Random Variable

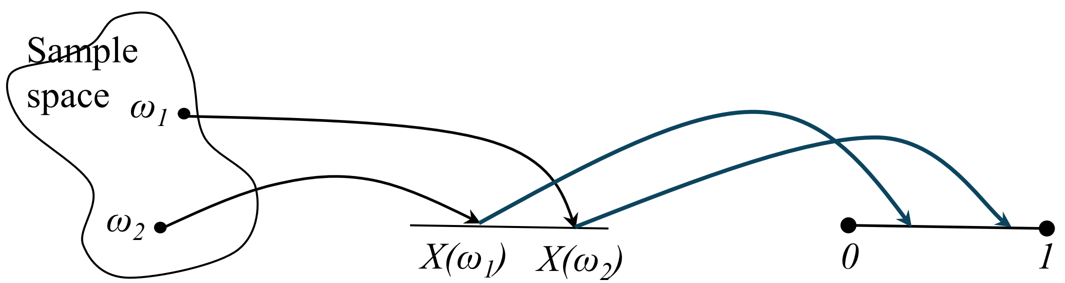
Random Variable Definition

- It frequently occurs that in performing an experiment we are mainly interested in some functions of the outcome as opposed to the outcome itself
  - Rotary, casino, game, etc.

- These quantities of interest, or more formally, these real-valued functions defined on the sample space, are known as random variables (RVs).

- More mathematical expression of RVs
  - Given a probability space (S, F, P), a random variable is a measurable function (mapping) from S to the real line X: S → R
    - X({H}) = 100, X({H}) = -50
    - X({1,2}) = 100, X({3,4,5,6}) = -50

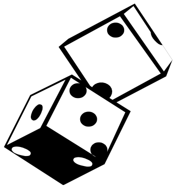
- Since the value of a random variable is defined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable

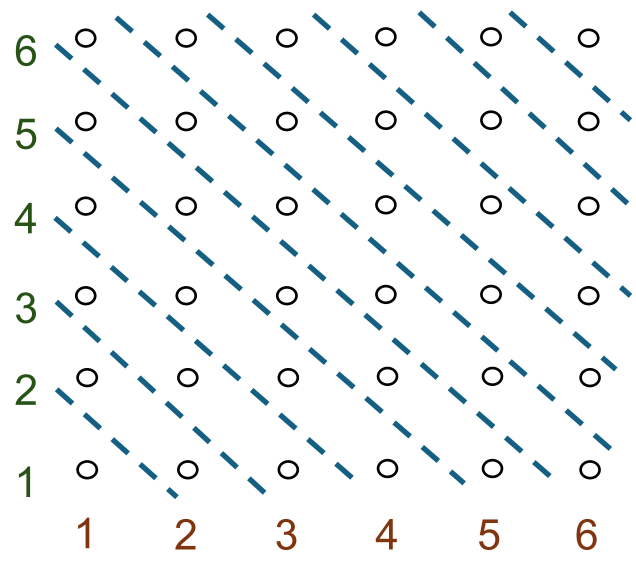


Random Variable: Examples

- Ex 2.1] Letting X denote the random variable that is defined as the sum of two fair dice
  - The outcomes of two dice = (ω_1, ω_2)
  - RV $X(\omega_1, \omega_2) = \omega_1 + \omega_2$
  - Possible values of X = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
  - Probabilities to the possible values of the random variable
    - P(X=2)=P{(1,1)}=1/36
    - P(X=3)=P{(1,2), (2,1)}=2/36

      ...
    - P(X=11)=P{(5,6), (6,5)}=2/36
    - P(X=12)=P{(6,6)}=1/36
    - $1 = P\{\cup_{n=2}^{12}\{X = n\}\}$
      $= \sum_{n=2}^{12} P(X = n)$





- Ex.2.3] Suppose that we toss a coin having a probability p of coming up heads, until the first head appears. Letting N denote the number of flips required, then assuming that the outcome of successive flips are independent, N is a random variable taking on one of the values 1, 2, 3, . . . , with respective probabilities

$$P\{N = 1\} = P\{H\} = p,$$
$$P\{N = 2\} = P\{(T, H)\} = (1 - p)p,$$
$$P\{N = 3\} = P\{(T, T, H)\} = (1 - p)^2p,$$
$$\ldots$$
$$P\{N = n\} = P\{(T, T, \ldots , T, H)\} = (1 - p)^{n-1}p, \; n \geq 1$$

- Indicator random variable

$$I_E(x) = 1 \text{ if } x \in E, \text{ 0 otherwise.}$$

- Ex. 2.5] Suppose that independent trials, each of which results in any of m possible outcomes with respective probabilities $p_1, \ldots, p_m$, $\sum_{i=1}^{m} p_i = 1$, are continually performed. Let X denote the number of trials needed until each outcome has occurred at least once.
  - Instead of $P\{X = n\}$, we will first determine $P\{X > n\}$ the probability that at least one of the outcomes has not yet occurred after n trials.
  - Letting A_i denote the event that outcome i has not yet occurred after n first trials, $n = 1, \ldots, m$, then
  - $P\{X > n\} = P(\bigcup_{i=1}^{m} A_i)$
  $= \sum_{i=1}^{m} P(A_i) - \sum\sum_{i<j} P\left(A_i A_j\right)$
  $+ \sum\sum\sum_{i<j<k} P\left(A_i A_j A_k\right) - \cdots + (-1)^{m+1} P(A_1 \ldots A_m)$

◆ $P\{A_i\}$ is the probability that each of the first *n* trials results in a non-*i* outcome

$$P(A_i) = (1 - p_i)^n$$

◆ $P\{A_iA_j\}$ is the probability that the first *n* trials all result in a non-*i* and non-*j* outcome

$$P(A_iA_j) = (1 - p_i - p_j)^n$$

$$P\{X > n\} = \sum_{i=1}^{m}(1 - p_i)^n - \sum_{i<j}\sum(1 - p_i - p_j)^n + \sum_{i<j<k}\sum\sum(1 - p_i - p_j - p_k)^n - \cdots$$

◆ $P\{X = n\} = P\{X > n - 1\} - P\{X > n\}$

$$P\{X = n\} = \sum_{i=1}^{m} p_i(1 - p_i)^{n-1} - \sum_{i<j}\sum(p_i + p_j)(1 - p_i - p_j)^{n-1} + \sum_{i<j<k}\sum\sum(p_i + p_j + p_k)(1 - p_i - p_j - p_k)^{n-1} - \cdots$$

CDF

- If a random variable takes on either a finite or a countable number of possible values, the RV is called discrete.

- If a continuum of possible values, continuous.

- The cumulative distribution function (CDF) F( ) of the random variable X is defined for any real number b, $-\infty<b<\infty$ by $F(b)=P\{X\leq b\}$.

- Properties of CDF F
  1. F(b) is a nondecreasing function of b
  2. $\lim_{b\to\infty}F(b)=F(\infty)=1$
  3. $\lim_{b\to-\infty}F(b)=F(-\infty)=0$

- Ex. $P\{a<X\leq b\}=F(b)-F(a)$ for all $a<b$

Types and Properties of RVs

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

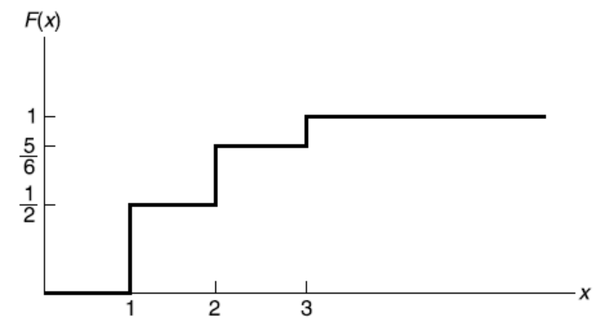
Discrete RVs

- A random variable that can take on at most countable number of possible values

- Probability Mass Function (pmf): for a discrete random variable X, pmf p(a) of X is defined a p(a) = P{X=a}.

- Properties of pmf
  1. $p(x_i)>0$, i=1,2,…
  2. p(x)=0 all other values of x
  3. $\sum_{i=\{1,..,\infty\}} p(x_i)=1$
  4. cdf vs. pmf: $F(a)=\sum_{x_i \le a} p(x_i)$

- Ex. p(1)=1/2, p(2)=1/3, p(3)=1/6
  - $F(a) = \begin{cases} 0 & for\ a < 1 \\ \frac{1}{2} & for\ 1 \le a < 2 \\ \frac{5}{6} & for\ 2 \le a < 3 \\ 1 & for\ 3 \le a \end{cases}$



- Discrete random variables are often classified according to their pmf
  - Ex: Bernoulli RV, Binomial RV, Geometric RV, Poisson RV

Example of Discrete RVs: The Bernoulli RV

- A random variable X is said to be a Bernoulli RV for given sample space $S=\{A,A^c\}$ and some $p \in (0,1)$ if its pmf is given by
  $P\{A\}=p$, $P\{A^c\}=1-p$

- For example, suppose that a trial, or an experiment, whose outcome can be classified as either a "success" or as a "failure" is performed. If we let X equal 1 if the outcome is a success and 0 if it is a failure, then the probability mass function of X is given by
  $p(0) = P\{X = 0\} = 1 - p,$
  $p(1) = P\{X = 1\} = p \qquad (2.2)$
  where p, $0 \le p \le 1$, is the probability that the trial is a "success."

Example of Discrete RVs: The Binomial RV

- Suppose that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability 1−p, are to be performed.

- If X represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p).

- The pmf of a binomial RV with (n,p)
  $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$ for i = 0, 1, . . . , n (2.3)
  where $\binom{n}{i} = \frac{n!}{(n-i)!i!}$.

- $\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + (1-p))^n = 1$

- Ex. 2.7] It is known that any item produced by a certain machine will be defective with probability 0.1, independently of any other item. What is the probability that in a sample of three items, at most one will be defective?
  - If X is the number of defective items in the sample, then X is a binomial random variable with parameters (3, 0.1). Hence, the desired probability is given by
  - P{X = 0} + P{X = 1} =(3,0)(0.1)0(0.9)3+(3,1)(0.1)1(0.9)2 = 0.972

Example of Discrete RVs: The Geometric RV

- Suppose that independent trials, each having probability p of being a success, are performed until a success occurs.

- If we let X be the number of trials required until the first success, then X is said to be a geometric random variable with parameter p.

- Its probability mass function is given by
  $p(n) = P\{X = n\} = (1 - p)^{n-1}p, \; n = 1, 2, \ldots$ (2.4)

- To check that p(n) is a probability mass function, we note that

$$\sum_{n=1}^{\infty} p(n) = p \sum_{n=1}^{\infty} (1 - p)^{n-1} = 1$$

Example of Discrete RVs: The Poisson RV

- A random variable X, taking on one of the values 0, 1, 2, . . . , is said to be a Poisson random variable with parameter λ, if for some λ > 0,
  $p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$, i=0,1,2,3,… (2.5)

- To check that p(n) is a probability mass function, we note that
$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

- Binomial vs. Poisson RVs
  - n>>1, p<<1, let λ=np
  - $P\{X = 1\} = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i} = \frac{n(n-1)\ldots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$
  - For large n and small p
    - $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$, $\left(1-\frac{\lambda}{n}\right)^i \approx 1$, $\frac{n(n-1)\ldots(n-i+1)}{n^i} \approx 1$
  - $P\{X = i\} \approx e^{-\lambda}\frac{\lambda^i}{i!}$

- Ex. 2.10] Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter λ = 1. Calculate the probability that there is at least one error on this page.
  - $P\{X \geq 1\} = 1 - P\{X=0\} = 1 - e^{-1} = 0.633$

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \dfrac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$

Continuous RVs

- RV X is said to be a continuous RV if there exists a non-negative function f(x), defined for all real x∈{-∞,∞} having the property that for any set B of real numbers
  $P\{X \in B\} = \int_B f(x)dx = 1$
- The function f(x) is called the probability density function (pdf) of the random variable X
- Properties of pdf of X
  - $P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx = 1$
  - $P\{a \leq X \leq b\} = \int_a^b f(x)dx$
  - $P\{X = a\} = \int_a^a f(x)dx = 0$
  - $F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$
  - $\frac{dF(a)}{da} = f(a)$
- Ex. Uniform Rv, Exponential Rv, Gamma RV, Normal RV

Example of Continuous RVs: Uniform RV

- A RV is said to be uniformly distributed over the interval (α,β) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, \alpha < x < \beta \\ 0, otherwise \end{cases}$$

- Ex. 2.14] X~U(0,10)
  - P{X<3}= ?
  - P{X>7}= ?
  - P{1<X<6}= ?

Example of Continuous RVs: Uniform RV

- A continuous RV is said to be uniformly distributed over the interval (α,β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, \alpha < x < \beta \\ 0, otherwise \end{cases}$$

- Ex. 2.14] X~U(0,10)
  - ◆ $P\{X<3\} = 1/10 \int_{X \in (0,3)} dx = 3/10$
  - ◆ $P\{X>7\} = 1/10 \int_{X \in (7,10)} dx = 3/10$
  - ◆ $P\{1<X<6\} = 1/10 \int_{X \in (1,6)} dx = 1/2$

Example of Continuous RVs: Exponential RV

- A continuous RV is said to be an exponential RV with parameter λ if its probability density function is given, for λ>0, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, if\ x \geq 0 \\ 0, otherwise \end{cases}$$

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}$$

Example of Continuous RVs: Gamma RV

- A continuous RV is said to be a gamma RV with parameter α, λ, if its probability density function is given, for α >0, λ>0, by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, if\ x \geq 0 \\ 0, otherwise \end{cases}$$

where

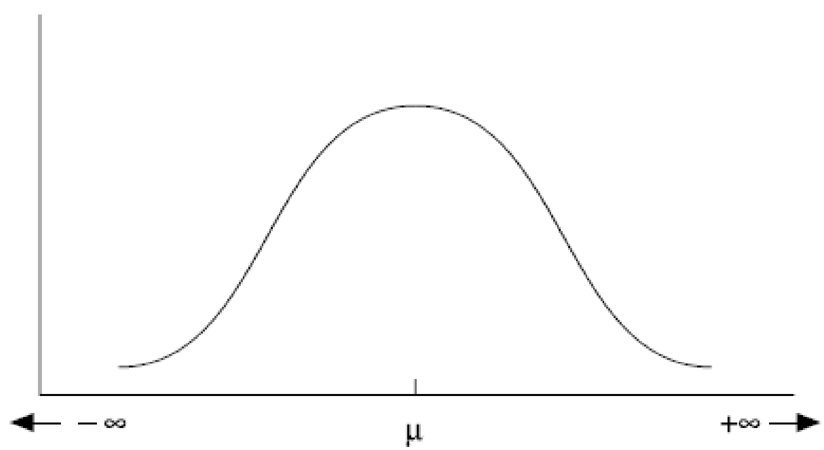
$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

Example of Continuous RVs: Normal RV

- A continuous RV is said to be a normal RV with parameter μ and σ^2, if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < \infty$.



Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(n, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\},$ $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Expectation of a function of a RV

- The expected value of X, E[X], is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes that value.

- If RV X is a discrete RV having a pmf p(x), then E[X] is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

  - Ex. 2.16~2.19] find the followings
    - Expectation of a Bernoulli Random Variable
    - Expectation of a Binomial Random Variable
    - Expectation of a Geometric Random Variable
    - Expectation of a Poisson Random Variable

- If RV X is a continuous RV having a pdf f(x), then E[X] is defined by

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

  - Ex. 2.20~2.22] find the followings
    - Expectation of a Uniform Random Variable
    - Expectation of a Exponential Random Variable
    - Expectation of a Normal Random Variable

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Expectation of a function of a RV

- Given a RV X and its probability distribution, what is the expectation of a function of X?
  1. Since $g(X)$ is itself a random variable, it must have a probability distribution, which should be computable from a knowledge of the distribution of X. Once we have obtained the distribution of $g(X)$, we can then compute $E[g(X)]$ by the definition of the expectation.
  2. Another way is to compute the expectation of a function of X from a knowledge of the distribution of X. See Proposition 2.1.

- Ex. 2.23] Suppose X has the following probability mass function: $p(0) = 0.2$, $p(1) = 0.5$, $p(2) = 0.3$. Calculate $E[X^2]$.
  - Letting $Y = X^2$, we have that Y is a random variable that can take on one of the values 0_2, 1_2, 2_2 with respective probabilities
    $p_Y(0) = P\{Y = 0^2\} = 0.2$,
    $p_Y(1) = P\{Y = 1^2\} = 0.5$,
    $p_Y(4) = P\{Y = 2^2\} = 0.3$
  - Hence, $E[X^2] = E[Y] = 0(0.2) + 1(0.5) + 4(0.3) = 1.7$
  - Note that $E[X^2] \neq E[X]^2$

Expectation of a function of a RV

- Proposition 2.1
  1. If *X* is a discrete random variable with probability mass function *p(x)*, then for any real-valued function *g*,
  $$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$
  2. If *X* is a continuous random variable with probability density function *f (x)*, then for any real-valued function *g*,
  $$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

- Ex 2.26] Let X be uniformly distributed over (0,1), E[X^3]?
  - $E[X^3] = \int_0^1 X^3\,dx = \frac{1}{4}$

- Corollary 2.2
  - If *a* and *b* are constants, then $E[aX + b] = aE[X] + b$

Expectation of a function of a RV

- The expected value of a RVX, E[X], is also referred to as the mean or the first moment of X.

- The quantity $E[X^n]$, $n \geq 1$, is called the n_{th} moment of X

- The variance of a RV X, denoted by Var(X), is defined by Var(X)=$E[(X-E[X])^2]$, deviation of X from the mean.

- $$\begin{aligned} Var(X) &= E[(X-E[X])^2] \\ &= E[X^2-2E[X]X+E[X]^2] \\ &= E[X^2]-E[2E[X]X]+E[E[X]^2] \\ &= E[X^2]-2E[X]^2+E[X]^2 \\ &= E[X^2]-E[X]^2 \end{aligned}$$

- Ex 2.27] Var(X) of the normal RV with μ and σ.
    - $Var(X) = E[(X-\mu)^2]$
    $$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy$$
    $$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$
    $$= \sigma^2$$

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \leq p \leq 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leq p \leq 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Jointly Distributed RVs

- For any two RVs X and Y, the joint cumulative probability distribution function of X and Y is defined as $F(a,b) = P\{X \le a, Y \le b\}$, -∞≤a,b≤∞
  - ◆ $F_X(a,b) = P\{X \le a\} = P\{X \le a, Y \le \infty\} = F(a,\infty)$
  - ◆ If X and Y are discrete RVs, the joint pmf of X and Y is defined as $$\mathrm{p}(x,y) = P\{X = x, Y = y\}$$

- The joint pdf of X and Y, f(x,y) is defined as

$$\frac{d^2}{dxdy}F(x,y) = f(x,y), F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy$$

$$P\{(x,y) \in D\} = \iint_D f(x,y)dxdy$$

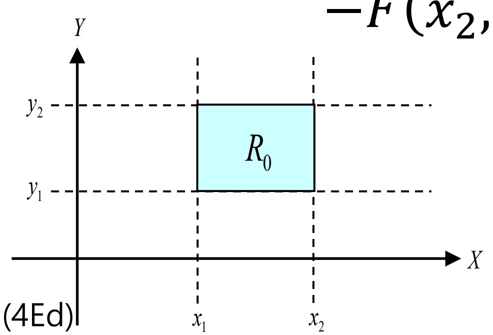
- A variation of Proposition 2.1
  - ◆ $E[g(X,Y)] = \sum_x \sum_y g(x,y)p(x,y)$ (discrete case)
    $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dxdy$ (continuous case)
  - ◆ E.g) E[aX+bY]=aE[X]+bE[Y]

Properties of Joint Distribution

1. The $F(x, y)$ is such that
$$F(-\infty, y) = 0, F(x, -\infty) = 0, F(\infty, \infty) = 1$$

2. The event $\{x_1 < X \le x_2, Y \le y\}$ consists of all points $\{X, Y\}$ in the vertical half-strip D_2 and the event $\{\boldsymbol{x} \le x, y_1 < Y \le y_2\}$ consists of all points $\{\boldsymbol{x}, \boldsymbol{y}\}$ in the vertical half-strip D_3. We maintain that
$$P\{x_1 < X \le x_2, Y \le y\} = F(x_2, y) - F(x_1, y)$$
$$P\{X \le x, y_1 < Y \le y_2\} = F(x, y_2) - F(x, y_1)$$

3. $$P\{x_1 < X \le x_2, Y \le y\} = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$



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Jointly Distributed RVs: Example

- **Example 2.30** As another example of the usefulness of Equation (2.11), let us use it to obtain the expectation of a binomial random variable having parameters n and p. Recalling that such a random variable X represents the number of successes in n trials when each trial has probability p of being a success, we have $X = X_1 + X_2 + \cdots + X_n$, where X_i is 1, if the ith trial is a success, otherwise 0.
  - ◆E[X]=np from a variation of Proposition 2.1

- **Example 2.31** At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who select their own hats.
  - ◆Letting X denote the number of men that select their own hats, we can best compute $E[X]$ by noting that $X = X_1 + X_2 + \cdots + X_N$ where X_i=1, if the i_{th} man selects his own hat, otherwise 0.
  - ◆Now, because the ith man is equally likely to select any of the N hats, it follows that $P\{X_i = 1\} = P\{i_{th}$ man selects his own hat$\}$ = **1/*N***
  - ◆$E[X]=E[X_1]+E[X_2]+ \cdots +E[X_N]=N(1/N)=1$

Independent RVs:

- The random variables X and Y are said to be *independent* if, for all a, b,
  $P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}$
  $\Leftrightarrow F(a, b) = F_X(a)F_Y(b)$ for all a, b
  $\Leftrightarrow f(x, y) = f_X(x)f_Y(y)$ for continuous cases
  $\Leftrightarrow p(x, y) = p_X(x)p_Y(y)$ for discrete cases

- **Proposition 2.3** If X and Y are independent, then for any functions h and g
  $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$
  - Proof] $E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \int_{-\infty}^{\infty} g(x)f_X(x)\,dx$$

$$= E[h(Y)]E[g(X)]$$

Covariance and Variance of Sums of Random Variables

- The covariance of any two random variables X and Y, denoted by Cov(X, Y), is defined by
  - Cov$(X, Y) = E[(X - E[X])(Y - E[Y])]$
    $= E[XY - YE[X] - XE[Y] + E[X]E[Y]]$
    $= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y]$
    $= E[XY] - E[X]E[Y]$

- Note that if X and Y are independent, then by Proposition 2.3 it follows that Cov$(X, Y) = 0$.

- **Example 2.33** The joint density function of X, Y is,
  $f(x, y) = \frac{1}{y} e^{-(y+x/y)}$, $0 < x, y < \infty$
  - Verify that the preceding is a joint density function.
  - Find Cov(X, Y).
  - By yourself

Properties of Covariance

- For any random variables X, Y,Z and constant c,
  1. Cov(X,X) = Var(X),
  2. Cov(X, Y) = Cov(Y,X),
  3. Cov(cX, Y) = c Cov(X, Y),
  4. Cov$(X, Y + Z)$ = Cov(X, Y) + Cov(X,Z).

  ◆Proof] Whereas the first three properties are immediate, the final one is easily proven as follows:
  Cov(X, Y + Z) = E[X(Y + Z)] − E[X]E[Y + Z]
  = E[XY] − E[X]E[Y] + E[XZ] − E[X]E[Z]
  = Cov(X, Y) + Cov(X,Z)

- A useful expression for the variance of the sum of random variables can be obtained as follows:
  ◆$Var(\sum_{i=1}^{n} X_i) = Cov\left(\sum_{i=1}^{n} X_i , \sum_{j=1}^{n} X_j\right)$ (2.16)

- If Xi, i = 1, . . . , n are independent random variables, then Equation (2.16) reduces to
  ◆$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

- **Definition 2.1** If $X_1, \ldots, X_n$ are independent and identically distributed, then the random variable $\bar{X} = \sum_{i=1}^{n} X_i$ is called the *sample mean*.

- **Proposition 2.4** Suppose that $X_1, \ldots, X_n$ are independent and identically distributed with expected value μ and variance σ^2. Then,
  a. $E[\bar{X}] = \mu$
  b. $\mathrm{Var}(\bar{X}) = \sigma^2/n$
  c. $\mathrm{Cov}(\bar{X}, X_i - \bar{X}) = 0$, i=1, …,n
  - Proofs…

- **Example 2.34 (Variance of a Binomial Random Variable)** Compute the variance of a binomial random variable X with parameters n and p.

One Function of Two RVs

- Given two random variables X and Y and a function $g(X,Y)$, we form a new random variable Z as $Z = g(X,Y)$.

- Given the joint p.d.f $f_{XY}(X,Y)$, how does one obtain $f_Z(Z)$ the p.d.f of Z?
  - $f_Z(Z) = P\{Z(\zeta) \leq z\} = P\{g(X,Y) \leq z\} = P\{(X,Y) \in D_z\} = \iint_{D_Z} f(x,y)dxdy$

- Problems of this type are of interest from a practical standpoint. For example, a receiver output signal usually consists of the desired signal buried in noise, and the above formulation in that case reduces to $Z = X + Y$.

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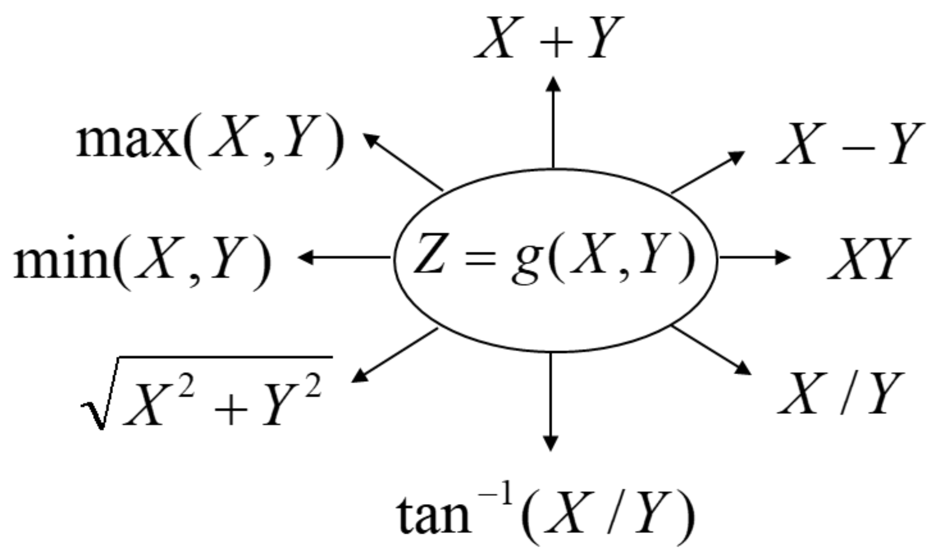
Example of One Function of Two RVs



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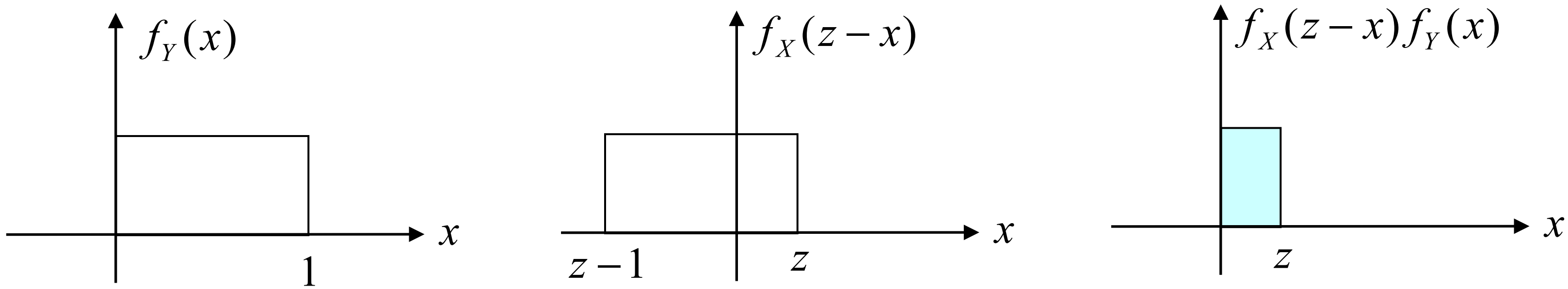
Distribution of X+Y

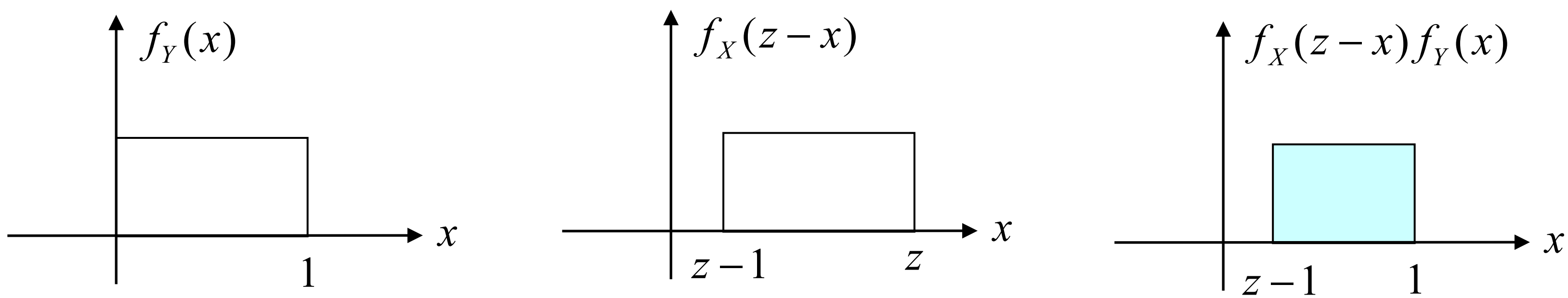
- Suppose first that X and Y are continuous, X having probability density f and Y having probability density g. Then, letting $F_{X+Y}(a)$ be the cumulative distribution function of $X+Y$, we have
  - $F_{X+Y}(a) = P\{X+Y \leq a\} = \int\int_{x+y\leq a} f(x)g(y)dx\,dy$
    $= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f(x)g(y)dx\,dy \ = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a-y} f(x)\,dx\right)g(y)dy$
    $= \int_{-\infty}^{\infty} F_X(a-y)g(y)dy$
    - The cumulative distribution function F_{X+Y} is called the *convolution* of the distributions F_X and F_Y.
  - $f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)g(y)dy$
    $= \int_{-\infty}^{\infty} \frac{d}{da} F_X(a-y)g(y)dy = \int_{-\infty}^{\infty} f(a-y)g(y)dy$

Example 1

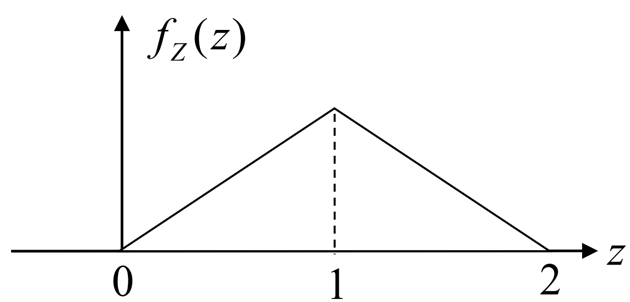
- **Example 2.36 (Sum of Two Independent Uniform Random Variables)** If X and Y are independent random variables both uniformly distributed on (0, 1), then calculate the probability density of $X + Y$.
  - Answer?



$(a)\quad 0 \le z < 1$



$(b)\quad 1 \le z < 2$



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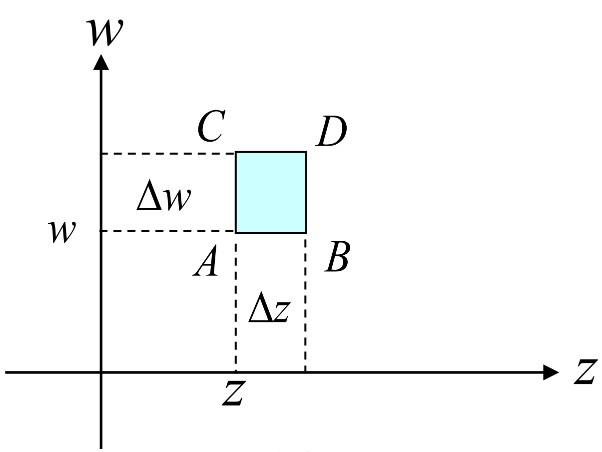
Joint Probability Distribution of Functions of Random Variables

- Let X_1 and X_2 be jointly continuous random variables with joint probability density function $f(x_1, x_2)$.

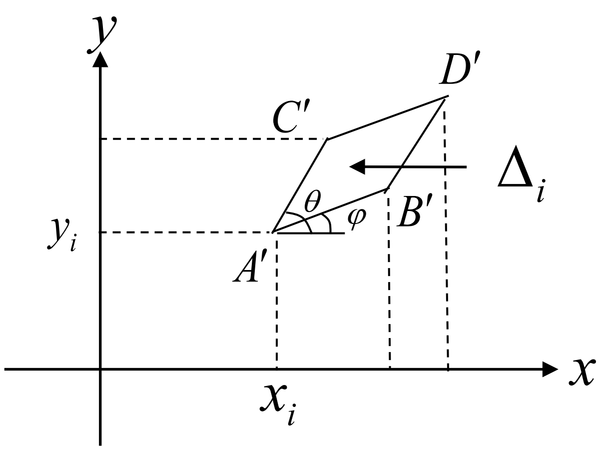
- Suppose $Y_1 = g_1(x_1, x_2)$, $Y_2 = g_2(x_1, x_2)$.
  - $F_{Y_1,Y_2}(y_1, y_2) = \iint_{D_{y_1,y_2}} f_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2$
    $= P\{Y_1 \le y_1, Y_2 \le y_2\} = P\{g_1(x_1, x_2) \le y_1, g_1(x_1, x_2) \le y_2\}$
    $= P\left\{(x_1, x_2) \in D_{Y_1,Y_2}\right\} = \iint_{D_{Y_1,Y_2}} f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$
    $= \iint_{D_{X_1,X_2}} f_{X_1,X_2}(x_1, x_2) |J(x_1, x_2)|^{-1} dx_1 dx_2$
  - $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) |J(x_1, x_2)|^{-1}$

    where $|J(x_1, x_2)| = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = |J(y_1, y_2)|^{-1}$

Meaning of $|J(x_1, x_2)|$



(a)



(b)

Moment Generating Functions

- The *moment generating function $\varphi(t)$* of the random variable X is defined for all values t by
  - $\phi(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx \end{cases}$
- Properties
  - $\phi'(t) = \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right] = E[Xe^{tX}]$
  - $\phi''(t) = \frac{d}{dt} \phi'(t) = E\left[\frac{d}{dt}(Xe^{tX})\right] = E[X^2 e^{tX}]$
  - $\phi^{(n)}(t) = E[X^n e^{tX}]$
    - $\phi'(0) = E[X]$, $\phi''(0) = E[X^2]$, $\phi^{(n)}(0) = E[X^n]$
- An important property of moment generating functions is that the *moment generating function of the sum of independent random variables is just the product of the individual moment generating functions*. To see this, suppose that X and Y are independent and have moment generating functions $\varphi_X(t)$ and $\varphi_Y(t)$, respectively. Then $\varphi_{X+Y}(t)$, the moment generating function of $X + Y$, is given by
  - $\varphi_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}] E[e^{tY}] = \varphi_X(t) \varphi_Y(t)$

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \leq p \leq 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leq p \leq 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geq 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Limit Theorems: Strong Law of Large Numbers

- **Theorem 2.1 (Strong Law of Large Numbers)** Let $X_1, X_2, \ldots$ be a sequence of independent random variables having a common distribution, and let $E[X_i] = \mu$. Then, with probability 1, $\frac{X_1+X_2+\cdots+X_n}{n} \to \mu$

- **Example:** suppose that a sequence of independent trials is performed. Let E be a fixed event and denote by $P(E)$ the probability that E occurs on any particular trial. Letting

$$X_i = \begin{cases} 1, if\ E\ occurs\ on\ the\ ith\ trial \\ 0, if\ E\ does\ not\ occur\ on\ the\ ith\ trial \end{cases}$$

we have by the strong law of large numbers that, with probability 1,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to E[X] = P(E)$$

Limit Theorems: Central Limit Theorem

- **Theorem 2.2 (Central Limit Theorem)** Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2. Then the distribution of $\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as n→∞. That is $\mathrm{P}\left\{\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \leq a\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{x^2}{2}}dx$ s n→∞.
- If X is binomially distributed with parameters n and p, then X has the same distribution as the sum of n independent Bernoulli random variables, each with parameters p. Hence, the distribution of
$$\frac{X-E[X]}{\sqrt{Var(X)}} = \frac{X-np}{\sqrt{np(1-p)}}$$
approaches the standard normal distribution as n approaches ∞.