MECH 360 Notes

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Overview

There exist four main ways to generate stresses in solid mechanics:

Normal Stress: Normal stress occurs when a force is applied to the normal face of an object.

$$\sigma = \frac{F}{A_\perp}$$

Shear Stress: Shear stress occurs when a force is applied perpendicular to the normal face of an object.

$$\tau = \frac{V}{A_\parallel}$$

Torsional Stress: Torsional stress occurs when an equitorial moment is applied about an object. It produces a shear stress.

$$\tau = \frac{Tr}{J} = \frac{G\phi}{l}$$

where J is the polar moment of area and r is the distance from the neutral axis.

Bending Stress: Bending stress occurs when an axial moment is applied about an object. It produces a normal stress.

$$\sigma = \frac{My}{I} = \frac{E}{\rho}$$

It is also important to note that $J = I_x + I_y$, so $I = \frac{1}{2}J$.

Bending Moment Diagrams

Bending moment diagrams appear to be the heart and soul of solid mechanics. There exists a standard convention drawing bending moment diagrams:

1. Define a coordinate system starting from one of the ends of the beam. Set counterclockwise rotation as positive, and downwards shear as positive.
2. Make cuts at different points of the beam where different forces behave differently, where the "segment" that you extract has one end at the coordinate system origin, and the other end of the cut.
3. At the non-origin side of the cut, p, draw a counterclockwise moment and a downwards shear let the moment be M and the shear be V.
4. Using the force equilibirum that $\sum M_p = 0$ and $\sum F_y = 0$ for the segment, derive formulae for M and V.
5. Use the formulae for M and V to plot as functions on a graph, and determine the max/min values.
6. Use the fact that $\frac{\partial M}{\partial x} = V$ to check your work.

Beam Deflection

The deflection of a beam can be calculated using the following formula:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

However, it is to be noted that this contains a second derivative, so two boundary conditions are required to solve the problem. Generally, these boundary conditions take the following forms:

- The deflection at a point a is known to be zero $y(a) = 0$.

- The deflection at a point a is known to be maximal $y'(a) = 0$.

Using the boundary conditions, we can solve the differential equation and solve for the overall deflection of the beam.

We can also use superposition to of various deflections to calculate the total deflection.

Buckling

For a column/object that is being loaded in compression, it can experience buckling if the load is large enough. To calculate this, we can follow a similar process as beam deflections, with one key difference: we assume there is a y deflection such that a moment is induced from the load and the y deflection.

1. Define a coordinate system starting from one of the ends of the beam. Set counterclockwise rotation to be positive.

2. Make a cut at some arbitrary point in the beam, where the “segment” starts at the origin, and ends at slicing point. and the other end of the cut.

3. At the slicing point of the cut, p, draw a counterclockwise moment and a downwards shear let the moment be M and the shear be V. Assume that the slicing point has a deflection, y, which produces a moment.

4. Using the force equilibirum that $\sum M_p = 0$ for the segment, derive formulae for $M(y)$.

5. Rearranging the formula from beam deflection: $\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$ we can obtain:

$$M(y) = EIy''.$$

   Thus, we are generally left with an equation of the form:

$$EIy'' + Py = 0$$

6. Given a second order homogenous ODE, one can guess a solution of the form

$$y(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

7. Solving this differential equation, we can employ the boundary conditions that:
   (a) $x = 0 \rightarrow y = 0$
   (b) $x = L \rightarrow y = L$

8. The solution of this ODE will bring the P_{cr}, or the critical load that results in bending.

<u>Off-Center Loads:</u> In the event of an off-center load, where the load P is off center by a distance d, there is a moment $M_0 = Pd$ induced at the top. This moment causes bending stress in the beam as well. This bending stress is important for calculating the normal stress for yield strength in the beam, However <u>P_{cr} stays the same</u>.

Shearing Stress and Shear Flow

When a transverse load is applied to a a beam, there exists a shear stress on each layer along of the beam. This is called the "shear flow" and is denoted by q, whose units are force/area. Shear flow is useful, because it can allow one to find two important quantities:

- Total shearing force on the layer of length l: $F = q \cdot l$
- Shearing stress at any point on the layer of **width** t: $\tau = \frac{q}{t}$

To calculate q, we can use the formula for shear flow.

$$q = \frac{VQ}{I}$$

- I: [Second] Moment of Area with respect to neutral axis
- V: Shearing force at the section of the beam desired
- Q: First moment of area, where $Q = (A)(L)$, where A is the area of a section, and L is its distance from the neutral axis.

From this concept of shear flow, we can also derive more precise values for the maximal shearing stress in various cross sections:

- Rectangular Cross Sections: $\tau_{\max} = \frac{3}{2}\frac{V}{A}$
- Cylindrical Cross Sections: $\tau_{\max} = \frac{4}{3}\frac{V}{A}$
- Thin-Walled Cylindrical Cross Sections: $\tau_{\max} = 2\frac{V}{A}$

Combined Loading

In the case that an object has both normal and shear stresses in multiple dimensions, we can define the stress tensor, which defines all of the stress at a point on an object.

2D Stress Tensor

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

3D Stress Tensor

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

It is important to note that the eigenvalues of the stress tensors correspond to principal stresses of the point of interest. The principal stresses are the maximum normal stresses of the point of interest.

If one orders the stresses from smallest to largest, we can define the maximum shearing stress as:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

Pressure Vessels

$$\sigma_{\text{hoop}} = \frac{Pr}{t} \qquad \sigma_{\text{axial}} = \frac{Pr}{2t}$$

Failiure Criteria

We can also define two types of failiure criteria for a body:

Von Mises: Failiure happens if σ_v reaches S_y:

$$f_s = \frac{S_y}{\sigma_V} \qquad \sigma_V = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}}$$

Tresca: Failiure happens if $\tau_{\max}$ reaches $0.5S_y$:

$$f_s = \frac{0.5S_y}{\tau_{\max}}$$

Final Exam

Castigliano's Theorem

Castigliano's theoerm originates from the existence of strain energy within a system. Since a form of Hooke's law exists for every single type of strain, whether that be axial, shear, bending, or torsional loading. Hence, we can write the energy in a system as $U = \frac{1}{2}kx^2$ where k is generally a modulus of some sort, and x is the strain. Hence, we can accumulate all of this strain energy over a volume usin integration. Hence, we define four different formulae for such energy:

Axial Loading	$U = \int_0^l \frac{P^2}{2AE}dx$
Bending	$U = \int_0^l \frac{M^2}{2EI}dx$
Torsion	$U = \int_0^l \frac{T^2}{2GJ}dx$
Shearing	$U \approx 0$

With these definitions, then we can introduce the most important part of Castigliano's theorem: the notion that the displacement of any point j on a structure, subjected to a load P_j, measured along the line of action of P_j can be expressed as the partial derivative of the strain energy of the strucutre with respect to the load P_j:

$$\delta_J = \frac{\partial U}{\partial P_J}$$

Hence, it often occurs that a load is not actually applied at a point, thus, we often apply an "imaginary" load at a point that agrees with a system, such that we can take the derivative at the end, which will send a lot of things to zero.

It is also important to take the infinitesimal quantities of the strain at each point.

Impact Loading

Given some loading with an initial height h and weight W, we can define the effective load P in terms of an impact factor f_i:

$$P = W\left(1 + \sqrt{1 + \frac{2h}{\delta_{\text{st}}}}\right) \qquad P = Wf_i$$

$$f_i = 1 + \sqrt{1 + \frac{2h}{\delta_{\text{st}}}}$$