State-feedback control systems

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1 Useful Lemmas

The following three lemmas will be used to obtain the LMI-based control design condition.

Lemma 1. For a positive scalar ϵ and any matrices $M \in \mathbb{R}^{r\times s}$ and $N \in \mathbb{R}^{s\times r}$, the following inequality holds:

$$\mathbf{He}\{MN\} \leq \epsilon^{-1}MM^T + \epsilon N^T N.$$

Lemma 2 (Schur complement). For any matrices $S = S^T$, $R = R^T$, and N, if the following condition is satisfied:

$$\left[\begin{array}{c|c} S & N \\ \hline N^T & R \end{array}\right] < 0 \left(\text{or} \left[\begin{array}{c|c} R & N^T \\ \hline N & S \end{array}\right] < 0 \right) \tag{1}$$

then it holds that

$$S - NR^{-1}N^T < 0,\ S < 0,\ R < 0. \tag{2}$$

Lemma 3 (Congruent transformation). For any nonsingular (i.e., invertible) matrix M, the following two conditions are equivalent:

- $\Omega < 0$ (or $\Omega \leq 0$) (3)
- $M^T \Omega M < 0$ (or $M^T \Omega M \leq 0$). (4)

Example 1.1. Is it true or false?

- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} < 0$
- $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} < 0$

2 State-feedback control

• Let us start with considering a continuous-time linear system of the following general form:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

where $x(t) \in \mathbb{R}^n$ represents the state and $u(t) \in \mathbb{R}^m$ represents the control input.

• The main objective of this chapter is to deal with the problem of designing the control input $u(t)$ that allows

❶ to stabilize unstable autonomous systems

❷ to improve system performance.

- In general, there are three prominent techniques for generating the control input:

  ❶ State-feedback control

  ❷ Observer-based control

  ❸ Dynamic output-feedback control

- State-feedback control is a technique that generates the control input $u(t)$ based on knowledge of the internal state $x(t)$.

- In detail, the state-feedback controller is typically designed as a linear function of the state variables:

$$u(t) = Kx(t) \tag{6}$$

where $K \in \mathbb{R}^{m \times n}$ is called the **control gain**.

- As a result, the **closed-loop (control) system** is described as follows:

$$\dot{x}(t) = \Big(A + BK\Big)x(t). \tag{7}$$

- Accordingly, the control gain $K \in \mathbb{R}^{m \times n}$ must be designed such that closed-loop control system (7) becomes stable.

The following lemma provides the state-feedback control design condition for (7).

Lemma 4. Closed-loop control system (7) is stable at the origin if there exit $P = P^T \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{m\times n}$ such that

$$P > 0 \tag{8}$$

$$\mathbf{He}\left\{PA + PBK\right\} < 0. \tag{9}$$

Remark 1. However, it should be noted that condition (9) is not formulated as a linear matrix inequality (LMI), because the term PBK in (9) contains two variables P and K simultaneously.

The following lemma provides the control design condition for (7), formulated in terms of linear matrix inequalities (LMIs).

Theorem 1. Closed-loop control system (7) is stable at the origin if there exist $\bar{P} = \bar{P}^T \in \mathbb{R}^{n\times n}$ and $\bar{K} \in \mathbb{R}^{m\times n}$ satisfying

$$\bar{P} > 0 \tag{10}$$

$$\mathbf{He}\left\{A\bar{P} + B\bar{K}\right\} < 0. \tag{11}$$

Moreover, the control gain K can be reconstructed as follows:

$$K = \bar{K}\bar{P}^{-1}.$$

Example 2.1. Using LMIs in Theorem 1, design a feasible state-feedback control gain K for (7) with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}.$$

3 LQR state-feedback control

• The Linear Quadratic Regulator (LQR) controller is a type of optimal control algorithm used in control theory to design feedback controllers for linear dynamical systems.

• This LQR controller is designed to minimize the following quadratic cost function:

$$\mathcal{J} = \int_0^\infty \big(x^T(t)Qx(t) + u^T(t)Ru(t)\big)dt \tag{14}$$

where $Q = Q^T \in \mathbb{R}^{p\times p} > 0$ is the state weighting matrix (given), and $R = R^T \in \mathbb{R}^{m\times m} > 0$ is the control weighting matrix (given).

Definition 1. The control input $u(t)$ becomes the LQR control input if it is guaranteed that

- closed-loop control system is stable
- $\mathcal{J} < \alpha < \infty$ (15)

where α is an arbitrary finite constant.

The following lemma provides a condition for designing the LQR control input in the sense of Lyapunov stability.

Lemma 5. System (5) is stable at the origin and satisfies the guaranteed cost given in (15), if it holds that

$$\dot{V}(t) + x^T(t)Qx(t) + u^T(t)Ru(t) < 0. \tag{16}$$

The following lemma provides the LQR control design condition for (7), formulated in terms of LMIs.

Theorem 2. For given $Q \in \mathbb{R}^{p\times p}$ and $R \in \mathbb{R}^{m\times m}$, closed-loop control system (7) is stable and satisfies (15) if there exists matrices $\bar{P} = \bar{P}^T$ and $\bar{K}$ satisfying

$$0 < \bar{P} \tag{17}$$

$$0 > \left[\begin{array}{c|c|c} \mathbf{He}\{A\bar{P} + B\bar{K}\} & \star & \star \\ \hline Q\bar{P} & -Q & \star \\ \hline R\bar{K} & 0 & -R \end{array}\right] \tag{18}$$

where $\bar{P} \in \mathbb{R}^{n\times n}$ and $\bar{K} \in \mathbb{R}^{m\times n}$. Furthermore, the control gain can be reconstructed as follows:

$$K = \bar{K}\bar{P}^{-1}.$$

Example 3.1. Using LMIs in Theorem 2, design a feasible LQR state-feedback control gain K for (7) with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 10$.

- As the value of Q in the LQR control increases, the state response converges to the origin more quickly.
- However, increasing Q too much can also lead to instability or excessive control effort, so it often requires a balance and may involve tuning through simulation or experimentation.

Example 3.2. Compare the result of Example 3.1 with that of the following case:

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad R = 10.$$

4 Robust state-feedback control

- The uncertain linear system is formulated as

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) + \big(B + \Delta B\big)u(t). \tag{21}$$

- The parameter uncertainties $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times m}$ are described as, respectively,

$$\Delta A = E\Upsilon H_1,\ \Delta B = E\Upsilon H_2. \tag{22}$$

where $E \in \mathbb{R}^{n\times p}$, $H_1 \in \mathbb{R}^{q\times n}$, and $H_2 \in \mathbb{R}^{q\times m}$ are known real constant matrices; and $\Upsilon \in \mathbb{R}^{p\times q}$ is an unknown matrix but satisfies

$$\Upsilon^T\Upsilon \leq I. \tag{23}$$

- Let us recall the following state-feedback control law:

$$u(t) = Kx(t) \tag{24}$$

where $K \in \mathbb{R}^{m\times n}$ denotes the control gain to be designed later.

- As a result, the closed-loop control system is described as follows:

$$\begin{aligned}\dot{x}(t) &= \big(A + E\Upsilon H_1\big)x(t) + \big(B + E\Upsilon H_2\big)Kx(t) \\ &= \big(A + BK + E\Upsilon(H_1 + H_2K)\big)x(t) \\ &= \big(\bar{A} + E\Upsilon\bar{H}\big)x(t) \end{aligned} \tag{25}$$

by defining

$$\bar{A} = A + BK,\ \ \bar{H} = H_1 + H_2K.$$

Theorem 3. System (21) is robustly stable at the origin if there exist matrices $\bar{P} = \bar{P}^T \in \mathbb{R}^{n\times n}$, $\bar{K} \in \mathbb{R}^{m\times n}$, and a scalar $\epsilon > 0$ satisfying

$$\bar{P} > 0 \tag{26}$$

$$\left[\begin{array}{c:c} \mathbf{He}\{A\bar{P} + B\bar{K}\} + \epsilon E E^T & \star \\ \hdashline H_1\bar{P} + H_2\bar{K} & -\epsilon I \end{array}\right] < 0. \tag{27}$$

Furthermore, the control gain can be reconstructed as follows:

$$K = \bar{K}\bar{P}^{-1}.$$