Series and Series Tests

Reese Critchlow, 2021

Geometric Series

Infinite Sum

General Form	Value if $\|r\| < 1$	Value if $r = 1$	Value if $\|r\| > 1$
$S = \sum_{n=1}^{\infty} ar^{n-1}$	$S = \frac{a}{1-r}$	$S = \text{Divergent}$	$S = \text{Divergent}$

Partial Sum

General Form	Value if $\|r\| \neq 1$	Value if $r = 1$
$S_N = \sum_{n=1}^{N} ar^{n-1}$	$S_N = a\frac{1-r^{N+1}}{1-r}$	$S_N = a(N+1)$

Telescoping Series

Infinite Sum

General Form	Value
$S = \sum_{n=1}^{\infty} a_n - a_{n+1}$	$S = a_1 - \lim_{n\to\infty} a_n$

Partial Sum

General Form	Value
$S_N = \sum_{n=1}^{N} a_n - a_{n+1}$	$S = a_1 - a_{N+1}$

Divergence Test – Best for when the n^th term in the series *fails* to converge to zero towards infinity

Theorem

if $\lim_{n\to\infty} a_n \neq 0$

The series diverges.

if $\lim_{n\to\infty} a_n = 0$

The test is completely fucking useless.

Conditions

No Conditions

The Integral Test – Best for when n can be easily substituted for x and integrated

Theorem

if $\int_{N_0}^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

- and -

if $\int_{N_0}^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Conditions

1. $f(x) \geq 0$ for all $x \geq N_0$
2. $f(x)$ is a decreasing function
3. $f(n) = a_n$ for all $n \geq N_0$

$$\int_{N}^{\infty} f(x)dx \leq \sum_{n=N}^{\infty} \leq a_N + \int_{N}^{\infty} f(x)dx \text{ or } \int_{N-1}^{\infty} f(x)dx$$

The P Test

Theorem

Given $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$

if $p > 1$, then S converges.
if not, S is divergent.

Conditions

No Conditions

The Comparison Test – Best when a_n can be easily simplified to a term, b_n at n very large

Theorem

Given $S = \sum_{n=1}^{\infty} a_n$

and some $S_c = \sum_{n=1}^{\infty} c_n$

1. if $|a_n| < c_n$ for all $n > N$, and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
2. if $a_n > c_n$ for all $n > N$, and $\sum_{n=1}^{\infty} c_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Note: it is always a good choice to compare with a p-test-able sum.

The Limit Comparison Test – Same applications as the *Comparison Test*, but with a more definitive result and more general use.

Theorem

Given

$$S_a = \sum_{n=1}^{\infty} a_n \qquad S_b = \sum_{n=1}^{\infty} b_n$$

and the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \neq \text{DNE}$$

1. if S_b converges, then S_a converges
2. if $\underline{L \neq 0}$ and S_b diverges, then S_a diverges.

   Note: The Condition $L \neq 0$ is crucial!
3. When $L = 0$, no information can be provided on divergence.

Alternating Series Test

Theorem

The sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

Converges if

1. $b_n \geq b_{n+1}$ for $n \geq N$
2. $\lim_{n\to\infty} b_n = 0$

Remainders

Partial Sum Error (Alternating)

$$|R_n| \leq a_{N+1}$$

Partial Sum Error – Integral Test

$$\int_{N+1}^{\infty} f(x)dx \leq S_{\infty} - S_N \leq \int_{N}^{\infty} f(x)dx$$

Conditional and Absolute Convergence

Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$, if $\sum_{n=1}^{\infty} |a_n|$ converges (the absolute value of that series), then $\sum_{n=1}^{\infty} a_n$ converges.

Conditional Convergence

If a series $\sum_{n=1}^{\infty} a_n$ converges, but its absolute value, $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series is conditionally convergent.

The Ratio Test

Given

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

Where $a_n \neq 0$

Conclusions

1. If $L < 1$, then the sum is converges (absolutely).
2. If $L > 1$ or DNE, the sum diverges.
3. IF $L = 1$, use a different test.

Power Series

Exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Geometric

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Geometric – Derivative

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

Geometric – Integral

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Geometric – Negative Integral

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Sine

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$$

Cosine

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Arctangent

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Taylor Series

General Formula

$$T_N(x) = \sum_{n=1}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Error Formula

$$R_N(x) \leq \frac{f^{(N+1)}(\bar{c})}{(N+1)!}(x-c)^{N+1}$$

Remark

If $R_N(x)$ reaches zero for some N, the series is said to be convergent. This can also be proven by using the ratio test on the series. When a Taylor Series converges, the approximation equals the function when $N \to \infty$.

Binomial Expansions

General Form

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Where:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$