Electrostatics Basics

Line Charge: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\lambda(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,dl'$

Surface Charge: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\sigma(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,da'$

Volume Charge: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||^2}\hat{r}_{\text{sep}}\,d\tau'$

Potential Difference: $V(\vec{b}) - V(\vec{a}) = -\displaystyle\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$

Potential of a Volume Charge: $V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||}d\tau' t$

Potential of a Collection of Point Charges: $V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\sum_{i=0}^{n}\dfrac{q_i}{||\vec{r_{\text{sep}}}||}$

Work to Move a Charge: $W = \displaystyle\int_{\mathbf{a}}^{\mathbf{b}} \vec{F}\cdot d\vec{l} = -Q\int_{\mathbf{a}}^{\mathbf{b}} \vec{E}\cdot d\vec{l}$

Work to Move a Charge (Potential): $W = Q[V(\vec{b}) - Q(\vec{a})]$

Energy of a Collection of Point Charges: $W = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} q_i V(\vec{r_i})$

Energy of a Continuous Charge Distribution: $W = \dfrac{\epsilon_0}{2}\displaystyle\int_{\text{univ}} E^2 d\tau$

Field at a Charged Surface: $\vec{E}_{\text{surf}} = \dfrac{1}{2}\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$

Parallel Plate Voltage: $V = \dfrac{Q}{A\epsilon_0}d$

Capacitance: $C \equiv \dfrac{Q}{V}$

Energy of a Capacitor: $W = \displaystyle\int_0^Q \left(\dfrac{q}{C}\right)dq = \dfrac{1}{2}\dfrac{Q^2}{C} = \dfrac{1}{2}CV^2$

Electrostatic Multipoles

Generalized Multipole Expansion: $V(r) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\sum_{n=0}^{\infty}\dfrac{1}{r^{n+1}}\int_V ||\vec{r'}||^n P_n(\cos\alpha)\rho(\vec{r'})d\tau'$

Monopole Voltage: $V(r) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$

Dipole Voltage: $V(r) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}\cdot\hat{r}}{r^2}$

Dipole Moment (Continuous): $\vec{p} \equiv \displaystyle\int_V \vec{r'}\rho(\vec{r'})d\tau'$

Dipole Moment (Discrete): $\vec{p} = \displaystyle\sum_{i=1}^{n} q_i\vec{r'}_i$

Dipole Moment (Change of Origin): $\vec{p'} = \vec{p} - Q\vec{a}$

Torque of a Dipole: $\vec{N} = \vec{p}\times\vec{E}$

Energy of a Dipole: $U = -\vec{p}\cdot\vec{E}$

Force on a Dipole: $\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$

Field of a Dipole: $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{p}{r^2}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$

Electrostatics in Matter

Electric Displacement: $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$

Bound Surface Charge: $\sigma_{\text{bound}} = \vec{P}\cdot\hat{n}$

Bound Volume Charge: $\rho_{\text{bound}} \equiv -\nabla\cdot\vec{P}$

Linear Dielectrics – Polarization: $\vec{P} = \epsilon_0\chi_e\vec{E}$

Linear Dielectrics – E-Field: $\vec{D} = \epsilon\vec{E}$

Electric Permittivity: $\epsilon = \epsilon_0(1+\chi_e) = \epsilon_0\epsilon_r$

Gauss's Law for $\vec{D}$ (Derivative): $\nabla\cdot\vec{D} = \rho_f$

Gauss's Law for $\vec{D}$ (Integral): $\displaystyle\oint \vec{D}\cdot d\vec{A} = Q_{f_{\text{encl}}}$

Force on a Dielectric: $F = -\nabla U$

Energy of a Dielectric: $W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau$

Techniques for Solving Problems

Cylindrical Laplacian Solution:

$$V(s,\phi) = A\ln s + B + \sum_{n=1}^{\infty}\left(A_n s^n + \frac{B_n}{s^n}\right)(C_n\cos(\phi n) + D_n\sin(\phi n))$$

Solution for Spherical Laplacians: $f(r,\theta) = \displaystyle\sum_{l=0}^{\infty}\left(A_l r^l + \dfrac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

Method of Images General Principle: $\displaystyle\sum \dfrac{q_i}{||\vec{r_{\text{sep}}}_i||} = 0$

Method of Images for Two Points: $-\dfrac{q_1}{q_2} = \dfrac{||\vec{r_{\text{sep}}}_1||}{||\vec{r_{\text{sep}}}_2||}$

Image Charge Surface Charge: $\sigma = -\epsilon_0\dfrac{\partial V}{\partial n}$

Binomial Expansion: $(1+x)^k = \displaystyle\sum_{n=0}^{\infty}\left(\dfrac{k!}{n!(k-n)!}\right)x^n$

Taylor Expansion: $f(x) \approx \displaystyle\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}$

Because I can't remember: $f(x) \approx f(0) + f'(0)\cdot x + f''(0)\cdot\dfrac{x^2}{2} + \cdots$

Legendre Polynomials:

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$

Magnetostatics

Lorentz Force Law: $\vec{F}_{\text{mag}} = Q(\vec{v}\times\vec{B})$

Surface Current Density: $\vec{K} \equiv \dfrac{d\vec{I}}{dl_\perp}$, $\vec{K} = \sigma\vec{v}$

Volume Current Density: $J \equiv \dfrac{d\vec{I}}{da_\perp}$, $\vec{J} = \rho\vec{v}$

Magnetic Force (General): $\vec{F}_{\text{mag}} = \displaystyle\int I(d\vec{I}\times\vec{B}) = I\int(d\vec{I}\times\vec{B})$

Magnetic Force (Surface): $\vec{F}_{\text{mag}} = \displaystyle\int(\vec{K}\times\vec{B})da$

Magnetic Force (Volume): $\vec{F}_{\text{mag}} = \displaystyle\int(\vec{J}\times\vec{B})d\tau$

Biot-Savart Law: $\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int\dfrac{\vec{I}\times\hat{r}_{\text{sep}}}{||\vec{r_{\text{sep}}}||^2}dl'$

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}I\displaystyle\int\dfrac{d\vec{l'}\times\hat{r}_{\text{sep}}}{||\vec{r_{\text{sep}}}||^2}$

Ampere's Law: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$, $\displaystyle\oint\vec{B}\cdot d\vec{l} = \mu_0 I_{\text{encl}}$

Magnetic Vector Potential: $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{\nabla}\cdot\vec{A} = 0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$

Vector Potential – Line: $\vec{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int\dfrac{\vec{I}}{||\vec{r_{\text{sep}}}||}dl' = \dfrac{\mu_0 I}{4\pi}\int\dfrac{1}{||\vec{r_{\text{sep}}}||}d\vec{l'}$

Vector Potential – Surface: $\vec{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int\dfrac{\vec{K}}{||\vec{r_{\text{sep}}}||}da'$

Vector Potential – Volume: $\vec{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int\dfrac{\vec{J}(\vec{r'})}{||\vec{r_{\text{sep}}}||}d\tau'$

Magnetic Dipole Moment: $\vec{m} \equiv I\displaystyle\int d\vec{a} = I\vec{a}$

Magnetic Dipole Potential Expansion: $\vec{A}_{\text{dip}}(\vec{r}) = \dfrac{\mu_0}{4\pi}\dfrac{\vec{m}\times\hat{r}}{r^2}$

Magnetic Torque: $\vec{N} = \vec{m}\times\vec{B}$

Force on a Magnetic Dipole: $\vec{F} = \nabla(\vec{m}\cdot\vec{B})$

Force Between Two Current-Carrying Wires: $\dfrac{F}{l} = \dfrac{\mu_0 I_1 I_2}{2\pi r}$

Magnetostatics in Matter

Paramagnets: magnetization is **parallel** to $\vec{B}$

Diamagnets: magnetization is **opposite** to $\vec{B}$

Ferromagnets: magnetization holds outside the presence of an external magnetic field.

Magnetization: $\vec{M} \equiv$ magnetic dipole per unit volume

Bound Volume Current: $\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}$

Bound Surface Current: $\vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$

Free Current: $\vec{J} = \vec{J}_b + \vec{J}_f$

Auxillary Field: $\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$

Ampere's Law for Auxillary Fields:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$
$$\oint \vec{H} \cdot d\vec{l} = I_{f_{\text{encl}}}$$

Linear Magnetics: $\vec{M} = \chi_m \vec{H}$

Permeability: $\mu \equiv \mu_0(1 + \chi_m)$

Linear Magnetics with E and H Fields: $\vec{B} = \mu\vec{H}$.

Electrodynamics

Ohm's Law: $V = IR$

Power in a Circuit: $P = VI = I^2R = \frac{V^2}{R}$

Electromotive Force:

$$\mathcal{E} = \frac{F_{\text{mag, tot}}}{Q} = \int (\vec{v} \times \vec{B}) d\vec{l}$$
$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$$
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Inductance:

$$\Phi = LI$$
$$\mathcal{E} = -L\frac{dI}{dt}$$

Work of a Magnetic Field:

$$\frac{1}{2}LI^2$$
$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau$$

Displacement Current: $\vec{J}_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Maxwell's Equations

Gauss's Law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$

(unnamed) $\vec{\nabla} \cdot \vec{B} = 0$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere's Law with Maxwell's Correction: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Maxwell's Equations in Matter

Gauss's Law: $\vec{\nabla} \cdot \vec{D} = \rho_f$

(unnamed) $\vec{\nabla} \cdot \vec{B} = 0$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere's Law with Maxwell's Correction: $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

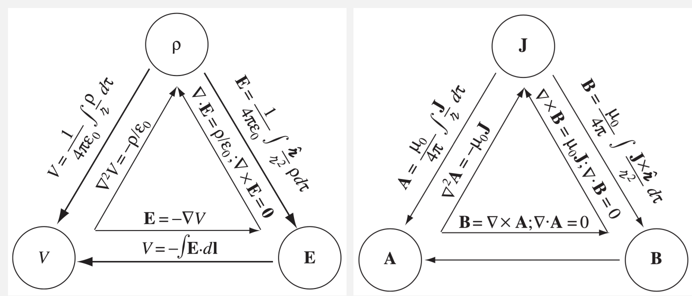
Boundary Conditions in Electrostatics

$$\vec{D}^{\perp}_{\text{above}} - \vec{D}^{\perp}_{\text{below}} = \sigma_f$$
$$\vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$$
$$\vec{E}^{\perp}_{\text{above}} - \vec{E}^{\perp}_{\text{below}} = \sigma_{\text{tot}}/\epsilon_0$$
$$\vec{E}^{\parallel}_{\text{above}} - \vec{E}^{\parallel}_{\text{below}} = \vec{0}$$
$$V_{\text{above}} = V_{\text{below}}$$
$$\epsilon_{\text{above}}\vec{E}^{\perp}_{\text{above}} - \epsilon_{\text{below}}\vec{E}^{\perp}_{\text{below}} = \sigma_f$$
$$\epsilon_{\text{above}}\frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}}\frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

Boundary Conditions in Magnetostatics

$$\vec{B}^{\parallel}_{\text{above}} - \vec{B}^{\parallel}_{\text{below}} = \mu_0\vec{K}$$
$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{\mathbf{n}})$$
$$\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$$
$$\frac{\partial A_{\text{above}}}{\partial n} - \frac{\partial A_{\text{below}}}{\partial n} = -\mu_0\vec{K}$$
$$\vec{H}^{\perp}_{\text{above}} - \vec{H}^{\perp}_{\text{below}} = -(\vec{M}^{\perp}_{\text{above}} - \vec{M}^{\perp}_{\text{below}})$$
$$\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_f \times \hat{\mathbf{n}}$$

Triangles



More Boundary Conditions

Equations 7.60-63 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

$$\left.\begin{array}{ll} \text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\} \quad (7.64)$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll} \text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}. \end{array} \quad (7.65)$$

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.

Takeaways from Practice

- Gauss's law inside a dielectric always includes both ρ_b and σ_b.
- Use the definition for displacement where ever possible: $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$.
- The potential is constant below an image plane and the field is always zero.

Names of Stuff

ϵ_0: Permittivity of Free Space
ϵ_r: Dielectric Constant or Relative Permittivity
ϵ: Permittivity of a Material