Advanced Linear Algebra

Chapter 4: Determinant

Jin-Ho Chung
School of IT Convergence
University of Ulsan

Table of Contents

Definition

- $M_n(F)$: the set of all $n \times n$ matrices defined over F

- A function $D : M_n(F) \rightarrow F$ is called a determinant function if the three conditions are satisfied.

  1) n-linearity:

$$D\left(\left[\begin{array}{c} \alpha\mathbf{a}_1 + \beta\mathbf{b}_1 \\ M \end{array}\right]\right) = \alpha D\left(\left[\begin{array}{c} \mathbf{a}_1 \\ M \end{array}\right]\right) + \beta D\left(\left[\begin{array}{c} \mathbf{b}_1 \\ M \end{array}\right]\right)$$

  2) alternating property: For $i \neq j$, let $B = P_{ij}A$ with row interchanging P_{ij}.

$$D(B) = -D(A).$$

  3) identity matrix:

$$D(I) = 1.$$

Additional Properties (1)

4) If the two rows of A are equal, then $\det(A) = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

5) Subtracting a multiple of one row from another row preserves $\det(A)$.

$$\begin{vmatrix} a - lc & b - ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6) If A has a zero row, then $\det(A) = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

Additional Properties (2)

7) If A is triangular, $\det(A)$ is the product of all the diagonal entries.
$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad.$$

8) $\det(A) = 0$ if and only if A is singular.

9) Product of two matrices:
$$\det(AB) = \det(A)\det(B).$$

10) Transpose:
$$\det(A^T) = \det(A).$$

Table of Contents

2×2 Case

$$\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
&= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
&= ad - bc.
\end{aligned}$$

3×3 Case

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}
$$

$$
+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}
$$

$$
\begin{aligned}
= \quad & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
& -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}.
\end{aligned}
$$

Permutation

- Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$.
- Expression: a permutation σ of order 5 can be represented in these ways -

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix},$$

or

$$\sigma(1) = 2,\ \sigma(2) = 1,\ \sigma(3) = 4,\ \sigma(4) = 5,\ \sigma(5) = 3.$$

- Any permutation can be represented as a product of inversions:

$$\sigma = (3\ 4)(3\ 5)(1\ 2).$$

(Note that there are multiple representations for a σ, but the parity of the $\#$ of inversions is invariant.)

Permutation Matrix

- A matrix P is called a permutation matrix if it is a $(0,1)$-matrix such that 1 appears exactly once at every row and every column.
- An $n \times n$ permutation matrix P_σ corresponding to a permutation σ on $\{1,\ldots,n\}$ is given by

$$(P_\sigma)_{i,j} = \begin{cases} 1, & \text{if } j = \sigma(i) \\ 0, & \text{otherwise.} \end{cases}$$

- Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \leftrightarrow \quad P_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

General Formula for Determinant

- Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$.
- The determinant function of an $n \times n$ matrix $A = (a_{i,j})_{1 \le i,j \le n}$ is given by

$$\det(A) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \det(P_\sigma)$$

- However, it may be too complex to compute the determinant of a large matrix in this way.

Table of Contents

Determinant for 3×3 Matrices

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} .
$$

(Exercise: Expand with respect to the second row.)

Formula for $n \times n$ Matrices

- M_{ij} (Minor of a_{ij} in A):
  the determinant of the $(n-1) \times (n-1)$ matrix after removing the ith row and the jth column from A
- A_{ij} (cofactor of a_{ij} in A):

$$A_{ij} = (-1)^{i+j} M_{ij}$$

- Laplace expansion for determinant:

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij}$$

Table of Contents

Application 1: Calculation of Inverse

- Note that

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \begin{cases} 0, & \text{if } i \neq k \\ \det(A), & \text{if } i = k. \end{cases}$$

- Therefore,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix}}_{\text{adj}(A)\text{: adjoint matrix of } A}$$

$$= \begin{bmatrix} \det(A) & & \\ & \ddots & \\ & & \det(A) \end{bmatrix}$$

Application 2: Cramer's Rule

- For an $n \times n$ matrix A, $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$ and $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$, assume that $A\mathbf{x} = \mathbf{b}$.
- Let $\mathbf{a}_i$ be the ith column of A. Insert $\mathbf{b}$ at the jth column.

$$
\begin{aligned}
&\det\left(\mathbf{a}_1 \ \cdots \ \mathbf{b} \ \cdots \ \mathbf{a}_n\right) \\
&\quad = \det\left(\mathbf{a}_1 \ \cdots \ \sum_{k=1}^{n} x_k \mathbf{a}_k \ \cdots \ \mathbf{a}_n\right) \\
&\quad = \sum_{k=1}^{n} \det\left(\mathbf{a}_1 \ \cdots \ x_k \mathbf{a}_k \ \cdots \ \mathbf{a}_n\right) \\
&\quad = x_j \cdot \det(A).
\end{aligned}
$$

- Therefore,

$$
x_j = \frac{\det\left[\mathbf{a}_1 \ \cdots \ \mathbf{b} \ \cdots \ \mathbf{a}_n\right]}{\det(A)}.
$$

General Form of Laplace Expansion (1)

- Fix m row in A. Then,

$$\begin{aligned}\det(A) &= \sum_{\{j_1,\dots,j_m\}\subseteq\mathbb{Z}_n} \det(A(i_1,\dots,i_m)(j_1,\dots,j_m)) \\ &\qquad \cdot\det(A^c(i_1,\dots,i_m)(j_1,\dots,j_m)) \\ &\qquad \cdot(-1)^{i_1+\cdots+i_m+j_1+\cdots+j_m}\end{aligned}$$

where $A(i_1,\dots,i_m)(j_1,\dots,j_m)$ is the $m\times m$ submatrix of A consisting of the intersections of the rows $i_1,\dots,i_m$ and the columns $j_1,\dots,j_m$, and A^c is the $(n-m)\times(n-m)$ submatrix consisting of the remained rows and columns.

General Form of Laplace Expansion (2)

- Let
$$A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

- Expansion with $i_1 = 1$ and $i_2 = 2$:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} (-1)^{1+2+1+2} + \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} (-1)^{1+2+1+3} \\ &+ \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} (-1)^{1+2+1+4} + \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} (-1)^{1+2+2+3} \\ &+ \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} (-1)^{1+2+2+4} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} (-1)^{1+2+3+4} \end{aligned}$$

- Exercise: Expand with $i_1 = 1$ and $i_2 = 3$, and calculate $\det(A)$.

Volume of a Parallelogram

- In the right-angled system, the area (or volume) of the parallelogram represented by $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is given by $\det(A)$, where

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}.$$