ELEC 221 Notes

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Notation

Continuous Signals	Discrete Signals
$x(t)$	$x(n)$
Continuous-Time Systems	Discrete-Time Systems
$x(t) \to y(t)$	$x[n] \to y[n]$

The arrow for systems means that for a given signal, say $x(t)$ when in putted into a system ($\to$) yields an output signal $y(t)$.

An equals sign ($=$) simply indicates equality between two signals.

System Properties

1. **Memory**
   A system is memoryless if the output at each time depends only on the input at the same time.
   Examples:

Example: Voltage Through a Resistor	$V(t) = RI(t)$
Counter Example: Voltage through a Capacitor	$V(t) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} I(\tau)d\tau$
Counter Example: A Delay System	$y[n] = x[n-1]$

   Generally, a system where the function has some sort of time depenence which is not in the instantaneous current time period has a memory.

2. **Invertibility**
   A system is invertible if distinct inputs lead to distinct outputs.
   An example of a noninvertible system is $y(t) = x^2(t)$, because the square destroys a negative sign that may have existed in the input signal.

3. **Causality**
   A system is causal if the output at *any time* depends only on the input at the present time or in the past. A comparison between causal systems and systems with memories may be able to be drawn, but they are not the same.

   Causal Systems

   - Dependent events can only be in the present or in the past.

   Memory-Bearing Systems

   - Dependent events can be in the present or in the future.

   A capacitor is causal, but a moving average, or time reversal is not.

4. **Stability**
   A system is stable if small changes in input do not cause the output to diverge.
   A stable system can also be described as one where bounded inputs lead to bounded outputs; essentially that the system never reaches an unbounded value.
   A system is also stable if the impulse response is absolutley integrable.

5. **Time Invariance**
   A system is time invariant if time shifts in the input lead to identical time shifts in the output.

   Proving Time Invariance: Shift the input signal by a time period a, and pass it through the system. Shift the un-shifted output signal of the system by the same time period a. If they are the same signal, the system is time invariant.

6. **Linearity** A linear system has the following properties:

Additivity	Homogeneity
$x_1(t) + x_2(t) \to y_1(t) + y_2(t)$	$ax_1(t) \to ay_1(t)$

   These two properties can also be combined:

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

Harmonics

A signal that has **only odd harmonics** requires that:

$$f\left(t + \frac{T}{2}\right) = -f(t)$$

Whereas a signal that has **only even harmonics** requires that:

$$f\left(t + \frac{T}{2}\right) = f(t)$$

Impulse Response

The **Impulse Response** of a system is the signal a system produces after a unit impulse signal is passed through it. The unit impulse response is denoted by:

$$h(n) \qquad\qquad h[n].$$

The impulse response is generally used to determine the ouput of a system by means of the convolution.

Convolution

The **Convolution** operation multiplies the entries of one signal by another in a systematic fashion. It is defined by:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad\qquad x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this context, the convolution operator is being used to determine the output of a signal through a system, given the system's impulse response.

Properties of the Convolution: The convolution is:

- Associative: $x * (y * z) = (x * y) * z$
- Commutative: $x * y = y * x$
- Distributive: $x * (y + z) = x * y + x * z$

Fourier Series (Continuous Time)

Given a correct periodic signal, one can describe it as a sum of complex exponentials, called a **Fourier Series**.

Signals that can be expressed as a fourier series must obey the Dirichlet Conditions:

If over one period, a signal $x(t)$:

1. Is single-valued
2. Is absolutley integrable
3. Has a finite number of maxima and minima
4. Has a finite number of discontinuities

The Dirichlet conditions are sufficient but not necessary, however. This being said, they can tell us that the Fourier series converges to:

- $x(t)$ where it is continuous
- half the value of the jump if it is discontinuous.

The Fourier Series has two parts to it:

Fourier Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

where ω is the fundamental frequency of the function.

Fourier Analysis Equation

$$c_k = \frac{1}{T} \int_T e^{-jk\omega t} x(t) dt$$

where T is the period of the function.

Gibbs Phenomena: If a function represented by a Fourier series has discontinuities, there will be "imperfections" or "ringings" in the Fourier series. This is known as the Gibbs Phenomena, where it is known that there will be "spikes" of about 9% the height of the discontinuity around the bounds of the discontinuity itself.

Operations on Fourier Series

Summation of Signals

Given two signals with the same period:

$$x_1(t) = \sum_{k=-\infty}^{\infty} c_{k(1)} e^{jk\omega t} \qquad x_2(t) = \sum_{k=-\infty}^{\infty} c_{k(2)} e^{jk\omega t}$$

Then we can find their sum as:

$$y(t) = Ax_1(t) + Bx_2(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = Ac_{k(1)} + Bc_{k(2)}$$

Time Shifting

Given a signal, its timeshift, $x(t) \to x(t - t_0)$ can be obtained by:

$$x(t - t_0) = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega t} \qquad c'_k = e^{-jk\omega t_0} \cdot c_k$$

Convolution of Signals Given two signals with the same fundamental frequency*:

$$x_1(t) = \sum_{k=-\infty}^{\infty} c_{k(1)} e^{jk\omega t} \qquad\qquad x_2(t) = \sum_{k=-\infty}^{\infty} c_{k(2)} e^{jk\omega t}$$

Then the convolution of the signals takes the form of:

$$y(t) = x_1(t) * x_2(t) = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega t} \qquad\qquad c'_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Signal Power and Energy

We can define the power and energy of a periodic function over a single period.

Energy of a Signal

$$E = \int_T |x(t)|^2 \, dt$$

Power of a Signal

$$P = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

Power and energy can also be used to show Parseval's Relation.

$$\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Discrete Time Fourier Series

Precursor – Differences between DT and CT periodic signals: There are two main differences between DT and CT periodic signals.

1. Frequency does not increase infinitley with ω.

   For some $x[n]$ with period T:

$$\begin{aligned} x[n] &= e^{\frac{2\pi n}{T} j} \\ &= e^{\frac{2\pi (n+N)}{T} j} \\ &= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi N}{T} j} \\ \text{But if } & N > T, \\ &= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi (T\alpha + N')}{T} j} \\ &= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi (N')}{T} j} e^{2\pi\alpha j} \\ &= e^{\frac{2\pi n}{T} j} e^{2\pi (N')} \end{aligned}$$

   It is important to note that this same result does not hold in CT because the resolution of α is not restricted to $\mathbb{Z}$.

2. $\frac{\omega}{2\pi}$ must be rational for the signal to be periodic.

   Note: If $\frac{\omega}{2\pi}$ is a fraction, the numerator is the period.

Now, we can define the Fourier synthesis and analysis equations for discrete time.

Fourier Synthesis Equation	Fourier Analysis Equation
$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}}$$	$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$
where N is the period or number of samples of the function.	where N is the period or number of samples of the function.

The Frequency Response

The **Frequency Response** or the **System Response** of a system, denoted by $H(j\omega)$ describes how frequencies are attenuated, amplified, and phase shifted in a system.
It is computed by:

$$H(j\omega) = \int_{-\infty}^{\infty} e^{j\omega\tau} h(\tau) d\tau$$

Question: Are there frequency responses in DT?

Using the Frequency Response: To use the frequency response, to compute how a system changes a signal, there are three steps.

1. Compute the frequency response (if it isn't already given).
2. Compute the Fourier coefficients of the signal.
3. Apply the frequency response to each of the Fourier coefficents to obtain the output signal.

This can be generalized to:

$$x(t) \to y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \qquad x[n] \to y[n] = \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}$$

Filters

We can characterize filters by their frequency responses, H.

Filters in Discrete Time: It is important to note that a filter in DT is mirrored across zero. Frequencies increase up until $\omega = \pi$ and then subsequently decrease.

Categories of Filters: Filters can be broken up into two main categories:

1. Infinite Impulse Response (IIR)
   Infinite impulse response filters are those whose impulse response does not become zero over a finite amount of time.
2. Finite Impulse Response (FIR)
   Finite impulse response filters are thos whose impulse response does become zero over a finite amount of time.

The Fourier Transform

The **Fourier Transform**, not to be confused with the *Fourier series* employs the principles of the *Fourier series* to aperiodic signals. Thus, the equations governing it are:

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Transform (Fourier Spectrum)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

A more physical intuition of the Fourier transform is that it takes a signal and decomposes it into a spectrum of its frequencies. This is called the Fourier Spectrum.

Fourier Transform and Impulse Response: The Fourier transform can be used to obtain the frequency response from the impulse response and vice versa. This forwards direction of this was already known however.

Continuous Time

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau$$

Discrete Time

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)d\omega$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{-j\omega n}h[n]$$

Midterm 2

Properties of Fourier Transforms

Multiplication Property: We can shift the frequency spectrum of a signal by using the multiplication property. This comes in the form:

$$x(t) \cdot e^{j\omega_c t} \overset{f}{\leftrightarrow} X(j(\omega - \omega_0))$$

Differentiation: Differentiation also is made easy under Fourier transforms.

$$\frac{dx(t)}{dt} \overset{f}{\leftrightarrow} j\omega X(jw)$$

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{f}{\leftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

We can also chain all of these together with successive multiplications in the frequency domain.

Differential Equations under Fourier Transforms: One can obtain the transfer function of a differential equation under a fourier transform. Given a differential equation of the form:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

We can obtain its transfer function:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

This formula is kind of confusing. It's important to just note it as:

$$H(j\omega) = \frac{\sum x \text{ coeffs} \cdot (j\omega)^k}{\sum y \text{ coeffs} \cdot (j\omega)^k}$$

Difference Equations under Fourier Transforms: We can also apply the same formulae to difference equations:

$$\sum_{k=0}^{N} \alpha_k y[n-k] = \sum_{k=0}^{M} \beta_k x[n-k]$$

Thus, resulting in a similar formula:

$$H(e^{j\omega}) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k e^{-jk\omega}}{\sum_{k=0}^{M} \alpha_k e^{-jk\omega}} = \frac{\sum x \text{ coeffs} \cdot e^{-jk\omega}}{\sum y \text{ coeffs} \cdot e^{-jk\omega}}$$

The Discrete Time Fourier Transform

The Discrete Time Fourier Transform is used for transforming **discrete, aperiodic signals**. It is as follows:

Inverse DTFT (Synthesis)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT (Analysis)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

We also require convergence criteria in the DTFT. It is as follows:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad\qquad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Only one of these criteria must be satisfied.

There exist the same properites for the CT Fourier transform, however, some more properties aries in the DTFT:

Conjugation:

$$\text{Even}(x[n]) \stackrel{f}{\longleftrightarrow} \text{Re}(X(e^{j\omega})) \qquad\qquad \text{Odd}(x[n]) \stackrel{f}{\longleftrightarrow} j \cdot \text{Im}(X(e^{j\omega}))$$

Periodicity:

$$X(e^{j(\omega+2\pi)} = X(e^{j\omega}))$$

Differentiation in Frequency: (Warning: this is very different.)

$$n \cdot x[n] \stackrel{f}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Differencing:

$$x[n] - x[n-1] \stackrel{f}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

Accumulating:

$$\sum_{m=-\infty}^{n} x[m] \stackrel{f}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(w - 2\pi k)$$

Parseval's Theorem: We can also define Parseval's theorem in the DTFT.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} |X(e^{j\omega})|^2 d\omega$$

The Fast Fourier Transform

It's cool. It has $N \log_2 N$ time complexity.

Frequency Responses of LTI Systems

We can look at the frequency responses of LTI systems with a slightly different prespective.

Since a transfer function, $H(j\omega)$ or $H(e^{j\omega})$ has both a magnitude and a phase, we can write a transfer function, or any frequency spectrum as:

$$H(j\omega) = |H(j\omega)|e^{\angle H(j\omega)}$$

Hence, we can write input and ouptut signals as follows:

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$
$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Group Delay: We can also define how much a system delays its input signal through the group delay.

$$\tau(\omega) = -\frac{d}{d\omega}\left(\angle H(j\omega)\right)$$

Step Response

Under an ideal filter, a step function attains some sort of "ringing" in the graph. Thus, we find it important to define the step response for systems.

Computing the step response is as follows:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Bode Plots

Often, we find ourselves in situations where plotting phase and magnitudes of transfer functions is difficult because of multiplication/division. Thus, we can use a Bode Plot. The process for obtaining a Bode Plot is as follows:

Magnitude

1. Take $-20\log_{10}|H(j\omega)|$
2. Identify the point(s), τ_n for which the function has an "inflection point".
3. Take $\omega \gg \tau_n$ and $\omega \ll \tau_n$, and plot the two behaviours.

Phase

1. Plot against $\log_{10}\omega$.

Sampling Theory

Given a signal $x(t)$, we can define an "impulse train" $p(t)$, which is a train of impulses that "sample" our signal at a given interval T. We can also define our sampling frequency to be:

$$\omega_s = \frac{2\pi}{T}.$$

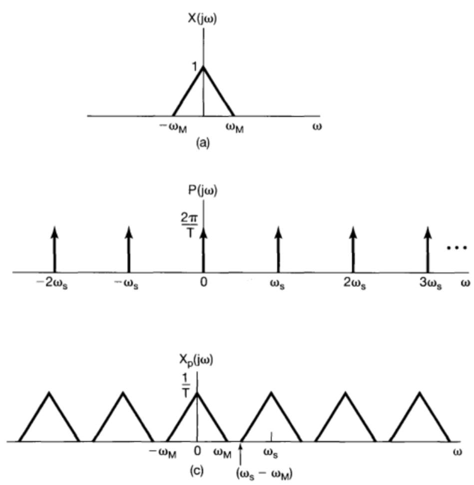
Hence, we can define our sampled signal $x_p(t) = x(t) \cdot p(t)$.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT).$$

Given a sampled signal, it is natural to investigate the frequency spectrum of $x(t)$, $p(t)$, and $x_p(t)$. Hence, we can derive the Fourier transforms for each:

$X(j\omega)$	$P(j\omega)$	$X_p(j\omega)$
$X(j\omega) = X(j\omega)$	$P(j\omega) = \frac{2\pi}{T} \sum_{k=\infty}^{\infty} \delta(\omega - k\omega_s)$	$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

It's also very important to get a grasp on the graphs that result from these functions:



Note that for the graph of $X_p(j\omega)$ has the $\omega_s - \omega_m$, which will be important later. ω_m is defined as the maximum frequency contained in the input signal.

Hence, the question may be posed: How can retrieve the original signal from the sampled signal? The answer to this question is the **Sampling Theorem**:

Sampling Theorem (condensed): Any signal can be reconstructed if the sampling frequency is *strictly* larger than twice the maximum frequency in the sampled signal. In math, this looks like:

$$\omega_s > 2\omega_m$$

Hence, we can also define:

- Nyquist Rate: $\omega_s = 2\omega_M$
- Nyquist Frequency: $\omega_M = \omega_s/2$

Interpolation

Often, we cannot always fully reconstruct our signal from the sampled signal, due to a sampling rate that is too low. Hence, there are some methods in which we can *interpolate* the sampled signal to attempt to reconstruct the original signal. Some of these methods include, but are not limited to:

- Zero Order Hold: Assume each point maintains the same value until the next sample is available.
- Band Limited Interpolation: Apply a lowpass filter to the system, with cutoff ω_c and gain T.

$$x_r(t) = \sum_{n=\infty}^{\infty} x(nT) \frac{\omega_c T \sin(\omega_c(t - nT))}{\pi\omega_c(t - nT)}$$

Aliasing

An important phenomena arises when you don't sample at a high enough rate: **aliasing**.

Essentially, the frequencies in $X_p(j\omega)$ begin to overlap, and the frequency spectrum of the original spectrum becomes distorted.

Conversions Between DT and CT

Many common computing practices see the conversion of a signal from CT to DT and then back to CT in order to employ computational methods of signal processing. For such a process, we define a standard process for such signal processing.

1. Original signal $x_c(t)$ is sampled by an impulse train $p(t)$ to form the sampled signal $x_p(t)$.
2. Sampled signal is converted to a discrete time signal $x_d[n]$.
3. Discrete signal is processed through $H_d(e^{j\omega})$.
4. Processed, discrete signal is transformed back into a CT signal.
5. CT signal is low-passed to recover the original signal.

Hence, when converting to a DT signal, we find ourselves with a new frequency defining our Fourier spectrum, Ω.

Thus, the Fourier spectrum for the transformed signal ends up being defined as:

$$X_d(e^{j\omega}) = X_p(j\Omega/T)$$

Thus, we can also define the new frequency, Ω.

$$\Omega = \omega T.$$

It's also important to define the signal in discrete time:

$$x_c[n] = x_c(nT)$$

Decimation/Downsampling

Downsampling occurs when samples are removed from a DT signal in a process as such:

$$x_p[n] = \begin{cases} x[n] & N \mid \text{n} \\ 0 & \text{otherwise} \end{cases}$$

Where N describes every Nth sample to take.

In the case of downsampling, we find that the frequency spectrum is altered.

$$X_b(e^{j\omega}) = X_p(e^{\frac{j\omega}{N}})$$

Upsampling also exists:

$$X_u(e^{j\omega}) = X_p(e^{j\omega N})$$

Final Exam

Amplitude Modulation

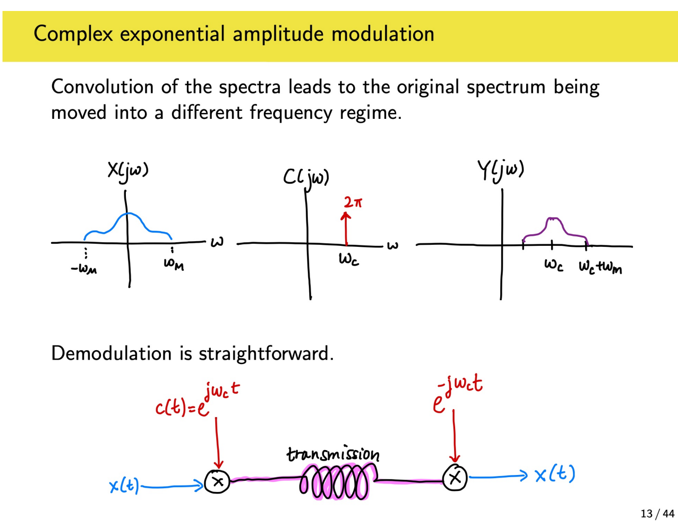
To send and recieve signals, a technique called amplitude modulation is often utilized. The main principle behinde amplitude modulation is that a band-limited frequency can be frequency-shifted up such that multiple different signals can be broadcasted on the same summed signal. Hence, for this technique, there generally is a "carrier signal" which is responsible for the modulation of frequency. Generally, these carrier signals (in this course) come in two forms:

- Complex Exponential Signals: $c(t) = e^{j(\omega_c t + \theta_c)}$
- Sinusoidal Signals: $c(t) = \cos(\omega_c t + \theta_c)$

The original signal is then multiplied with the carrier signal $x(t)c(t)$, since multiplication in the time domain implies convolution in the frequency domain.

Amplitude Modulation using Complex Exponentials: Amplutde modulation using complex exponentials is relativley straightfoward. Since the amplitude is preserved, then the frequency is only shifted up by ω_c. Hence, we can outline a process:

1. Original signal $x(t)$ is multiplied by the carrier signal $c(t) = e^{j\omega_c t}$, $y(t) = x(t)c(t)$. The frequency domain of $y(t)$ is then simply the original signal shifted up by ω_c.
2. Signal is broadcasted.
3. Signal is demodulated using a similar carrier signal $c'(t) = e^{-j\omega_c t}$, $x(t) = c'(t)y(t)$.
4. The original signal is recovered.



Amplitude Modulation using Sinusoids: When using sinusoids, there exist some more uances. Since the frequency spectrum of a sinusoid has both a positive and a negative part, then two signals are outputs of the convolution of the carrier signal and the original signal. It also is more challenging since a the frequency spectrum of a sinusoid is also scaled down by $\frac{1}{2}$. Hence, when demodulating the signal, we require that it is subsequently band-pass filtered and applied a gain of 2.