Mathematical Description of Linear Systems with various constraints

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Assume that f has $\mathcal{C}^1$ continuity and $\phi(t)$ is a given solution of (8) for the given specific function $\psi(t)$.
Then we can linearize the nonlinear system (8) about $\phi(t)$ in the following manner.

$$\dot{z}(t) = A(t)z(t) + B(t)v(t)$$

where $z(t) = x(t) - \phi(t)$, $v(t) = u(t) - \psi(t)$,

$$A(t) = \frac{\partial f}{\partial x}(t, x, u)\Big|_{x=\phi(t), u=\psi(t)}$$

$$B(t) = \frac{\partial f}{\partial u}(t, x, u)\Big|_{x=\phi(t), u=\psi(t)}.$$

Examples

Example 3.1. Let us consider a simple pendulum described by

$$\ddot{\theta}(t) + k \sin \theta(t) = 0$$

where $k > 0$ is a constant.

Defining $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$, we can obtain

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -k \sin x_1(t). \end{cases} \tag{9}$$

That is,

$$\dot{x}(t) = f(t, x(t)) = \begin{bmatrix} x_2(t) \\ -k \sin x_1(t) \end{bmatrix} \in \mathbb{R}^2.$$

Let us select $\phi(t)$ as

$$\phi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 0, x_2 = 0 \rightarrow \dot{x}_1 = 0, \dot{x}_2 = 0.$$

Then

$$A = \frac{\partial f}{\partial x}(t, x)\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -k\cos x_1(t) & 0 \end{bmatrix}\Bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}.$$

Thus, the linearized equation of (10) about the solution $\phi(t) = 0$ is given by

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} z(t).$$

Example 3.2. Let us consider the following nonlinear system:

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$

$$\ddot{\theta}(t) = -2\frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t).$$

Then defining that $x_1(t) = r(t)$, $x_2(t) = \dot{r}(t)$, $x_3(t) = \theta(t)$, and $x_4(t) = \dot{\theta}(t)$, we can obtain

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_1(t)x_4^2(t) - \dfrac{k}{x_1^2(t)} + u_1(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = -2\dfrac{x_2(t)x_4(t)}{x_1(t)} + \dfrac{1}{x_1(t)}u_2(t). \end{cases} \tag{10}$$

Furthermore, we can choose a solution $\phi(t)$ of the following form:

$$\phi(t) = \begin{bmatrix} r_0 \\ 0 \\ w_0 t + \theta_0 \\ w_0 \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

under the assumption:

$$w_0^2 = \frac{k}{r_0^3}.$$

Then the linearized system matrices are given by

$$A = \frac{\partial f}{\partial x}(t,x)\Big|_{x=\phi, u=\psi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w_0^2 & 0 & 0 & 2r_0w_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2w_0}{r_0} & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(t,x)\Big|_{x=\phi, u=\psi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}.$$

Example 3.3. Let us consider the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}$$

where $g = 10\ (m/sec^2)$ and $l = 1\ (m)$. Then, a linearized model of this system about the solution $\phi(t) = [0, 0]^T$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) For $x(0) = [x_0, 0]^T$ with $x_0 = \pi/18, \pi/12, \pi/6,$ and $\pi/3$, plot the states for $t \geq 0$, for the nonlinear model, using MATLAB.

(b) Repeat (a) for the linear model.

(c) Compare the results in (a) and (b).

Assignment 3.1. Let us consider the following nonlinear system:

$$\begin{aligned}
&\ddot{\varphi} - \left(\frac{g}{L'}\right)\sin\varphi + \left(\frac{1}{L'}\right)\ddot{S}\cos\varphi = 0 \\
&M\ddot{S} + F\dot{S} = \mu(t).
\end{aligned}$$

Linearize this nonlinear system about $\phi(t) = 0$.

4 Uncertain linear systems

4.1 Parameter Uncertainty

In general, the uncertain linear systems are described as

$$\dot{x}(t) = \big(A + \Delta A\big)x(t) + \big(B + \Delta B\big)u(t) \tag{11}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times m}$ denote the parameter uncertainties.

Specifically, ΔA and ΔB are expressed as

$$\Delta A = E\Upsilon H_1,\ \Delta B = E\Upsilon H_2$$

where

- $\Upsilon \in \mathbb{R}^{p\times q}$: an unknown matrix such that

$$\Upsilon^T \Upsilon \leq I.$$

- $E \in \mathbb{R}^{n\times p}$, $H_1 \in \mathbb{R}^{q\times n}$, $H_2 \in \mathbb{R}^{q\times m}$: known real constant matrices.

Examples

Example 4.1. In Example 2.1, let us consider the case where

$$\frac{1}{L} = \frac{1}{L_0} + \delta,\ \delta \in [-0.1,\ 0.1].$$

Please define A, B, E, H_1, and H_2.

4.2 Actuator failure

Let us consider the following actuator failure model:

$$\dot{x}(t) = Ax(t) + Bu^F(t) \tag{12}$$

with

$$u^F(t) = \alpha u(t) \in \mathbb{R}^m,\ \alpha = \mathbf{diag}\big(\alpha_1, \alpha_2, \cdots, \alpha_m\big)$$

where the variables α_i quantify the failures of the actuators and

$$0 \le \underline{\alpha}_i \le \alpha_i \le \overline{\alpha}_i \le 1,\ i = 1, 2, \cdots, m.$$

- When $\underline{\alpha}_i = \overline{\alpha}_i = 1$, it corresponds to the normal fully operating case, that is, $u^F(t) = u(t)$.
- When $\underline{\alpha}_i = \overline{\alpha}_i = 0$, the failure model is equivalent to the case of the ith actuator outage.
- When $0 < \underline{\alpha}_i < \overline{\alpha}_i < 1$, it corresponds to the case where the intensity of the signal from actuator may vary.

The failure rate is rewritten as

$$\alpha_i = \frac{\overline{\alpha}_i + \underline{\alpha}_i}{2} + \delta_i,$$

where δ_i is unknown but satisfies

$$\delta_i \in \left[-\frac{\overline{\alpha}_i - \underline{\alpha}_i}{2}, \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2} \right].$$

Let us define

$$e_i = \frac{\overline{\alpha}_i + \underline{\alpha}_i}{2},\ h_i = \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2}$$

and let us normalize δ_i as $\bar{\delta}_i = \delta_i / h_i$. Then $\bar{\delta}_i \in [-1, 1]$ holds. Furthermore, since $\delta_i = \bar{\delta}_i h_i$, it follows that

$$\begin{aligned} \alpha &= \mathbf{diag}(e_1 + \delta_1, \cdots, e_m + \delta_m) \\ &= \mathbf{diag}(e_1 + \bar{\delta}_1 h_1, \cdots, e_m + \bar{\delta}_m h_m) = E + \Upsilon H \end{aligned}$$

where

$$\begin{aligned} E &= \mathbf{diag}\left(\frac{\overline{\alpha}_1 + \underline{\alpha}_1}{2}, \frac{\overline{\alpha}_2 + \underline{\alpha}_2}{2}, \cdots, \frac{\overline{\alpha}_m + \underline{\alpha}_m}{2}\right) : \text{known} \\ \Upsilon &= \mathbf{diag}(\bar{\delta}_1, \bar{\delta}_2, \cdots, \bar{\delta}_m) : \text{unknown} \\ H &= \mathbf{diag}\left(\frac{\overline{\alpha}_1 - \underline{\alpha}_1}{2}, \frac{\overline{\alpha}_2 - \underline{\alpha}_2}{2}, \cdots, \frac{\overline{\alpha}_m - \underline{\alpha}_m}{2}\right) : \text{known.} \end{aligned}$$

As a result, it is obtained that

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(E + \Delta)u(t) \\ &= Ax(t) + (BE + B\Upsilon H)u(t)\end{aligned}$$

where $\Upsilon^T \Upsilon \leq 1$.