\end{aligned}$$