Electrostatics Basics

Line Charge:
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,dl'$$

Surface Charge:
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,da'$$

Volume Charge:
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,d\tau'$$

Potential Difference:
$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$$

Potential of a Volume Charge:
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||}d\tau' t$$

Potential of a Collection of Point Charges:
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=0}^{n} \frac{q_i}{||\vec{r_{\text{sep}}}||}$$

Work to Move a Charge:
$$W = \int_{\mathbf{a}}^{\mathbf{b}} \vec{F}\cdot d\vec{l} = -Q\int_{\mathbf{a}}^{\mathbf{b}} \vec{E}\cdot d\vec{l}$$

Work to Move a Charge (Potential):
$$W = Q[V(\vec{b}) - Q(\vec{a})]$$

Energy of a Collection of Point Charges:
$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r_i})$$

Energy of a Continuous Charge Distribution:
$$W = \frac{\epsilon_0}{2}\int_{\text{univ}} E^2 d\tau$$

Field at a Charged Surface:
$$\vec{E}_{\text{surf}} = \frac{1}{2}\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$$

Parallel Plate Voltage:
$$V = \frac{Q}{A\epsilon_0}d$$

Capacitance:
$$C \equiv \frac{Q}{V}$$

Energy of a Capacitor:
$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

Electrostatic Multipoles

Generalized Multipole Expansion:
$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty} \frac{1}{r^{n+1}}\int_V ||\vec{r'}||^n P_n(\cos\alpha)\rho(\vec{r'})d\tau'$$

Monopole Voltage:
$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Dipole Voltage:
$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

Dipole Moment (Continuous):
$$\vec{p} \equiv \int_V \vec{r'}\rho(\vec{r'})d\tau'$$

Dipole Moment (Discrete):
$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r'}_i$$

Dipole Moment (Change of Origin):
$$\vec{p'} = \vec{p} - Q\vec{a}$$

Torque of a Dipole:
$$\vec{N} = \vec{p}\times\vec{E}$$

Energy of a Dipole:
$$U = -\vec{p}\cdot\vec{E}$$

Force on a Dipole:
$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

Field of a Dipole:
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

Electrostatics in Matter

Electric Displacement:
$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$$

Bound Surface Charge:
$$\sigma_{\text{bound}} = \vec{P}\cdot\hat{n}$$

Bound Volume Charge:
$$\rho_{\text{bound}} \equiv -\nabla\cdot\vec{P}$$

Linear Dielectrics – Polarization:
$$\vec{P} = \epsilon_0\chi_e\vec{E}$$

Linear Dielectrics – E-Field:
$$\vec{D} = \epsilon\vec{E}$$

Electric Permittivity:
$$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0\epsilon_r$$

Gauss's Law for $\vec{D}$ (Derivative):
$$\nabla\cdot\vec{D} = \rho_f$$

Gauss's Law for $\vec{D}$ (Integral):
$$\oint \vec{D}\cdot d\vec{A} = Q_{f_{\text{encl}}}$$

Force on a Dielectric:
$$F = -\nabla U$$

Energy of a Dielectric:
$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau$$

Names of Stuff

ϵ_0: Permittivity of Free Space
ϵ_r: Dielectric Constant or Relative Permittivity
ϵ: Permittivity of a Material

Techniques for Solving Problems

Cylindrical Laplacian Solution:
$$V(s,\phi) = A\ln s + B + \sum_{n=1}^{\infty}\left(A_n s^n + \frac{B_n}{s^n}\right)(C_n\cos(\phi n) + D_n\sin(\phi n))$$

Solution for Spherical Laplacians:
$$f(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Method of Images General Principle:
$$\sum \frac{q_i}{||\vec{r_{\text{sep}_i}}||} = 0$$

Method of Images for Two Points:
$$-\frac{q_1}{q_2} = \frac{||\vec{r_{\text{sep}_1}}||}{||\vec{r_{\text{sep}_2}}||}$$

Image Charge Surface Charge:
$$\sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

Binomial Expansion:
$$(1+x)^k = \sum_{n=0}^{\infty}\left(\frac{k!}{n!(k-n)!}\right)x^n$$

Taylor Expansion:
$$f(x) \approx \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}$$

Because I can't remember:
$$f(x) \approx f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2} + \cdots$$

Legendre Polynomials:
$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$
$$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$$

Magnetostatics

Lorentz Force Law:
$$\vec{F}_{\text{mag}} = Q(\vec{v}\times\vec{B})$$

Surface Current Density:
$$\vec{K} \equiv \frac{d\vec{I}}{dl_{\perp}}$$
$$\vec{K} = \sigma\vec{v}$$

Volume Current Density:
$$\vec{J} \equiv \frac{d\vec{I}}{da_{\perp}}$$
$$\vec{J} = \rho\vec{v}$$

Magnetic Force (General):
$$\vec{F}_{\text{mag}} = \int I(d\vec{I}\times\vec{B}) = I\int(d\vec{I}\times\vec{B})$$

Magnetic Force (Surface):
$$\vec{F}_{\text{mag}} = \int(\vec{K}\times\vec{B})da$$

Magnetic Force (Volume):
$$\vec{F}_{\text{mag}} = \int(\vec{J}\times\vec{B})d\tau$$

Biot-Savart Law:
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}_{\text{sep}}}{||\vec{r_{\text{sep}}}||^2}dl'$$
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}I\int \frac{d\vec{I'}\times\hat{r}_{\text{sep}}}{||\vec{r_{\text{sep}}}||^2}$$

Ampere's Law:
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{encl}}$$

Magnetic Vector Potential:
$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{\nabla}\cdot\vec{A} = 0$$
$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

Vector Potential – Line:
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{||\vec{r_{\text{sep}}}||}dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{||\vec{r_{\text{sep}}}||}d\vec{l'}$$

Vector Potential – Surface:
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{||\vec{r_{\text{sep}}}||}da'$$

Vector Potential – Volume:
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'})}{||\vec{r_{\text{sep}}}||}d\tau'$$

Magnetic Dipole Moment:
$$\vec{m} \equiv I\int d\vec{a} = I\vec{a}$$

Magnetic Dipole Potential Expansion:
$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

Magnetic Torque:
$$\vec{N} = \vec{m}\times\vec{B}$$

Force on a Magnetic Dipole:
$$\vec{F} = \nabla(\vec{m}\cdot\vec{B})$$

Force Between Two Current-Carrying Wires:
$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Maxwell's Equations

Gauss's Law:
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$$

(unnamed)
$$\vec{\nabla}\cdot\vec{B} = 0$$

Faraday's Law:
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's Law with Maxwell's Correction:
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\,\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Magnetostatics in Matter

Paramagnets:	magnetization is **parallel** to $\vec{B}$
Diamagnets:	magnetization is **opposite** to $\vec{B}$
Ferromagnets:	magnetization holds outside the presence of an external magnetic field.
Magnetization:	$\vec{M} \equiv$ magnetic dipole per unit volume
Bound Volume Current:	$\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}$
Bound Surface Current:	$\vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$
Free Current:	$\vec{J} = \vec{J}_b + \vec{J}_f$
Auxillary Field:	$\vec{H} \equiv \dfrac{1}{\mu_0} \vec{B} - \vec{M}$
Ampere's Law for Auxillary Fields:	$\vec{\nabla} \times \vec{H} = \vec{J}_f$ $\oint \vec{H} \cdot d\vec{l} = I_{f_{\text{encl}}}$
Linear Magnetics:	$\vec{M} = \chi_m \vec{H}$
Permeability:	$\mu \equiv \mu_0(1 + \chi_m)$
Linear Magnetics with E and H Fields:	$\vec{B} = \mu \vec{H}.$

Electrodynamics

Ohm's Law:	$V = IR$
Power in a Circuit:	$P = VI = I^2 R = \dfrac{V^2}{R}$
Electromotive Force:	$\mathcal{E} = \dfrac{F_{\text{mag, tot}}}{Q} = \int (\vec{v} \times \vec{B}) d\vec{l}$ $\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$ $\mathcal{E} = -\dfrac{d\Phi}{dt}$
Faraday's Law:	$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
Inductance:	$\Phi = LI$ $\mathcal{E} = -L\dfrac{dI}{dt}$
Work of a Magnetic Field:	$\dfrac{1}{2} LI^2$ $W = \dfrac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau$
Displacement Current:	$\vec{J}_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Maxwell's Equations in Matter

Gauss's Law:	$\vec{\nabla} \cdot \vec{D} = \rho_f$
(unnamed)	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's Law:	$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
Ampere's Law with Maxwell's Correction:	$\vec{\nabla} \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t}$

Triangles
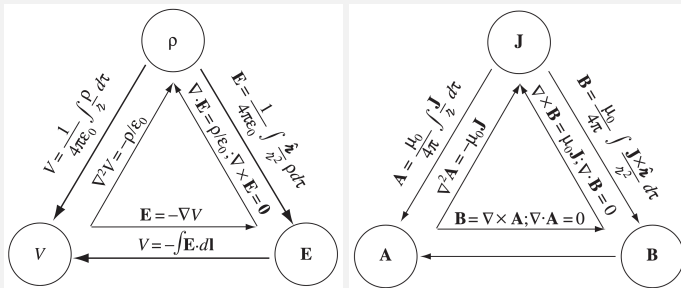
Takeaways from Practice

- Gauss's law inside a dielectric always includes both ρ_b and σ_b.
- Use the definition for displacement where ever possible: $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$.
- The potential is constant below an image plane and the field is always zero.

Boundary Conditions in Electrostatics

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$
$$\vec{D}_{\text{above}}^{\parallel} - \vec{D}_{\text{below}}^{\parallel} = \vec{P}_{\text{above}}^{\parallel} - \vec{P}_{\text{below}}^{\parallel}$$
$$\vec{E}_{\text{above}}^{\perp} - \vec{E}_{\text{below}}^{\perp} = \sigma_{\text{tot}}/\epsilon_0$$
$$\vec{E}_{\text{above}}^{\parallel} - \vec{E}_{\text{below}}^{\parallel} = \vec{0}$$
$$V_{\text{above}} = V_{\text{below}}$$
$$\epsilon_{\text{above}} \vec{E}_{\text{above}}^{\perp} - \epsilon_{\text{below}} \vec{E}_{\text{below}}^{\perp} = \sigma_f$$
$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

Boundary Conditions in Magnetostatics

$$\vec{B}_{\text{above}}^{\parallel} - \vec{B}_{\text{below}}^{\parallel} = \mu_0 \vec{K}$$
$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$
$$\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$$
$$\frac{\partial A_{\text{above}}}{\partial n} - \frac{\partial A_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$
$$\vec{H}_{\text{above}}^{\perp} - \vec{H}_{\text{below}}^{\perp} = -(\vec{M}_{\text{above}}^{\perp} - \vec{M}_{\text{below}}^{\perp})$$
$$\vec{H}_{\text{above}}^{\parallel} - \vec{H}_{\text{below}}^{\parallel} = \vec{K}_f \times \hat{n}$$

More Boundary Conditions

Equations 7.60-63 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$, (iii) $\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}$,

(ii) $B_1^{\perp} - B_2^{\perp} = 0$, (iv) $\dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$ (7.64)

In particular, if there is no free charge or free current at the interface, then

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0$, (iii) $\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}$,

(ii) $B_1^{\perp} - B_2^{\perp} = 0$, (iv) $\dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}.$ (7.65)

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.