an Adaptive Controller for Manipulator

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

The dynamics of a two-link planar arm

$$M(q) = \begin{bmatrix} (m_1 + m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2\cos\theta_2 & m_2a_2^2 + m_2a_1a_2\cos\theta_2 \\ m_2a_2^2 + m_2a_1a_2\cos\theta_2 & m2a_2^2 \end{bmatrix}$$

$$V(q,\dot{q}) = \begin{bmatrix} -m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\dot{\theta}_2 \\ m_2a_1a_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} (m_1 + m_2)ga_1\cos\theta_1 + m_2ga_2\cos(\theta_1 + \theta_2) \\ m_2ga_2\cos(\theta_1 + \theta_2) \end{bmatrix}$$

Linear Time-Invariant Case

THEOREM 1.4-6: *Given a linear time-invariant system*

$$\dot{x}(t) = Ax(t),$$

the system is stable if and only if there exists a positive-definite solution P to the equation

$$A^T P + PA = -Q,$$

where Q is an arbitrary positive-definite matrix. ■

$$V = x^T P x$$

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = -\dot{x}^T Q x$$

- **Choose a positive definite matrix Q**
- **Solve for P from the Lyapunov equation** $A^T P + PA = -Q$
- **Check whether P is positive definite**

$$\tau = M(\theta)\ddot{\theta} + V_m(\theta,\dot{\theta})\dot{\theta} + G(\theta)$$

Coriolis/Centripetal Structural Matrices

$$V_m = \tfrac{1}{2}(\dot{M} + U^T - U) = \begin{bmatrix} -\dot{\theta}_2 m_2 a_1 a_2 \sin\theta_2 & -(\dot{\theta}_1 + \dot{\theta}_2) m_2 a_1 a_2 \sin\theta_2 \\ \dot{\theta}_1 m_2 a_1 a_2 \sin\theta_2 & 0 \end{bmatrix}$$

The Skew Symmetry Property

The skew-symmetric matrix $S(q,\dot{q})$:

$$S = \dot{M} - 2V_m(\theta,\dot{\theta})$$

Linearized Dynamic Model: $\tau = W(q,\dot{q},\ddot{q})\varphi$

The dynamics, including friction, can be written as

$$\tau_1 = [(m_1 + m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2\cos\theta_2]\ddot{\theta}_1 + [m_2a_2^2 + m_2a_1a_2\cos\theta_2]\ddot{\theta}_2 - m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 + (m_1 + m_2)ga_1\cos\theta_1 + m_2ga_2\cos(\theta_1 + \theta_2) + v_1\dot{\theta}_1 + k_1\,\mathrm{sgn}(\dot{\theta}_1)$$

$$\tau_2 = [m_2a_2^2 + m_2a_1a_2\cos\theta_2]\ddot{\theta}_1 + m_2a_2^2\ddot{\theta}_2 + m_2a_1a_2\dot{\theta}_1^2\sin\theta_2 + m_2ga_2\cos(\theta_1 + \theta_2) + v_2\dot{\theta}_2 + k_2\,\mathrm{sgn}(\dot{\theta}_2).$$

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & 0 & 0 \\ 0 & w_{22} & 0 & 0 & w_{25} & w_{26} \end{bmatrix}$$

$$\varphi = [m_1 \quad m_2 \quad k_1 \quad v_1 \quad k_2 \quad v_2]^T.$$

$$w_{11} = a_1^2\ddot{\theta}_1 + ga_1\cos\theta_1$$

$$w_{12} = [a_1^2 + a_2^2 + 2a_1a_2\cos\theta_2]\ddot{\theta}_1 + [a_2^2 + a_1a_2\cos\theta_2]\ddot{\theta}_2 - a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 + ga_1\cos\theta_1 + ga_2\cos(\theta_1 + \theta_2)$$

$$w_{13} = \mathrm{sgn}(\dot{\theta}_1)$$

$$w_{14} = \dot{\theta}_1$$

$$w_{22} = [a_2^2 + a_1a_2\cos\theta_2]\ddot{\theta}_1 + a_2^2\ddot{\theta}_2 + a_1a_2\dot{\theta}_1^2\sin\theta_2 + ga_2\cos(\theta_1 + \theta_2)$$

$$w_{25} = \mathrm{sgn}(\dot{\theta}_2)$$

$$w_{26} = \dot{\theta}_2.$$

an Adaptive Computed Torque Controller

From (1) we can rewrite :

$$\tau = W(\theta, \dot{\theta}, \ddot{\theta})\varphi \qquad (2)$$

Where φ is unknown parameter vector

The approximate computed-torque controller would have

$$\tau = \hat{M}(\theta)\left(\ddot{\theta}_d + K_v \dot{e} + K_p e\right) + \hat{V}\left(\theta, \dot{\theta}\right) + \hat{G} \qquad (3)$$

From (1) (3)we can rewrite :

$$\ddot{e} + K_v \dot{e} + K_p e = \hat{M}^{-1}[(M - \hat{M})\ddot{\theta} + (V - \hat{V}) + (G - \hat{G})] \qquad (4)$$

From (2) (4)we can rewrite :

$$\ddot{e} + K_v \dot{e} + K_p e = \hat{M}^{-1}(W\varphi - W\hat{\varphi}) = \hat{M}^{-1}W(\varphi - \hat{\varphi}) \qquad (5)$$

Where $\tilde{\varphi}$ is parameter error $\tilde{\varphi} = \varphi - \hat{\varphi}$

Rewrite (5) in the state-space from :

$$\dot{E} = AE + B\hat{M}^{-1}W\tilde{\varphi} = AE + B\xi \qquad (6)$$

Where:

$$E = \begin{bmatrix} e \\ \dot{e} \end{bmatrix};$$

$$A = \begin{bmatrix} 0_n & I_n \\ -K_p & -K_v \end{bmatrix};$$

$$B = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}; \qquad \xi = \hat{M}^{-1}W\tilde{\varphi}$$

I_n is the identity matrix, O_n is zero matrix

an Adaptive Computed Torque Controller

use Lyapunov stability analysis to show that the tracking error vector E is asymptotically stable with the right choice of adaptive update law.
The first we select a Lyapunov function:

$$V = E^T P E + \tilde{\varphi}^T \Gamma^{-1} \tilde{\varphi} \qquad \textbf{(7)}$$

Where:
P is 2nx2n positive definite, constant, symmetric matrix and Γ is a diagonal, positive matrix

$$\begin{aligned}
\dot{V} &= E^T P \dot{E} + \dot{E}^T P E + 2\tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}} \\
&= E^T P (AE + B\xi) + (E^T A^T + \xi^T B^T) P E + 2\tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}} \qquad \textbf{(8)} \\
&= E^T (PA + A^T P) E + E^T P B \xi + \xi^T B^T P E + 2\tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}} \\
&= E^T (PA + A^T P) E + 2\xi^T B^T P E + 2\tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}} \\
&= E^T (PA + A^T P) E + 2\tilde{\varphi}^T W^T \hat{M}^{-1} B^T P E + 2\tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}}
\end{aligned}$$

$$\left[\tilde{\varphi}^T \Gamma^{-1} \tilde{\varphi}\right]^T = \tilde{\varphi}^T \Gamma^{-1} \tilde{\varphi} \qquad E^T P B \xi = \xi^T B^T P E = \tilde{\varphi}^T W^T \hat{M}^{-T} B^T P E \qquad \xi = \hat{M}^{-1} W \tilde{\varphi}$$

an Adaptive Computed Torque Controller

From (8) and (9) we have:

$$\dot{V} = -E^T Q E + 2\tilde{\varphi}^T \underbrace{(\Gamma^{-1}\dot{\tilde{\varphi}} + W^T \hat{M}^{-1} B^T P E)}_{\text{Part I}} \qquad \textbf{(10)}$$

Where Q is positive definite, symmetric matrix that satisfies the Lyapunov equatio Measurement of $\ddot{q}$ is required.

$$A^T P + P A = -Q$$

The time derivative of V is at least negative semidefinite when part I in (10) equal 0.

$$\dot{\tilde{\varphi}} = \dot{\varphi} - \dot{\hat{\varphi}} = -\Gamma W^T \hat{M}^{-1} B^T P E \qquad \textbf{(11)}$$

$$\dot{\varphi} = 0, \qquad \dot{\hat{\varphi}} = \Gamma W^T \hat{M}^{-1} B^T P E$$

This is the adaptive update rule

TABLE 5.2-1 Adaptive Computed-Torque Controller

Torque Controller:
$$\tau = \hat{M}(q)(\ddot{q}_d + K_v\dot{e} + K_p e) + \hat{V}_m(q,\dot{q})\dot{q} + \hat{G}(q) + \hat{F}(\dot{q})$$
Update Rule:
$$\dot{\hat{\phi}} = \Gamma W^T(q,\dot{q},\ddot{q})\hat{M}^{-1}(q)B^T P\mathbf{e}$$
where
$$\mathbf{e} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \quad B = \begin{bmatrix} O_n \\ I_n \end{bmatrix}, \quad A = \begin{bmatrix} O_n & I_n \\ -K_p & -K_v \end{bmatrix}$$
$$W(q,\dot{q},\ddot{q})\hat{\phi} = \hat{M}(q)\ddot{q} + \hat{V}_m(q,\dot{q})\dot{q} + \hat{G}(q) + \hat{F}(\dot{q})$$
$$A^T P + PA = -Q$$
for some positive-definite, symmetric matrices P and Q.
Stability:
Tracking error vector $\mathbf{e}$ is asymptotically stable.
Restrictions:
Parameter resetting method is required. Measurement of $\ddot{q}$ is required.

Serious disadvantage: Measurement of $\ddot{q}$ is required.

an Adaptive Computed Torque Controller

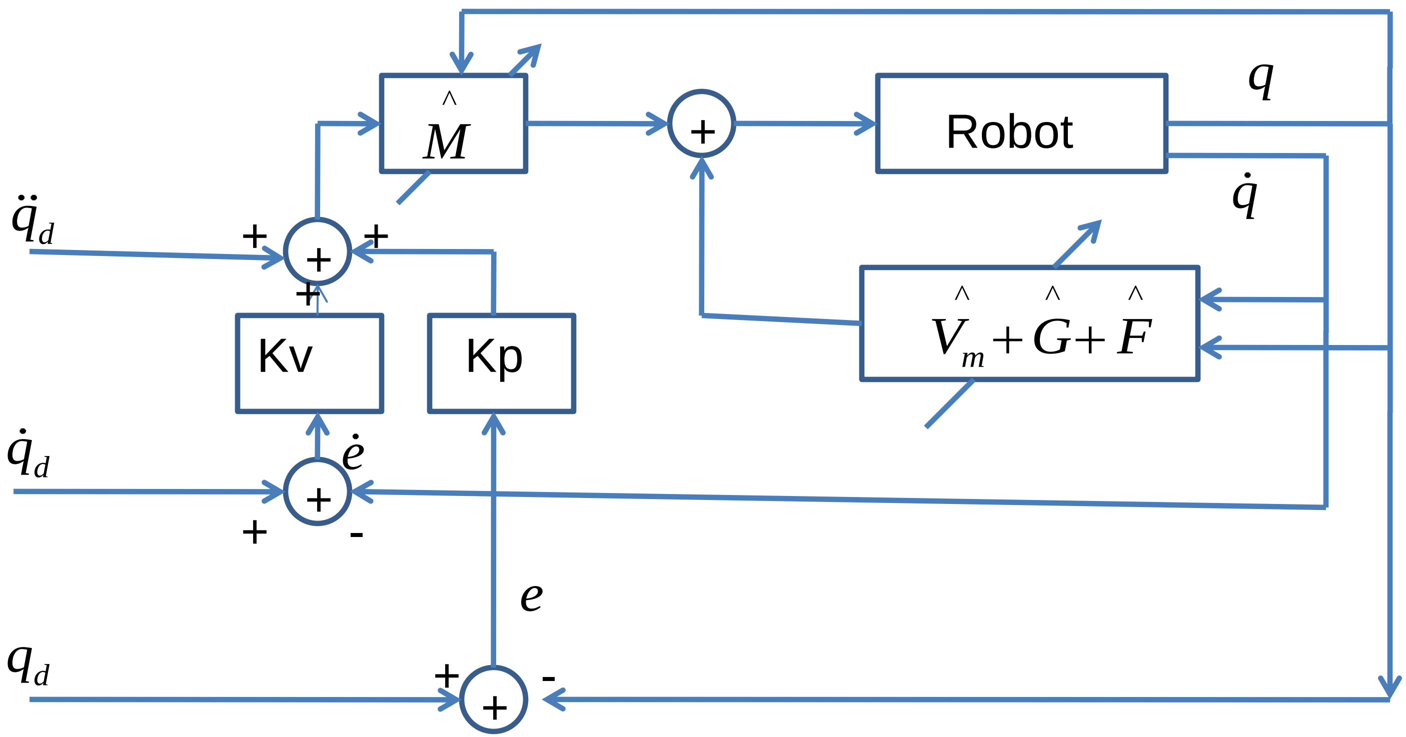

Fig 2.*The block diagram of the adaptive computed-torque controller*

an Adaptive Computed Torque Controller

$$\ddot{e} + K_v\dot{e} + K_p e = \hat{M}^{-1}(q)W(q,\dot{q},\ddot{q})\tilde{\varphi}$$

$$\begin{aligned}\tau_1 = {} & (2\hat{m}_2 l_1 l_2 c_2 + \hat{m}_2 l_2^2 + (\hat{m}_1 + \hat{m}_2) l_1^2)(\ddot{q}_{d1} + k_{v1}\dot{e}_1 + k_{p1}e_1) \\ & + (\hat{m}_2 l_1 l_2 c_2 + \hat{m}_2 l_2^2)(\ddot{q}_{d2} + k_{v2}\dot{e}_2 + k_{p2}e_2) - \hat{m}_2 l_1 l_2 s_2 \dot{q}_2^2 \\ & - 2\hat{m}_2 l_1 l_2 s_2 \dot{q}_1 \dot{q}_2 + \hat{m}_2 l_2 g c_{12} + (\hat{m}_1 + \hat{m}_2) l_1 g c_1\end{aligned}$$

$$W(q,\dot{q},\ddot{q}) = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

$$\begin{aligned}\tau_2 = {} & (\hat{m}_2 l_2^2 + \hat{m}_2 l_1 l_2 c_2)(\ddot{q}_{d1} + k_{v1}\dot{e}_1 + k_{p1}e_1) + \hat{m}_2 l_2 g c_{12} \\ & + \hat{m}_2 l_2^2(\ddot{q}_{d2} + k_{v2}\dot{e}_2 + k_{p2}e_2) + \hat{m}_2 l_1 l_2 s_2 \dot{q}_1^2,\end{aligned}$$

$$\begin{aligned}W_{11} &= l_1^2\ddot{q}_1 + l_1 g c_1, \\ W_{12} &= l_2^2(\ddot{q}_1 + \ddot{q}_2) + l_1 l_2 c_2(2\ddot{q}_1 + \ddot{q}_2) + l_1^2\ddot{q}_1 - l_1 l_2 s_2 \dot{q}_2^2 - 2 l_1 l_2 s_2 \dot{q}_1 \dot{q}_2 + l_2 g c_{12} + l_1 g c_1, \\ W_{21} &= 0, \\ W_{22} &= l_1 l_2 c_2\, \ddot{q}_1 + l_1 l_2 s_2 \dot{q}_1^2 + l_2 g c_{12} + l_2^2(\ddot{q}_1 + \ddot{q}_2).\end{aligned}$$

Simulation and Result

The real parameter of robot

No	Length(m)	Length's mass	Mass(kg)
1	0.5	0.25	0.88
2	0.4	0.2	0.72

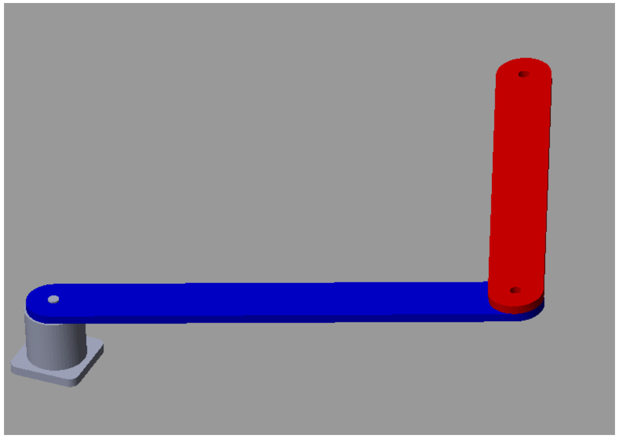

Fig 2. *Robot 2 DOF manipulator*

The assume parameter of robot

No	Length(m)	Length's mass	Mass(kg)
1	0.5	0.25	0.1
2	0.4	0.2	0.1

The desired trajectory

$$\begin{cases} x = 0.3 + 0.1 * \cos(\pi / 4) \\ y = 0.45 + 0.1 * \sin(\pi / 4) \end{cases}$$

Parameter's controller:

$$K_v = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}; K_p = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix};$$

$$\Gamma = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T;$$

$$P = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

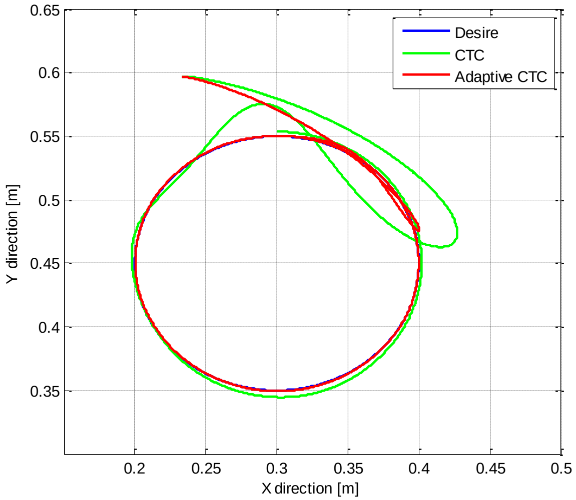

Fig 2. *The trajectory planning*

Result:

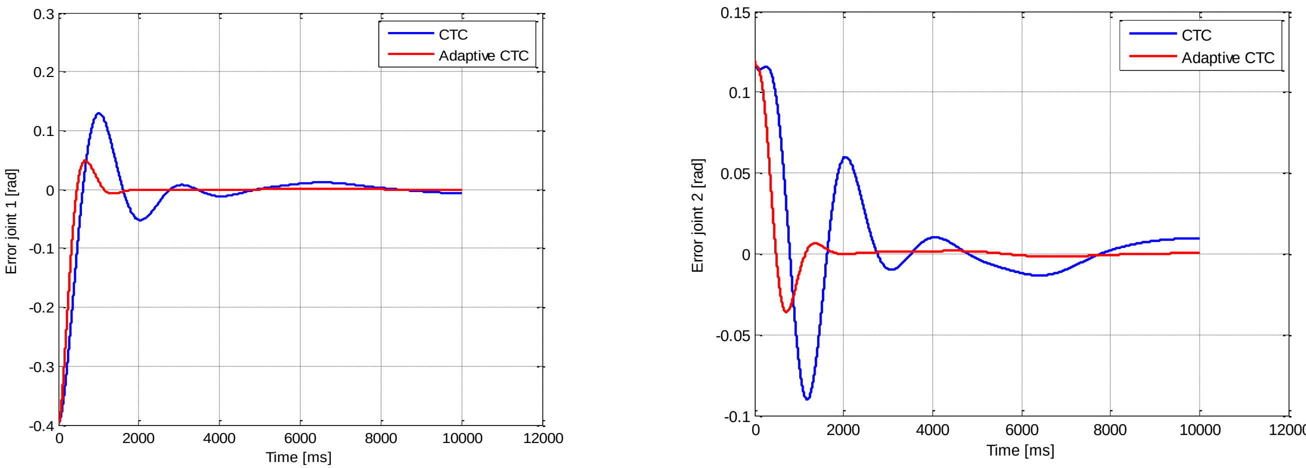

Fig 2. *The error of joints*

$$M(\theta)\ddot{\theta} + V_m(\theta,\dot{\theta})\dot{\theta} + \mathrm{G}(\theta) = \tau = \hat{M}(\theta)\ddot{\theta}_r + \hat{V}_m(\theta,\dot{\theta})\dot{\theta}_r + \hat{\mathrm{G}}(\theta) + K_D r$$

$$r = \dot{e} + \Lambda e$$ **Filtered Tracking Error**

$$\dot{\theta}_r = \dot{\theta}_d + \Lambda e \qquad \dot{\theta}_r - \dot{\theta} = \dot{\theta}_d - \dot{\theta} + \Lambda e = \dot{e} + \Lambda e = r$$

$$\ddot{\theta}_r = \ddot{\theta}_d + \Lambda \ddot{e} \qquad \ddot{\theta}_r - \ddot{\theta} = \ddot{\theta}_d - \ddot{\theta} + \Lambda \dot{e} = \ddot{e} + \Lambda \dot{e} = \dot{r}$$

$$\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Slotine selected the inertia-related Lyapunov-like function

$$V = \tfrac{1}{2} r^T M(q) r + \tfrac{1}{2} \tilde{\varphi}^T \Gamma^{-1} \tilde{\varphi}$$

$$\dot{V} = r^T M(q)\dot{r} + \tfrac{1}{2} r^T \dot{M}(q) r + \tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}}.$$

$$M(\theta)\ddot{\theta}+V_m(\theta,\dot{\theta})\dot{\theta}+\mathrm{G}(\theta)=\tau=\hat{M}(\theta)\ddot{\theta}_r+\hat{V}_m(\theta,\dot{\theta})\dot{\theta}_r+\hat{\mathrm{G}}(\theta)+K_D r$$

$$M\ddot{\theta}+V_m\dot{\theta}+\mathrm{G}=M\ddot{\theta}_r+V_m\dot{\theta}_r+\mathrm{G}(\theta)-\{M\ddot{\theta}_r+V_m\dot{\theta}_r+\mathrm{G}(\theta)$$
$$-(\hat{M}\ddot{\theta}_r+\hat{V}_m\dot{\theta}_r+\hat{\mathrm{G}})\}+K_D r$$

$$M(\ddot{\theta}_r-\ddot{\theta})+V_m(\dot{\theta}_r-\dot{\theta})+K_D r=(M-\hat{M})\ddot{\theta}_r+(V_m-\hat{V}_m)\dot{\theta}_r+(G-\hat{\mathrm{G}})$$

$$M\dot{r}=-V_m r-K_D r+(M-\hat{M})\ddot{\theta}_r+(V_m-\hat{V}_m)\dot{\theta}_r+(G-\hat{\mathrm{G}})$$

$$M\dot{r}=-V_m r-K_D r+Y(.)\varphi-Y(.)\hat{\varphi}$$

$$M\dot{r}=-V_m r-K_D r+Y(.)\tilde{\varphi}$$

$$Y(.)\varphi=Y(\theta,\dot{\theta},r,\dot{r})\varphi=M(\theta)\ddot{\theta}_r+V_m(\theta,\dot{\theta})\dot{\theta}_r+\mathrm{G}(\theta)$$

$$\dot{V} = r^T M(q)\dot{r} + \tfrac{1}{2} r^T \dot{M}(q) r + \tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}}.$$

$$= -r^T K_D r + r^T \left(\frac{1}{2}\dot{M} - V_m\right) r + r^T Y(.)\tilde{\varphi} + \tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}}$$

$$S = (\dot{M} - 2V_m) \quad skew\ symmetric\ matrix$$

$$= -r^T K_D r + \tilde{\varphi}^T Y^T r + \tilde{\varphi}^T \Gamma^{-1} \dot{\tilde{\varphi}}$$

$$= -r^T K_D r + \tilde{\varphi}^T (\Gamma^{-1} \dot{\tilde{\varphi}} + Y^T r) \leq 0 \qquad when \quad \Gamma^{-1} \dot{\tilde{\varphi}} + Y^T r = 0$$

the adaptive update rule is

$$\dot{\tilde{\varphi}} = \dot{\varphi} - \dot{\hat{\varphi}} = -\Gamma Y^T r$$

$$\dot{\hat{\varphi}} = \Gamma Y^T r$$

TABLE 5.3-1 Adaptive Inertia-Related Controller

Torque Controller:

$$\tau = Y(\cdot)\hat{\varphi} + K_v\dot{e} + K_v\Lambda e$$

Update Rule:

$$\dot{\hat{\varphi}} = \Gamma Y^T(\cdot)(\Lambda e + \dot{e})$$

where

$$Y(\cdot)\hat{\varphi} = \hat{M}(q)(\ddot{q}_d + \Lambda\dot{e}) + \hat{V}_m(q,\dot{q})(\dot{q}_d + \Lambda e) + \hat{G}(q) + \hat{F}(\dot{q})$$

Stability:

Tracking error e and $\dot{e}$ are asymptotically stable. Parameter estimate $\hat{\varphi}$ is bounded.

an Adaptive Inertia Related Controller

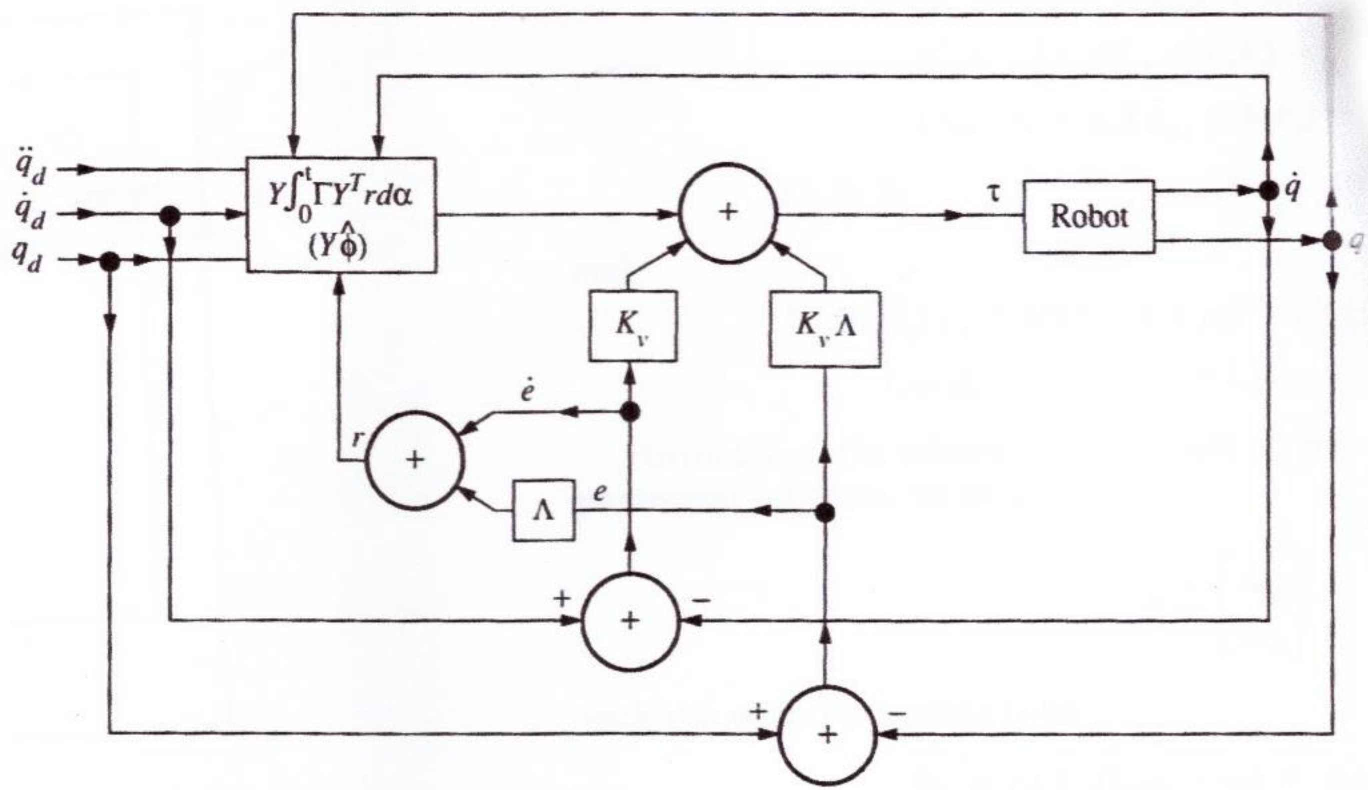

FIGURE 5.3-1 Block diagram of the adaptive inertia-related controller.

For a 2 link manipulator, the adaptive inertia related control law is

$$\tau_1 = Y_{11}\hat{m}_1 + Y_{12}\hat{m}_2 + k_{v1}\dot{e}_1 + k_{v1}\lambda_1 e_1$$

$$\tau_2 = Y_{21}\hat{m}_1 + Y_{22}\hat{m}_2 + k_{v2}\dot{e}_2 + k_{v2}\lambda_2 e_2.$$

$$\dot{\hat{\varphi}} = \Gamma Y^T r$$

$$Y(\ddot{q}_d, \dot{q}_d, q_d, q, \dot{q}) = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

$$\dot{\hat{m}}_1 = \gamma_1[Y_{11}(\lambda_1 e_1 + \dot{e}_1) + Y_{21}(\lambda_2 e_2 + \dot{e}_2)]$$

$$\dot{\hat{m}}_2 = \gamma_2[Y_{12}(\lambda_1 e_1 + \dot{e}_1) + Y_{22}(\lambda_2 e_2 + \dot{e}_2)].$$

$$Y_{11} = l_1^2(\ddot{q}_{d1} + \lambda_1\dot{e}_1) + l_1 g c_1,$$

$$\begin{aligned} Y_{12} = {} & (l_2^2 + 2l_1 l_2 c_2 + l_1^2)(\ddot{q}_{d1} + \lambda_1\dot{e}_1) \\ & + (l_2^2 + l_1 l_2 c_2)(\ddot{q}_{d2} + \lambda_2\dot{e}_2) - l_1 l_2 s_2 \dot{q}_2(\dot{q}_{d1} + \lambda_1 e_1) \\ & - l_1 l_2 s_2(\dot{q}_1 + \dot{q}_2)(\dot{q}_{d2} + \lambda_2 e_2) + l_2 g c_{12} + l_1 g c_1, \end{aligned}$$

$$Y_{21} = 0,$$

$$\begin{aligned} Y_{22} = {} & (l_1 l_2 c_2 + l_2^2)(\ddot{q}_{d1} + \lambda_1\dot{e}_1) + l_2^2(\ddot{q}_{d2} + \lambda_2\dot{e}_2) \\ & l_1 l_2 s_2 \dot{q}_1(\dot{q}_{d1} + \lambda_1 e_1) + l_2 g c_{12}. \end{aligned}$$