2016 Estimation Theory Midexam

Warning: There is a penalty for the wrong answer (minus score). Be careful!

(1) Suppose X and Y are uncorrelated random variables, where $X \sim N(0, \sigma_x^2)$ and $Y \sim N(0, \sigma_y^2)$. A random process $Z(t)$ is defined as follows:

$$Z(t) = X + Y$$

- Is $Z(t)$ ergodic?
- What is $R_Z(\tau)$ (the autocorrelation function of $Z(t)$).

(2) Consider the following continuous system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

where $w(t)$ is zero-mean white Gaussian noise with $\mathrm{E}\{w^2(t)\} = 1$. Suppose the $x(t)$ is sampled with the period 1 sec. When the discretized system is given by

$$x_{k+1} = \Phi x_k + w_k$$

where $\mathrm{E}\{w_k w_k'\} = Q$, find Φ and Q.

(3) Suppose Z is given by

$$Z = X + w$$

where X is a random variable whose distribution is $N(1, 1)$. Let w be a random variable whose distribution is $N(0, 2)$. Suppose X and w are uncorrelated. If $Z = 2$, what is your optimal $\hat{X}$ (estimated value of X), which minimizes $\mathrm{E}\{(X - \hat{X})^2\}$? And what is $\mathrm{E}\{(X - \hat{X})^2\}$ for the optimal estimate?

(4) Consider the following system:

$$\begin{aligned} x_{k+1} &= 0.5 x_k + w_k \\ z_k &= x_k + v_k \end{aligned}$$

where w_k and v_k are white Gaussian noises whose variances are $Q = 1$ and $R = 1$. We know that $z_0 = 10$, $z_1 = 6$, $z_2 = 3$. You can start Kalman filter by choosing $\hat{x}_0 = z_0 = 10$ and $P_0 = R = 1$. Now compute $\hat{x}_1$ and $\hat{x}_2$.

Estimation Theory Midexam: score ________/ (total 15)

(1) ☐ not ergodic

☐ ☐ ☐ $R_x(\tau) = \sigma_x^2 + \sigma_y^2$

(2) ☐ ☐ $\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$

☐ ☐ $Q = \begin{bmatrix} \frac{1}{3}T^3 + T^2 + R & T + 0.5T^2 \\ T + 0.5T^2 & T \end{bmatrix}$

(3) ☐ ☐ $\hat{x} = \frac{4}{3}$

☐ ☐ $P = \frac{2}{3}$

(4) ☐ ☐ ☐ $\hat{x}_1 = \frac{50}{9}$

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