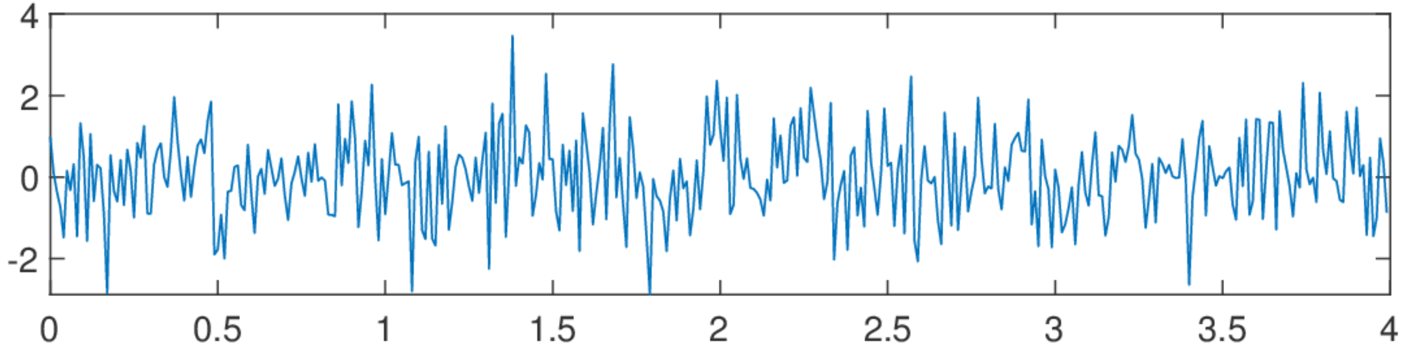
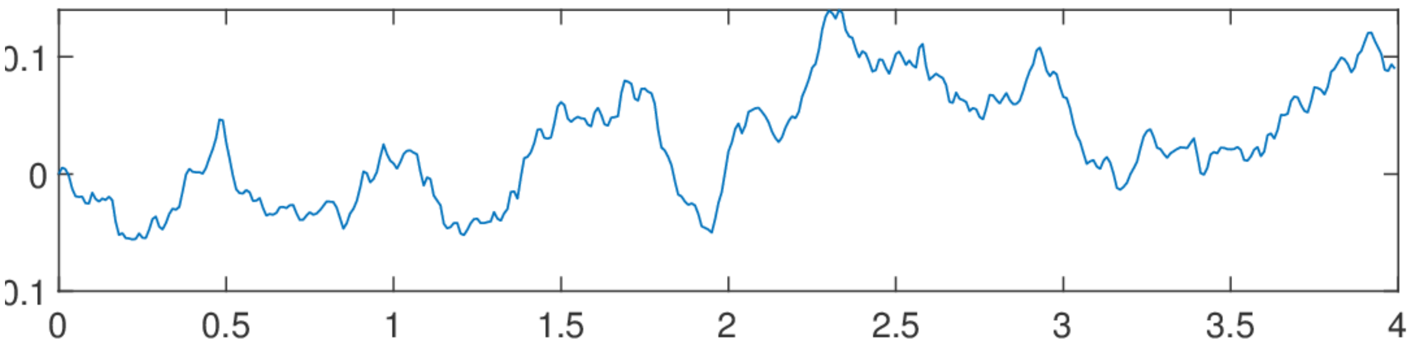
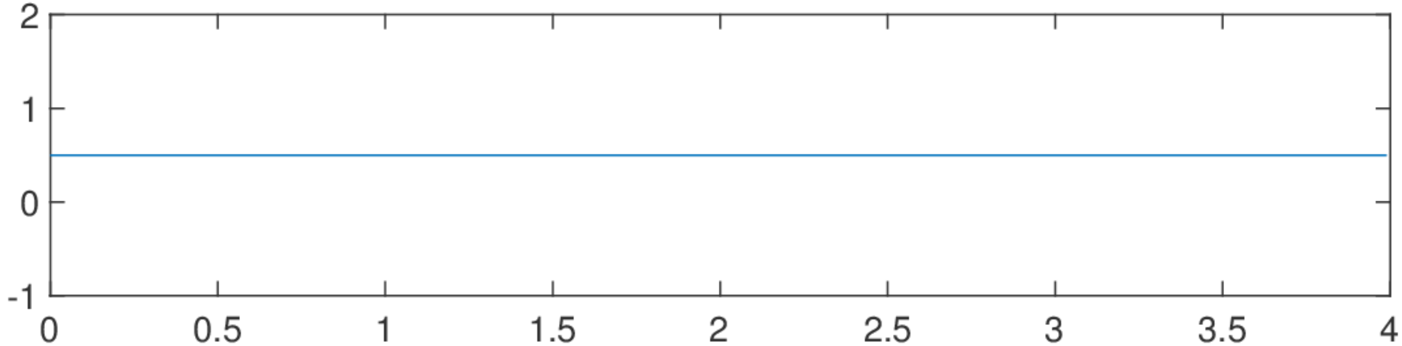
which one is Gaussian random process



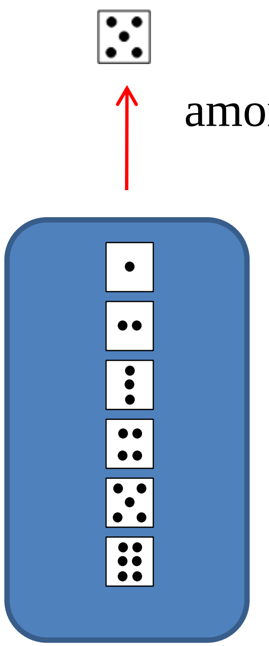




Chap 2. Random process

- Random Process X(s,t) : (Sample space, time) → (real numbers)
  - fix t : X(s) becomes Random Variable
  - fix s : X(t) become a time domain function

- RV X (dice example) : a dice was tossed and the result was 5



among 6 possibilities, the result was accidentally 5

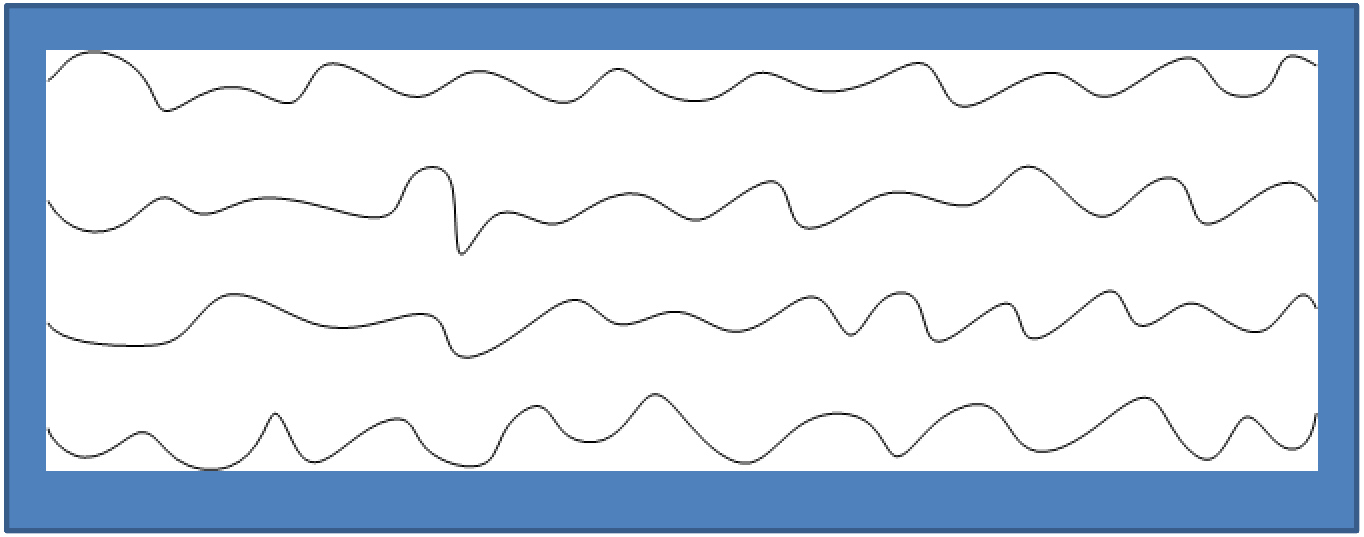
X = 5

X=5 is just a number, nothing random here

random process

- Random Process example : this waveform is experiment result

among infinitely many possibilities, the result was this waveform accidentally

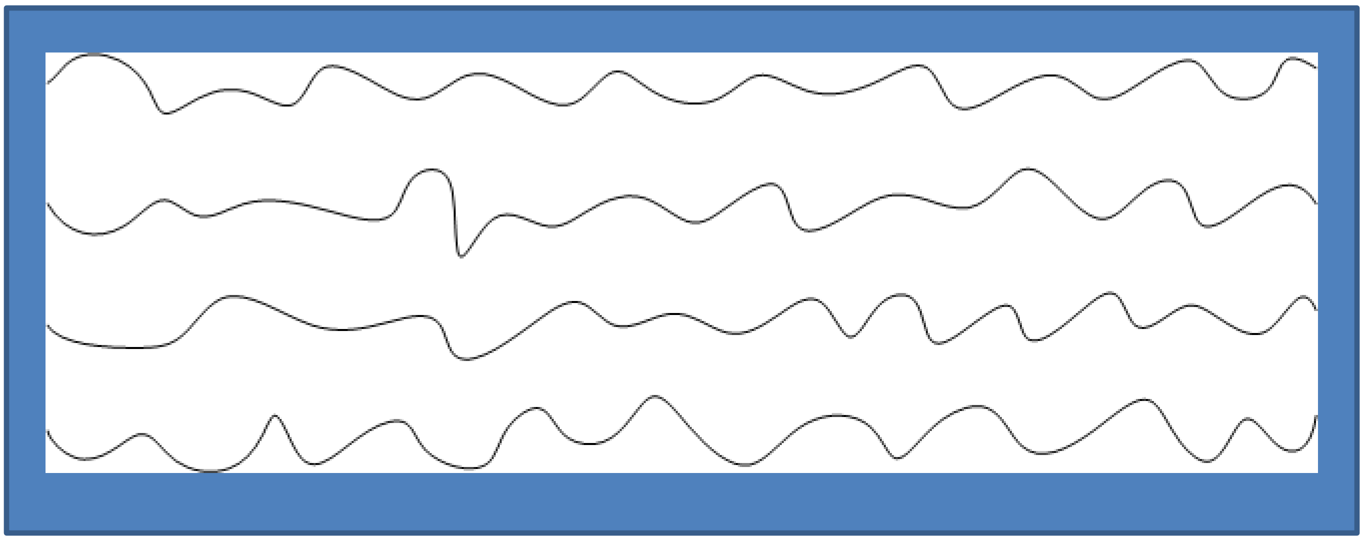


- Random Process example

time

time related things (such as autocorrelation function) should be considered in this direction

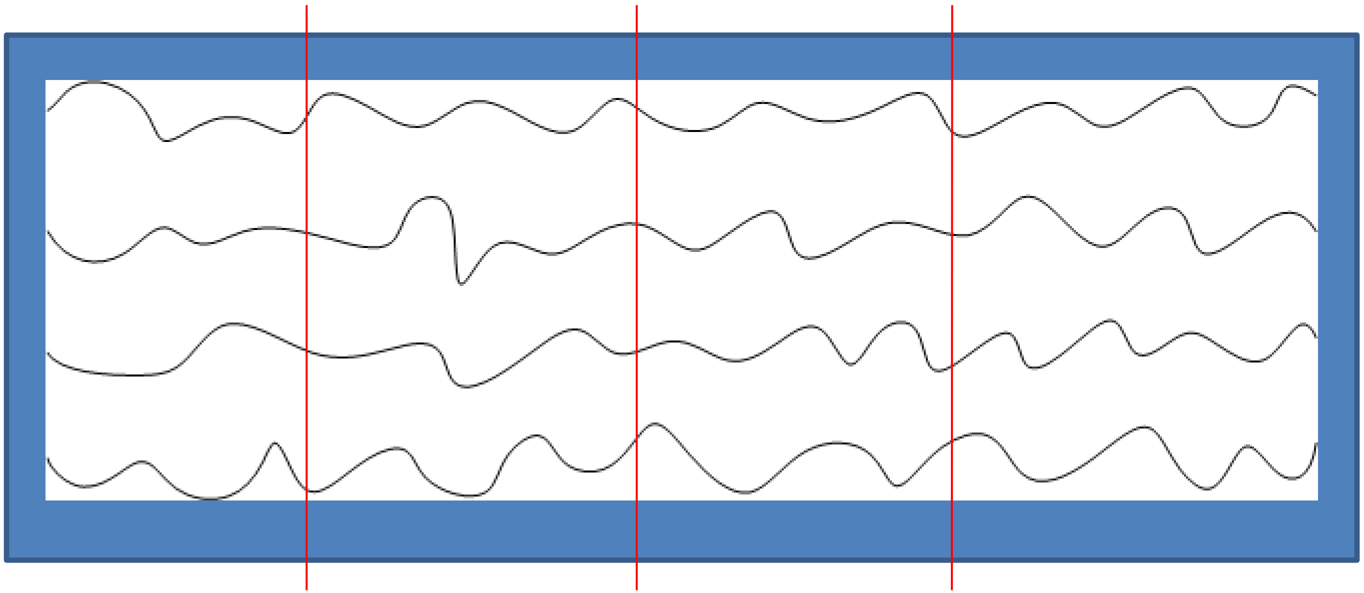
sample space



RV related things (such as mean) should be considered in this direction

mean (Ensemble average)

- mean should be computed each time t : E[X(t)]

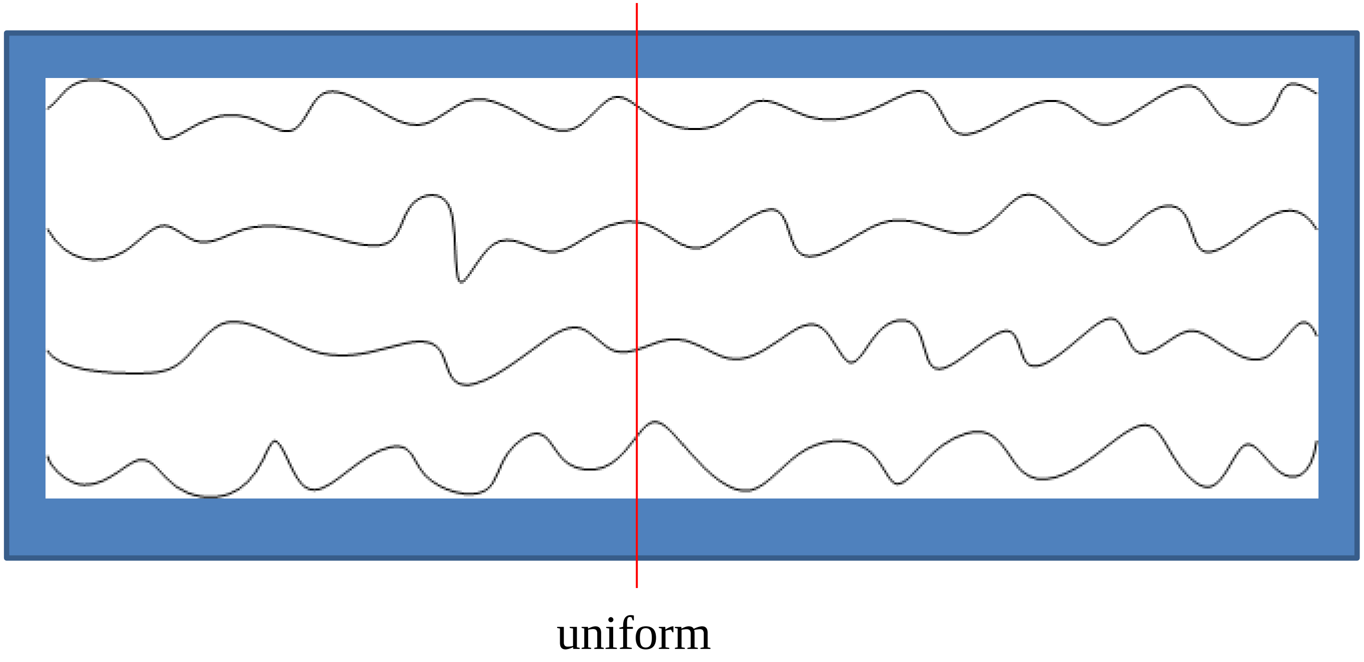


E[X(t1)]　　E[X(t2)]　　E[X(t3)]

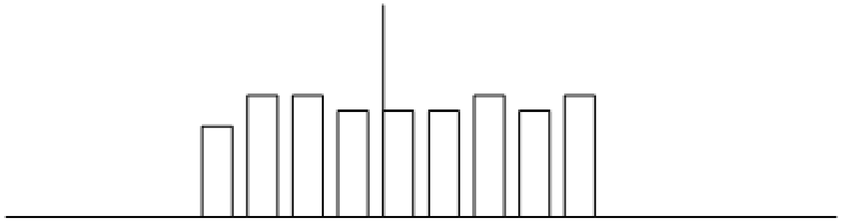
  - Mean (Ensemble average) : $E[x(t)]$
  - Time average $\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt$

Random process: RV property

- Example: uniform [-1,1] random process X(t)

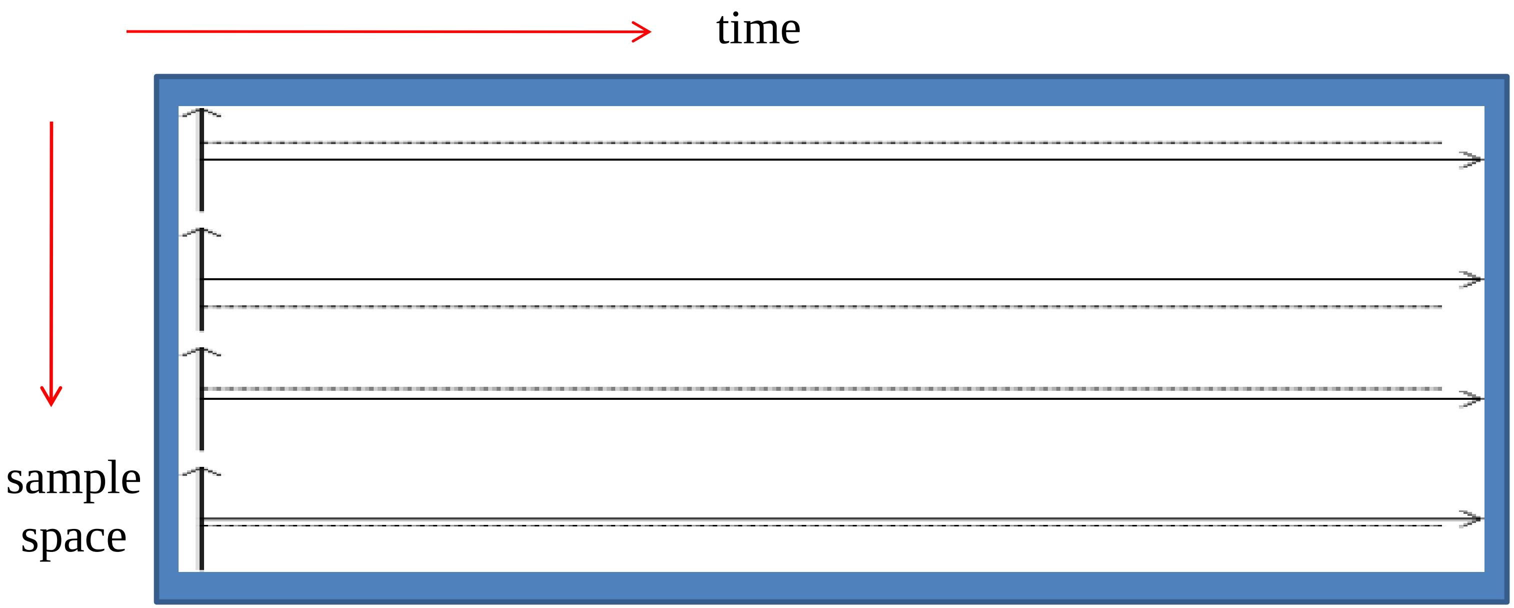


- Perform many experiments and draw histogram of X(t1)



Random process : example

- gyroscope bias
  - when the angular velocity is zero, the output should be zero
  - In practice, the sensor has small bias



- Is this Gaussian or uniform?

Autocorrelation function

- Autocorrelation function of Random process X(t)

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\}$$

- Example: Let X(t) = A, where A~N(0,1)
  – What is the autocorrelation function of X(t)?

- Stationary random process
  – E[X(t)] is independent of t (i.e., constant)

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\} = R_X(t_1 - t_2) = R_X(\tau)$$

    • autocorrelation function depends on time difference only

$$R_X(\tau) = \mathrm{E}\{X(t)X(t+\tau)\}$$

Autocorrelation function

- Autocorrelation function of Random process X(t)

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\}$$

- Example: Let X(t) = A, where A~N(0,1)
  - What is the autocorrelation function of X(t)?

$$\mathrm{E}\{X(t_1)X(t_2)\} = \mathrm{E}\{AA\} = \mathrm{E}\{A^2\} = 1$$

– autocorrelation function is just constant value in this case.

stationary and autocorrelation function

- Let X(t) be a random process defined by

$$
\begin{aligned}
X(t) &= A, \quad 0 \le t \le 1 \\
&= 0, \quad \text{otherwise}
\end{aligned}
$$

  - is this stationary random process? If so, find the autocorrelation function.

Properties of Autocorrelation function

- $R_X(\tau) = R_X(-\tau)$

$$R_X(\tau) = \mathrm{E}\{X(t)X(t+\tau)\} = \mathrm{E}\{X(t+\tau)X(t)\} = R_x(-\tau)$$

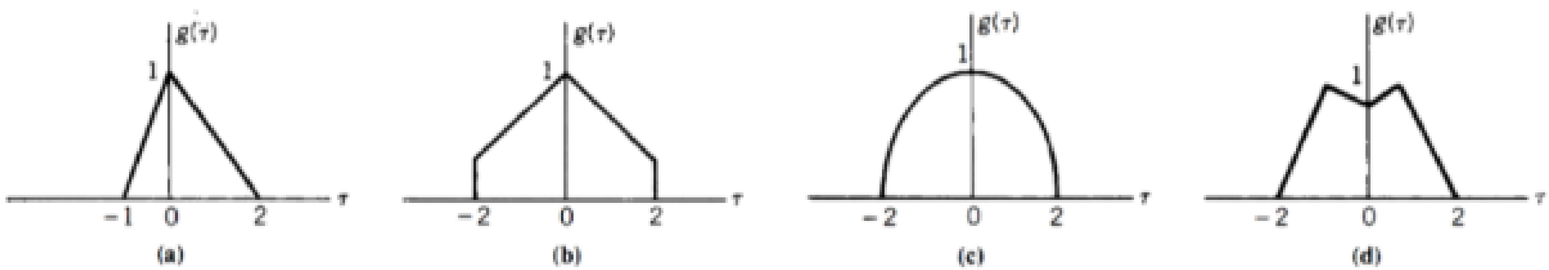
- $|R_X(\tau)| \leq R_X(0)$

$$\mathrm{E}\{(X(t)+X(t+\tau))^2\} = \mathrm{E}\{X(t)^2 + X(t+\tau)^2 \pm 2X(t)X(t+\tau)\} \geq 0$$

$$2R_X(0) = \mathrm{E}\{X^2(t) + X(t+\tau)^2\} \geq |2\mathrm{E}\{X(t)X(t+\tau)\}| = 2|R_X(\tau)|$$

- $R_X(\infty) = 0 \Rightarrow \mathrm{E}\{X(t)\} = 0$

- Autocorrelation function or not?



Source: Cooper and McGillem, Problem 6-3.1

Properties of Autocorrelation function

- $R_X(\tau) = R_X(-\tau)$

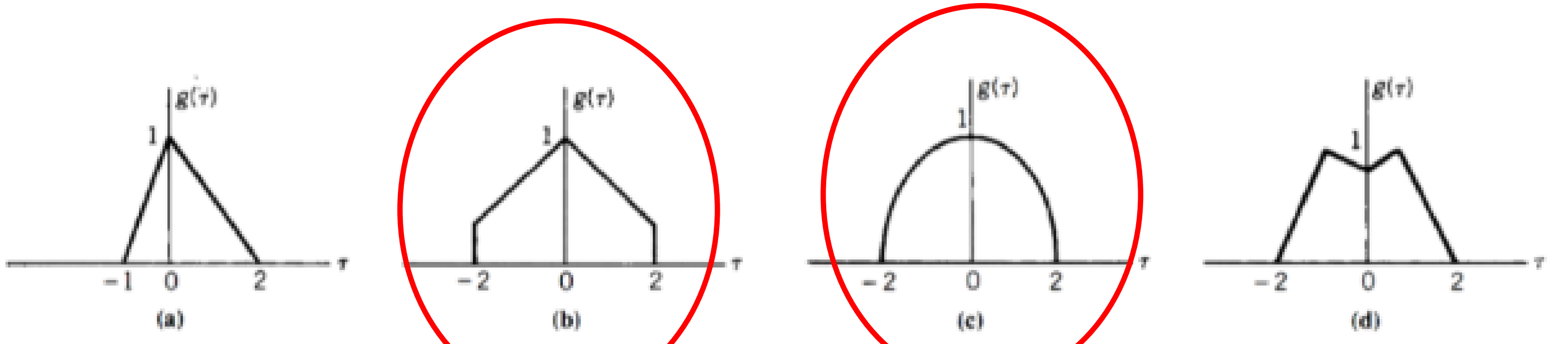
$$R_X(\tau) = \mathrm{E}\{X(t)X(t+\tau)\} = \mathrm{E}\{X(t+\tau)X(t)\} = R_x(-\tau)$$

- $|R_X(\tau)| \leq R_X(0)$

$$\mathrm{E}\{(X(t) + X(t+\tau))^2\} = \mathrm{E}\{X(t)^2 + X(t+\tau)^2 \pm 2X(t)X(t+\tau)\} \geq 0$$

$$2R_X(0) = \mathrm{E}\{X^2(t) + X(t+\tau)^2\} \geq |2\mathrm{E}\{X(t)X(t+\tau)\}| = 2|R_X(\tau)|$$

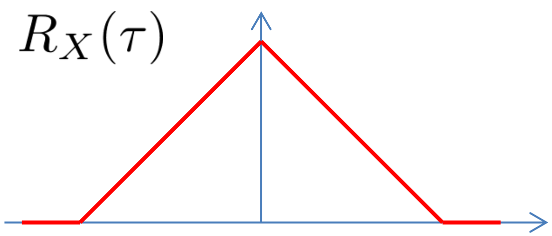
- Autocorrelation function or not?

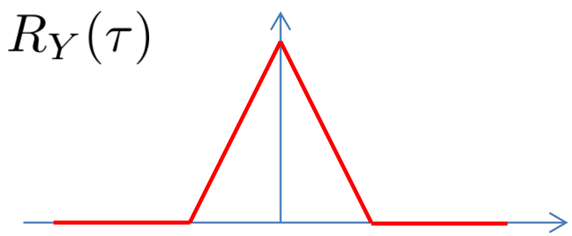


Source: Cooper and McGillem, Problem 6-3.1

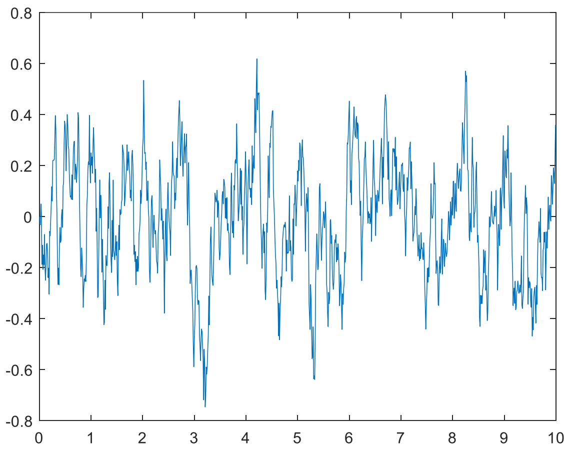
autocorrelation function

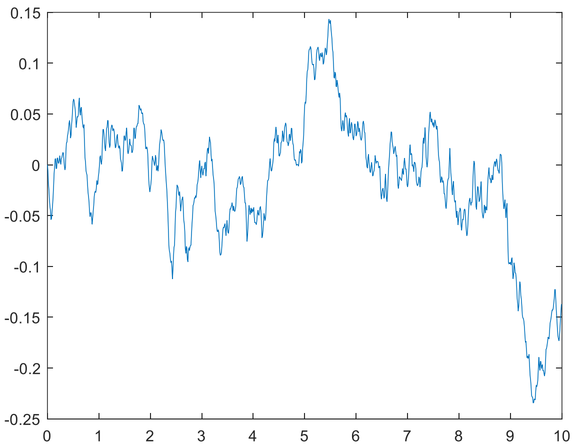
- Let X(t) and Y(t) be Gaussian random processes with autocorrelation functions





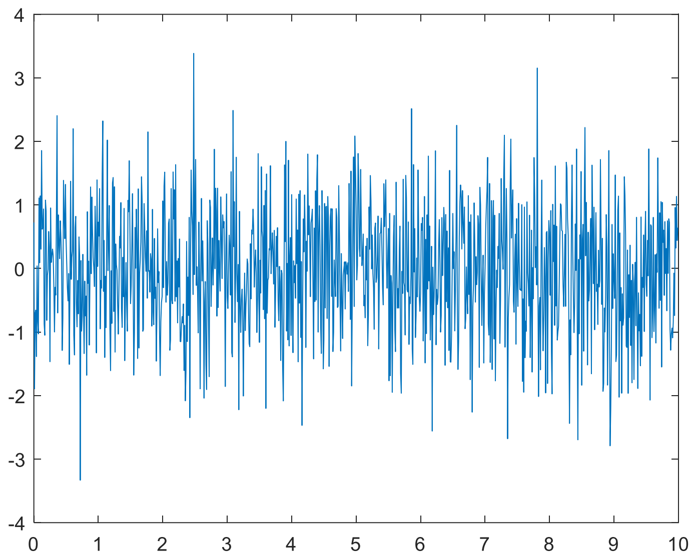
- In the following signals, which one is X(t)?





extreme case

- autocorrelation function is impulse function $R_X(\tau) = \delta(\tau)$



- X(t) and X(t+0.0000001) does not have any correlation
  - from X(t), you can't guess X(t+0.0000001)!
  - crazy signal!
  - this random process is called "white"

white Gaussian noise

- From some journal papers:

$$
\begin{aligned}
y_a &= C(q)\tilde{g} + a_b + b_a + n_a \\
y_g &= \omega + b_g + n_g
\end{aligned}
$$

Let T be the sampling period of the sensor output. The sampled outputs of y_a and y_g are denoted by $y_{a,k} = y_a(kT)$ and $y_{g,k} = y_g(kT)$. The same notation is used for q_k, r_k and v_k. Let N be the final discrete time index of the sensor output, where the starting index is one. The discrete sensor noise $n_{a,k}$ and $n_{g,k}$ are assumed to be zero mean white Gaussian noises, whose covariances are given by:

- noise n(t) is white Gaussian
  - Random variable property(Gaussian) $n(t) \sim N(0, \sigma^2)$
  - Time property (white) $R_n(\tau) = \delta(\tau)$

Autocorrelation function example

- Let X(t) be a stationary random process with $R_X(\tau)$
- What is the autocorrelation function of Y(t): $Y(t) = aX(t) + b$
  - a, b are constants

Autocorrelation function example

- Let X(t) be a stationary random process with $R_X(\tau) = \sigma^2 \exp(-\beta|\tau|)$

- What is the autocorrelation function of Y(t): $Y(t) = aX(t) + b$
  - a, b are constants

$$
\begin{aligned}
R_Y(\tau) &= \mathrm{E}\{Y(t)Y(t+\tau)\} = \mathrm{E}\{(aX(t)+b)(aX(t+\tau)+b)\} \\
&= a^2\mathrm{E}\{X(t)X(t+\tau)\} + ab\mathrm{E}\{X(t)\} + ab\mathrm{E}\{X(t+\tau)\} + \mathrm{E}\{b^2\} \\
&= a^2 R_X(\tau) + b^2
\end{aligned}
$$

  - $\mathrm{E}\{X(t)\} = 0$ since $R_X(\infty) = 0$

Autocorrelation function example

- Let X(t) be a stationary random process with $R_X(\tau) = \exp(-\tau^2)$

- Find the autocorrelation function of Z(t): $Z(t) = X(t) + X(t+a)$

Autocorrelation function example

- Let X(t) be a stationary random process with $R_X(\tau) = \exp(-\tau^2)$

- Find the autocorrelation function of Z(t): $Z(t) = X(t) + X(t+a)$

$$\begin{aligned} R_Z(\tau) &= \mathrm{E}\{Z(t)Z(t+\tau)\} = \mathrm{E}\{(X(t)+X(t+a))(X(t+\tau)+X(t+a+\tau)\} \\ &= R_X(\tau) + R_X(a+\tau) + R_X(a-\tau) + R_X(\tau) \\ &= \exp(-\tau^2) + \exp(-(a+\tau)^2) + \exp(-(a-\tau)^2) \end{aligned}$$

2.7 Power spectral density function

- (Power) spectral density function

$$S_X(j\omega) = \mathfrak{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

Fourier transform

- Example: Random process X(t) with $R_X(\tau) = \sigma^2 \exp(-\beta|\tau|)$

Gauss-Markov process

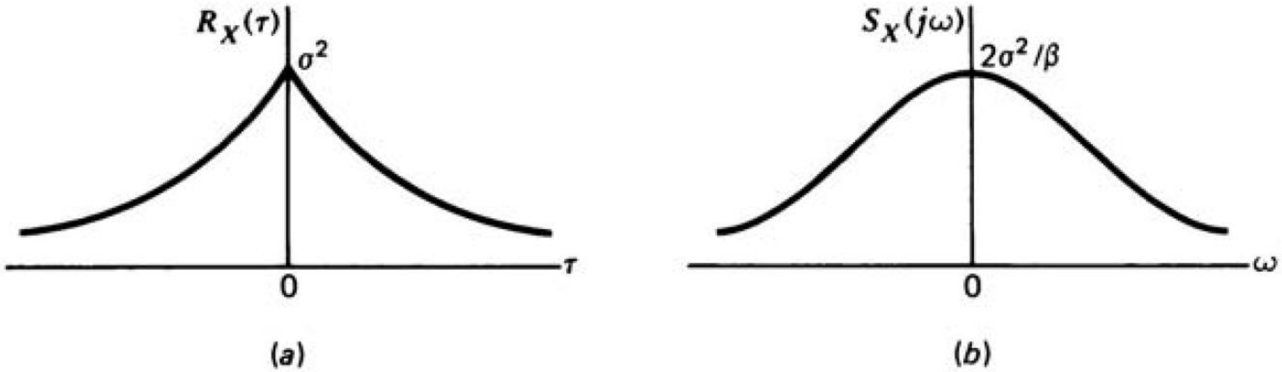


Figure 2.11 Autocorrelation and spectral density functions for Example 2.9. (*a*) Autocorrelation function. (*b*) Spectral function.

power spectral density function of white noise

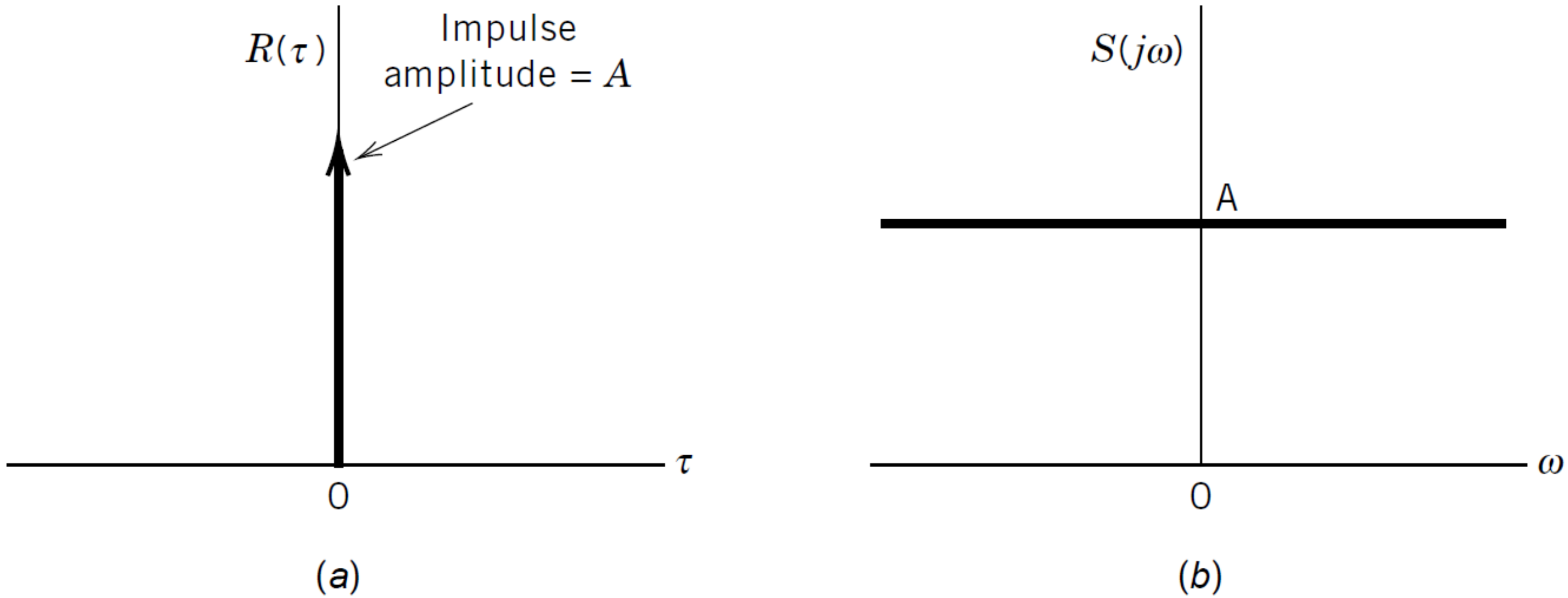


Figure 2.12 White noise. (*a*) Autocorrelation function. (*b*) Spectral density function.

band limited white noise

- power spectral density function

$$S_{bwn}(j\omega) = \begin{cases} A, & |\omega| \le 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$

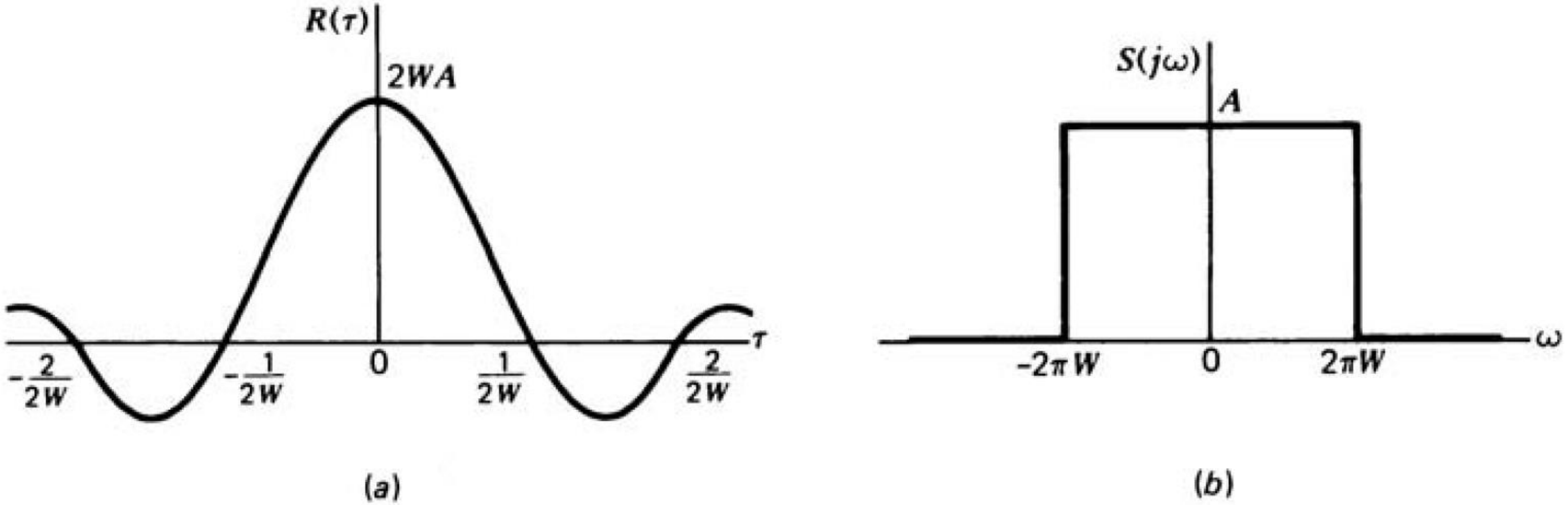
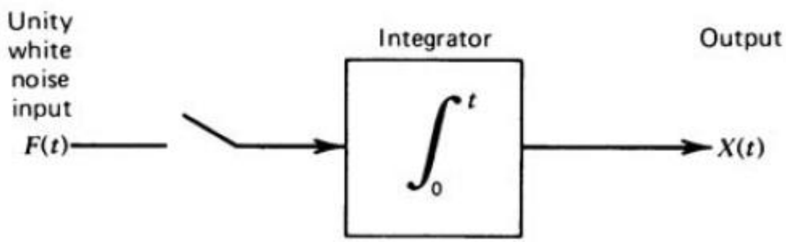


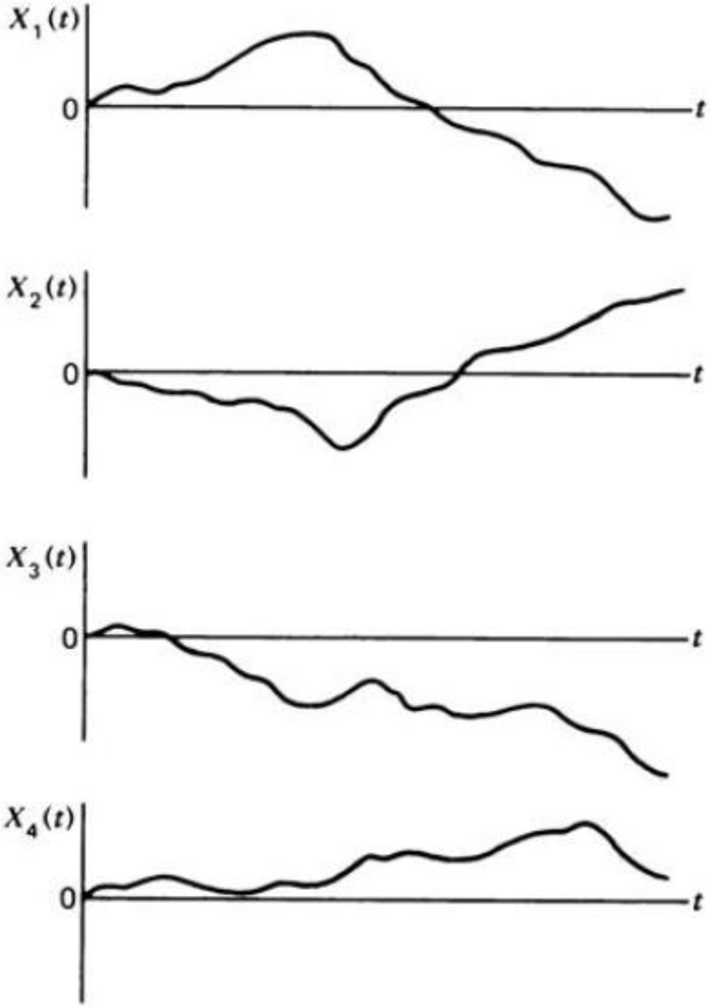
Figure 2.13 Baseband bandlimited white noise. (*a*) Autocorrelation function. (*b*) Spectral density function.

Wiener or Brownian-motion process

- integration of white noise

$$X(t) = \int_0^t F(u)\, du$$





Wiener process

- mean $$\mathrm{E}\{X(t)\} = \mathrm{E}\{\int_0^t F(r)\,dr\} = \int_0^t \mathrm{E}\{F(r)\}\,dr = 0$$

- variance

$$E[X^2(t)] = E\left[\int_0^t F(u)\,du \int_0^t F(\upsilon)\,d\upsilon\right] = \int_0^t \int_0^t E[F(u)F(\upsilon)]\,du\,d\upsilon \qquad (2.11.3)$$

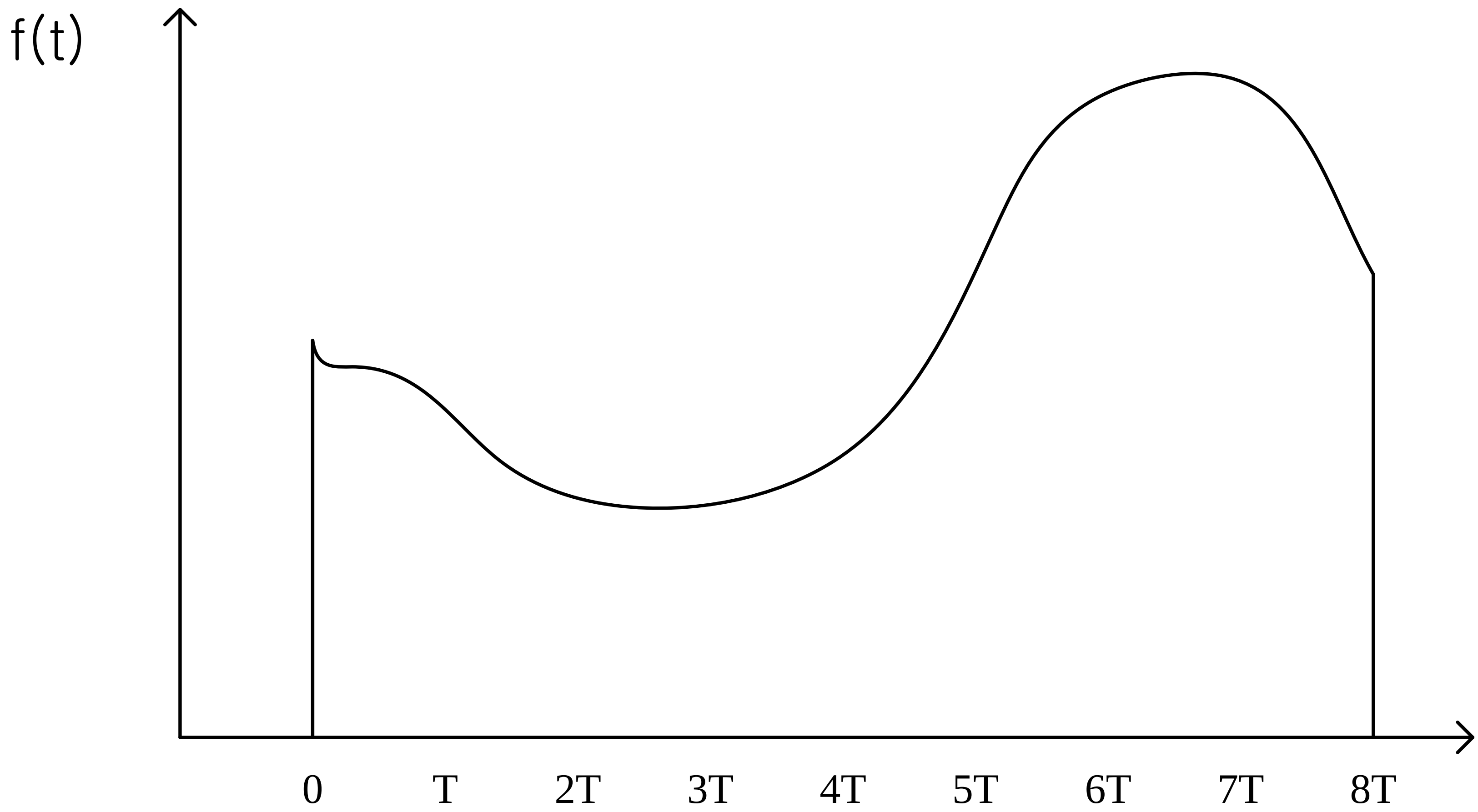
But $E[F(u)F(\upsilon)]$ is just the autocorrelation function $R_F(u-\upsilon)$, which in this case is a Dirac delta function. Thus,

$$E[X^2(t)] = \int_0^t \int_0^t \delta(u-\upsilon)\,du\,d\upsilon = \int_0^t d\upsilon = t \qquad (2.11.14)$$

- Nonstationary!

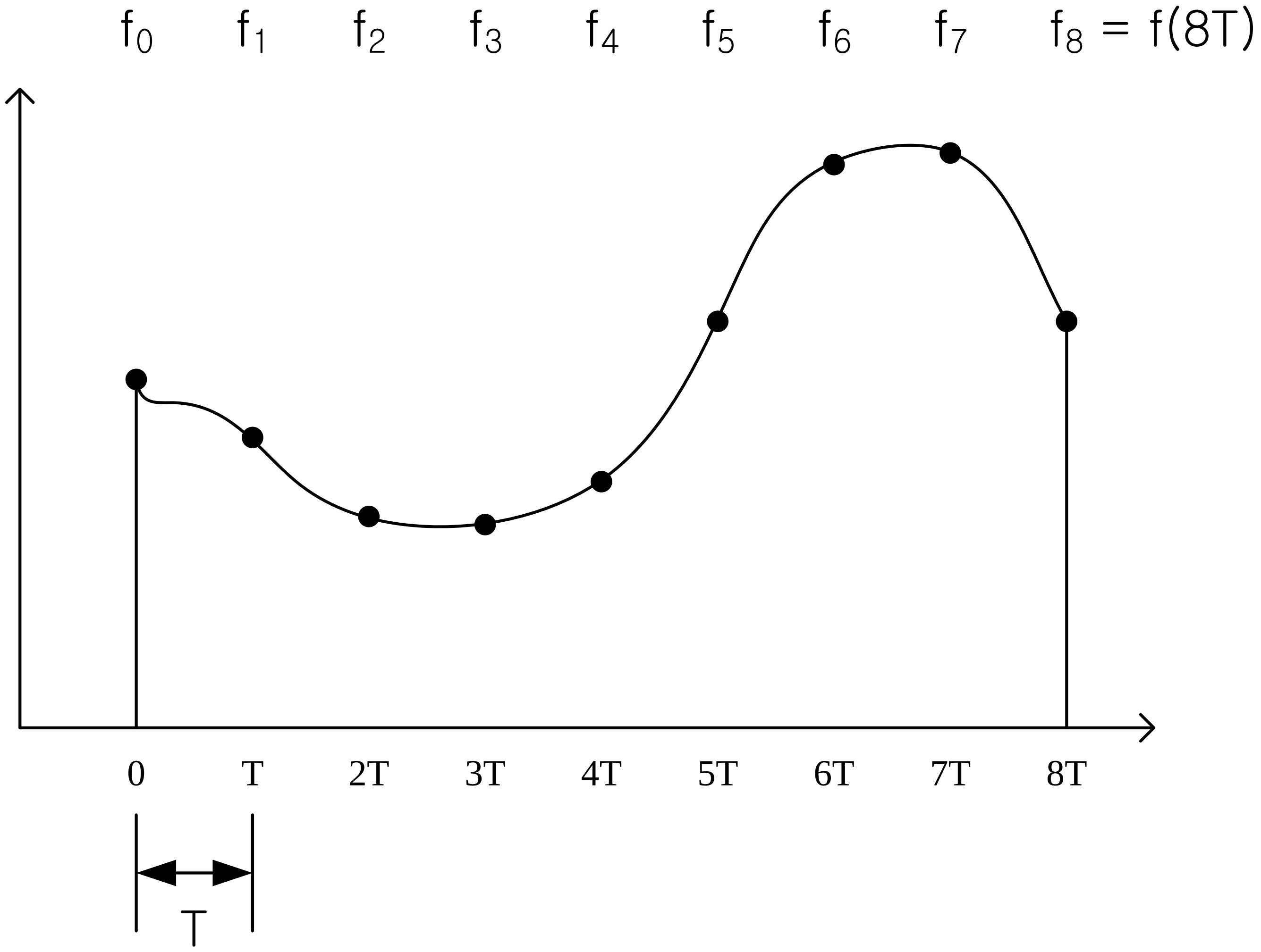
Wiener process generation in matlab (2)

- numerical integration example: $$\int_0^{8T} f(r)\,dr$$



Wiener process generation in matlab (3)

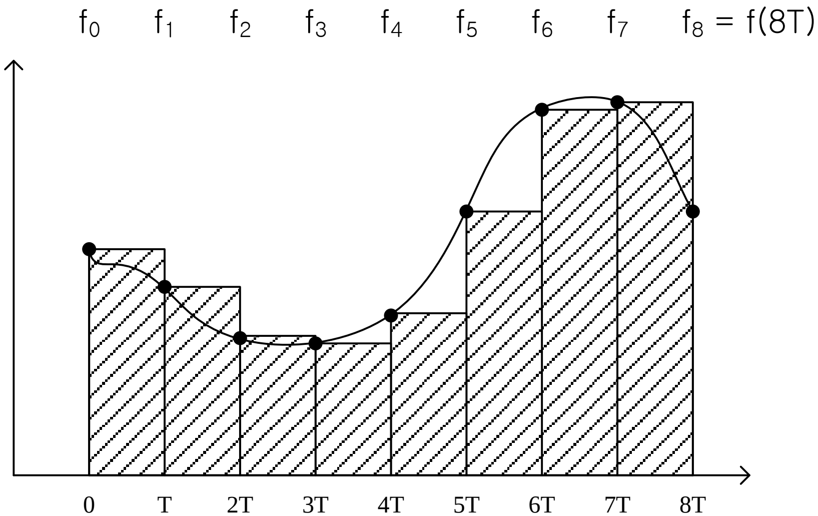
- numerical integration using f0~f8

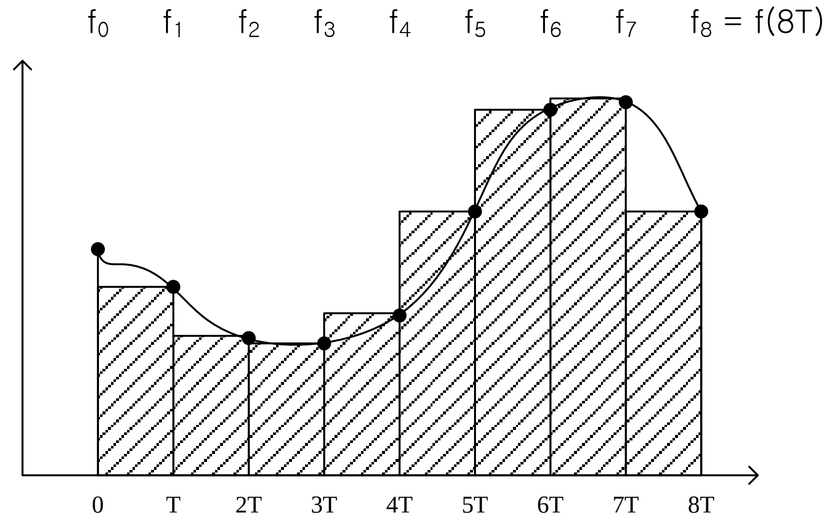


forward or backward integration

- simple numerical inegration
  - method 1: forward

    method 2: backward

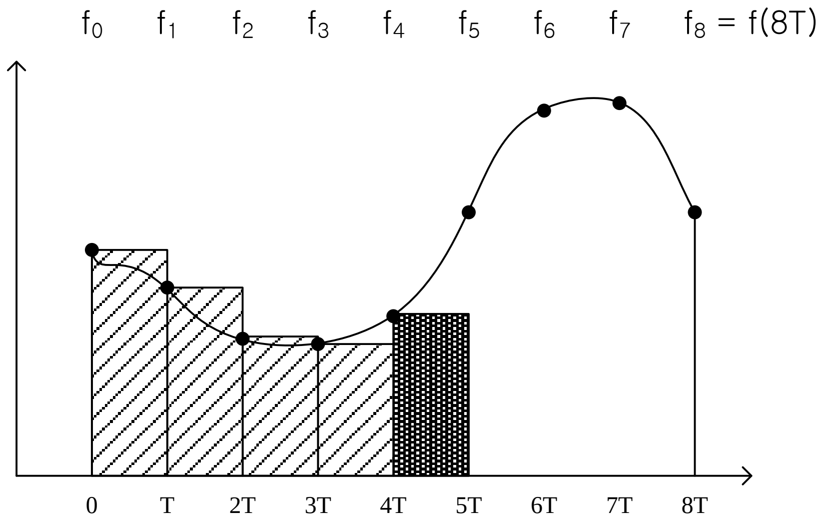




One-shot solution vs recursive solution

- one-shot solution
  - when f0~f8 are given $$I_8 = f_0 T + f_1 T + \ldots + f_8 T$$

- recursive solution $$I_4 = \int_0^{4T} f(r)dr \quad \Rightarrow \quad I_5 = I_4 + f_4 T$$



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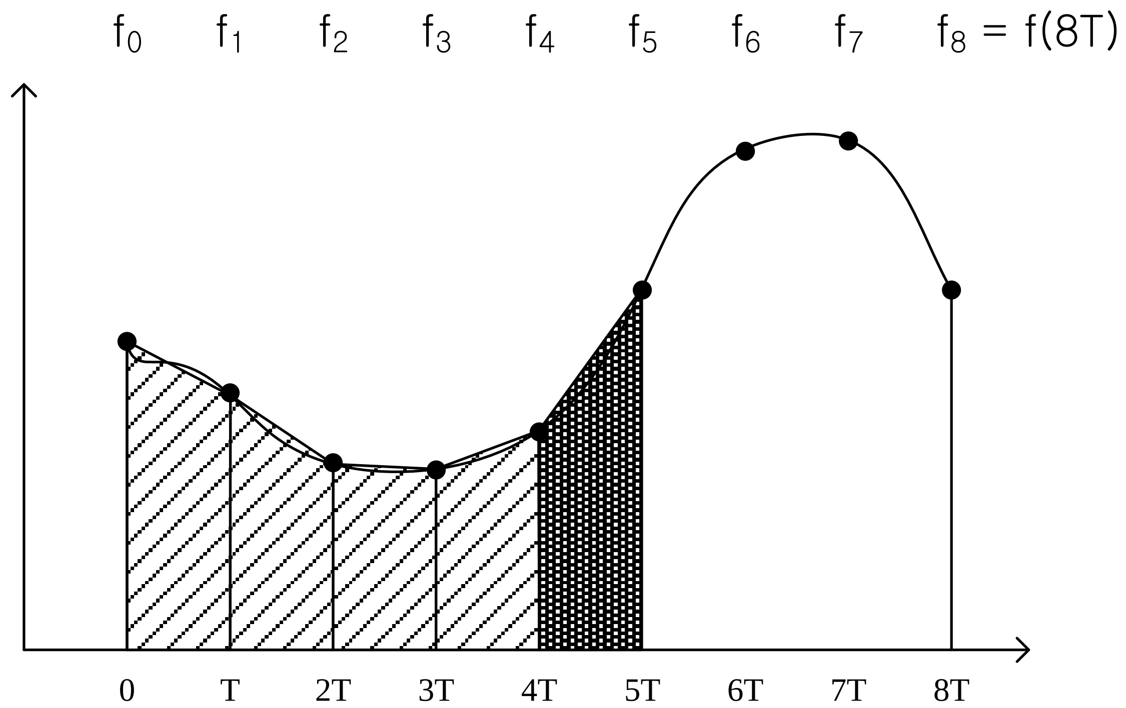
method 3

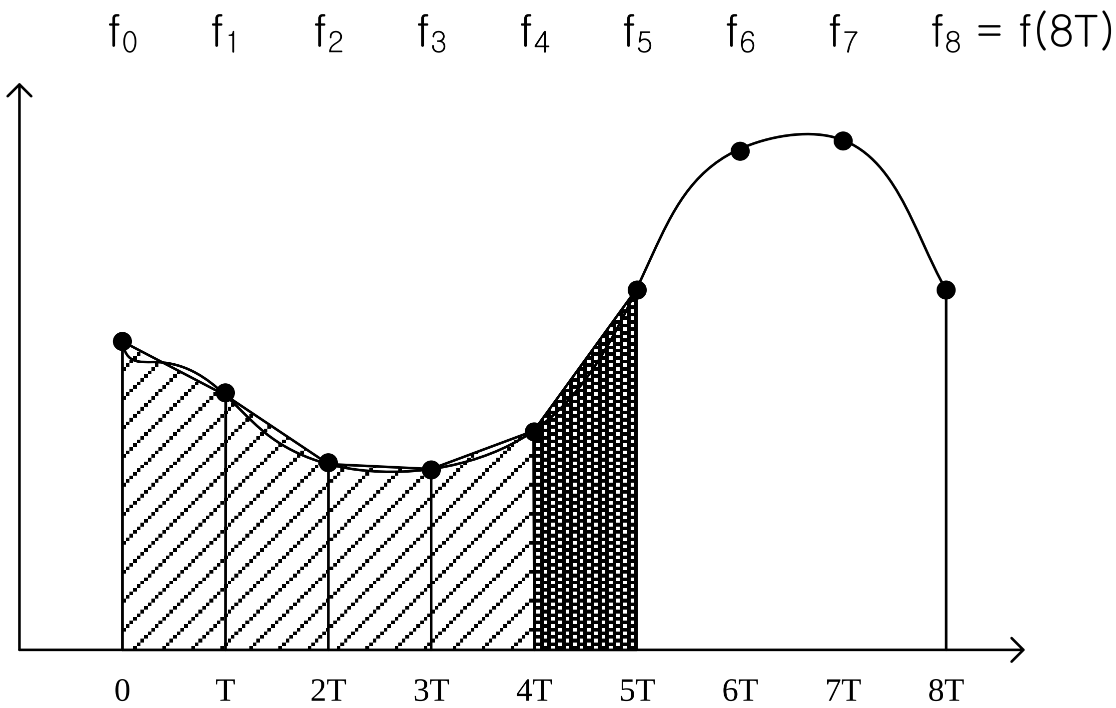
- trapezoidal rule : a little bit more accurate method



- What is the recursive equation: $I_4 = \int_0^{4T} f(r)dr \quad \Rightarrow \quad I_5 =$ ____________

method 3

- trapezoidal rule : a little bit more accurate method

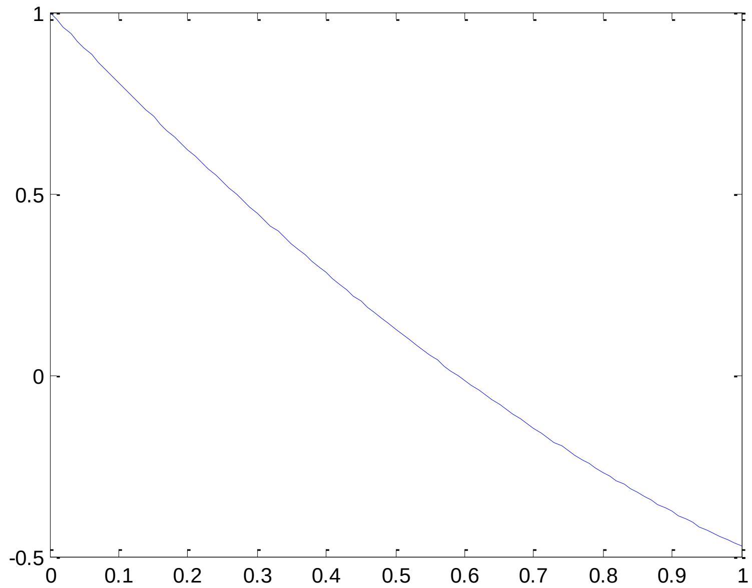


- What is the recursive equation: $$I_4 = \int_0^{4T} f(r)dr \quad \Rightarrow \quad I_5 = I_4 + \frac{f_4 + f_5}{2}$$

Example: numerical integration

- integrate the following on the interval [0,1]

$$f(t) = \exp(-t) - \sin(t)$$

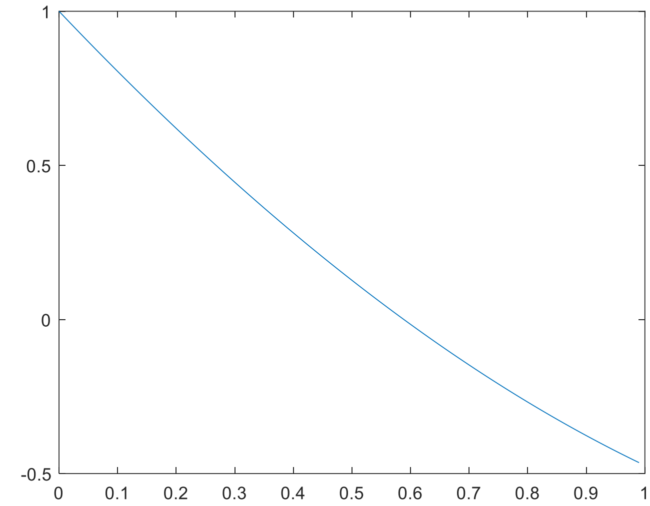


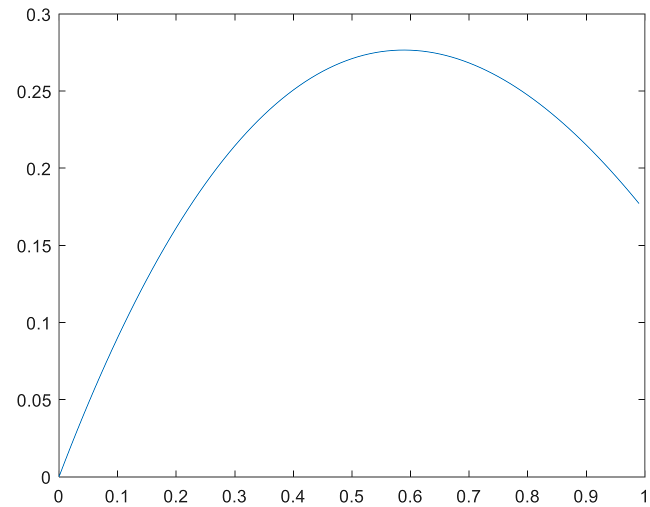
- Note that the analytical solution is given by

$$\int_0^1 \exp(-t) - \sin(t)dt = \left[-\exp(-t) + \cos(t)\right]_0^1 = \cos(1) - \exp(-1)$$

Example: f generation

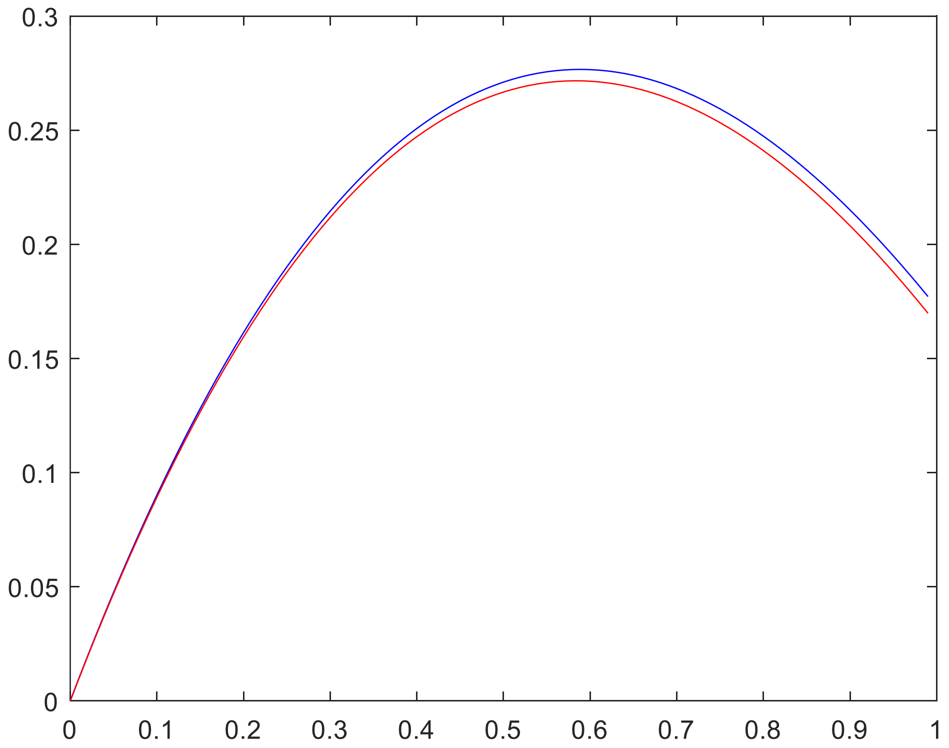
```
N = 100;
T = 0.01;
tt = 0:T:(N-1)*T;
f = zeros(1,N);
g = zeros(1,N);
for i = 1:N
    t = (i-1)*T;
    f(i) = exp(-t) - sin(t);
    g(i) = -exp(-t) + cos(t);
end
figure(1);  plot(tt,f);
figure(2);  plot(tt,g);
```





numerical integration

- blue line: true integrated value, red line: numerically integrated value



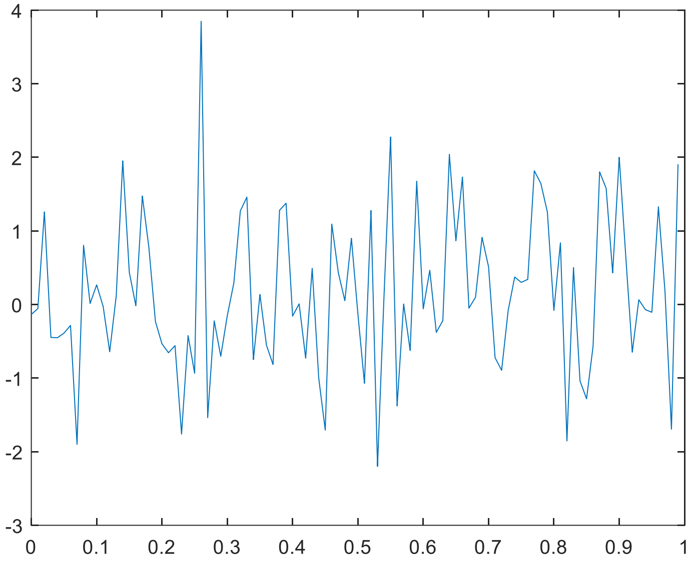
matlab: numerical integration

```
x = zeros(1,N);
for i = 2:N
    x(i) = x(i-1) + T * f(i);
end


figure(3)
plot(tt,g,'b',tt,x,'r')
```

Wiener process generation in matlab (1)

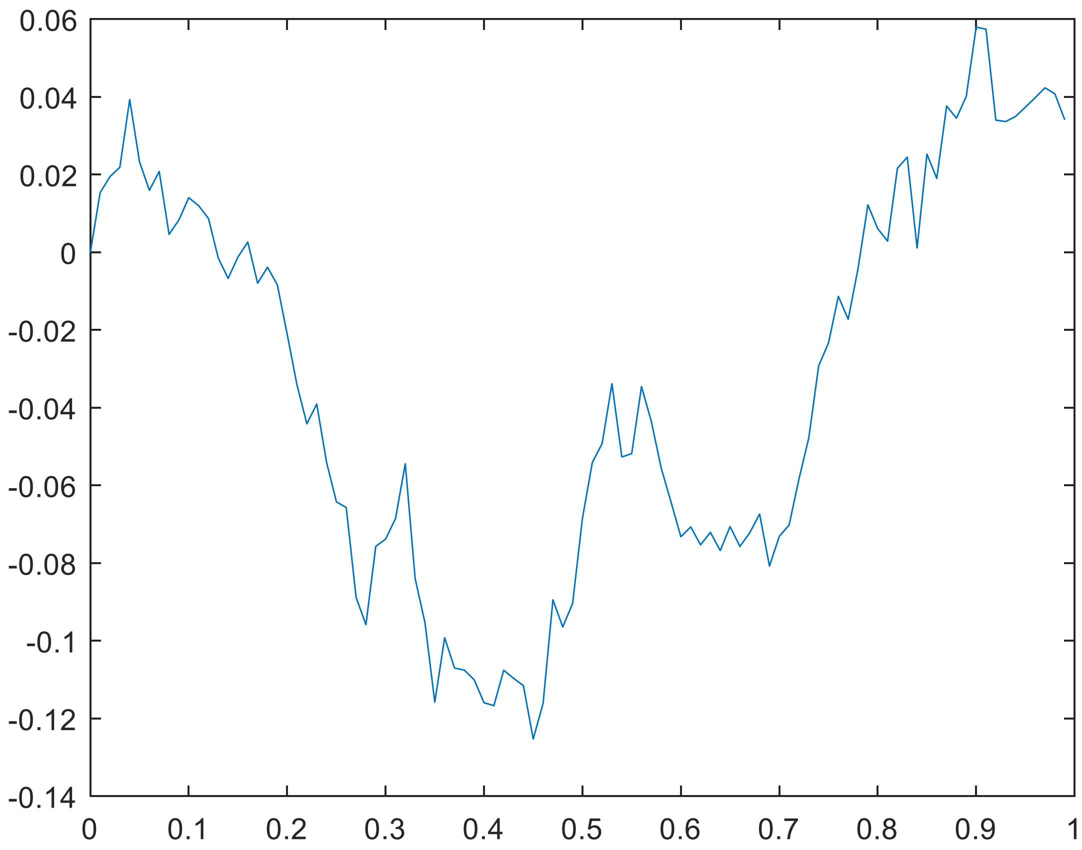
- white Gaussian noise generation

```
N = 100;

T = 0.01;

tt = 0:T:(N-1)*T;

f = randn(1,N);

plot(tt,f);
```



Wiener process generation in matlab (2)

- generate wiener process by integrating a white noise



matlab code

```
N = 100;
T = 0.01;
tt = 0:T:(N-1)*T;
f = randn(1,N);
```

```
x = zeros(1,N);
for i = 2:N
    x(i) = x(i-1) + T * f(i);
end
plot(tt,x)
```

ergodic

- ergodic random process: time average = Ensemble average

- $X(t) = A$, $A \sim N(0,1)$ → not ergodic

- $X(t)$ is a white Gaussian noise → erogidc

which one is Gaussian random process