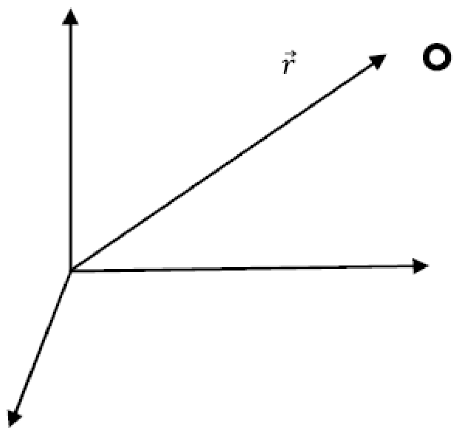
Ch. 5 Velocities and Static Forces: Jacobian

The motion of a particle

No Consideration of rotation

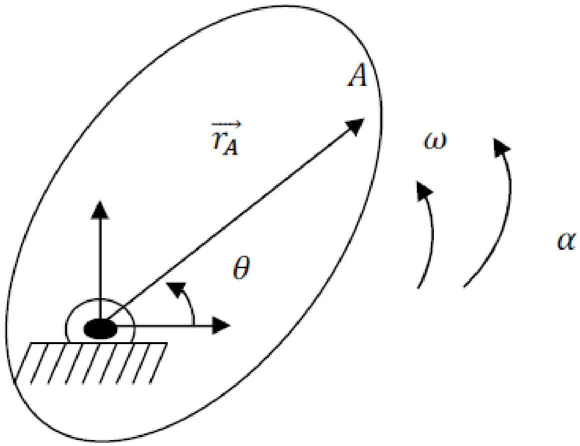


$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$: Position

$\dot{r} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$: Velocity

$\ddot{r} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$: Acceleration

The angular motion of rigid body about fixed pt.



i)

$$\vec{r_a} = rcos\theta\hat{\imath} + rsin\theta\hat{\jmath}$$

$$\dot{\vec{r_a}} = -rsin\theta\dot{\theta}\hat{\imath} + rcos\theta\dot{\theta}\hat{\jmath}$$

$$= r\dot{\theta}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})$$

$$\ddot{\vec{r_a}} = r\ddot{\theta}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath}) + r\dot{\theta}^2(-cos\theta\hat{\imath} - sin\theta\hat{\jmath})$$

Where $\dot{\theta} = \omega$ and $\ddot{\theta} = \alpha$

ii) Vector notation

$$\vec{r_A} = \vec{r_A},\ \vec{\omega},\ \vec{\alpha}$$

$$\dot{\vec{r_A}} = \vec{\omega} \times \vec{r_A}$$

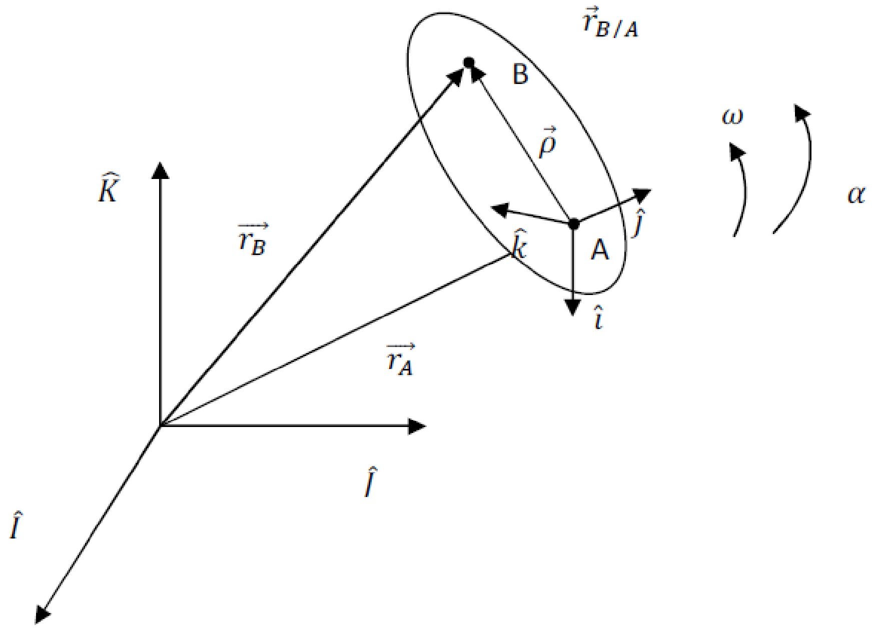
$$= \dot{\theta}\hat{k} \times (rcos\theta\hat{\imath} + rsin\theta\hat{\jmath})$$

$$= r\dot{\theta}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})$$

$$\ddot{\vec{r_A}} = \vec{\alpha} \times \vec{r_A} + \vec{\omega} \times (\vec{\omega} \times \vec{r_A})$$

Where $\vec{\omega} = \dot{\theta}\hat{k}$ and $\vec{\alpha} = \ddot{\theta}\hat{k}$

The motion of a Rigid body



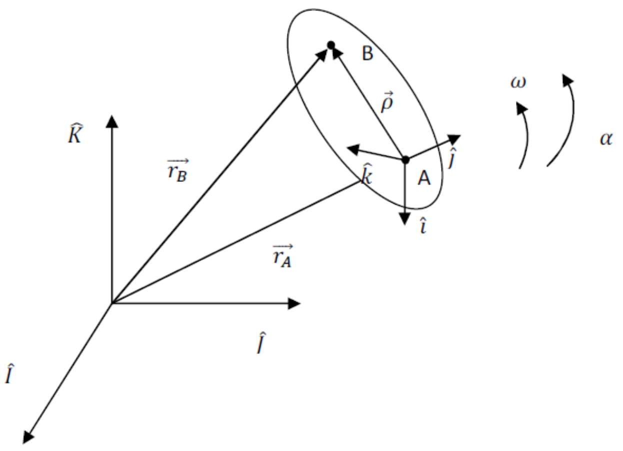
$$\overrightarrow{r_B} = \overrightarrow{r_A} + \vec{r}_{B/A} = \overrightarrow{r_A} + \vec{\rho}$$

$$\dot{\vec{r}}_B = \overrightarrow{v_B} = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \overrightarrow{v_A} + \vec{v}_{B/A}$$

$$\overrightarrow{v_B} = \overrightarrow{v_A} + \vec{\omega} \times \vec{\rho}$$

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A + \ddot{\vec{\rho}}$$

Rigid body motion with rotating axes



$$\overrightarrow{\boldsymbol{r_B}} = \overrightarrow{\boldsymbol{r_A}} + \vec{\boldsymbol{\rho}}|_{xyz}$$

$$= r_x\hat{I} + r_y\hat{J} + r_z\hat{K} + \rho_x\hat{\imath} + \rho_y\hat{\jmath} + \rho_z\hat{k}$$

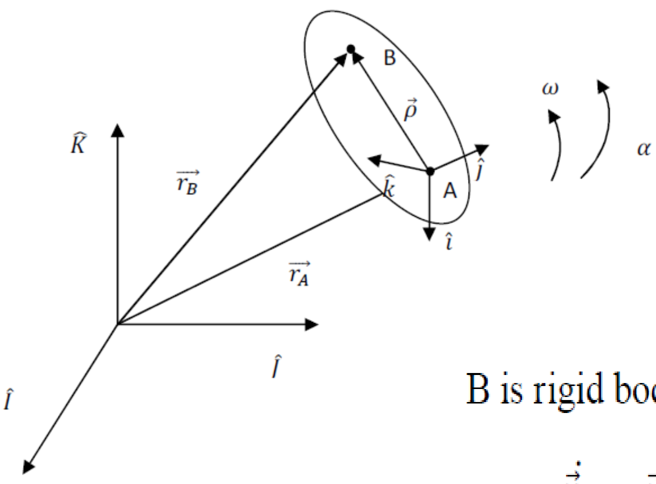
$$\dot{\vec{\boldsymbol{r}}}_{\boldsymbol{B}} = \dot{\vec{\boldsymbol{r}}}_{\boldsymbol{A}} + \dot{\vec{\boldsymbol{\rho}}}$$

$$= \dot{r}_x\hat{I} + \dot{r}_y\hat{J} + \dot{r}_z\hat{K} + \dot{\rho}_x\hat{\imath} + \dot{\rho}_y\hat{\jmath} + \dot{\rho}_z\hat{k} + \rho_x\dot{\hat{\imath}} + \rho_y\dot{\hat{\jmath}} + \rho_z\dot{\hat{k}}$$

$$= \dot{\vec{r}}_A + \dot{\vec{\rho}}_{rel} + \vec{\omega} \times \vec{\rho}$$

$$with\ \ \dot{\hat{\imath}} = \omega \times \hat{\imath}, \qquad \dot{\hat{\jmath}} = \omega \times \hat{\jmath}, \qquad \dot{\hat{k}} = \omega \times \hat{k}$$

Rigid body motion with rotating axes



B is rigid body $\dot{\vec{\rho}}_{rel} = 0$

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{\rho}}_{rel} + \vec{\omega} \times \vec{\rho}$$

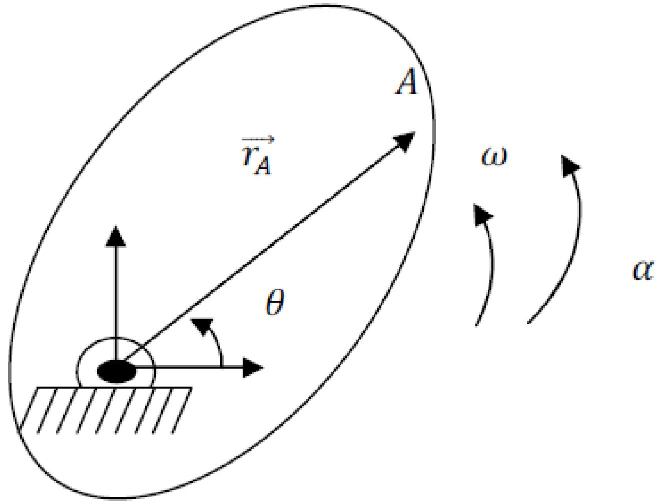
$$= \dot{r}_x\hat{I} + \dot{r}_y\hat{J} + \dot{r}_z\hat{K} + \dot{\rho}_x\hat{\imath} + \dot{\rho}_y\hat{\jmath} + \dot{\rho}_z\hat{k} + \vec{\omega} \times (\rho_x\hat{\imath} + \rho_y\hat{\jmath} + \rho_z\hat{k})$$

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \underbrace{\ddot{\rho}_x\hat{\imath} + \ddot{\rho}_y\hat{\jmath} + \ddot{\rho}_z\hat{k}}_{\ddot{\rho}|_{rel}} + \dot{\rho}_x\dot{\hat{\imath}} + \dot{\rho}_y\dot{\hat{\jmath}} + \dot{\rho}_z\dot{\hat{k}} + \underbrace{\vec{\alpha} \times \left(\rho_x\hat{\imath} + \rho_y\hat{\jmath} + \rho_z\hat{k}\right)}_{\vec{\alpha} \times \rho|_{rel}} +$$

$$+\underbrace{\vec{\omega} \times \left(\dot{\rho}_x\hat{\imath} + \dot{\rho}_y\hat{\jmath} + \dot{\rho}_z\hat{k}\right)}_{\vec{\omega} \times \dot{\rho}|_{rel}} + \vec{\omega} \times (\rho_x\dot{\hat{\imath}} + \rho_y\dot{\hat{\jmath}} + \rho_z\dot{\hat{k}})$$

$$= \ddot{\vec{r}}_A + \underbrace{\vec{\alpha} \times \vec{\rho}}_{\text{tangential}} \quad \underbrace{+ \vec{\omega} \times (\vec{\omega} \times \vec{\rho})}_{\text{centrifugal}} \quad + \ddot{\vec{\rho}}|_{rel} \quad \underbrace{+ 2\vec{\omega} \times \dot{\vec{\rho}}|_{rel}}_{\text{Coriolis}}$$

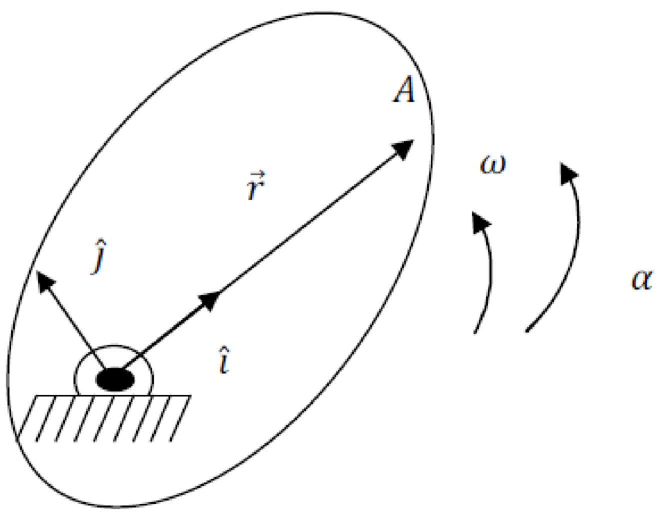
Ex



$$\vec{r_a} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath}$$

$$\vec{v}_a = \vec{\omega} \times \vec{r}_a$$

$$\vec{a}_A = \vec{\alpha} \times \vec{r}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r}_A)$$



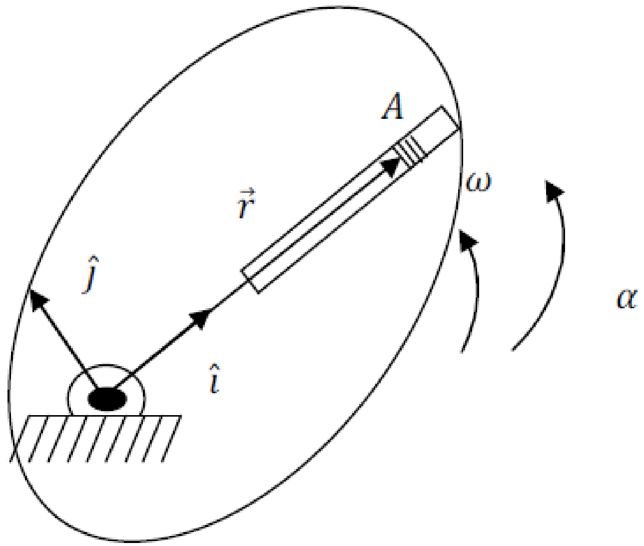
$$\vec{r} = \vec{\rho} = r\hat{\imath}$$

$$\vec{v}_A = \vec{\omega} \times \vec{\rho} + \dot{\vec{\rho}}|_{rel}$$

$$\dot{\vec{\rho}}|_{rel} = \dot{r}\hat{\imath} = 0$$

$$\vec{v}_A = \vec{\omega} \times \vec{\rho}$$

$$a_A = \vec{\alpha} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho}) + \underbrace{\ddot{\vec{\rho}}|_{rel} + 2\vec{\omega} \times \dot{\vec{\rho}}|_{rel}}_{=0}$$



$$\vec{r} = \vec{\rho} = r\hat{\imath}$$

$$\vec{v}_A = \vec{\omega} \times \vec{\rho} + \dot{\vec{\rho}}|_{rel}$$

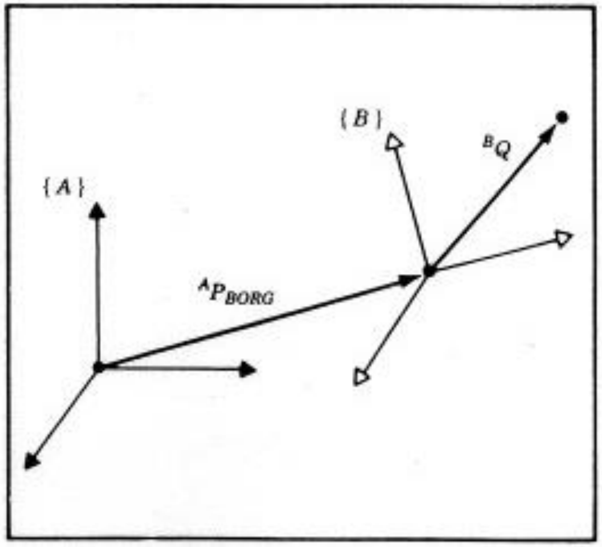
$$\dot{\vec{\rho}}|_{rel} = \dot{r}\hat{\imath} \neq 0$$

$$\vec{v}_A = \vec{\omega} \times \vec{\rho}$$

$$a_A = \vec{\alpha} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho}) + \ddot{\vec{\rho}}|_{rel} + 2\vec{\omega} \times \dot{\vec{\rho}}|_{rel}$$

$\omega > 0,\ \dot{r} > 0,\quad \vec{\omega} \times \dot{\vec{\rho}}|_{rel} \qquad \hat{\jmath}\ direction$

$\omega > 0,\ \dot{r} < 0,\quad \vec{\omega} \times \dot{\vec{\rho}}|_{rel} \qquad -\hat{\jmath}\ direction$



Notation for time-varying passion and orientation

^BQ: *Position expressed in* $\{B\}$

^BV_Q: *Velocity of Q expressed in* $\{B\}$

$^A(^BV_Q)$: *Velocity of Q expressed in* $\{A\}$

$$^A(^BV_Q) = {}^A_BR\,^BV_Q$$

$^B(^BV_Q)$: ^BV_Q

* Simultaneous linear and rotational velocity

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \vec{\omega}\times\vec{\rho} + \dot{\vec{\rho}}_{rel}$$

$$^AV_Q = {}^A\vec{V}_{BORG} + {}^A_BR\,^BV_Q + {}^A\Omega_B \times {}^A_BR\,^BQ$$

참조) $^A\vec{r}_Q = r_B + \vec{\rho} = r_B + {}^A_BR\,^BQ$

$$^A\dot{r}_Q = {}^A\dot{V}_Q = \dot{r}_B + \underline{\frac{d}{dt}({}^A_BR\,^BQ)}$$

$$= {}^A\vec{V}_{BORG} + {}^A_BR\,^BV_Q + \underline{{}^A\Omega_B \times {}^A_BR\,^BQ}$$

Acceleration of a rigid body

$^{B}\dot{V}_{Q}$: *Acceleration of Q expressed in {B}*

$^{B}\dot{\Omega}_{B}$: *angular Acceleration of {B} expressed in {A}*

^{B}Q fixed in {B}

$$^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q$$

$$\ddot{\vec{r}}_{Q} = \vec{\alpha} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})$$

$$^{A}\dot{V}_{Q} = {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q)$$

${}^{B}Q$ Changing $\omega.r.t$ {B}

$${}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q$$

$$\ddot{\vec{r}}_Q = \vec{\alpha} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho}) + \ddot{\vec{\rho}}|_{rel} + 2\vec{\omega} \times \dot{\vec{\rho}}|_{rel}$$

$$= {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q) + {}^{A}_{B}R\,{}^{B}\dot{V}_Q + 2\,{}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q$$

Simultaneous linear and rotational acceleration

$$\begin{cases} {}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BORG} + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q) + {}^{A}_{B}R\,{}^{B}\dot{V}_Q + \\ \qquad + 2\,{}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q \\ {}^{B}Q \text{ Fixed, } {}^{B}\dot{V}_Q = 0 = {}^{B}V_Q \end{cases}$$

- **More on angular velocity: Mathematical approach**

  - A property of the derivative of an orthonormal matrix

    We can derive an interesting relationship between the derivative of an orthonormal matrix and a certain skew symmetric matrix as follows. For any $n \times n$ orthonormal matrix, R, we have

$$RR^T = I_n \tag{5.14}$$

where I_n is the $n \times n$ identity matrix. Our interest, by the way, is for the case $n = 3$ and R a *proper* orthonormal matrix, or rotation matrix. Differentiating (5.14) yields

$$\dot{R}R^T + R\dot{R}^T = 0_n \tag{5.15}$$

where 0_n is the $n \times n$ zero matrix. Eq. (5.15) may also be written

$$\dot{R}R^T + (\dot{R}R^T)^T = 0_n. \tag{5.16}$$

Defining

$$S = \dot{R}R^T \tag{5.17}$$

So, we see that S is a skew-symmetric matrix. Hence, a property relating the derivative of orthonormal matrices with skew-symmetric matrices exists and may be stated as

$$S = \dot{R}R^{-1}. \tag{5.19}$$

Velocity of a point due to rotating reference frame

Consider a fixed vector ${}^{B}P$ unchanging with respect to frame $\{B\}$. It's description in another frame $\{A\}$ is given as

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P. \tag{5.20}$$

If frame $\{B\}$ is rotating (i.e., the derivative ${}^{A}_{B}\dot{R}$ is nonzero) then ${}^{A}P$ will be changing even though ${}^{B}P$ is constant; that is

$${}^{A}\dot{P} = {}^{A}_{B}\dot{R}\,{}^{B}P, \tag{5.21}$$

or, using our notation for velocity,

$${}^{A}V_{P} = {}^{A}_{B}\dot{R}\,{}^{B}P. \tag{5.22}$$

Now, rewrite (5.22) by substituting for ${}^{B}P$ to obtain

$$ {}^{A}V_{P} = {}^{A}_{B}\dot{R}\; {}^{A}_{B}R^{-1}\; {}^{A}P. \tag{5.23} $$

Making use of our result (5.19) for orthonormal matrices, we have

$$ {}^{A}V_{P} = {}^{A}_{B}S\; {}^{A}P, \tag{5.24} $$

where we have adorned S with sub- and superscripts to indicate that it is the skew-symmetric matrix associated with the particular rotation matrix ${}^{A}_{B}R$. Because of its appearance in (5.24) and for other reasons to be seen shortly, the skew-symmetric matrix we have introduced is called the **angular velocity matrix**.

Skew-symmetric matrices and the vector cross product

If we assign the elements in a skew-symmetric matrix, S, as

$$ S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \tag{5.25} $$

and define the 3×1 column vector

$$ \Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}, \tag{5.26} $$

then it is easily verified that

$$ SP = \Omega \times P, $$

Hence, our relation (5.24) may be written

$$ {}^{A}V_{P} = {}^{A}\Omega_{B} \times {}^{A}P, \tag{5.28} $$

Gaining physical insight concerning the angular velocity vector

$$\dot{R} = lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t}. \qquad (5.29)$$

$$R(t + \Delta t) = R_K(\Delta\theta) R(t), \qquad (5.30)$$

$$\dot{R} = lim_{\Delta t \to 0} \left(\frac{R_k(\Delta\theta) - I_3}{\Delta t} R(t) \right), \qquad (5.31)$$

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{R_k(\Delta\theta) - I_3}{\Delta t} \right) R(t), \qquad (5.32)$$

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_z k_y v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}. \quad (2.80)$$

Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^A\breve{K} = [k_x \quad k_y \quad k_z]^T$.

$$R_K(\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix}$$

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{\begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 0 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 0 \end{bmatrix}}{\Delta t} \right) R(t)$$

$$\dot{R} = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t).$$

$$\dot{R}R^{-1} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{K}. \quad (5.37)$$

instantaneos axis of rotation

Other representations of angular velocity

the angular velocity of a rotating body is available as rates of the set of Z-Y-Z Euler angles:

$$\dot{\Theta}_{Z'Y'Z'} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}. \qquad \dot{R}R^T = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

$$\Omega_x = \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23},$$

$$\Omega_y = \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33},$$

$$\Omega_z = \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13}.$$

$$\Rightarrow \quad \Omega = E_{Z'Y'Z'}(\Theta_{Z'Y'Z'})\dot{\Theta}_{Z'Y'Z'}.$$

$E(\cdot)$ is a Jacobian relating an angle set velocity vector and the angular velocity

Z-Y-Z Euler Angles

$$
{}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta cr - s\alpha sr & -c\alpha c\beta sr - s\alpha cr & c\alpha s\beta \\ s\alpha c\beta cr + c\alpha sr & -s\alpha c\beta sr + c\alpha cr & s\alpha s\beta \\ -s\beta cr & s\beta sr & c\beta \end{bmatrix}
$$

$$
R_{z-y-z} = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\overrightarrow{W} = \dot{\alpha} k_1 + \dot{\beta} j_2 + \dot{\gamma} k_3
$$

$$
\begin{bmatrix} {}^{0}W_x \\ {}^{0}W_y \\ {}^{0}W_z \end{bmatrix} = \begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \qquad \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = inv\left(\begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix} \right) \begin{bmatrix} {}^{0}W_x \\ {}^{0}W_y \\ {}^{0}W_z \end{bmatrix}
$$

$$
\begin{cases} \alpha_{i+1} = \alpha_i + \dot{\alpha}_i \square t \\ \beta_{i+1} = \beta_i + \dot{\beta}_i \square t \\ \gamma_{i+1} = \gamma_i + \dot{\gamma}_i \square t \end{cases}
$$

Transformation matrices:

$$ {}^{0}_{1}R = R(Z,\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^{1}_{0}R = R^{T}(Z,\alpha) $$

$$ {}^{1}_{2}R = R(Y,\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}, \qquad {}^{2}_{1}R = R^{T}(Y,\beta) $$

$$ {}^{2}_{B}R = R^{T}(Z,\gamma) = \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^{B}_{2}R = R^{T}(Z,\gamma) $$

$$ {}^{B}_{2}R = R^{T}(Z,\gamma) = \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}\omega = \dot{\alpha}\,{}^{B}\hat{k}_0 + \dot{\beta}\,{}^{B}\hat{j}_1 + \dot{\gamma}\,{}^{B}\hat{k}_2 = \begin{bmatrix} {}^{B}\hat{k}_0 & {}^{B}\hat{j}_1 & {}^{B}\hat{k}_2 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} $$

$$ {}^{B}\omega = \dot{\alpha}\,{}^{B}\hat{k}_1 + \dot{\beta}\,{}^{B}\hat{j}_2 + \dot{\gamma}\,{}^{B}k_B = \begin{bmatrix} {}^{B}\hat{k}_1 & {}^{B}\hat{j}_2 & {}^{B}k_c \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} $$

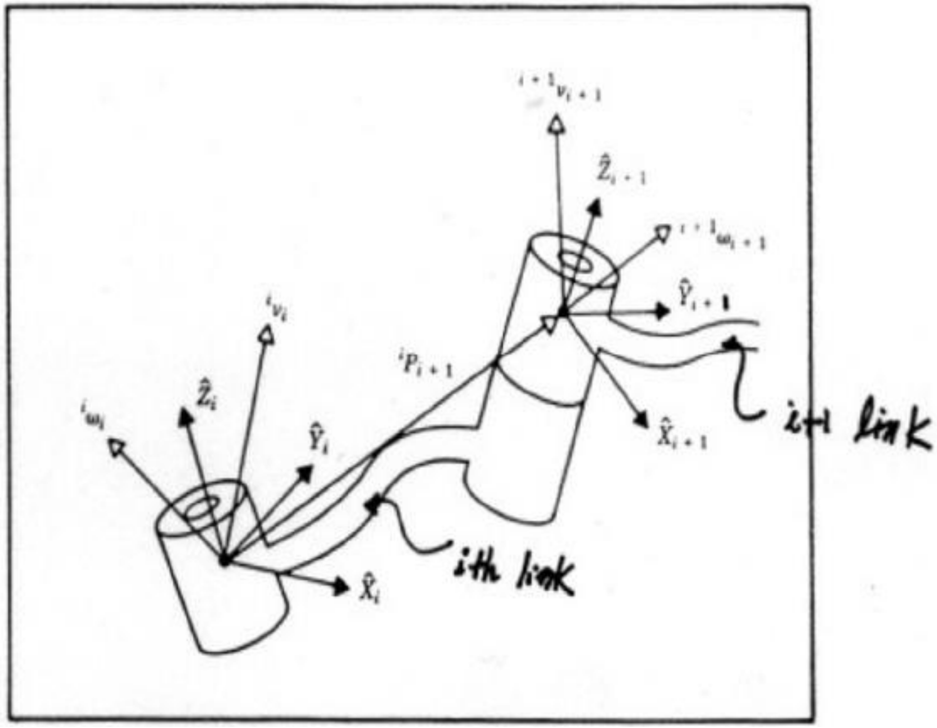
$$ {}^{B}_{1}R = \begin{bmatrix} c\beta c\gamma & s\gamma & -s\beta c\gamma \\ -c\beta s\gamma & c\gamma & s\beta s\gamma \\ s\beta & 0 & c\beta \end{bmatrix} $$

$$ {}^{B}_{0}R = {}^{0}_{B}R^{T} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\beta s\gamma \\ c\alpha s\beta & s\alpha s\beta & c\beta \end{bmatrix} $$

$$ {}^{B}E(\Theta) = \begin{bmatrix} -s\beta c\gamma & s\gamma & 0 \\ s\beta s\gamma & c\gamma & 0 \\ c\beta & 0 & 1 \end{bmatrix} $$

Motion of the links of a robot

Velocity "propagation" from link to link



For i+1: Revolute Joint

$$ {}^{i}w_{i+1} = {}^{i}w_i + {}_{i+1}^{\ \ i}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}w_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}w_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}p_{i+1} $$

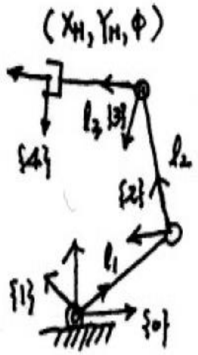
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}p_{i+1}\right) $$

For i+1: Prismatic Joint

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i, $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

ExD 3 link Manipulator



For revolute joint

$${}^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)$$

$${}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad {}^{1}v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^{1}\vec{\omega}_1 = {}^{2}_{1}R\,{}^{1}\omega_1 + \dot{\theta}_2\,{}^{2}\hat{Z}_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$${}^{2}\vec{v}_2 = {}^{2}_{1}R({}^{1}\vec{v}_1 + {}^{1}\omega_1 \times {}^{1}p_2) = \begin{bmatrix} c_2 & -s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 S_2 \dot{\theta}_1 \\ l_1 C_2 \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$ {}^{3}\vec{\omega}_3 = {}^{3}_{2}R\,{}^{2}\omega_2 + \dot{\theta}_3\,{}^{3}\hat{Z}_3 = \begin{bmatrix} c_3 & -s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} $$

$$ {}^{3}\vec{v}_3 = {}^{3}_{2}R({}^{2}\vec{v}_2 + {}^{2}\omega_2 \times {}^{2}p_3) = \begin{bmatrix} c_3 & -s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \left\{ \begin{matrix} 0 \\ l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{matrix} \right\} \right\} $$

$$ {}^{3}\vec{v}_4 = {}^{3}\vec{v}_3 + {}^{3}\omega_3 \times {}^{3}p_4 = \begin{bmatrix} l_1 s_{23}\dot{\theta}_1 + l_2 s_3(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_{23}\dot{\theta}_1 + l_2 c_3(\dot{\theta}_1 + \dot{\theta}_2) + l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \\ 0 \end{bmatrix} $$

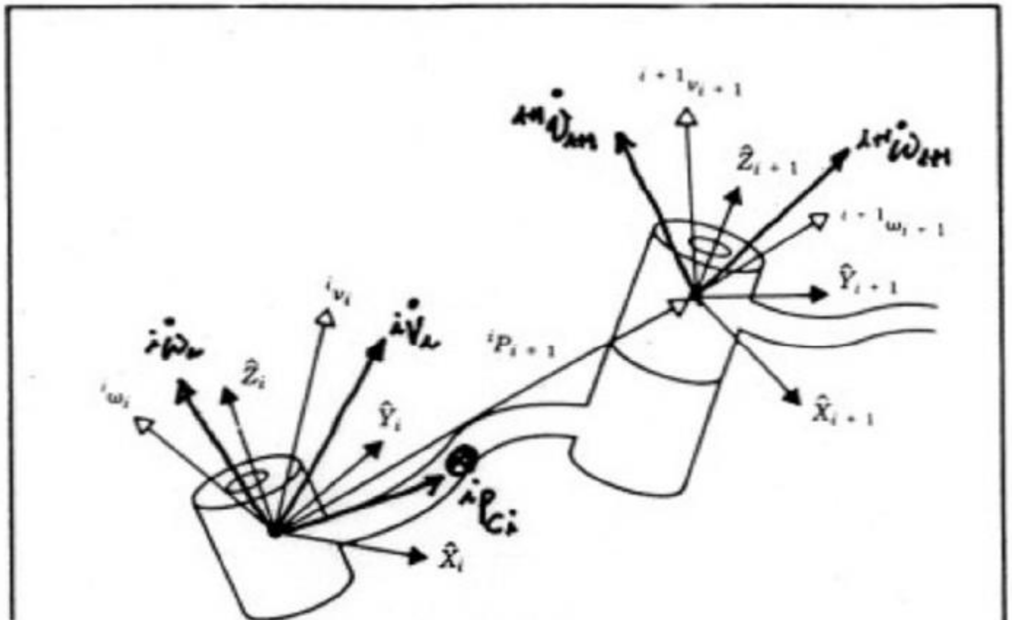
$$ {}^{3}\dot{\mathrm{X}}_{\mathrm{H}} = l_1 s_{23}\dot{\theta}_1 + l_2 s_3(\dot{\theta}_1 + \dot{\theta}_2) $$

$$ {}^{3}\dot{\mathrm{Y}}_{\mathrm{H}} = l_1 c_{23}\dot{\theta}_1 + l_2 c_3(\dot{\theta}_1 + \dot{\theta}_2) + l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) $$

$$ {}^{3}\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 $$

$$ \begin{bmatrix} {}^{0}\dot{\mathrm{X}}_{\mathrm{H}} \\ {}^{0}\dot{\mathrm{Y}}_{\mathrm{H}} \\ {}^{0}\dot{\phi} \end{bmatrix} = {}^{0}_{3}\mathrm{R} \begin{bmatrix} {}^{3}\dot{\mathrm{X}}_{\mathrm{H}} \\ {}^{3}\dot{\mathrm{Y}}_{\mathrm{H}} \\ {}^{3}\dot{\phi} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{123} & -\mathrm{s}_{123} & 0 \\ \mathrm{s}_{123} & \mathrm{c}_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{3}\dot{\mathrm{X}}_{\mathrm{H}} \\ {}^{3}\dot{\mathrm{Y}}_{\mathrm{H}} \\ {}^{3}\dot{\phi} \end{bmatrix} $$

Acceleration “propagation” from link to link



For i+1: Revolute Joint

$$ {}^iw_{i+1} = {}_{i+1}^{\;\;i}R\,{}^iw_i + \dot{\theta}_{i+1}\,\hat{Z}_{i+1} $$

$$ {}^i\dot{w}_{i+1} = {}^i\dot{w}_i + {}^iw_i \times {}_{i+1}^{\;\;i}R\dot{\theta}_{i+1}\,\hat{Z}_{i+1} + {}_{i+1}^{\;\;i}R\ddot{\theta}_{i+1}\,\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{w}_{i+1} = {}^{i+1}_{\;\;\;i}R\,{}^i\dot{w}_i + {}^{i+1}_{\;\;\;i}R\,{}^iw_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{\;\;\;i}R\left[{}^i\dot{w}_i \times {}^ip_{i+1} + {}^iw_i \times \left({}^iw_i \times {}^ip_{i+1}\right) + {}^i\dot{v}_i\right] $$

For i+1: Prismatic Joint

$$ {}^{i+1}\dot{w}_{i+1} = {}^{i+1}_{\;\;\;i}R\,{}^i\dot{w}_i $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{\;\;\;i}R\left({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{i+1}\right) + {}^i\dot{v}_i\right) + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. $$

$$ {}^i\dot{v}_{C_i} = {}^i\dot{\omega}_i \times {}^iP_{C_i} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{C_i}\right) + {}^i\dot{v}_i $$

Jacobians

$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6),$$
$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6),$$
$$\vdots$$
$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6).$$

$$Y = F(X).$$

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \quad \cdots \quad + \frac{\partial f_1}{\partial x_6}\delta x_6,$$
$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \quad \cdots \quad + \frac{\partial f_2}{\partial x_6}\delta x_6,$$
$$\vdots$$
$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \quad \cdots \quad + \frac{\partial f_6}{\partial x_6}\delta x_6,$$

$$\begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}$$

$$\mathrm{dy} = \frac{\partial f}{\partial x} dx$$

$$\dot{y} = J\dot{x}$$

Where J is Jacobian.

In field of Robotics

$$\dot{V}_H = \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = J(\theta)\dot{\theta}$$

$${}^{0}\dot{V}_H = {}^{0}J\underline{\dot{\theta}}$$

$${}^{3}\dot{V}_H = {}^{3}J\underline{\dot{\theta}}$$

Back to our example,

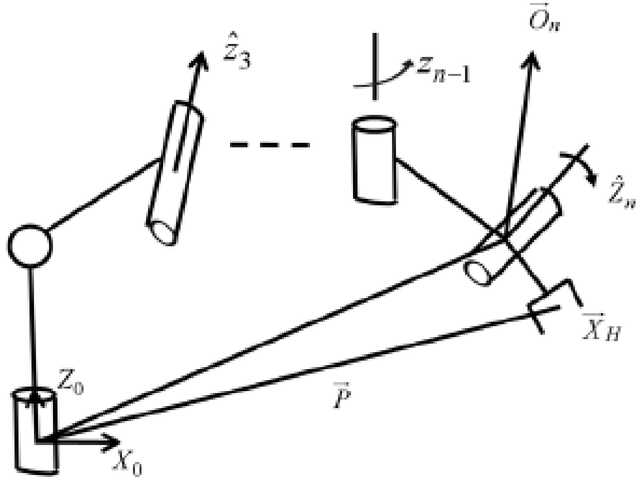
$${}^{0}J = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

$${}^{3}J = \begin{bmatrix} l_1 s_{23} + l_2 s_3 & l_2 s_3 & 0 \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_3 + l_3 & l_3 \\ 1 & 1 & 1 \end{bmatrix}$$

For general n dof manipulator,

$$\underline{\dot{X}} = J(\theta)\dot{\theta}$$

• How to get Jacobian numerically (Thomas)



$$\dot{X}_H = \begin{bmatrix} \vec{V}_H \\ \vec{\omega}_H \end{bmatrix}$$

$\vec{\omega}_{i+1} = \vec{\omega}_i + \dot{\theta}_{i+1}\hat{z}_{i+1}$ for revolute joint (i+1)

(1) Angular velocity

$\vec{\omega}_{i+1} = \vec{\omega}_i$ for prismatic joint (i+1)

$$\vec{\omega}_H = \vec{\omega}_n = \vec{\omega}_{n-1} + \dot{\theta}_n\hat{z}_n$$

$$= \vec{\omega}_{n-2} + \dot{\theta}_{n-1}\hat{z}_{n-1} + \dot{\theta}_n\hat{z}_n$$

.

.

.

$$= \vec{\omega}_{n-2} + \dot{\theta}_{n-1}\hat{z}_{n-1} + \dot{\theta}_n\hat{z}_n$$

• $$\vec{\omega}_H = \underbrace{\begin{bmatrix} \hat{z}_1 \vdots & \hat{z}_2 \vdots & \vec{0} \vdots & \dots & \vdots \hat{z}_n \end{bmatrix}}_{\text{3xn matrix}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

- Linear Velocity

$$\vec{v}_{i+1} = \vec{v}_i + \vec{\omega}_i \times {}^{i}\vec{P}_{i+1} \text{ for } (i+1)th \text{ revolute joint}$$

$$\vec{v}_{i+1} = \vec{v}_i + \vec{\omega}_i \times {}^{i}\vec{P}_{i+1} + \dot{d}_{i+1}\hat{z}_{i+1} \text{ for } (i+1)th \text{ prismatic joint}$$

$$\begin{aligned}
\vec{v}_H &= \vec{v}_n + \vec{\omega}_n \times (\vec{P} - \vec{0}_n) \\
&= \vec{v}_{n-1} + \vec{\omega}_{n-1} \times (\vec{0}_n - \vec{0}_{n-1}) + \vec{\omega}_n(\vec{P} - \vec{0}_n) \\
&= {}^{0}\vec{v}_1 + \vec{\omega}_1(\vec{0}_2 - \vec{0}_1) + \vec{\omega}_2(\vec{0}_3 - \vec{0}_2) + \dot{d}_3\hat{z}_3 \\
&\quad + \vec{\omega}_2(\vec{0}_4 - \vec{0}_3) + \cdots + \vec{\omega}_n(\vec{P} - \vec{0}_n)
\end{aligned}$$

$$\begin{aligned}
&= \dot{\theta}_1\hat{z}_1 \times (\vec{0}_2 - \vec{0}_1) + (\theta_1{}^{0}\hat{z}_1 + \theta_2{}^{1}\hat{z}_2) \times (\vec{0}_3 - \vec{0}_2) + \dot{d}_3\hat{z}_3 \\
&\quad + (\dot{\theta}_1{}^{0}\hat{z}_1 + \dot{\theta}_2\hat{z}_2) \times (\vec{0}_4 - \vec{0}_3) + \cdots + (\dot{\theta}_1\hat{z}_1 + \dot{\theta}_2\hat{z}_2 + \dot{\theta}_4\hat{z}_4 + \cdots \dot{\theta}_n\hat{z}_n) \times (\vec{P} - \vec{0}_n)
\end{aligned}$$

$$\begin{aligned}
&= \dot{\theta}_1{}^{0}\hat{z}_1 \times (\vec{0}_2 - \vec{0}_1 + \vec{0}_3 - \vec{0}_2 + \vec{0}_4 - \vec{0}_3 + \cdots \vec{P} - \vec{0}_n \\
&\qquad + \dot{\theta}_2{}^{0}\hat{z}_2 \times (\vec{0}_3 - \vec{0}_2 + \vec{0}_4 - \vec{0}_3 + \cdots \vec{P} - \vec{0}_n) + \cdots
\end{aligned}$$

$$\vec{v}_H = \dot{\theta}_1 {}^0\hat{z}_1 \times (\vec{P} - \vec{0}_1) + \dot{\theta}_2 {}^0\hat{z}_2 \times (\vec{P} - \vec{0}_2) + \dot{d}_3 \hat{z}_3 + \dot{\theta}_4 \hat{z}_4 (\vec{P} - \vec{0}_n)$$

$$+ \cdots + \dot{\theta}_n {}^0\hat{z}_n \times (\vec{P} - \vec{0}_n)$$

$$= \begin{bmatrix} {}^0\hat{z}_1 \times (\vec{P} - \vec{0}_1) & {}^0\hat{z}_2 \times (\vec{P} - \vec{0}_2) & \hat{z}_3 & \ldots & {}^0\hat{z}_n \times (\vec{P} - \vec{0}_n) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

i^{th} Joint: ith column of Jacobian

(1) Revolute

$$\begin{bmatrix} \hat{z}_2 \times (\vec{P} - \vec{0}_i \\ \hat{z}_i \end{bmatrix} \dot{\theta}_i$$

(2) Prismatic

$$\begin{bmatrix} \hat{z}_i \\ 0 \end{bmatrix} \dot{d}_i$$

Easy way to get Jacobian Numerically

$$ {}^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\vec{\omega}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{z}_{i+1} \quad -① $$

$$ {}^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right) \quad -② $$

For general n dof manipulator,

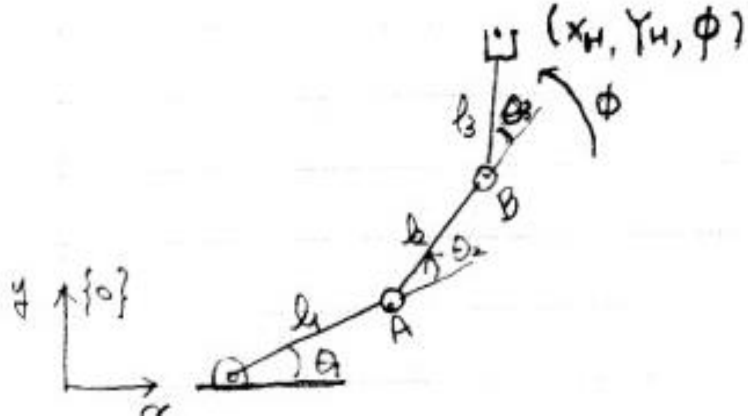
$\dot{\theta} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ get ${}^{N+1}\vec{\omega}_{N+1}$, ${}^{N+1}\vec{v}_{N+1}$ $\Rightarrow$ first column of Jacobian

$\dot{\theta} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ get ${}^{N+1}\vec{\omega}_{N+1}$, ${}^{N+1}\vec{v}_{N+1}$ $\Rightarrow$ 2nd column of Jacobian

$\vdots$

$\dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ get ${}^{N+1}\vec{\omega}_{N+1}$, ${}^{N+1}\vec{v}_{N+1}$ $\Rightarrow$ nth column of Jacobian

- 3 link manipulator : Jacobian



$$x_H = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$Y_H = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

$$\dot{x}_H = -l_1 \sin\theta_1\, \dot{\theta}_1 - l_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$
$$\dot{Y}_H = l_1 \cos\theta_1\, \dot{\theta}_1 + l_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$
$$\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

$$\begin{bmatrix} \dot{X}_H \\ \dot{Y}_H \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_3 \sin(\theta_1+\theta_2+\theta_3) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_3 \cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\dot{\underline{X}} = J(\theta)\, \dot{\underline{\theta}}$$

$$\ddot{\underline{X}} = J(\theta)\, \ddot{\underline{\theta}} + \dot{J}(\theta)\, \dot{\underline{\theta}}$$ for acceleration

- Inverse instantaneous Kinematics

$\underline{\dot{X}} = J(\theta)\underline{\dot{\theta}}$ for n dof Manipulator where $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ and $\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\theta}_n \end{bmatrix}$

Generally, $n = 6$ $J(\theta)$: 6×6 square matrix

$$\left.\begin{array}{c} \underline{\dot{\theta}} = J^{-1}(\theta)\underline{\dot{X}} \text{ and } \underline{\theta} = f^{-1}(X) \\ \Big\downarrow \; \dot{\theta}_d, \theta_d \\ \tau = K_p(\theta_d - \theta) + K_p(\dot{\theta}_d - \dot{\theta}) \end{array}\right\} \text{Resolved Motion rate control}$$

Cf) $\ddot{\theta} = J^{-1}(\ddot{X} - \dot{J}\dot{\theta})$ $\longleftarrow$ Resolved Acceleration control

As $\underline{\theta}$ changes, $J(\theta)$ changes $\Rightarrow$ Need get $J(\theta)$ efficiently

- Singularity $\quad J^{-1}(\theta)$ does not exist. : Singular configuration
  - ➢ $Det(J) = 0$
  - ➢ One of eigenvalue of (J) is equal to zero
  - ➢ J does not have full rank.

Manipulator가 어느 방향으로는 움직일 수 없다.

- Singularity

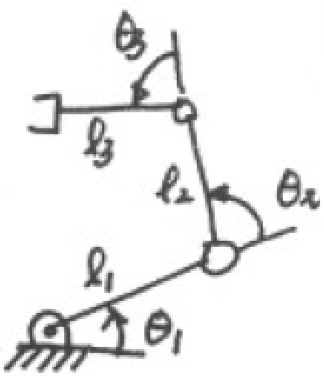
Workspace boundary singularities:

Stretched out of folder back position near or at the boundary of the workspace

Workspace interior singularities:

generally caused by two or more joint axes lining up

Singularity for 3 link manipulator



$$^0J = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} - l_3 S_{123} & -l_2 S_{12} - l_2 S_{123} & -l_3 S_{123} \\ l_1 C_1 + l_2 C_{12} + l_3 C_{123} & l_2 C_{12} + l_3 C_{123} & l_3 C_{123} \\ 1 & 1 & 1 \end{bmatrix} \quad Det(^0J) = l_1 l_2 \sin\theta_2$$

$$^3J = \begin{bmatrix} l_1 S_{23} + l_3 S_3 & l_2 S_3 & 0 \\ l_1 C_{23} + l_2 C_3 + l_3 & l_2 C_3 + l_3 & l_3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$Det(^3J) = l_1 l_2 \sin\theta_2$$

Singularity position $\theta_2 = 0, 180°$

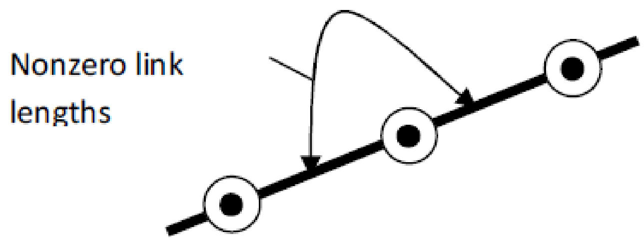
0: stretched out

$180°$: folded back

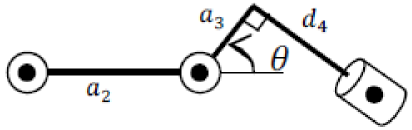
Exercise 5.6)

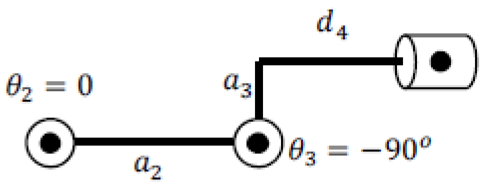
"Any mechanism with 3 revolute joints and nonzero link lengths must be a locus of singular points interior to its workspace"

(See B. Shimano, "The kinematic design and force control of Computer controlled Manipulators", 1978)



5.14 [18] If the link parameter a_3 of the PUMA 560 were zero, a workspace boundary singularity would occur when $\theta_3 = -90.0°$. Give an expression for the value of θ_3 where the singularity occurs and show that if a_3 were zero, the result would be $\theta_3 = -90.0°$. *Hint*: In this configuration a straight line passes through joint axes 2, 3, and the point where axes 4, 5, and 6 intersect.

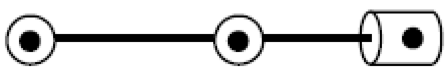




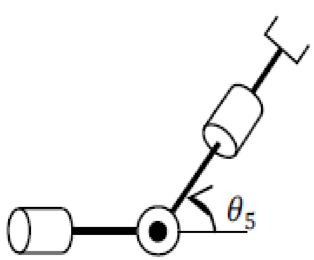
$$tan\theta = \frac{d_4}{a_3}, or\ \theta = Atan2(d_4, a_3)$$

➔ Workspace interior singular position

$$\theta_3 = -Atan2\ (d_4, a_3) \approx -90^o\ when\ a_3 = 0$$



If $a_3 = 0$, workspace boundary singularity

	 $\sin\theta_5 = 0, \theta_5 = 0$	Work space interior singularity

A Study of the Jacobian Matrix of Serial Manipulators

K. J. Waldron
Mem. ASME

Shih-Liang Wang

S. J. Bolin

Department of Mechanical Engineering,
The Ohio State University,
Columbus, Ohio 43210

Inversion of the Jacobian matrix is the critical step in rate decomposition which is used to solve the so-called "inverse kinematics" problem of robotics. This is the problem of achieving a coordinated motion relative to the fixed reference frame. In this paper a general methodology is presented for formulation and manipulation of the Jacobian matrix. The formulation is closely tied to the geometry of the system and lends itself to simplification using appropriate coordinate transformations. This is of great importance since it gives a systematic approach to the derivation of efficient, analytical inverses. The method is also applied to the examination of geometrically singular positions. Several important general results relating to the structure of the singularity field are deducible from the structure of the algebraic system.

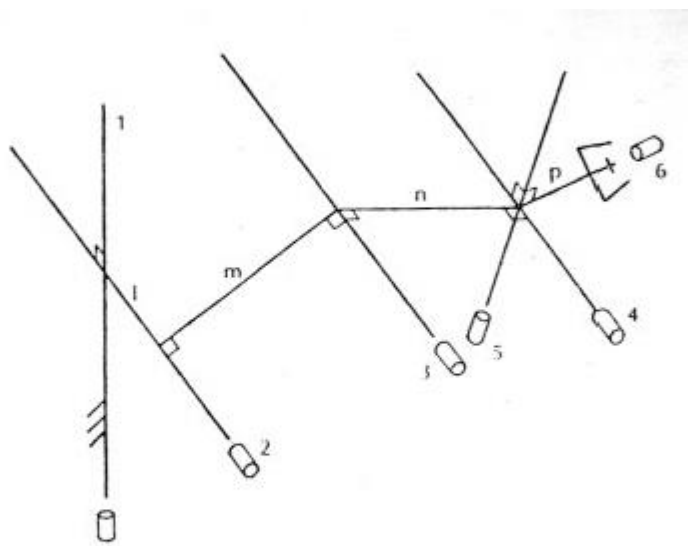


Fig. 3 Geometry of manipulator chain of Example 1. Geometric parameters are summarized in Table 1.

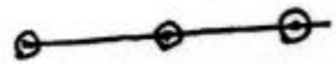
$$\Gamma = \begin{bmatrix} S(\theta_2+\theta_3) & 0 & 0 & 0 & S\theta_4 & C\theta_4 S\theta_5 \\ C(\theta_2+\theta_3) & 0 & 0 & 0 & -C\theta_4 & S\theta_4 S\theta_5 \\ 0 & 1 & 1 & 1 & 0 & -C\theta_5 \\ lC(\theta_2+\theta_3) & mS\theta_3 & 0 & 0 & 0 & 0 \\ -lS(\theta_2+\theta_3) & (n+mC\theta_3) & n & 0 & 0 & 0 \\ -(mC\theta_2+nC(\theta_2+\theta_3)) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega_x = \dot\theta_1 S(\theta_2+\theta_3) + \dot\theta_5 S\theta_4 + \dot\theta_6 C\theta_4 S\theta_5$$
$$\omega_y = \dot\theta_1 C(\theta_2+\theta_3) - \dot\theta_5 C\theta_4 + \dot\theta_6 S\theta_4 S\theta_5$$
$$\omega_z = \dot\theta_2 + \dot\theta_3 + \dot\theta_4 - \dot\theta_6 C\theta_5$$
$$\mu_x = \dot\theta_1 l C(\theta_2+\theta_3) + \dot\theta_2 m S\theta_3$$
$$\mu_y = -\dot\theta_1 l S(\theta_2+\theta_3) + \dot\theta_2(n + mC\theta_3) + \dot\theta_3 n$$
$$\mu_z = -\dot\theta_1 (mC\theta_2 + nC(\theta_2+\theta_3))$$

$$\dot\theta_1 = \frac{-\mu_z}{mC\theta_2 + nC(\theta_2+\theta_3)}$$
$$\dot\theta_2 = \frac{1}{mS\theta_3}\{\mu_x - \dot\theta_1 l C(\theta_2+\theta_3)\}$$
$$\dot\theta_3 = \frac{1}{n}\{\dot\theta_1 l S(\theta_2+\theta_3) - \dot\theta_2(n+mC\theta_3) + \mu_y\}$$
$$\dot\theta_5 = \omega_x S\theta_4 - \omega_y C\theta_4 + \dot\theta_1 C(\theta_2+\theta_3+\theta_4)$$
$$\dot\theta_6 = \frac{1}{S\theta_5}\{\omega_x C\theta_4 + \omega_y S\theta_4 - \dot\theta_1 S(\theta_2+\theta_3+\theta_4)\}$$
$$\dot\theta_4 = \omega_z - \dot\theta_2 - \dot\theta_3 + \dot\theta_6 C\theta_5$$

Singularity Analysis

i) $S\theta_3 = 0$



펴져있어, singularity (yaw angle = 0 인 경우)

ii) $S\theta_5 = 0$

iii) $mC\theta_2 + nC(\theta_2+\theta_3) = 0$

wrist 점이 이 축의 선상에 오면 singularity

popular industrial robot geometry

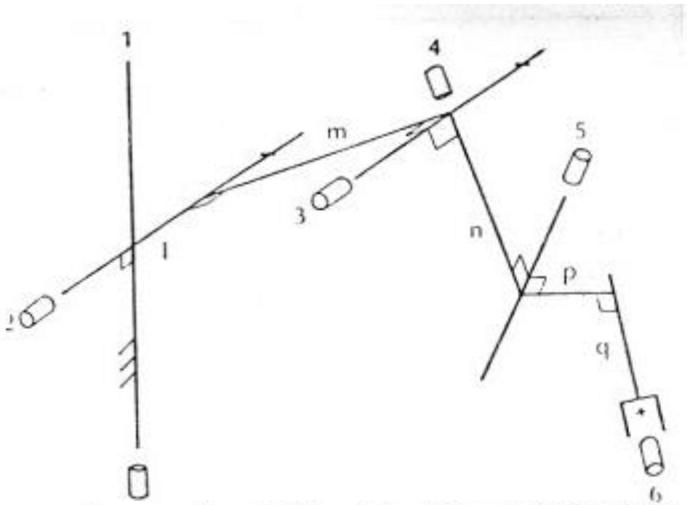


Fig. 4 Geometry of manipulator chain of Example 2. Geometric parameters are summarized in Table 2.

$$\Gamma = \begin{bmatrix} S(\theta_2+\theta_3) & 0 & 0 & 0 & S\theta_4 & C\theta_4 S\theta_5 \\ 0 & 1 & 1 & 0 & -C\theta_4 & S\theta_4 S\theta_5 \\ -C(\theta_2+\theta_3) & 0 & 0 & 1 & 0 & -C\theta_5 \\ lC(\theta_2+\theta_3) & mS\theta_3 & 0 & 0 & nC\theta_4 & -S\theta_4(p+nS\theta_5) \\ -mC\theta_2 & 0 & 0 & 0 & nS\theta_4 & C\theta_4(p+nS\theta_5) \\ lS(\theta_2+\theta_3) & -mC\theta_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- θ_1, θ_6 does not change the relative placement of Joint axes

  → no relation to geometric singularity

- q : no relation

• If $p=0$,

a) $S\theta_5=0$ θ_4 와 θ_5 가 평행

b) $C\theta_3=0$

c) $mC\theta_2+nS(\theta_2+\theta_3)=0$

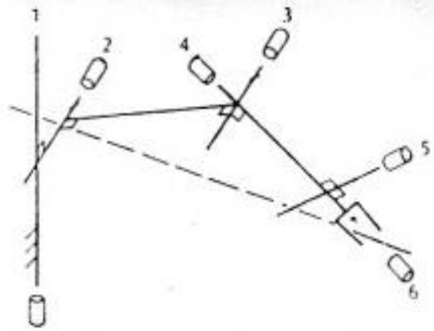


Fig. 6(a) Reciprocal axis given by $S\theta_5 = 0$. Axis satisfies same location equations as general case ($P \neq 0$); reciprocal axis has nonzero pitch.

θ_4, θ_6 colinear

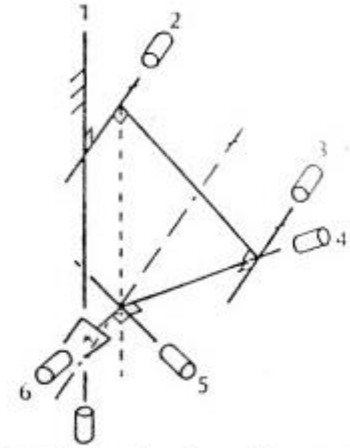


Fig. 6(c) Reciprocal axis given by $mC\theta_2 + nS(\theta_2 + \theta_3) = 0$. Reciprocal axis intersects joint axes 1, 4, 5, and 6 and is parallel to joint axes 2 and 3. It has zero pitch.

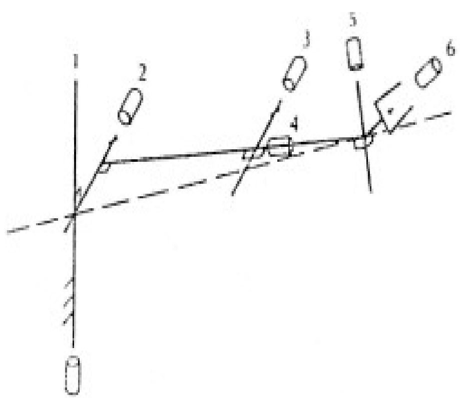


Fig. 6(b) Reciprocal axis given by $C\theta_3 = 0$. Reciprocal axis intersects all six joint axes and has zero pitch.

$(\hat{z}_1, \hat{z}_2)$

θ_1, θ_2 가 만드는

평면에 wrist origin

이 놓여진 상태

◎ Infinitesimal Rotations (Text: Asada & Slotine)

- mathematical tools for representing the spatial orientation of a rigid body : 3×3 rotation matrix, Euler angles — using rotations of finite angles

- Infinitesimal rotations or time derivatives of orientations are substantially different from finite angle rotations and orientations.

- want to explain the difference between finite and infinitesimal rotations

We begin by writing the 3×3 rotation matrix representing infinitesimal rotation $d\phi_x$ about the x axis:

$$\mathbf{R}_x(d\phi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(d\phi_x) & -\sin(d\phi_x) \\ 0 & \sin(d\phi_x) & \cos(d\phi_x) \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix}. \qquad (3-9)$$

Let $\mathbf{R}_y(d\phi_y)$ be the 3×3 infinitesimal rotation matrix about the y axis; then the result of consecutive rotations about the x and y axes is given by

$$\begin{aligned} \mathbf{R}_x(d\phi_x)\mathbf{R}_y(d\phi_y) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & 0 \\ -d\phi_y & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & d\phi_y \\ d\phi_x d\phi_y & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix} \end{aligned} \qquad (3-10)$$

$$\mathbf{R}_x(d\phi_x)\mathbf{R}_y(d\phi_y) = \mathbf{R}_y(d\phi_y)\mathbf{R}_x(d\phi_x) \qquad (3-11)$$

not true for finite angle

Therefore, infinitesimal rotations do not depend on the order of rotations; in other words, they commute.

* Commutative

For $d\phi_x, d\phi_y, d\phi_z$ rotations,

$$\mathrm{R}(d\phi_x, d\phi_y, d\phi_z) = \begin{bmatrix} 1 & -d\phi_z & d\phi_y \\ d\phi_z & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix} \tag{3-12}$$

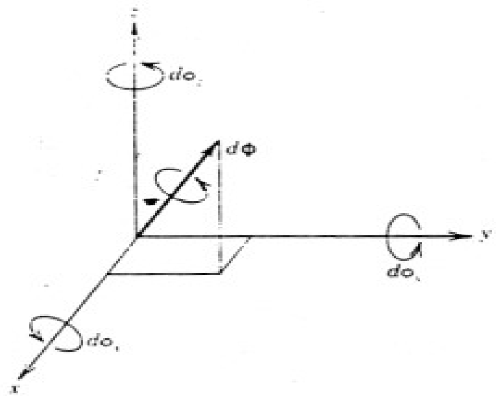


Figure 3-2 : Infinitesimal rotation vector.

Let $\mathrm{R}(d\phi_x, d\phi_y, d\phi_z)$ and $\mathrm{R}(d\phi'_x, d\phi'_y, d\phi'_z)$ be two infinitesimal rotation matrices. the consecutive rotations of the two yield

$$\mathrm{R}(d\phi_x, d\phi_y, d\phi_z)\mathrm{R}(d\phi'_x, d\phi'_y, d\phi'_z)$$

$$= \begin{bmatrix} 1 & (d\phi_z+d\phi'_z) & -(d\phi_y+d\phi'_y) \\ -(d\phi_z+d\phi'_z) & 1 & (d\phi_x+d\phi'_x) \\ (d\phi_y+d\phi'_y) & -(d\phi_x+d\phi'_x) & 1 \end{bmatrix} \tag{3-13}$$

$$= \mathrm{R}(d\phi_x+d\phi'_x, d\phi_y+d\phi'_y, d\phi_z+d\phi'_z)$$ * additive

Infinitesimal rotations are additive and commutative.

⇒ treat them as vectors

$$d\phi = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix}$$

- Vector representation is not allowed for finite rotations.

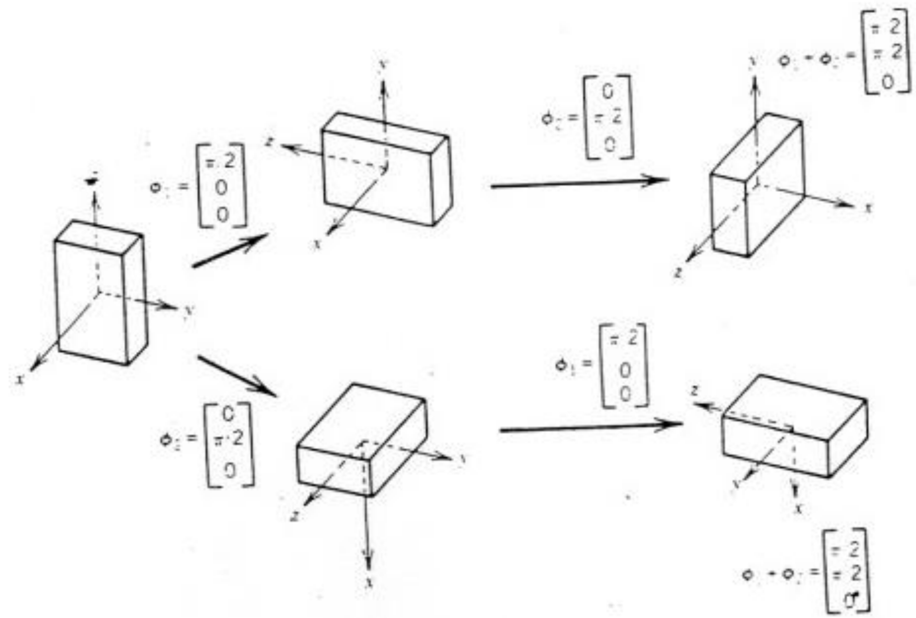


Figure 3-3 : Finite angle rotations.

Finite Rotations are not vectors. ⇐ Not additive, Not commutative

◎ Differential Translation and Rotation (R.P. Paul Text)

Case 1) Based on base coord. frame

$$T = \begin{bmatrix} {}^{0}_{T}R & {}^{0}P_{T} \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

Base coordinate에서의 rotation

$$\underline{T + dT = \text{Trans}(dx, dy, dz)\,\text{Rot}(k, d\theta)\,T} \tag{4.5}$$

where

$\text{Trans}(dx, dy, dz)$	is a transformation representing a translation of dx, dy, dz in base coordinates.
$\text{Rot}(k, d\theta)$	is a transformation representing a differential rotation $d\theta$ about a vector **k** also in base coordinates.

dT is given by

$$dT = (\text{Trans}(dx, dy, dz)\,\text{Rot}(k, d\theta) - I)\,T \tag{4.6}$$

Case 2) Based on a given coordinate frame T

$$T + dT = T\,\text{Trans}(dx, dy, dz)\,\text{Rot}(k, d\theta) \tag{4.7}$$

where

$\text{Trans}(dx, dy, dz)$	is now a transformation representing the differential translation with respect to coordinate frame **T**
$\text{Rot}(k, d\theta)$	represents the differential rotation $d\theta$ about a vector **k** described in coordinate frame **T**.

dT is now given by

$$dT = T(\underline{\text{Trans}(dx, dy, dz)\,\text{Rot}(k, d\theta)} - I) \tag{4.8}$$

Now, define Δ as

$$\Delta = \mathrm{Trans}(dx, dy, dz)\mathrm{Rot}(\mathbf{k}, d\theta) - \mathbf{I} \tag{4.9}$$

Hence,

$$dT = {}^{0}\Delta\, T = T\, {}^{T}\Delta$$

Base Coord. 에서 본 Δ

T coord 에서 본 Δ

$$\mathrm{Rot}(\mathbf{k}, \theta) =$$

$$\begin{bmatrix} k_x k_x \operatorname{vers}\theta + \cos\theta & k_y k_x \operatorname{vers}\theta - k_z \sin\theta & k_z k_x \operatorname{vers}\theta + k_y \sin\theta & 0 \\ k_x k_y \operatorname{vers}\theta + k_z \sin\theta & k_y k_y \operatorname{vers}\theta + \cos\theta & k_z k_y \operatorname{vers}\theta - k_x \sin\theta & 0 \\ k_x k_z \operatorname{vers}\theta - k_y \sin\theta & k_y k_z \operatorname{vers}\theta + k_x \sin\theta & k_z k_z \operatorname{vers}\theta + \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.13}$$

θ is in this case finite. For a differential change $d\theta$ the corresponding trigonometric functions become

$$\lim_{\theta \to 0} \sin\theta \to d\theta$$

$$\lim_{\theta \to 0} \cos\theta \to 1$$

$$\lim_{\theta \to 0} \operatorname{vers}\theta \to 0$$

and Equation 4.13 becomes

$$\mathrm{Rot}(\mathbf{k}, d\theta) = \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta & 0 \\ k_z d\theta & 1 & -k_x d\theta & 0 \\ -k_y d\theta & k_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.14}$$

Equation 4.9 then becomes

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta & 0 \\ k_z d\theta & 1 & -k_x d\theta & 0 \\ -k_y d\theta & k_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -k_z d\theta & k_y d\theta & d_x \\ k_z d\theta & 0 & -k_x d\theta & d_y \\ -k_y d\theta & k_x d\theta & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.15}$$

$$\mathrm{Rot}(x, \delta_x)\,\mathrm{Rot}(y, \delta_y)\,\mathrm{Rot}(z, \delta_z) = \begin{bmatrix} 1 & -\delta_z & \delta_y & 0 \\ \delta_z & 1 & -\delta_x & 0 \\ -\delta_y & \delta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.26}$$

$$k_x d\theta = \delta_x$$
$$k_y d\theta = \delta_y$$
$$k_z d\theta = \delta_z$$

We may then rewrite Equation 4.15 as

$$\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.28}$$

Example 4.1

Given a coordinate frame **A**

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

what is the differential transformation **dA** corresponding to a differential translation $\mathbf{d} = 1\mathbf{i} + 0\mathbf{j} + 0.5\mathbf{k}$ and rotation $\delta = 0\mathbf{i} + 0.1\mathbf{j} + 0\mathbf{k}$ made with respect to base coordinates?

Solution:

We first construct the differential translation and rotation transformation Δ, as in Equation 4.28.

$$\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and then use Equation 4.10 of subsection 4.3.1 to solve for **dA**

$$\mathbf{dA} = \Delta\mathbf{A}$$

$$\mathbf{dA} = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{dA} = \Delta\mathbf{A}$$

$$\mathbf{dA} = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{dA} = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

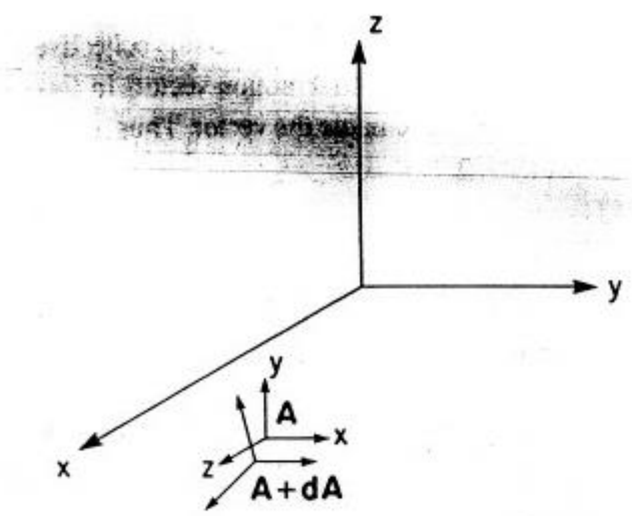


Figure 4.1. The Differential Change in Coordinate Frame A

⊙ Transforming Differential Changes between Coordinate Frames

$$dT = {}^{0}\Delta T = T\,{}^{T}\Delta = \underbrace{\Delta}_{\text{기준의 coord. frame.}} T$$

$${}^{T}\Delta = T^{-1}\,{}^{0}\Delta\, T \quad \text{or} \quad T^{-1}\Delta T$$

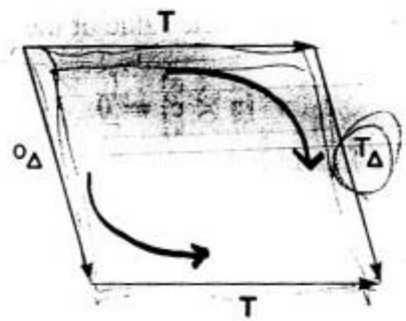


Figure 4.2. Transform Graph for Differential Changes

$$\Delta T = \begin{bmatrix} (\delta\times n)_x & (\delta\times o)_x & (\delta\times a)_x & ((\delta\times p)+d)_x \\ (\delta\times n)_y & (\delta\times o)_y & (\delta\times a)_y & ((\delta\times p)+d)_y \\ (\delta\times n)_z & (\delta\times o)_z & (\delta\times a)_z & ((\delta\times p)+d)_z \\ 0 & 0 & 0 & \underline{0} \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ \underline{0}^T & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = \mathrm{Trans}(d)\,\mathrm{Rot}(\delta)$$

$$T^{-1}\Delta T = \begin{bmatrix} n\cdot(\delta\times n) & n\cdot(\delta\times o) & n\cdot(\delta\times a) & n\cdot((\delta\times p)+d) \\ o\cdot(\delta\times n) & o\cdot(\delta\times o) & o\cdot(\delta\times a) & o\cdot((\delta\times p)+d) \\ a\cdot(\delta\times n) & a\cdot(\delta\times o) & a\cdot(\delta\times a) & a\cdot((\delta\times p)+d) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.35}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}). \tag{4.36}$$

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{c}) = 0 \tag{4.37}$$

Thus the diagonal terms of Equation 4.35 are all zero.

Rearranging the terms of the triple products in Equation 4.35 we obtain

$$\boxed{{}^{T}\Delta} = \begin{bmatrix} 0 & -\delta\cdot(\mathbf{n}\times\mathbf{o}) & \delta\cdot(\mathbf{a}\times\mathbf{n}) & \delta\cdot(\mathbf{p}\times\mathbf{n})+\mathbf{d}\cdot\mathbf{n} \\ \delta\cdot(\mathbf{n}\times\mathbf{o}) & 0 & -\delta\cdot(\mathbf{o}\times\mathbf{a}) & \delta\cdot(\mathbf{p}\times\mathbf{o})+\mathbf{d}\cdot\mathbf{o} \\ -\delta\cdot(\mathbf{a}\times\mathbf{n}) & \delta\cdot(\mathbf{o}\times\mathbf{a}) & 0 & \delta\cdot(\mathbf{p}\times\mathbf{a})+\mathbf{d}\cdot\mathbf{a} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.38}$$

Further, as

$$\begin{aligned} \mathbf{n}\times\mathbf{o} &= \mathbf{a}; \\ \mathbf{a}\times\mathbf{n} &= \mathbf{o}; \\ \mathbf{o}\times\mathbf{a} &= \mathbf{n}. \end{aligned} \tag{4.39}$$

we may finally write Equation 4.35 as

$${}^{T}\Delta = \begin{bmatrix} 0 & -\delta\cdot\mathbf{a} & \delta\cdot\mathbf{o} & \delta\cdot(\mathbf{p}\times\mathbf{n})+\mathbf{d}\cdot\mathbf{n} \\ \delta\cdot\mathbf{a} & 0 & -\delta\cdot\mathbf{n} & \delta\cdot(\mathbf{p}\times\mathbf{o})+\mathbf{d}\cdot\mathbf{o} \\ -\delta\cdot\mathbf{o} & \delta\cdot\mathbf{n} & 0 & \delta\cdot(\mathbf{p}\times\mathbf{a})+\mathbf{d}\cdot\mathbf{a} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.40}$$

However, ${}^{T}\Delta$ is defined to be

$${}^{T}\Delta = \begin{bmatrix} 0 & -{}^{T}\delta_z & {}^{T}\delta_y & {}^{T}d_x \\ {}^{T}\delta_z & 0 & -{}^{T}\delta_x & {}^{T}d_y \\ -{}^{T}\delta_y & {}^{T}\delta_x & 0 & {}^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.41}$$

$$\begin{aligned} {}^{T}d_x &= \delta\cdot(\mathbf{p}\times\mathbf{n})+\mathbf{d}\cdot\mathbf{n} \\ {}^{T}d_y &= \delta\cdot(\mathbf{p}\times\mathbf{o})+\mathbf{d}\cdot\mathbf{o} \\ {}^{T}d_z &= \delta\cdot(\mathbf{p}\times\mathbf{a})+\mathbf{d}\cdot\mathbf{a} \end{aligned}$$

$$\begin{aligned} {}^{T}\delta_x &= \delta\cdot\mathbf{n} \\ {}^{T}\delta_y &= \delta\cdot\mathbf{o} \\ {}^{T}\delta_z &= \delta\cdot\mathbf{a} \end{aligned}$$

$$\begin{bmatrix} {}^{T}d_x \\ {}^{T}d_y \\ {}^{T}d_z \\ {}^{T}\delta_x \\ {}^{T}\delta_y \\ {}^{T}\delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (\mathbf{p}\times\mathbf{n})_x & (\mathbf{p}\times\mathbf{n})_y & (\mathbf{p}\times\mathbf{n})_z \\ o_x & o_y & o_z & (\mathbf{p}\times\mathbf{o})_x & (\mathbf{p}\times\mathbf{o})_y & (\mathbf{p}\times\mathbf{o})_z \\ a_x & a_y & a_z & (\mathbf{p}\times\mathbf{a})_x & (\mathbf{p}\times\mathbf{a})_y & (\mathbf{p}\times\mathbf{a})_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

Example 4.2

Given the same coordinate frame and differential translation and rotation as in Example 4.1

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{d} = 1\mathbf{i} + 0\mathbf{j} + 0.5\mathbf{k}$$

$$\delta = 0\mathbf{i} + 0.1\mathbf{j} + 0\mathbf{k}$$

what is the equivalent differential translation and rotation in coordinate frame A?

Solution:

With

$$\mathbf{n} = 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k};$$
$$\mathbf{o} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k};$$
$$\mathbf{a} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k};$$
$$\mathbf{p} = 10\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}.$$

$$\delta \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.1 & 0 \\ 10 & 5 & 0 \end{vmatrix}$$

$$\delta \times \mathbf{p} = 0\mathbf{i} + 0\mathbf{j} - 1\mathbf{k}$$

$$\delta \times \mathbf{p} + \mathbf{d} = 1\mathbf{i} + 0\mathbf{j} - 0.5\mathbf{k}$$

$$^{A}\mathbf{d} = 0\mathbf{i} - 0.5\mathbf{j} + 1\mathbf{k};$$
$$^{A}\delta = 0.1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

$$^{A}\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can check this result by using Equation 4.11 to evaluate dA

$$dA = A\,{}^{A}\Delta$$

$$dA = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$dA = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

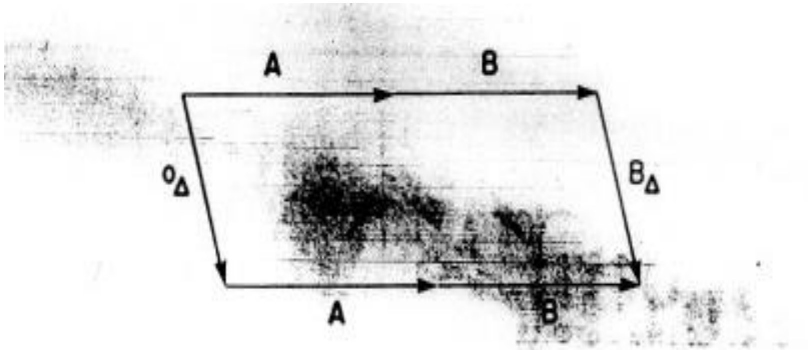


Figure 4.4. Differential Changes between Two Coordinate Frames

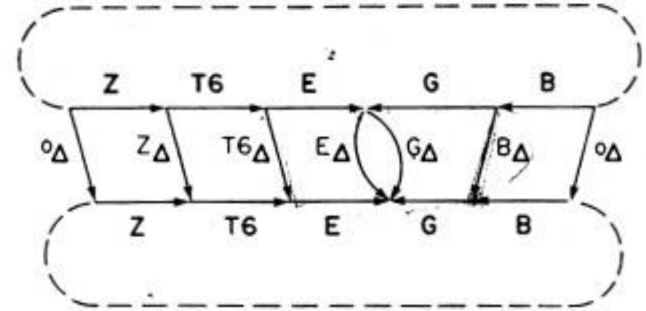


Figure 4.5. General Differential Change Graph

Known $A, B, {}^{B}\Delta$

Find ${}^{0}\Delta$

$${}^{0}\Delta A B = A B\, {}^{B}\Delta$$

$${}^{0}\Delta = A B\, {}^{B}\Delta (AB)^{-1}$$

$$= ((AB)^{-1})^{-1}\, {}^{B}\Delta (AB)^{-1}$$

with $T = (AB)^{-1}$

$${}^{0}\Delta Z T_6 = Z T_6\, {}^{T_6}\Delta$$

$${}^{0}\Delta = Z T_6\, {}^{T_6}\Delta (Z T_6)^{-1}$$

$$= ((Z T_6)^{-1})^{-1}\, {}^{T_6}\Delta (Z T_6)^{-1}$$

Example 4.3

A camera is attached to link 5 of a manipulator. The connection is defined b

$$^{T_5}CAM = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} dx \\ dy \\ dz \\ \end{matrix}$$

The last link of the manipulator is described, in its current position, by

$$A_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An object ^{CAM}O is observed and differential changes in CAM coordinates are given in order to bring the end effector into contact with the object.

$$^{CAM}d = -1i + 1j + 0k \qquad ^{CAM}\delta = 0i + 0j + 0.1k$$

What are the required changes in T_6 coordinates?

Known : $^{CAM}\Delta$ Find $^{T_6}\Delta$

$$T_5 A_6 EX = T_5 \, CAM \, O$$

differential

$$T_5 A_6 \, ^{T_6}\Delta \, EX = T_5 \, CAM \, ^{CAM}\Delta \, O$$

$$^{T_6}\Delta \, EX = A_6^{-1} CAM \, ^{CAM}\Delta \, O$$

$$= A_6^{-1} CAM \, ^{CAM}\Delta \, CAM^{-1} A_6 \, EX$$

$$\therefore \; ^{T_6}\Delta = (CAM^{-1} A_6)^{-1} \, ^{CAM}\Delta \, CAM^{-1} A_6$$

$$CAM^{-1} = \begin{bmatrix} 0 & 0 & -1 & 10 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

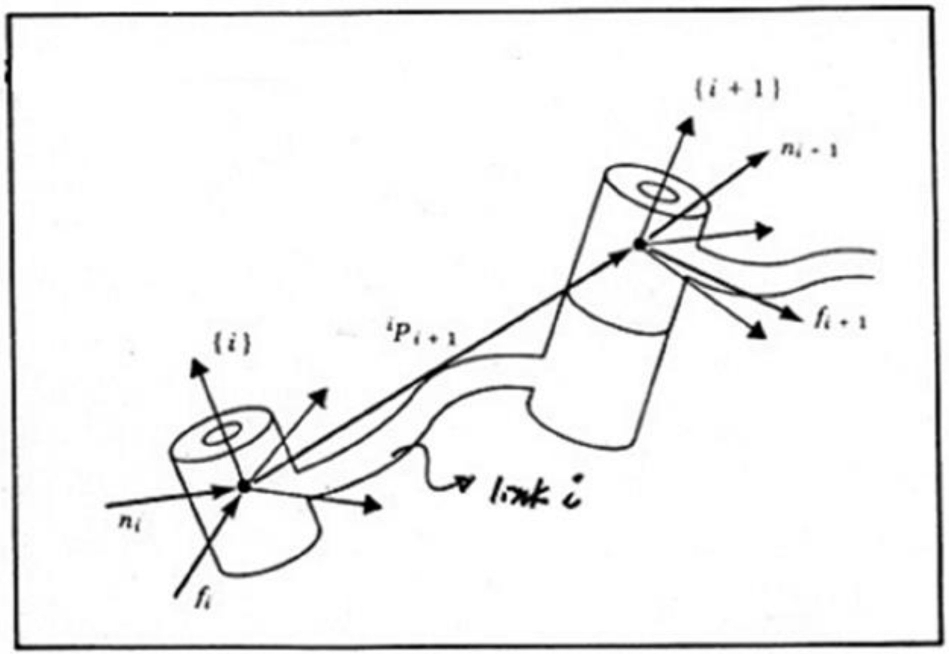
$$T = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \delta \times p = \begin{vmatrix} i & j & k \\ 0 & 0 & 0.1 \\ 2 & 0 & 5 \end{vmatrix} = 0i + 0.2j + 0k$$

$$\delta \times p + d = -1i + 0.2j + 0k$$

- Static forces in manipulators

Wish to solve for the joint torques which must be acting to keep the system in static equilibrium

Neglects gravity in this case



Set f_i= force exerted on link i by link i-1,

N_i= torque exerted on link i by link i-1.

For the equilibrium,

$$^if_i - {^if_{i+1}} = 0$$

Summing torques about the origin of frame {i} we have

$$^in_i - {^in_{i+1}} - {^iP_{i+1}} \times {^if_{i+1}} = 0$$

$$^if_i = {^if_{i+1}}$$

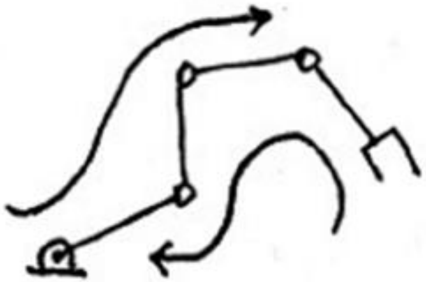
$$^in_i = {^in_{i+1}} + {^iP_{i+1}} \times {^if_{i+1}}$$

static force "propagation" from link to link:

$$^if_i = {^i_{i+1}R}\,{^{i+1}f_{i+1}}$$

$$^in_i = {^i_{i+1}R}\,{^{i+1}n_{i+1}} + {^iP_{i+1}} \times {^if_i}$$

Forward velocity propagation

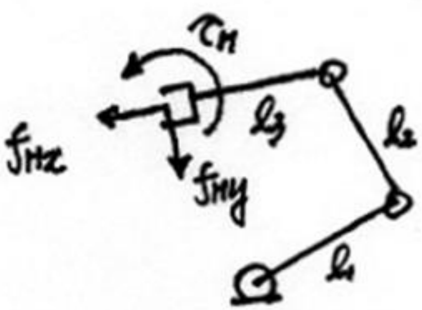


Backward force propagation

$$\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i$$ For i: Revolute Joint

$$\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i$$ For i: Prismatic Joint

Ex)



$$^3f_3 = \begin{bmatrix} ^3f_{Hx} \\ ^3f_{Hy} \\ 0 \end{bmatrix} \qquad ^3\boldsymbol{n}_3 = \begin{bmatrix} 0 \\ 0 \\ \tau_H \end{bmatrix} + l_3 \, {}^3\hat{x}_3 \times \begin{bmatrix} ^3f_{Hx} \\ ^3f_{Hy} \\ 0 \end{bmatrix}$$

$$^3\boldsymbol{n}_3 = \begin{bmatrix} 0 \\ 0 \\ \tau_H \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_3 \, {}^3f_{Hy} \end{bmatrix}$$

$$^3\boldsymbol{n}_3 = \begin{bmatrix} 0 \\ 0 \\ \tau_H + l_3 \, {}^3f_{Hy} \, 0 \end{bmatrix}$$

$$^{2}\boldsymbol{f}_{2} = {}^{2}_{3}R\,{}^{3}\boldsymbol{f}_{3} = \begin{bmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{3}f_{Hx} \\ {}^{3}f_{Hy} \\ 0 \end{bmatrix} = \begin{bmatrix} C_3\,{}^{3}f_{Hx} - S_3\,{}^{3}f_{Hy} \\ S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy} \\ 0 \end{bmatrix}$$

$$^{2}\boldsymbol{n}_{2} = {}^{2}_{3}R\,{}^{3}n_{3} + l_2\,{}^{2}\hat{x}_{2} \times {}^{2}\boldsymbol{f}_{2}$$

$$^{2}\boldsymbol{n}_{2} = \begin{bmatrix} 0 \\ 0 \\ \tau_H + l_3\,{}^{3}f_{Hy} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_2(S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy}) \end{bmatrix}$$

$$^{1}\boldsymbol{f}_{1} = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{2}\boldsymbol{f}_{2} = \begin{bmatrix} C_2(C_3\,{}^{3}f_{Hx} - S_3\,{}^{3}f_{Hy}) - S_2(S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy}) \\ S_2(C_3\,{}^{3}f_{Hx} - S_3\,{}^{3}f_{Hy}) + C_2(S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy}) \\ 0 \end{bmatrix}$$

$$^{1}\boldsymbol{n}_{1} = {}^{1}_{2}R\,{}^{2}n_{2} + l_1\,{}^{1}\hat{x}_{1} \times {}^{1}\boldsymbol{f}_{1}$$

$$^{1}\boldsymbol{n}_{1} = \begin{bmatrix} 0 \\ 0 \\ \tau_H + l_3\,{}^{3}f_{Hy} + l_2(S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1 S_2(C_3\,{}^{3}f_{Hx} - S_3\,{}^{3}f_{Hy}) + l_1 C_3(S_3\,{}^{3}f_{Hx} + C_3\,{}^{3}f_{Hy}) \end{bmatrix}$$

$$\tau_3 = l_3\, f_{Hy} + \tau_H$$

$$\tau_2 = l_2\, S_3 f_{Hx} + l_2\, C_3 f_{Hy} + l_3\, f_{Hy} + \tau_H$$

$$\tau_1 = l_2 S_3 f_{Hx} + (l_2\, C_3 + l_3) f_{Hy} + l_3\, f_{Hy} + \tau_H + l_1 S_2\, S_3 f_{Hx} - l_1 S_2\, S_3 f_{Hy} + l_1 C_2 S_3 f_{Hx} + l_1 C_2 C_3 f_{Hy}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} l_1 s_{23} + l_2 s_3 & l_1 c_{23} + l_2 c_3 + l_3 & 1 \\ l_2 s_3 & l_2 c_3 + l_3 & 1 \\ 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} {}^3f_{Hx} \\ {}^3f_{Hy} \\ {}^3\tau_H \end{bmatrix} \Rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = {}^3J^T \, {}^3\underline{f}_H$$

$${}^3\dot{V}_H = {}^3J\underline{\dot{\theta}}$$

$${}^3J = \begin{bmatrix} l_1 s_{23} + l_2 s_3 & l_2 s_3 & 0 \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_3 + l_3 & l_3 \\ 1 & 1 & 1 \end{bmatrix}$$

- Jacobian in the force domain

The principle of virtual work

$$\mathcal{F} \cdot \delta \mathcal{X} = \tau \cdot \delta \Theta,$$

$$\mathcal{F}^T \delta \mathcal{X} = \tau^T \delta \Theta. \qquad \delta \mathcal{X} = J \delta \Theta,$$

$$\mathcal{F}^T J \delta \Theta = \tau^T \delta \Theta,$$

$$\mathcal{F}^T J = \tau^T.$$

$$\tau = J^T \mathcal{F}.$$

◎ Redundancy

$$\dot{p} = J\dot{q}$$

$\dot{p}$ m-dimensional space , $p \in R^m$

$\dot{q}$ n-dimensional space , $q \in R^n$

$n > m \rightarrow$ Redundant Manipulator

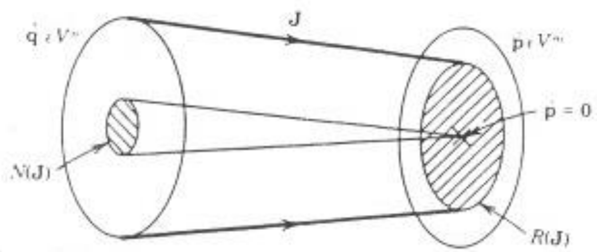


Figure 3-9 : Linear mapping diagram of instantaneous kinematics

R(J) : Range space of linear mapping

N(J) : Null space of linear mapping

- $J\dot{q} = 0$ 을 만족하는 모든 element들이 구성하는 Subspace

If Jacobian is of full rank, $\dim(N(J)) = n - m$

즉, $\dim R(J) + \dim N(J) = n$

$$J\dot{q}^* = \dot{P}$$

$$k \times J\dot{q}_0 = 0$$ 이라면

$$J(\dot{q}^* + k\dot{q}_0) = \dot{P}$$; k: arbitrary scalar quantity

그러므로, $k\dot{q}_0$ can be chosen arbitrarily within the null space.

Instantaneous Inverse Kinematics 에서 $k\dot{q}_0$ 즉 null space term을 로보트의 성능 향상을 위하여 결정할 수 있다.

$$J\dot{q}^* = \dot{P}$$

$k \times J\dot{q}_0 = 0$ 이라면

$$J(\dot{q}^* + k\dot{q}_0) = \dot{P} \quad ; \; k : \text{arbitrary scalar quantity}$$

그러므로, $k\dot{q}_0$ can be chosen arbitrarily within the null space.

Instantaneous Inverse Kinematics 에서 $k\dot{q}_0$ 즉 null space term을 로보트의 성능 향상을 위하여 결정할 수 있다.

Cost function $f(\underline{x})$ with constraint $G(\underline{x}) = 0$

$$P(\underline{x}, \lambda) = f(\underline{x}) + \lambda^T G(\underline{x})$$

$$\frac{\partial P}{\partial \underline{x}} = 0, \quad \frac{\partial P}{\partial \lambda} = 0 \quad : \text{necessary cond.}$$

이것을 이용하여 Redundant manipulator의 Instantaneous Inverse Kinematics 를 수행한다.

$\dot{p} = J\dot{q}$ $\quad n > m$

$\dot{q} = \,?\,\dot{p}$ $\quad$ minimizing $\quad G(\dot{q}) = \dot{q}^T W \dot{q}$

where $\mathbf{W}$ is an $n \times n$ symmetric positive definite weighting matrix. The problem is to find the $\dot{\mathbf{q}}$ that satisfies equation (3-40) for a given $\dot{\mathbf{p}}$ and $\mathbf{J}$ while minimizing the cost function $G(\dot{\mathbf{q}})$. Let us solve this problem using Lagrange multipliers. To this end we use a modified cost function of the form

$$G(\dot{\mathbf{q}},\boldsymbol{\lambda}) = \dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}} - \boldsymbol{\lambda}^T(\mathbf{J}\dot{\mathbf{q}}-\dot{\mathbf{p}}) \tag{3-44}$$

where $\boldsymbol{\lambda}$ is an $m \times 1$ unknown vector of Lagrange multipliers. The necessary conditions that the optimal solution must satisfy are

$$\frac{\partial G}{\partial \dot{\mathbf{q}}} = \mathbf{0} \quad , \quad \text{that is,} \quad 2\mathbf{W}\dot{\mathbf{q}} - \mathbf{J}^T\boldsymbol{\lambda} = \mathbf{0} \tag{3-45}$$

and

$$\frac{\partial G}{\partial \boldsymbol{\lambda}} = 0 \quad , \quad \text{that is,} \quad \mathbf{J}\dot{\mathbf{q}} - \dot{\mathbf{p}} = 0 \tag{3-46}$$

which is of course identical to (3-40). Now matrix $\mathbf{W}$ is positive definite, hence invertible. Thus, we obtain from equation (3-45)

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{W}^{-1}\mathbf{J}^T\boldsymbol{\lambda} \tag{3-47}$$

Substituting the above into (3-46) yields

$$(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)\boldsymbol{\lambda} = 2\dot{\mathbf{p}} \tag{3-48}$$

Since J is assumed to be of full row-rank, matrix product $\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T$ is a full-rank square matrix, and is therefore invertible. Eliminating the Lagrange multiplier vector $\boldsymbol{\lambda}$ in equations (3-47) and (3-48), we obtain the optimal solution

$$\dot{\mathbf{q}} = \underbrace{\mathbf{W}^{-1}\mathbf{J}^T(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}}\dot{\mathbf{p}} \tag{3-49}$$

→ weighted pseudo inverse solution (J_W^+)

Clearly, the above solution satisfies the original velocity relationship (3-40). Indeed, we can obtain equation (3-40) by premultiplying equation (3-49) by the Jacobian matrix J. When the weighting matrix W is the $m \times m$ identity matrix, the above solution reduces to

Instantaneous

$$\dot{\mathbf{q}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\mathbf{p}} \tag{3-50}$$

√ velocity norm minimization solution

The matrix product $\mathbf{J}^{+} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}$ is known as the *pseudo-inverse* of the Jacobian matrix.

$$\dot{q} = J_W^{+} \dot{p} \quad \leftarrow \quad \dot{p} = J\dot{q}$$

$$J_W^{+} = W^{-1}J^{T}(JW^{-1}J^{T})^{-1}$$

when $W = I, \quad J^{+} = J^{T}(JJ^{T})^{-1}$

$W = M(\theta), \quad J_M^{+} = M^{-1}J^{T}(JM^{-1}J^{T})^{-1}$: Kinetic energy minimization

$$G(\dot{\theta}) = \dot{q}^{T}M(\theta)\dot{q}$$

$$JJ^{+} = JJ^{T}(JJ^{T})^{-1} = I \quad \text{or} \quad JJ_W^{+} = JW^{-1}J^{T}(JW^{-1}J^{T})^{-1} = I$$

$$JJ^{+}J = JI_{n\times n} \Rightarrow J(I_{n\times n} - J^{+}J) = [0]_{m\times n}$$

$$J\underbrace{(I_{n\times n} - J^{+}J)\,\varepsilon_{n\times 1}}_{k\dot{q}_0} = 0_{m\times 1}$$

$$\therefore \quad \dot{q} = J^{+}\dot{p} + (I_{n\times n} - J^{+}J)\underline{\varepsilon}$$ ▲ general intantaneous inverse kinematics ①

cf) $\ddot{q} = J^{+}(\ddot{p} - \dot{J}\dot{q}) + (I_{n\times n} - J^{+}J)\underline{\dot{\varepsilon}}$ → acceleration level inverse kinematics ②

Ex)

① : singularity avoidance scheme (Yoshikawa)

② : torque optimization scheme (Kang)