Mathematical Description of Linear Systems with various constraints

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3 Linearization

"Linearization" is a method of converting a nonlinear system into a linear system based on an equilibrium point or a limited operating region.

- Case I: without control input
- Case II: with control input

Definition 1.

$$y(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_m(x) \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \frac{dy}{dx} := \begin{bmatrix} \frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_n} \\ \vdots & \frac{dy_i}{dx_j} & \vdots \\ \frac{dy_m}{dx_1} & \cdots & \frac{dy_m}{dx_n} \end{bmatrix}.$$

3.1 For nonlinear systems without control input

Our goal is to obtain a linear state-space model from a nonlinear system without control input (please see Example 3.1).

The first thing to do is to covert the given nonlinear system into the nonlinear state-space form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{bmatrix} \in \mathbb{R}^n. \tag{7}$$

In other words,

$$\dot{x}(t) = f(t,x) \in \mathbb{R}^n.$$

Assumption 1. The nonlinear function $f_i(t, x)$ has $\mathcal{C}^1$ continuity and $x = \phi(t)$ is an equilibrium point of (7).

Definition 2. A function $f_i(t, x)$ is said to be $\mathcal{C}^1$ continuous at a point $x = a$ if the following conditions are met:

1) (Smoothness) The function $f_i(t, x)$ is continuous at $x = a$.

2) (Differentiability) The derivative $f_i'(t, x)$ exists at $x = a$.

Based on Assumption 1, we can linearize the nonlinear system (7) about $\phi(t)$ in the following manner.

After choosing an equilibrium point $\phi(t)$, define $z(t) = x(t) - \phi(t)$ and obtain

$$A(t) = \frac{\partial f}{\partial x}(t,x)\Big|_{x=\phi(t)} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1}(t,x) & \cdots & \frac{\partial f_1}{\partial x_n}(t,x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(t,x) & \cdots & \frac{\partial f_n}{\partial x_n}(t,x) \end{array}\right]\Bigg|_{x\,=\,\phi(t)} .$$

Then, the linearized state-space model is given by

$$\dot{z}(t) = A(t)z(t).$$

Remark 1. $\dfrac{\partial f}{\partial x}(t,x)$ is called the Jacobian matrix of $f(t,x)$ with respect to x.

Remark 2. If the solution $\phi(t) = \mathbf{0} \in \mathbb{R}^n$, then $z(t) = x(t)$. Accordingly, it holds that

$$\dot{x}(t) = A(t)x(t).$$

Example 3.1. Let us consider a simple pendulum described by

$$\ddot{\theta}(t) + k \sin \theta(t) = 0$$

where $k > 0$ is a constant.

3.2 For nonlinear systems with control input

Let us consider the following nonlinear state-space model with control input $u(t)$, given by

$$\dot{x} = f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix} \in \mathbb{R}^n. \tag{9}$$

Assumption 2. The nonlinear function $f_i(t,x,u)$ has $\mathcal{C}^1$ continuity and $(x,u) = (\phi(t),\psi(t))$ is an equilibrium point of (9).

Then we can linearize the nonlinear system (9) about $(\phi(t), \psi(t))$ in the following manner.

After choosing an equilibrium point $(\phi(t), \psi(t))$, define

$$z(t) = x(t) - \phi(t),\ \ v(t) = u(t) - \psi(t)$$

and obtain

$$A(t) = \frac{\partial f}{\partial x}(t, x, u)\Big|_{x = \phi(t), u = \psi(t)}$$

$$B(t) = \frac{\partial f}{\partial u}(t, x, u)\Big|_{x = \phi(t), u = \psi(t)}.$$

Then, the linearized state-space model is given by

$$\dot{z}(t) = A(t)z(t) + B(t)v(t).$$

Example 3.2. Let us consider the following nonlinear system:

$$\begin{aligned}\ddot{r}(t) &= r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)\\ \ddot{\theta}(t) &= -2\frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t).\end{aligned}$$

Example 3.3 (On-line). Let us consider the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}$$

where $g = 10$ (m/sec^2) and $l = 1$ (m). Then, a linearized model of this system about the solution $\phi(t) = [0, 0]^T$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) For $x(0) = [x_0, 0]^T$ with $x_0 = \pi/18, \pi/12, \pi/6$, and $\pi/3$, plot the states for $t \geq 0$, for the nonlinear model, using MATLAB.

(b) Repeat (a) for the linear model.

(c) Compare the results in (a) and (b).