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HOMEWORK #1

1. (Fixed angle)

Rotation ϕ about $\hat{X}_A$: $R_X(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$

Rotation θ about $\hat{Z}_A$: $R_Z(\theta)$ $\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a). $^A_BR_{XZ} = R_Z(\theta)\ R_X(\phi)$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & -s\theta c\phi & s\theta s\phi \\ s\theta & c\theta c\phi & -c\theta s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

(b). $\phi = 30^\circ$ and $\cancel{\phi = 45^\circ}$ $\theta = 45$: $\cos 45^\circ \simeq 0.7$ $\sin 45^\circ \simeq 0.7$

$\cos 30^\circ \simeq 0.87$ $\sin 30 \simeq 0.5$

$$^A_BR_{XZ} = \begin{bmatrix} 0.7 & -0.612 & 0.353 \\ 0.7 & 0.612 & -0.353 \\ 0 & 0.5 & 0.87 \end{bmatrix}$$

2. (Euler)

Rotation θ about $\hat{Z}_B \rightarrow {}^{A}_{A'}R_{Z_B}(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation ϕ about $\hat{X}_B \rightarrow {}^{A'}_{B}R_{X}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$

(a). ${}^{A}_{B}R_{ZX} = {}^{A}_{A'}R_Z(\theta)\, {}^{A'}_{B}R_X(\phi)$

$$= \begin{bmatrix} c\theta & -s\theta c\phi & s\theta s\phi \\ s\theta & c\theta\, c\phi & -c\theta s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

(b). $\phi = 30^\circ$ and $\theta = 45^\circ$

$${}^{A}_{B}R_{ZX} = \begin{bmatrix} 0.7 & -0.612 & 0.353 \\ 0.7 & 0.612 & -0.353 \\ 0 & 0.5 & 0.87 \end{bmatrix}$$

c) ${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} \Rightarrow {}^{A}\hat{Y}_B = \begin{bmatrix} -0.612 \\ 0.612 \\ 0.5 \end{bmatrix}$

3).

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{B}_{A}R & & {}^{B}P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} \Rightarrow {}^{A}_{B}R = {}^{B}_{A}R^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}^{T} $$

$$ {}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix} $$

$$ {}^{A}p = {}^{A}_{B}T \cdot {}^{B}p $$

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & -{}^{B}_{A}R^{T} \cdot {}^{B}P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{A}P_{BORG} = -{}^{B}_{A}R^{T}\,{}^{B}P_{AORG} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 3 \\ -1.4 \\ -0.48 \end{bmatrix} $$

$$ {}^{A}p = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.6 & -0.8 & -1.4 \\ 0 & 0.8 & 0.6 & -0.48 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6.4 \\ 9.52 \\ 1 \end{bmatrix} $$

4)

$$R = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(a). $$R^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1\sqrt{2}}{2} & 1/2 \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -1/2 \end{bmatrix}$$

$$R^T . R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ so R is a rotation matrix

(b). Equivalent Angle-Axis. $\rightarrow \hat{k}, \theta$. $R_k(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$\theta = A\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$= A\cos\left(\frac{0 + \frac{1}{2} - \frac{1}{2} - 1}{2}\right) = 120^\circ \text{ or } -60^\circ.$$

with $\theta = 120^\circ$: $$\hat{k} = \frac{1}{2\sin 120^\circ}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \frac{\sqrt{3}}{3}\begin{bmatrix} 1/4 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

(c). $\varepsilon_0 = \cos\frac{\theta}{2} = 0.5$

$$\vec{\varepsilon} = \hat{k}\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{3}\begin{bmatrix} 1/4 \\ \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/8 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \varepsilon_1 = 1/8 \\ \varepsilon_2 = \sqrt{2}/2 \\ \varepsilon_3 = 0 \end{matrix}$$

5) Rotation Matrix (Euler Parameter)

$$R = 2\begin{bmatrix} e_0^2+e_1^2-1/2 & e_1e_2-e_0e_3 & e_1e_3+e_0e_2 \\ e_1e_2+e_0e_3 & e_0^2+e_2^2-\frac{1}{2} & e_2e_3-e_0e_1 \\ e_1e_3-e_0e_2 & e_2e_3+e_0e_1 & e_0^2+e_3^2-\frac{1}{2} \end{bmatrix}$$

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{k}\sin\frac{\theta}{2} \Rightarrow \hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \frac{1}{\sin\frac{\theta}{2}}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\, e_0 \Rightarrow \frac{1}{\sin\frac{\theta}{2}} = \frac{2e_0}{\sin\theta}$$

$$\Rightarrow \hat{k} = \frac{2}{\sin\theta}\begin{bmatrix} e_0e_1 \\ e_0e_2 \\ e_0e_3 \end{bmatrix}$$

$$k_x^2+k_y^2+k_z^2 = 1 \Rightarrow \frac{4e_0^2(e_1^2+e_2^2+e_3^2)}{\sin^2\theta} = 1$$

$$\cancel{\sin^2\theta = 1-\cos^2\theta = v\theta\Big(1+\underbrace{2\cos^2\tfrac{\theta}{2}}_{e_0^2}-1\Big) \Rightarrow v\theta = 2\frac{(e_1^2+e_2^2+e_3^2)}{e_0}}$$

$$\cos\theta = 2e_0^2-1 \Rightarrow v\theta = 1-\cos\theta = 2-2e_0^2$$

$$\sin\theta = \pm\sqrt{4e_0^2(e_1^2+e_2^2+e_3^2)} \Rightarrow \hat{k} = \begin{bmatrix} e_1/\sqrt{e_1^2+e_2^2+e_3^2} \\ e_2/\sqrt{e_1^2+e_2^2+e_3^2} \\ e_3/\sqrt{e_1^2+e_2^2+e_3^2} \end{bmatrix}$$

$$R_k(\theta) = \begin{bmatrix} \frac{e_1^2(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_0^2-1 & \frac{e_1e_2(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}-2e_3e_0 & \frac{e_1e_3(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_2e_0 \\ \frac{e_1e_2(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_3e_0 & \frac{e_2^2(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_0^2-1 & \frac{e_2e_3(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}-2e_1e_0 \\ \frac{e_1e_3(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}-2e_2e_0 & \frac{e_2e_3(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_1e_0 & \frac{e_3^2(2-2e_0^2)}{e_1^2+e_2^2+e_3^2}+2e_0^2-1 \end{bmatrix}$$