104B Computational Project 2

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1 Introduction

The Black-Scholes equation, a cornerstone of modern financial mathematics, provides a theoretical estimate for the price of European-style options. Mathematically, it is a partial differential equation that describes the price evolution of an option over time. The equation is based on several assumptions, including constant volatility and interest rates, and it ignores dividends. Its significance lies in its ability to model the dynamics of an option's price by factoring in the underlying asset's current price, time to expiration, volatility, and the risk-free interest rate. The Black-Scholes formula revolutionized the field of financial derivatives by providing a systematic method to price options, leading to the growth of financial markets in options and other derivative securities. The main purpose of the project is to investigate the stability of numerical solutions for the Black-Scholes Equation given different initial conditions or financial settings.

2 Methodology

The Black-Scholes equation, which is fundamental in financial mathematics, especially for option pricing, is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where we define the variables to be V(S, t): Price of option (we will be using call options), S(t): Underlying stock Price, t: Time, σ: Volatility, r: Risk Free Interest Rate, and T: Final time/Option maturity

The finite differences we used for approximating the first and second derivatives are as such:

$$\frac{\partial V}{\partial t}(S,t) \approx \frac{1}{k}(V(S,t+k) - V(S,t)) \tag{2}$$

$$\frac{\partial V}{\partial S}(S,t) \approx \frac{1}{2h}(V(S+h,t) - V(S-h,t)) \tag{3}$$

$$\frac{\partial^2 V}{\partial S^2}(S,t) \approx \frac{1}{h^2}(V(S+h,t) - 2V(S,t) + V(S-h,t)) \tag{4}$$

Where one can replace k with Δt and $2h$ and h^2 with $2\Delta S$, ΔS^2 respectively if they prefer different notation.

In numerical approximation we are using centered difference for the second order term, as well as the centered difference for the first derivative with respect to the stock price. For the time step, we cannot use a backward or implicit method because we know the price of the stock ahead of time, thus our method cannot be implicit.

3 Computational Setup

The computational setup involves defining a grid over the domain of interest and marching through time to solve the equation at each grid point. The Boundary conditions:

$$C(0,t) = 0 \text{ for all } t \tag{5}$$

$$C(S,t) \sim S - Ke^{-r(T-t)} \text{ as } S \to \infty \tag{6}$$

$$C(S,T) = \max\{S - K, 0\} \tag{7}$$

where K is the strike price, volatility defined as σ and the risk-free rate as R are specified. The spatial domain is defined from the left boundary, A to right boundary, B, and the temporal domain is from initial time 0 to the final marching time, T . The number of spatial grid points, N and time grid points, K are chosen based on the desired resolution. The step size in the stock price domain, ΔS and time domain, ΔT are ...

By replacing Eqn. (1) with (2),(3), and (4), we end up with the following discretization using backward finite difference for the time derivative,

$$\frac{1}{k}(w_{i,j} - w_{i,j-1}) + S_i^2 \frac{\sigma^2}{2h^2}(w_{i+1,j} - 2w_{ij} + w_{i-1,j}) + rS_i \frac{1}{2h}(w_{i+1,j} - w_{i-1,j}) - rw_{i,j} = 0 \tag{8}$$

Then with some simple algebra techniques, solving for $w_{i,j+1}$ and separating the spatial steps,

$$\begin{aligned} w_{i,j-1} = &\left(\frac{k}{2h^2} S_i^2 \sigma^2 - rS_i \frac{k}{2h}\right) w_{i-1,j} \\ &+ \left(1 - kr - \frac{k}{h^2}\sigma^2 S_i^2\right) w_{i,j} \\ &+ \left(\frac{k}{2h^2}\sigma^2 S_i^2 + \frac{k}{2h} rS_i\right) w_{i+1,j} \end{aligned} \tag{9}$$

Then we can define the following for simpler notation $\delta := \frac{k}{h^2}\sigma^2 S_i^2$, $\rho := kr$, $\phi := \frac{k}{2h} r S_i$. Thus with another strategy of simple algebraic arrangement, we end up with the system in matrix form.

$$\begin{bmatrix} w_{1,j-1} \\ \vdots \\ w_{m,j-1} \end{bmatrix} = \begin{bmatrix} 1-\rho-\delta_1 & \frac{1}{2}\delta_1+\phi_1 & 0 & \cdots & 0 \\ \frac{1}{2}\delta_2-\phi_2 & 1-\rho-\delta_2 & \frac{1}{2}\delta_2+\phi_2 & \ddots & \vdots \\ 0 & \frac{1}{2}\delta_3-\phi_3 & 1-\rho-\delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2}\delta_{m-1}+\phi_{m-1} \\ 0 & \cdots & 0 & \frac{1}{2}\delta_m-\phi_m & 1-\rho-\delta_m \end{bmatrix} \begin{bmatrix} w_{1j} \\ \vdots \\ w_{mj} \end{bmatrix}$$
$$+ \begin{bmatrix} (\frac{1}{2}\delta_0-\phi_0)w_{0,j} \\ 0 \\ \vdots \\ 0 \\ (\frac{1}{2}\delta_{m+1}+\phi_{m+1})w_{m+1,j} \end{bmatrix}$$

This allows us to visualize a linear representation of the problem, which can either be solved by updating the option price iteratively via (9), or in matrix form, we can use matrix solvers such as Conjugate Gradient Descent method for sparse matrices like this one.

4 Results

Present the main results of your simulations. This includes the graphical representations of the grid domain and the stencil polygon. Discuss the numerical solution obtained for the option price over time, including both the explicit solution and the matrix form.

5 Stability Analysis

Detail any observations made regarding the stability of the solution. If applicable, discuss the challenges encountered in stability analysis and how they were addressed or why they were beyond the scope of the project.

6 Conclusion

Summarize the findings, focusing on the stability of the numerical solutions for the Black-Scholes Equation. Discuss potential implications of these findings and suggest directions for future research.

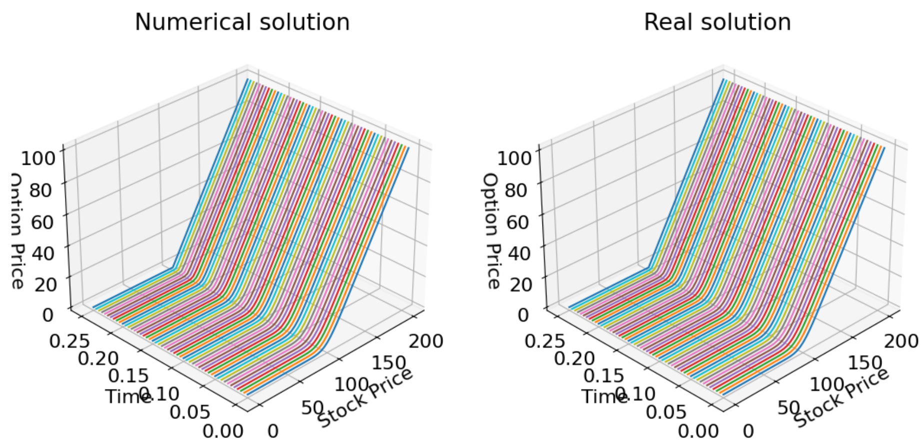

Figure 1: Black-Scholes Computational Results