# MATH 400 Cookbook

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## First Order PDEs

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Given some PDE of the form:

$$\begin{cases} u_t + c(x,t,u)u_x = s(x,t,u) \\ u(x,0) = f(x) \end{cases}$$

We solve by the following:

1. Obtain the system of ODEs corresponding to the original PDE:
$$\frac{dx}{dt} = c(x,t,u) \qquad \frac{du}{dt} = s(x,t,u).$$
2. Solve whichever equation has no variables that do not appear in derivative term.
3. Depending on which equation was solved in step 2, impose the corresponding initial conditions:
$$x(0) = \xi \qquad u(x,0) = f(\xi).$$
4. Use the result of step 3 to substitute back into the incomplete equation.
5. Solve for $\xi$, and substitute it back into the result for $u$.

<u>Alternate Strategy:</u>

Given some PDE of the form:

$$\begin{cases} a(x,t,u)u_t + b(x,t,u)u_x = c(x,t,u) \\ u(x,0) = f(x) \text{ on } t = g(x). \end{cases}$$

We solve the following system of ODEs:

$$\begin{aligned} \frac{dt}{ds} &= a(x,t,u) & t(0) &= g(\xi) \\ \frac{dx}{ds} &= b(x,t,u) & x(0) &= \xi \\ \frac{du}{ds} &= c(x,t,u) & u(x=0) &= f(\xi) \end{aligned}$$

<u>Data Curves and Characteristic Curves:</u> Through solving these PDEs, we obtain the following:

- Data Curve: the intitial condition $t = g(x)$ is called the *data curve*, as it contains the original data from the IC.
- Characteristic Curves: the characteristic curves are given by $\xi = h(x,y)$. They describe the curves on the space where the PDE reduces to the ODE $\frac{du}{dt} = 0$.

There exist some complications when special cases of the data curves and the characteristic curves:

- Two Characteristics Intersect
  - Cannot solve $s(x,y), \xi(x,y)$.
  - $u$ may be multi-valued.

- Characteristic Intersects the Data Curve Twice
  - No solution in general.
- Data Curve is a Characteristic Curve
  - Need to choose some special $u = f(\xi)$ to avoid contradiction, and cannot determine $u$ outside the data curve.
  - $u$ cannot be determined outside the data curve.

## Breaking

For some problems, we get that multiple characteristics intersecting at some certain point. The first point at which characteristics intersect is called the "breaking point". This generally is most seen in PDEs of the form:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Breaking [allegedly] occurs when $u_x \to \infty$, so to find this, we take

$$\frac{d}{dx} x(\xi, t),$$

and solve for the condition which makes $\xi_x \to \infty$.

## Traffic Flow Problems

Traffic flow problems arise from integral conservation laws. They assume the general form:

$$\begin{cases} \rho_t + c(\rho)\rho_x = 0 \\ \rho(x,0) = f, \end{cases}$$

where $f$ is generally a piecewise constant function. For demonstration, let

$$f(x) = \begin{cases} A, & x < a \\ B, & a < x < b \\ C, & x > b \end{cases}$$

<u>General Solution Strategy:</u>

1. Convert the PDE into a system of ODEs:

$$\frac{dx}{dt} = c(\rho) \qquad\qquad \frac{d\rho}{dt} = 0.$$

   The latter of the two equations implies that $\rho =$ constant, and with the condition that $u(x,0) = f$, then $u(x,t) = f$. However, we still need to re-translate the bounds of the piecewise function.

2. Hence, we then solve the first equation in the ODE with initial condition $x(0) = x_0 = \xi$ to obtain the form:

$$x(t, \xi) = c(\rho)t + \xi.$$

   Actually, it is more elegant to view this in terms of $x_0$.

$$x(t, x_0) = c(\rho(x_0))t + x_0$$

We can then use this to readjust all of the terms in the piecewise function. Looking back at $f(x)$, since these points are all at $t = 0 \implies x = x_0$, then, we can rewrite the bounds of $f$:

$$\begin{aligned} x < a &\implies x_0 < a \\ &\implies x < c(\rho(a))t + a \end{aligned}$$

Hence, we get a full solution with these results, and so on.

3. Lastly, it is often the case that there are discontinuities in the result $u$. There are two cases in which this happens:

   (a) Fanlike Characteristics: In this case, one will end up with a solution where there are regions with undefined values in the solution. To find the full solution, we do the following:

      i. Identify the points of $(x_0)_i$ from where the discontinuity arises from. (i.e. $a$, $b$)

      ii. Sub in $(x_0)_i$ into the $x$ equation and solve for $f(x_0) = \rho((x_0)_i)$:

      $$x = c(\rho((x_0)_i))t + x_0.$$

   (b) Shock Characteristics: In this case, one will end up with a solution where characteristics cross. This also corresponds to solutions which are piecewise and boundaries overlap. In these cases one must do the following:

      i. Determine the shock point, which is the point where the shock first occurs ($s_0$).

      ii. Determine the shock function, which can be determined by the following equation:

      $$\frac{ds}{dt} = \frac{Q(\rho_+) - Q(\rho_-)}{\rho_+ - \rho_-},$$

      where $q_x(\rho) = \rho_x c(\rho)$ (generally have to solve that before).

## Second Order Linear PDEs

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Well-posed-ness: There are three conditions that need to hold for an Initial Boundary Value Problem (IBVP) for a PDE that need to hold in order for it to be well-posed. They are as follows:

1. Existence: There exists at least one solution $u(x,t)$, satisfying all conditions.

2. Uniqueness: There exists at most one solution $u(x,t)$, satisfying all conditions.

3. Stability: The solution depends in a stable manner on the initial data, that is if the data changes by a small amount, then the solution changes by only a small amount as well.

Methods for Proving Stability:

1. Let $u(x,t)$ be a solution for the IBVP with data $f(x)$. Let $\tilde{u}(x,t)$ be a solution for the IBVP with data $\tilde{f}(x)$. Show that if $\max_x |f(x) - \tilde{f}(x)|$ is bounded by a small amount, then $\max_x |u(x,t) - \tilde{u}(x,t)|$ is also bounded by a small amount, for any $t > 0$.

2. If a sequence of inital data of the form $f_n(x)$ has that $\max_x |f_n(x)| \to 0$ as $n \to \infty$, then prove that $\max_x |u_n(x,t)| \to 0$, as $n \to \infty$, $\forall t > 0$.

## Classifications and Transformations of PDEs

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Second Order Linear PDEs of the following form:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y)$$

with determinant $D = b^2 - ac$ can be classified into the following forms:

1. Elliptic ($D < 0$): $u_{xx} + u_{yy} =$ Lower Order Terms

2. Hyperbolic ($D > 0$): $u_{xx} - u_{yy} =$ Lower Order Terms

3. Parabolic ($D = 0$): $u_{xx} =$ Lower Order Terms

General Form for Classifying and Transforming PDEs:

1. Classify the PDE using the determinant.
2. Rewrite the PDE into an operator form, i.e.:
$$u_{xx} + 8u_{xy} = (\partial_x^2 + 8\partial_x\partial_y)u.$$
3. Complete the square with the partials.
4. Substitute operators such that the PDE takes the classified form, with new variables.
5. With the coordinate transform from $x, y \to \xi, \eta$, write the matrix $B^T$, which fits the following form:
$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = B^T \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}.$$
6. Use the matrix $B$ to obtain $\xi$ and $\eta$ in terms of $x, y$:
$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}$$

## Wave Equation

General Form Solution:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x). \end{cases}$$

For a problem of this type, the solution is of the form:

$$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.$$

□

There are other forms of the wave equation for which the solution is not as trivial.

Constant BC:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0, t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x > 0 \\ u(0,t) = c & t > 0. \end{cases}$$

To solve this form, we know that the general solution to the wave equation is always:

$$u(x,t) = f(x+ct) + g(x-ct)$$

Hence, we can set up two systems of equations to solve:

$$\begin{cases} u(x,0) = u_0(x) = f(x) + g(x) \\ u_t(x,0) = u_1(x) = cf'(x) + c'g(x) \end{cases} \qquad \text{for } x \geq ct$$

$$\begin{cases} u(x,0) = u_0 = f(x) + g(x) \\ u(0,t) = c = f(ct) + g(-ct) = f(ct = x) + g(-ct = -x) \end{cases} \qquad \text{for } 0 < x < ct.$$

Factored Wave Equation:
Given some wave equation of the form:

$$\begin{cases} (a\partial_x + b\partial_y)(c\partial_x - d\partial_y)u = 0, \\ u(x,0) = u_0(x), \\ u_t(x,0) = e^x \end{cases}$$

The general solution reduces to:

$$u(x,t) = f(ax+bt) + g(cx-dt).$$

Which can be solved using a system of equations similar to the prior case.

Energy and Waves: Since energy in a system should be conserved, we can calculate the total energy of a wave PDE with the general form:

$$\rho u_{tt} = T u_{xx},$$

the energy is given by:

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2(x,0) + T u_x^2(x,0)dx$$

Method of Reflections:
For equations of the form:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > 0 \\ u(x,0) = u_0(x), & x > 0 \\ u_t(x,0) = u_1(x), & x > 0 \end{cases}$$

The solution is of the form:

$$u(x,t) = \begin{cases} \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds, & x > ct \\ \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] & \\ +\frac{1}{2c}\left(\int_{x-ct}^{0} u_1^{\text{ext}}(y)dy + \int_0^{x+ct} u_1^{\text{ext}}(y)dy\right) & \\ +\frac{1}{2c}\left(\int_0^{t-\frac{x}{c}} \int_{-x+c(t-s)}^{x+c(t-s)} f^{\text{ext}}(y,s)dyds + \int_{t-\frac{x}{c}}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f^{\text{ext}}(y,s)dyds\right), & 0 < x < ct \end{cases}$$

In the case that the equation has the form:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > 0 \\ u(x,0) = u_0(x), & x > 0 \\ u_t(x,0) = u_1(x), & x > 0 \\ u(0,t) = h(t) & \end{cases}$$

Then it is the same as the standard method of reflections, just with the addition of an extra term:

$$u(x,t) = u_0 \text{ term } + u_1 \text{ term } + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\iint_{\Delta} f.$$

Reflections on the Finite Interval: For reflections on the finite interval, we take the odd periodic extension of all functions and proceed the same as prior. Generally, questions of this nature will ask for a value at a point instead of a general solution.

## Heat Equation

General Form: Given a heat equation of the form:

$$\begin{cases} u_t = ku_{xx}, \\ u(0,t) = u_0(x), \end{cases}$$

a solution to the heat equation is:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)u_0(y)dy.$$

Heat Equation with a Source: Given a heat equation of the following form:

$$\begin{cases} u_t = ku_{xx}, \\ u(0,t) = u_0(x), \end{cases}$$

a solution to the heat equation is:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)u_0(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds.$$

Well-posedness: Given a heat equation on the whole line:

$$\begin{cases} u_t = ku_{xx}, \\ u(0,t) = u_0(x), \end{cases}$$

it follows that the following criteria must hold for a IBVP to be *well-posed*:

1. Existence: There is at least one solution.
2. Uniqueness: The solution is unique and bounded.
3. Stability: We prove stability by the following method: let

   $$M = \max_{y\in\mathbb{R}} |u_0(y)|, \qquad F = \max_{y\in\mathbb{R}, 0\leq s\leq T} |f(y,s)|,$$

   then, it follows that:

   $$\max_{0\leq t\leq T, x\in\mathbb{R}} |u(x,t)| \leq M + FT.$$

   This implies stability, such that if $u_0 - \tilde{u_0}$ and $f - \tilde{f}$ are small, then $u - \tilde{u}$ is also small, as:

   $$\max_{x\in\mathbb{R},\ 0\leq t\leq T} |u(x,t) - \tilde{u}(x,t)| \leq \max_{x\in\mathbb{R}} |u_0(x) - \tilde{u_0}(x)| + T \max_{x\in\mathbb{R},\ 0\leq t\leq T} |f(x,t) - \tilde{f}(x,t)|.$$

Maximal Principle for Heat Equation: If some $u(x,t)$ satisfies the heat equation in a rectangle, then the maximum value of $u(x,t)$ is assumed either initially or on the lateral sides ($x = 0$ or $x = l$).

Comparison Principle: If, for the domain $U = (a,b) \times (0,T)$, then if

$$u_t \leq ku_{xx} \text{ (subsolution)}, \qquad v_t \geq ku_{xx} \text{ (supersolution)}$$

and $u \leq v$ on $\partial U$, then

$$u(x,t) \leq v(x,t)$$

for all of $U$.