2018 Estimation Theory Midexam

Warning: There is a penalty for the wrong answer (minus score). Be careful!

(1) Suppose X and Y are independent random variables, where X is a uniform random variable on $[0, 1]$ and Y is a uniform random variable on $[1, 2]$.

- What is the joint probability density function $f_{XY}(x, y)$?
- Compute $E[XY]$.
- Let $Z = X + Y$. What is the probability that $1 \leq Z \leq 1.5$?

(2) Let X be a uniform random variable on $[-1, 1]$. Let $Y(t)$ be a random process defined by $Y(t) = 3X + 1$.

- What is the autocorrelation function $R_Y(\tau)$ of $Y(t)$?
- Is $Y(t)$ ergodic?

(3) Consider the following system:

$$\begin{aligned} x_{k+1} &= 2x_k + w_k \\ z_k &= x_k + v_k \end{aligned}$$

where w_k and v_k are white Gaussian noises whose variances are $q = 1$ and $r = 5$, respectively. Suppose $\hat{x}_0 = 1$ and $P_0 = 1$.

- When $z_1 = 3$, compute $\hat{x}_1$ and P_1.
- What is the probability that $|x_1 - \hat{x}_1| \leq 5$?

(4) Let $Y = X + V$ where $X \sim N(a, 1)$ and $V \sim N(0, 1)$. Assume that X and V are uncorrelated. Suppose your estimator is given by

$$\hat{X} = \alpha a + \beta Y$$

where α and β are constant weighting values.

- Find the condition that the estimator becomes an unbiased estimator (that is, $\mathrm{E}\{e\} = \mathrm{E}\{X - \hat{X}\} = 0$).
- What is $\mathrm{E}\{e^2\}$ when $\alpha = \beta = 0.5$.

O: X: Total:

(1) ☐ ☐

$$f_{XY}(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \text{ and } 1 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

☐ ☐ $\mathrm{E}\{XY\} = \mathrm{E}\{X\}\mathrm{E}\{Y\} = \frac{1}{2}\frac{3}{2} = \frac{3}{4}$

☐ ☐ $\frac{1}{8}$

(2) ☐ ☐ $R_Y(\tau) = \mathrm{E}\{(3X+1)^2\} = 4$

☐ ☐ not ergodic

(3) ☐ ☐ $\hat{x}_1 = 2.5$ and $P_1 = 2.5$

☐ ☐ 0.99?

(4) ☐ ☐ $\alpha = 1 - \beta$

☐ ☐ $\frac{1}{2}$

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