Common Boundary Conditions

Given a PDE with a separable solution $u(p,q) = P(p) \cdot Q(q)$, which can be rearranged to fit the form

$$\frac{Q^{(n)}(q)}{Q(q)} + C = \frac{P''(p)}{P(p)} = -\lambda$$

and with boundary conditions in a, example: $u(0,q), u(L,q)...$, we can identify a set of common solutions to the initial boundary value problem that is $P''(p) + \lambda P(p) = 0$

1. **Dirichlet Boundary Conditions**
   Form
   $u(0,q) = u(L,q) = 0$
   Solutions
   $P_n = \sin\left(\frac{n\pi}{L}p\right)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\mu_n = \frac{n\pi}{L}$, $n \geq 1$
2. **Neumann Boundary Conditions**
   Form
   $u_p(0,q) = u_p(L,q) = 0$
   Solutions
   $P_0 = 1$, $\lambda_0 = 0$
   $P_n = \cos\left(\frac{n\pi}{L}p\right)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\mu_n = \frac{n\pi}{L}$, $n \geq 1$
3. **Periodic Boundary Conditions**
   Form
   $u(0,q) = u(L,q)$ and $u_p(0,q) = u_p(L,q)$
   Solutions
   $P_0 = 1$, $\lambda_0 = 0$
   $P_n = \sin\left(\frac{n\pi}{L}p\right) + \cos\left(\frac{n\pi}{L}p\right)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n \geq 1$
4. **Mixed Type 1 Boundary Conditions**
   Form
   $u(0,q) = u_p(L,q) = 0$
   Solutions
   $P_n = \sin\left(\frac{2n-1}{2L}\pi p\right)$, $\lambda_n = \left(\frac{2n-1}{2L}\right)^2$, $n \geq 1$
5. **Mixed Type 2 Boundary Conditions**
   Form
   $u_p(0,q) = u(L,q) = 0$
   Solutions
   $P_n = \cos\left(\frac{2n-1}{2L}\pi p\right)$, $\lambda_n = \left(\frac{2n-1}{2L}\right)^2$, $n \geq 1$

Fourier Series

Given a function $f(x)$, can describe it in terms of an infinite series of sine and cosine functions with varying frequencies.

$$f(p) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\mu_n p) + \sum_{n=1}^{\infty} B_n \sin(\mu_n p)$$

To find the arbitrary constants A_n and B_n that satisfy the Fourier series, we can apply orthogonality.

$$\int_{-L}^{L} \sin(\mu_m p)\sin(\mu_n p)dp = \begin{cases} 0 & \mu_m \neq \mu_n \\ L & \mu_m = \mu_n \end{cases}$$

$$\int_{-L}^{L} \cos(\mu_m p)\cos(\mu_n p)dp = \begin{cases} 0 & \mu_m \neq \mu_n \\ L & \mu_m = \mu_n \neq 0 \\ 2L & \mu_m = \mu_n = 0 \end{cases}$$

$$\int_{-L}^{L} \cos(\mu_m p)\sin(\mu_n p)dp = 0$$

everywhere, if μ_n and μ_m are of the form $\frac{n\pi}{a}$, $a \in \mathbb{Z}$.

If orthogonality fails, we must integrate to find our constants.

Cosine Fourier Series (Even Extension)

$$A_0 = \frac{1}{L}\int_0^L f(p)dp$$

$$A_n = \frac{2}{L}\int_0^L f(p)\cos(\mu_n p)dp$$

Sine Fourier Series (Odd Extension)

$$B_n = \frac{2}{L}\int_0^L f(p)\sin(\mu_n p)dp$$

Common Forms of Partial Differential Equations

1. Inhomogeneous, Time Independent Dirichlet Boundary Conditions

   Form:

   $u_t = \alpha^2 u_{xx}$
   $u(0,t) = u_0$
   $u(L,t) = u_1$

   Solving Tactic

   (a) Choose a solution of the form $u(x,t) = w(x) + v(x,t)$
   (b) Choose $w(x)$ of the form $w(x) = Ax + B$
   (c) Directly apply the BCs to find the arbitrary constants A and B.
   (d) Take the necessary derivatives of $w(x) = Ax + B$ and $v(x,t)$ and insert them into the original PDE. You should be left with $v_t = \alpha^2 v_{xx}$
   (e) Solve the initial boundary value problem in x.
   (f) Use λ from the IBVP to find $T(t)$

   General form should be reached, rest is left to Fourier series.

   Expected Solution

   $u(x,t) = \frac{u_1 - u_0}{L} + u_0 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$

2. Inhomogeneous, Time Independent Neumann Boundary Conditions

   Form:

   $u_t = \alpha^2 u_{xx}$
   $u_x(0,t) = q_0$
   $u_x(L,t) = q_1$

   Solving Tactic

   (a) Choose a solution of the form $u(x,t) = w(x,t) + v(x,t)$
   (b) Choose $w(x,t)$ of the form $w(x,t) = Ax^2 + Bx + Dt$

   Remark: We choose a solution of the form $w(x,t) = Ax^2 + Bx + Dt$ because:

   i. $w(x) = Ax + B$ has x derivative $w_x(x) = A$, which would mean that A would take on values of q_0 and q_1. If $q_0 \neq q_1$, then this is impossible.
   ii. A next logical guess would be $w(x) = Ax^2 + Bx + C$, however, this fails because $w_{xx} = 2A$ in this case, which does not satisfy the PDE, as $w_t = 0$, so the original PDE $u_t = \alpha^2 u_{xx}$ is not satisfied.
   iii. Therefore, we add a time-dependent term to get $w(x,t) = Ax^2 + Bx + C + Dt$, which holds, however, because we deal with no equations that don't take an x or t derivative, C is never used, and therefore can be assumed to be zero.

   (c) Directly apply the BCs to find the arbitrary constants A and B, (but not D yet).
   (d) Insert $w(x,t)$ into the PDE ($w_t = \alpha^2 w_{xx}$) to find the constant D.
   (e) Insert the entire $u(x,t) = w(x,t) + v(x,t)$ ($v(x,t)$ still unknown) into the original PDE. All the w terms should cancel, and you should be left with $v_t = \alpha^2 v_{xx}$
   (f) Solve the IBVP in x to find X_n and λ_n
   (g) Knowing that the initial condition for $v(x,t)$ is $g(x) = f(x) - w(x,0)$, solve the fourier series.

   Expected Solution

   $u(x,t) = \frac{q_1 - q_0}{2L} x^2 + q_1 x + \alpha^2 \frac{q_1 - q_0}{L} t + A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$

   $A_0 = \frac{1}{L} \int_0^L g(x) dx$, $A_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx$

3. Inhomogeneous (or Homogeneous!) Boundary Conditions with a Forcing Function

   Form:

   $u_t = \alpha^2 u_{xx} \pm \gamma u + g(x)$
   $u(0,t)$ or $u_x(0,t) = a$ or 0
   $u(L,t)$ or $u_x(L,t) = b$ or 0

   Note: I totally may have overstepped on this problem and how general I made the initial form. At a minimum, the PDE should be expected to be $u_t = \alpha^2 - u + x$

   Solving Tactic

   (a) Choose a solution of the form $u(x,t) = w(x) + v(x,t)$
   (b) Insert $w(x)$ into the PDE to find an equation of the form $w'' \pm \gamma w = -g(x)$
   (c) Use a characteristic equation and the method of undetermined coefficients, identify the homogeneous and particular solutions to find the general form of $w(x)$.
   (d) Directly employ the boundary conditions from the initial problem to find the coefficients of $w(x)$.
   (e) Insert $u(x,t) = w(x) + v(x,t)$ into the original PDE. The w terms should cancel, leaving you with $v_t = v_{xx} - v$.
   (f) Using the relationship $u(x,t) = w(x) + v(x,t)$, identify the boundary and initial conditions for $v(x,t)$
   (g) Employ separation of variables to solve $v(x,t) = X(x) \cdot T(t)$
   (h) Insert into the modified v PDE to obtain $XT' = X''T - \gamma XT$
   (i) Divide by XT to get $\frac{T'(t)}{T(t)} + \gamma = \frac{X''(x)}{X(x)} = -\lambda$
   (j) Solve the boundary value problem in x.
   (k) Use the λ_n identified in (j) to solve for $T(t)$
   (l) Combine results to get the general solution, use Fourier series to obtain the arbitrary constants.

4. Time Dependent Homogeneous Boundary Conditions

   Form:

   $u_t = \alpha^2 u_{xx}$
   $u(0,t)$ or $u_x(0,t) = At$
   $u(L,t)$ or $u_x(L,t) = Bt$

   Solving Tactic

   - Choose a solution of the form $u(x,t) = w(x,t) + v(x,t)$
   - Choose $w(x,t) = A(t)x + B(t)$
   - Solve the problem like you would in (1) or (2)

5. Time-Dependent Source/Sink

<u>Form:</u>

$u_t = \alpha^2 u_{xx} + e^{-\beta t}\cos(\gamma\pi x) + g(x)$
- or -
$u_t = \alpha^2 u_{xx} + e^{-\beta t}\sin(\gamma\pi x) + g(x)$

where the extra term is the source/sink ($S(x,t)$)

<u>Solving Tactic</u>

(a) Choose the appropriate $w(x)$ or $w(x,t)$ for $u(x,t) = w(x,t) + v(x,t)$ to satisfy the inhomogeneous boundary conditions (if the BCs are inhomogeneous).

(b) Insert $u(x,t) = w(x,t) + v(x,t)$ into the original PDE to obtain a PDE of the form $v_t = \alpha^2 v_{xx} + e^{-\beta t}\sin(\gamma\pi x)$

(c) Transform the original boundary and initial conditions to get boundary conditions for $v(x,t)$ given the relationship $u(x,t) = w(x,t) + v(x,t)$

(d) Solve the [homogeneous] initial boundary value problem for $v_t = \alpha^2 v_{xx}$ (no source/sink term) to obtain X_n and λ_n (finding $T(t)$ is unnecessary).

(e) Use the method of eigenfunction expansion to derive an expression for the source term in terms of a Fourier series with a time-dependent arbitrary constant

$$S(x,t) = e^{-\beta t}\sin(\gamma\pi x) = \sum_{n=1}^{\infty} S_n(t)\sin(\mu_n x)$$

(f) $S_n(t)$ will take on the form of a delta function: $\delta_{n\epsilon}$, where ϵ is the n which satisfies the equality that $\gamma = \mu_n$. If it doesn't you will have to integrate and cry.

(g) Because the source term denies the correctness of the homogeneous solution in time, we don't know the time dependence of our solution from solving the homogeneous problem. Therefore, we must describe $v(x,t)$ in terms of a Fourier series with a time-dependent arbitrary function.

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin(\mu_n x)$$

(h) From this, we can take the derivatives of the $v(x,t)$ series as well as our source series and insert them into our modified PDE for v.

$$v_t = \sum_{n=1}^{\infty} V_n'(t)\sin(\mu_n x)$$

$$v_{xx} = \sum_{n=1}^{\infty} -V_n(t)\mu_n^2\sin(\mu_n x)$$

(i) Substituting this all into the modified PDE and rearranging all terms to be in a single sum yields a series of the form:

$$\sum_{n=1}^{\infty}\left[V_n'(t) - \alpha^2\mu_n^2 V_n(t) - \delta_{n\epsilon}e^{-\beta t}\right]\sin(\mu_n x) = 0$$

(j) Knowing that sending the sinusoidal function to zero would yield a trivial solution, we can solve the differential equation for $V_n(t)$, using the integrating factor method.

$$V_n'(t) - \alpha^2\mu_n^2 V_n(t) = \delta_{n\epsilon}e^{-\beta t}$$

(k) Armed $V_n(t)$, we obtain an expression for $v(x,t)$, which we can impose our initial conditions on.

$$v(x,t) = \sum_{n=1}^{\infty}\left[\frac{\delta_{n\epsilon}}{\mu_n^2 - \beta}e^{-\beta t} + C_n e^{-\mu_n^2 t}\right]\sin(\mu_n x) \;\rightarrow\; v(x,0) = \sum_{n=1}^{\infty}\left[\frac{\delta_{n\epsilon}}{\mu_n^2 - \beta} + C_n\right]^{\star}\sin(\mu_n x) = \sin(a\pi x)$$

(l) We can consider the $\star$ term to be it's own arbitrary constant D_n, which we can solve for using orthogonality, then solve for C_n to find our general solution.

(m) C_n will likely take the form of another delta (δ) function, so we can eliminate our sum and set the appropriate n terms to those agreeing with our δ functions.

6. Wave Equations

Form

$u_{tt} = c^2 u_{xx}$
... but unlike heat equations, they have two initial conditions.

Solving Tactics

(a) Solve your IBVPs in x as per usual
(b) Solve $T(t)$ with the λ_n that was obtained in (a)
(c) Combine $X(x)$ and $T(t)$ and impose the two initial conditions to find the two arbitrary constants in T(t).

7. D'Alembert's Solution to the Wave Equation

Given a wave equation of the form $u_{tt} = c^2 u_{xx}$, and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, the solution to the *unbounded* problem is:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

8. Graphical Solutions to Wave Equations

Given a wave equation of the form $u_{tt} = c^2 u_{xx}$, and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x) = 0$, with $f(x)$ of the form:

$$u(x,0) = f(x) = \begin{cases} g(x) & a \le x < b \\ h(x) & b \le x \le c \\ 0 & x < a,\ x > c \end{cases}$$

We know that by D'Alembert's Solution that the solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] = \frac{1}{2}\left[f(\zeta) + f(\eta)\right]$$

Therefore, we can break $f(x)$ into two pieces:

$$u(x,t) = \frac{1}{2}\left[\begin{cases} g(\zeta) & a \le \zeta < b \\ h(\zeta) & b \le \zeta \le c \\ 0 & \zeta < a, \zeta > c \end{cases} + \begin{cases} g(\eta) & a \le \eta < b \\ h(\eta) & b \le \eta \le c \\ 0 & \eta < a, \eta > c \end{cases}\right]$$

From the equation above, we can create a graph that has axes of x and t, but also has constant lines in ζ and η corresponding to regions of dependency for the solutions. Overlapping regions in these plots correspond to regions of dual dependency (dependency on both ζ and η functions).

We can also insert $\zeta = x - ct$ and $\eta = x + ct$ into our functions and rearrange them to get a solution that corresponds to a real solution space.

$$u(x,t) = \frac{1}{2}\left[\begin{cases} g(x-ct) & a \le x-ct < d \\ h(x-ct) & d \le x-ct \le b \\ 0 & x-ct < a, x-ct > b \end{cases} + \begin{cases} g(x+ct) & a \le x+ct < d \\ h(x+ct) & d \le x+ct \le b \\ 0 & x+ct < a, x+ct > b \end{cases}\right]$$

$$u(x,t) = \frac{1}{2}\left[\begin{cases} g(x-ct) & a+ct \le x < d+ct \\ h(x-ct) & d+ct \le x \le b+ct \\ 0 & x < a+ct, x > b+ct \end{cases} + \begin{cases} g(x+ct) & a-ct \le x < d-ct \\ h(x+ct) & d-ct \le x \le b-ct \\ 0 & x-ct < a, x > b-ct \end{cases}\right]$$

This provides us with an easily attainable graphical interpretation of the solution.

9. 2D Cartesian Laplace Equations Given a rectangular domain on $0 \leq x \leq a$ and $0 \leq y \leq b$, with the partial differential equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ with any of the boundary condition

$$u(x,0) = f_1(x) \qquad u(0,y) = g(y) \qquad u(x,b) = f_2(x) \qquad u(a,y) = g_2(y)$$

- or -

$$u_y(x,0) = f_1(x) \qquad u_x(0,y) = g(y) \qquad u_y(x,b) = f_2(x) \qquad u_x(a,y) = g_2(y)$$

We can divide up the problem into n subproblems, where n is the number of inhomogeneous boundary conditions in the problem.

To solve one of these subproblems, u^N, we set all of the other inhomogeneous boundary conditions to be homogeneous (zero), and solve the problem in the secondary coordinate to the inhomogeneous boundary condition.

Reese's Laplacian Shortcut

Take a solution to a subproblem to always be of the form:

$$u^N(x,y) = \sum_{n=1}^{\infty} \frac{a_n^{g_m}}{h(\pm\mu_n a)} g(\mu_n(x - a^\dagger)) f(\mu_n y)$$

- or -

$$u^N(x,y) = \sum_{n=1}^{\infty} \frac{b_n^{f_m}}{h(\pm\mu_n b)} g(\mu_n(y - b^\dagger)) f(\mu_n x)$$
$$0 \leq x \leq a \qquad 0 \leq y \leq b$$

Given a subproblem with four boundary conditions, we can characterize each one of the boundary conditions:

(a) 1st homogeneous secondary-axis $u_?(x,0) = 0$
(b) 2nd homogeneous secondary-axis $u_?(x,b) = 0$
(c) Homogeneous primary-axis $u_?(\sigma,y) = 0$
(d) Inhomogeneous primary-axis $u_?(\rho,y) = g(y)$

Now, knowing that we find our λ_n and μ_n from (a) and (b), we know that μ_n must be the μ_n that satisfies the boundary value problem in w, produced by (a) and (b). In addition, f will be the sinusoidal function in Y_n.

We also know that g is either sinh or cosh based on the (c) boundary condition, with the given criteria:

i. If $\sigma = a$, then the function represented by g is cosh, and $a^\dagger = a$

   If $\sigma = 0$, $a^\dagger = 0$ and g can take two forms:

ii. If (c) is a Neumann boundary condition (derivative), then $g = \sinh$
iii. If (c) is a Dirichlet boundary condition (no derivative), then $g = \cosh$

We can also know h based on the (d) boundary condition, with the given criteria.

i. If (d) is a Dirichlet boundary condition (no derivative), then h is the same as g
ii. If (d) is a Neumann boundary condition (derivative), then h will be the opposite function of g.

Finally, the sign inside h is based on the following criteria:

i. $(+)$ if $a^\dagger = 0$
ii. $(-)$ if $a^\dagger = a$

10. General Laplacian Equations on Polar Coordinates

Given a partial differential equation on polar coordinates of the form:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$a \leq r \leq b \qquad 0 \leq \theta \leq \alpha$$

With standard boundary conditions, and the addition of three additional possible boundary conditions:

$$u(0,\theta) = \gamma, \ \ \gamma \in \mathbb{R} \qquad\qquad \lim_{\rho \to \infty} u(\rho,\theta) = \gamma, \ \ \gamma \in \mathbb{R}$$

$$u(r,\pi) = u(r,-\pi) \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}(r,\pi) = \frac{\partial u}{\partial \theta}(r,-\pi)$$

We can assume a separable form of the equation to be

$$u(r,\theta) = R(r) \cdot \Theta(\theta)$$

Which when inserted into the original PDE, multiplied by r^2 and divided by $R\Theta$ we obtain:

$$-\left(r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)}\right) = \frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda$$

Because the equations in r appear to be hellish, we can only solve the IVBP in θ.

From solving the IBVP, we get a Θ_n, as well as a λ_n and μ_n, which can be used to solve $R(r)$:

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = \lambda$$

$$r^2R'' + rR' - \lambda R = 0$$

This equation has the form of a Cauchy-Euler Equation, which has a characteristic equation of the form $s(s-1)+\alpha s+\beta = 0$, where α is the R' coefficient, and β is the R coefficient.

In all (hopefully) cases of this problem type, our characteristic equation will be of the form:

$$s^2 - \lambda_n = 0$$

Therefore, our corresponding $R_n(r)$ will be of the following form:

$$R_n(r) = \begin{cases} C_0 + D_0 \ln r & \lambda = 0 \\ C_n r_n^{\mu} + D_n r^{-\mu_n} & \lambda_n \neq 0,\, \lambda \in \mathbb{R} \end{cases}$$

We can then use the homogeneous boundary condition in r to simplify our forms of $R(r)$.
(same BC for both R_0 and R_n)

Finally, we can combine $R(r)$ and $\Theta(\theta)$ to find our general solution, which should have only one arbitrary constant, for which we can use orthogonality, a Fourier series, and the final inhomogeneous boundary condition to find.

11. Inhomogeneous Laplacian Equations

Form

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

$u(r, \alpha) = u_1$

Solving Tactic

(a) Choose a solution of the form $u(r,\theta) = w(\theta) + v(r,\theta)$, where $w(\theta)$ satisfies the inhomogeneous boundary conditions.

(b) Choose $w(\theta) = A\theta + B$, and solve for A and B using the direct boundary conditions in θ (use the inhomogeneous).

(c) Solve the initial boundary value problem $\Theta'' + \lambda\Theta = 0$

(d) Use the μ_n and λ_n found from the IBVP in θ to solve the corresponding Cauchy-Euler equation.

(e) Given the relationship $u(r,\theta) = w(\theta) + v(r,\theta)$, a second inhomogeneous boundary condition in θ will present itself.

(f) Using the two inhomogeneous boundary conditions in θ, we can solve for a single arbitrary constant in each equation (P_n and Q_n).

$$P_n = \frac{2}{L_\theta}\int_0^{L_\theta} f(\theta)\cos(\mu_n\theta)d\theta = A_n b^{\mu_n} + B_n b^{-\mu_n}$$

$$Q_n = \frac{2}{L_\theta}\int_0^{L_\theta} g(\theta)\cos(\mu_n\theta)d\theta = A_n a^{\mu_n} + B_n a^{-\mu_n}$$

Where $g(\theta)$ is the "synthetic inhomogenous boundary condition"
and $f(\theta)$ is the original inhomogeneous boundary condition

(g) Knowing that the A_n and B_n terms in both equations are the same, we can identify that there exists a matrix solution for A_n and B_n.

$$\begin{bmatrix} b^{\mu_n} & b^{-\mu_n} \\ a^{\mu_n} & a^{-\mu_n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} P_n \\ Q_n \end{bmatrix}$$

(h) Employing matrix algebra ($A^{-1}b = \vec{N}$), we can find A_n and B_n, and insert them into our general solution to find the solution to the initial PDE.

12. Forwards/Backwards Finite Difference Approximations

Given a series of equal steps away from a reference point, we can Taylor expand them to find an approximation for a chosen derivative.

If we have $f(x+\Delta x), f(x+2\Delta x), \cdots, f(x+N\Delta x)$ terms, then we can Taylor expand them up until the N^{th} derivative, with the addition of a $f(x) = f(x)$ equation.

$$f(x) = f(x) \tag{a}$$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \cdots + \frac{f^{(N)}(x)\Delta x^N}{N!} + \mathcal{O}(\Delta x)^{(N+1)} \tag{b}$$

$$\vdots$$

$$f(x+\alpha\Delta x) = f(x) + \alpha f'(x)\Delta x + \frac{\alpha^2}{2}f''(x)\Delta x^2 + \cdots + \frac{\alpha^N}{N!}f^{(N)}(x)\Delta x^N + \mathcal{O}(\Delta x)^{(N+1)} \tag{η}$$

where $\alpha = N$ in the last equation

We can then find a linear combination of all of these terms to obtain a finite difference approximation for the M^{th} derivative about x.

To do this, we can form a matrix algebra problem of the form:

$$\begin{bmatrix} a_0 & b_0 & c_0 & \cdots & \eta_0 \\ a_1 & b_1 & c_1 & \cdots & \eta_1 \\ & \vdots & & & \vdots \\ a_N & b_N & c_N & \cdots & \eta_N \end{bmatrix} \begin{bmatrix} a \\ b \\ \vdots \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1_M \\ \vdots \\ 0 \end{bmatrix}$$

Where $a_n, b_n, c_n, \cdots, \eta_n$ are the coefficients of the n-th derivative in the equation labelled by $a, b, c, \cdots, \eta$, and 1_M is the number 1 in the the position of whichever M-th derivative we are searching for. (In this case it's the 1st derivative)

To find the order of this solution, we know that we expanded up until the Nth derivative, so our higher order terms are of N+1th order. However, in order to fully find our solution, we remove whatever ΔX^M coefficient that is attached to the derivative that we are searching for. In this case, we divide our higher order terms as well by ΔX^M, reducing our order of accuracy by M. Therefore, we can say our order of accuracy is:

$$N + 1 - M$$

13. Central Finite Difference Approximations

Central schemes are essentially the same, with the exception that when finding the even-th derivatives, can use half the number of equations that we would normally require, because the alternating terms cancel.

<u>Common Difference Approximations</u>

Forward: $f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$

Backward: $f'(x_0) = \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$

Center: $f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$

2nd Derivative: $f''(x_0) = \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{\Delta x^2} + \mathcal{O}(\Delta x^2)$

Heat Equation FDA: $u_i^{k+1} = \alpha^2 \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) + u_i^k$ Coefficient of Fluid Dynamics: $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$

<u>Note:</u> You can use a central scheme with $f'(x_0) = 0$ to find a "ghost cell" point if the boundary condition is Neumann at x_0.

Cauchy-Euler Equations

Given an ODE of the Form:

$$x^2y'' + \alpha xy' + \beta y = 0$$

We can extrapolate a characteristic equation of the form:

$$s(s-1) + \alpha s + \beta = 0$$

The roots to the characteristic equation can be used to find the general solution of the problem.

Solution Cases

s_1, s_2 are distinct real roots

$$y(x) = Ax^{s_1} + Bx^{s_2}$$

s_1, s_2 are distinct complex roots of the form $\lambda \pm \mu i$

$$y(x) = x^{\lambda}\left[A\cos\left(\mu\ln(x)\right) + B\sin\left(\mu\ln(x)\right)\right]$$

s_1, s_2 are repeated real roots

$$y(x) = Ax^s + B\ln(x)x^s$$

Integrating Factor Method of ODEs

Given an ODE of the Form:

$$y' + p(x)y = g(x)$$

We can write an "integrating factor" of the form:

$$\mu(x) = e^{\int p(x)dx}$$

Which can be used to find a solution of the form

$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$$

Linear Second Order ODEs

Given an ODE of the Form:

$$ay'' + by' + cy = 0$$

We can extrapolate a characteristic equation of the form:

$$ar^2 + br + c = 0$$

The roots to the characteristic equation can be used to find the general solution of the problem.

Solution Cases

r_1, r_2 are distinct real roots

$$y(x) = Ae^{r_1x} + Be^{r_2x}$$

r_1, r_2 are distinct complex roots of the form $\lambda \pm \mu i$

$$y(x) = e^{\lambda x}\left[A\sin(\mu x) + B\cos(\mu x)\right]$$

r_1, r_2 are repeated real roots

$$y(x) = Ae^{rx} + Bxe^{rx}$$

Method of Particular Solutions

Given an ODE of the Form:

$$Ay'' + By' + Cy = g(t)$$

We can determine a particular solution to the problem of the form:

$$y_p(t) = -y_1\int\frac{y_2g(t)}{W(y_1,y_2)}dt + y_2\int\frac{y_1g(t)}{W(y_1,y_2)}dt$$

Where y_1 and y_2 are homogeneous solutions to the ODE, and $W(y_1, y_2)$, or the Wronksian is:

$$W(y_1,y_2) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1y_2' = y_2y_1'$$

Method of Reduction of Order

1. Assuming $y_1(t)$ is a solution to a second order linear ODE, we can say that $y_2(t) = v(t)y_1(t)$
2. If we take the derivatives of $y_2(t)$ and subsitute them into the original ODE, we can get an equation in terms of v'' and v'.
3. To solve this, we can call $v' = w$, solve for w, then re-integrate to get $v(x)$

Appendix

2x2 Inverse Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$