PHYS 301 Notes

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Midterm 1

Calculus Review

Spherical Coordinates:

Sometimes, spherical coordinates are easier to work with than cartesian coordinates. When using spherical coordinates, there are some key things to note:

1. The coordinates themselves::
   ϕ: Equatorial Azimuth $[0, 2\pi]$ (from the "x" axis)
   θ: Axial Azimuth $[0, \pi]$ (from the "z" axis)
   r: Radial Distance
2. The infinitesimal displacement is different:
   $d\vec{l} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi}$
3. The volume element is different:
   $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Cylindrical Coordinates:: See textbook, this would be too redundant.

Rules for Irrotational Fields/Conservative Fields:

1. $\vec{\nabla} \times \vec{F} = \vec{0}$ everywhere.
2. $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.
3. $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
4. $\vec{F}$ is the gradient of some scalar function: $\vec{F} = -\nabla V$.

It is important to note that since $\vec{E} = -\nabla V$, then all electrostatic fields are irrotational.

Rules for Divergence-less fields.:

1. $\vec{\nabla} \cdot \vec{F} = 0$ everywhere.
2. $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line.
3. $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface.
4. $\vec{F}$ is the curl of some vector function.

Taylor Expansions: Often, it is difficult to obtain limits for when one quantity gets significantly larger than another in equations describing fields or potentials. Thus, the Taylor Expansion is helpful for this. Take for example a quantity a and a quantity b. For $b >> a$, it is useful to try to bring the equation into a form such that $\left(\frac{a}{b}\right)^n, n \in \mathbb{N}$. Since $b >> a$, then we can say that $\frac{a}{b} \approx 0$ Thus, a Taylor series centered around $x = 0$ is obtained, which can be evaluated with the expression:

$$f(x) \approx f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2} + \cdots + \frac{f^{(n)} x^n}{n!}$$

Generally, it is best practice to evaluate the Taylor series up until the degree of the largest polynomial in the original function.

Electric Fields

It is important to recognize the notation in the Griffith's textbook, which is used primarily for this course when working with Fields and directions:

$\vec{r}$: distance from the origin to a "field point".
$\vec{r'}$: distance from the origin to the charge
$\vec{r_{\text{sep}}}$: distance from the charge to the "field point"
Thus, it is given that $\vec{r_{\text{sep}}} = \vec{r} - \vec{r'}$.

Now, while working with Coulomb's law, we can define the electric field due to a point charge to be:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{||\vec{r_{\text{sep}}}||^2} \vec{r_{\text{sep}}}$$

Types of Field Integrations: There exist three main types of field integrations:

Line Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, dl'$$

Surface Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, da'$$

Volume Charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||^2} \hat{r}_{\text{sep}}\, d\tau'$$

Gauss's Law: Gauss's law is as follows:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

By knowing the geometry and symmetry of a situation however, Gauss's law can require no integration whatsoever. This is when we know that field is the same for every point on a surface.

Electric Potential

The potential of a point in a field relative to another is defined as:

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{r} \vec{E} \cdot d\vec{l}$$

Which is often referred to as the potential difference between two points:

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

It is also to be noted the relation between field and potential:

$$\vec{E} = -\nabla V$$

Remark: it is important to note that unlike electric field, electric potential is a scalar. It has no direction, and should be handled accordingly.

Potential can also be obtained for a volume charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{||\vec{r_{\text{sep}}}||} d\tau'$$

Similar formulae can be extrapolated for line and surface charge distributions, analogous to those for electric fields.

However, the formula for point charges is important to note:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^{n} \frac{q_i}{||\vec{r_{\text{sep}}}||}$$

Work and Energy in Electrostatics

To calculate the work that it takes to get from one point to another in a field, we can use a work integral. We can also employ the potential difference.

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \vec{F} \cdot d\vec{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \vec{E} \cdot d\vec{l} = Q[V(\vec{b}) - Q(\vec{a})]$$

It is important to note that for the second form of this integral uses **negative** Q, not positive. This is because the integral by default describes the amount of work that the field does, not the work that is required. One can use a positive value of Q if trying to find out how much work the field does.

It is also to be noted that the potential of a system is the work that is required to create the system per unit charge.

Work and Point Charges: We can also describe the amount of work that it takes to assemble a collection of point charges:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r_i})$$

Work of a Continuous Charge Distribution: To describe the amount of energy that a charge distribution has, or the amount of energy required to create it in empty space, the following formula can be used:

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Boundary Conditions

One of the important equations in the relationships between ρ, V, and E is $\nabla^2 V = -\frac{\rho}{\epsilon_0}$. However, this equation often leaves one with a number of undetermined coefficients from the integration. Thus, it is important to investigate the boundary conditions of fields and potentials.

From this, a set of important boundary conditions emerges for a boundary at a.

- Continuity of Potential: Since $\nabla V = -\vec{E}$, it is known that potential should be continuous along all space. Thus,
$$V_{\text{above}}(a) = V_{\text{below}}(a)$$
- Preservation of Field: For any field interacting with a surface charge, by Gauss's law, it is known that the difference between the field above and the field below is simply the surface charge density over the permittivity of free space:
$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}$$
Where $\hat{n}$ is the normal vector of the boundary surface. The presence of the normal vector is important, because it requires the boundary surface to have symmetry with the field above and the field below.
- Bounding of Potential (not general): Since it is known that potential should generally be bounded over all space (except in some cases), it is appropriate to say that
$$V(0) \text{ is bounded.}$$
This is generally more true for configurations with spherical symmetry.
- Zero Potential at Infinity (not general): For many spherically-symmetric examples, it is known that the potential at infinity is zero. Thus, it can be said that:
$$V(\infty) = 0.$$

Uniqueness of Solutions

Given the relationship $\nabla^2 V = -\frac{\rho}{\epsilon_0}$, there exists the case that $\rho = 0$, for which a distinct set of attributes arise:

1. V has no local maxima nor minima inside. The maxima and minima are located on the surrounding boundaries.
2. V is smooth and continuous, everywhere.
3. $V(\vec{r})$ is the average of V over surface of any surrounding sphere: $V(\vec{r}) = \frac{1}{4\pi R^2} \oint V dA$.
4. V is unique, as the solution of the Laplace equation is uniquely determined if V is specified on the boundary surface around the volume.

Earnshaw's Theorem: As a sidenote, is important to note Earnshaw's Theorem, which states that:

> A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.

This can be analyzed using divergence amongst other methods of analysis.

Conductors

At a base level, conductors are materials in which electrons are free to move inside of the material. Because of this, [ideal] conductors are able to rearrange their charges, allowing them to have special properties:

1. $\vec{E} = 0$ inside a conductor. In short, charges move to oppose any external $\vec{E}$ fields. This can also be interpreted under the principle that if there was any field inside a conductor, the free electrons would be moving, and hence not *electrostatic*.
2. Any net charges reside on the surface of a conductor.
3. $\rho = 0$ inside a conductor. Since there is no field inside a conductor, Gauss's law requires that there is no enclosed charge, thus, there is no charge density. One may make the argument for the surface charges, however, since they are equal in magnitude, they cancel.
4. A conductor is an equipotential. Since there is no field inside a conductor, given the relationship $\nabla V = -\vec{E}$, the potential V must be a constant.
5. $\vec{E}$ is normal to the surface of a conductor. This is because any non-normal direction of field would result in fields inside the conductor. (Unsure of this)

Capacitance

As a purely geometrical property, we can define **capacitance** as the ratio of charge to voltage. Thus, it is defined as:

$$C \equiv \frac{Q}{V}.$$

As a practical device, capacitors are capable of storing energy. Thus, it is important to define the amount of energy that a capacitor can store for a given charge Q and capacitance C:

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

Midterm 2

Multipole Expansion

It can be said that every charge distribution has characteristics of various multipoles at large distances. Thus, we can utilize the multipole expansion to determine voltages at large distances.

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_V ||\vec{r'}||P_n(\cos\alpha)\rho(\vec{r'})d\tau'$$

Where α is the angle between the reference and separation vectors.

However, it is to be noted that this formula is often very difficult to use. Thus, we can isolate the first two terms, which are generally of the most use to us.

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} + \cdots$$

The first term in the sequence is the monopole term and the second term in the sequence is the dipole term. Thus, we also define Q as the total charge of the charge distribution under analysis and $\vec{p}$ as the dipole moment of the distribution. The dipole moment is described by:

$$\vec{p} \equiv \int_V \vec{r'}\rho(\vec{r'})d\tau'$$

We can also describe dipole moment for a collection of point charges as well:

$$\vec{p} = \sum_{i=1}^{n} q_i\vec{r'}_i$$

Lastly, we can also define the dipole moment for a dipole that is not centered at the origin.

$$\vec{p'} = \vec{p} - Q\vec{a}$$

for a shift of $\vec{a}$ from the origin.

Polarization and Dielectrics

Polariztion occurs when an electric field passes through a dielectric, which is an insulator. Because the charges in a dielectric are not free to roam like in a conductor, the charges tend to "strech" and "rotate", which produce different behaviours.

In atoms with no permanent dipole, the electrons and nucleus will rearrange such that a dipole is induced into the atom. In molecules with permanent dipoles, however, they experience a torque, which can be described by

$$\vec{N} = \vec{p} \times \vec{E}$$

That being said, there could also be a nonuniform field, and thus a nonuniform force distribution, resulting in a net force, described by:

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}.$$

We can also describe the energy of an ideal dipole in an electric field:

$$U = -\vec{p}\cdot\vec{E}.$$

Bound and Free Charges

As a result of polarization, we can describe the phenomena known as bound charges and free charges.

Bound Charges are charges that are produced when the "stretching" effect of an electric field results in charges moving to the outer surfaces of an object. We can define such surface charges as:

$$\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}.$$

Additionally, we can also define the charge density of the bound charges:

$$\rho_{\text{bound}} \equiv -\nabla \cdot \vec{P}$$

It is important to note that such bound charges create electric fields within the dielectric itself. Thus, the total electric field inside of a dielectric is described by:

$$\vec{E}_d = \vec{E}_{\text{outside}} + \vec{E}_{\text{inside}}$$

Free Charge is the charge on an object that is "externally applied". It is not generated by polarization.

Electric Displacement

Electric displacement is the vector field representing how bound charges separate within a material. It is defined as:

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}.$$

Electric displacement can also be described using Gauss's law, which follows the form of:

$$\nabla \cdot \vec{D} = \rho_f \qquad\qquad \oint \vec{D} \cdot d\vec{A} = Q_{f_{\text{encl}}}$$

These expressions are convenient because free charges are the ones that "we control".

Remark: There is an important condition that comes into play when using Gauss's law with displacement fields. This has to do with the fact that the curl of a displacement field is not always zero. This is because the way in which the field is generated is not in the same irrorational sense as electric fields. Thus, since Gauss's law is application of Divergence Theorem, Gauss's law for electric displacement does not hold when $\nabla \times \vec{D} \neq 0$.

Avoiding Error: Per Griffiths, if the problem has spherical, cyclindrical, or planar symmetry, $\vec{D}$ can be obtained by by the usual Gauss's law methods. Additionally, due to the definition, instead of checking if $\nabla \times \vec{D} \neq 0$, we can also check that:

$$\nabla \times \vec{P} \neq 0.$$

Boundary Conditions: We can also define a new boundary condition for electric displacement questions:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}.$$

Linear Dielectrics

It is known that for some materials, the polarization can be described as proportional to the electric field that the material is experiencing. The relationship, which is described by:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

introduces the constant χ_e, which is the electric susceptibility of the material. Materials that obey this relationship are called linear dielectrics.

This relationship can be also rewritten as:

$$\vec{D} = \epsilon \vec{E}$$

where,

$$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0 \epsilon_r.$$

We also define the quantity ϵ_r, as the relative permittivity or dielectric constant of the material.

Laplace Equations in Cartesian Coordinates

Laplace equations are helpful to solve problems where planar boundary conditions exist. They exploit the fact that the solution follows a separable form. Since we've seen these before [reference MATH 257 notes], we only require some basic review.

1. Solutions will have a hyperbolic trig function or a sinusoidal trig function. the difference between the two depends on boundary conditions.
   (a) Boundaries that start and end at the same value generally will be sinusoidal function solutions.
   (b) Boundaries that start and end at different values generally will beam hyperbolic trigonometric function solutions.
   - $\sinh(0) = 0$
   - $\cosh(0) = 1$
2. To find coefficients in a Fourier series, we use the length of that dimension to integrate over.

Laplace Equations in Spherical Coordinates

Laplace equations in spherical coordinates are helpful for solving problems with complicated external factors and spherical symmetry. They yield solutions that take the form of:

$$f(r, \theta) = R(r)\Theta(\theta).$$

Working through the steps of the laplace equation, we obtain that this form looks like:

$$f(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$$

Where $P_l(\cos\theta)$ are associated Legendre polynomials, which have the form:

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_1(\cos\theta) &= \cos\theta \\ P_2(\cos\theta) &= \frac{3}{2}\cos^2\theta - \frac{1}{2} \\ P_3(\cos\theta) &= \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta \end{aligned}$$

Boundary Conditions: There exist a particular set of boundary conditions that are often helpful when solving problems involving laplace equations in spherical coordinates:

1. Bounding of Potential at Zero: $V(r = 0) =$ finite
2. Bounding of Potential at Infinity: $V(r \to \infty) = V_0$
3. Continuity of Field: (Linear, Internal Dielectrics) $\epsilon_r \frac{\partial V_{\text{in}}}{\partial r}\big|_{r=R} = \frac{\partial V_{\text{out}}}{\partial r}\big|_{r=R}$ or $\frac{\partial V_{\text{above}}}{\partial r}\big|_{r=R} - \frac{\partial V_{\text{below}}}{\partial r}\big|_{r=R} = \frac{\sigma}{\epsilon_0}$
4. Continuity of Potential: $V_{\text{in}} = V_{\text{out}}$

Solving Spherical Laplacian Problems:

1. Employ bounding boundary conditions [(1) and (2)].
2. Match terms with continuity of potential.
3. Match terms with continuity of electric field.

Method of Images

The method of images is used to solve problems involving charge distributions and large, grounded sheets. Generally, a "mirror charge" is required in such cases, although the following formula may be of use:

$$V(r=a,\theta)=\frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{||\vec{r_{1\text{sep}}}||}+\frac{q_2}{||\vec{r_{2\text{sep}}}||}\right)=0$$

Solving the equation of

$$-\frac{q_1}{q_2}=\frac{||\vec{r_{1\text{sep}}}||}{||\vec{r_{2\text{sep}}}||}$$

will allow one to determine the location and charge magnitude of corresponding image charges.

It is also important to note that the following formula may be useful in finding the surface charge of the grounded plane:

$$\sigma=-\epsilon_0\frac{\partial V}{\partial n}$$

k where n is the normal direction of the surface which is being integrated over.