Stability analysis of linear systems

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1 Preliminaries

- The state-space model of linear time-invariant systems is given as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state vector and the control input vector, respectively.

- When $u(t) \equiv 0$, the state-space model reduces to

$$\dot{x}(t) = Ax(t) \tag{2}$$

which is called the **autonomous** system.

2 Stability analysis

Definition 1. **Stability** is the ability of an autonomous system to drive its state towards a desired equilibrium point (often denoted as the origin) over time, even if the system starts from non-zero initial conditions.

More formally, if for every initial state $x(0)$, the following condition holds when $u(t) \equiv 0$:

$$\lim_{t \to \infty} x(t) = 0 \tag{3}$$

then system (2) can be said to be **stable** at the origin.

2.1 Hurwitz stability criterion

• It is worth noting that the eigenvalue of matrix A corresponds to the pole of the transfer function.

Example 2.1. Consider

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = u(t).$$

• The autonomous system is stable if and only if matrix A has all its eigenvalues in the open left-half plane $\mathbb{C}^-$.

$$\text{“system is stable”} \;\Leftrightarrow\; \mathbf{Re}\Big[\lambda(A)\Big] < 0,\; \forall i = 1, 2, \cdots, n.$$

Example 2.2. Find the eigenvalue of A given by

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

and assess the stability of $\dot{x}(t) = Ax(t)$.

Example 2.3. Find the range of k such that $\dot{x}(t) = Ax(t)$ is stable, where

$$A = \begin{bmatrix} -1+k & -2 \\ 1 & -4 \end{bmatrix}.$$

2.2 Lyapunov stability

- The Lyapunov stability is a well-known method for evaluating the stability of

$$\dot{x}(t) = Ax(t).$$

- To employ the Lyapunov stability method effectively, our first step is to select a proper **Lyapunov function**, described as $V(t)$.

- In linear systems, the standard choice for a Lyapunov function is as follows:

$$V(t) = x^T(t)Px(t) \tag{4}$$

and $P \in \mathbb{R}^{n\times n}$ must be symmetric and positive definite.

- The Lyapunov function becomes a positive real-valued function:

$$V(t) = x^T(t)Px(t) > 0, \; \forall x(t) \neq 0.$$

Lemma 1 (Lyapunov stability criterion). If it holds that

$$\dot{V}(t) < 0 \tag{5}$$

system (2) can be said to be stable at the origin.

Proof: If (5) holds, then $V(t)$ is strictly monotonically decreasing. Accordingly, $V(t)$ converges to zero as time goes to infinity:

$$\lim_{t\to\infty} x^T(t)Px(t) = 0.$$

Since $P > 0$, this limitation is achieved when the state $x(t)$ converges to the origin, written as follows:

$$\lim_{t\to\infty} x(t) = 0.$$

As a result, according to Definition 1, we can claim that system (2) is stable at the origin.

- Now, the Lyapunov stability criterion in (5) will be transformed into a set of linear matrix inequalities (LMIs).
- This is because LMIs can be solved through numerical tools that use various optimization techniques.

With this aim, the following lemma provides a set of LMIs to assess the stability of (2).

Lemma 2. Suppose that there exists a symmetric matrix P that satisfies

$$P > 0 \tag{6}$$

$$A^T P + PA < 0. \tag{7}$$

Then, system (2) is stable at the origin.

Proof: From (4), it follows that

$$\begin{aligned}\dot{V}(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)\\ &= x^T(t)A^TPx(t) + x^T(t)PAx(t)\\ &= x^T(t)\Big(A^TP + PA\Big)x(t).\end{aligned}$$

Thus, if (7) holds, it is satisfied that $\dot{V}(t) < 0$.

Therefore, according to Lemma 1, we can claim that system (2) is stable at the origin. ∎

Example 2.4. Using the robust toolbox in MATLAB, analyze the stability of (2) with

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$