Mathematical Description of Linear Systems with various constraints

University of Ulsan

Prof. KIM

4 Uncertain linear systems

4.1 Parameter Uncertainty

• In **uncertain linear systems**, the system matrices contain additional **parameter uncertainties** ΔA and ΔB, described as follows:

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) + \big(B + \Delta B\big)u(t) \tag{5}$$

• $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times m}$ are expressed such that they have the same dimensions as $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, respectively.

• The **nominal** term provides a reference (or center) point around which uncertainties are characterized and analyzed.

Example 4.1. In Example 2.1, let us consider the case where

$$\frac{1}{L} = \frac{1}{L_0} + \delta,\ \ \delta \in \big[-0.1,\ 0.1\big].$$

Please define ΔA and ΔB; and express them in the following form:

$$\Delta A = E\Upsilon H_1,\ \ \Delta B = E\Upsilon H_2$$

subject to

$$\Upsilon^T\Upsilon \leq I.$$

• As shown in Example 4.1, the parameter uncertainties ΔA and ΔB can be expressed as follows:

$$\Delta A = E\Upsilon H_1, \ \Delta B = E\Upsilon H_2$$

by defining E, H_1, H_2, and Υ.

• The **unknown** matrix $\Upsilon \in \mathbb{R}^{p\times q}$ must meet

$$\Upsilon^T\Upsilon \leq I.$$

• Furthermore, the **known** real constant matrices have the following dimensions: $E \in \mathbb{R}^{n\times p}$, $H_1 \in \mathbb{R}^{q\times n}$, and $H_2 \in \mathbb{R}^{q\times m}$.

- As a result, the uncertain linear system in (5) can be described as

$$\dot{x}(t) = \big(A + E\Upsilon H_1\big)x(t) + \big(B + E\Upsilon H_2\big)u(t) \tag{6}$$

subject to

$$\Upsilon^T \Upsilon \leq I.$$

- This model will later be used to design a **robust controller**.

4.2 Actuator failure (faults)

- In practice, the control input $u(t) \in \mathbb{R}^m$ is transmitted to the linear system through **actuators** (: $u(t) \rightarrow u^F(t)$).
- However, actuators can fail due to various reasons such as
  - Mechanical wear and tear: Over time, the mechanical components of an actuator can wear down due to constant use, leading to failure.
  - Electrical issues: Electrical faults such as short circuits, voltage spikes, or wiring problems can cause actuator failure.
- The linear system with **actuator failure** can be formulated as an uncertain linear model.

- For instance, let us consider the following case for $m = 3$:
  - The first component $u_1(t)$ is transmitted to the system through an actuator **without loss**.
  - The second component $u_2(t)$ is transmitted to the system through an actuator with $20 \sim 40\%$ **loss**.
  - The third component $u_3(t)$ is not completely transmitted to the system due to **actuator outage**.
- Then, letting

$$\alpha_1 = 1,\ \alpha_2 \in [0.6, 0.8],\ \alpha_3 = 0$$

this case can be described as

$$u_1^F(t) = \alpha_1 u_1(t),\ u_2^F(t) = \alpha_2 u_2(t),\ u_3^F(t) = \alpha_3 u(t). \tag{7}$$

- Accordingly, the faulty control input in (7) can be rearranged as

$$\begin{bmatrix} u_1^F(t) \\ u_2^F(t) \\ u_3^F(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}}_{\alpha :=} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \Leftrightarrow u^F(t) = \alpha u(t)$$

where

$$\alpha_1 = 1,\ \alpha_2 \in [0.6, 0.8],\ \alpha_3 = 0.$$

• As shown above, the linear state-space model with actuator failure can be generally described as follows:

$$\dot{x}(t) = Ax(t) + B\alpha u(t) \tag{8}$$

$$\text{with } \alpha = \mathbf{diag}\big(\alpha_1, \alpha_2, \cdots, \alpha_m\big) \in \mathbb{R}^{m\times m}$$

where $(1-\alpha_i)$ denotes the loss (failure) rate of the ith actuator, and α_i satisfies

$$0 \le \underline{\alpha}_i \le \alpha_i \le \overline{\alpha}_i \le 1,\ \ i = 1, 2, \cdots, m.$$

- The parameter $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$ can be rewritten as

$$\alpha_i = \underbrace{\frac{\overline{\alpha}_i + \underline{\alpha}_i}{2}}_{e_i :=} + \delta_i = e_i + \delta_i$$

where δ_i is unknown but satisfies

$$\delta_i \in \left[-\frac{\overline{\alpha}_i - \underline{\alpha}_i}{2}, \; \underbrace{\frac{\overline{\alpha}_i - \underline{\alpha}_i}{2}}_{h_i :=} \right] = \Big[-h_i, h_i \Big]. \tag{9}$$

- By employing a normalized parameter $\bar{\delta}_i = \delta_i / h_i$, (9) is described as

$$\delta_i = h_i \cdot \bar{\delta}_i, \;\; \bar{\delta}_i \in [-1, 1].$$

- Since $\delta_i = \bar{\delta}_i h_i$, we can obtain

$$\begin{aligned}\alpha &= \mathbf{diag}(e_1 + \delta_1, \cdots, e_m + \delta_m)\\ &= \mathbf{diag}(e_1 + \bar{\delta}_1 h_1, \cdots, e_m + \bar{\delta}_m h_m) = E + \Upsilon H\end{aligned}$$

where

$$E = \mathbf{diag}\left(\frac{\overline{\alpha}_1 + \underline{\alpha}_1}{2}, \frac{\overline{\alpha}_2 + \underline{\alpha}_2}{2}, \cdots, \frac{\overline{\alpha}_m + \underline{\alpha}_m}{2}\right) : \text{known}$$

$$\Upsilon = \mathbf{diag}(\bar{\delta}_1, \bar{\delta}_2, \cdots, \bar{\delta}_m) \text{: unknown}$$

$$H = \mathbf{diag}\left(\frac{\overline{\alpha}_1 - \underline{\alpha}_1}{2}, \frac{\overline{\alpha}_2 - \underline{\alpha}_2}{2}, \cdots, \frac{\overline{\alpha}_m - \underline{\alpha}_m}{2}\right) : \text{known.}$$

- As a result, it is obtained that

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\alpha u(t) \\ &= Ax(t) + B(E + \Upsilon H)u(t)\end{aligned}$$

subject to

$$\Upsilon^T \Upsilon \leq 1.$$