PHYS 401 Relativity Notes

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Lorentz Transformations

For any frame $\mathcal{S}'$ travelling at velocity v relative to some other frame $\mathcal{S}$ along the x-axis, the new coordinates of the new frame $\mathcal{S}'$ are given by the following:

$$
\begin{aligned}
t' &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \\
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z
\end{aligned}
$$

We can then also derive the inverse Lorentz transformations:

$$
\begin{aligned}
t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \\
x &= \gamma(x' + vt') \\
y &= y' \\
z &= z'
\end{aligned}
$$

<u>Lorentz Invariance</u> are any quantities that remain the same under a Lorentz transformation. These include:

- The speed of light.
- The spacetime interval between any two points in Minkowski space.
- The scalar product of any two four-vectors.
- $\vec{E} \cdot \vec{B}$ is invariant.
- $\frac{1}{c^2}\vec{E} - \vec{B}$

Where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$.

This can also be represented using matricies:

$$
\begin{pmatrix} ct' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

4-Vectors

In the notion of Minkowski space, we have that each point in the universe has a position and a time. Hence, like in simple positional/3-space, we can define a 4-vector to capture both space and time.

$$
\begin{aligned}
A_\mu = \mathbf{A} &= (A_t, A_x, A_y, A_z) \\
&= (A_t, A_i) \\
&= (a_t, \mathbf{a})
\end{aligned}
$$

Thus, we define the dot product for a four-vector to be the following:

$$a_\mu b_\mu = a_t b_t - \mathbf{a} \cdot \mathbf{b}.$$

This dot product makes it such that the 4-vector is invariant under rotations or frame changes.
Common 4-vectors:

Quantity	Form
4-velocity	$X_\mu = (ct, \vec{x}) = (ct, x, y, z)$
4-velocity	$\gamma(c, \vec{u}) = \gamma\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$
4-momentum	$P_\mu = (\gamma mc, \gamma m\vec{v}) = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right)$
4-current	$J_\mu = (c\rho, \vec{j}) = (c\rho, j_x, j_y, j_z)$
4-potential	$\left(\frac{V}{c}, \vec{A}\right) = \left(\frac{V}{c}, A_x, A_y, A_z\right)$

Four-Dimensional Gradient:
We can define the four-dimensional gradient to be the following:

$$\nabla_\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right),$$

which, when applied to a 4-vector yields the following:

$$\nabla_\mu a_\mu = \frac{\partial}{\partial t} a_t + \boldsymbol{\nabla} \cdot \mathbf{a}.$$

Derivative	Form
4-divergence	$\nabla_\mu a_\mu = \frac{\partial a_t}{\partial t} + \boldsymbol{\nabla} \cdot a$
4-gradient	$\nabla_\mu \lambda = \left(\frac{\partial \lambda}{\partial t}, -\boldsymbol{\nabla}\lambda\right)$
4-laplacian (d'Alembertian)	$\nabla_\mu \nabla_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2$

Applying 4-Dimensional Quantities to Electrodynamics

As we knew from the 3-dimensional case:

$$\Box^2 V = \frac{\rho}{\epsilon_0} \qquad \Box^2 \mathbf{A} = \frac{\mathbf{J}}{\epsilon_0}.$$

Hence, we can generalize to get the following form:

$$\Box^2 A_\mu = \frac{j_\mu}{\epsilon_0} \tag{1}$$

We can also say that these equations hold only if the Lorentz gauge is obeyed. This is called the Lorenz Condition and can be written $\nabla_\mu A_\mu = 0$. The Lorenz Condition is said to be an invariant condition and thus, for all frames (1) holds.

We can also say that the charage conservation equation can be written in similar terms:

$$\nabla_\mu j_\mu = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

and that gauge invariance has that:

$$A'_\mu = A_\mu - \nabla_\mu f$$

Maxwell's equations can also be rewritten in terms of four-vectors:

$$\Box^2 A_\mu = j_\mu \mu_0$$

We can also look at the lorentz transformations for V and A_x:
Note: these give V' and A'_x in the unprimed coordinates, not the primed ones.

$$\frac{V'}{c} = \gamma \left(\frac{V}{c} - \beta A_x \right) \qquad\qquad A'_x = \gamma \left(A_x - \beta \frac{V}{c} \right)$$

For a moving point charge in the x direction, we can say that:

$$\begin{aligned}
V &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-vt)^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2+z^2)}} \\
A_x &= \frac{v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-vt)^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2+z^2)}} \\
\vec{E} &= \frac{q}{4\pi\epsilon_0} \left(1 - \frac{v^2}{c^2}\right) \left[\frac{(x-vt)\hat{x} + y\hat{y} + z\hat{z}}{\left[(x-vt)^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2+z^2)\right]^{\frac{3}{2}}} \right] \\
\vec{B} &= \frac{\vec{v} \times \vec{E}}{c^2} \\
\vec{S} &= \frac{-\vec{E}(\vec{E} \cdot \vec{v}) + \vec{v}\vec{E}^{\,2}}{\mu_0 c^2}
\end{aligned}$$

Field Tensor: Since we cannot possibly contain all of the information of the fields within a simple 4-vector, we use a tensor to describe the fields. This tensor F is an antisymmetric tensor such that $-F = F$. Each entry in F is given by:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

and thus, we can define the tensor as:

$$F = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & F_{0x} & F_{0y} & F_{0z} \\ F_{x0} & 0 & F_{xy} & F_{xz} \\ F_{y0} & F_{yx} & 0 & Fyz \\ F_{z0} & F_{zx} & F_{zy} & 0 \end{bmatrix}$$

Generalized Relativistic Transformations:
Generalizing this, we can write the transforms for **E** and **B** fields in a generalized sense, given some $\mathcal{S}'$ frame with velocity v_x relative to some $\mathcal{S}$ frame. Again, these all give results in the *unprimed* coordinates.

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}$$

We can generalize this further using the following forms:

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_y & B'_y &= \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E})_y \\ E'_z &= \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_z & B'_y &= \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E})_z \end{aligned}$$

4-Force:

$$f_\mu = \left(\frac{\mathbf{F} \cdot \frac{\mathbf{v}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{F}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

Can be generalized:

$$ds = dt\sqrt{1 - \frac{v^2}{c^2}}$$

and we can rewrite as:

$$\frac{dx_\mu}{ds} = u_\mu$$

as well as:

$$\frac{dp_\mu}{ds} = f_\mu$$

Or, lastly:

$$m_0 \frac{d^2 x_\mu}{ds^2} = f_\mu = q u_\nu F_{\mu\nu}$$

Generalizing, even further:
For some:

$$F_{\mu\nu} = a_\mu b_\nu - a_\nu b_\mu$$

We get that:

$$\begin{aligned} a'_0 &= \frac{a_0 - \frac{v}{c}a_x}{\sqrt{1 - \frac{v^2}{c^2}}} & b'_0 &= \frac{b_0 - \frac{v}{c}b_x}{\sqrt{1 - \frac{v^2}{c^2}}} \\ a'_x &= \frac{a_x - \frac{v}{c}a_0}{\sqrt{1 - \frac{v^2}{c^2}}} & b'_x &= \frac{b_x - \frac{v}{c}b_0}{\sqrt{1 - \frac{v^2}{c^2}}} \\ a'_{y,z} &= a_{y,z} & b'_{y,z} &= b_{y,z} \end{aligned}$$