

CSCI 132:

Basic Data Structures and Algorithms

Sorting (Merge Sort)

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Fall 2023

Announcements

Program 4 Due **Friday @ 11:59 PM**

Lab 11 due **tomorrow @ 11:59 PM**

Optional help session on Friday (no lecture)



Merge Sort is a sorting algorithm that works by dividing an array into smaller subarrays, sorting each subarray, and then merging the subarrays back together to form the final sorted array

Merge sort is a **Divide and Conquer** algorithm, which involves dividing the problem into smaller sub-problems (divide), recursively solving the smaller problems (conquer), and combining the sub problems to get the final solution for the original problem

Merge sort and the next sorting algorithm we will discuss next week are rather complex. I don't expect you to memorize the code, and if you don't fully understand the code, *that is fine!*

You should, however, be able to describe how merge sort works from a high level, and be able to draw out the steps if given an example array

You should also know the time complexity of the sorting algorithms that we talk about



38	27	43	3	9	82	10
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Goal: Divide array into two subarrays using recursion:

Base Case:

If an array is of size 1, **return**

Recursive Case:

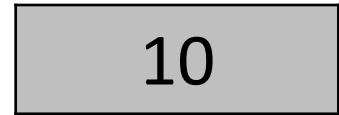
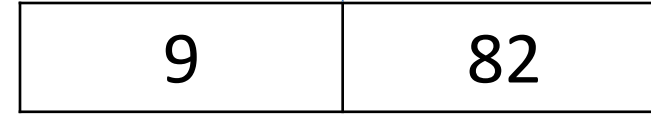
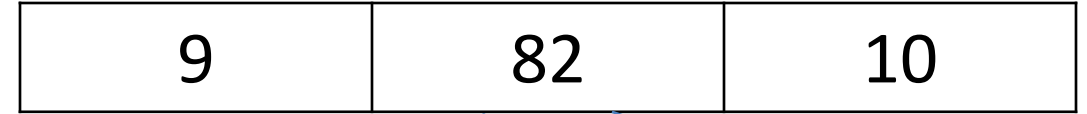
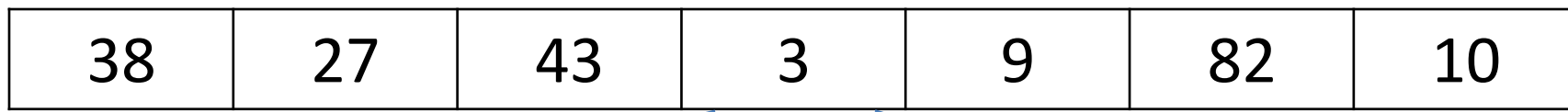
Generate two subarrays `leftArray`, and `rightArray`
`mergeSort(leftArray)`, `mergeSort(rightArray)`

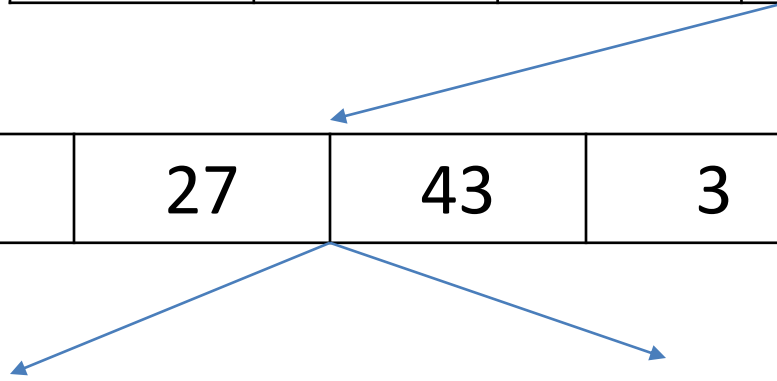
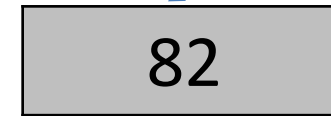
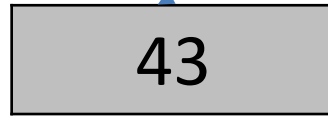
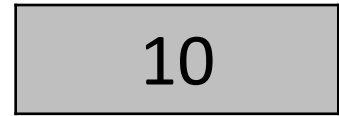
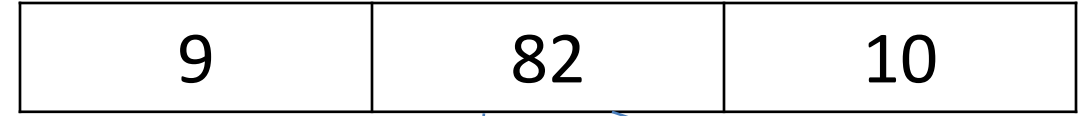
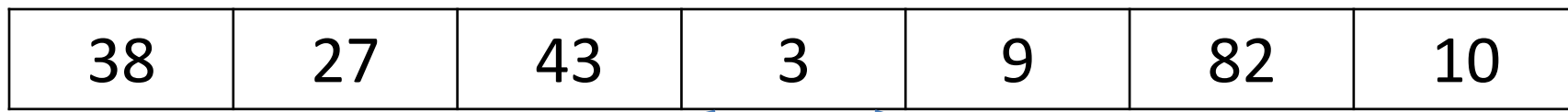
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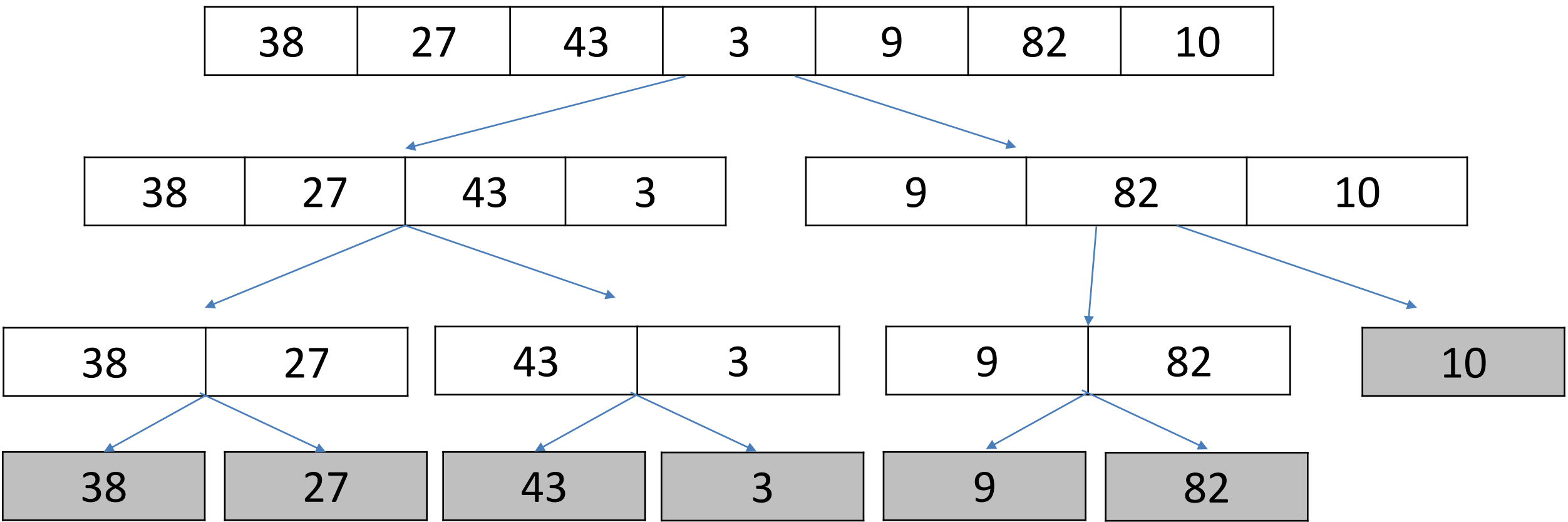


38	27	43	3
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9	82	10
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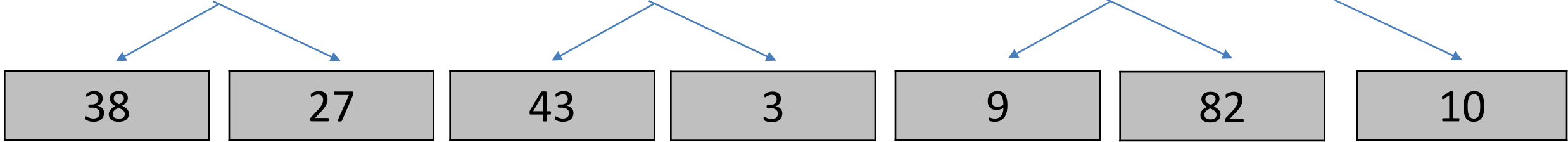






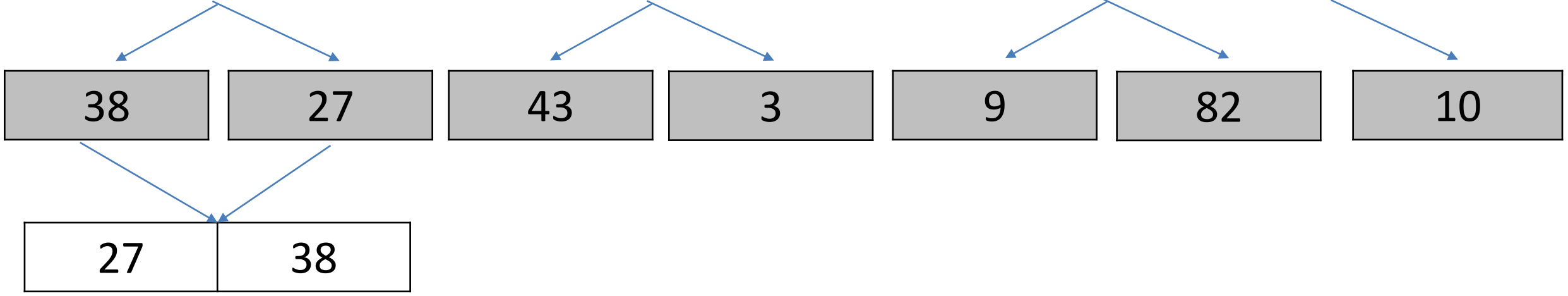
We've hit all our base cases (arrays of size 1), now we will begin to **merge** the subarrays

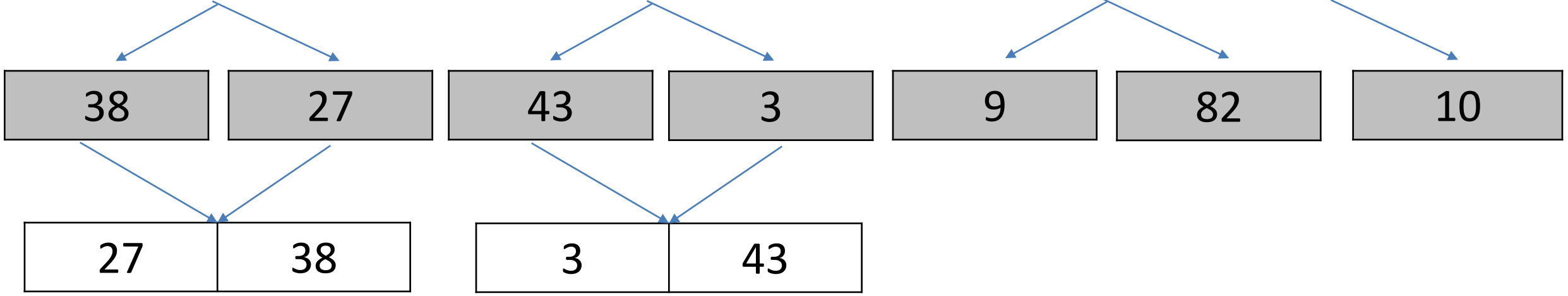
When we merge, our merged array will be sorted

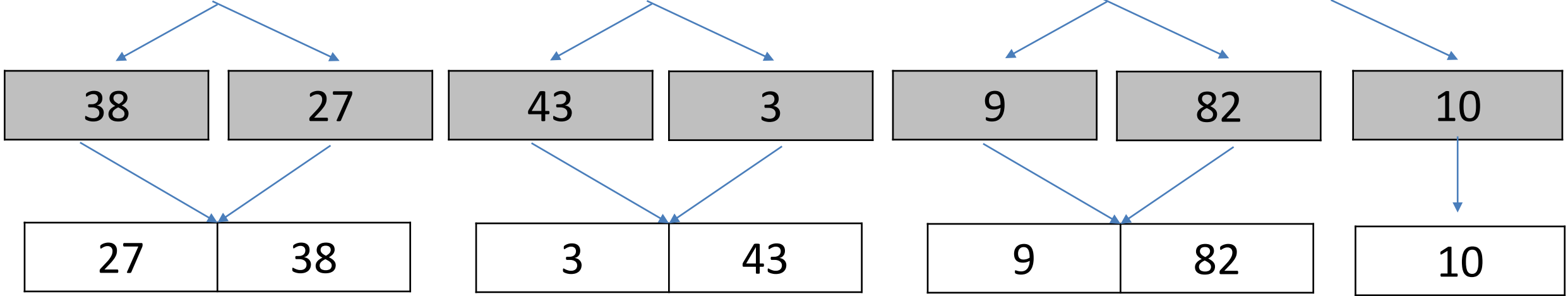


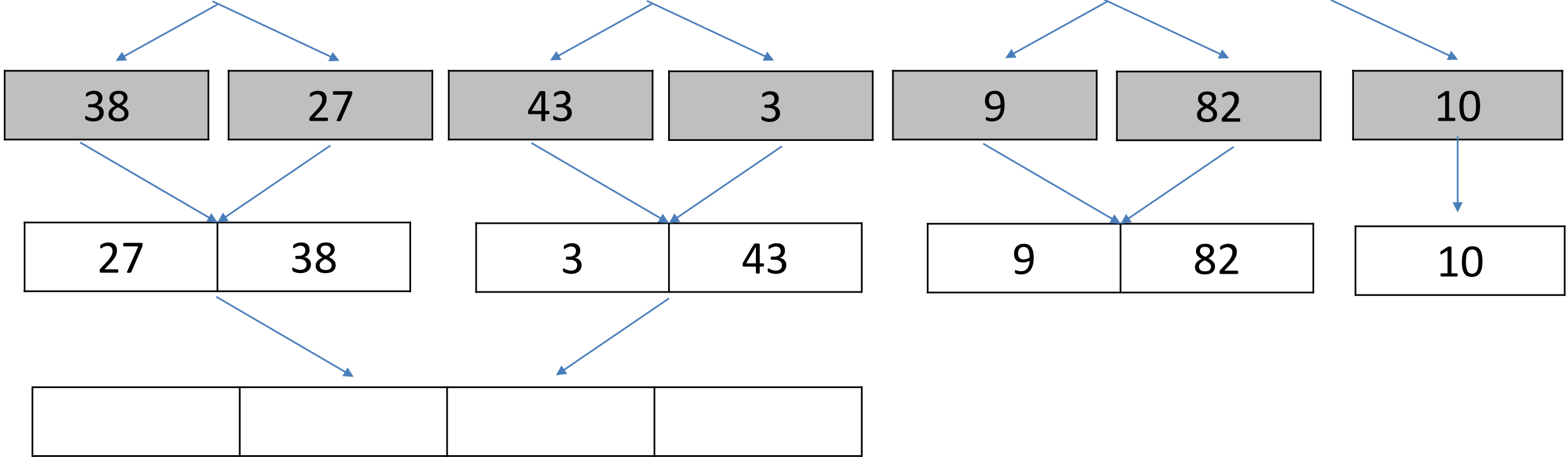
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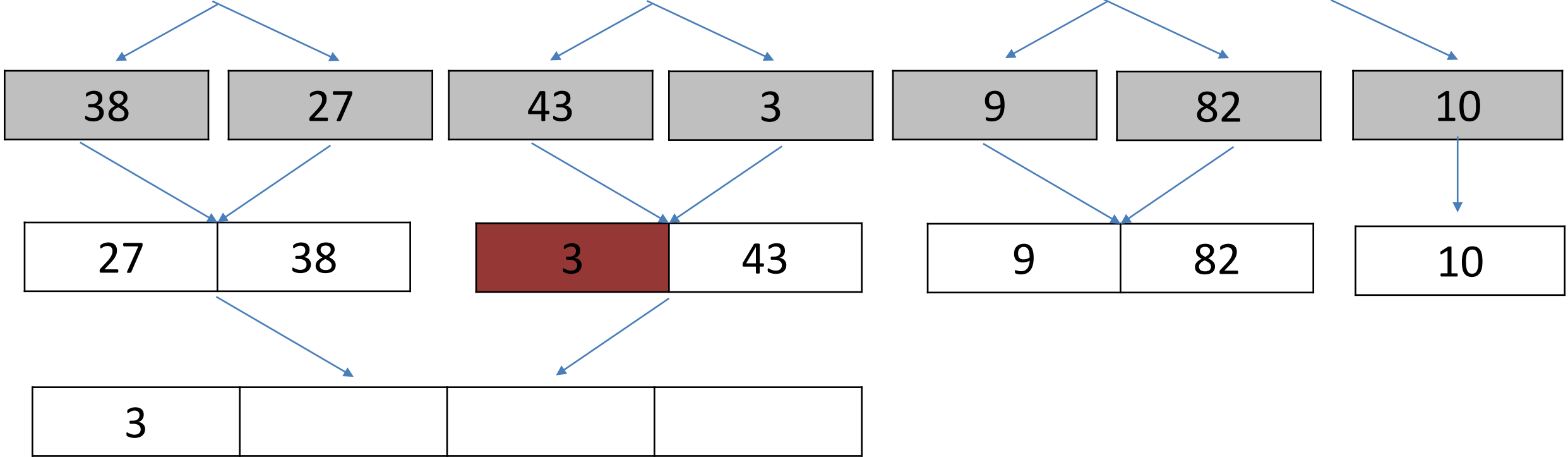
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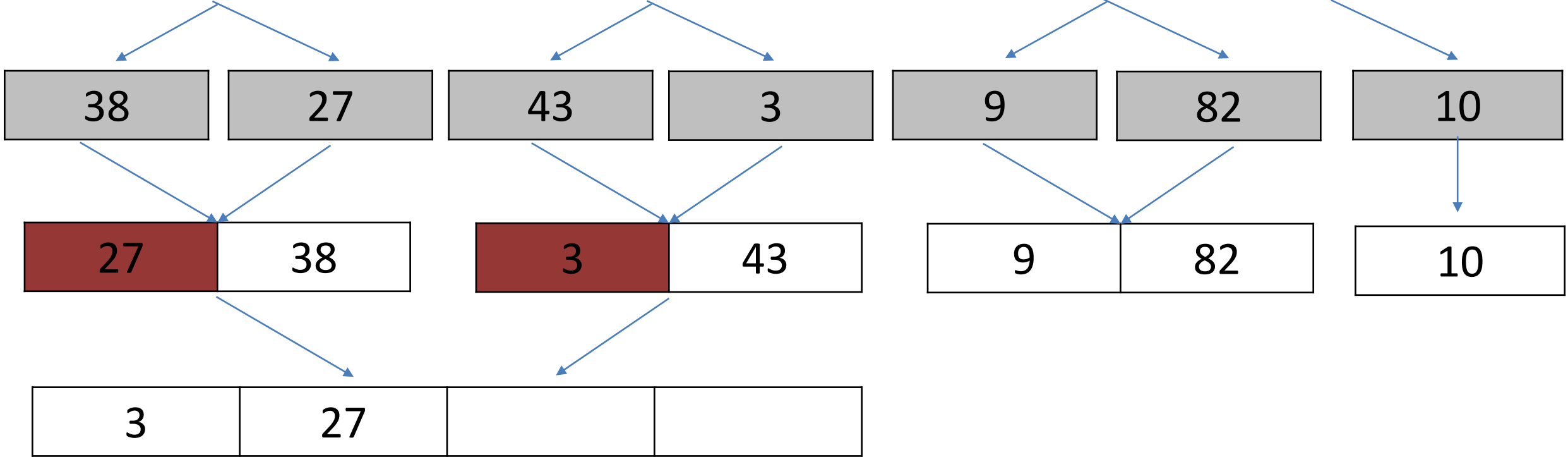


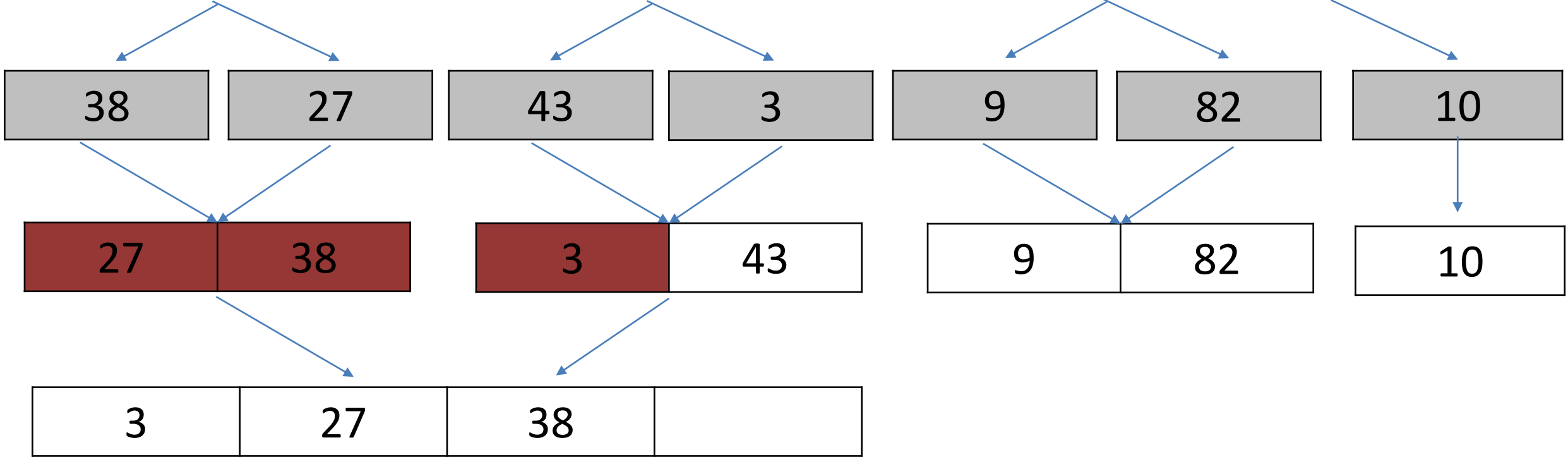


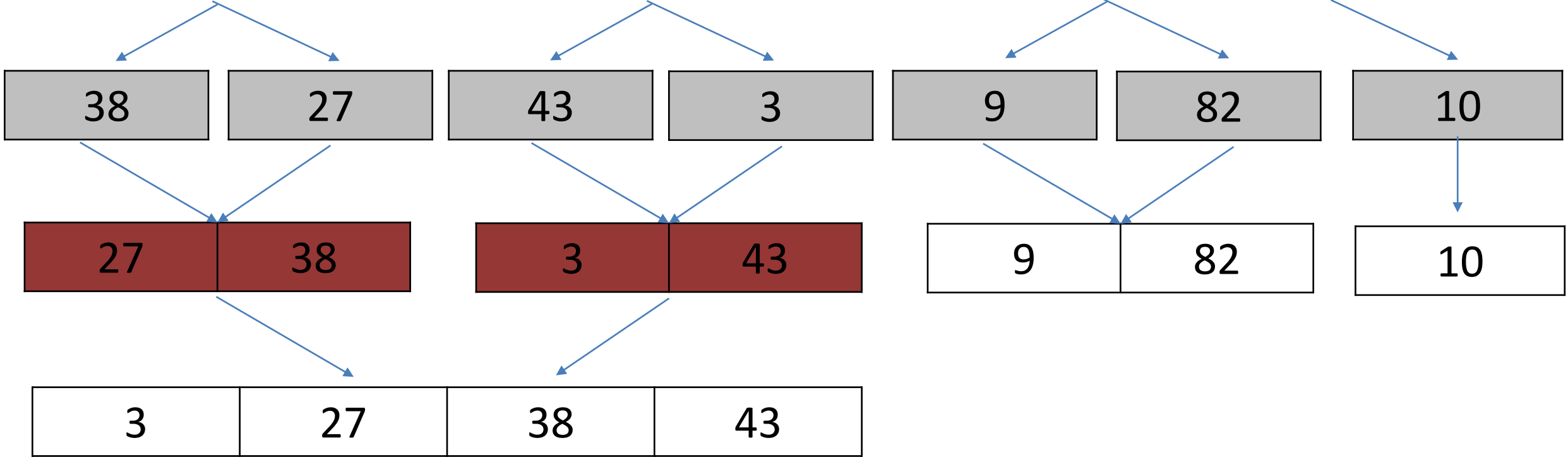


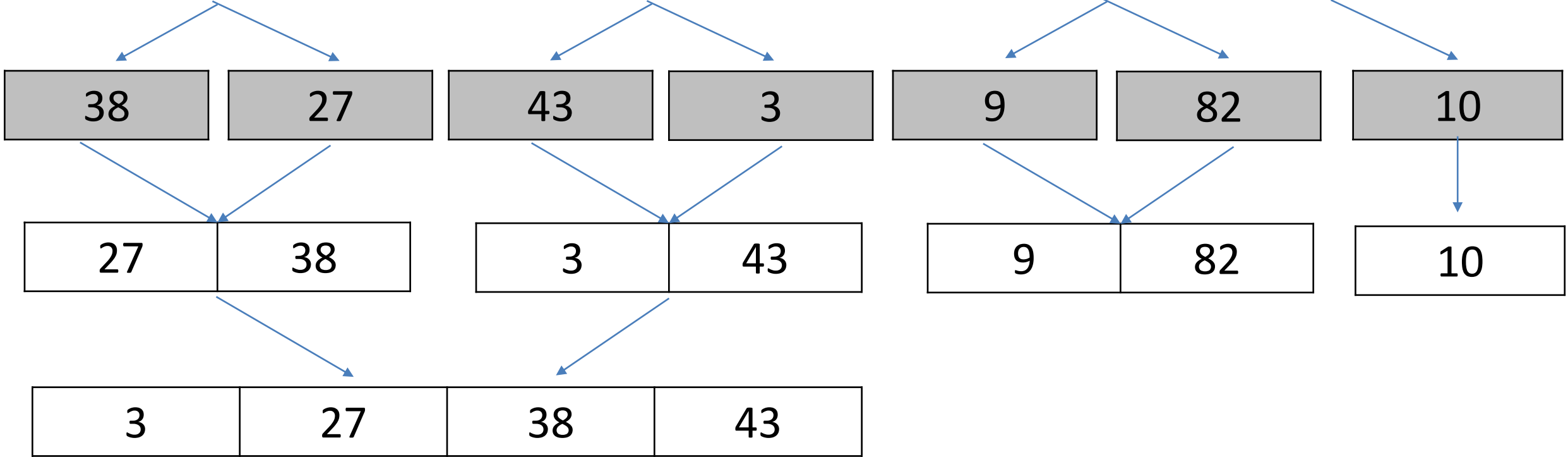


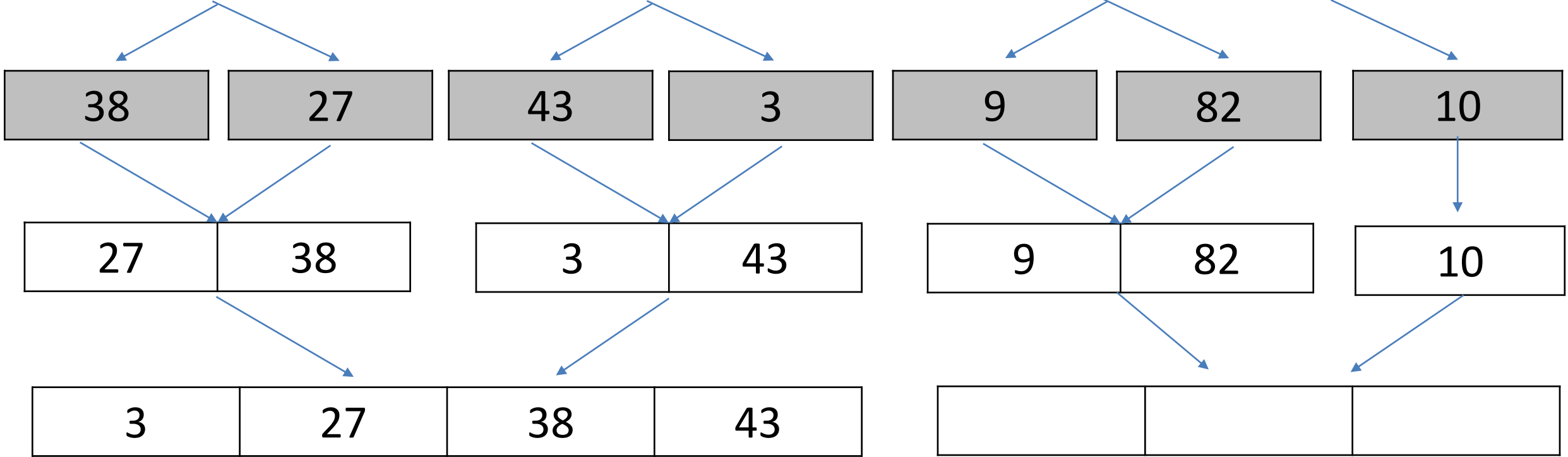


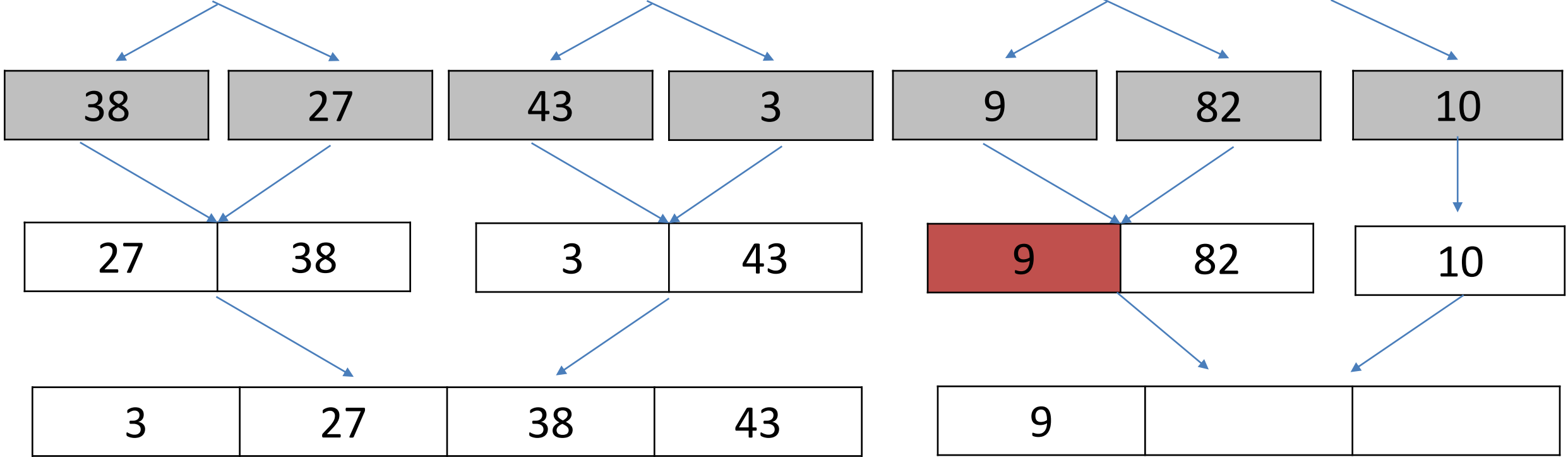


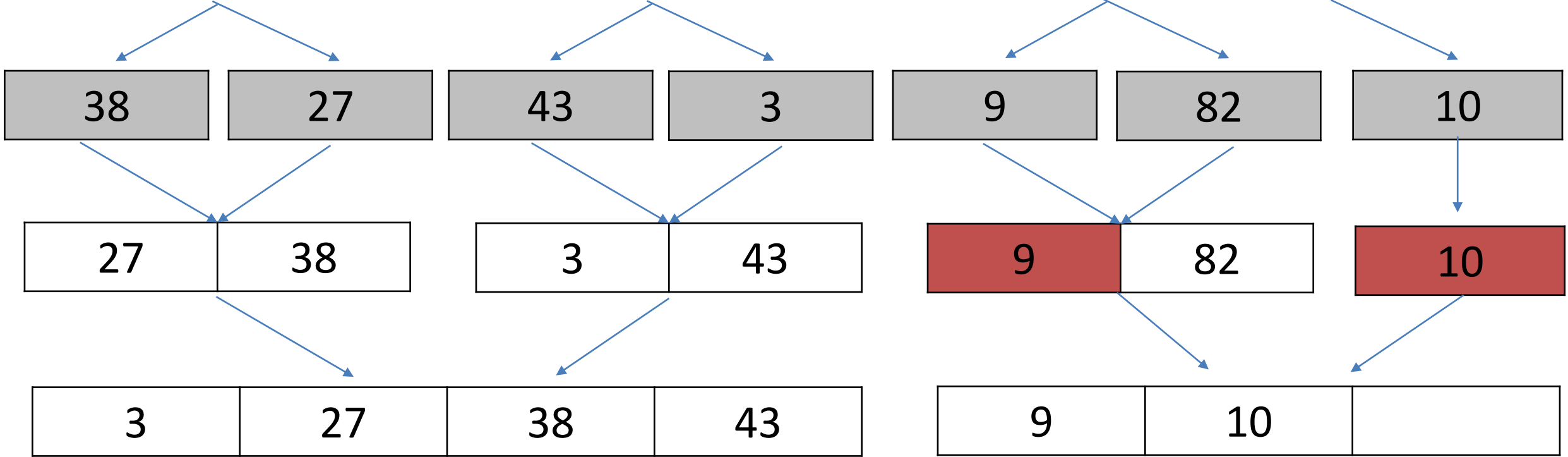


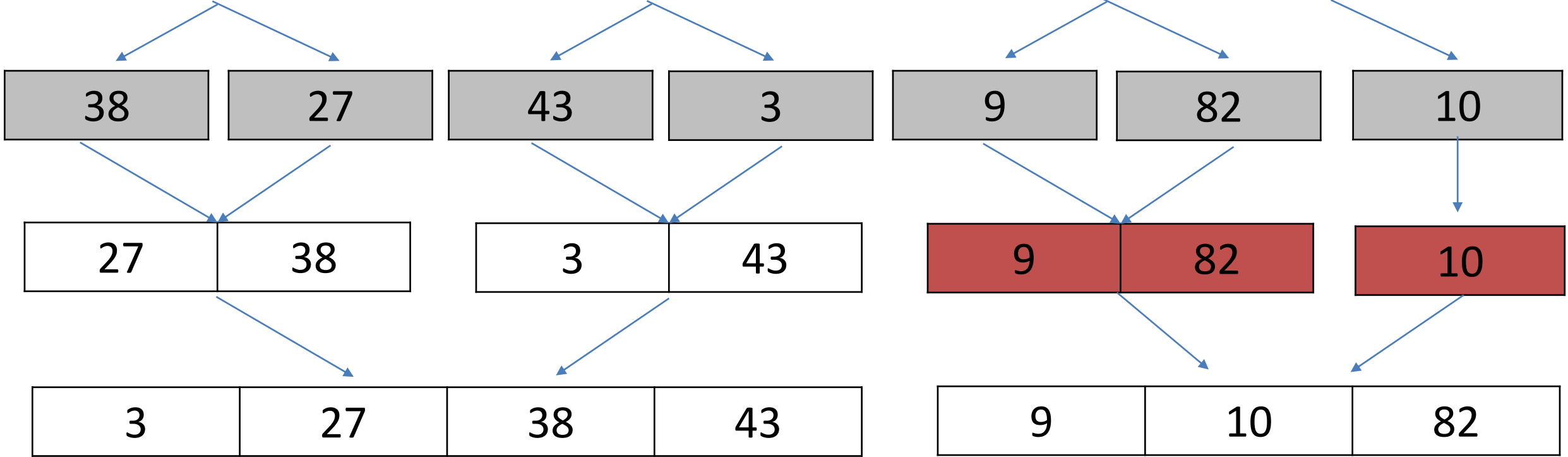


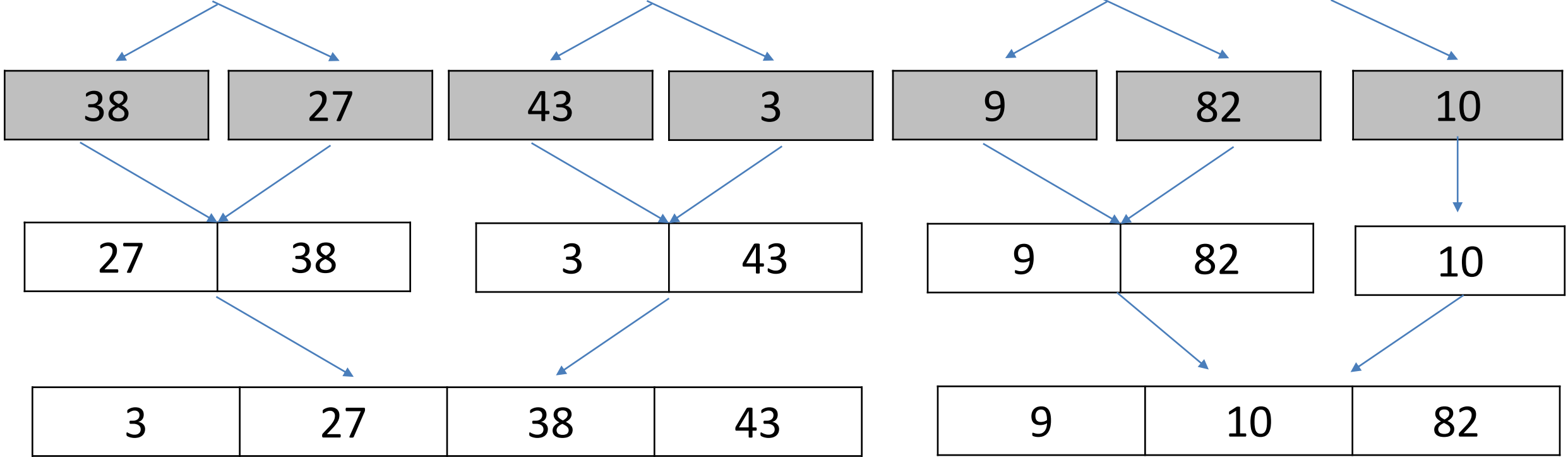


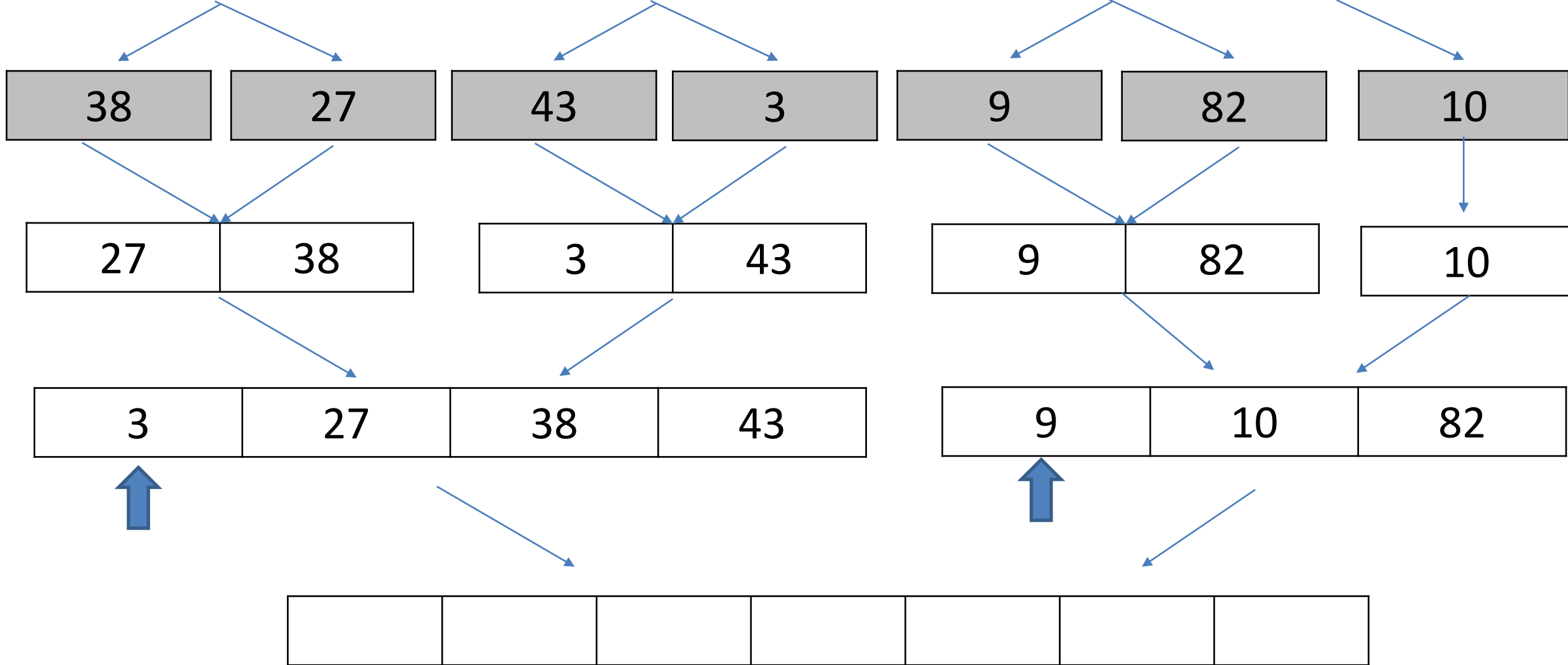




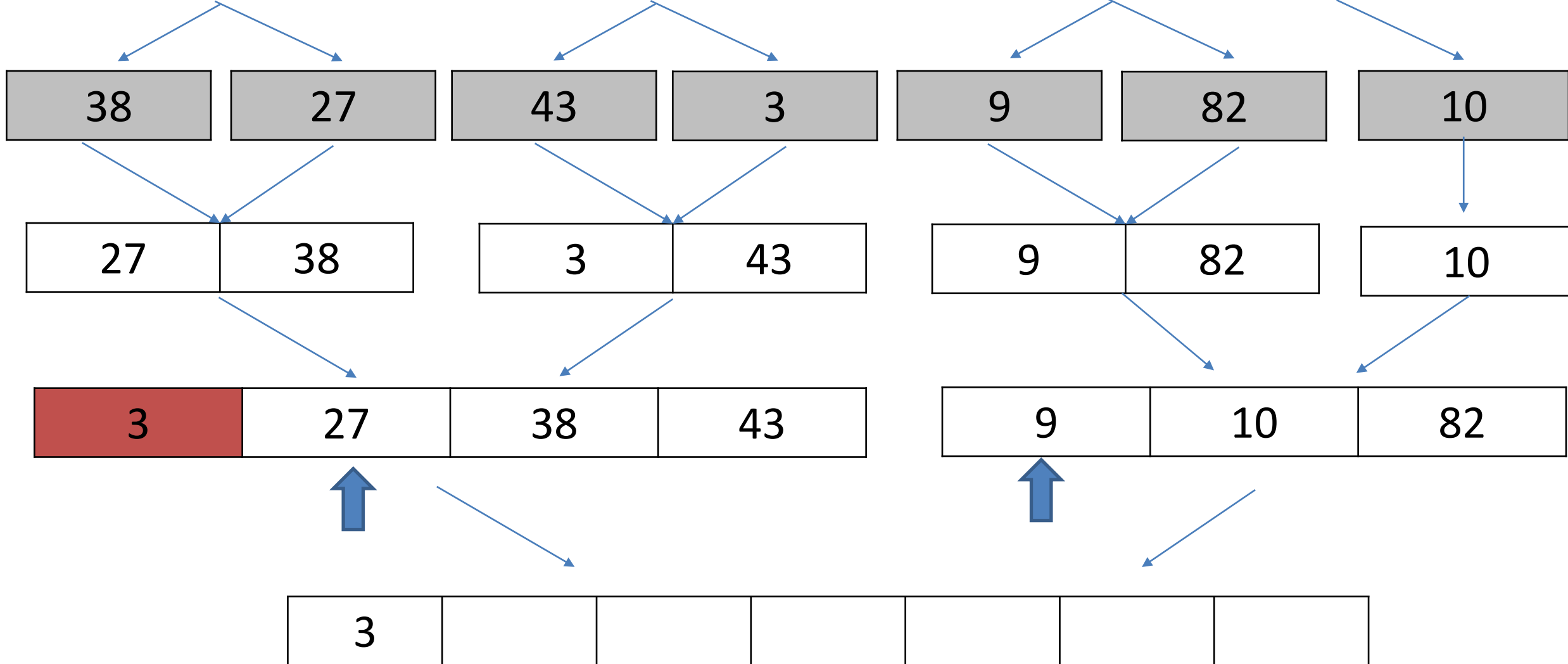




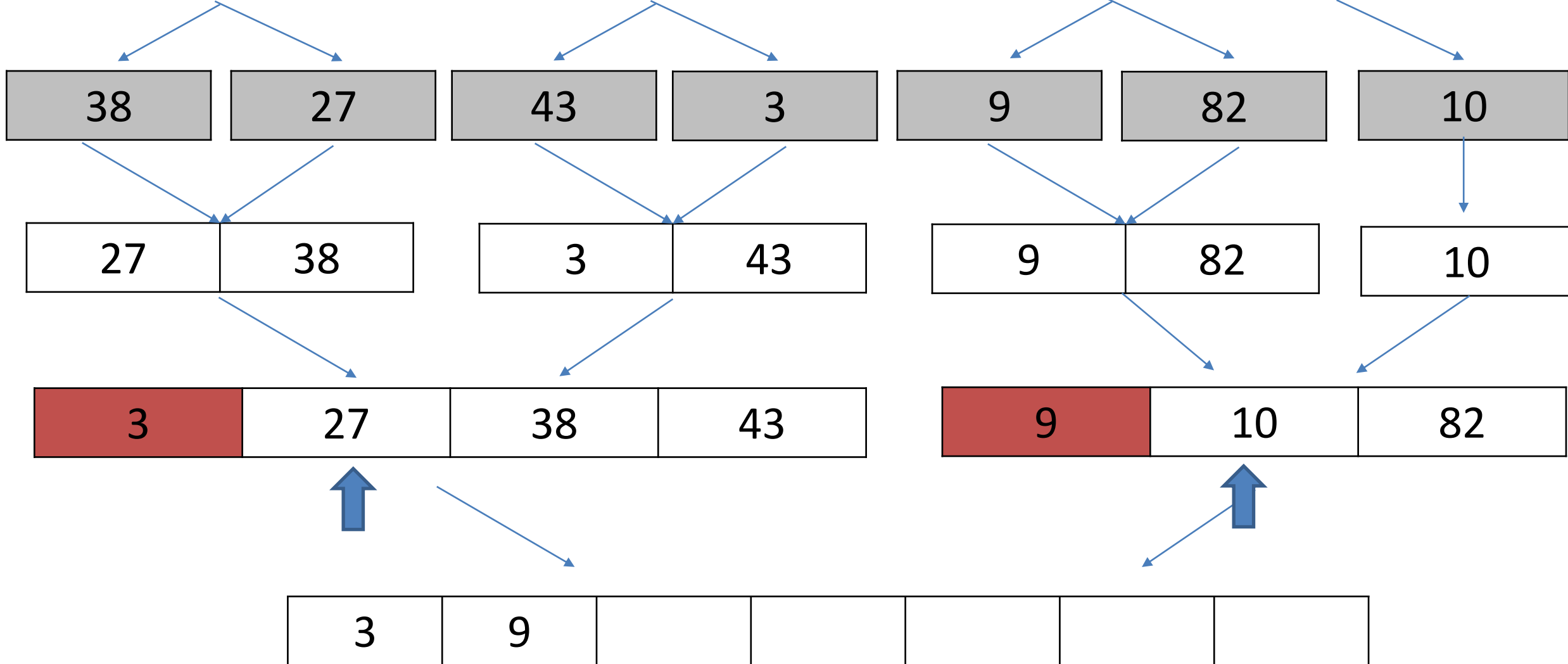




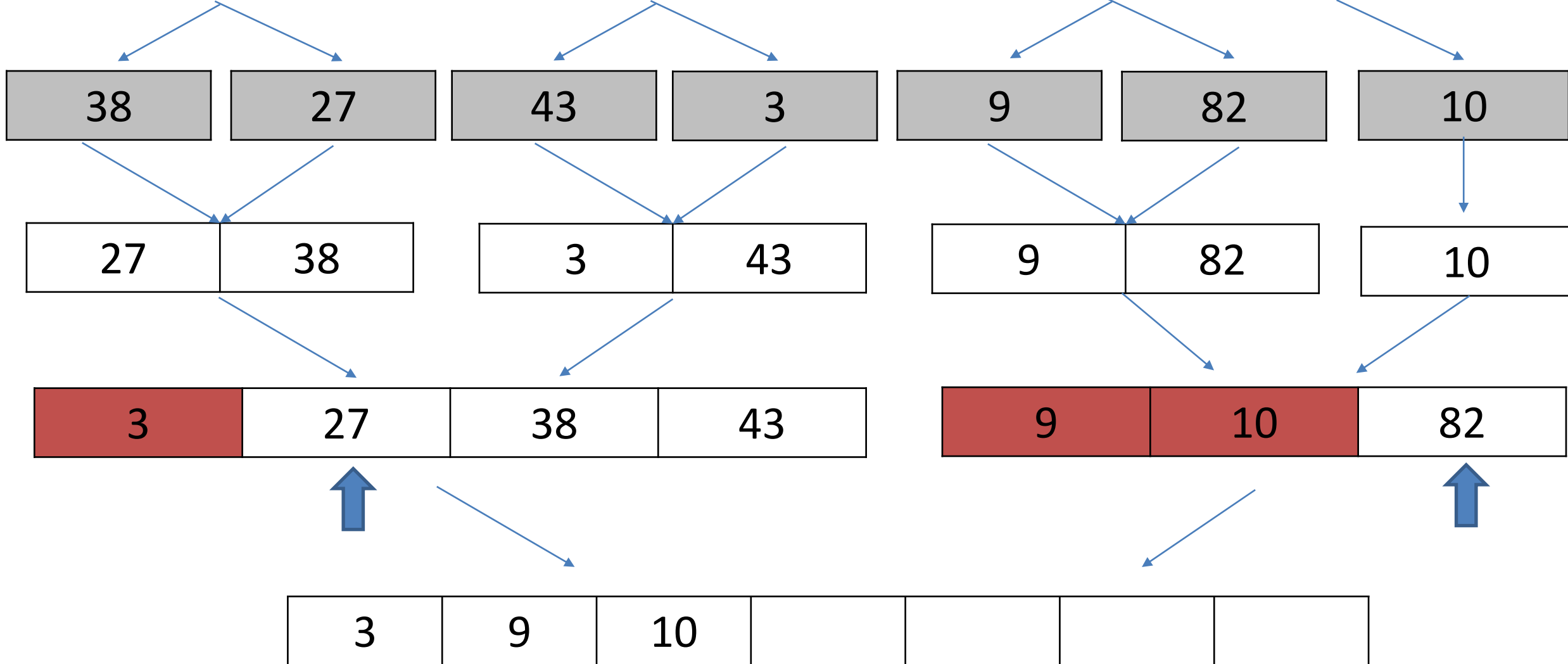
Because we know the subarray that we are merging are already sorted, the smallest element will always be at the first index



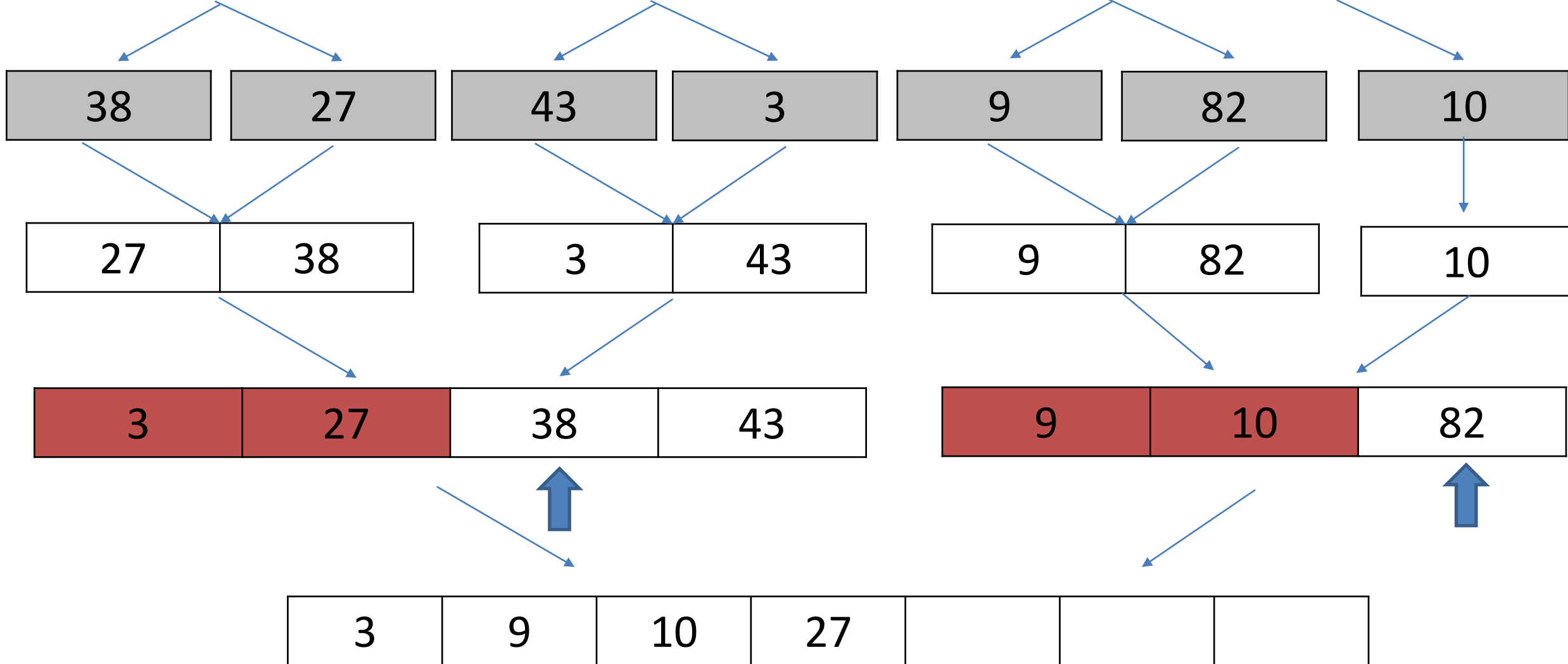
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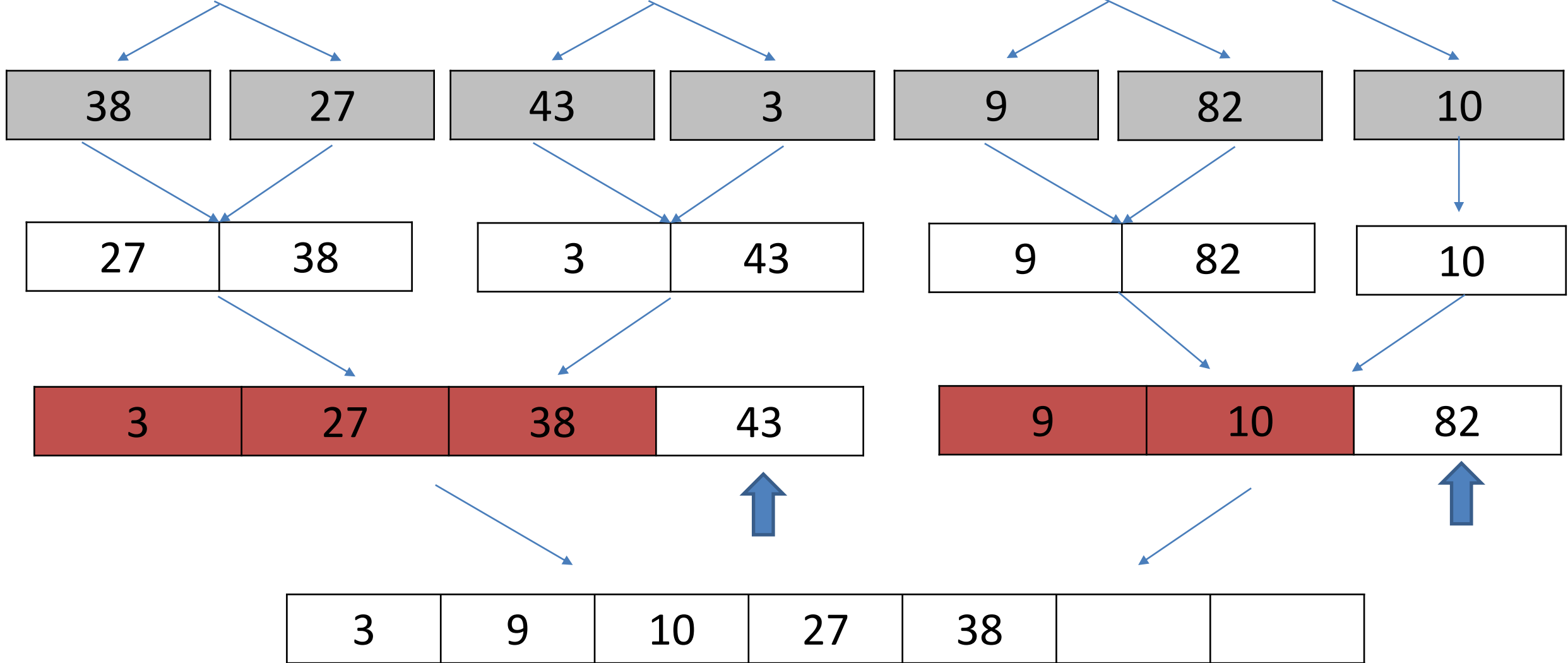
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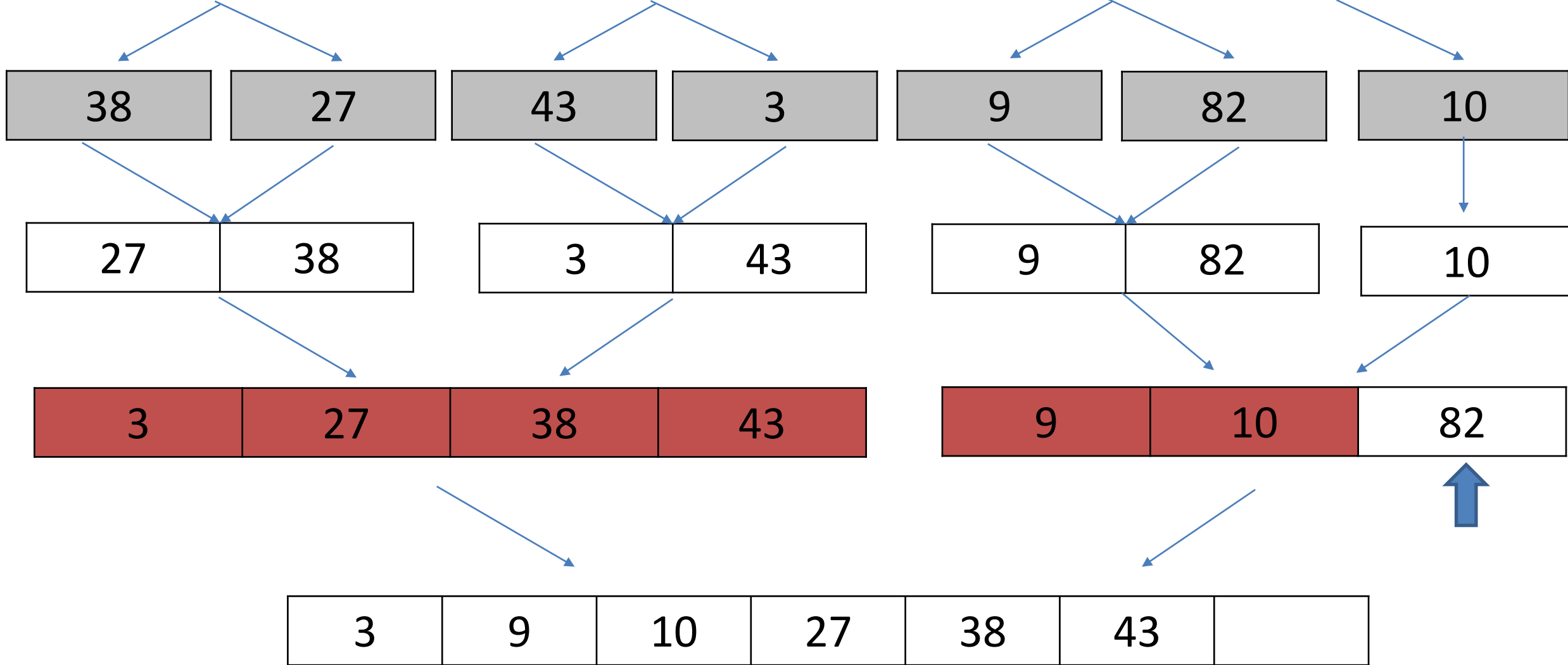
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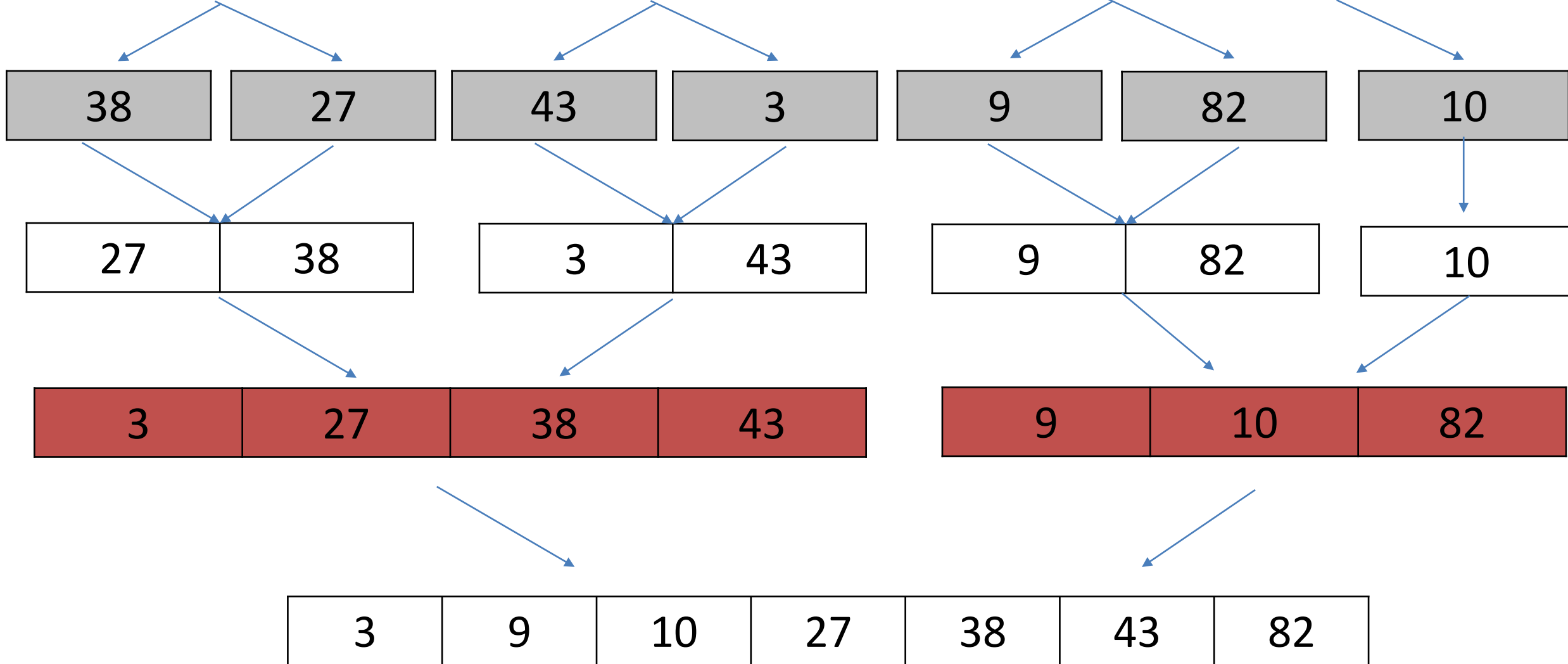
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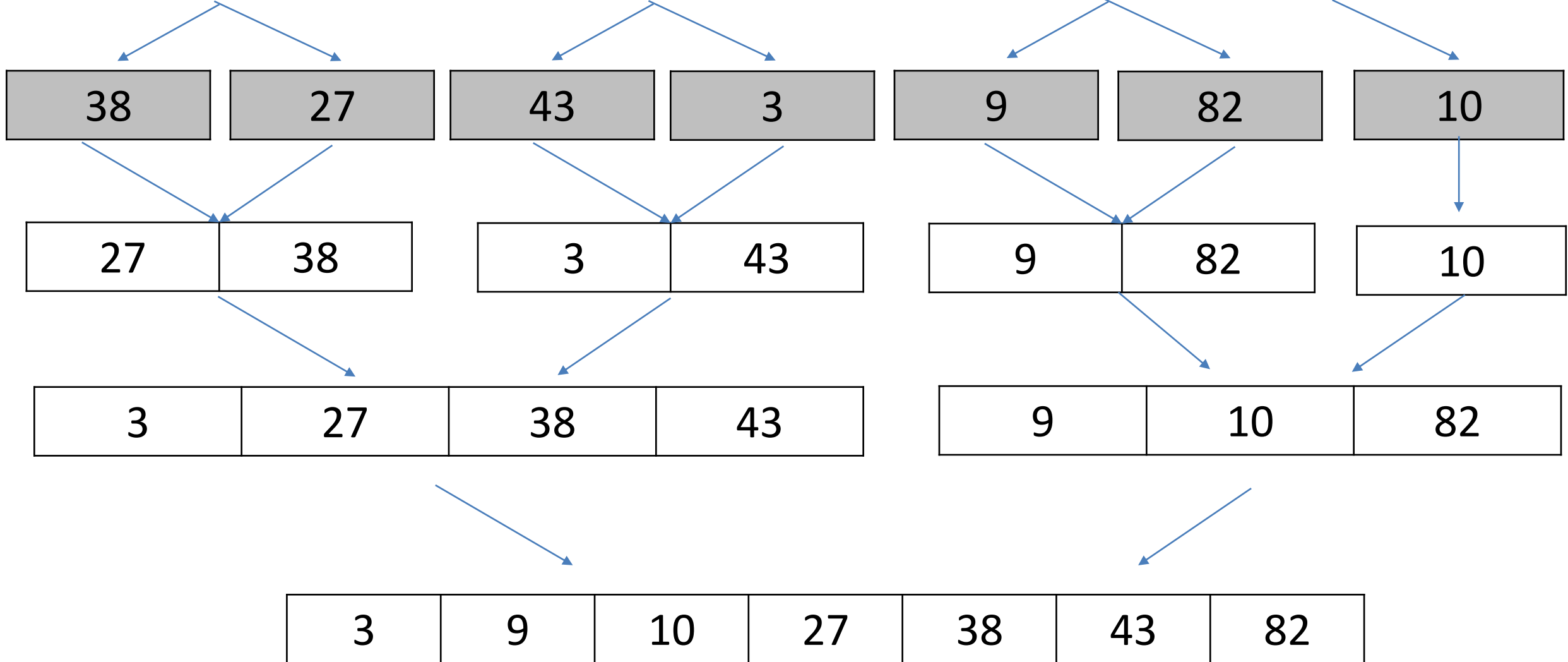
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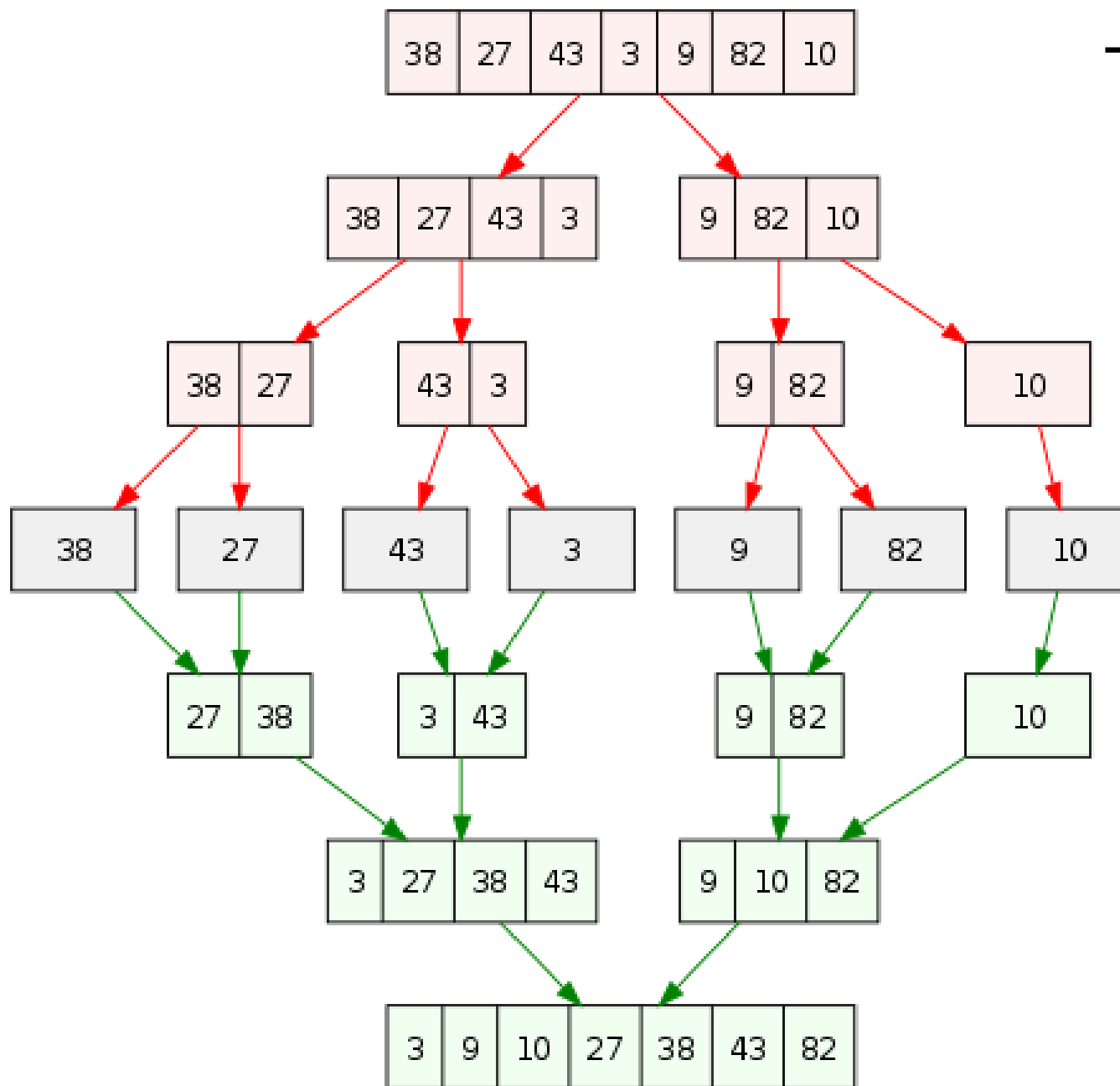
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Our original array is now sorted!!

The entire merge sort process

Divide



(base case)

Merge

38	27	43	3	9	82	10
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In our Java code, this will actually be the order of how things are done...

In practice, we will always prioritize solving the “left” tree first

38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3	9	82	10
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38	27	43	3	9	82	10
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38	27	43	3
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38	27
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38

38	27	43	3	9	82	10
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38	27	43	3
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38	27
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38	27
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38	27	43	3	9	82	10
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38	27	43	3
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38	27
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38	27
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27	38
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27
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27	38
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43
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27	38
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43	3
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27	38
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43	3
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27	38	3	43
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43	3
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27	38	3	43
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43	3
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27	38	3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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38	27	43	3
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38	27	43	3
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27	38	3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27	43	3
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38	27	43	3
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27	38	3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27
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43	3
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9	82
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38	27	43	3
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27	38
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3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27
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43	3
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9	82
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38

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27	38
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3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27
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43	3
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27	38
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3	43
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27
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38

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27	38
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27	43	3	9	82	10
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38	27	43	3	9	82
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27	38	3	43	9	82
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3	27	38	43
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27	43	3	9	82	10
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38	27	43	3	9	82
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27	38	3	43	9	82
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3	27	38	43
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9	10	82
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38	27	43	3	9	82	10
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38	27	43	3
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9	82	10
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38	27	43	3	9	82	10
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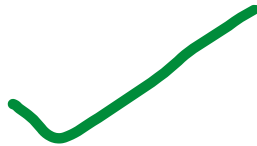
38	27	43	3	9	82
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27	38	3	43	9	82
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3	27	38	43
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9	10	82
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3	9	10	27	38	43	82
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38	27	43	3	9	82	10
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38	
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10

38	
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10

38	
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27	
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3

82

3	9	10	27	38	43	82
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Let's code this!!



Running time of merge sort??



Running time of merge sort??

Running time = number of recursive calls made* · amount of work done in each call

*for merge sort, this won't lead us to the correct answer

```
public static int[] merge_sort(int[] inputArray) {
    int inputLength = inputArray.length;
    if (inputLength < 2) {
        return inputArray;
    }
    int midIndex = inputLength / 2;
    int[] leftHalf = new int[midIndex];
    int[] rightHalf = new int[inputLength - midIndex];
    for (int i = 0; i < midIndex; i++) {
        leftHalf[i] = inputArray[i];
    }
    for (int i = midIndex; i < inputLength; i++) {
        rightHalf[i - midIndex] = inputArray[i];
    }
    merge_sort(leftHalf);
    merge_sort(rightHalf);
    inputArray = merge(inputArray, leftHalf, rightHalf);
    return inputArray;
}
```

```
public static int[] merge_sort(int[] inputArray) {  
    int inputLength = inputArray.length; O(1)  
    if (inputLength < 2) {  
        return inputArray; O(1)  
    }  
    int midIndex = inputLength / 2; O(1)  
    int[] leftHalf = new int[midIndex];  
    int[] rightHalf = new int[inputLength - midIndex];  
    for (int i = 0; i < midIndex; i++) {  
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    int[] leftHalf = new int[midIndex]; O(n/2)  
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    for (int i = 0; i < midIndex; i++) {  
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    for (int i = 0; i < midIndex; i++) { O(n/2)  
        leftHalf[i] = inputArray[i];  
    }  
    for (int i = midIndex; i < inputLength; i++) { O(n/2)  
        rightHalf[i - midIndex] = inputArray[i];  
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    merge_sort(leftHalf);  
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    int[] rightHalf = new int[inputLength - midIndex]; O(n/2)  
    for (int i = 0; i < midIndex; i++) { O(n/2)  
        leftHalf[i] = inputArray[i];  
    }  
    for (int i = midIndex; i < inputLength; i++) { O(n/2)  
        rightHalf[i - midIndex] = inputArray[i];  
    }  
    merge_sort(leftHalf); O(1)  
    merge_sort(rightHalf); O(1)  
    inputArray = merge(inputArray, leftHalf, rightHalf); O(???)  
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        rightHalf[i - midIndex] = inputArray[i];  
    }  
    merge_sort(leftHalf); O(1)  
    merge_sort(rightHalf); O(1)  
    inputArray = merge(inputArray, leftHalf, rightHalf); O(???)  
    return inputArray;  
}
```



```
private static int[] merge (int[] inputArray, int[] leftHalf, int[] rightHalf) {
    int leftSize = leftHalf.length;
    int rightSize = rightHalf.length;
    int i = 0, j = 0, k = 0;
    while (i < leftSize && j < rightSize) {
        if (leftHalf[i] <= rightHalf[j]) {
            inputArray[k] = leftHalf[i];
            i++;
        }
        else {
            inputArray[k] = rightHalf[j];
            j++;
        }
        k++;
    }
    while (i < leftSize) {
        inputArray[k] = leftHalf[i];
        i++;
        k++;
    }
    while (j < rightSize) {
        inputArray[k] = rightHalf[j];
        j++;
        k++;
    }
    return inputArray;
}
```

```

private static int[] merge (int[] inputArray, int[] leftHalf, int[] rightHalf) {
    int leftSize = leftHalf.length; O(1)
    int rightSize = rightHalf.length; O(1)
    int i = 0, j = 0, k = 0;
    while (i < leftSize && j < rightSize) { O(n)
        if (leftHalf[i] <= rightHalf[j]) {
            inputArray[k] = leftHalf[i];
            i++; O(1)
        }
        else {
            inputArray[k] = rightHalf[j];
            j++; O(1)
        }
        k++;
    }
    while (i < leftSize) {
        inputArray[k] = leftHalf[i];
        i++;
        k++;
    }
    while (j < rightSize) {
        inputArray[k] = rightHalf[j];
        j++;
        k++;
    }
    return inputArray;
}

```

amount of work done in each call?

```
private static int[] merge (int[] inputArray, int[] leftHalf, int[] rightHalf) {  
    int leftSize = leftHalf.length; O(1)  
    int rightSize = rightHalf.length; O(1)  
    int i = 0, j = 0, k = 0;  
    while (i < leftSize && j < rightSize) { O(n)  
        if (leftHalf[i] <= rightHalf[j]) {  
            inputArray[k] = leftHalf[i];  
            i++; O(1)  
        }  
        else {  
            inputArray[k] = rightHalf[j];  
            j++; O(1)  
        }  
        k++;  
    }  
    while (i < leftSize) { O(n/2)  
        inputArray[k] = leftHalf[i];  
        i++; O(1)  
        k++;  
    }  
    while (j < rightSize) { O(n/2)  
        inputArray[k] = rightHalf[j];  
        j++; O(1)  
        k++;  
    }  
    return inputArray; O(1)  
}
```


$$O(n) + O(n/2) + O(n/2) = O(2n)$$

Running time of merge subroutine

$O(n)$

amount of work done in each call?

```
public static int[] merge_sort(int[] inputArray) {  
    int inputLength = inputArray.length; O(1)  
    if (inputLength < 2) {  
        return inputArray; O(1)  
    }  
    int midIndex = inputLength / 2; O(1)  
    int[] leftHalf = new int[midIndex]; O(n/2)  
    int[] rightHalf = new int[inputLength - midIndex]; O(n/2)  
    for (int i = 0; i < midIndex; i++) { O(n/2)  
        leftHalf[i] = inputArray[i];  
    }  
    for (int i = midIndex; i < inputLength; i++) { O(n/2)  
        rightHalf[i - midIndex] = inputArray[i];  
    }  
    merge_sort(leftHalf); O(1)  
    merge_sort(rightHalf); O(1)  
    inputArray = merge(inputArray, leftHalf, rightHalf); O(n)  
    return inputArray; O(1)  
}
```



amount of work done in each call?

```
public static int[] merge_sort(int[] inputArray) {  
    int inputLength = inputArray.length; O(1)  
    if (inputLength < 2) {  
        return inputArray; O(1)  
    }  
    int midIndex = inputLength / 2; O(1)  
    int[] leftHalf = new int[midIndex]; O(n/2)  
    int[] rightHalf = new int[inputLength - midIndex]; O(n/2)  
    for (int i = 0; i < midIndex; i++) { O(n/2)  
        leftHalf[i] = inputArray[i];  
    }  
    for (int i = midIndex; i < inputLength; i++) { O(n/2)  
        rightHalf[i - midIndex] = inputArray[i];  
    }  
    merge_sort(leftHalf); O(1)  
    merge_sort(rightHalf); O(1)  
    inputArray = merge(inputArray, leftHalf, rightHalf); O(n)  
    return inputArray; O(1)  
}
```

$O(n) + O(n/2) + O(n/2) + O(n/2) + O(n)$

Total running time of a
single merge_sort call:

$O(n)$

Running time of merge sort??

Running time = number of recursive calls made* · amount of work done in each call

Running time = ??? * $O(n)$

*for merge sort, this won't lead us to the correct answer

Running time of merge sort??

Running time = number of recursive calls made* · amount of work done in each call

Running time = ??? * O(n)

When we recursively call our method when dividing, we give a problem **that is half the size** of the original problem

--	--	--	--	--	--	--	--

$c(n)$

Suppose that the cost of solving a problem of size n can be expressed as $c(n)$



$c(n)$

$c(n)$

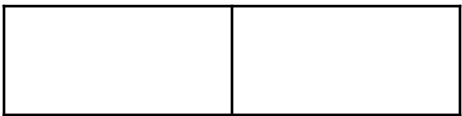


$c(n/2)$



$c(n/2)$

$c(n)$



$c(n/4)$



$c(n/4)$



$c(n/4)$



$c(n/4)$

$c(n)$



$c(n/8)$



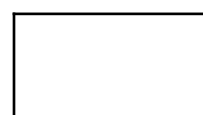
$c(n/8)$



$c(n/8)$



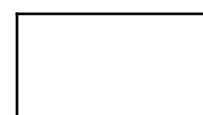
$c(n/8)$



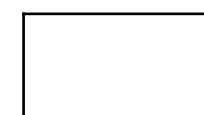
$c(n/8)$



$c(n/8)$



$c(n/8)$



$c(n/8)$

$c(n)$



$c(n)$

$c(n)$

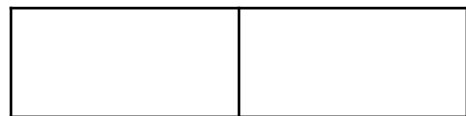


$c(n/2)$

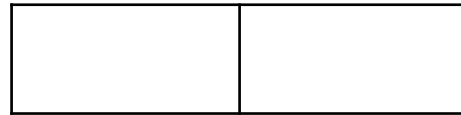


$c(n/2)$

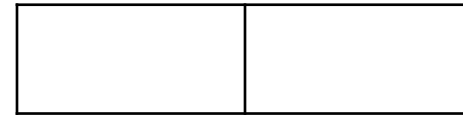
$c(n)$



$c(n/4)$



$c(n/4)$

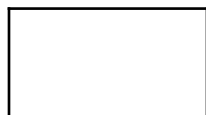


$c(n/4)$



$c(n/4)$

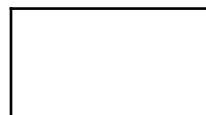
$c(n)$



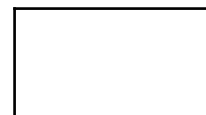
$c(n/8)$



$c(n/8)$



$c(n/8)$



$c(n/8)$



$c(n/8)$



$c(n/8)$



$c(n/8)$

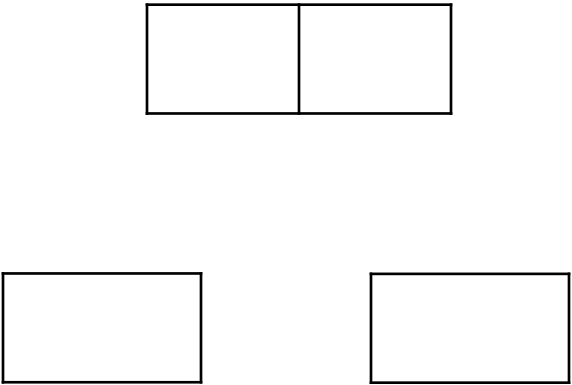


$c(n/8)$

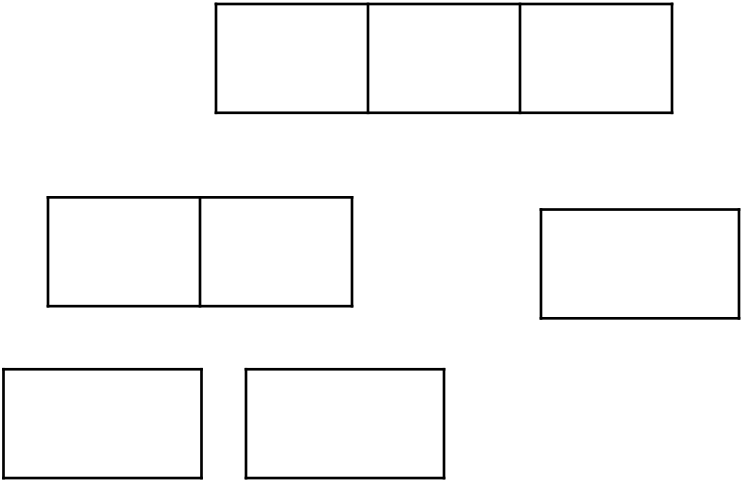
$c(n)$

How much do we divide (*in regards to n*)? AKA **what is the height of the recursion tree?**

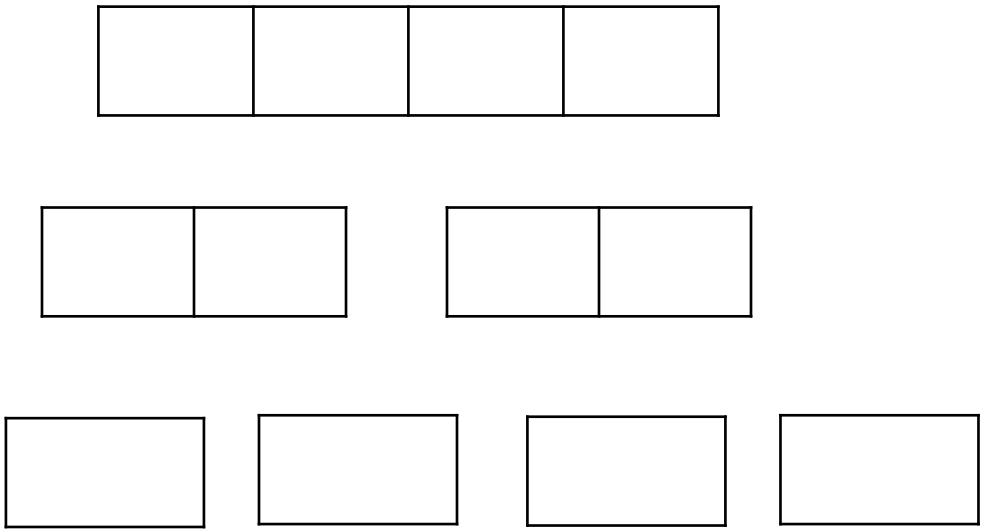
When $n = 2$, the height is 2



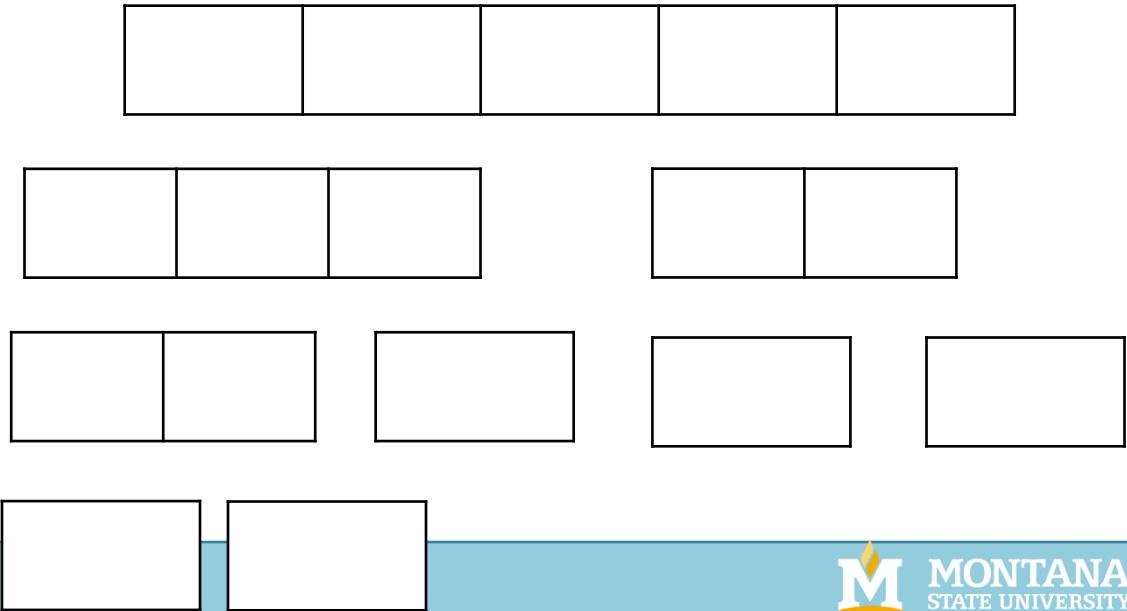
When $n = 3$, the height is 3



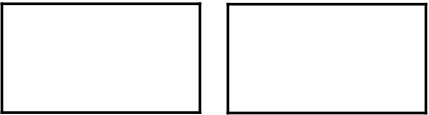
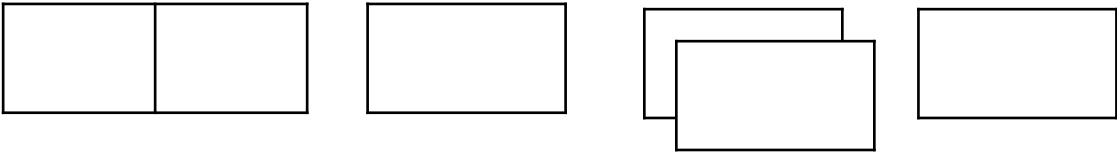
When $n = 4$, the height is 3



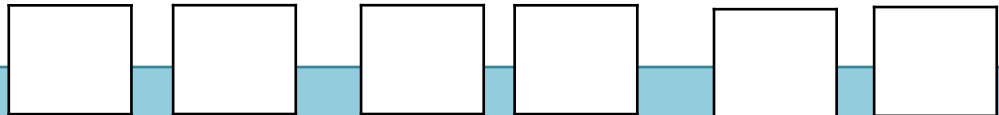
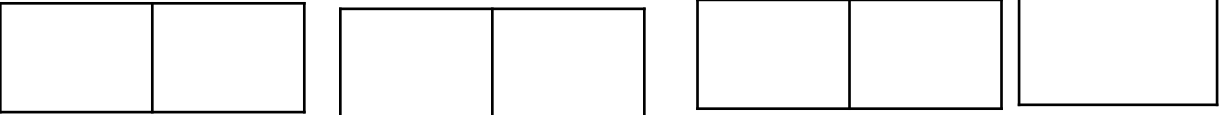
When $n = 5$, the height is 4



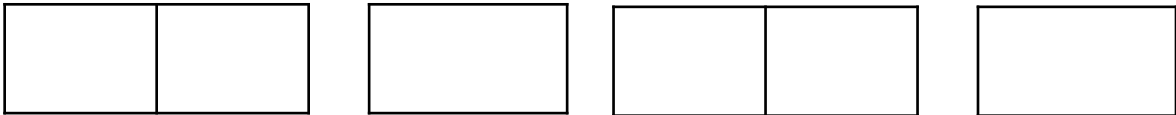
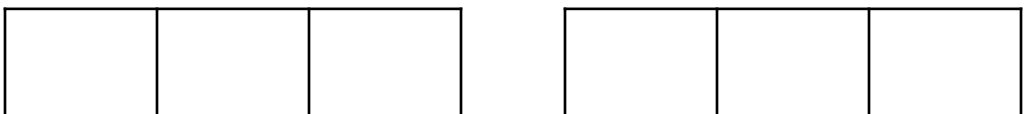
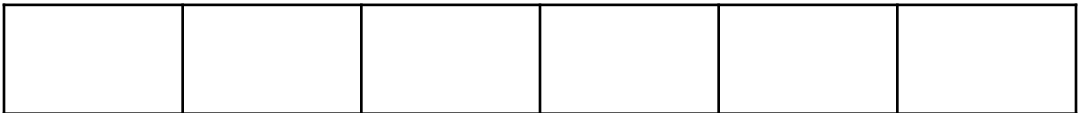
When $n = 5$, the height is 4



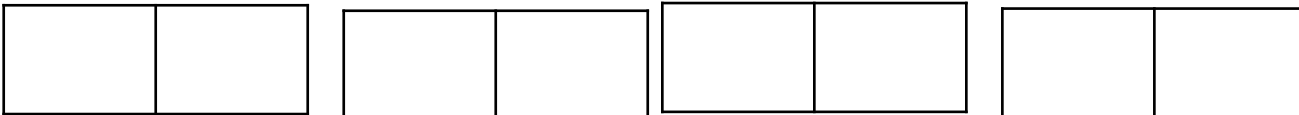
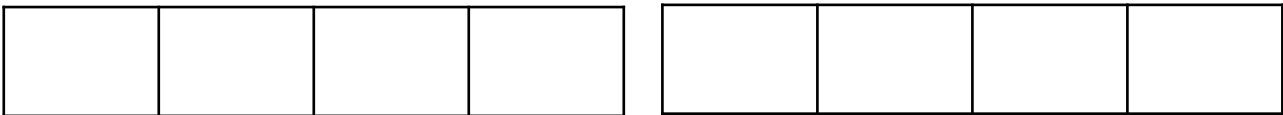
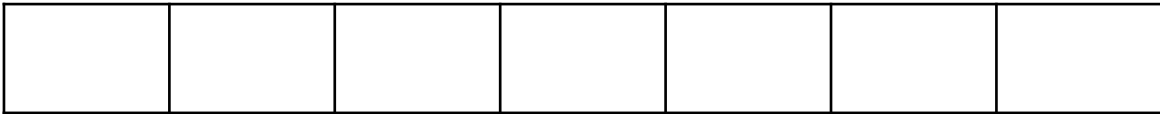
When $n = 7$, the height is 4



When $n = 6$, the height is 4

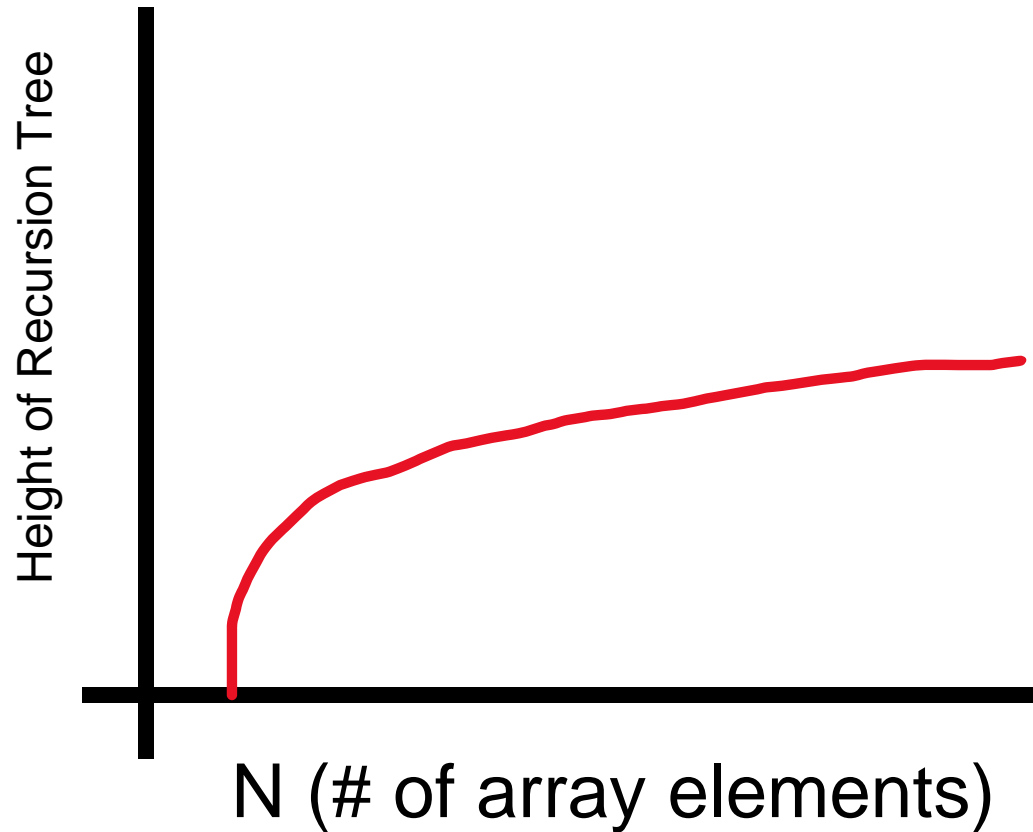


When $n = 8$, the height is 4



What is the growth rate of the height of our recursion tree (the # of recursive calls made) ?

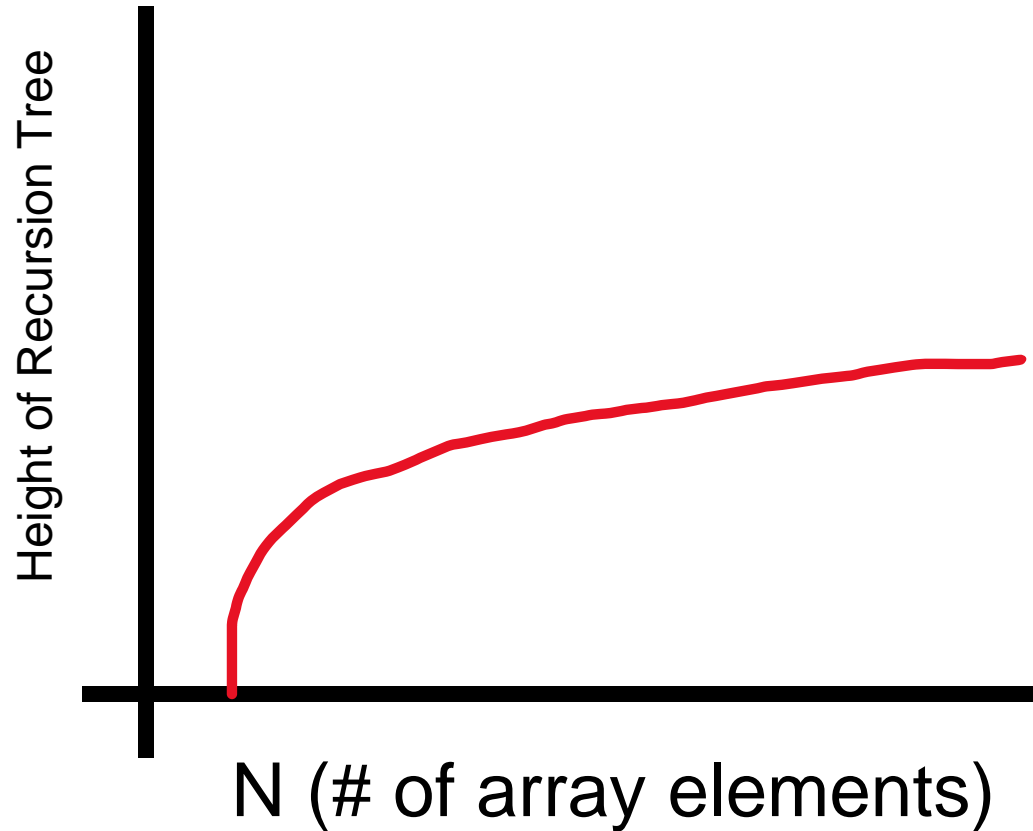
If we were to continue this counting, the graph would look something like this:



???

What is the growth rate of the height of our recursion tree (the # of recursive calls made) ?

If we were to continue this counting, the graph would look something like this:

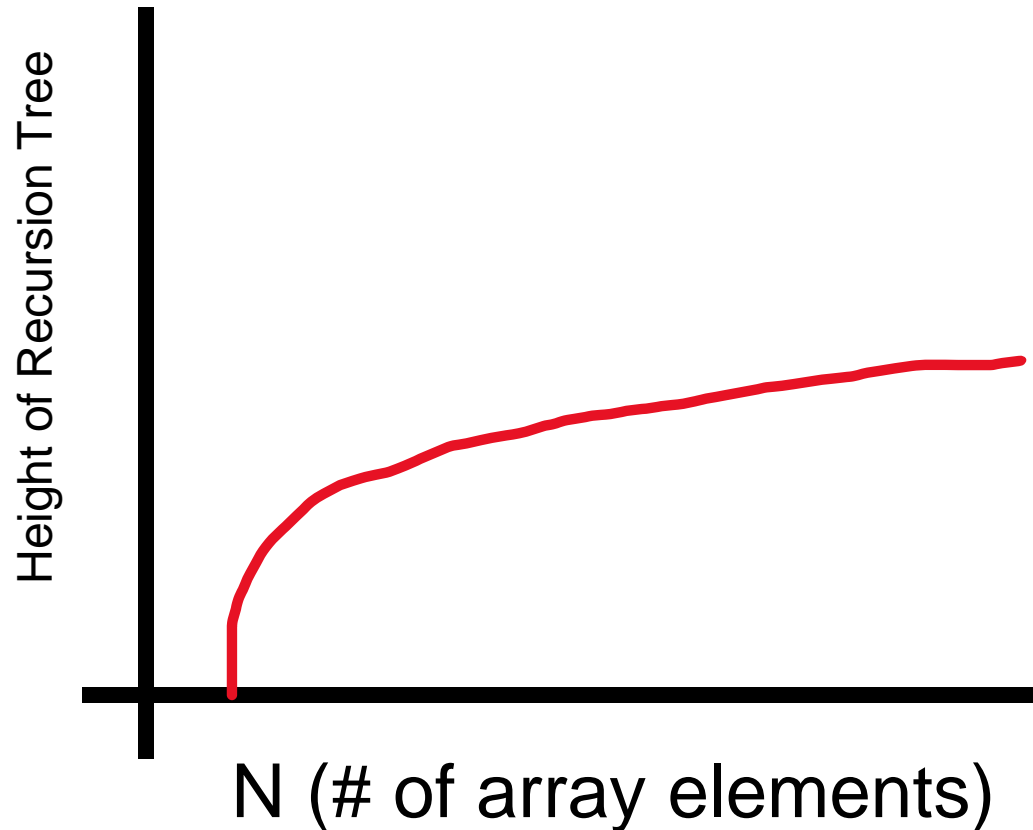


Logarithmic !!

$\log(n)$

What is the growth rate of the height of our recursion tree (the # of recursive calls made) ?

If we were to continue this counting, the graph would look something like this:



Logarithmic !!

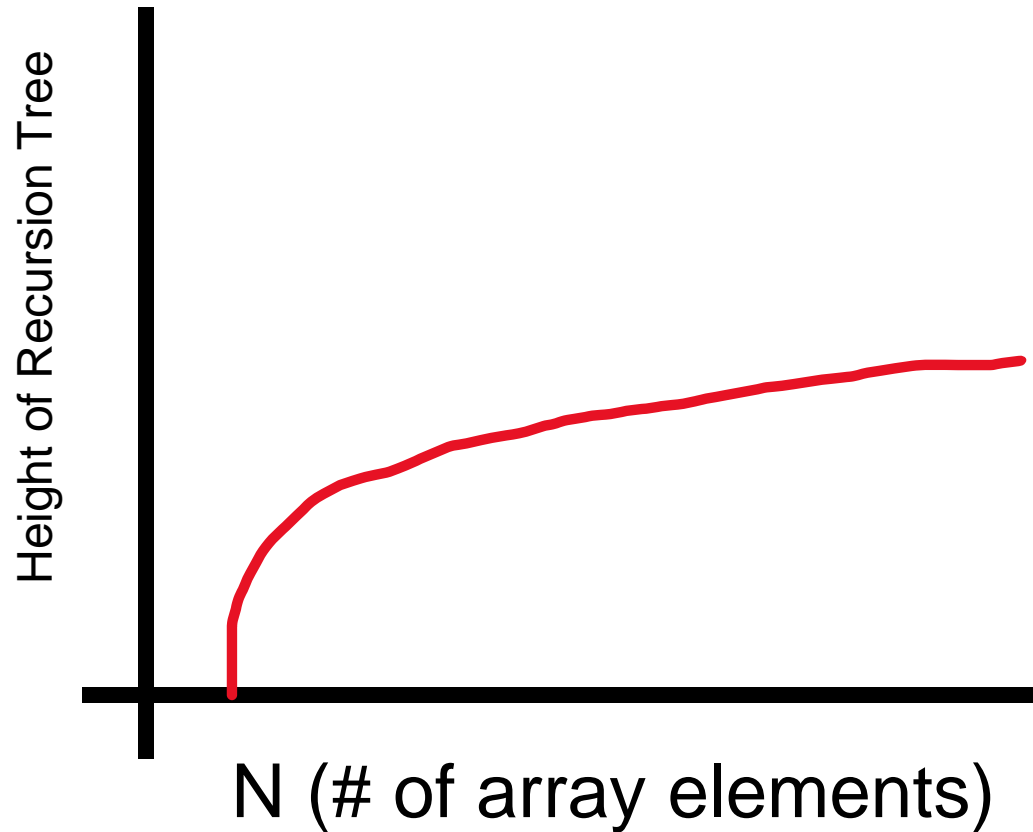
$$\log_2(n)$$

It will actually be log base 2, because we are dividing our array in half in each recursive call

However, in computer science, all logarithms are to the base 2 unless specified otherwise

What is the growth rate of the height of our recursion tree (the # of recursive calls made) ?

If we were to continue this counting, the graph would look something like this:

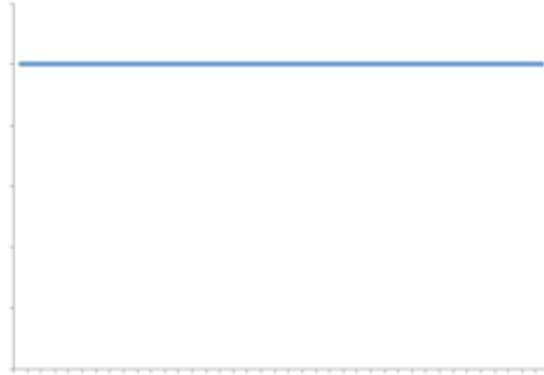


Logarithmic !!

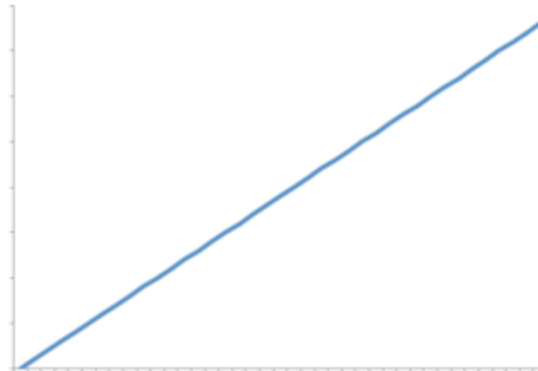
$\log(n)$

Growth Rates

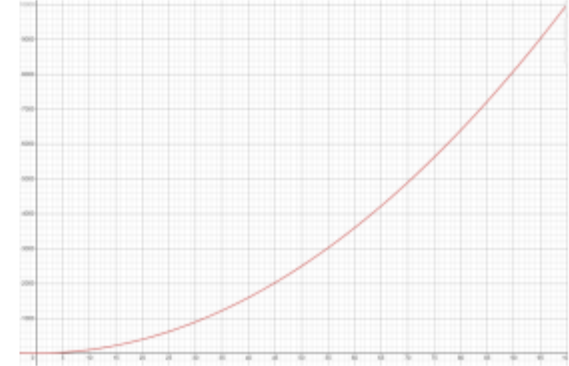
Constant



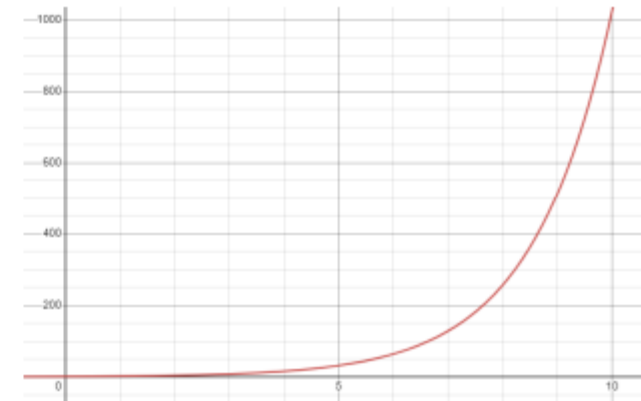
Linear



Quadratic



Exponential



We have a new
member of the
family!

Growth Rates

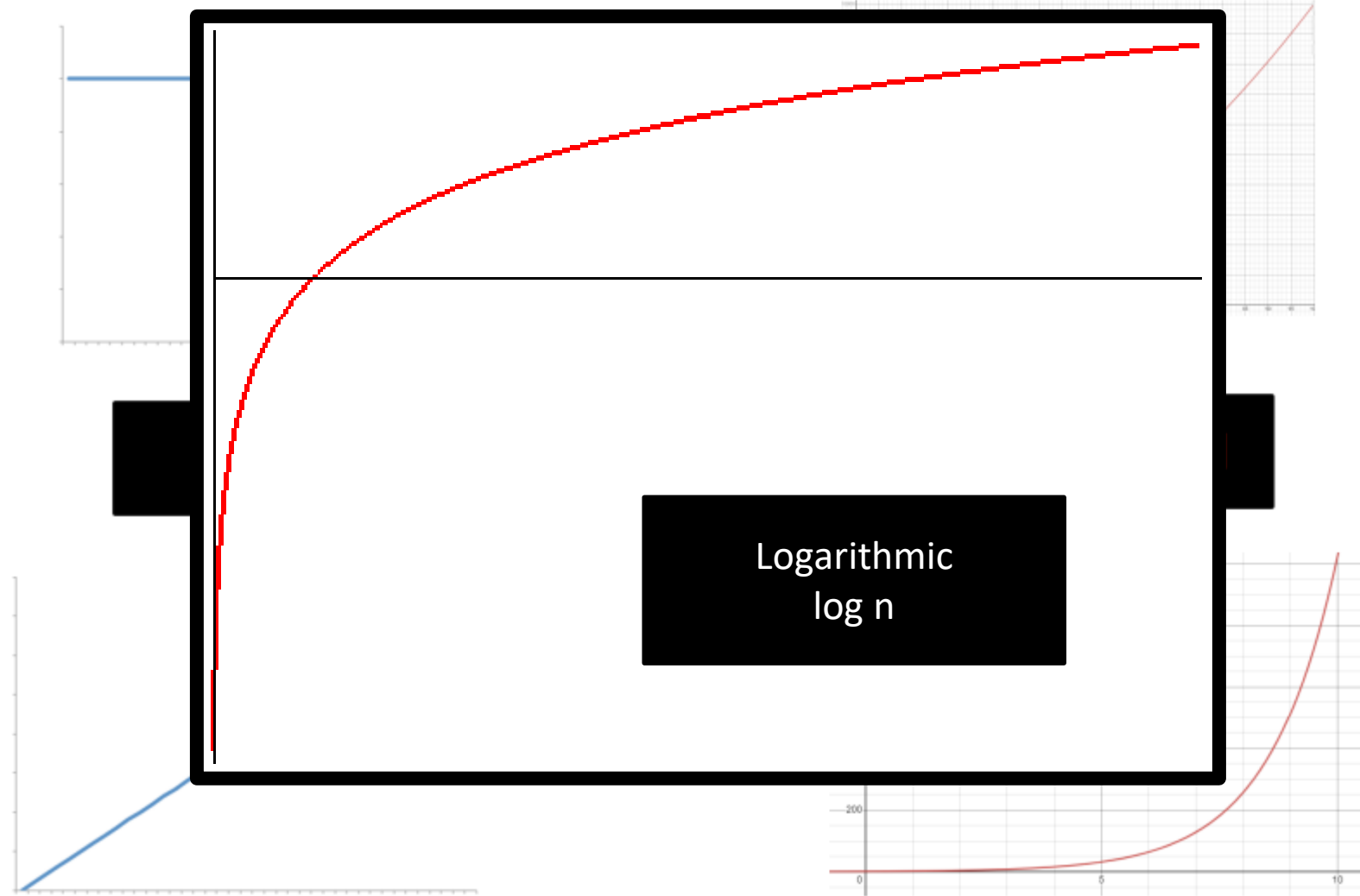
Constant

Quadratic

We have a new
member of the
family!

$\log n$ is *smaller*
than n

algorithms that run
in **$O(\log n)$** time
are good!



Running time of merge sort??

Height of recursive tree

Running time = ~~number of recursive calls made~~ · amount of work done in each call

Running time =

???

*

$O(n)$

Running time of merge sort??

Height of recursive tree

Running time = ~~number of recursive calls made~~ · amount of work done in each call

Running time = $O(\log n)$ * $O(n)$

Running time of merge sort = $O(n * \log n)$

Running time of merge sort??

Height of recursive tree

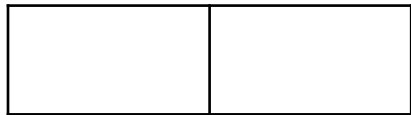
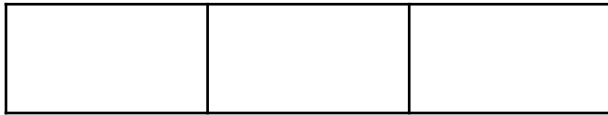
Running time = ~~number of recursive calls made~~ · amount of work done in each call

Running time = $O(\log n)$ * $O(n)$

Running time of merge sort = $O(n * \log n)$

This is **much** faster than $O(n^2)$

What about stack overflow errors?



We still have to worry about stack overflow errors if our input is reallyyyyyyyyyyyyyyy big

However, because merge sorts works from “left to right”, we won’t have n recursive calls active, and we are much more efficient with how many method calls we put on call stack