Advanced Linear Algebra

Chapter 1: Matrices and Gaussian Elimination

Jin-Ho Chung
School of IT Convergence
University of Ulsan

Central problem of linear algebra: Solve linear equations

Example: Given

$$\begin{aligned} 2x + 3y &= 7 \\ 4x - 5y &= 11, \end{aligned}$$

solve the equations:

Method (a) **Elimination of variables**

$$2x + 3y = 7 \quad \Longrightarrow \quad 2x = 7 - 3y \quad \Longrightarrow \quad x = (7 - 3y)/2$$

So $4x - 5y = 11$ becomes $4\left(\dfrac{7-3y}{2}\right) - 5y = 11.$

$$\begin{aligned} &\Rightarrow\ (14 - 6y) - 5y = 11 \\ &\Rightarrow\ 3 = 11y \\ &\therefore\ y = 3/11. \end{aligned}$$

$$\begin{aligned} &\Rightarrow\ x = \frac{7 - 9/11}{2} \\ &\therefore\ x = \frac{34}{11}. \end{aligned}$$

Method (b) **Cramer's Rule** (by determinants)

In matrix notation

$$\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}.$$

$$D = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -10 - 12 = -22.$$

$$x = \frac{\begin{vmatrix} 7 & 3 \\ 11 & -5 \end{vmatrix}}{D} = \frac{-35-33}{-22} = \frac{68}{22} = \frac{34}{11}.$$

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 4 & 11 \end{vmatrix}}{D} = \frac{22-28}{-22} = \frac{6}{22} = \frac{3}{11}.$$

Method (c) **Matrix reduction** (by Elementary Row Operations)

(1) permuting rows;

(2) multiplying any row by constant $c \neq 0$;

(3) adding one row to another.

In the example

$$\underbrace{\begin{bmatrix} 2 & 3 & \vdots & 7 \\ 4 & -5 & \vdots & 11 \end{bmatrix}}_{\text{"augmented matrix"}} \longrightarrow \underbrace{\begin{bmatrix} \underline{2} & 3 & 7 \\ 0 & \underline{-11} & -3 \end{bmatrix}}_{\text{pivots}} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{3}{11} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & \frac{34}{11} \\ 0 & 1 & \frac{3}{11} \end{bmatrix} \iff \begin{array}{l} 1 \cdot x + 0 \cdot y = \frac{34}{11} \\ 0 \cdot x + 1 \cdot y = \frac{3}{11} \end{array}$$

- What happens when we do each of the elementary row operations?

(1) *Permuting the equations still leaves us with the same system*, so we still have the same solution(s).

(2) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, then for any $c \neq 0$

$$ca_1x_1 + ca_2x_2 + \cdots + ca_nx_n = cb$$

has the same solution(s).

(3) Consider the two systems

$$\mathrm{I} \ : \ \left[\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= b \\ a_1'x_1 + a_2'x_2 + \cdots + a_n'x_n &= b' \end{aligned} \right.$$

$$\mathrm{II} \ : \ \left[\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= b \\ (a_1 + a_1')x_1 + (a_2 + a_2')x_2 + \cdots + (a_n + a_n')x_n &= b + b' \end{aligned} \right.$$

The systems I and II have the same solution(s).

■ In general, *a system of n linear equations in n unknowns* looks like:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

The system in (1) is said to be *independent* if

$$D \triangleq \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Otherwise, it is said to be *dependent*.

The system in (1) is said to be *homogeneous* if $b_1 = b_2 = \cdots = b_n = 0.$ Otherwise, it is said to be *inhomogeneous* (i.e., if at least one of the c_i's is non-zero).

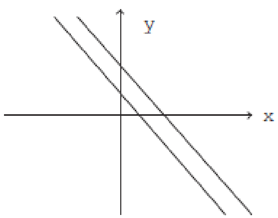
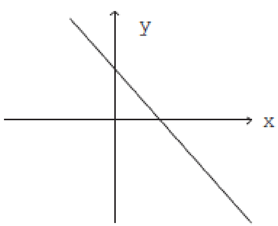
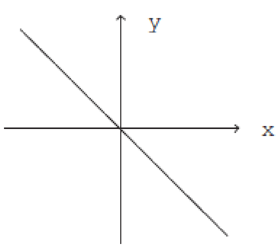
	independent	dependent
inhomogeneous	Exactly one solution exists and is unique. It is not $(0, 0, \ldots 0)$.	may be no solutions; may be infinitely many; but not $(0, 0, \ldots 0)$.
homogeneous	The $(0, 0, \ldots 0)$ is the unique solution.	$(0, 0, \ldots 0)$ is a solution, but not the only solution.

There is one and only one solution.

$$(x_1, x_2, \cdots x_n) = (k_1, k_2, \cdots k_n)$$

homogeneous $\implies (x_1, x_2, \cdots x_n) = (0, 0, \ldots 0)$ is always a solution.

Example:

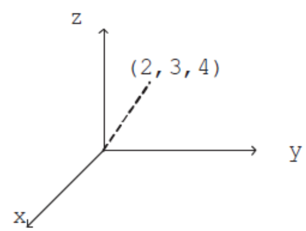
$x + y = 2$ $x + 2y = 5$ $\Longrightarrow (x, y) = (-1, 3)$	$\left.\begin{aligned} x + y &= 1 \\ 3x + 3y &= 2 \end{aligned}\right\}$	no solution	
	$\left.\begin{aligned} x + y &= 1 \\ 3x + 3y &= 3 \end{aligned}\right\}$	infinitely many solutions	
$x + y = 0$ $x + 2y = 0$ $\Longrightarrow (x, y) = (0, 0)$	$\left.\begin{aligned} x + y &= 0 \\ 3x + 3y &= 0 \end{aligned}\right\}$	$y = -x$ is the general solution	

☐ **Vectors** (in a narrow sense)

- A vector is *an ordered array of numbers*.
  - Numbers are real, otherwise specified.
  - Can be extended to an abstract object.

Example:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$



- Row vectors and column vectors $\longleftarrow$ conventional meaning

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

- We can identify a vector having n components with a vector in n-dimensional space.

- Addition

  If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then we mean by $\mathbf{x}+\mathbf{y}$:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

  (Similarly, with subtraction)

☐ Fields

A set $\mathcal{F}$ together with two binary operations "$+$" and "$\cdot$" is called a *field* if the followings are satisfied:

a1) closure $\quad a+b \in \mathcal{F}, \quad \forall\, a, b \in \mathcal{F}$

a2) associativity $\quad (a+b)+c = a+(b+c), \quad \forall\, a, b, c \in \mathcal{F}$

a3) identity $\quad \exists 0 \in \mathcal{F}$ such that $0+a = a+0 = a, \quad \forall\, a \in \mathcal{F}$

a4) inverse $\quad \forall a \in \mathcal{F}, \exists -a \in \mathcal{F}$ such that $a+(-a) = (-a)+a = 0$

a5) commutativity $\quad a+b = b+a \quad \forall\, a, b \in \mathcal{F}$

m1) closure $\quad a \cdot b \in \mathcal{F}, \quad \forall\, a, b \in \mathcal{F}$

m2) associativity $\quad (a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \forall\, a, b, c \in \mathcal{F}$

m3) identity $\quad \exists 1 \in \mathcal{F}$ such that $1 \cdot a = a \cdot 1 = a, \quad \forall\, a \in \mathcal{F}$

m4) inverse $\quad \forall a \in \mathcal{F} \backslash \{0\}, \exists a^{-1} \in \mathcal{F}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$

m5) commutativity $\quad a \cdot b = b \cdot a \quad \forall\, a, b \in \mathcal{F}$

6) distributivity $\quad a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall\, a, b, c \in \mathcal{F}$

Example:

$\mathcal{F} = \mathbb{Q} \triangleq$ set of rational numbers

$\mathcal{F} = \mathbb{R} \triangleq$ set of real numbers

$\mathcal{F} = \mathbb{C} \triangleq$ set of complex numbers

$\mathcal{F} = \mathbb{Z}_3 \triangleq \{0, 1, 2, \}$ with addition and multiplication mod 3

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Note:

1) $(R,+)$ satisfies the axioms a1) $\sim$ a4).
$\Rightarrow R$ is a *group* under the operation '$+$'

2) $(R,+)$ satisfies the axioms a1) $\sim$ a5).
$\Rightarrow R$ is a *abelian group* or *commutative group*.

3) $(R,+,\cdot)$ satisfies the axioms a1) $\sim$ a5), and
+ m1) , m2) , 6) $\Rightarrow$ *ring*
+ m3) $\Rightarrow$ *ring with unity*
+ m5) $\Rightarrow$ *commutative ring*

4) $\mathbb{Z}_n \triangleq \{0,1,\cdots,n-1\}$ ($\triangleq$ the set of integers modulo n) is a ring under multiplication and addition modulo n.

5) $\mathbb{Z}_n$ is a field $\Leftrightarrow n$ is prime.

6) $|\mathcal{F}| < \infty \Rightarrow \mathcal{F}$ is called a *finite field*.

7) $\mathcal{F}$ is a finite field $\Rightarrow |\mathcal{F}| = p^m$ for some prime p and an integer m.

Conversely, *there exist a finite field with p^m elements for any prime p and any positive integer m.*

Notation: Galois field

$$\begin{aligned} \mathcal{F} &= \mathrm{GF}(q) \text{ or } \mathbb{F}_q \\ &= \text{a finite field with } q \text{ elements.} \end{aligned}$$

8) $|\mathcal{F}| = p \;\Rightarrow\; \mathcal{F} \cong \mathbb{Z}_p$, called a *prime field*.

$p =$ *characteristic* of $\mathcal{F}$.

□ Matrices

- An $m \times n$ matrix (over a field $\mathcal{F}$) is *an ordered rectangular array* *m rows and n columns*, (where the entries are in $\mathcal{F}$).

<u>Notation</u>: parenthesis ( ) or square brackets [ ]

Example: $\mathcal{F} = \mathbb{R}$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & -6 \\ -3 & -1 & -1 \end{bmatrix}$$

2×3 matrix $\qquad$ 4×3 matrix

- If $m = n$, the matrix is said to be *a square matrix*.

Example:

$$C = \begin{bmatrix} 2 & 4 \\ 16 & -1 \end{bmatrix} \qquad 2 \times 2$$

- square matrix of order $n \iff n \times n$ square matrix.
- $1 \times n$ matrix $\Longrightarrow$ row vector
  $m \times 1$ matrix $\Longrightarrow$ column vector (conventionally, vector)
- *Component notation* for matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \triangleq (a_{ij}); \quad m \times n$$

where a_{ij} = the element in the ith row and jth column

- **Matrix addition**

If $A = (a_{ij}), B = (b_{ij})$, and $C = (c_{ij})$ have the same size (i.e., al them are $m \times n$ matrices), then

$$C = A + B \iff c_{ij} = a_{ij} + b_{ij} \;\; \forall i, j$$

"componentwise"

Example:

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 1+(-1) & 2+0 & -4+2 \\ 0+1 & 3+(-5) & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & -2 \\ 1 & -2 & 2 \end{bmatrix}$$

- *Zero matrix or null matrix*

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}; \quad m \times n$$

The matrix O is the additive identity

$$A + O = O + A = A$$

- **Scalar multiplication**

Let $A = (a_{ij})$ be an $m \times n$ matrix over the field $\mathcal{F}$ and c be an element in $\mathcal{F}$. Then $B = cA$ is the $m \times n$ matrix, where

$$b_{ij} = ca_{ij} \qquad\qquad \forall i, j.$$

Example:

$$2\begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot(-2) & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 6 & 10 \end{bmatrix}$$

Note: $A - B = A + (-1)B$.

- **Transpose**

The matrix $B = A^t$ is the transpose of the matrix A, where

$$b_{ij} = a_{ji} \qquad \forall i, j.$$

Example:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & 7 \end{bmatrix} \qquad B = A^t = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ 5 & 7 \end{bmatrix}$$

Note:

If $B = A^t$, then $A = B^t$.

If $A = A^t$, then A is said to be *symmetric*.

Example:

$$C = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 1 & -1 \\ 0 & -1 & 4 \end{bmatrix} \quad \text{is symmetric.} \quad (\because C = C^t)$$

• Matrix multiplication

Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times s$ matrix. Then the *matrix product* AB is the $(m \times s)$ matrix $C = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2s} \\ & & & \vdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix} \Longrightarrow (i,j)^{th}\text{entry}$$

Note: Inner (dot) product of two vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{a} \cdot \mathbf{b} \triangleq \mathbf{a}^t \mathbf{b} = \begin{bmatrix} a_1 & a_2 \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i.$$

Note: Two other viewpoints of multiplication.

i) $$\underset{m\times n}{\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \dots & \mathbf{a}_n \end{bmatrix}} \underset{n\times s}{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ b_{21} & b_{22} & \dots & b_{2s} \\ & & \vdots & \\ b_{n1} & b_{n2} & \dots & b_{ns} \end{bmatrix}} = \underset{m\times s}{\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \dots & \mathbf{c}_s \end{bmatrix}}$$

Then

$$\begin{aligned} \mathbf{c}_1 &= b_{11}\mathbf{a}_1 + b_{21}\mathbf{a}_2 + \dots + b_{n1}\mathbf{a}_n \\ &= \sum_{l=1}^{n} b_{l1}\mathbf{a}_l \end{aligned}$$

In general,

$$\mathbf{c}_j = \sum_{l=1}^{n} \mathbf{a}_l b_{lj} \quad \longrightarrow \quad j\text{th column of } C$$

i.e., *each column of AB is a linear combination of the columns of A.*

ii) $$\underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times s}{\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}} = \underset{m\times s}{\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_m \end{bmatrix}}$$

$$\mathbf{c}_i = \sum_{l=1}^{n} a_{il}\mathbf{b}_l \quad \longrightarrow \quad i\text{th row of } C$$

Each row of AB is a linear combination of the row of B.

Example:

1) $$AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2\cdot 1 + 3\cdot 5 & 2\cdot 2 + 3\cdot(-1) & 2\cdot 0 + 3\cdot 0 \\ 4\cdot 1 + 0\cdot 5 & 4\cdot 2 + 0\cdot(-1) & 4\cdot 0 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}$$

2) $$AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix}$$
$$\mathbf{c}_1 = 1\cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 5\cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \end{bmatrix}$$

3) $$AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$
$$\mathbf{c}_2 = 4\cdot \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} + 0\cdot \begin{bmatrix} 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \end{bmatrix}$$

Theorem 1 *Matrix multiplication is associative, i.e.,*

$$(AB)C = A(BC)$$

Note: Matrix multiplication is not commutative. (in general)

Theorem 2 *Matrix multiplication is distributive over addition.*

$$A(B+C) = AB + AC$$

Theorem 3 $(AB)^t = B^t A^t$

- The $n \times n$ *identity matrix* I_n is a square matrix, where

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{etc} \qquad (\text{simply, } I)$$

Theorem 4 *(Assuming sizes allow multiplication)*

$$IA = AI = A$$

Definition 5 Let A be an $n \times n$ square matrix. If there exists an $n \times n$ square matrix B such that

$$AB = BA = I_n$$

then B is called the *inverse of A*, written by A^{-1}.

Theorem 6 *If $AB = I = CA$, then $B = C = A^{-1}$.*

Note:

- If A is an $n \times n$ square matrix and $AB = I_n$, then $BA = I_n$, so $B = A^{-1}$.
- Similarly, $BA = I_n \Longrightarrow AB = I_n$, so $B = A^{-1}$.

Exercise: Assume A and B are all $n \times n$ matrices with inverses. Prove that

1) $(AB)^{-1} = B^{-1}A^{-1}$

2) $(A^t)^{-1} = (A^{-1})^t$

☐ Example of Gauss-Jordan Method

$$\begin{aligned} 2x - 4y + 6z &= 18 \\ -3x + 9y &= -12 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

(using augmented matrices)

Wilhelm Jordan

(cf. Camille Jordan: JCF, Jordan measure,...)

$$\begin{bmatrix} \underline{2} & -4 & 6 & : & 18 \\ -3 & 9 & 0 & : & -12 \\ 2 & -5 & 5 & : & 17 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -4 & 6 & 18 \\ 0 & \underline{3} & 9 & 15 \\ 0 & -1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{2} & -4 & 6 & 18 \\ 0 & \underline{3} & 9 & 15 \\ 0 & 0 & \underline{2} & 4 \end{bmatrix}$$

"forward elimination"

↑

At this stage
we can use
"back substitution"

$$\longrightarrow \begin{bmatrix} 2 & -4 & 0 & 6 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

Example: Exchange rows

$$u + v + w = 3$$
$$2u + 2v + 5w = 9$$
$$4u + 6v + 8w = 18$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 9 \\ 4 & 6 & 8 & 18 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{1} & 1 & 1 & 3 \\ 0 & \underline{2} & 4 & 6 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

Example: Singular Case

$$u + v + w = 3$$
$$2u + 2v + 5w = 9$$
$$4u + 4v + 8w = 16$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 9 \\ 4 & 4 & 8 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$\Longrightarrow$ infinitely many solutions

$$w = 1; \quad u + v = 2$$

$$u + v + w = 3$$
$$2u + 2v + 5w = 9$$
$$4u + 4v + 8w = 20$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 9 \\ 4 & 4 & 8 & 20 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\Longrightarrow$ "No solutions"

☐ Elementary Matrices

Definition 7 An $n \times n$ *permutation matrix* is a matrix all of whose entries are either 0 or 1 and in addition each column and each row of the matrix contains precisely one 1.

Example: $n = 3$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Exchange rows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \end{bmatrix}$$; Row 2 and Row 3 are exchanged.

Exercise: Prove that for any permutation matrix P

$$P^{-1} = P^t. \qquad (\text{i.e., } PP^t = I)$$

Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 8 A *diagonal matrix* D is a square matrix all of whose off-diagonal elements are zero, i.e.,

$$(D)_{ij} = 0 \quad \text{if} \;\; i \neq j.$$

Example:

$$\begin{bmatrix} d_1 & & 0 & \\ & d_2 & & \\ & & \ddots & \\ & 0 & & d_n \end{bmatrix}$$

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} d_1\mathbf{r}_1 \\ d_2\mathbf{r}_2 \\ \vdots \\ d_n\mathbf{r}_n \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} = \begin{bmatrix} d_1\mathbf{c}_1 & d_2\mathbf{c}_2 & \cdots & d_n\mathbf{c}_n \end{bmatrix}$$

Definition 9 A *lower* (*upper*) *triangular matrix* L (U) is a square matrix in which

$$l_{ij} = 0 \ \text{ for } j > i \quad (u_{ij} = 0 \ \text{ for } j < i, \ \text{respectively}).$$

Example:

$$\begin{bmatrix} 1 & \underline{0} & \underline{0} \\ 5 & 6 & \underline{0} \\ 7 & 9 & -3 \end{bmatrix} = L$$

Theorem 10 *This product of two lower(upper) triangular matrices is once again a lower(upper) matrix.*

Exercise: Show that the inverse of a lower (upper) triangular matrix is also lower(upper) triangular.

- **Elementary row operations and elementary matrices**

1) Permute two rows of A: multiply A by a permutation matrix.
   For example, let

$$P = \begin{array}{c} \begin{bmatrix} 1 & & & \vdots & & & & \vdots & & & \\ & \ddots & & \vdots & & & & \vdots & & & \\ & & 1 & \vdots & & & & \vdots & & & \\ & & & 0 & & \cdots & & 1 & & & \\ & & & \vdots & 1 & & & \vdots & & & \\ & & & \vdots & & \ddots & & \vdots & & & \\ & & & \vdots & & & 1 & \vdots & & & \\ & & & 1 & & \cdots & & 0 & & & \\ & & & \vdots & & & & \vdots & 1 & & \\ & & & \vdots & & & & \vdots & & \ddots & \\ & & & \vdots & & & & \vdots & & & 1 \end{bmatrix} \\ \qquad\qquad i \qquad\qquad\quad j \qquad\qquad\qquad \end{array}.$$

PA : interchange ith and jth rows of A.

2) Multiply one row of A by a nonzero constant c: multiply A by a diagonal matrix.

Example:

$$D = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow i \\ \\ \\ \\ \end{array}$$

DA : multiply ith row of A by c.

3) rth row of $A \longrightarrow$ (rth row of A) $+ c \cdot$ (sth row of A): multiply A by a lower triangular matrix.

Example:

$$L = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & & \ddots & & & \\ & & c & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \longleftarrow r \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ s \end{array} & \end{array}$$

□ LDU Factorization

Example:

$$\underbrace{\begin{bmatrix} 2 & -4 & 6 \\ -3 & 9 & 0 \\ 2 & -5 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 18 \\ -12 \\ 17 \end{bmatrix}}_{\mathbf{b}}$$

Gaussian Elimination: (Forward elimination)

$$\begin{bmatrix} \underline{2} & -4 & 6 & : & 18 \\ -3 & 9 & 0 & : & -12 \\ 2 & -5 & 5 & : & 17 \end{bmatrix} \xrightarrow[R_2 + \frac{3}{2}R_1]{} \begin{bmatrix} 2 & -4 & 6 & : & 18 \\ 0 & 3 & 9 & : & 15 \\ 2 & -5 & 5 & : & 17 \end{bmatrix}$$

$$\xrightarrow[R_3 + (-1)R_1]{} \begin{bmatrix} \underline{2} & -4 & 6 & : & 18 \\ 0 & \underline{3} & 9 & : & 15 \\ 0 & -1 & -1 & : & -1 \end{bmatrix} \xrightarrow[R_3 + \frac{1}{3}R_2]{} \begin{bmatrix} \underline{2} & -4 & 6 & : & 18 \\ 0 & \underline{3} & 9 & : & 15 \\ 0 & 0 & \underline{2} & : & 4 \end{bmatrix}$$

(By back substitution, $z = 2, y = -1, x = 1$)

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 6 \\ -3 & 9 & 0 \\ 2 & -5 & 5 \end{bmatrix} = \begin{bmatrix} \underline{2} & -4 & 6 \\ 0 & \underline{3} & 9 \\ 0 & 0 & \underline{2} \end{bmatrix}$$

$$E_{32} \qquad E_{31} \qquad E_{21} \qquad A \qquad U$$

Therefore,

$$A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U.$$

Note: When only the (i, j)-entry α_{ij} of an elementary lower triangular matrix is nonzero $(i < j)$,

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \alpha_{ij} & \ddots & \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\alpha_{ij} & \ddots & \\ & & & 1 \end{bmatrix}$$

Example:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the above example for the Gaussian elimination procedure, we have an LDU factorization:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}}_{E_{32}^{-1}} \begin{bmatrix} \underline{2} & -4 & 6 \\ 0 & \underline{3} & 9 \\ 0 & 0 & \underline{2} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \underline{2} & -4 & 6 \\ 0 & \underline{3} & 9 \\ 0 & 0 & \underline{2} \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Theorem 11 *The LDU decomposition is unique. In other words, if $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, then $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$.*

Note: In this case, it is assumed that L and U have '1' on the diagonals.

- **LDU factorization of symmetric matrices**

  * If A is symmetric and $A = LDU$, then

$$A = A^t = (LDU)^t = U^t D^t L^t = \underbrace{U^t}_{\text{lower triangular}} \quad D \quad \underbrace{L^t}_{\text{upper triangular}}$$

From the uniqueness,

$$L = U^t \quad \text{and} \quad U = L^t.$$

The factorization of a symmetric matrix A is:

$$A = LDL^t$$

- **Cholesky Decomposition**

  For symmetric matrices with positive pivots, we have the Cholesky Decomposition:

$$A = SS^t$$

where $S = L\sqrt{D}$. $\square$

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \text{ with all } d_i > 0 \quad (\text{Note } d_i \neq 0)$$

Let

$$\sqrt{D} \triangleq \begin{bmatrix} \sqrt{d_1} & & & 0 \\ & \sqrt{d_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{d_n} \end{bmatrix}$$

Then $D = \sqrt{D} \cdot \sqrt{D}$ and $(\sqrt{D})^t = \sqrt{D}$.

$$A = (L\sqrt{D})(\sqrt{D}^{\,t} L^t) = (L\sqrt{D})(L\sqrt{D})^t$$

• LDU decomposition with row exchange

Instead of $A = LDU$, we will now have a $PA = LDU$ decomposition.

Definition 12 If a suitable interchange of equations results in all the pivots being nonzero, then the $n \times n$ coefficient matrix A is said to be *nonsingular*.

Therefore, a nonsingular matrix can always be factored into the form:

$$\underset{\substack{\uparrow \\ \text{permutation matrix}}}{P}A = LDU$$

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad PA = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 8 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 8 \\ 2 & 2 & 5 \end{bmatrix} \underset{R_2 - 4R_1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 2 & 2 & 5 \end{bmatrix} \underset{R_3 - 2R_1}{\longrightarrow} \begin{bmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{2} & 4 \\ 0 & 0 & \underline{3} \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

□ Gauss-Jordan method for computing inverses

- **Finding an inverse by Gauss-Jordan method**
  - To find B such that $AB = I$, we apply elementary row operations C such that $CA = I$. Then

$$CAB = CI, \quad \text{so } B = C.$$

In general, $\underbrace{DE_l \cdots E_2 P E_1}_{C} A = I$.

  - The procedure can be applied to the augmented matrix $[A : I]$.

$$C\,[A : I] \;\longrightarrow\; [I : C]$$

Example: In order to find the inverse of A, we consider

$$\underbrace{A}_{3\times 3}\underbrace{\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}}_{B} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3$ "standard basis vectors"

Then we have three systems:

$$\left[\begin{array}{l} A\mathbf{x}_1 = \mathbf{e}_1 \\ A\mathbf{x}_2 = \mathbf{e}_2 \\ A\mathbf{x}_3 = \mathbf{e}_3 \end{array}\right.$$

The three systems can be solved simultaneously in the following:

$$\left[\begin{array}{rrrcrrr} \underline{1} & 3 & -2 & : & 1 & 0 & 0 \\ 2 & 5 & -3 & : & 0 & 1 & 0 \\ -3 & 2 & -4 & : & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrcrrr} 1 & 3 & -2 & : & 1 & 0 & 0 \\ 0 & \underline{-1} & 1 & : & -2 & 1 & 0 \\ 0 & 11 & -10 & : & 3 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrcrrr} 1 & 3 & -2 & : & 1 & 0 & 0 \\ 0 & -1 & 1 & : & -2 & 1 & 0 \\ 0 & 0 & \underline{1} & : & -19 & 11 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrcrrr} 1 & 3 & 0 & : & -37 & 22 & 2 \\ 0 & \underline{-1} & 0 & : & 17 & -10 & -1 \\ 0 & 0 & 1 & : & -19 & 11 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrcrrr} 1 & 0 & 0 & : & 14 & -8 & -1 \\ 0 & -1 & 0 & : & 17 & -10 & -1 \\ 0 & 0 & 1 & : & -19 & 11 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrcrrr} 1 & 0 & 0 & : & 14 & -8 & -1 \\ 0 & 1 & 0 & : & -17 & 10 & 1 \\ 0 & 0 & 1 & : & -19 & 11 & 1 \end{array}\right]$$

$$\downarrow$$

$$B = A^{-1}$$