Mathematical Description of Linear Systems with various constraints

University of Ulsan

Prof. KIM

5 Linear systems with external disturbances

5.1 Norms of signals

Let us consider a vector-valued signal $v(t) \in \mathbb{R}^w$.

• The $\mathcal{L}_1$-norm of $v(t)$ (namely, $||v(t)||_1$) is obtained by

$$||v(t)||_1 = \sum_{i=1}^{w} \int_0^{\infty} |v_i(t)|dt.$$

• If $||v(t)||_1$ is finite, it can be said that $v(t)$ belongs to the $\mathcal{L}_1$-space defined as

$$\mathcal{L}_1 := \left\{ v(t) \in \mathbb{R}^w \;\middle|\; ||v(t)||_1 < \infty \right\}.$$

Example 5.1. Find the $\mathcal{L}_1$-norm of $v(t)$ and show that $v(t)$ belongs to the $\mathcal{L}_1$-space:

$$v(t) = \begin{bmatrix} 3e^{-t} \\ -3e^{-3t} \end{bmatrix} \in \mathbb{R}^2.$$

- The $\mathcal{L}_2$-norm of $v(t)$ (namely, $||v(t)||_2$) is obtained by

$$||v(t)||_2 = \left(\sum_{i=1}^{w} \int_0^\infty v_i^2(t)dt\right)^{\frac{1}{2}} = \left(\int_0^\infty v^T(t)v(t)dt\right)^{\frac{1}{2}}.$$

- If $||v(t)||_2$ is finite, it can be said that $v(t)$ belongs to the $\mathcal{L}_2$-space defined as

$$\mathcal{L}_2 := \left\{ v(t) \in \mathbb{R}^w \;\middle|\; ||v(t)||_2 < \infty \right\}.$$

Example 5.2. Find the $\mathcal{L}_2$-norm of $v(t)$ and show that $v(t)$ belongs to the $\mathcal{L}_2$-space:

$$v(t) = \begin{bmatrix} 3e^{-t} \\ -3e^{-3t} \end{bmatrix} \in \mathbb{R}^2.$$

- The $\mathcal{L}_\infty$-norm of $v(t)$ (namely, $||v(t)||_\infty$) is obtained by

$$||v(t)||_\infty := \max_{i=1,2,\ldots,w}\left(\sup_{t\geq 0}|v_i(t)|\right).$$

- If $||v(t)||_\infty$ is finite, it can be said that $v(t)$ belongs to the $\mathcal{L}_\infty$-space defined as

$$\mathcal{L}_\infty := \left\{v(t)\in\mathbb{R}^w \;\middle|\; ||v(t)||_\infty < \infty\right\}.$$

Example 5.3. Find the $\mathcal{L}_\infty$-norm of $v(t)$ and show that $v(t)$ belongs to the $\mathcal{L}_\infty$-space:

$$v(t) = \begin{bmatrix} 3\sin\dfrac{\pi}{2}t \\ -5\cos\pi t \end{bmatrix} \in \mathbb{R}^2.$$

5.2 External disturbances

• In general, external disturbances $w(t) \in \mathbb{R}^d$ are included in the linear state-space model as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + Nw(t) \tag{10}$$

$$z(t) = Gx(t) + Hu(t) \tag{11}$$

• To define the performance index later, the **performance output** $z(t) \in \mathbb{R}^q$ must be described as $x(t)$ and $u(t)$ along with (10).

• The **performance index** plays an important role in showing the influence of $w(t)$ on $z(t)$.

• One of the well-known performance indices is the $\underline{\mathcal{H}_\infty \text{ performance}}$, and it will later be used in the system analysis and control synthesis.

Definition 3 ($\mathcal{H}_\infty$ performance). For $w(t) \in \mathcal{L}_2$, if the response $z(t)$ satisfies

$$||z(t)||_2^2 \leq \gamma^2 \cdot ||w(t)||_2^2$$

that is,

$$\int_0^\infty z^T(\tau)z(\tau)d\tau \leq \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau < \infty$$

it can be said that system (10)-(11) has the $\mathcal{H}_\infty$ performance level γ.