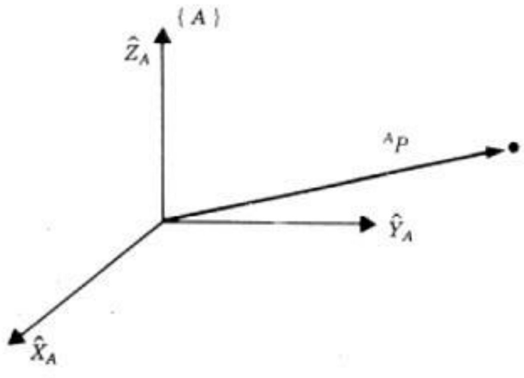
Ch. 2

- Description: positions, orientations, and frames

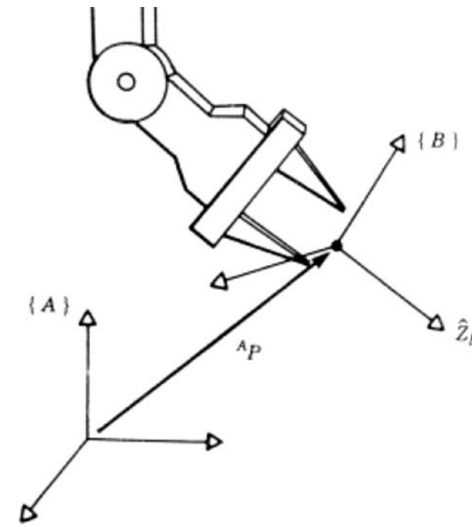
  - Description of a position



$$^AP = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

  - Description of orientation

A reference coordinate frame {A}
A body fixed coordinate frame {B}



,

- In order to describe the orientation of body, need to find a description of the coordinate system {B} relative to the reference coordinate system {A}.

$$\{A\}-\left(\hat{x}_A, \hat{y}_A, \hat{z}_A\right) \qquad \{B\}-\left(\hat{x}_B, \hat{y}_B, \hat{z}_B\right)$$

- Now, define "*rotation matrix*"

$$\{A\} \rightarrow \{B\} \qquad {}^A_BR = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\{B\} \rightarrow \{A\} \qquad {}^B_AR = \begin{bmatrix} {}^B\hat{x}_A & {}^B\hat{y}_A & {}^B\hat{z}_A \end{bmatrix}$$

- Meaning of ${}^{A}_{B}R$: **Direction Consine**

  Projects the unit vector of {B} on {A}

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} \hat{X}_{B}.\hat{X}_{A} & \hat{Y}_{B}.\hat{X}_{A} & \hat{Z}_{B}.\hat{X}_{A} \\ \hat{X}_{B}.\hat{Y}_{A} & \hat{Y}_{B}.\hat{Y}_{A} & \hat{Z}_{B}.\hat{Y}_{A} \\ \hat{X}_{B}.\hat{Z}_{A} & \hat{Y}_{B}.\hat{Z}_{A} & \hat{Z}_{B}.\hat{Z}_{A} \end{bmatrix} $$

In same way,

$$ {}^{B}_{A}R = \begin{bmatrix} {}^{B}\hat{X}_{A} & {}^{B}\hat{Y}_{A} & {}^{B}\hat{Z}_{A} \end{bmatrix} = \begin{bmatrix} \hat{X}_{A}.\hat{X}_{B} & \hat{Y}_{A}.\hat{X}_{B} & \hat{Z}_{A}.\hat{X}_{B} \\ \hat{X}_{A}.\hat{Y}_{B} & \hat{Y}_{A}.\hat{Y}_{B} & \hat{Z}_{A}.\hat{Y}_{B} \\ \hat{X}_{A}.\hat{Z}_{B} & \hat{Y}_{A}.\hat{Z}_{B} & \hat{Z}_{A}.\hat{Z}_{B} \end{bmatrix} = {}^{A}_{B}R^{T} $$

$$ {}^{B}_{A}R . {}^{A}_{B}R = {}^{A}_{B}R^{T}\, {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{x}_{B}^{T} \\ {}^{A}\hat{y}_{B}^{T} \\ {}^{A}\hat{z}_{B}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} & {}^{A}\hat{z}_{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times 3} $$

$$ {}^{A}_{B}R^{T} = {}^{A}_{B}R^{-1} = {}^{B}_{A}R $$

$\rightarrow$ Roation matrix is "orthogonal matrix".

$$ R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

9 parameters $\rightarrow$ 3 independent parameters

- Since R is orthogonal matrix, 6 constraints exist

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = 1$$

$$r_{21}^2 + r_{22}^2 + r_{23}^2 = 1$$

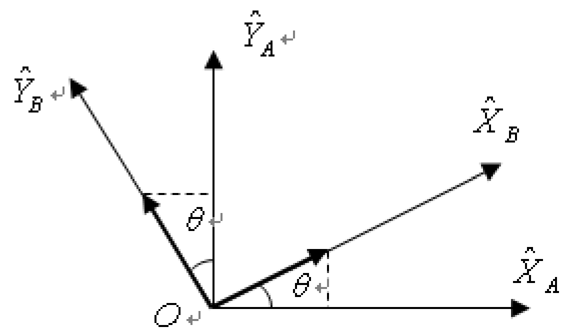
$$r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$$

$$r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} = 0$$

$$r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} = 0$$

$$r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} = 0$$

Example,



$${}^{A}\hat{X}_B = \cos\theta.{}^{A}\hat{X}_A + \sin\theta.{}^{A}\hat{Y}_A$$

$${}^{A}\hat{Y}_B = -\sin\theta.{}^{A}\hat{X}_A + \cos\theta.{}^{A}\hat{Y}_A$$

$${}^{A}\hat{Z}_B = {}^{A}\hat{Z}_A$$

Builds the rotation matrix

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}\hat{X}_A = \cos\theta.{}^{B}\hat{X}_B - \sin\theta.{}^{B}\hat{Y}_B$$

$${}^{B}\hat{Y}_A = \sin\theta.{}^{B}\hat{X}_B + \cos\theta.{}^{B}\hat{Y}_B$$

$${}^{B}\hat{Z}_A = {}^{B}\hat{Z}_B$$

$${}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.3. **Mappings**: Changing descriptions from frame to frame

- Translation

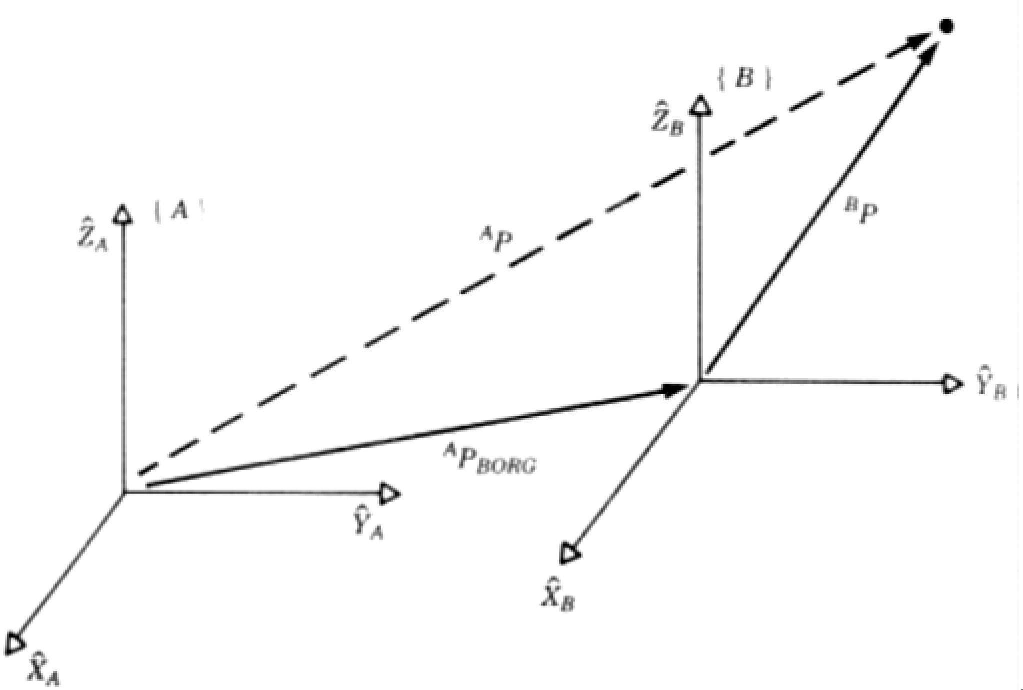


Figure 2.4. Translational mapping.

$$^AP = {}^AP_{BORG} + {}^BP$$

- Rotation

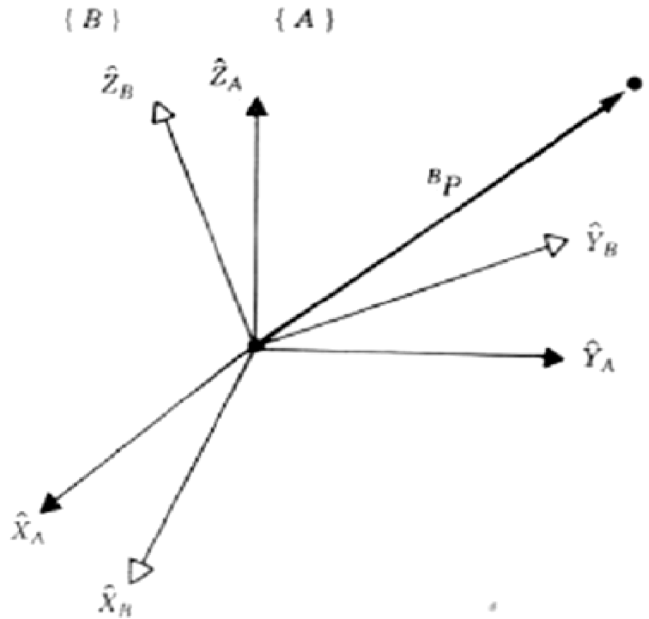


Fig. Rotating the description of a vector.

$$ {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} $$

$$ {}^Ap_x = {}^B\hat{X}_A \cdot {}^BP, $$
$$ {}^Ap_y = {}^B\hat{Y}_A \cdot {}^BP, $$
$$ {}^Ap_z = {}^B\hat{Z}_A \cdot {}^BP. $$

$$ \therefore \quad {}^AP = {}^A_BR \cdot {}^BP $$

Ex 2.1)

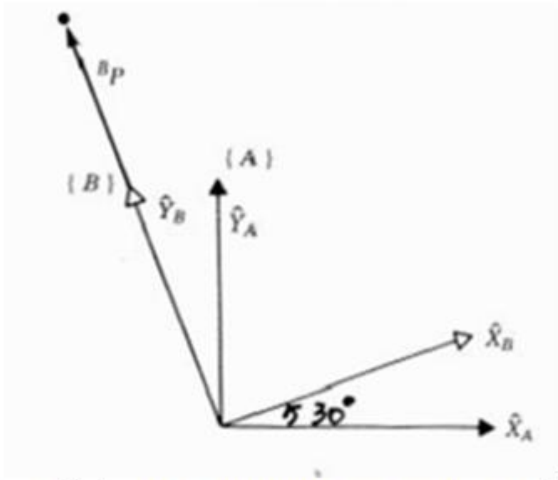


Fig. $\{B\}$ rotated 30 degree about $\hat{Z}$.

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \tag{2.14} $$

Given

$$ {}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} \tag{2.15} $$

We calculate ${}^{A}P$ as

$$ {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} \tag{2.16} $$

- Mapping involving general frames

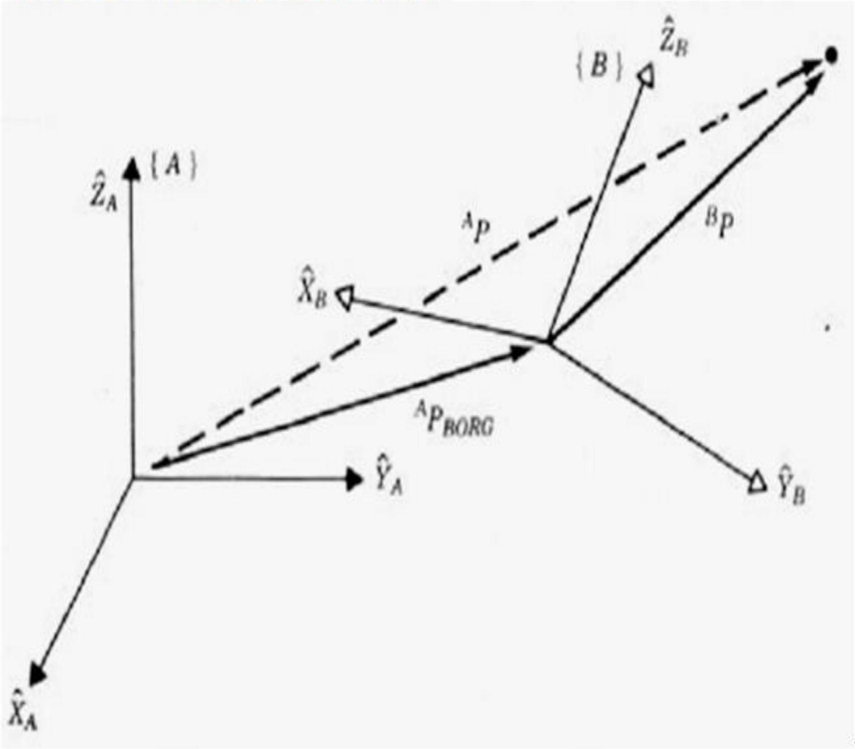


Figure 2.7. General transform of a vector.

$$ {}^{A}P = {}^{A}P_{BORG} + {}^{A}({}^{B}P) = {}^{A}P_{BORG} + {}^{A}_{B}R \cdot {}^{B}P $$

$$ \begin{bmatrix} {}^{A}P \\ --- \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ --- & --- & --- & --- \\ 0 & 0 & 0 & \end{array}\right] \begin{bmatrix} {}^{B}P \\ --- \\ 1 \end{bmatrix} $$

$$ = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ --- \\ 1 \end{bmatrix} $$: Homogeneous transformation.

Ex 2.2)

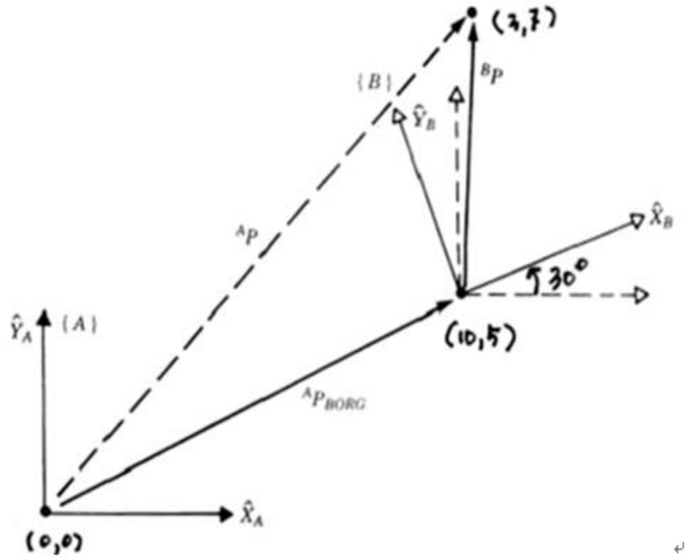


Fig. Frame $\{B\}$ rotated and translated.

The definition of frame $\{B\}$ is

$$ {}^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

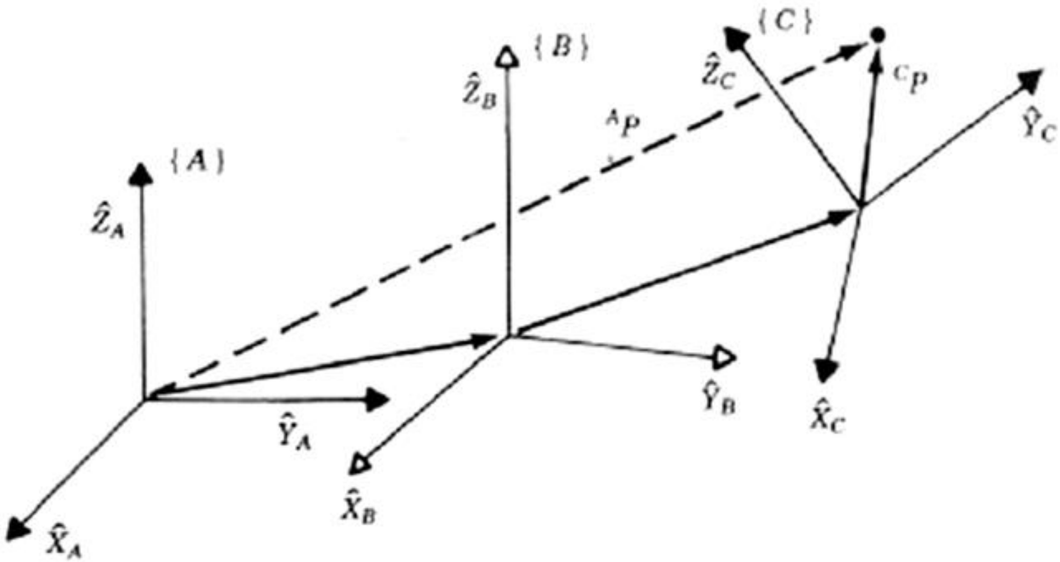
Given

$$ {}^BP = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix} $$

We use the definition of $\{B\}$ given above as a transformation,

$$ {}^AP = {}^A_BT \cdot {}^BP = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix} $$

- Compound transformation



Known : ^{C}P and Transformations

Find : ^{A}P

$$^{B}P = {}^{B}_{C}T\,{}^{C}P,$$

From which we could define

$$^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T.$$

And then transform ^{B}P into ^{A}P as

$$^{A}P = {}^{A}_{B}T\,{}^{B}P.$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P,$$

$$^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R\,{}^{B}_{C}R & & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$
{}^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R\,{}^{B}_{C}R & & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

- Inverting Transform

Want to compute ${}^{B}_{A}T$

1) Simple way: ${}^{A}P = {}^{A}_{B}T\,{}^{B}P \qquad {}^{B}P = {}^{A}_{B}T^{-1}\,{}^{A}P = {}^{B}_{A}T\,{}^{A}P$

${}^{A}_{B}T = {}^{B}_{A}T^{-1}$: 4×4 matrix inversion: cumbersome

2) Efficient way

$$ {}^{A}_{B}T\,{}^{B}_{A}T = I_{4\times4} $$

$$ \begin{bmatrix} {}^{A}_{B}R & | & {}^{A}P_{BORG} \\ --- & | & --- \\ \underline{0} & | & 1 \end{bmatrix} \begin{bmatrix} A & B \\ & \\ C & D \end{bmatrix} = \begin{bmatrix} I & | & \underline{0} \\ --- & | & --- \\ \underline{0} & | & 1 \end{bmatrix} $$

$$ {}^{A}_{B}RA + {}^{A}P_{BORG}C = I \qquad (1) $$

$$ {}^{A}_{B}RA + {}^{A}P_{BORG}C = 0 \qquad (2) $$

$$ C = 0 $$

$$ D = 1 $$

From (2) $\quad {}^{A}P_{BORG} = -{}^{A}_{B}RB \qquad \therefore B = -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG}$

From (1) $\quad A = {}^{A}_{B}R^{T}$

$$ \therefore \quad {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & | & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ --- & | & ------ \\ \underline{0} & | & 1 \end{bmatrix} $$

Ex 2.5)

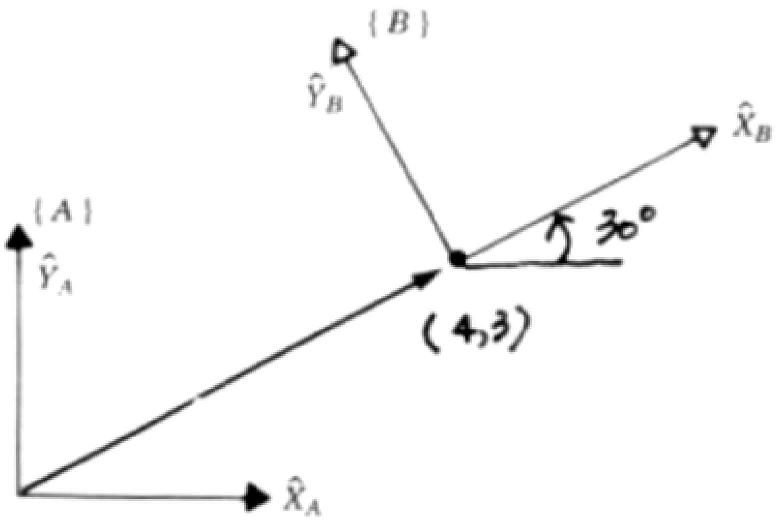
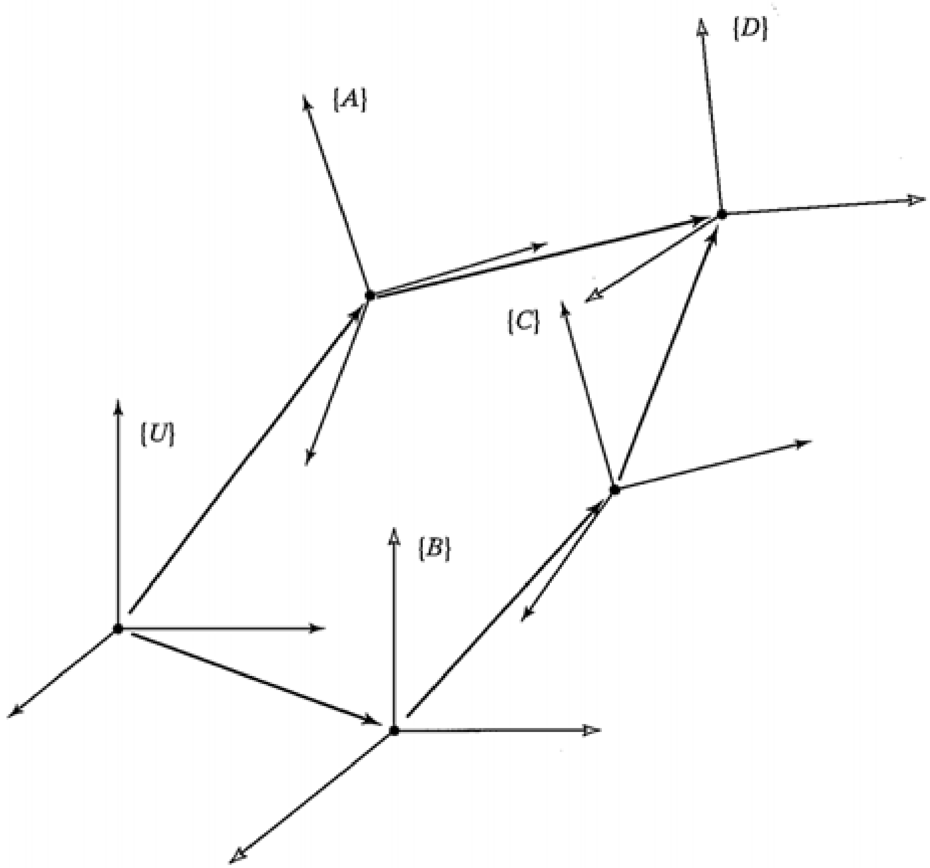


Figure2.13. {B} relative to {A}

$^{A}_{B}T = ?$ $^{B}_{A}T = ?$

2.7 Transform Equations:



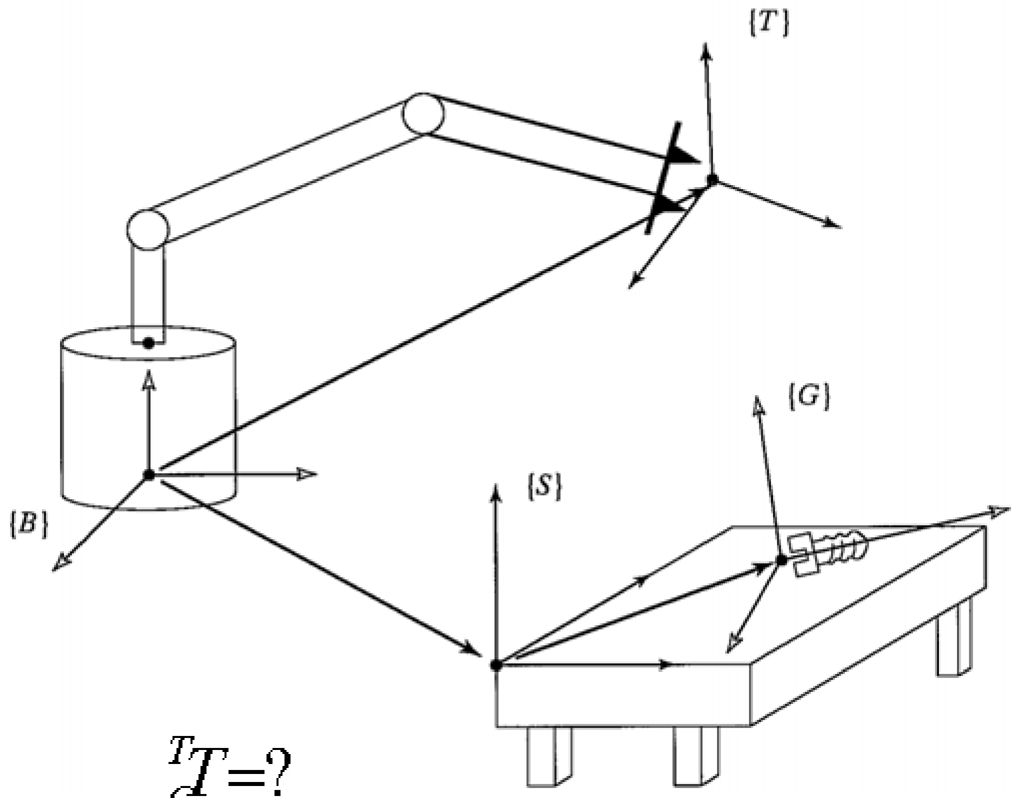
$$ {}^{U}_{D}T = {}^{U}_{A}T\, {}^{A}_{D}T $$

$$ {}^{U}_{D}T = {}^{U}_{B}T\, {}^{B}_{C}T\, {}^{C}_{D}T $$

$$ {}^{U}_{A}T\, {}^{A}_{D}T = {}^{U}_{B}T\, {}^{B}_{C}T\, {}^{C}_{D}T $$

Fig. Set of transforms forming a loop

Ex 2.6)



$${}^{T}_{G}T = ?$$

Fig. Manipulator reaching for a bolt.

2.8 More on representation of Orientation:

$$R = [\hat{X} \quad \hat{Y} \quad \hat{Z}]$$

there are six constraints on the nine matrix elements:

$$|\hat{X}| = 1$$

$$|\hat{Y}| = 1$$

$$|\hat{Z}| = 1$$

$$\hat{X} \bullet \hat{Y} = 0$$

$$\hat{X} \bullet \hat{Z} = 0$$

$$\hat{Y} \bullet \hat{Z} = 0$$

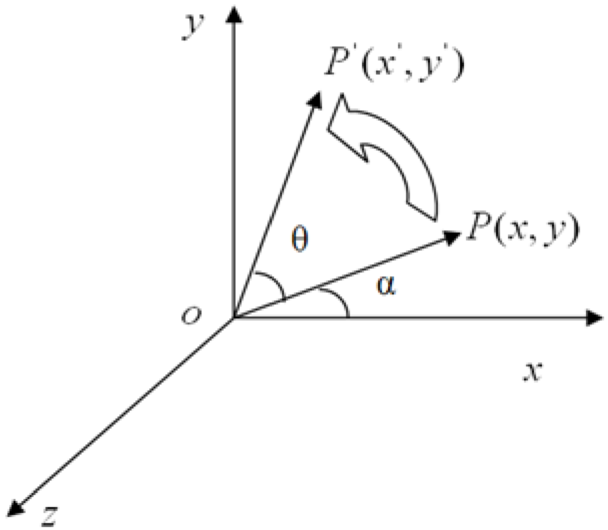
9 parameters $\rightarrow$ 6 constraints $\rightarrow$

3 independence Parameters $\rightarrow$

"3 parameter expression for rotation matrix"

- **Point Rotation VS. Coordinate Rotation**

- Point Rotation



rotate P about z to get P'

$$|\overline{OP}| = l$$

$$x = l\cos\alpha,\ y = l\sin\alpha$$

$$x' = l\cos(\theta+\alpha),\ y' = l\sin(\theta+\alpha)$$

$$x' = l\cos\theta\cos\alpha - l\sin\theta\sin\alpha$$

$$y' = l\sin\theta\cos\alpha + l\cos\theta\sin\alpha$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Likewise, rotate P' about x to get P''

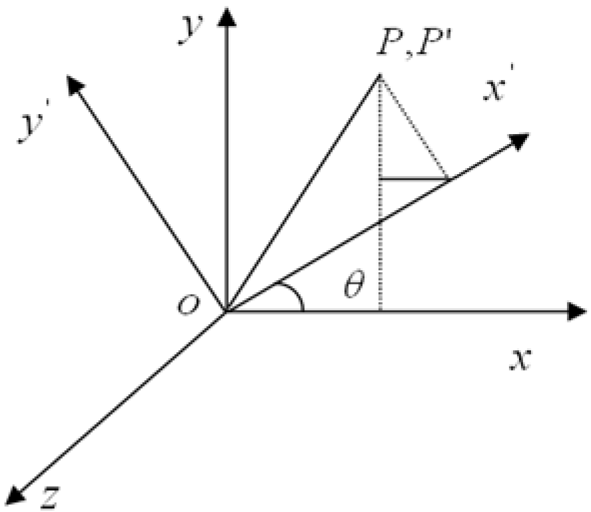
$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

To get the result after two consecutive rotations

$$\underbrace{\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}}_{\text{Unknown}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{Known}}$$

Don't bother with this!

- Body fixed Coordinate Rotation



$$P' = (x', y'), P = (x, y)$$
$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$
$$z = z'$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$${}^{A}P = {}^{A}_{B}R . {}^{B}P$$

- $x^{'}, y^{'}, z^{'}$ ....

$$\underbrace{P^{''} = (x^{''}, y^{''})}_{\textbf{Known}}, P^{'} = (x^{'}, y^{'})$$

$$\begin{bmatrix} x^{'} \\ y^{'} \\ z^{'} \end{bmatrix} = {}^{B}_{C}R \begin{bmatrix} x^{''} \\ y^{''} \\ z^{''} \end{bmatrix}$$

- $x^{'}, y^{'}, z^{'}$ ...

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\textbf{Unknow}} = {}^{A}_{B}R\,{}^{B}_{C}R \underbrace{\begin{bmatrix} x^{''} \\ y^{''} \\ z^{''} \end{bmatrix}}_{\textbf{Known}}$$

- **X-Y-Z fixed angle rotation**

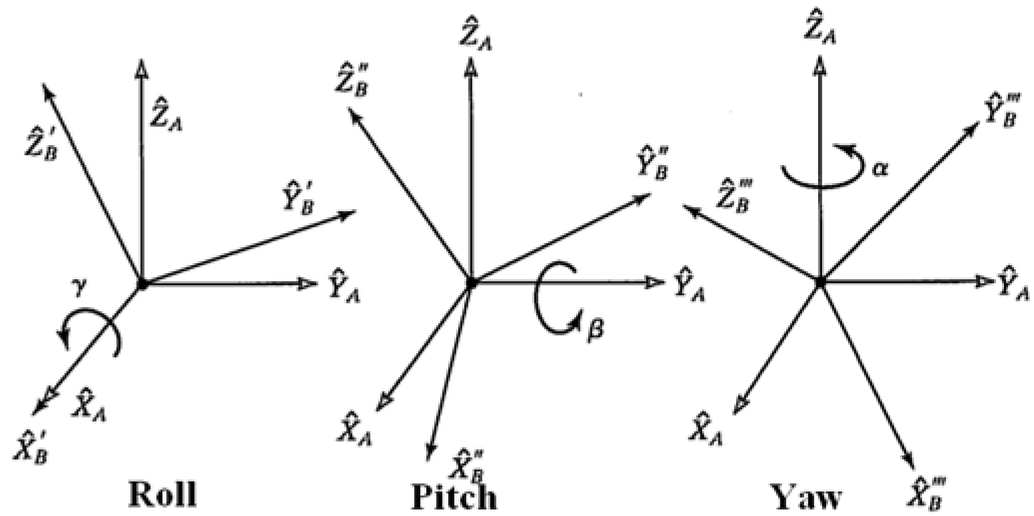


FIGURE 2.17 X-Y-Z fixed angles. Rotations are performed in the order

$R_X(\gamma), R_Y(\beta), R_Z(\alpha)$

$\{A\} \longrightarrow \{B\}$

Rotation γ about $\hat{X}_A$: Roll

Rotation β about $\hat{Y}_A$: Pitch

Rotation α about $\hat{Z}_A$: Yaw

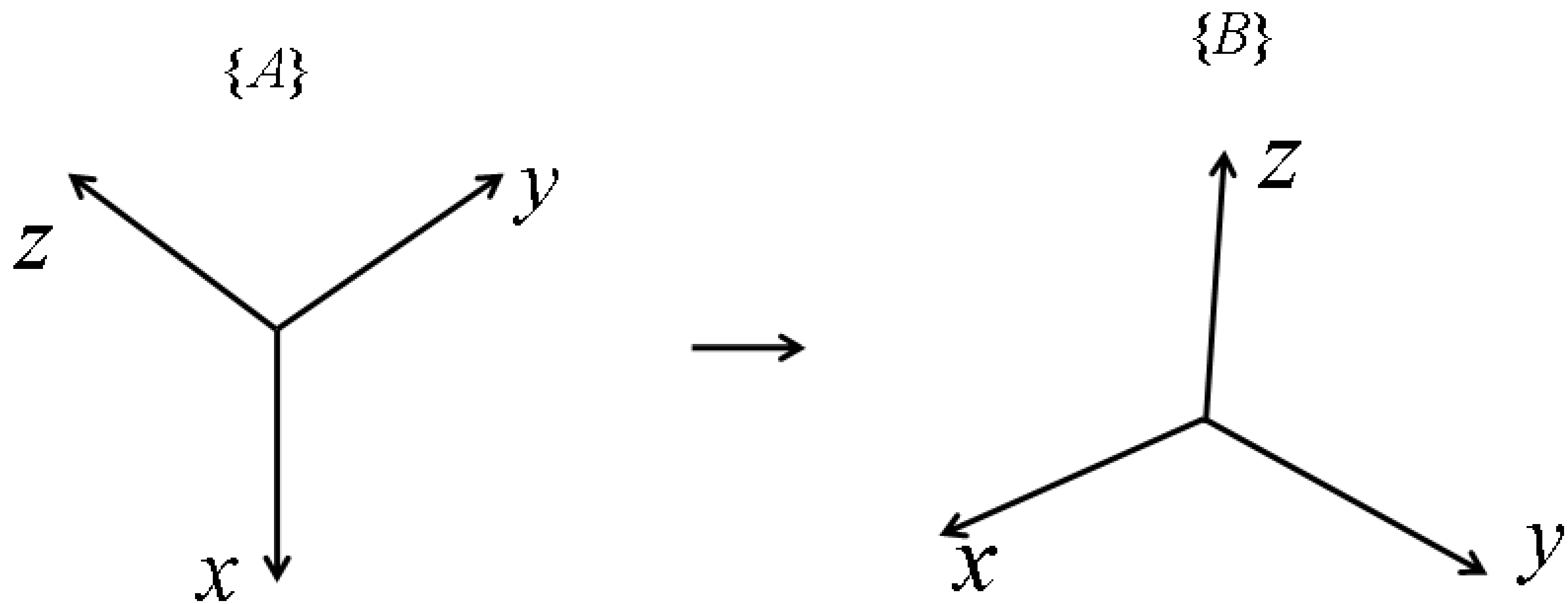
- Fixed axes….

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(\alpha)\,R_Y(\beta)\,R_X(\gamma)
$$

$$
=\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\alpha \\ 0 & s\alpha & c\gamma \end{bmatrix}
$$

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta s\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

- **Inversion of x-y-z fixed angle rotation**
  - Rotation matrix x-y-z fixed angle
  - Why we need this?



$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta s\gamma \end{bmatrix}
$$

Find: α, β, γ

- Procedure:

$$r_{11}^2 + r_{21}^2 = c\beta^2(c\alpha^2 + s\alpha^2) = c\beta^2 \;\rightarrow\; c\beta = \pm\sqrt{r_{11}^2 + r_{21}^2}$$

$$\rightarrow\; c\beta = \sqrt{r_{11}^2 + r_{21}^2} \qquad -90^\circ \le \beta \le 90^\circ$$

- $r_{31} = s\beta$

$$\tan\beta = \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}} \;\rightarrow\; \beta = A\tan^2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

Def) $A\tan 2(x, y) = \tan^{-1}(\frac{y}{x})$ with signed x,y

$$A\tan 2(-2,-2) = -135^\circ$$

$$A\tan 2(2,2) = 45^\circ$$

$$r_{11} = c\alpha c\beta \qquad c\alpha = \frac{r_{11}}{c\beta}$$

$$r_{21} = s\alpha s\beta \qquad s\alpha = \frac{r_{21}}{s\beta}$$

$$\rightarrow \tan\alpha = \frac{r_{11}/c\beta}{r_{21}/s\beta}$$

- $\alpha = A\tan 2(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta})$

$$r_{32} = c\beta s\gamma \qquad s\gamma = \frac{r_{32}}{c\beta}$$

$$r_{33} = c\beta c\gamma \qquad c\gamma = \frac{r_{33}}{c\beta}$$

$$\rightarrow \tan\gamma = \frac{r_{32}/c\beta}{r_{33}/s\beta}$$

- $\gamma = A\tan 2(\frac{\gamma_{32}}{c\beta}, \frac{\gamma_{33}}{c\beta})$

when $\beta = \pm 90^\circ$ (singularity)

$$\beta = 90.0^\circ$$

$$\alpha = 0.0$$

$$\gamma = A\tan 2(r_{12}, r_{22})$$

- **Z-Y-X Euler angles**

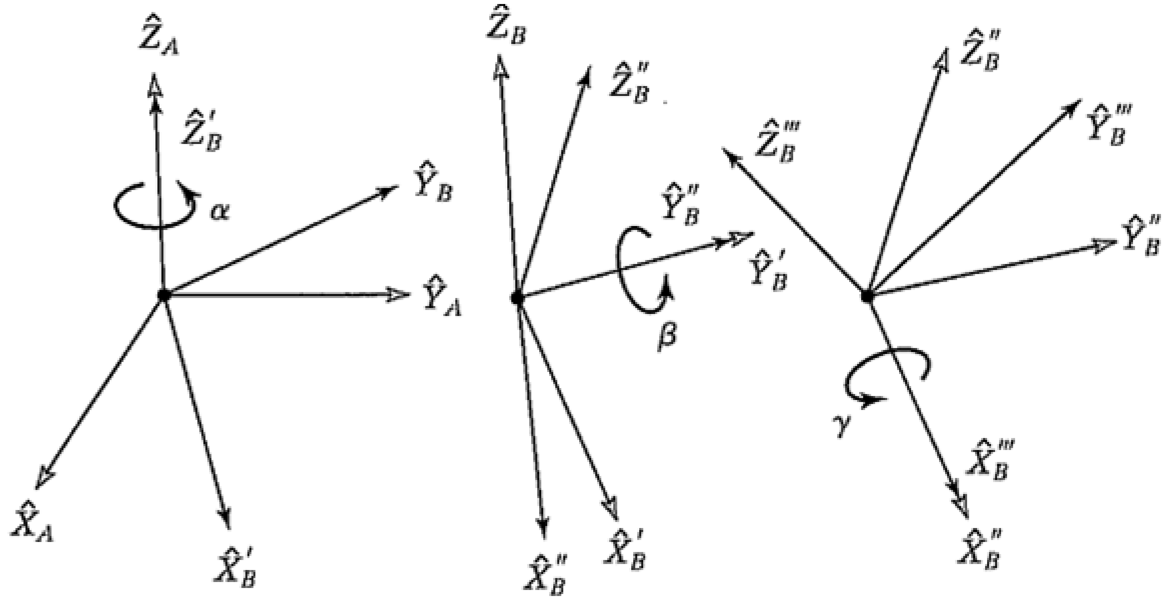


Figure 2.18. Z-Y-X Euler angles.

Rotation α about $\hat{z}_A$ $\rightarrow$ $^{A}P = {}^{A}_{A}R_Z(\alpha)\,{}^{A}P$

Rotation β about $\hat{Y}_A$ $\rightarrow$ $^{A}P = {}^{A}_{A'}R_Y(\beta)\,{}^{A'}P$

Rotation γ about $\hat{x}_{A'}$ $\rightarrow$ $^{A'}P = {}^{A}_{A'}R_x(\gamma)\,{}^{B}P$

$$^{A}P = {}^{A}_{A}R_Z(\alpha)\,{}^{A'}_{A'}R_y(\beta)\,{}^{A'}_{B}R_x(\gamma)\,{}^{B}P$$

$$
{}^{A}_{B}R_{Z'Y'X'} = R_Z(\alpha)R_Y(\beta)R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \qquad (2.70)
$$

Where $c\alpha = \cos\alpha$ and $s\alpha = \sin\alpha$, etc. Multiplying out, we obtain

$$
{}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \qquad (2.71)
$$

$$
= {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) \qquad (roll - pitch - yaw)
$$

Inversion Prob. is the same as XYZ fixed angle case

- **Z-Y-Z Euler angles**

Rotation α about $\hat{Z}_A$

Rotation β about $\hat{Y}_{A}$

Rotation γ about $\hat{Z}_{A'}$

$$ {}^{A}_{B}R_{ZYZ}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma & c\alpha c\beta \\ s\alpha c\beta c\gamma & -s\alpha c\beta s\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} $$

If $\beta = 0.0$, then a solution may be calculated as

$$ \beta = 0.0 $$
$$ \alpha = 0.0 $$
$$ \gamma = A\tan 2(-r_{12}, r_{11}) $$

If $\beta = 0.0$, then a solution may be calculated as

$$ \beta = 180.0 $$
$$ \alpha = 0.0 $$
$$ \gamma = A\tan 2(r_{12}, -r_{11}) $$

➢ Inversion Prob.

$$ {}^{A}_{B}R_{ZYZ}(\alpha,\beta,\gamma)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

If $\sin\beta \neq 0$, then

$$ \beta = A\tan 2(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}) $$
$$ \alpha = A\tan 2(r_{23}/s\beta, r_{13}/s\beta) $$
$$ \gamma = A\tan 2(r_{32}/s\beta, -r_{31}/s\beta) $$

- Other angle set conventions

  Fixed angle set

  $$R_{XYZ}(\gamma, \beta, \alpha)$$

  $$R_{XZY}(\gamma, \beta, \alpha)$$

  $$R_{XYX}(\gamma, \beta, \alpha)$$

  $$R_{XZX}(\gamma, \beta, \alpha)$$

  $$\begin{array}{ccccc} X & — & Y & — & X \\ & & Z & & Z \end{array}$$

  Y

  Z

  $$3 \times 2 \times 2 = 12$$

  Euler angle set

  12 conventions are for fixed angle sets, and 12 are for Euler angle sets.

Note that because of the duality of fixed angle sets and Euler angle sets, there are really only 12 unique sets exist.

12오일러 각도법은 다음과 같다.

$$R_{X'Y'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}$$

$$R_{X'Z'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{Y'X'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix}$$

$$R_{Y'Z'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{Z'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix}$$

$$R_{Z'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$R_{X'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}$$

$$R_{X'Z'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{Y'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}$$

$$R_{Y'Z'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{Z'X'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}$$

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

12고정 각도법은 다음과 같다.

$$R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$R_{XZY}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{YXZ}(\gamma,\beta,\alpha)=\begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix}$$

$$R_{YZX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{ZXY}(\gamma,\beta,\alpha)=\begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix}$$

$$R_{ZYX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}$$

$$R_{XYX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}$$

$$R_{XZX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$R_{YXY}(\gamma,\beta,\alpha)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}$$

$$R_{YZY}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

Equivalent Angle-Axis

3 angle parameter $(\alpha, \beta, \gamma) \longrightarrow \hat{k}, \theta$ *with* $k_x^2 + k_y^2 + k_z^2 = 1$

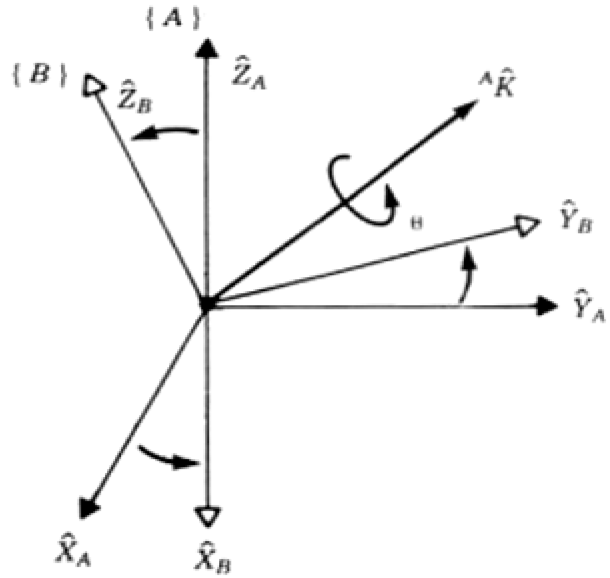


Fig. Equivalent angle-axis representation.

$$R_k(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

Where $c\theta = cos\theta, s\theta = sin\theta, v\theta = 1 - cos\theta$, and $\hat{K} = [k_x \quad k_y \quad k_z]^T$.

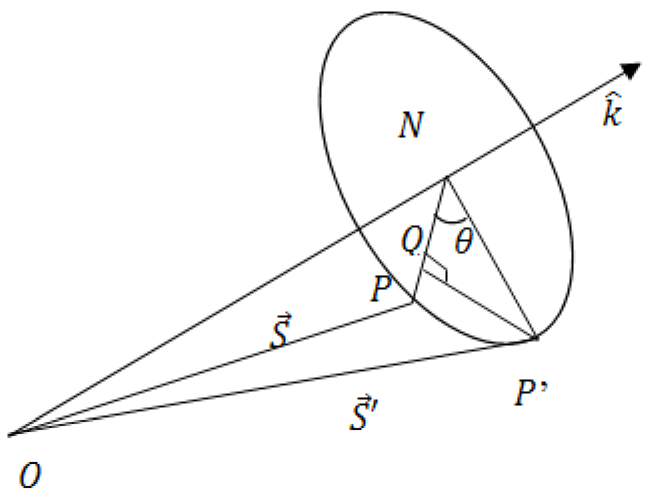
Inversion problem

$$ {}^{A}_{B}R_{K}(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ \theta = Acos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) $$

$$ \hat{K} = \frac{1}{2sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} $$

Euler Parameters (Quaternion) : Rodrigue's Formula



Given $\vec{S}$

Unit vector $\hat{k}$

Rotation by θ about $\hat{k}$

Find $\vec{S}'$ in terms of $\vec{S}$, $\hat{k}$, θ

$$\overrightarrow{S'} = \overrightarrow{ON} + \overrightarrow{NQ} + \overrightarrow{QP'}$$

$$\overrightarrow{ON} = (\vec{S}.\hat{k})\hat{k} \qquad \textbf{(1)}$$

$$\overrightarrow{NP} = \vec{S} - \overrightarrow{ON} = \vec{S} - (\vec{S}.\hat{k})\hat{k}$$

$$|\overrightarrow{NP}| = |\overrightarrow{NP'}|, \qquad |\overrightarrow{NQ}| = |\overrightarrow{NP'}|cos\theta = |\overrightarrow{NP}|cos\theta$$

$$NQ = [\vec{S} - (\vec{S}.\hat{k})\hat{k}]cos\theta \qquad \textbf{(2)}$$

$$|\hat{k} \times \vec{S}| = |\overrightarrow{NP}|$$

$$\overrightarrow{QP'} = (\hat{k} \times \vec{S})sin\theta \qquad \textbf{(3)}$$

From (1), (2) and (3)

$$\overrightarrow{S'} = (\vec{S}.\hat{k})\hat{k} + [\vec{S} - (\vec{S}.\hat{k})\hat{k}]cos\theta + (\hat{k} \times \vec{S})sin\theta$$

$$\overrightarrow{S'} = \vec{S}cos\theta + k(\vec{S}.\hat{k})(1 - cos\theta) + (\hat{k} \times \vec{S})sin\theta$$

$$\overrightarrow{S'} = \vec{S}cos\theta + k(\vec{S}.\hat{k})(1 - cos\theta) + (\hat{k} \times \vec{S})sin\theta$$

Define:

$$\vec{e} = \hat{k}sin\frac{\theta}{2}$$

$$e_0 = cos\frac{\theta}{2}$$

$$cos\theta = 2cos^2\frac{\theta}{2} - 1, \quad sin\theta = 2\,sin\frac{\theta}{2}cos\frac{\theta}{2}, \quad 1 - cos\theta = 2sin^2\frac{\theta}{2}$$

$$\overrightarrow{S'} = \vec{S}cos\theta + k(\vec{S}.\hat{k})(1 - cos\theta) + (\hat{k} \times \vec{S})sin\theta$$

$$cos\theta = 2e_0^2 - 1, \quad \hat{k} = \frac{\vec{e}}{sin\frac{\theta}{2}}, \quad \vec{S}.\hat{k} = \frac{\vec{e}.\vec{S}}{sin\frac{\theta}{2}}$$

$$\overrightarrow{S'} = (2e_0^2 - 1)\vec{S} + 2\vec{e}(\vec{e}.\vec{S}) + 2e_0\vec{e} \times \vec{S}$$

$$\vec{S'} = (2e_0^2 - 1)\vec{S} + 2\vec{e}\left(\vec{e}.\vec{S}\right) + 2e_0\vec{e} \times \vec{S}$$

Where $\vec{e}\times = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$

$$\vec{S'} = [(2e_0^2 - 1)I + 2\vec{e}\vec{e}^T + 2e_0\vec{e}\times]\vec{S}$$

$\vec{S'} = RS$: Point rotation matrix

$$R = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

Quaternion expression for the rotation

Let $\vec{u}$ be a unit vector (the rotation axis) and let $q = \cos\frac{\alpha}{2} + \vec{u}\sin\frac{\alpha}{2}$.

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})\sin\frac{\theta}{2}$$

$$\mathbf{q}^{-1} = e^{-\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} - (u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})\sin\frac{\theta}{2}.$$

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

Quaternion-derived rotation matrix [edit]

A quaternion rotation $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ (with $\mathbf{q} = q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$) can be algebraically manipulated into a matrix rotation $\mathbf{p}' = \mathbf{R}\mathbf{p}$, where R is the rotation matrix given by:[7]

$$\mathbf{R} = \begin{bmatrix} 1-2s(q_j^2+q_k^2) & 2s(q_iq_j-q_kq_r) & 2s(q_iq_k+q_jq_r) \\ 2s(q_iq_j+q_kq_r) & 1-2s(q_i^2+q_k^2) & 2s(q_jq_k-q_iq_r) \\ 2s(q_iq_k-q_jq_r) & 2s(q_jq_k+q_iq_r) & 1-2s(q_i^2+q_j^2) \end{bmatrix}$$

Here $s = ||q||^{-2}$ and if q is a unit quaternion, $s = 1$

2.10 Computational Considerations

Order of Multiplication

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P $$

i) $ {}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P $

$$ (3 \qquad 2) \times 9 = 21 \qquad 18 $$

$$ (3 \qquad 2) \times 9 = 21 \qquad 18 $$

$$ (3 \qquad 2) \times 3 = 9 \qquad 6 $$

Total 63 Muls and 42 adds

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P $$

$$ (3 \qquad 2) \times 3 \times 3 = 21\,Muls \qquad 18\,adds $$