Taylor Expansion

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}$$

$$f(x) \approx f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2} + \cdots$$

Field Integrals

Line Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, dl'$$

Surface Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, da'$$

Volume Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, d\tau'$$

Potential Difference

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

Potentials

Volume Charge

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||} d\tau'$$

Collection of Point Charges

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^{n} \frac{q_i}{||\vec{r_{\text{sep}}}||}$$

Work to Move a Charge

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \vec{F} \cdot d\vec{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \vec{E} \cdot d\vec{l}$$

$$W = Q[V(\vec{b}) - Q(\vec{a})]$$

Energy of a Charge Distributions

Collection of Point Charges

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r_i})$$

Continuous Charge Distribution

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Parallel Plate Voltage

$$V = \frac{Q}{A\epsilon_0} d$$

Capacitance

$$C \equiv \frac{Q}{V}$$

Energy of a Capacitor

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$$

Common Boundary Conditions

Continuity of Potential

$$V_{\text{above}}(a) = V_{\text{below}}(a)$$

Preservation of Field
(Symmetry Required)

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}$$

Bounding of Origin
(Generally Spherical Distributions)

$V(0)$ is bounded.

Bounding of Infinity
(Generally Spherical Distributions)

$V(\infty) = 0$.

Uniqueness of Solutions

For a charge distribution with $\nabla^2 V = 0$:

1. V has no local maxima nor minima inside. The maxima and minima are located on the surrounding boundaries.
2. V is smooth and continuous, everywhere.
3. $V(\vec{r})$ is the average of V over surface of any surrounding sphere: $V(\vec{r}) = \frac{1}{4\pi R^2} \oint V dA$.
4. V is unique, as the solution of the Laplace equation is uniquely determined if V is specified on the boundary surface around the volume.

Earnshaw's Theorem

A charged particle cannot be held in a stable equilibrium by electrostatic forces alone. This can be analyzed using divergence amongst other methods of analysis.

Properties of Conductors

1. $\vec{E} = 0$ inside a conductor. In short, charges move to oppose any external $\vec{E}$ fields. This can also be interpreted under the principle that if there was any field inside a conductor, the free electrons would be moving, and hence not *electrostatic*.
2. Any net charges reside on the surface of a conductor.
3. $\rho = 0$ inside a conductor. Since there is no field inside a conductor, Gauss's law requires that there is no enclosed charge, thus, there is no charge density. One may make the argument for the surface charges, however, since they are equal in magnitude, they cancel.
4. A conductor is an equipotential. Since there is no field inside a conductor, given the relationship $\nabla V = -\vec{E}$, the potential V must be a constant.
5. $\vec{E}$ is $\perp$ to the surface of a conductor.

Rules for Irrotational Fields

Since all electrostatic fields are conservative and therefore irrotational:

1. $\vec{\nabla} \times \vec{F} = \vec{0}$ everywhere.
2. $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.
3. $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
4. $\vec{F}$ is the gradient of some scalar function: $\vec{F} = -\nabla V$.

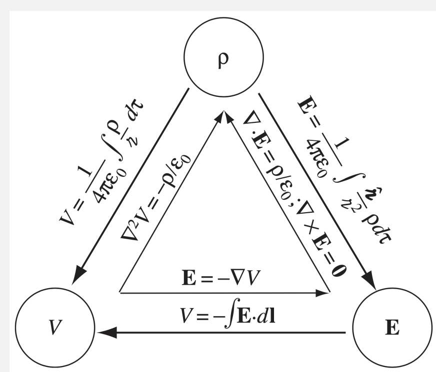
Problem Solving Stragegy – Coulomb Integrals

1. Choose a coordinate system.
2. Identify $\vec{r}$, the vector from the origin to the point of interest.
3. Identify $\vec{r'}$, the vector from the origin to the charge distribution. Convert this to your target coordinate system.
4. Identify $\vec{r_{\text{sep}}}$, the vector separating the charge distribution and the point of interest: $(\vec{r_{\text{sep}}} = \vec{r} - \vec{r'})$
5. If possible, identify any symmetries to try to solve the problem with.
6. Integrate.

Notes:

- Switch your coordinate systems as early as possible
- The $\vec{r'}$ vector should be in the alternate coordinate system
- The values in the $\vec{r}$ vector are **constants**
- Leave everything in its purest form.

The Electricity and Magnetism Triangle



Other Notes

- For conductor problems, **always** check with Gauss's Law afterwards!

Surface Field

$$\vec{E}_{\text{surf}} = \frac{1}{2}\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$$

Generalized Multipole Expansion

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_V ||\vec{r'}||^n P_n(\cos\alpha)\rho(\vec{r'})d\tau'$$

Monopole Voltage

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Dipole Voltage

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

Dipole Moment – Continuous Charge Distribution

$$\vec{p} \equiv \int_V \vec{r'}\rho(\vec{r'})d\tau'$$

Dipole Moment – Point Charge Distribution

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r'}_i$$

Dipole Moment – Change of Origin

$$\vec{p'} = \vec{p} - Q\vec{a}$$

Force on a Dipole

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

Torque of a Dipole

$$\vec{N} = \vec{p}\times\vec{E}$$

Energy of a Dipole in an Electric Field

$$U = -\vec{p}\cdot\vec{E}$$

Bound Surface Charge Density

$$\sigma_{\text{bound}} = \vec{P}\cdot\hat{n}$$

Image Charge Surface Charge

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

Linear Dielectrics

$$\vec{P} = \epsilon_0\chi_e\vec{E}$$
$$\vec{D} = \epsilon\vec{E}$$
$$\epsilon = \epsilon_0(1+\chi_e) = \epsilon_0\epsilon_r$$

Bound Volume Charge Density

$$\rho_{\text{bound}} \equiv -\nabla\cdot\vec{P}$$

Electric Displacement

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$$

Gauss's Law for Electric Displacement

$$\nabla\cdot\vec{D} = \rho_f$$

$$\oint \vec{D}\cdot d\vec{A} = Q_{f_{\text{encl}}}$$

Displacement Boundary Condition

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}$$

Force on a Dielectric

$$F = -\nabla U$$

Solution for For Spherical Laplacians

$$f(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Legendre Polynomials

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$
$$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$$

Work of a Dielectric

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau$$

Names of Stuff

ϵ_0: Permittivity of Free Space
ϵ_r: Dielectric Constant or Relative Permittivity
ϵ: Permittivity of a Material

Field of a Dipole

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

Common Boundary Conditions

- $\vec{D}^{\perp}_{\text{above}} - \vec{D}^{\perp}_{\text{below}} = \sigma_f$
- $\vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$
- $\vec{E}^{\perp}_{\text{above}} - \vec{E}^{\perp}_{\text{below}} = \frac{\sigma_{\text{surf}}}{\epsilon_0}$
- $V_{\text{above}} = V_{\text{below}}$
- $\epsilon_{\text{above}}E^{\perp}_{\text{above}} - \epsilon_{\text{below}}E^{\perp}_{\text{below}} = -\sigma_f$
- $\epsilon_{\text{above}}\frac{\partial V_a}{\partial n}^{\perp} - \epsilon_{\text{below}}\frac{\partial V_b}{\partial n}^{\perp} = \sigma_f$
- $V(r=0) =$ finite
- $V(r\to\infty) =$ finite or zero.

Method of Images Voltage

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{||\vec{r}_{\text{sep}_1}||} + \frac{q_2}{||\vec{r}_{\text{sep}_2}||}\right)$$
$$\frac{-q_1}{q_2} = \frac{||\vec{r}_{\text{sep}_1}||}{||\vec{r}_{\text{sep}_2}||}$$

Binomial Expansion

$$(1+x)^k = \sum_{n=0}^{\infty}\left(\frac{k!}{n!(k-n)!}\right)x^n$$

Takeaways from Practice

- Gauss's law inside a dielectric always includes both ρ_b and σ_b.
- Use the definition for displacement where ever possible: $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$.
- The potential is constant below an image plane and the field is always zero.

Cylindrical Laplacian Solution

$$V(s,\phi) = A\ln s + B + \sum_{n=1}^{\infty}\left(A_n s^n + \frac{B_n}{s^n}\right)(C_n\cos(\phi n) + D_n\sin(\phi n))$$

General Problem Solving Strategy for Dielectrics

1. Check that your situation has irrotational displacement. Use $\nabla\times\vec{D} = 0$ or $\nabla\times\vec{P} = 0$. If your displacement is not irrotational, this is useless.
2. If there are regions that have dielectrics, find the displacement using $\oint\vec{D}\cdot d\vec{A} = Q_{\text{free, encl}}$
3. For regions without dielectrics, find the $\vec{E}$ field using Gauss's law.
4. Using the results from (2) and (3), find $\vec{E}$ and $\vec{D}$ (whichever is lacking) using principles of linear dielectrics.
5. Use the definition of electric displacement to find $\vec{P}$: $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$.