ELEC 221 Notes

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Notation

Continuous Signals	Discrete Signals
$x(t)$	$x(n)$
Continuous-Time Systems	Discrete-Time Systems
$x(t) \to y(t)$	$x[n] \to y[n]$

The arrow for systems means that for a given signal, say $x(t)$ when in putted into a system ($\to$) yields an output signal $y(t)$.

An equals sign ($=$) simply indicates equality between two signals.

System Properties

1. **Memory**
   A system is memoryless if the output at each time depends only on the input at the same time.
   Examples:

Example: Voltage Through a Resistor	$V(t) = RI(t)$
Counter Example: Voltage through a Capacitor	$V(t) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} I(\tau)d\tau$
Counter Example: A Delay System	$y[n] = x[n-1]$

   Generally, a system where the function has some sort of time depenence which is not in the instantaneous current time period has a memory.

2. **Invertibility**
   A system is invertible if distinct inputs lead to distinct outputs.
   An example of a noninvertible system is $y(t) = x^2(t)$, because the square destroys a negative sign that may have existed in the input signal.

3. **Causality**
   A system is causal if the output at *any time* depends only on the input at the present time or in the past. A comparison between causal systems and systems with memories may be able to be drawn, but they are not the same.

   Causal Systems

   - Dependent events can only be in the present or in the past.

   Memory-Bearing Systems

   - Dependent events can be in the present or in the future.

   A capacitor is causal, but a moving average, or time reversal is not.

4. **Stability**
   A system is stable if small changes in input do not cause the output to diverge.
   A stable system can also be described as one where bounded inputs lead to bounded outputs; essentially that the system never reaches an unbounded value.
   A system is also stable if the impulse response is absolutley integrable.

5. **Time Invariance**
   A system is time invariant if time shifts in the input lead to identical time shifts in the output.

   Proving Time Invariance: Shift the input signal by a time period a, and pass it through the system. Shift the un-shifted output signal of the system by the same time period a. If they are the same signal, the system is time invariant.

6. **Linearity** A linear system has the following properties:

Additivity	Homogeneity
$x_1(t) + x_2(t) \to y_1(t) + y_2(t)$	$ax_1(t) \to ay_1(t)$

   These two properties can also be combined:

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

Harmonics

A signal that has **only odd harmonics** requires that:

$$f\left(t + \frac{T}{2}\right) = -f(t)$$

Whereas a signal that has **only even harmonics** requires that:

$$f\left(t + \frac{T}{2}\right) = f(t)$$

Impulse Response

The **Impulse Response** of a system is the signal a system produces after a unit impulse signal is passed through it. The unit impulse response is denoted by:

$$h(n) \qquad\qquad h[n].$$

The impulse response is generally used to determine the ouput of a system by means of the convolution.

Convolution

The **Convolution** operation multiplies the entries of one signal by another in a systematic fashion. It is defined by:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad\qquad x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this context, the convolution operator is being used to determine the output of a signal through a system, given the system's impulse response.

Properties of the Convolution: The convolution is:

- Associative: $x * (y * z) = (x * y) * z$
- Commutative: $x * y = y * x$
- Distributive: $x * (y + z) = x * y + x * z$

Fourier Series (Continuous Time)

Given a correct periodic signal, one can describe it as a sum of complex exponentials, called a **Fourier Series**.

Signals that can be expressed as a fourier series must obey the Dirichlet Conditions:

If over one period, a signal $x(t)$:

1. Is single-valued
2. Is absolutley integrable
3. Has a finite number of maxima and minima
4. Has a finite number of discontinuities

The Dirichlet conditions are sufficient but not necessary, however. This being said, they can tell us that the Fourier series converges to:

- $x(t)$ where it is continuous
- half the value of the jump if it is discontinuous.

The Fourier Series has two parts to it:

Fourier Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

where ω is the fundamental frequency of the function.

Fourier Analysis Equation

$$c_k = \frac{1}{T}\int_T e^{-jk\omega t} x(t) dt$$

where T is the period of the function.

Gibbs Phenomena: If a function represented by a Fourier series has discontinuities, there will be "imperfections" or "ringings" in the Fourier series. This is known as the Gibbs Phenomena, where it is known that there will be "spikes" of about 9% the height of the discontinuity around the bounds of the discontinuity itself.

Operations on Fourier Series

Summation of Signals

Given two signals with the same period:

$$x_1(t) = \sum_{k=-\infty}^{\infty} c_{k(1)} e^{jk\omega t} \qquad x_2(t) = \sum_{k=-\infty}^{\infty} c_{k(2)} e^{jk\omega t}$$

Then we can find their sum as:

$$y(t) = Ax_1(t) + Bx_2(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad c_k = Ac_{k(1)} + Bc_{k(2)}$$

Time Shifting

Given a signal, its timeshift, $x(t) \to x(t - t_0)$ can be obtained by:

$$x(t - t_0) = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega t} \qquad c'_k = e^{-jk\omega t_0} \cdot c_k$$

Convolution of Signals Given two signals with the same fundamental frequency*:

$$x_1(t) = \sum_{k=-\infty}^{\infty} c_{k(1)} e^{jk\omega t} \qquad\qquad x_2(t) = \sum_{k=-\infty}^{\infty} c_{k(2)} e^{jk\omega t}$$

Then the convolution of the signals takes the form of:

$$y(t) = x_1(t) * x_2(t) = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega t} \qquad\qquad c'_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Signal Power and Energy

We can define the power and energy of a periodic function over a single period.

Energy of a Signal

$$E = \int_T |x(t)|^2 \, dt$$

Power of a Signal

$$P = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

Power and energy can also be used to show Parseval's Relation.

$$\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Discrete Time Fourier Series

Precursor – Differences between DT and CT periodic signals: There are two main differences between DT and CT periodic signals.

1. Frequency does not increase infinitley with ω.

   For some $x[n]$ with period T:

$$\begin{aligned}
x[n] &= e^{\frac{2\pi n}{T} j} \\
&= e^{\frac{2\pi (n+N)}{T} j} \\
&= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi N}{T} j} \\
&\text{But if } N > T, \\
&= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi (T\alpha + N')}{T} j} \\
&= e^{\frac{2\pi n}{T} j} e^{\frac{2\pi (N')}{T} j} e^{2\pi \alpha j} \\
&= e^{\frac{2\pi n}{T} j} e^{2\pi (N')}
\end{aligned}$$

   It is important to note that this same result does not hold in CT because the resolution of α is not restricted to $\mathbb{Z}$.

2. $\frac{\omega}{2\pi}$ must be rational for the signal to be periodic.

   Note: If $\frac{\omega}{2\pi}$ is a fraction, the numerator is the period.

Now, we can define the Fourier synthesis and analysis equations for discrete time.

Fourier Synthesis Equation

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}}$$

where N is the period or number of samples of the function.

Fourier Analysis Equation

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$

where N is the period or number of samples of the function.

The Frequency Response

The **Frequency Response** or the **System Response** of a system, denoted by $H(j\omega)$ describes how frequencies are attenuated, amplified, and phase shifted in a system.

It is computed by:

$$H(j\omega) = \int_{-\infty}^{\infty} e^{j\omega\tau} h(\tau) d\tau$$

Question: Are there frequency responses in DT?

Using the Frequency Response: To use the frequency response, to compute how a system changes a signal, there are three steps.

1. Compute the frequency response (if it isn't already given).
2. Compute the Fourier coefficients of the signal.
3. Apply the frequency response to each of the Fourier coefficents to obtain the output signal.

This can be generalized to:

$$x(t) \to y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \qquad\qquad x[n] \to y[n] = \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}$$

Filters

We can characterize filters by their frequency responses, H.

Filters in Discrete Time: It is important to note that a filter in DT is mirrored across zero. Frequencies increase up until $\omega = \pi$ and then subsequently decrease.

Categories of Filters: Filters can be broken up into two main categories:

1. Infinite Impulse Response (IIR)
   Infinite impulse response filters are those whose impulse response does not become zero over a finite amount of time.
2. Finite Impulse Response (FIR)
   Finite impulse response filters are thos whose impulse response does become zero over a finite amount of time.

The Fourier Transform

The **Fourier Transform**, not to be confused with the *Fourier series* employs the principles of the *Fourier series* to aperiodic signals. Thus, the equations governing it are:

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Transform (Fourier Spectrum)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

A more physical intuition of the Fourier transform is that it takes a signal and decomposes it into a spectrum of its frequencies. This is called the Fourier Spectrum.

Fourier Transform and Impulse Response: The Fourier transform can be used to obtain the frequency response from the impulse response and vice versa. This forwards direction of this was already known however.

Continuous Time

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau$$

Discrete Time

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)d\omega$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{-j\omega n}h[n]$$

Midterm 2

Properties of Fourier Transforms

Multiplication Property: We can shift the frequency spectrum of a signal by using the multiplication property. This comes in the form:

$$x(t) \cdot e^{j\omega_c t} \overset{f}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Differentiation: Differentiation also is made easy under Fourier transforms.

$$\frac{dx(t)}{dt} \overset{f}{\longleftrightarrow} j\omega X(jw)$$

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{f}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

We can also chain all of these together with successive multiplications in the frequency domain.

Differential Equations under Fourier Transforms: One can obtain the transfer function of a differential equation under a fourier transform. Given a differential equation of the form:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

We can obtain its transfer function:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k}$$

This formula is kind of confusing. It's important to just note it as:

$$H(j\omega) = \frac{\sum x \text{ coeffs} \cdot (j\omega)^k}{\sum y \text{ coeffs} \cdot (j\omega)^k}$$

Difference Equations under Fourier Transforms: We can also apply the same formulae to difference equations:

$$\sum_{k=0}^{N} \alpha_k y[n-k] = \sum_{k=0}^{M} \beta_k x[n-k]$$

Thus, resulting in a similar formula:

$$H(e^{j\omega}) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k e^{-jk\omega}}{\sum_{k=0}^{M} \alpha_k e^{-jk\omega}} = \frac{\sum x \text{ coeffs} \cdot e^{-jk\omega}}{\sum y \text{ coeffs} \cdot e^{-jk\omega}}$$

The Discrete Time Fourier Transform

The Discrete Time Fourier Transform is used for transforming **discrete, aperiodic signals**. It is as follows:

Inverse DTFT (Synthesis)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT (Analysis)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

We also require convergence criteria in the DTFT. It is as follows:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad\qquad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Only one of these criteria must be satisfied.

There exist the same properites for the CT Fourier transform, however, some more properties aries in the DTFT:

Conjugation:

$$\text{Even}(x[n]) \overset{f}{\longleftrightarrow} \text{Re}(X(e^{j\omega})) \qquad\qquad \text{Odd}(x[n]) \overset{f}{\longleftrightarrow} j\cdot\text{Im}(X(e^{j\omega}))$$

Periodicity:

$$X(e^{j(\omega+2\pi)} = X(e^{j\omega}))$$

Differentiation in Frequency: (Warning: this is very different.)

$$n\cdot x[n] \overset{f}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Differencing:

$$x[n]-x[n-1] \overset{f}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$$

Accumulating:

$$\sum_{m=-\infty}^{n} x[m] \overset{f}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)$$

Parseval's Theorem: We can also define Parseval's theorem in the DTFT.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}|X(e^{j\omega})|^2 d\omega$$

The Fast Fourier Transform

It's cool. It has $N\log_2 N$ time complexity.

Frequency Responses of LTI Systems

We can look at the frequency responses of LTI systems with a slightly different prespective.

Since a transfer function, $H(j\omega)$ or $H(e^{j\omega})$ has both a magnitude and a phase, we can write a transfer function, or any frequency spectrum as:

$$H(j\omega) = |H(j\omega)|e^{\angle H(j\omega)}$$

Hence, we can write input and ouptut signals as follows:

$$\begin{aligned} |Y(j\omega)| &= |H(j\omega)||X(j\omega)| \\ \angle Y(j\omega) &= \angle H(j\omega) + \angle X(j\omega) \end{aligned}$$

Group Delay: We can also define how much a system delays its input signal through the group delay.

$$\tau(\omega) = -\frac{d}{d\omega}\left(\angle H(j\omega)\right)$$

Step Response

Under an ideal filter, a step function attains some sort of "ringing" in the graph. Thus, we find it important to define the step response for systems.

Computing the step response is as follows:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Bode Plots

Often, we find ourselves in situations where plotting phase and magnitudes of transfer functions is difficult because of multiplication/division. Thus, we can use a Bode Plot. The process for obtaining a Bode Plot is as follows:

Magnitude

1. Take $-20\log_{10}|H(j\omega)|$
2. Identify the point(s), τ_n for which the function has an "inflection point".
3. Take $\omega \gg \tau_n$ and $\omega \ll \tau_n$, and plot the two behaviours.

Phase

1. Plot against $\log_{10}\omega$.

Sampling Theory

Given a signal $x(t)$, we can define an "impulse train" $p(t)$, which is a train of impulses that "sample" our signal at a given interval T. We can also define our sampling frequency to be:

$$\omega_s = \frac{2\pi}{T}.$$

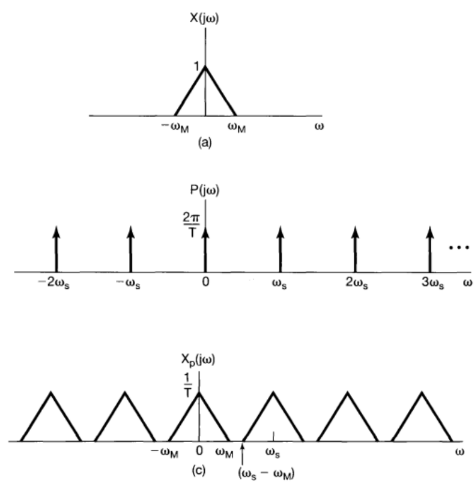
Hence, we can define our sampled signal $x_p(t) = x(t) \cdot p(t)$.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT).$$

Given a sampled signal, it is natural to investigate the frequency spectrum of $x(t)$, $p(t)$, and $x_p(t)$. Hence, we can derive the Fourier transforms for each:

$X(j\omega)$	$P(j\omega)$	$X_p(j\omega)$
$X(j\omega) = X(j\omega)$	$P(j\omega) = \frac{2\pi}{T} \sum_{k=\infty}^{\infty} \delta(\omega - k\omega_s)$	$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

It's also very important to get a grasp on the graphs that result from these functions:



Note that for the graph of $X_p(j\omega)$ has the $\omega_s - \omega_m$, which will be important later. ω_m is defined as the maximum frequency contained in the input signal.

Hence, the question may be posed: How can retrieve the original signal from the sampled signal? The answer to this question is the **Sampling Theorem**:

Sampling Theorem (condensed): Any signal can be reconstructed if the sampling frequency is *strictly* larger than twice the maximum frequency in the sampled signal. In math, this looks like:

$$\omega_s > 2\omega_m$$

Hence, we can also define:

- Nyquist Rate: $\omega_s = 2\omega_M$
- Nyquist Frequency: $\omega_M = \omega_s/2$

Interpolation

Often, we cannot always fully reconstruct our signal from the sampled signal, due to a sampling rate that is too low. Hence, there are some methods in which we can *interpolate* the sampled signal to attempt to reconstruct the original signal. Some of these methods include, but are not limited to:

- Zero Order Hold: Assume each point maintains the same value until the next sample is available.
- Band Limited Interpolation: Apply a lowpass filter to the system, with cutoff ω_c and gain T.

$$x_r(t) = \sum_{n=\infty}^{\infty} x(nT)\frac{\omega_c T \sin(\omega_c(t-nT))}{\pi\omega_c(t-nT)}$$

Aliasing

An important phenomena arises when you don't sample at a high enough rate: **aliasing**.

Essentially, the frequencies in $X_p(j\omega)$ begin to overlap, and the frequency spectrum of the original spectrum becomes distorted.

Conversions Between DT and CT

Many common computing practices see the conversion of a signal from CT to DT and then back to CT in order to employ computational methods of signal processing. For such a process, we define a standard process for such signal processing.

1. Original signal $x_c(t)$ is sampled by an impulse train $p(t)$ to form the sampled signal $x_p(t)$.
2. Sampled signal is converted to a discrete time signal $x_d[n]$.
3. Discrete signal is processed through $H_d(e^{j\omega})$.
4. Processed, discrete signal is transformed back into a CT signal.
5. CT signal is low-passed to recover the original signal.

Hence, when converting to a DT signal, we find ourselves with a new frequency defining our Fourier spectrum, Ω.

Thus, the Fourier spectrum for the transformed signal ends up being defined as:

$$X_d(e^{j\omega}) = X_p(j\Omega/T)$$

Thus, we can also define the new frequency, Ω.

$$\Omega = \omega T.$$

It's also important to define the signal in discrete time:

$$x_c[n] = x_c(nT)$$

Decimation/Downsampling

Downsampling occurs when samples are removed from a DT signal in a process as such:

$$x_p[n] = \begin{cases} x[n] & N \mid \text{n} \\ 0 & \text{otherwise} \end{cases}$$

Where N describes every Nth sample to take.

In the case of downsampling, we find that the frequency spectrum is altered.

$$X_b(e^{j\omega}) = X_p(e^{\frac{j\omega}{N}})$$

Upsampling also exists:

$$X_u(e^{j\omega}) = X_p(e^{j\omega N})$$

Final Exam

Amplitude Modulation

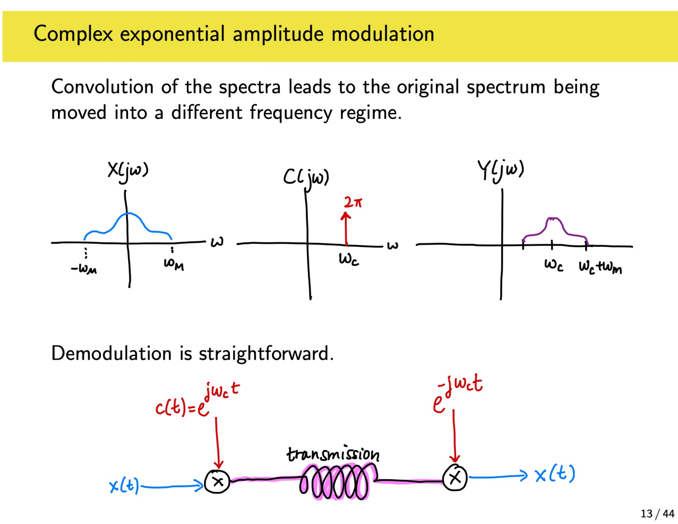
To send and recieve signals, a technique called amplitude modulation is often utilized. The main principle behinde amplitude modulation is that a band-limited frequency can be frequency-shifted up such that multiple different signals can be broadcasted on the same summed signal. Hence, for this technique, there generally is a "carrier signal" which is responsible for the modulation of frequency. Generally, these carrier signals (in this course) come in two forms:

- Complex Exponential Signals: $c(t) = e^{j(\omega_c t + \theta_c)}$
- Sinusoidal Signals: $c(t) = \cos(\omega_c t + \theta_c)$

The original signal is then multiplied with the carrier signal $x(t)c(t)$, since multiplication in the time domain implies convolution in the frequency domain.

Amplitude Modulation using Complex Exponentials: Amplutde modulation using complex exponentials is relativley straightfoward. Since the amplitude is preserved, then the frequency is only shifted up by ω_c. Hence, we can outline a process:

1. Original signal $x(t)$ is multiplied by the carrier signal $c(t) = e^{j\omega_c t}$, $y(t) = x(t)c(t)$. The frequency domain of $y(t)$ is then simply the original signal shifted up by ω_c.
2. Signal is broadcasted.
3. Signal is demodulated using a similar carrier signal $c'(t) = e^{-j\omega_c t}$, $x(t) = c'(t)y(t)$.
4. The original signal is recovered.



Amplitude Modulation using Sinusoids: When using sinusoids, there exist some more uances. Since the frequency spectrum of a sinusoid has both a positive and a negative part, then two signals are outputs of the convolution of the carrier signal and the original signal. It also is more challenging since a the frequency spectrum of a sinusoid is also scaled down by $\frac{1}{2}$. Hence, when demodulating the signal, we require that it is subsequently band-pass filtered and applied a gain of 2.

1. Original signal $x(t)$ is multiplied by the carrier signal $c(t) = \cos(\omega_c t)$ $y(t) = x(t)c(t)$.
2. Frequency spectrum of the broadcast $y(t)$ has two parts, one centered at ω_c and $-\omega_c$. Both parts have amplitude of $A/2$, where A was the original amplitude of $x(t)$.
3. Modulated signal is broadcasted/transmitted.
4. Modulated signal is demodulated with a signal $c'(t) = \cos(\omega_c t)$, which produces a signal with frequency bands centered at $-2\omega_c$, 0, and $2\omega_c$. Bands centered at $\pm 2\omega_c$ have amplitude $A/4$, whereas the frequency band centered at 0 has amplitude $A/2$.
5. To elimate the unwanted bands and return the signal to the correct amplitude, the frequencies are low passed with a cutoff frequency of ω_C, and applied a gain of 2.

For calculations involving sinusoidal amplitude modulations, sine and cosine functions are often multiplied together. Hence, it is important to remember some key trigonemetric identities:

$$\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta) \qquad \cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$$

Asynchronous Demodulation

For all of the instances above, we assumed that there was no phase shift in the signal $\theta_c = 0$. Hence, since phase cannot always be synchronized, we must consider the case in which there exists some phase shift. An analysis of asynchronous demodulation is as follows, for a carrier signal $c(t)$ with phase shift θ_c, and a demudlating signal $c'(t)$ with a phase shift of ϕ_c.

$$\begin{aligned} w(t) &= \cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c)x(t) \\ &= \left[\frac{1}{2}\cos(\theta_c - \phi_c) + \frac{1}{2}\cos\left(2\omega_c t + \omega_c + \phi_c\right)\right]x(t) \\ &\text{(after application of the low pass filter)} \\ x_r(t) &= \frac{1}{2}\cos(\theta_c - \phi_c)x(t) \end{aligned}$$

Since this yields undesireable effects throughout a system to deal with inconsistent gains, which could decrease signal fidelity, we introduce the idea of amplitude modulation, which involves sending a copy of the carrier signal alongside the original signal. Given some assumptions:

- $x(t)$ is always positive
- ω_c is much larger than ω_m,

an "envelope detector" can be constructed, which has the effect of detecting the carrier signal sent along with the modulated signal, so that the modulated signal can be effectively demodulated using the original carrier signal. This synchronizes the modulation and demodulation processes so no signal is lost.

Modulation Index: For a transmission signal of the form $(A + x(t))\cos(\omega_c t)$. Then the *modulation index* is defined as:

$$m = \frac{K}{A}$$

Where K is the maximal amplitude of $x(t)$ such that $|x(t)| \leq K$.

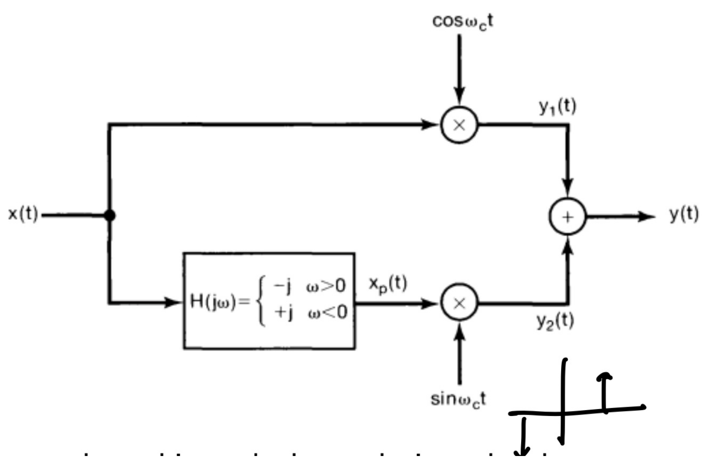
Frequency Division Multiplexing

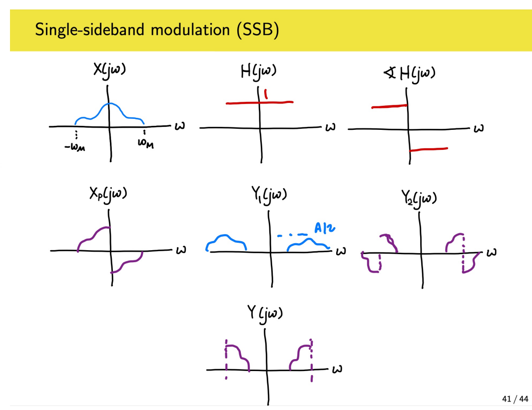
A common implementation of modulation is the concept of frequency division multiplexing. Frequency division multiplexing is where band limited signals are modulated to different frequencies, such that they can all occupy different, distinct, and non-overlapping regions of a frequency spectrum of a single signal. Hence, thre needs to be some way of extracting certain signals out of the composite signal. For this there are generally two phases:

1. A bandpass filter before demodulation.
2. A low pass filter after demodulation.

Single-Sideband Modulation

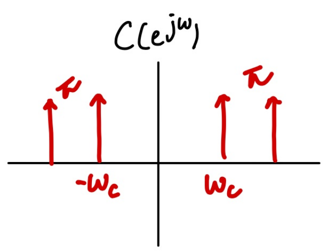
When sending multiple signals on one signal, modulation by a sinusoid results in the transmission of both positive and negative frequencies, and hence a waste in energy required to send both frequencies. Hence, we introduce the concept of single-sideband modulation. In single sideband modulation, we eliminate half of the frequencies of a given signal, such that only one half of the band is actually transmitted. The mechanism for how this happens can be observed:





Discrete Time Sinusoidal Amplitude Modulation

There exist some important differences when modulating in discrete time. The first is that due to everything being in discrete time, one must be cognisant of frequency replication and aliasing. It is also important to note what a cosine wave looks like in the frequency domain:



Thus, we are required to impose some conditions on ω_c and ω_n:

- $\omega_c > \omega_n$
- $\omega_c < \pi - \omega_n$

without these conditions, the signal will suffer from aliasing effects.

The Laplace Transform

As we saw with the Fourier transform, there were places in which the Fourier transform was not always possible since the integral defining the Fourier transform diverged. Hence, we can introduce the Laplace Transform, which outlines a more general case of the Fourier transform, adding a real constant instead of the purely imaginary $j\omega$ which defined the Fourier transform. With this additional real term σ, more signals/functions can be transformable, however, there are additional conditions which arise, which will be explored while discussing regions of convergence. Hence, we can define the Laplace transform as:

$$X(s) = \mathscr{L}\{x(t)\} \equiv \int_{-\infty}^{\infty} e^{-st}x(t)dt$$

Where $s = \sigma + j\omega$.

Since the Fourier and Laplace transforms differ by only the σ term in the complex exponential, we can take out the complex exponential, and write the Laplace transform in terms of the the Fourier transform.

$$\mathscr{L}\{x(t)\} = \mathcal{F}\left\{e^{-\sigma t}x(t)\right\}$$

Regions of Convergence: Due to the "adjustment factor" that σ is, it allows the Laplace transform to exist in some regions, where a Fourier transform does not. Hence, we introduce the concept of a region of convergence of a function, that describes where a function's Laplace transform may exist. However, it is important to note that functions may have the same algebraeic Laplace transform, but differing regions of convergence.

It is also important to note that if a function is a sum of functions, then the region of convergence is the region of convergence that allows all of the functions' Laplace transforms to exist.

Pole-Zero Plots: Oftentimes, the Laplace transform results in rational polynomials of s. Generally, under factorization, the roorts of these polynomials are shown on a pole-zero plot. In such plots:

- Numerator factors are called **zeroes** and symbolized as $\circ$ on plots.
- Denominator factors are called **poles** and symbolized as $\times$ on plots.

The region of convergence also has some nice properties:

- If the region of convergence does not contain the $j\omega$ axis, then the Fourier tranform does not converge.
- The region of convergence of a rational Laplace transform contains no poles.

We can also extrapolate signal attributes from various patterns on a region of convergence plot (pole-zero plot):

- Finite Length Signals: The reigon of convergence plot spans the entire real axis.
- Left Sided Signals: For signals that only exist on the left side of some boundary (they are infinite in the negative direction), then the region of convergence exists only to the left of some value $s < \sigma_L$.
- Right Sided Signals: For signals that only exist on the right side of some boundary (they are infinite in the positive direction), then the region of convergence exists only to the right of some value $s > \sigma_R$.
- Infinite Signals: For signals with no bounds, there *may* exist a region of convergence between some $\sigma_L < s < \sigma_R$.

Since these are all biconditionals, then the converse is also true.

Since a region of convergence can never contain a pole, then for each region between poles, there is a possible region of convergence for a given signal, and thus a different time domain signal that could represent a Laplace transform.

Properties of the Laplace Transform:

Property	Unmodified Transform	Modified Transform	ROC
Linearity	$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$	$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s)$	contains $R_1 \cap R_2$
Time Shifting	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_o}X(s)$	R
s Shifting	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$e^{s_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0)$	$R + \text{Re}(s_0)$
Time Scaling	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	aR
Time Reversal	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s)$	$-R$
Conjugation	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$	R
Convolution	$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$	$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s)$	contains $R_1 \cap R_2$
Time Differentiation	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$	contains R
s Differentiation	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$	$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dx(s)}{ds}$	R

Causality: For a signal to be causal, the transfer function $H(s)$ of its impulse resoponse $h(t)$ has to be right sided. That is, the ROC is the right-half plane to the right of the right-most pole.

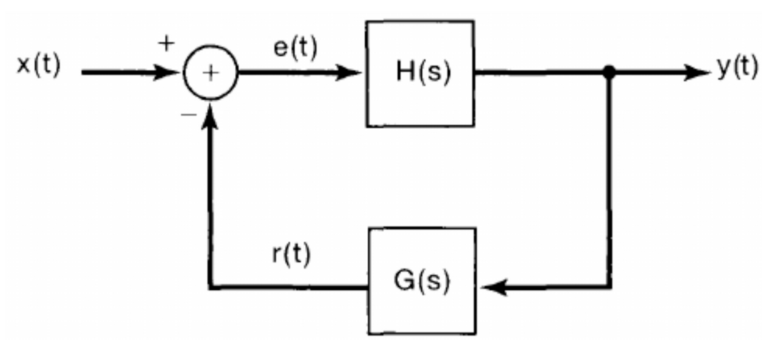
Stability: For a signal to be stable, the transfer function $H(s)$ must contain the entire $j\omega$ axis $\text{Re}(s) = 0$, **and** there are not more zeros than poles.

Differential Equations: The same principle for differential equations holds in the Laplace domain as the Fourier domain:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_k x \text{ coeffs} \cdot s^k}{\sum_k y \text{ coeffs} \cdot s^k}$$

Applications of Laplace Transforms

An important application of Laplace tranforms are in the implmentation of Feedback systems. A feedback system takes the following form:

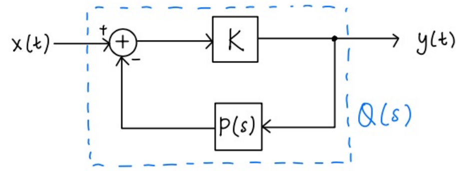


Hence, through some algebra, we cand define the transfer function for the overall system $Q(s)$ to be:

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

Given the form of this transfer function $Q(s)$, we can create systems that solve important problems:

Inverse Systems: An inverse system has the ability to reverse a system. They are generally laid out in the following form:



Where K denotes a gain of K. Hence, under analysis of the transfer function $Q(s)$, we can deduce that:

$$\lim_{K \gg P(s)} Q(s) = \lim_{K \gg P(s)} \frac{K}{1 + KP(s)} \approx \frac{1}{P(s)}.$$

Hence, this has the effect of "inverting the signal".

Stabilizing Systems: The other system that is useful is a stabilizing system. Given some signal with a pole $H(s) = \frac{b}{s-a}$, then it follows that:

$$\begin{aligned} Q(s) &= \frac{H(s)}{1 + KH(s)} \\ &= \frac{b}{(s-a)(1 + K\frac{b}{s-a})} \\ &= \frac{b}{s - a + Kb} \\ Q(s) &= \frac{b}{s - (a - Kb)}. \end{aligned}$$

Hence, if we make $kB > a$, we can move the pole to be on the left side of the $j\omega$ axis, such that the system is stable.

Z-Transforms

Like much of the techniques in this course, there is a discrete time analouge for the Laplace transform, called the Z-transform. We define the Z transform as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

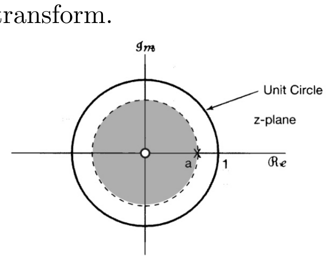
Where $z = re^{j\omega}$.

This can also be put in terms of a discrete-time Fourier transform:

$$X(re^{j\omega}) = X(z) = \mathcal{F}\left\{x[n]r^{-n}\right\}.$$

Regions of Convergence

However, it is to be said that regions of convergence look different in Z-transforms as opposed to Laplace transforms. Since z is in the form of a complex exponential, the regions of convergence find themselves on a circular domain. Hence, the unit circle is the z axis, which is analogous to the $j\omega$ axis in the Laplace transform.

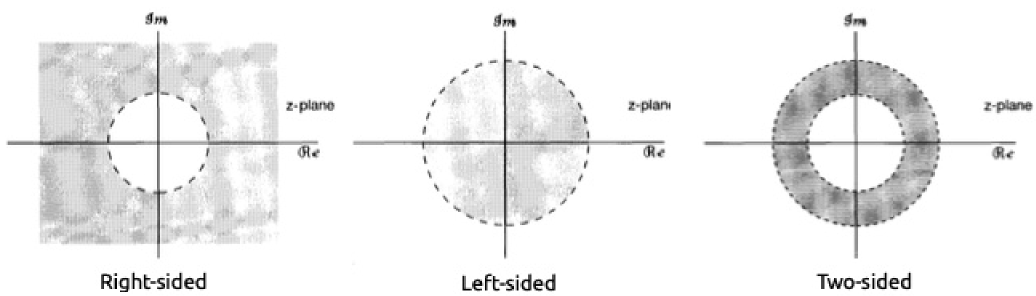


Hence, we can define some properties for the regions of convergence of the Z-transform:

- The ROC must contain the unit circle for the DTFT to converge.
- All ROCs do not contain poles.
- All ROCs are rings in the z-plane centered around the origin.

Patterns of convergence are also similar for the Z-Transform:

- If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and $z = \infty$.
- If $x[n]$ is a right-sided sequence, and the circle $|z| = r_0$ is in the ROC then all finite values of z for which $|z| > r_0$ will also be in the ROC.
- if $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.
- if $x[n]$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.



Inverse Z-Transforms: Generally, these are done with a Z-Transform table, but power series expansions are often a helpful tool, such as for log functions.

Properties of the Z-Transform:

Property	Form	ROC
Linearity	$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$	contains $R_1 \cap R_2$
Time Shift	$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} x^{-n_0}X(z)$	R, may add/delete 0 or ∞
Time Reversal	$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right)$	$\frac{1}{R}$
Time Expansion (insertion of $k - 1$ zeroes)	$X_{(k)}[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^k)$	$R^{\frac{1}{k}}$
z Scaling	$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right)$	$\|z_0\| \cdot R$
Conjugation	$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$	R

Z-Transform and Causality: A Discrete Time LTI system with a rational Z-Transform is causal if:

- The ROC is the exterior of circle outside the outermost pole.
- The order of the numerator in the transfer function $H(z)$ does not exceed the denominator.

Z-Transform and Stability: A Discrete Time LTI system is stable if:

- The ROC includes the unit circle $|z| = 1$.

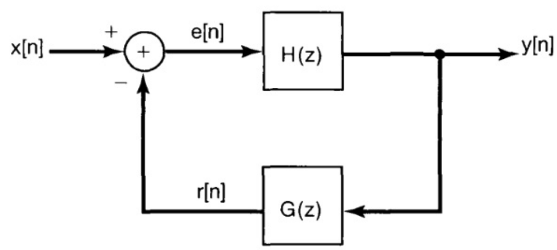
Difference Equations: Similar to everything else:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_k x \text{ coeffs} \cdot z^{-k}}{\sum_k y \text{ coeffs} \cdot z^{-k}}$$

Remeber: the sign of k is important.

Feedback Systems with Z-Tranforms

Very similarly to Laplace transforms, we can have feedback systems in Z-Transforms. They generally take the following form:



Hence, the transfer function takes the same form:

$$Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + G(z)H(z)}.$$