Math 305 Review Notes

Reese Critchlow

Complex Numbers

At this point in the course, there we define only one representation of the the imaginary unit, i:

$$i^2 = -1.$$

For the remainder of this document, we will represent complex numbers z as:

$$z = x + iy,$$

where i is the imaginary unit.

Hence, with i, we can define some other important properties:

1. $\overline{\overline{z}} = z$
2. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
3. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

<u>Inequalities and Complex Numbers:</u> Building off of the triangle inequality, we can define inequalities for complex numbers:

$\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$	$\|z_1 - z_2\| \geq \|\|z_1\| - \|z_2\|\|$
(triangle inequality)	$\|z_1 + z_2\| \geq \|\|z_1\| - \|z_2\|\|$

As a result, we can bound the modulus of the sum of two complex numbers as:

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

If trying to obtain a bound for the sum of multiple complex numbers, it is important to always obtain the maximal/minimal bounds for each case when aggregating complex numbers.

<u>Representations of Planar Sets in Complex Numbers:</u> Taking the prior definition of $z = x+iy$, and interpreting the x value as an x coordinate, and the same for y, then we can define planar sets in terms of complex numbers. To start off, we first define how to obtain the x and y values of a complex number:

$$\text{Re}(z) = x = \frac{z + \overline{z}}{2} \qquad \text{Im}(z) = x = \frac{z - \overline{z}}{2i}$$

With this definition, we can define some common representations:

1. <u>Circles in $\mathbb{R}^2$:</u>

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2 \iff |z - z_0| = r_0$$

2. <u>Lines in $\mathbb{R}^2$:</u>

$$ax + by = c \iff a\frac{z + \overline{z}}{2} + b\frac{z - \overline{z}}{2i} = c$$

3. <u>Ellipses in $\mathbb{R}^2$:</u>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff |z - F| + |z + F| = 2a$$

Where $F = \sqrt{a^2 - b^2}$.

It is to be noted that this representation only allows for horizontal shifts. One can get vertical shifts by using an alternate form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff |z - Fi| + |z + Fi| = 2b.$$

Shifting can be observed when some representation $|z - F_1| + |z + F_2| = 2a$, where $F_1 \neq F_2$. The shift can be obtained by averaging F_1 and F_2.

A common example of a coordinate transform is to square both sides and convert into a circle. This is because we require r_0 to be squared.

Polar Coordinates and Arguments

One can express a complex number z in polar coordinates using Euler's formula:

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

Hence, it is implied that a representation of a complex number can be as follows:

$$z = re^{i\phi}.$$

Where r is known as the modulus of z and ϕ is known as the modulus of z.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \phi = \arg(z)$$

Arguments: There exist two different types of arguments:

1. Principal Argument: $\text{Arg}(z) = \phi \in (-\pi, \pi]$.
   The principal argument is single-valued and unique.
2. General Argument*: $\arg(z) = \text{Arg}(z) \pm 2\pi k, k \in \mathbb{Z}$.
   The general argument can attain infinite different values.

Finding the Principal Argument: The Principal Argument, as a general form, is given by:

$$\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) + m\pi, m \in \{-1, 0, 1\}.$$

Hence, we define cases for the correct value of m, based on the location of z.

1. Quadrant I: $m = 0$, $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$
2. Quadrant II: $m = 1$, $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) + \pi$
3. Quadrant III: $m = -1$, $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) - \pi$.
4. Quadrant IV: $m = 0$, $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$

It is important to note that the argument of zero is undefined.

Properties of Complex Numbers in Polar Forms:

1. $e^{2k\pi i} = 1$
2. $e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)}$
3. $\overline{e^{i\phi}} = e^{-i\phi}$
4. $|e^{i\phi}| = 1$

Properties of Arguments:

1. $\text{Arg}(z_1 \cdot z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$
2. $\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi, k \in \mathbb{Z}$
3. $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

   It is important to note that (3) is *set equality*, not value equality.

Powers of Complex Numbers

Certain properties arise from taking powers of complex numbers.

De Moirre's Formula:

$$(\cos\phi + i\sin\phi)^N = \cos(N\phi) + i\sin(N\phi)$$

This can often be combined with the binomial theorem to derive trig identities.

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Roots of z: We can define the roots of z, $z^{\frac{1}{n}}$ as:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\left(\frac{\phi_0}{n} + \frac{2k\pi}{n}\right)} \qquad k = 0, 1, \ldots, n-1$$

We also define the principle value of a root to be the one corresponding to $k = 0$.

Raising Complex Numbers as Powers: There are also interesting consequences of raising complex numbers as powers. Take a complex number $z = x + iy$:

$$\begin{aligned} e^z &= e^{x+iy} \\ e^z &= e^x \cdot e^{iy} \\ e^z &= e^x(\cos y + i\sin y). \end{aligned}$$

Thus, we also obtain some properties:

1. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$
2. $e^{z_1-z_2} = \dfrac{e^{z_1}}{e^{z_2}}$
3. $e^{\overline{z}} = \overline{e^z}$
4. $|e^z| = e^x$

Functions of Complex Numbers

We can describe functions of complex numbers as:

$$w = f(z),$$

where $w = u + iv$ and $z = x + iy$. Different functions have different images. To determine the image of a given function, we can use the following approach:

1. Solve for z in terms of w from $f(z) = w$.
2. Substitute each z for the $z(w)$ expression in the set notation.

Different Transforms:

1. $w = Az + B$ transforms circles to circles, lines to lines.
2. $w = \frac{1}{z}$ transforms {circles or lines} to {circles or lines}.
3. Mobius Transform, $w = \frac{az+b}{cz+d}$ transforms {circles or lines} to {circles or lines}.
4. $w = z^n$ Power Transforms generally rotate sets.
5. $w = e^z$ Exponential Transofrms generally turn lines into circles and vice versa.

Derivatives of Complex Functions

Like single-variable calculus, there are some basic definitions that need to be highlighted in complex-variable calculus.

Limits: We can define the limit of a complex function as:

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

However, an additional layer of complexity is introduced here. $(x, y) \to (x_0, y_0)$ implies that any path can be taken to get to (x, y). Thus, strange behaviours occur.

As an aside, we can note that usual properties of limits generally still hold with complex functions:

1. $\lim \dfrac{f(z)}{g(z)} = \dfrac{\lim f(z)}{\lim g(z)}$
2. $\lim(c_1 f + c_2 g) = c_1 \lim f(z) + c_2 \lim g(z)$.

Continuity of Functions

$$f(z) \text{ is continuous at } z_0 \iff \lim_{z\to z_0} f(z) = f(z_0) \iff \left(\lim_{z\to z_0} u(z_0) = u(z_0) \wedge \lim_{z\to z_0} v(z_0) = v(z_0)\right)$$

It is important to note that the choice of path is important here. If two different paths lead to different values in the limit, the limit does not exist.

Differentiability:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \iff \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \iff f(z) \text{ is differentiable.}$$

Where the inclusion of a limit means that the limits must exist.

Cauchy-Riemann Equation: A helpful tool for governing differentiability is the Cauchy-Riemann Equation:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Implications of the CRE:

- ($\implies$): If some $f(z)$ is differentiable at some z_0, then the CRE is satisifed.
- ($\impliedby$): If some function $f(z)$ satisfies the CRE at some z_0, AND the derivative is continuous, then $f(z)$ is differentiable.

There are some additional definitions here too:

1. A function f is analytic at a point z_0 if f is differentiable in a neigbourhood of z_0.
2. A function f is entire if it is analytic everywhere.

Being differentiable at a single point is not analytic.

Taking the derivative of complex functions is the same as in non-complex functions. The product, quotient, and chain rules all hold. Remark:

$$f \text{ is differentiable } \implies \frac{\partial f}{\partial \bar{z}} = 0 \text{ at } z_0$$

Consequences of the Cauchy-Riemann Equation:

1. The CRE guarantees the satisfaction of the laplace equation if a function is entire. Thus, we can say that v is the harmonic conjugate of u, and vice versa.

Functions and Mappings

We can define a variety of different functions which map various domains to various images. Expanding off of what was already stated in the "Functions of Complex Numbers" section, we can state some more functions:

Name	Function	Mapping	Expanded Form
Square	$w = z^2$	Scales modulus and argument	$w = \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$
Exponential	$w = e^z$	Transforms planes/circles into planes/circles	$w = \begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$

Inverse Functions

Function	Inverse
$\sin(z)$	$\sin^{-1} z = -i \cdot \text{Log}(iz + (1-z^2)^{\frac{1}{2}})$ where $(1-z^2)^{\frac{1}{2}} = i\lvert z-1\rvert^{\frac{1}{2}}\lvert z+1\rvert^{\frac{1}{2}} e^{i\left(\frac{\phi_1+\phi_2}{2}\right)}$ and $2\pi < \phi_1 < 4\pi$ $-\pi < \phi_2 < \pi$
$w = e^z$	$z = \text{Log}(w)$
$w = z^\alpha$	$e^{\frac{1}{\alpha}\text{Log} z} = z^{\frac{1}{\alpha}}$