Special Topics in Communication Networks

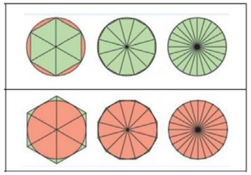
Lecture 1

Why is the understanding of randomness important?

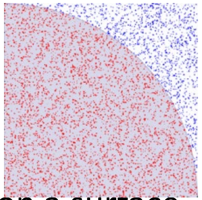
Why we should study the probability?

How can we find π?

1. **Geometric Methods**
   - **Circumscribed and Inscribed Polygons:** Archimedes used this method with a 96-sided polygon to estimate π as being between 3.1408 and 3.1429.



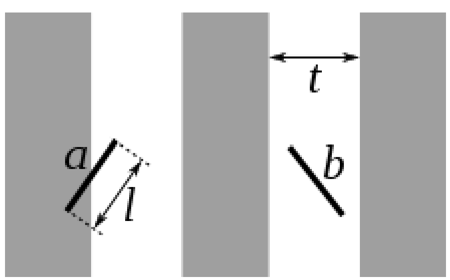
2. **Infinite Series**
   - **Leibniz Formula for π:** $\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$ simple, yet slow-converging series
   - **Nilakantha's Series:** $\pi = 4 \times \left(3 + \frac{4}{2\times3\times4} - \frac{4}{4\times5\times6} + \frac{4}{6\times7\times8} \ldots\right)$ A faster-converging series from the 15th century
3. **Probabilistic Methods**
   - **Monte Carlo Simulations:** Randomly generating points, the ratio of points that fall inside the quarter circle to the total number of points

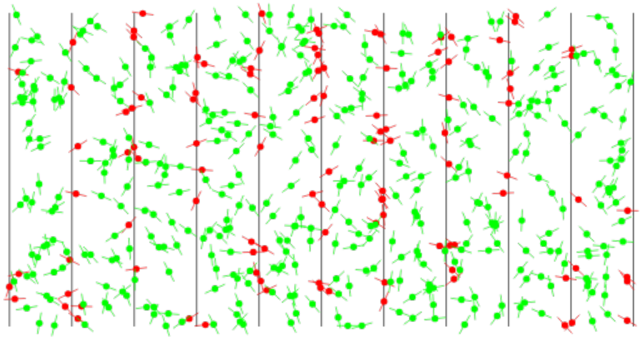


4. **Analytical Methods**
   - **Buffon's Needle:** This is a **probability-based** method involving dropping needles on a surface marked with parallel lines and observing the number of times the needles cross the lines.
5. **Computational Algorithms**
   - **Machin-like Formulas:** These are formulas that express π as sums of arctangents, which can be computed to high precision using Taylor series expansions. $\frac{\pi}{4} = atan1, \frac{\pi}{4} = 4\,\mathrm{atan}\left(\frac{1}{5}\right) + \mathrm{atan}\left(\frac{1}{239}\right)$
   - **BBP (Bailey-Borwein-Plouffe) Formula:** This formula allows for the calculation of the nth binary digit of π without needing to calculate the preceding n−1digits, facilitating high-precision calculations of π on computers.
6. **Using Built-in Mathematical Functions**

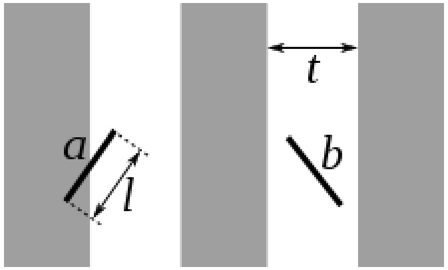
Buffon's Needle

- Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?





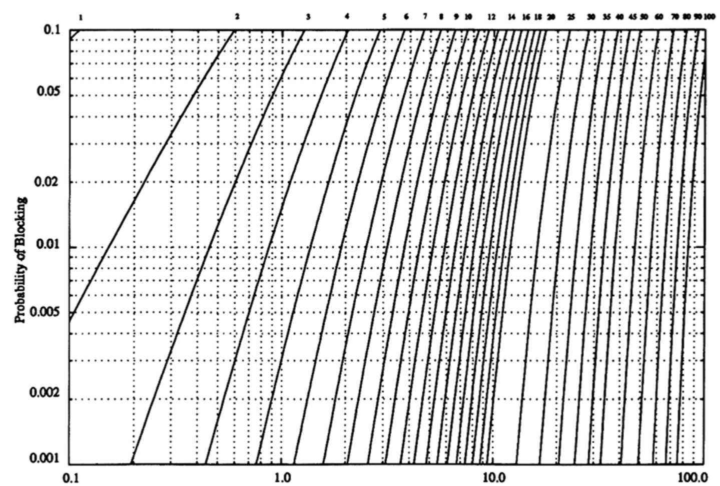
Solution: Buffon's Needle



- Let x be the distance from the center of the needle to the closest parallel line, and let θ be the acute angle between the needle and one of the parallel lines.
- The uniform probability density function (PDF) of x between 0 and $\frac{t}{2}$ is
  - $f_X(x) = \begin{cases} \frac{2}{t}, 0 \le x \le \frac{t}{2} \\ 0, otherwise \end{cases}$
- The uniform probability density function of θ between 0 and $\frac{\pi}{2}$ is
  - $f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, 0 \le x \le \frac{\pi}{2} \\ 0, otherwise \end{cases}$
- The two random variables, x and θ, are independent,[4] so the joint probability density function is the product
  - $f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{t\pi}, 0 \le x \le \frac{t}{2}, 0 \le x \le \frac{\pi}{2} \\ 0, otherwise \end{cases}$
- The needle crosses a line if $x \le \frac{l}{2}\sin\theta$
- When $l \le t$, $p = \int_{\theta=0}^{\frac{\pi}{2}} \int_{x=0}^{\frac{l}{2}\sin\theta} \frac{4}{t\pi} dx d\theta = \frac{2}{\pi} \cdot \frac{l}{t}$
- How about $l > t$?

How to define a capacity of trunk lines?

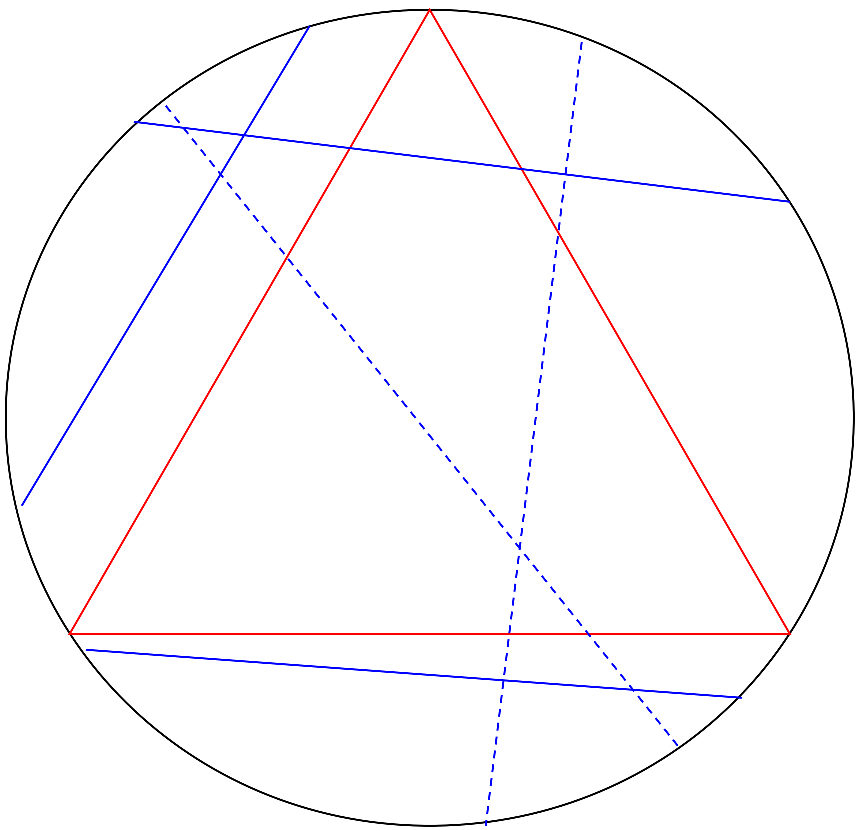
- Blocked calls cleared
- Calls arrive as determined by a Poisson process
- The duration of the time that a user occupies a channel is exponentially distributed
- How many users can be supported for 2% blocking probability for ten of trunked channels in a blocked calls cleared system?
- How many users can be supported 2% blocking probability for ten of trunked channels with an average calling time of 2 mins in an hour?



Bertrand Paradox

•Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?

1. A third
2. A half
3. A quarter



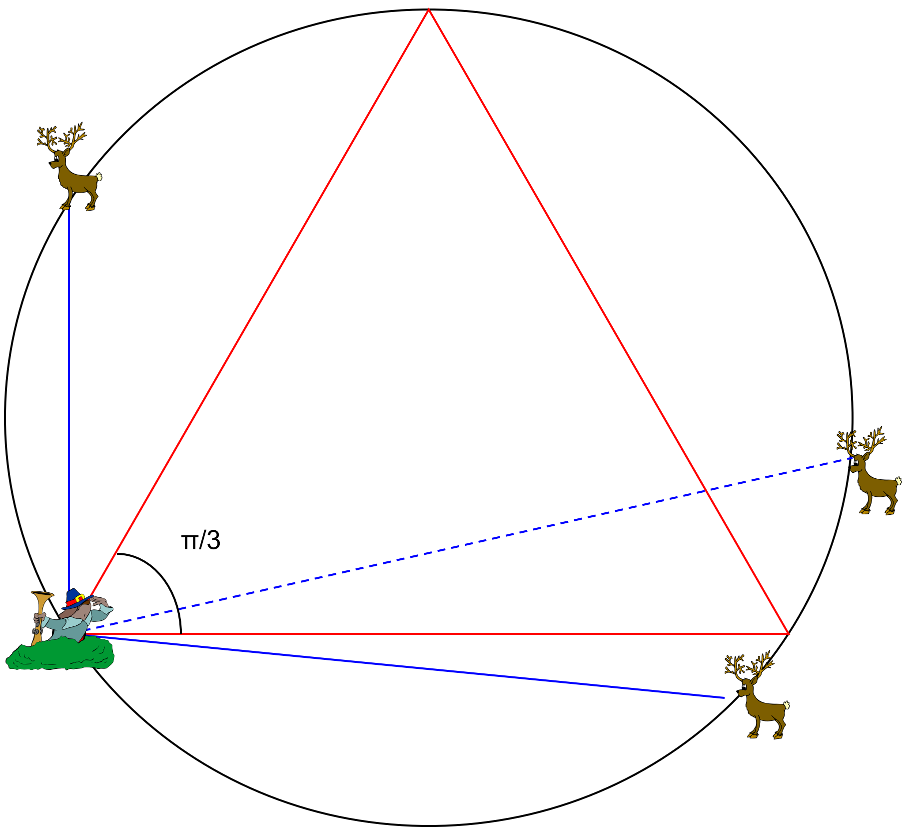
Solution 1: Bertrand Paradox

- Choose a point on the circumference and rotate the triangle so that the point is at one vertex.

- Choose another point on the circle and draw the chord joining it to the first point.

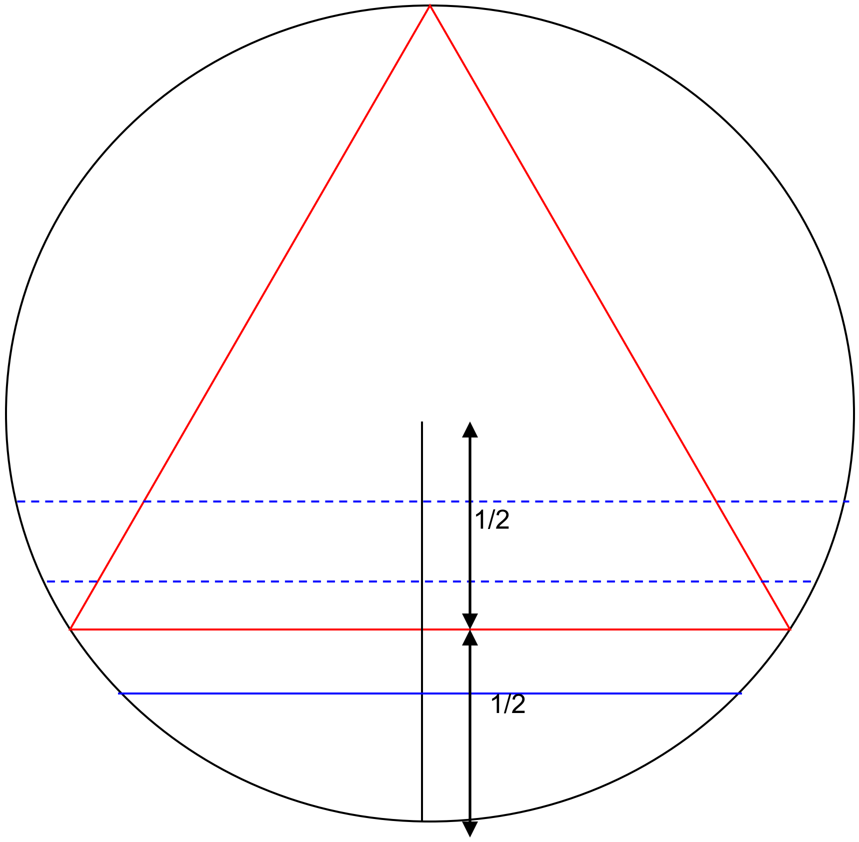
- For points on the arc between the endpoints of the side opposite the first point, the chord is longer than a side of the triangle.

- The length of the arc is one third of the circumference of the circle, therefore the probability a random chord is longer than a side of the inscribed triangle is one third.



Solution 2: Bertrand Paradox

- Choose a radius of the circle and rotate the triangle so a side is perpendicular to the radius.
- Choose a point on the radius and construct the chord whose midpoint is the chosen point.
- The chord is longer than a side of the triangle if the chosen point is nearer the center of the circle than the point where the side of the triangle intersects the radius.
- Since the side of the triangle bisects the radius, it is equally probable that the chosen point is nearer or farther.
- Therefore the probability a random chord is longer than a side of the inscribed triangle is one half.

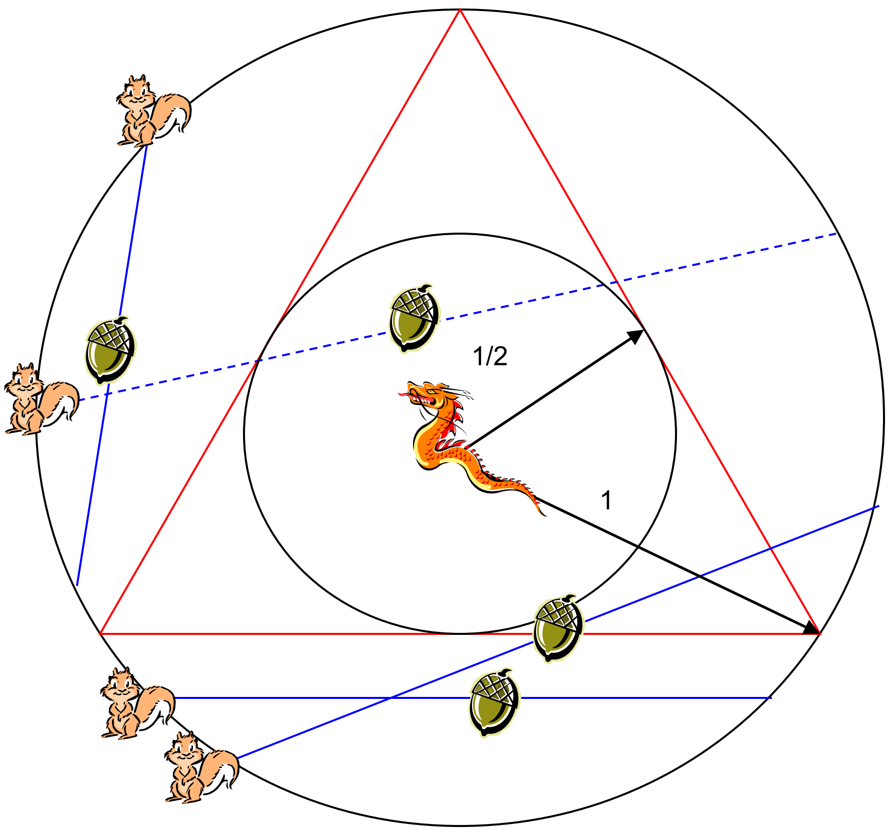


Solution 3: Bertrand Paradox

•Choose a point anywhere within the circle and construct a chord with the chosen point as its midpoint.

•The chord is longer than a side of the inscribed triangle if the chosen point falls within a concentric circle of radius 1/2.

•The area of the smaller circle is one fourth the area of the larger circle, therefore the probability a random chord is longer than a side of the inscribed triangle is one fourth.



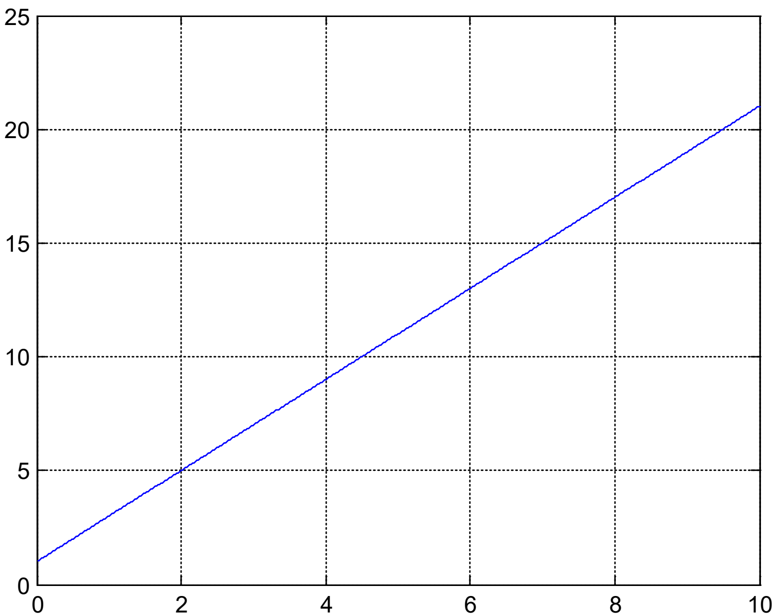
Our Interest and Goals

- Study special topics in communications regarding probabilities and stochastic processes
  - Reinforced learning
  - Queueing, etc.

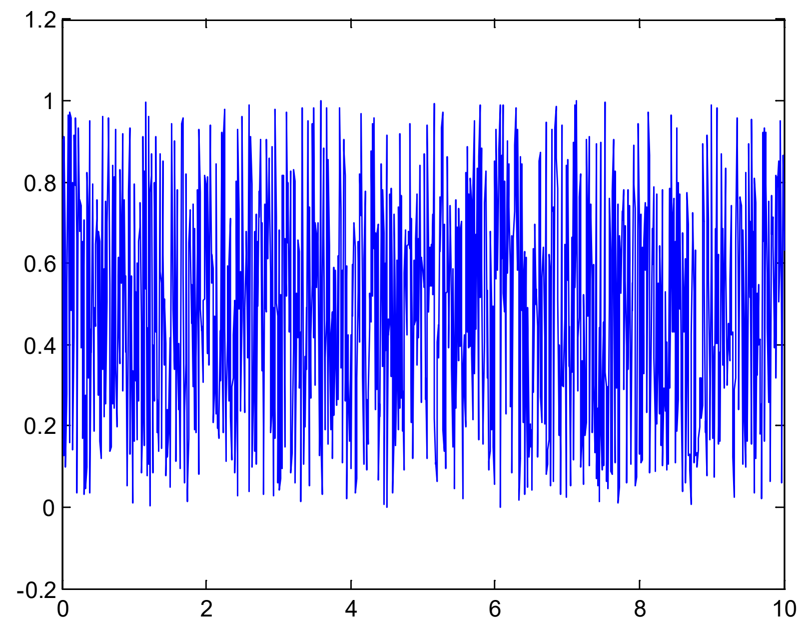
- Study related tools to characterize non-deterministic signals
  - Random processes, statistics

- Study tools to characterize uncertainty
  - Probability theory, random events, random variables

Random versus Deterministic



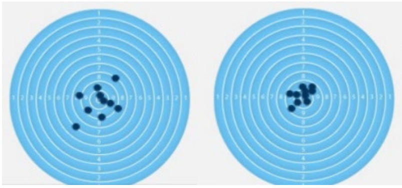
Y=2X+1



Random Noise

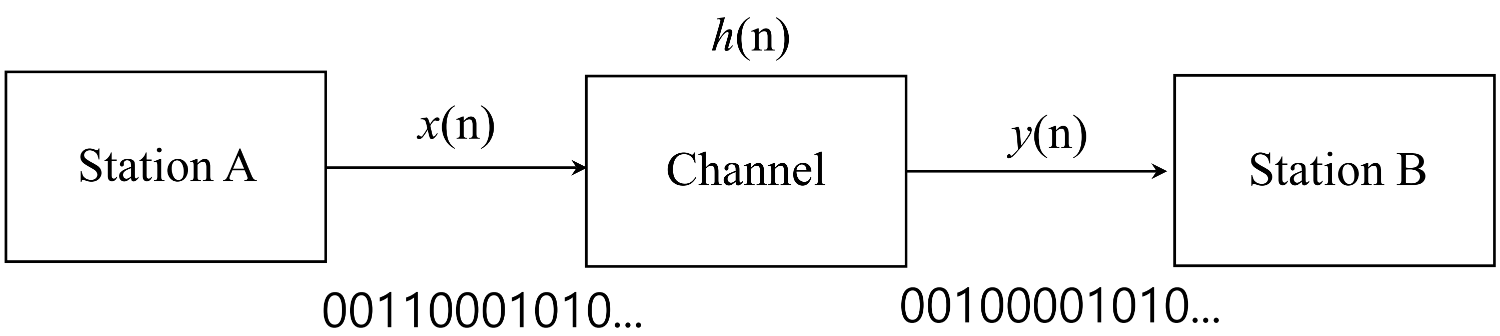
Randomness around us

- At bus stop
  - Waiting time for bus
  - The number of people to take a bus
- At a bank
  - Waiting time for service
  - Service time
  - The number of customers to wait for
- Bet/Game
  - Toss a coin: Head or Tail
  - Roll a dice: 1, 2, 3, 4, 5, 6
- Shooting game



Typical Electrical Engineering Problems (1/3)

- A noisy binary communication channel
  - The channel can be twisted pair, coaxial cable, fiber optic cable, or wireless medium.
  - The channel introduces noise and thereby bit errors.



Typical Electrical Engineering Problems (2/3)

- Desired target signal is buried in noise.

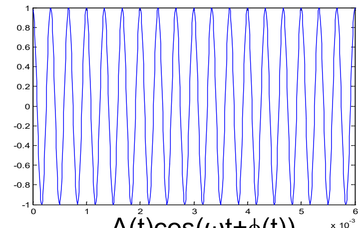
$$x(t) = A(t)\cos(\omega t + \varphi(t)) + n(t)$$

signal signal noise

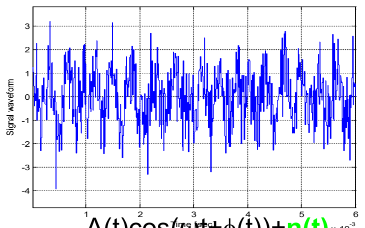
- Determine the presence or absence of the desired signal.

- Filter the signal out of noise.

- Demodulate the signal.



$A(t)\cos(\omega t+\phi(t))$



$A(t)\cos(\omega t+\phi(t))+n(t)$

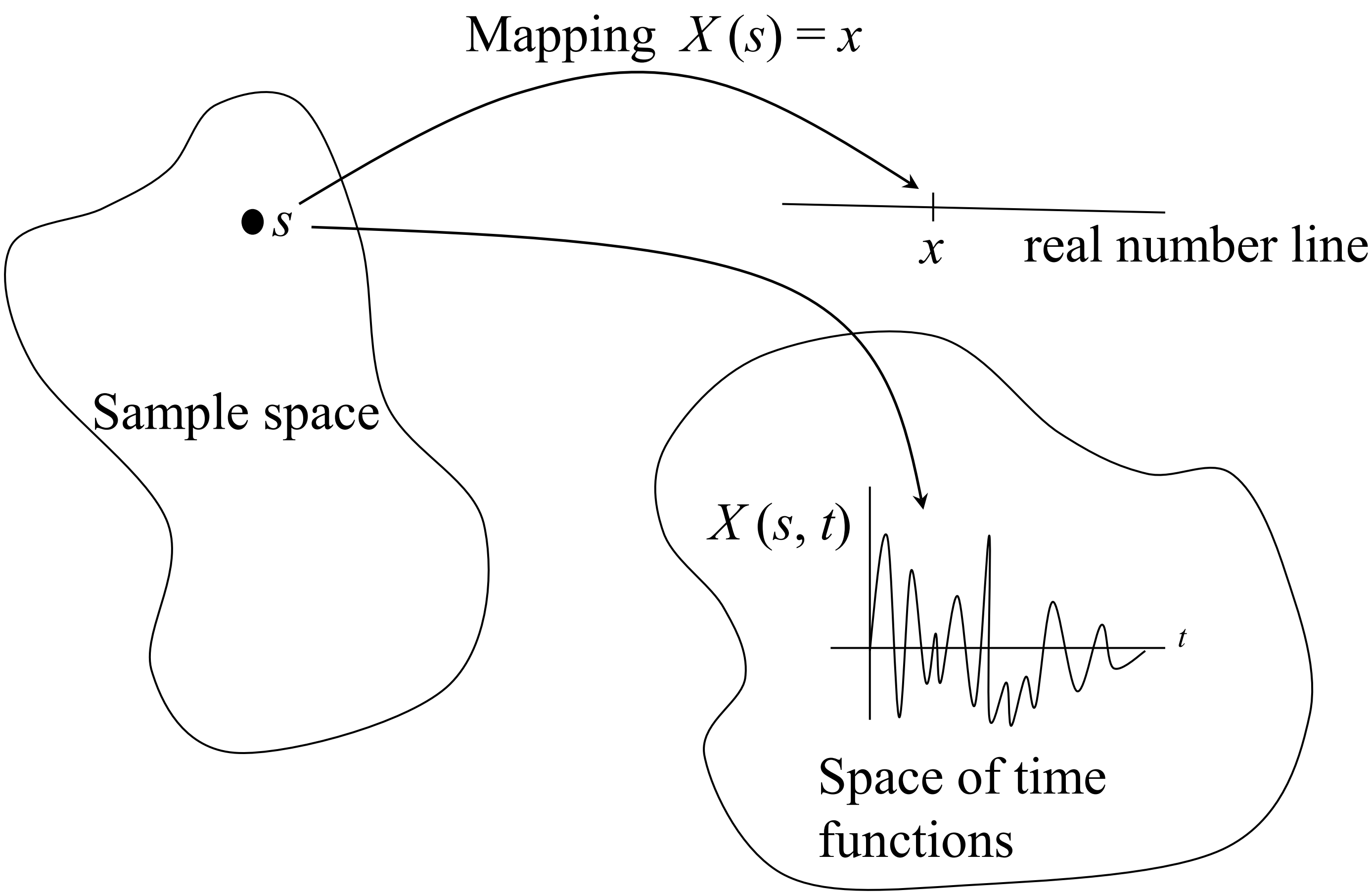
Typical Electrical Engineering Problems (3/3)

- In large computer networks, there are limited resources (e.g., bandwidth, routers, switches, printers and other devices) that need to be shared by the users.
  - User jobs/packets are queued and assigned service based on predefined criteria.
  - Demand is uncertain and service time is also uncertain.
  - Delay from the time the service is requested to the time it is completed is uncertain.

- Similar considerations exist for telephone networks, multiuser computer networks, and other communication networks.

Random Variables and Random Processes

- We will study them details later



Probability Space

- A probability space (S, F(S), P) is a triple made up of three elements
  1. S: Sample space
  2. F(S): A collection of sets S, event space
  3. The probabilities of P(E) for each E∈F(S)

     P(.): F(S) → [0, 1]

Sample Space (S, F(S), P)

- Experimental outcomes are unpredictable

- The set of all possible outcomes is known

- The sample space S of an experiment is defined as the set of all possible outcomes of the experiment.
  - The flipping of a coin: $S = \{H, T\}$
  - The rolling of a die: $S = \{1,2,3,4,5,6\}$
  - The flipping of two coins: $S=\{(H,H),(H,T),(T,H),(T,T)\}$
  - The lifetime of a car: $S=[0,\infty)$

- An Event is any subset E of the sample space S ($E\subset S$ or $E\in F(S)$)
  - $E = \{H\}$, $E = \{1\}$, $E = \{1,3,5\}$ $E =\{(H,H),(T,T)\}$

Basic Set Theory Definitions

- The set containing all possible elements of interest is called the universe, universal set or space S.

- The set containing no elements is called the empty set or null set.

- For any two events E and F⊂S, we define the event
  - E∪F: the event E∪F will occur if either E or F (the union of E and F)
  - E∩F (or EF): the event EF consists of all outcomes which are both in E and in F
    - E={1,3,5},F={1,2,3}, E∪F=? E∩F=?
  - If EF = 0, then E and F are said to be mutually exclusive.
  - For more than tow events, we can define unions and intersections such as $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$, where E_ns are events

- E^c : the complement of E, E^c will occur if and only if E does not occur
  - S = {1,2,3,4,5,6}, E={1,3,5}, E^c={2,4,6}, S^c = 0 or {}

- Two sets E and F are equal if they contain exactly the same elements.

Event Space (S, F(S), P)

- Intuitively, is a collection of events which we are interested in computing the probability of.

- Mathematically, is a family of subset of sample space S closed under certain set operations as below:
    1. If $E \in F(S)$, then $E^c \in F(S)$
    2. If $E_1, E_2 \in F(S)$, then $E_1 \cup E_2 \in F(S)$
    3. If $E_1, E_2, E_3 \ldots \in F(S)$, then $\cup_{i=\{1,\ldots,\infty\}} E_i \in F(S)$
  - A family of sets satisfying these three properties is called a σ-field

Probabilities Defined on Events (S, F(S), P)

- For each event E of a sample space S, we assume that a number P(E) is defined and satisfies the following three conditions:
  1. $0 \le P(E) \le 1$
  2. $P(S)=1$
  3. For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive, that is , events for which $E_n E_m = 0$ when $n \neq m$, then $p(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$

- We refer to P(E) as the probability of the event E.
  - E.g) the coin flipping P({H})=P({T}) = ½
  - The die rolling:
    P({1})= P({2})= P({3})= P({4})= P({5})= P({6})=1/6
    P({1,3,5})= P({1})+P({3})+P({5})= 1/2

Properties of Probability (1/2)

- $P(S)=1$
  - $P(E)+P(E^c)=P(S)$
  - $P(E) = 1-P(E^c)$
  - $P(\varnothing)=0$
  - If $EF = \varnothing$, then $P(EF) = 0$

- $P(E \cup F)=P(E)+P(F)-P(EF)$
  - If $E \subset F$, then $P(E) \leq P(F)$

- $P(E \cup F \cup G)$
  $=P(E)+P(F)-P(EF)+P(G)-P(EG \cup FG)$
  $=P(E)+P(F)-P(EF)+P(G)-P(EG)-P(FG)+P(EFG)$
  $=P(E)+P(F)+P(G)-P(EF) -P(EG)-P(FG)+P(EFG)$

- $P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n)$
  $=\sum_i P(E_i)-\sum_{i<j} P(E_i E_j)+\sum_{i<j<k} P(E_i E_j E_k)+\ldots+(-1)^{n+1} P(E_1 E_2 \ldots E_n)$

The Principle of Total Probability

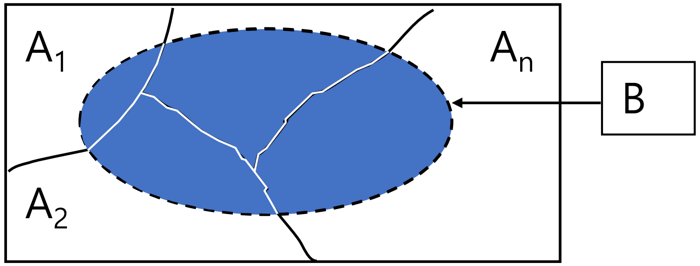
Let $A_1, A_2, \ldots, A_n$ be a set of mutually exclusive and collectively exhaustive events:

$$A_k A_j = 0 \qquad k \neq j$$

$$\bigcup_{j=1}^{n} A_j = S \qquad \text{then } \sum_{j=1}^{n} \Pr[A_j] = 1$$

Now let B be <u>any</u> event in S. Then,

$$\Pr[B] = \Pr[BA_1] + \Pr[BA_2] + \ldots + \Pr[BA_n]$$



Conditional Probability

- If we let E and F denote the envents, then the probability P(E|F) is **the conditional probability** that E occurs given that F has occurred
  - Because we know that F has occurred, it follows that F becomes our new sample space and hence the event EF occurs will equal the probability of EF relative to the probability of F. That is
    $P(E|F) = P(EF)/P(F)$ (1.5)

- Ex) A family has two children. What is the conditional probability that boys given that at least one of them is a boy? Assume that the sample space S is given by $S=\{(b,b),(b,g),(g,b),(g,g)\}$, and all outcomes are equally likely
  - $P(B|A)=P(BA)/P(A)=P(\{(b,b)\})/P(\{(b,b),(b,g),(g,b)\})$
    $=(1/4)/(3/4)=1/3$

Independent Events

- Two events E and F are said to **independent** if P(EF)=P(E)P(F)
- By (1.5) this implies that E and F are independent if P(E|F)=P(E)
  - P(E|F)=P(EF)/P(F)=P(E)P(F)/P(F)=P(E)
- Two event E and F that are not independent are said to be **dependent**
- The definition of independence can be extended to more than two events. The events $E_1, E_2, \ldots, E_n$ are said to be independent if for every subset $E_1, E_2, \ldots, E_r$ $r \le n$ of these events $P(E_1E_2\ldots E_r)=P(E_1)P(E_2)P(E_r)$
- Ex) Let E={1,2}, F={1,3}, G={1,4}, P({1})=P({2})=P({3})=P({4})
  - P(EF)=P(E)P(F) = ¼
  - P(EG)=P(E)P(G) = ¼
  - P(FG)=P(F)P(G) = ¼ (Mutually independent)
  - However, ¼=P(EFG)≠P(E)P(F)P(G)=1/8. Hence, the events EFG are not jointly independent.

Bayes' Formula

- Suppose that F_1, F_2, …, F_n are mutually exclusive events such that $S = \bigcup_{i=1}^{n} F_i$, then
  - $E = ES = E(\bigcup_{i=1}^{n} F_i) = \bigcup_{i=1}^{n} EF_i$
  - $P(E) = \sum_{i=1}^{n} P(EF_i)$
    $= \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$ from (1.5) (1.8)
  - $P(F_i|E) = \frac{P(E|F_i)}{P(E)}$ from (1.5)
    $= \frac{P(E|F_i)P(F_i)}{\sum_{i=1}^{n} P(EF_i)P(F_i)}$ from (1.8) (1.9)

- Equation (1.9) is known as Bayes' Formula

- Ex) $F \cup F^c = S$
  - $P(E)=P(ES)=P(E(F \cup F^c))=P(EF \cup EF^c)=P(EF)+P(EF^c)$
    $=P(E|F)P(F)+P(E|F^c)\ P(F^c)=P(E|F)P(F)+P(E|F^c)\ (1-P(F))$
  - $P(F|E)=P(E|F)P(F)/P(E)=P(E|F)P(F)/(P(E|F)P(F)+P(E|F^c)\ P(F^c))$

Example of Bayes'Formula

- Ex. 1.13] In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the answer and $1-p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?
  - Let C and K denote respectively the event that the student answers the question correctly and the event that she actually knows the answer. Now
  - $P(K|C) = P(KC)/P(C)$
    $= P(C|K)P(K)/(P(C|K)P(K) + P(C|K^c)P(K^c))$
    $= p/(p + (1/m)(1 - p))$
    $= mp/(1 + (m - 1)p)$
  - Thus, for example, if $m = 5$, $p = ½$, then the probability that a student knew the answer to a question she correctly answered is 5/6.

Example 1 of Probability

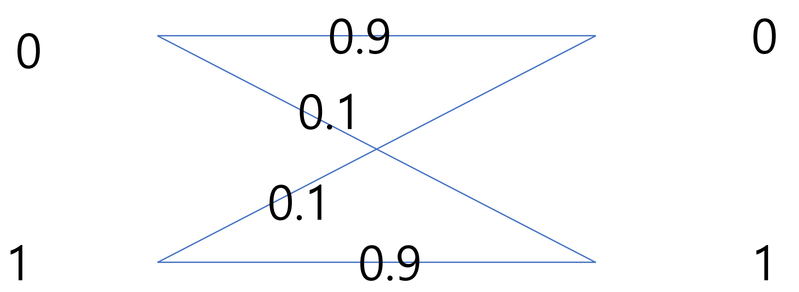
- Consider an experiment with a coin and a die. We define event A as an outcome with “head” of the coin and an even number of the die. Even B is defined as an outcome with a number greater than 3. Find the following answers
  - P(A)
  - P(B)
  - P(AB)
  - P(A|B)
  - P(B|A)

Answer: Example 1

- $P(A)=3/12=1/4$
- $P(B)=6/12=1/2$
- $AB = \{H4, H6\}$, $P(AB)=2/12=1/6$
- $P(A|B)=P(AB)/P(B)=1/3$
- $P(B|A)=P(AB)/P(B)=2/3$

Example 2 of Probability

- Let A_1 (A_0) and B_1 (B_0) be the event that 1 (0) is sent and the event that 1(0) is received, respectively.
- Assumption
  - $P(A_0) = 0.8$, $P(A_1) = 1 - P(A_0) = 0.2$,
  - The probability of error, i.e., $p=P(B_1|A_0)= P(B_0|A_1)$, is 0.1



- Find
  - The error probability at the receiver
  - The probability that 1 is sent when the receiver decides 1.

Answer: Example 2

- Parameters
  - $P(A_0) = 0.8$, $P(A_1) = 1 - P(A_0) = 0.2$, $P(B_1|A_0) = P(B_0|A_1) = 0.1$
- The error probability at the receiver
  - $P(\text{error}) = P(A_0 B_1) + P(A_1 B_0) = P(B_1|A_0)P(A_0) + P(B_0|A_1)P(A_1) = 0.1$
- The probability that 1 is sent when the receiver decides 1.
  - $P(A_m|B) = \frac{P(B|A_m)P(A_m)}{P(B)} = \frac{P(B|A_m)P(A_m)}{\sum_{n=1}^{N} P(B|A_n)P(A_n)}$
  - $P(A_1|B_1) = \frac{P(B_1|A_1)P(A_1)}{P(B_1)} = \frac{P(B_1|A_1)P(A_1)}{P(B_1|A_1)P(A_1) + P(B_1|A_0)P(A_0)} = \frac{0.9\times 0.2}{0.9\times 0.2 + 0.1\times 0.8} =$ 0.69