Math 305 Review Notes

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Complex Numbers

At this point in the course, there we define only one representation of the the imaginary unit, i:

$$i^2 = -1.$$

For the remainder of this document, we will represent complex numbers z as:

$$z = x + iy,$$

where i is the imaginary unit.

Hence, with i, we can define some other important properties:

1. $\overline{\overline{z}} = z$
2. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
3. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

<u>Inequalities and Complex Numbers:</u> Building off of the triangle inequality, we can define inequalities for complex numbers:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \text{(triangle inequality)}$$

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$
$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

As a result, we can bound the modulus of the sum of two complex numbers as:

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

If trying to obtain a bound for the sum of multiple complex numbers, it is important to always obtain the maximal/minimal bounds for each case when aggregating complex numbers.

<u>Representations of Planar Sets in Complex Numbers:</u> Taking the prior definition of $z = x+iy$, and interpreting the x value as an x coordinate, and the same for y, then we can define planar sets in terms of complex numbers. To start off, we first define how to obtain the x and y values of a complex number:

$$\text{Re}(z) = x = \frac{z + \overline{z}}{2} \qquad \text{Im}(z) = x = \frac{z - \overline{z}}{2i}$$

With this definition, we can define some common representations:

1. <u>Circles in $\mathbb{R}^2$:</u>

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2 \iff |z - z_0| = r_0$$

2. <u>Lines in $\mathbb{R}^2$:</u>

$$ax + by = c \iff a\frac{z + \overline{z}}{2} + b\frac{z - \overline{z}}{2i} = c$$

3. <u>Ellipses in $\mathbb{R}^2$:</u>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff |z - F| + |z + F| = 2a$$

Where $F = \sqrt{a^2 - b^2}$.

It is to be noted that this representation only allows for horizontal shifts. One can get vertical shifts by using an alternate form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff |z - Fi| + |z + Fi| = 2b.$$

Shifting can be observed when some representation $|z - F_1| + |z + F_2| = 2a$, where $F_1 \neq F_2$. The shift can be obtained by averaging F_1 and F_2.

A common example of a coordinate transform is to square both sides and convert into a circle. This is because we require r_0 to be squared.

Polar Coordinates and Arguments

One can express a complex number z in polar coordinates using Euler's formula:

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

Hence, it is implied that a representation of a complex number can be as follows:

$$z = re^{i\phi}.$$

Where r is known as the modulus of z and ϕ is known as the modulus of z.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \phi = \arg(z)$$

Arguments: There exist two different types of arguments:

1. Principal Argument: $\operatorname{Arg}(z) = \phi \in (-\pi, \pi]$.
   The principal argument is single-valued and unique.
2. General Argument*: $\arg(z) = \operatorname{Arg}(z) \pm 2\pi k, k \in \mathbb{Z}$.
   The general argument can attain infinite different values.

Finding the Principal Argument: The Principal Argument, as a general form, is given by:

$$\operatorname{Arg}(z) = \arctan\left(\frac{y}{x}\right) + m\pi, m \in \{-1, 0, 1\}.$$

Hence, we define cases for the correct value of m, based on the location of z.

1. Quadrant I: $m = 0$, $\operatorname{Arg}(z) = \arctan\left(\frac{y}{x}\right)$
2. Quadrant II: $m = 1$, $\operatorname{Arg}(z) = \arctan\left(\frac{y}{x}\right) + \pi$
3. Quadrant III: $m = -1$, $\operatorname{Arg}(z) = \arctan\left(\frac{y}{x}\right) - \pi$.
4. Quadrant IV: $m = 0$, $\operatorname{Arg}(z) = \arctan\left(\frac{y}{x}\right)$

It is important to note that the argument of zero is undefined.

Properties of Complex Numbers in Polar Forms:

1. $e^{2k\pi i} = 1$
2. $e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)}$
3. $\overline{e^{i\phi}} = e^{-i\phi}$
4. $|e^{i\phi}| = 1$

Properties of Arguments:

1. $\text{Arg}(z_1 \cdot z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$
2. $\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi, k \in \mathbb{Z}$
3. $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

   It is important to note that (3) is *set equality*, not value equality.

Powers of Complex Numbers

Certain properties arise from taking powers of complex numbers.

De Moirre's Formula:

$$(\cos\phi + i\sin\phi)^N = \cos(N\phi) + i\sin(N\phi)$$

This can often be combined with the binomial theorem to derive trig identities.

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Roots of z: We can define the roots of z, $z^{\frac{1}{n}}$ as:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\left(\frac{\phi_0}{n} + \frac{2k\pi}{n}\right)} \qquad k = 0, 1, \ldots, n-1$$

We also define the principle value of a root to be the one corresponding to $k = 0$.

Raising Complex Numbers as Powers: There are also interesting consequences of raising complex numbers as powers. Take a complex number $z = x + iy$:

$$\begin{aligned} e^z &= e^{x+iy} \\ e^z &= e^x \cdot e^{iy} \\ e^z &= e^x(\cos y + i\sin y). \end{aligned}$$

Thus, we also obtain some properties:

1. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$
2. $e^{z_1-z_2} = \dfrac{e^{z_1}}{e^{z_2}}$
3. $e^{\overline{z}} = \overline{e^z}$
4. $|e^z| = e^x$

Functions of Complex Numbers

We can describe functions of complex numbers as:

$$w = f(z),$$

where $w = u + iv$ and $z = x + iy$. Different functions have different images. To determine the image of a given function, we can use the following approach:

1. Solve for z in terms of w from $f(z) = w$.
2. Substitute each z for the $z(w)$ expression in the set notation.

Different Transforms:

1. $w = Az + B$ transforms circles to circles, lines to lines.
2. $w = \frac{1}{z}$ transforms {circles or lines} to {circles or lines}.
3. Mobius Transform, $w = \frac{az+b}{cz+d}$ transforms {circles or lines} to {circles or lines}.
4. $w = z^n$ Power Transforms generally rotate sets.
5. $w = e^z$ Exponential Transofrms generally turn lines into circles and vice versa.

Derivatives of Complex Functions

Like single-variable calculus, there are some basic definitions that need to be highlighted in complex-variable calculus.

Limits: We can define the limit of a complex function as:

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

However, an additional layer of complexity is introduced here. $(x, y) \to (x_0, y_0)$ implies that any path can be taken to get to (x, y). Thus, strange behaviours occur.

As an aside, we can note that usual properties of limits generally still hold with complex functions:

1. $\lim \dfrac{f(z)}{g(z)} = \dfrac{\lim f(z)}{\lim g(z)}$
2. $\lim(c_1 f + c_2 g) = c_1 \lim f(z) + c_2 \lim g(z)$.

Continuity of Functions

$$f(z) \text{ is continuous at } z_0 \iff \lim_{z\to z_0} f(z) = f(z_0) \iff \left(\lim_{z\to z_0} u(z_0) = u(z_0) \wedge \lim_{z\to z_0} v(z_0) = v(z_0)\right)$$

It is important to note that the choice of path is important here. If two different paths lead to different values in the limit, the limit does not exist.

Differentiability:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \iff \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \iff f(z) \text{ is differentiable.}$$

Where the inclusion of a limit means that the limits must exist.

Cauchy-Riemann Equation: A helpful tool for governing differentiability is the Cauchy-Riemann Equation:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Implications of the CRE:

- ($\implies$): If some $f(z)$ is differentiable at some z_0, then the CRE is satisifed.
- ($\impliedby$): If some function $f(z)$ satisfies the CRE at some z_0, AND the derivative is continuous, then $f(z)$ is differentiable.

There are some additional definitions here too:

1. A function f is analytic at a point z_0 if f is differentiable in a neigbourhood of z_0.
2. A function f is entire if it is analytic everywhere.

Being differentiable at a single point is not analytic.

Taking the derivative of complex functions is the same as in non-complex functions. The product, quotient, and chain rules all hold. Remark:

$$f \text{ is differentiable} \implies \frac{\partial f}{\partial \bar{z}} = 0 \text{ at } z_0$$

Consequences of the Cauchy-Riemann Equation:

1. The CRE guarantees the satisfaction of the laplace equation if a function is entire. Thus, we can say that v is the harmonic conjugate of u, and vice versa.

Functions and Mappings

We can define a variety of different functions which map various domains to various images. Expanding off of what was already stated in the "Functions of Complex Numbers" section, we can state some more functions:

Name	Function	Mapping	Expanded Form
Square	$w = z^2$	Scales modulus and argument	$w = \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$
Exponential	$w = e^z$	Transforms planes/circles into planes/circles	$w = \begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$

Inverse Functions

Function	Inverse
$\sin(z)$	$\sin^{-1} z = -i \cdot \text{Log}(iz + (1-z^2)^{\frac{1}{2}})$ where $(1-z^2)^{\frac{1}{2}} = i\lvert z-1\rvert^{\frac{1}{2}}\lvert z+1\rvert^{\frac{1}{2}} e^{i\left(\frac{\phi_1+\phi_2}{2}\right)}$ and $2\pi < \phi_1 < 4\pi$ $-\pi < \phi_2 < \pi$
$w = e^z$	$z = \text{Log}(w)$
$w = z^\alpha$	$e^{\frac{1}{\alpha}\text{Log} z} = z^{\frac{1}{\alpha}}$

Course Review

Elementary Functions and Mapping Properties

Name	Function	Description of Transform	Expanded Form
Linear	$w = az + b$	Lines $\to$ Lines Circles $\to$ Circles	$w = \begin{cases} u = ax + b \\ v = ay + b \end{cases}$
Mobius Transform	$w = \frac{az+b}{cz+d}$	Lines/Circles $\to$ Lines/Circles	$w = \begin{cases} u = \frac{ax+b}{cx+d} \\ v = \frac{ay+b}{cy+d} \end{cases}$
Exponential	$w = e^z$	Lines $\to$ Cirlces Circles $\to$ Lines	$w = \begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$
Joukowsky Map	$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$	Transforms a semi-circle of radius 1, centered at the origin to a flat line along the x-axis	
Sine	$w = \sin z$	Transforms an infinite 3-sided box with sides at $-\pi/2, \pi/2$ centered at 0 into a line on the x-axis.	$w = \begin{cases} u = \sin x \cosh y \\ v = \cos x \sinh y \end{cases}$
Cosine	$w = \cos z$	I forgor	$w = \begin{cases} u = \cos x \cosh y \\ v = -\sin x \sinh y \end{cases}$
Phase Rotation	$w = z \cdot e^{i\phi}$	Rotates the plane by ϕ radians.	$w = \begin{cases} u = x\cos\phi - y \sin\phi \\ v = x \sin\phi + y\cos\phi \end{cases}$
Polynomial	$w = z^n$	Scales angles and magnitudes by n.	$w = \begin{cases} u = x^n \\ v = y^n \end{cases}$

Inverse Functions and Branch Cuts

Multi-Valued Function	Single-Valued Function	Inverse of:
$\arg(z)$	$\text{Arg}(z), -\pi < \phi < \pi$	–
$\log(z) = \ln(\lvert z\rvert) + i(\text{Arg}(z) + 2\pi k)$	$\text{Log}(z) = \ln(\lvert z\rvert) + i \cdot \text{Arg}(z)$	$e^w = z$
$\sin^{-1}(z)$	$\sin^{-1}(z) = -i \cdot \text{Log}(iz + i(z^2-1)^{\frac{1}{2}})$ $(z^2-1)^{\frac{1}{2}} = \lvert z-1\rvert^{\frac{1}{2}}\lvert z+1\rvert^{\frac{1}{2}} e^{i\frac{\phi_1+\phi_2}{2}}$ $2\pi < \phi_1 < 4\pi$ $-\pi < \phi_2 < \pi$	$\sin(w) = z$
$\cos^{-1}(z)$	$\cos^{-1}(z) = -i \cdot \text{Log}(z + (z^2-1)^{\frac{1}{2}})$ $(z^2-1)^{\frac{1}{2}} = \lvert z-1\rvert^{\frac{1}{2}}\lvert z+1\rvert^{\frac{1}{2}} e^{i\frac{\phi_1+\phi_2}{2}}$ $0 < \phi_1 < 2\pi$ $-\pi < \phi_2 < \pi$	$\cos(w) = z$

Inequality Reminders

- $|e^z| \leq e^{\text{Re}(z)}$
- $|z^n + \cdots| \leq |z|^n + \cdots$
- $|z^n + \cdots| \geq |z|^n - \cdots$

Differentiability, Analyticity, Cauchy-Riemann Equations

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Limits: We can define the limit of a complex function as:

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Taking the derivative of complex functions is the same as in non-complex functions. The product, quotient, and chain rules all hold.
Remark:

$$f \text{ is differentiable} \implies \frac{\partial f}{\partial \bar{z}} = 0 \text{ at } z_0$$

Consequences of the Cauchy-Riemann Equation:

1. The CRE guarantees the satisfaction of the laplace equation if a function is entire. Thus, we can say that v is the harmonic conjugate of u, and vice versa.
2. A function is said to be harmonic if the laplace equation is satisfied, that is if:

$$\Delta u = u_{xx} + u_{yy} = 0 \qquad \text{or} \qquad \Delta v = v_{xx} + v_{yy} = 0$$

First Steps into Integrals: Path Integrals and the Fundamental Theorem of Calculs

Path Integrals: We can take the path integral of a complex function $f(z)$ over some path C:

$$\int_C f(z)dz = \int_{t_1}^{t_2} f(z(t)) \cdot z'(t)dt$$

Where $f(t)$ is some parametrization of the path C.

Fundamental Theorem of Calculus: Like in real calculus, we can use antiderivatives to integrate a function. Given if an antiderivative $F(z)$ exists for some function $f(z)$, then

$$\int_C f(z)dz = F(z_f) - F(z_i)$$

It is to be noted that in order for the antiderivative to exist, $f(z)$ must be analytic on C.

The Cauchy Integral Formula

Cauchy's Theorem: Before the introduction of the Cauchy Integral Formula, we first introduce Cauchy's Theorem. It is as follows:

Let a path C be a simple closed loop. Suppose a function $f(z)$ is analytic on and inside C. Then:

$$\int_C f(z)dz = 0$$

Consequently, we can also obtain the fact that for some function $f(z)$ with a discontinuity at z_0, then it follows that:

$$\int_{|z-z_0|=r} f(z)dz = 2\pi i$$

Deformation of Path: If there are two paths C_0 and C_1 on/in some domain C, if C is analytic and C_0 is contained in C_1 or vice versa, then:

$$\int_{C_0} f(z)dz = \int_{C_1} f(z)dz$$

Expanding on this, given some domain C, with N "holes" in it $C_1, \dots, C_N$, then the integral over C can be expressed as:

$$\int_C f(z)dz = \sum_{j=1}^{N} \int_{C_j} f(z)dz$$

Cauchy Integral Formula: Finally, we arrive at the Cauchy Integral Formula. It is as follows:

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

Consequently, we can rearrange this to be more applicable for integrals.

$$\int_C \frac{f(z)}{(z-z_0)^{m+1}} = 2\pi i \cdot \frac{f^{(m)}(z_0)}{m!}$$

Finally, we can generalize:

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{N} \frac{f_j^{(m_j)}(z_0)}{m_j!}$$

Where each $f_j(z)$ is the original function $f(z)$ with the discontinuity component $\frac{1}{z-z_k}$ divided out.

It is important to note that the Cauchy Integral Formula is only really good for functions of the form:

$$f(z) = \frac{g(z)}{(z-z_0)^{m_0} \cdot (z-z_1)^{m_1} \cdot \ldots}$$

Applications and Consequences of the Cauchy Integral Formula

Pointwise Estimate: For some function $f(z)$ on a circular domain $C : |z - z_0| = R$, then we can establish an upper bound for the function and its derivatives evaluated at $z = z_0$:

$$f^{(m)}(z_0) \leq \frac{m!}{R^m} \max_{|z-z_0|=R} |f(z)|$$

Liouville Theorem: Given some function $f(z)$ that is entire and bounded, then it is a constant function.

$$|f(z)| \leq M \implies f(z) \equiv C$$

There exist some important variants of the Liouville Theorem:

1. $f = u + iv \wedge (u \geq -M \vee u \leq K) \implies f \equiv C$
   Proof Mechanism: $g(z) = e^{-f(z)}$
2. $f = u + iv \wedge au + bv \geq 0 \implies f \equiv D$
   Proof Mechanism: $g = e^{\alpha f}, |g| = |e^{\alpha f}| = e^{-(au+bv)}$
3. $|f(z)| \leq C \cdot (1 + |z|)^n \implies$ f(z) must be a polynomial of degree lesser than n.

Maximum Principle: For some function $f(z)$ which is analytic on some domain and its boundary $D \cup \partial D$, its maximum value is found on its boundary:

$$\max_{\overline{D}} |f(z)| = \max_{\partial D} |f(z)|.$$

Minimum Principle: For some function $f(z)$ which is analytic on some domain and its boundary $D \cup \partial D$, its minimum value is either zero, or it is found on the boundary of the domain ∂D:

$$\left(\min_{\overline{D}} |f(z)| = 0\right) \vee \left(\min_{\overline{D}} |f(z)| = \min_{\partial D} |f(z)|\right).$$

Argument Principle: For some function $f(z)$ which is analytic both on and inside C, with a finite number of roots inside C with multiplicity n_j, then:

$$\int_C \frac{f'(z)}{f(z)} = 2\pi i \cdot \sum_{j=1}^{k} n_j$$

Conequently, the argument principle follows that:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = N = \frac{arg(f)|_{f(C)}}{2\pi}$$

However, this theorem is not very applicable, but its consequenting theorems are.

Rouche's Theorem (Application of Argument Principle): Given some function $f(z) = g(z) + h(z)$, if one of the terms inside $f(z)$ dominates the others, that is $|f(z) - g(z)| < |g(z)|$ on C, then the number of zeroes on C for $f(z)$ is the same as that of the number of zeroes on C for $g(z)$, $N_f = N_g$.

Nyquist Criterion (Application of Argument Principle): Given some polynomial function $p(z)$ with degree n, the number of zeroes on the right half plane $\{Re(z) > 0\}$ is equal to:

$$N = \frac{1}{2\pi}\left(n\pi + 2\cdot \arg(p(z))|_{\Gamma_1}\right)$$

Where $\Gamma_1 = \{z = iy, 0 < y < +\infty\}$.

Taylor and Laurent Series

In real calculus, the Taylor Series was introduced:

$$f(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{|z-z_0=r} \frac{f(\eta)}{(\eta - z_0)^{n+1}} d\eta$$

We can state that the Taylor Series held for a function $f(z)$ which was analytic on the domain $|z - z_0| < r_2$.

Laurent Series: Complex analysis introduces the Laurent Series, which serves two purposes:

1. Computes the negative power components of a series expansion of a function (e.g. z^-n)
2. Computes an expansion for an annular domain: $C : r_1 < |z - z_0| < r_2$.

Hence, the laurent series is as follows:

$$\begin{aligned} f(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots \\ + a_{-1}(z - z_0)^{-1} + \cdots + a_{-n}(z - z_0)^{-n} \end{aligned}$$

Where $a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(\eta)}{(\eta - z_0)^{n+1}} d\eta$.

We can use the Laurent Series to also classify singularities: for a singularity $z = z_0$ for some function $f(z)$, it follows that:

1. If there are no negative terms in the series $a_{-1} = \cdots = a_{-n} = 0$, then we classify z_0 as a removable singularity.
2. If the negative terms in the series terminate after some a_{-m}, then we classify z_0 as a singularity of order m.
3. If the negative terms never terminate, then z_0 is classified as an essential singularity.

Residues and the Cauchy Residue Theorem

Residue: In the Laurent series, we call the a_{-1} term the resiude of a function $f(z)$ and a singularity z_0. We can compute it in a multitude of ways:

Removable Singularities: $\text{Res}(f(z), z_0) = \dfrac{P(z_0)}{Q'(z_0)}$

Poles with Order ≥ 2: There are two options:

1. $\text{Res}(f(z), z_0) = \dfrac{1}{(m-1)!} f_0^{(m-1)}(z_0)$ (like prior)
2. Laurent Series

Essential Singularities: Laurent Series

Determining Pole Order:

1. If the pole is in an "easy" form $(z - z_j)^{m_j}$ in denominator, then the order of the pole is simply m_j.
2. If not, then there are two options:
   (a) Take the Laurent Series and find the order of the largest negative exponent.
   (b) (Hacky) Take successive derivatives of the denominator. The first n-th derivative which does not evaluate to zero is the order.

Cauchy Residue Theorem:

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{N} \text{Res}(f(z), z_j)$$

Real Integrals

There exist some real integrals which are easily solvable using complex analysis. For these integrals, a simple problem solving framework exists:

1. Choose a contour that will make computation simple, based on the form of the integrand.
2. Write out the integral over each segment of the contour. Terms that evaluate to the original integral, I, will appear.
3. Set this equal to the Cauchy Residue theorem for the domain that the contour encircles, and solve for I.

Fourier and Laplace Transforms

(under construction)