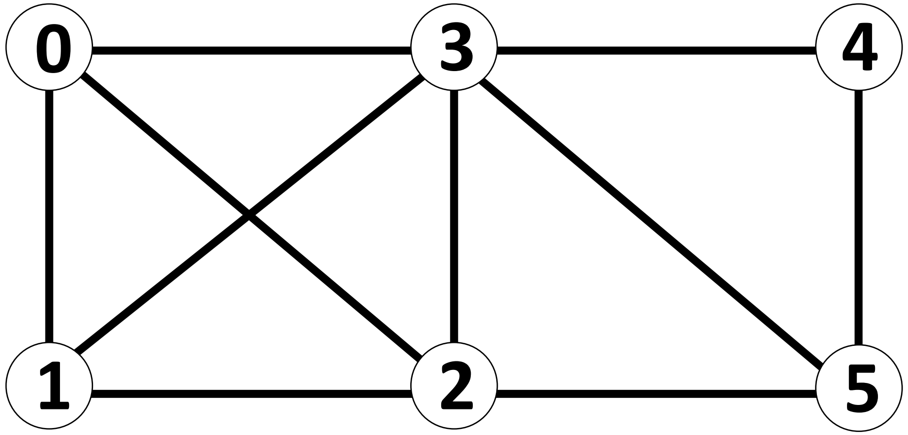
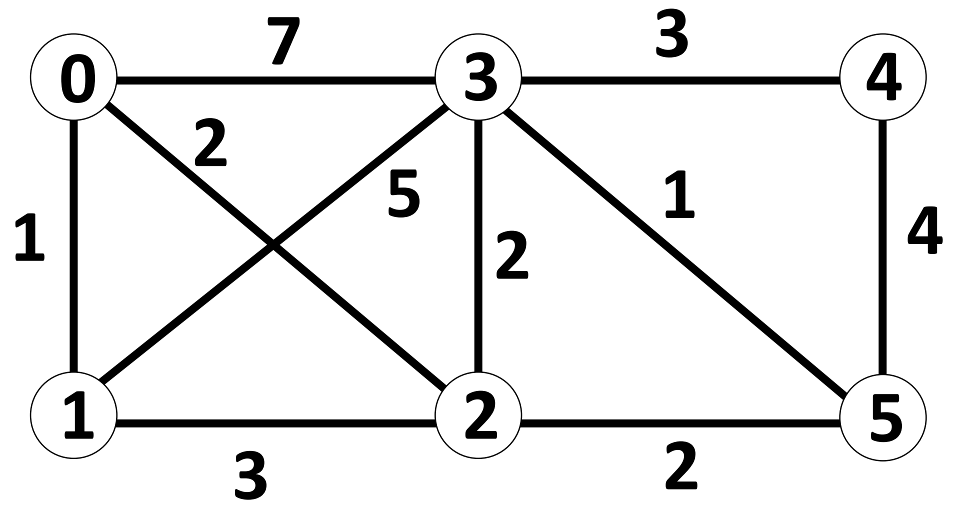
Minimum Spanning Trees

CSCI 232

Minimum Spanning Tree

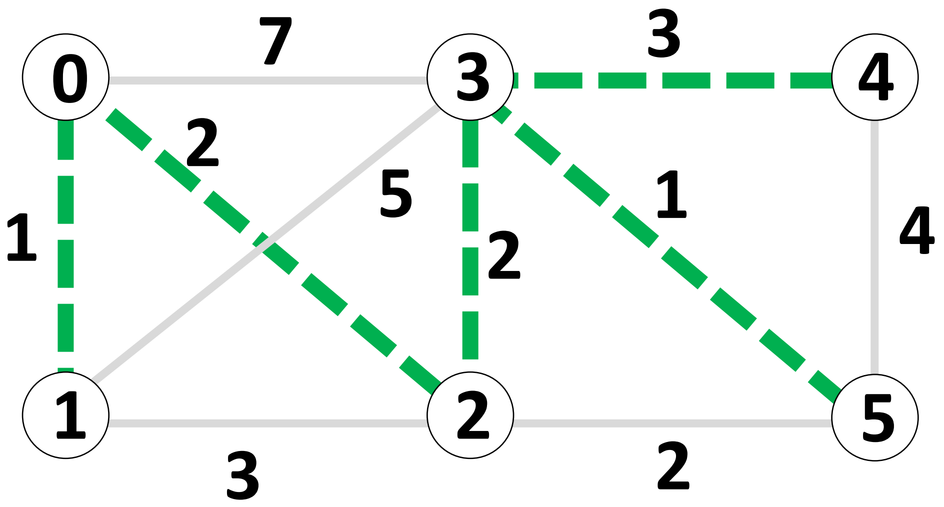


Minimum Spanning Tree



Edge-weighted graph: A graph where each edge has a weight (cost).

Minimum Spanning Tree

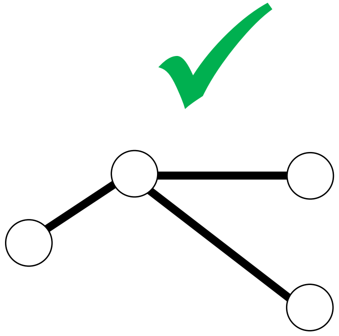


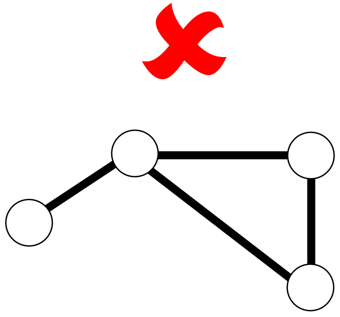
Edge-weighted graph: A graph where each edge has a weight (cost).

MST Goal: Connect all vertices to each other with a minimum weight subset of edges.

Minimum Spanning Tree

Tree – connected graph with no loops.





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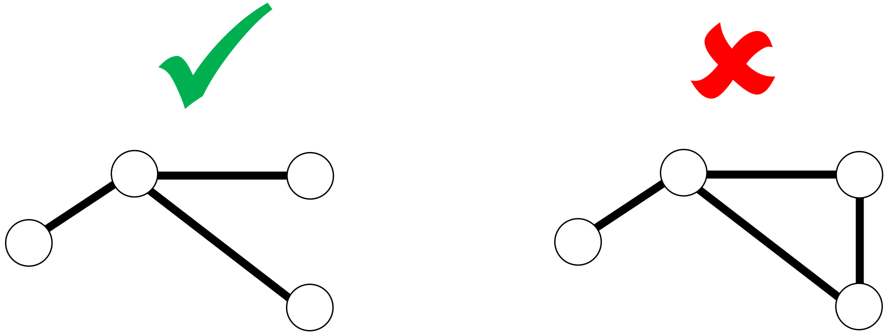
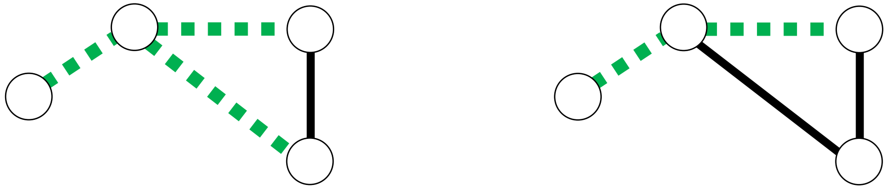


Spanning tree – tree that includes all vertices in a graph.



Minimum Spanning Tree

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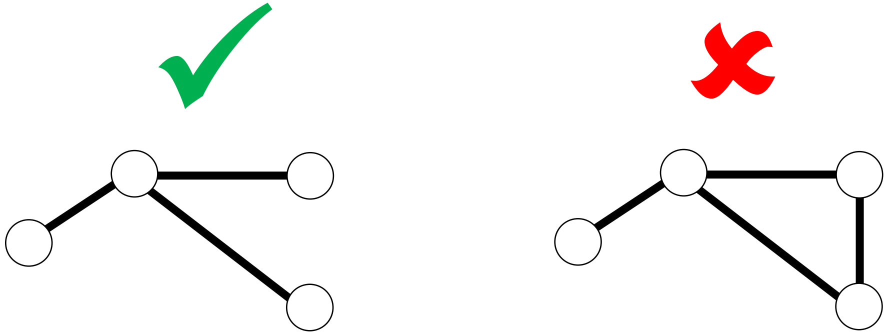

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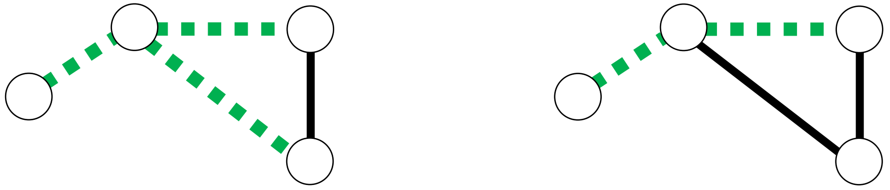
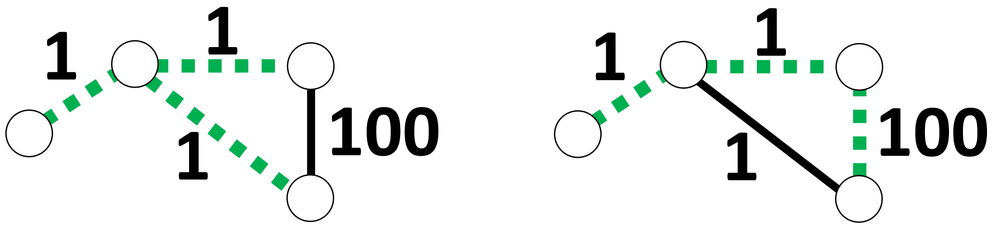
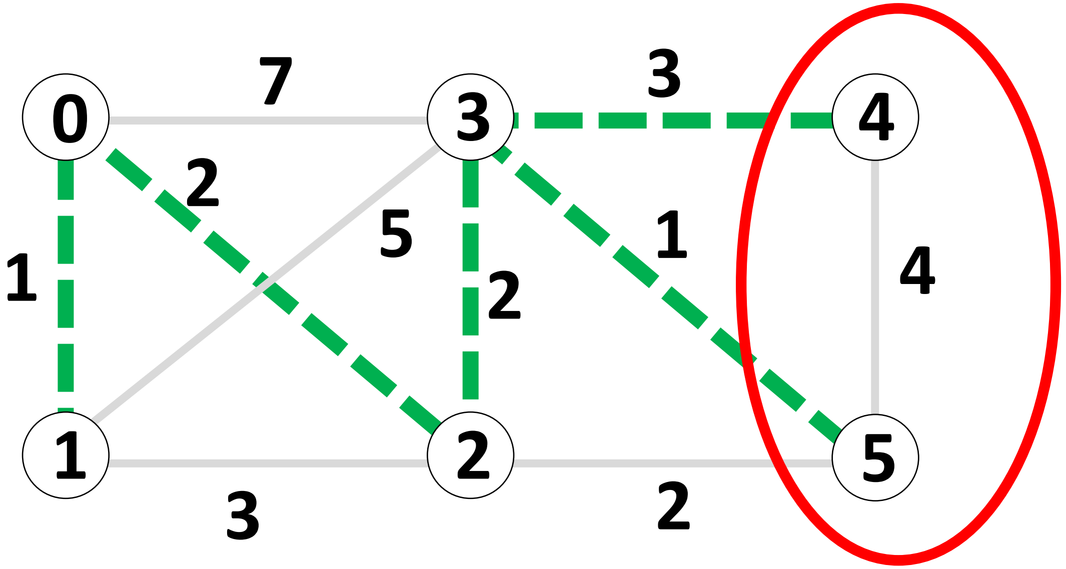


Minimum spanning tree – spanning tree whose sum of edge costs is the minimum possible value.



Minimum Spanning Tree

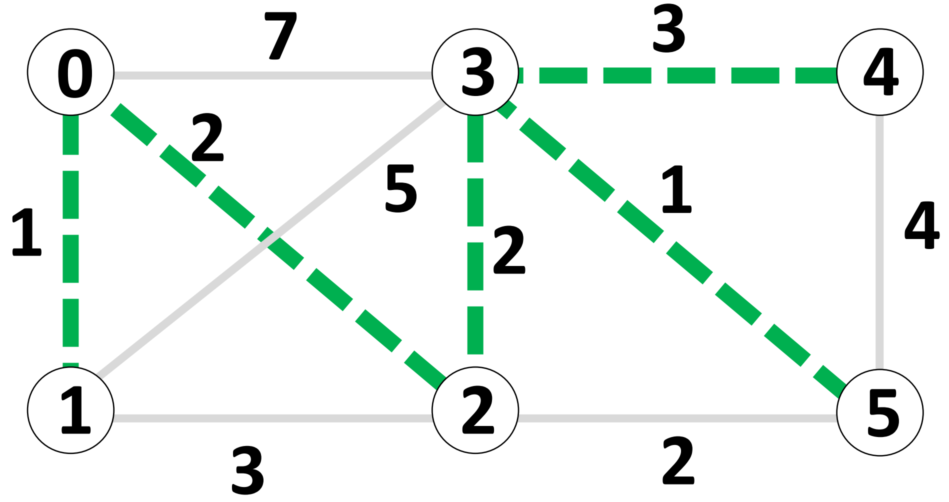


Does it ever make sense to have a cycle?

MST Goal: Connect all vertices to each other with a minimum weight subset of edges.

Must be a tree!

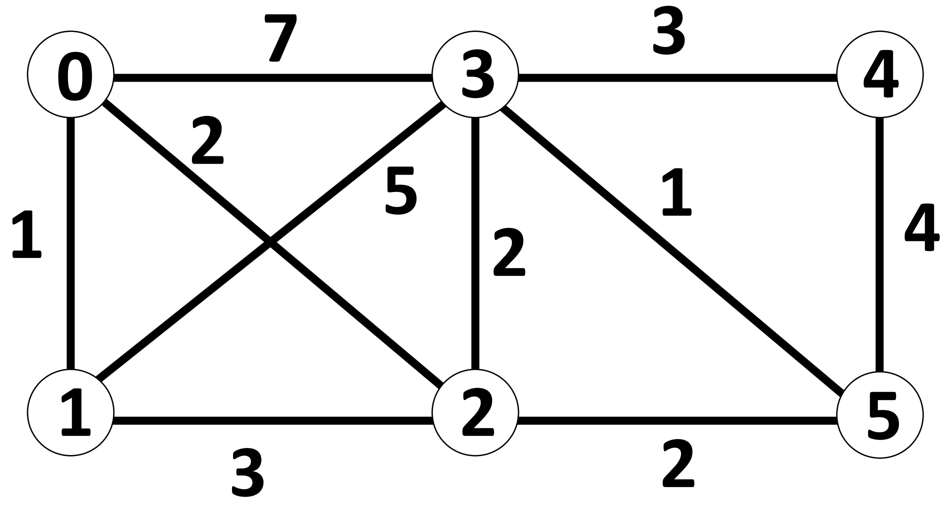
Minimum Spanning Tree



How to find MSTs?

Kruskal’s MST Algorithm

At each iteration, add the edge with smallest weight, that does not create a cycle.



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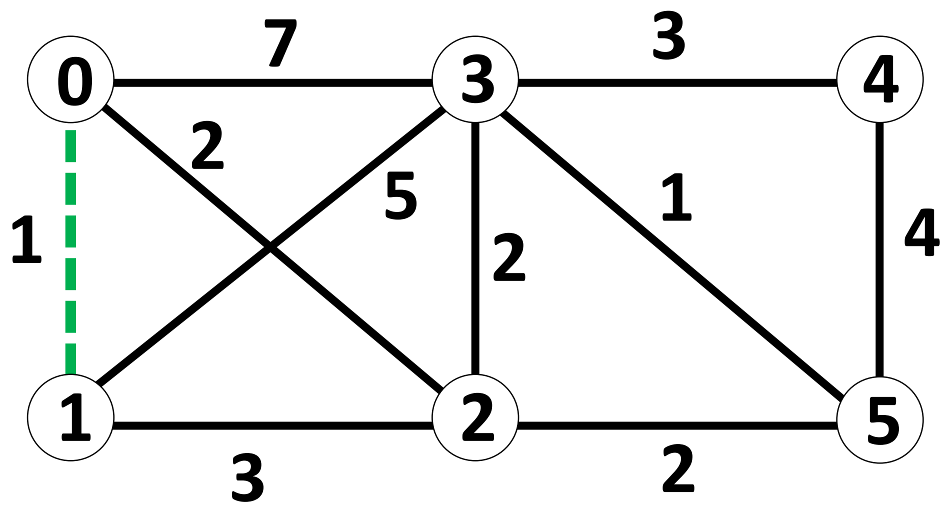

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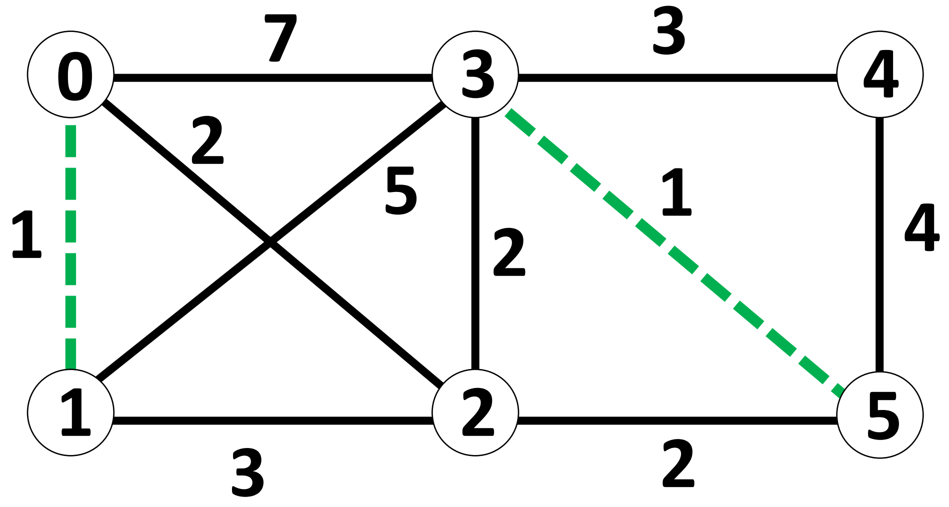

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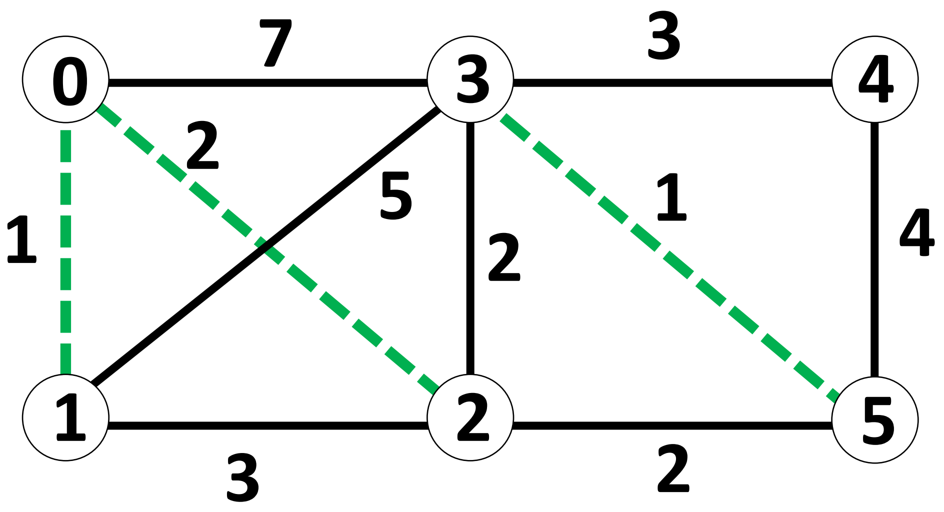

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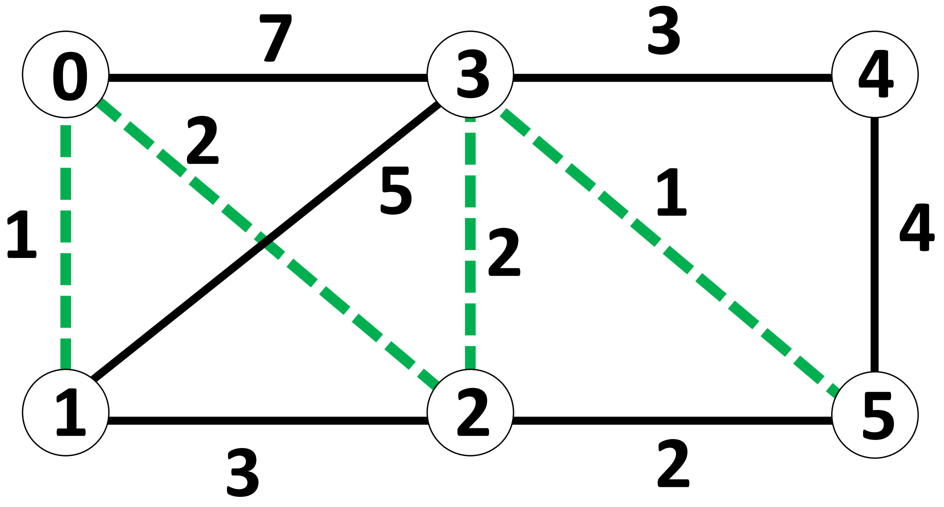

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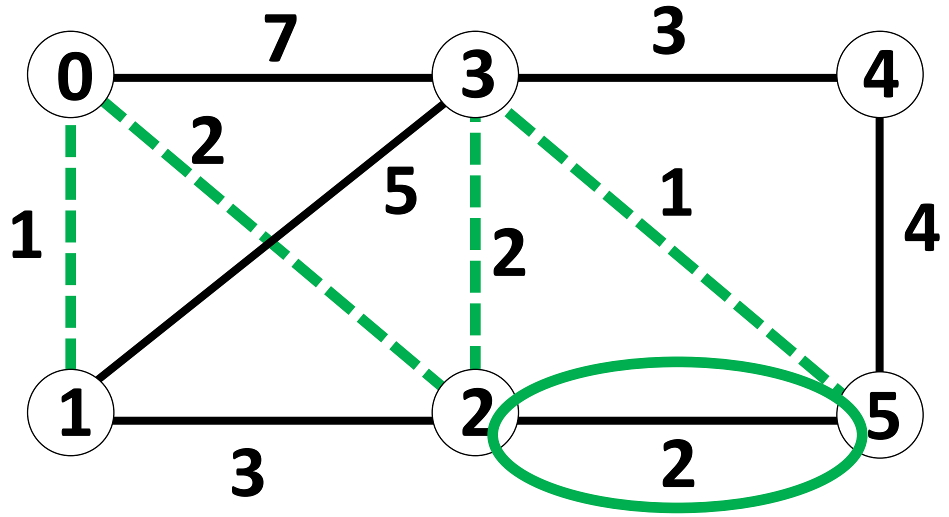

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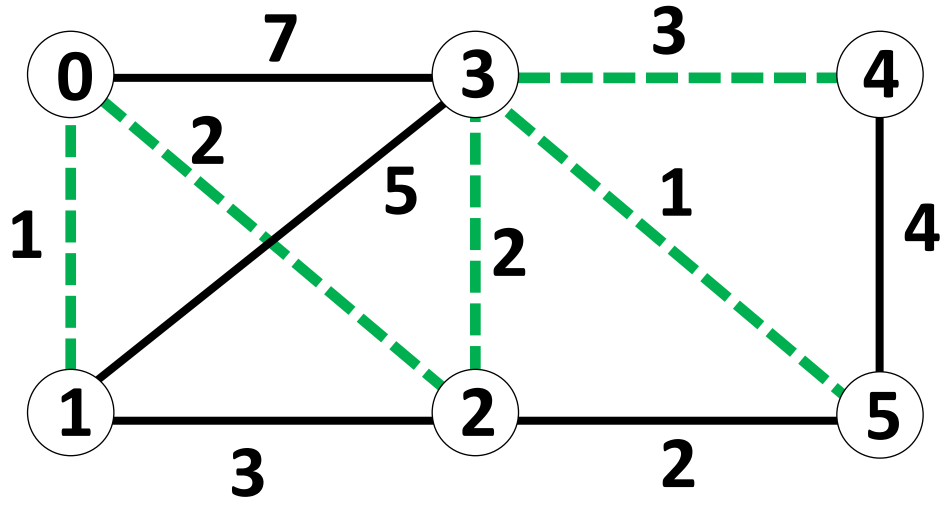

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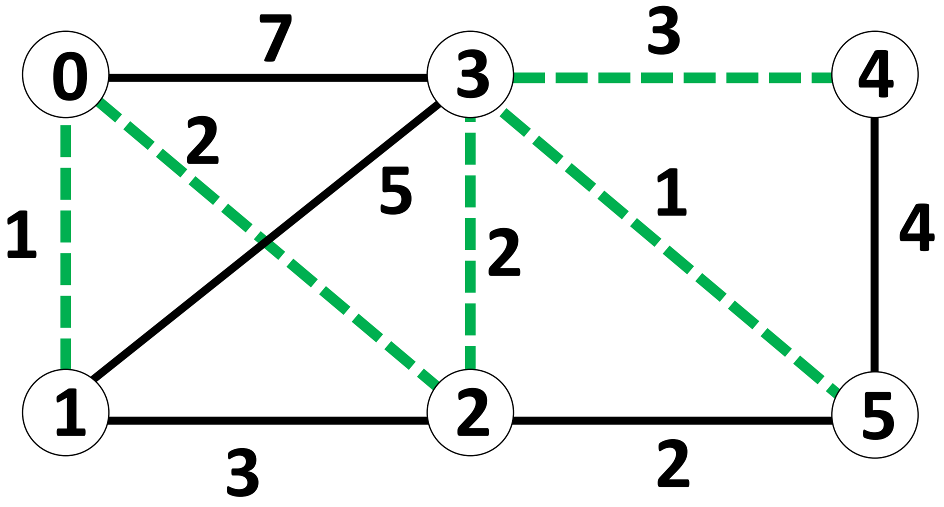

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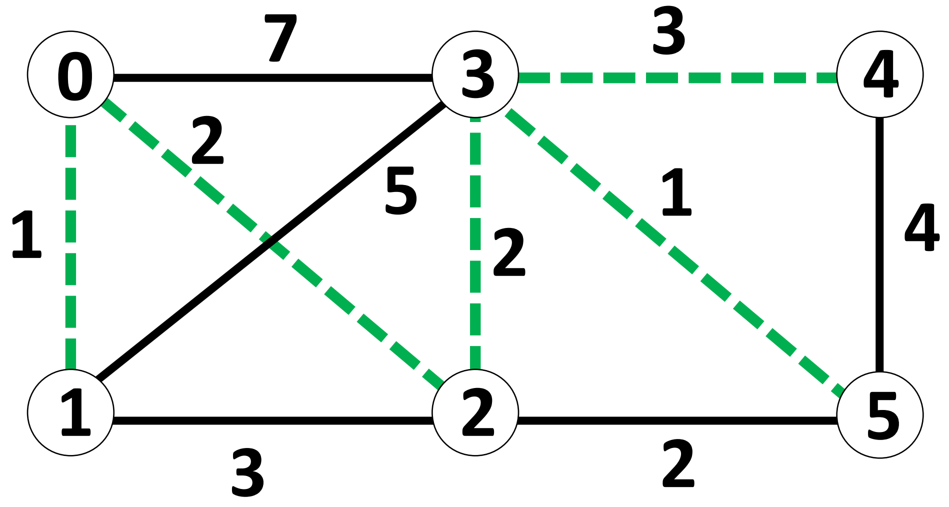
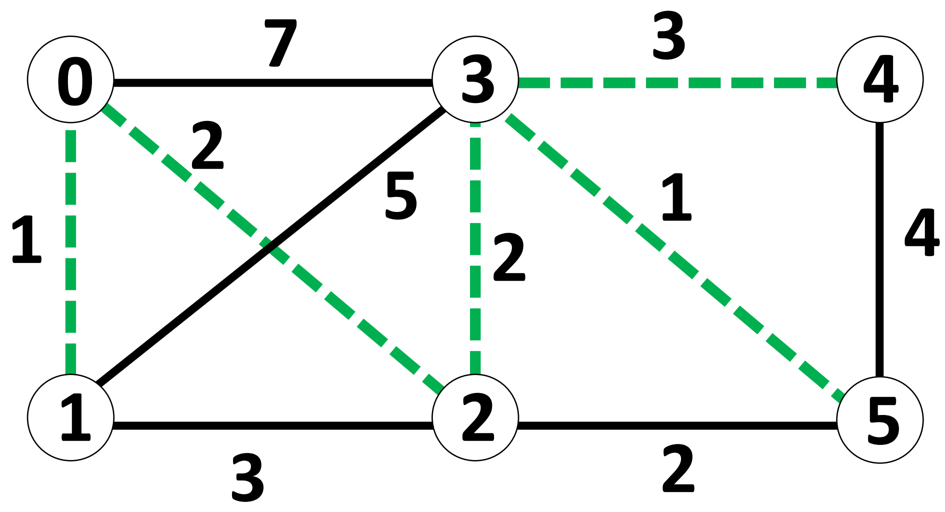


Three questions for any algorithm:

Kruskal’s MST Algorithm

At each iteration, add the edge with smallest weight, that does not create a cycle.

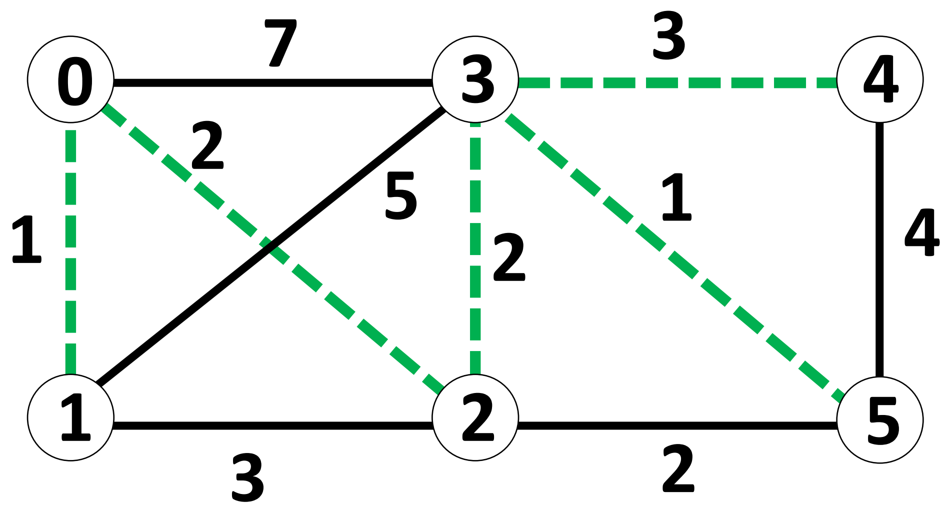


Three questions for any algorithm:

1. **Running time?**
2. **Validity/correctness?**
3. **Performance/accuracy?**

Kruskal’s MST Algorithm

At each iteration, add the edge with smallest weight, that does not create a cycle.



Three questi for any al m:

1. Run e?

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How to implement this?

Minimum Spanning Trees

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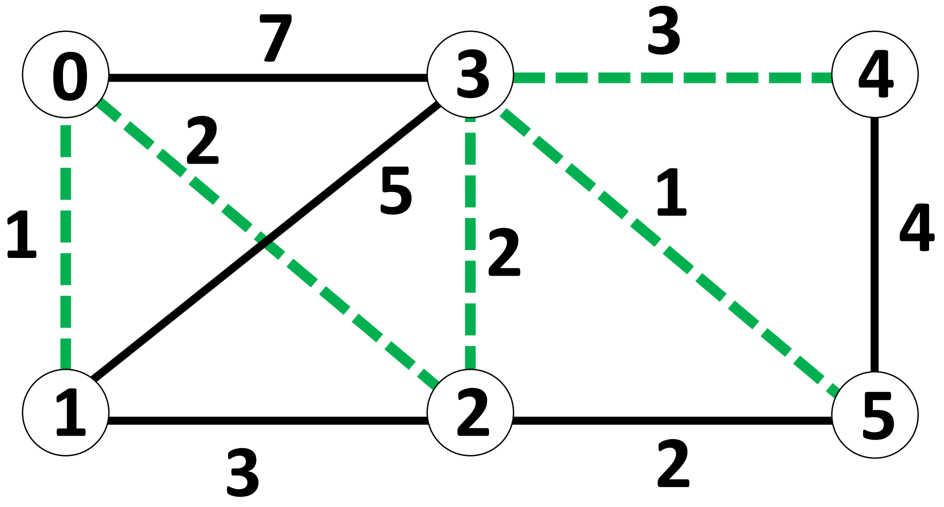

```
public MinimumSpanningTree(EdgeWeightedGraph graph) {

}
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public MinimumSpanningTree(EdgeWeightedGraph graph) {
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    // Get the set of edges.















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public MinimumSpanningTree(EdgeWeightedGraph graph) {
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    // Get the set of edges.
    HashSet<Edge> Edges = graph.getEdges();












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public MinimumSpanningTree(EdgeWeightedGraph graph) {
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    // Get the set of edges, in order of increasing weight.
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public MinimumSpanningTree(EdgeWeightedGraph graph) {
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    // Get the set of edges, in order of increasing weight.
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        edgeQueue.add(edge);
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    HashSet<Edge> mst = new HashSet<>();




    PriorityQueue<Edge> edgeQueue = new PriorityQueue<>();
    for (Edge edge : graph.getEdges()) {
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    }

    // Run Kruskal’s algorithm.




}
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    PriorityQueue<Edge> edgeQueue = new PriorityQueue<>();
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    while (!edgeQueue.isEmpty()) {








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Need to check if adding edge adds a loop!

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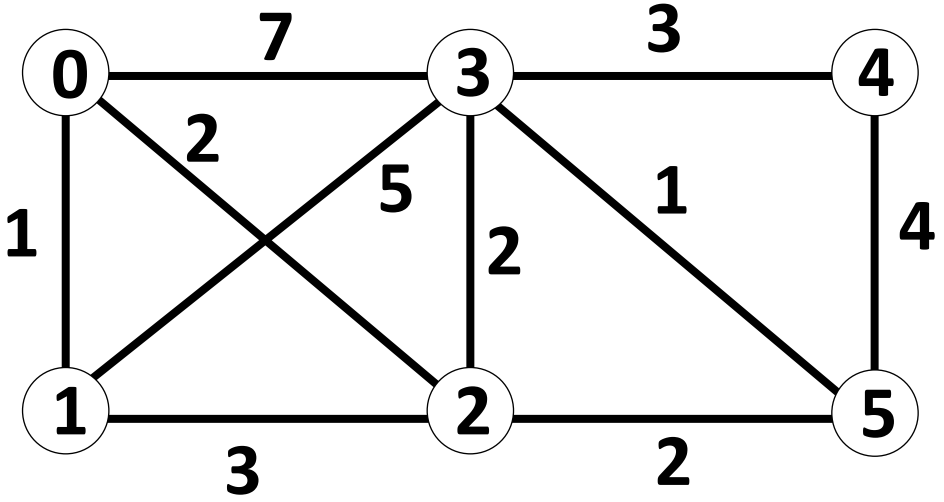
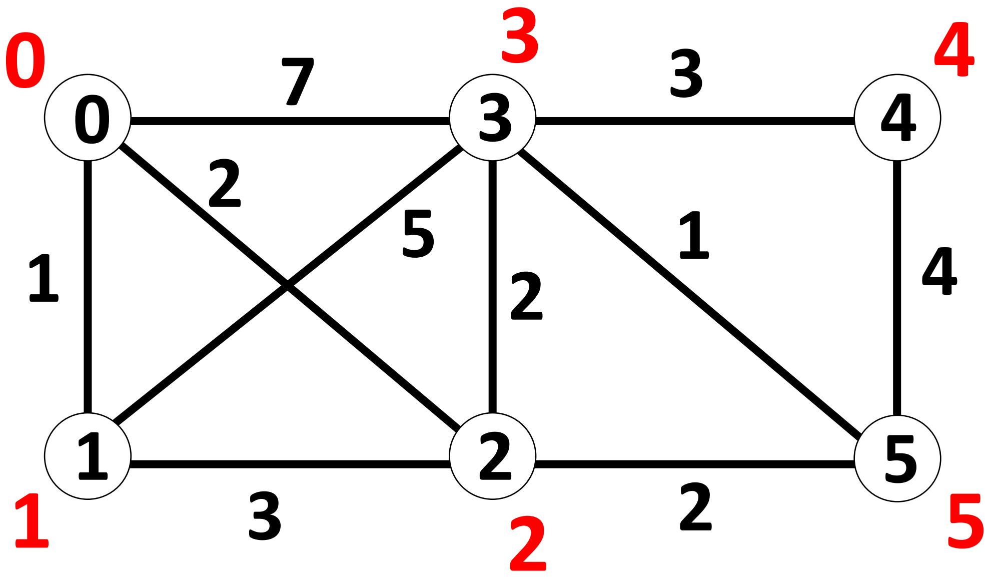
How?

Cycle Finding



How can we determine if adding an edge puts in a cycle?

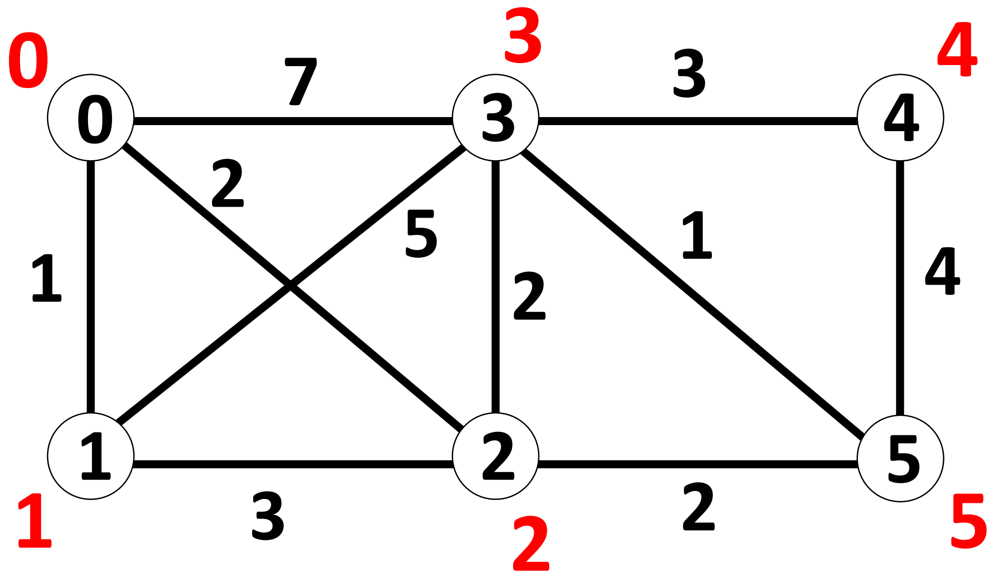
Cycle Finding



Rules: Only add edge if vertices have different connected component markers.

Connected component (in the tree) marker.

Cycle Finding



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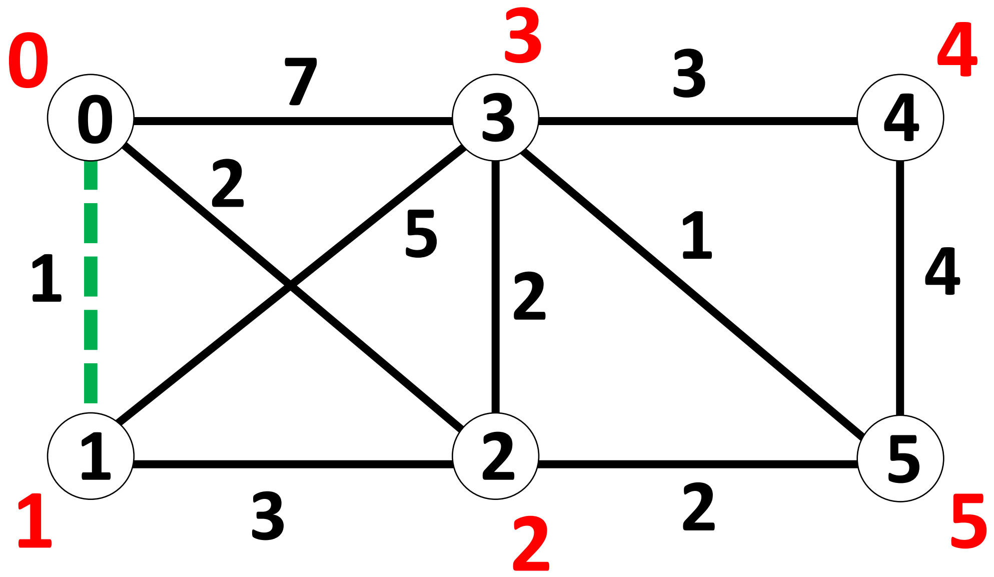

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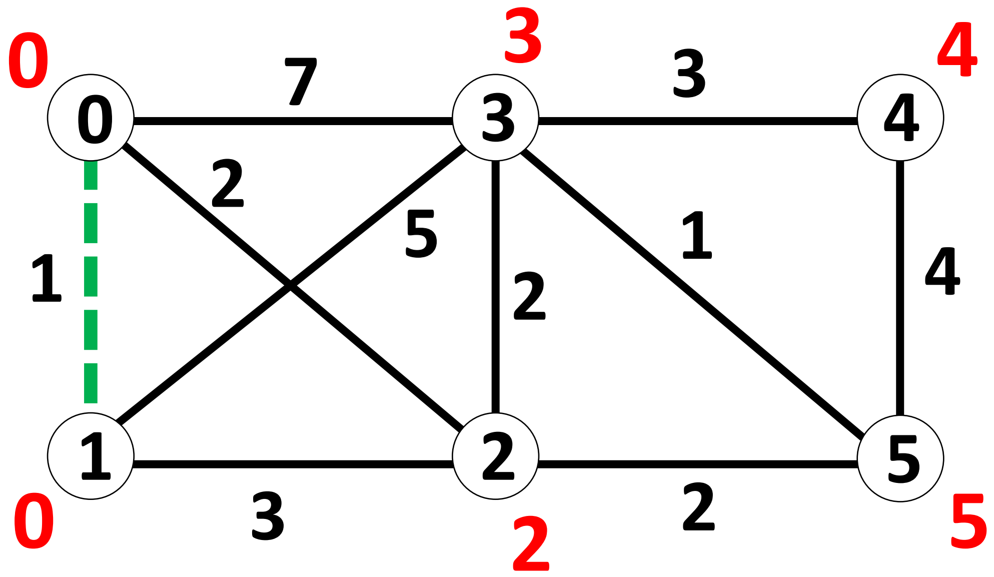

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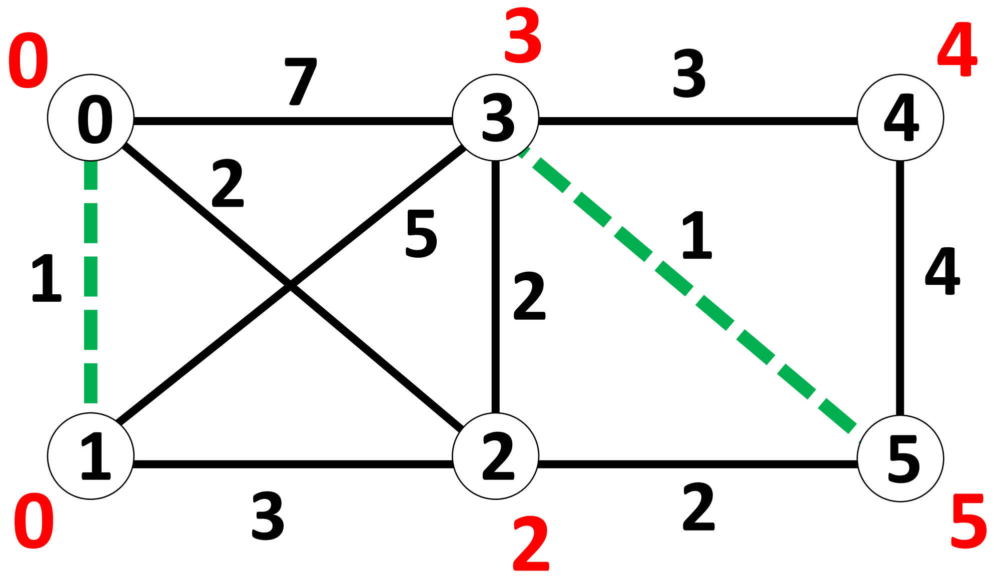

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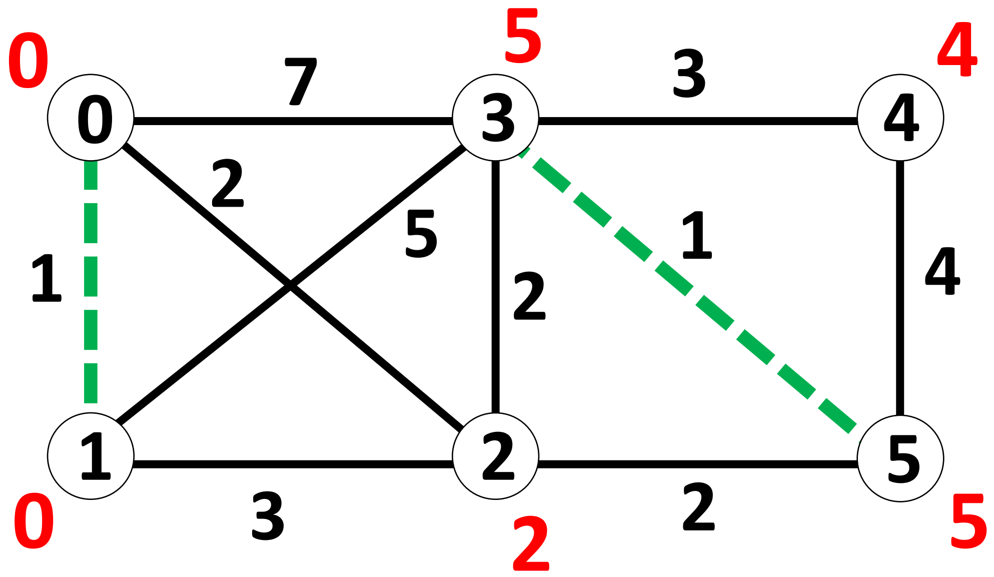

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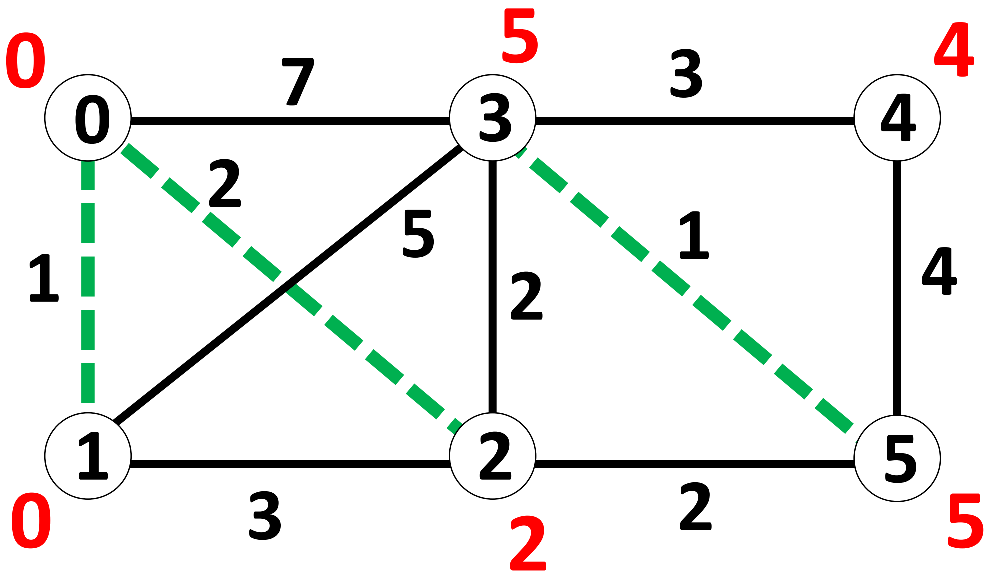

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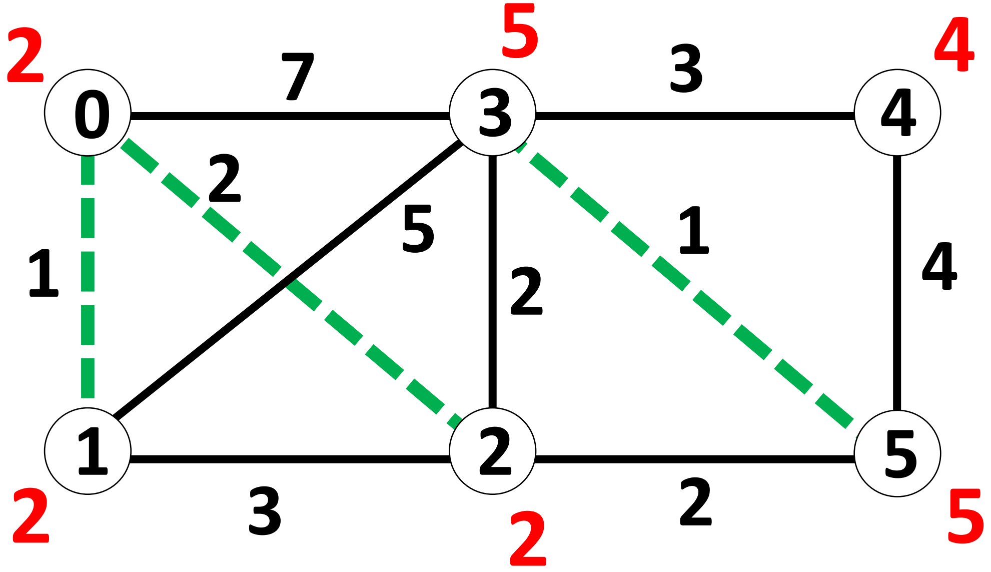

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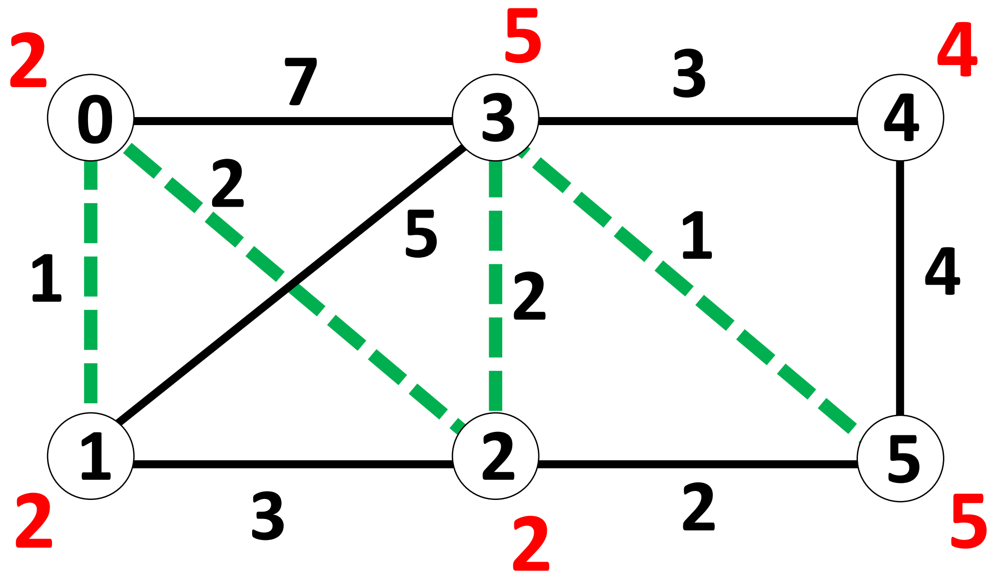

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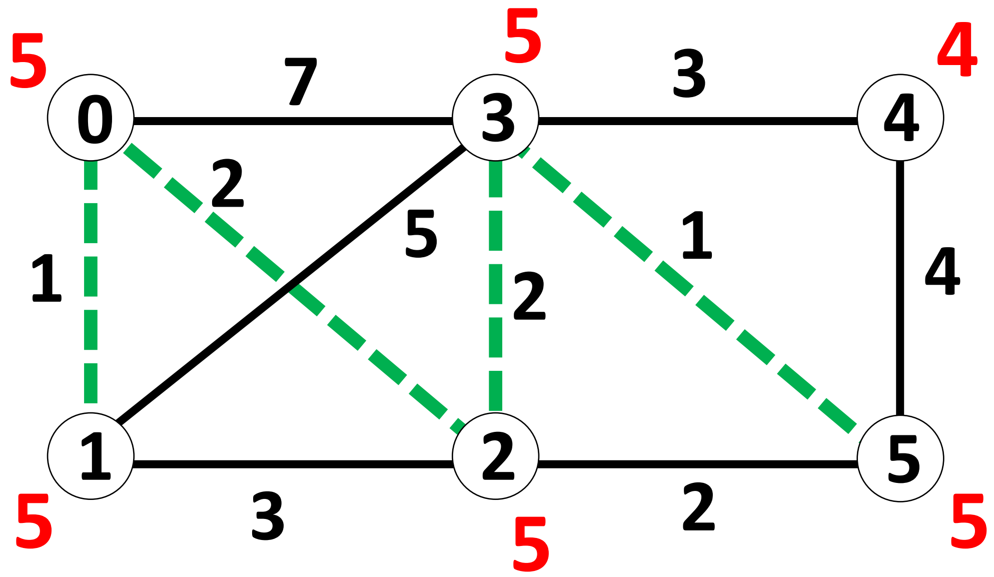

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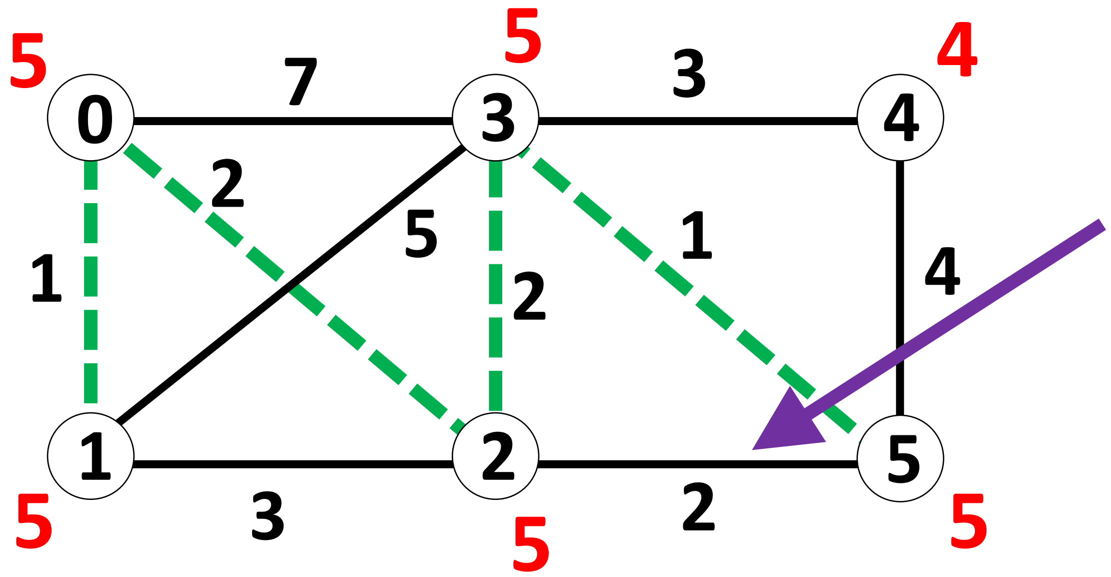

Why is adding this edge a bad idea?

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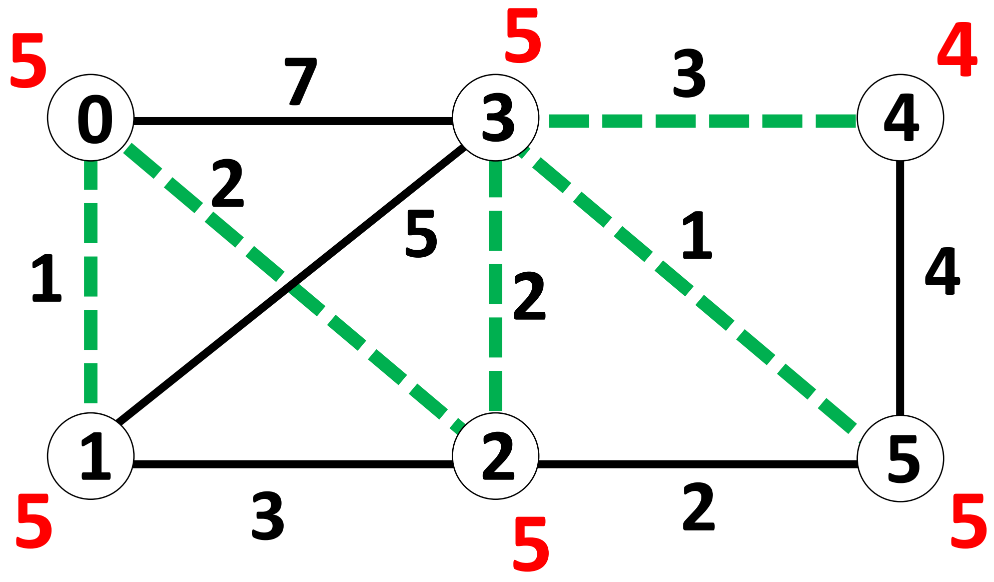

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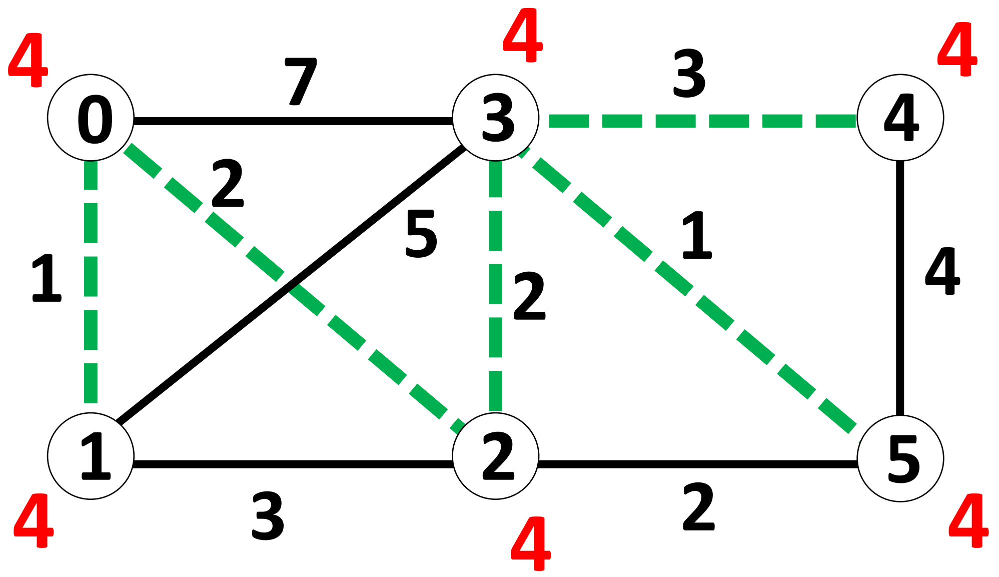

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public MinimumSpanningTree(EdgeWeightedGraph graph) {
    HashSet<Edge> mst = new HashSet<>();

    PriorityQueue<Edge> edgeQueue = new PriorityQueue<>();
    for (Edge edge : graph.getEdges()) {
        edgeQueue.add(edge);
    }

    while (!edgeQueue.isEmpty()) {
        Edge edge = edgeQueue.poll();
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}   }
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Need to check if adding edge adds a loop!

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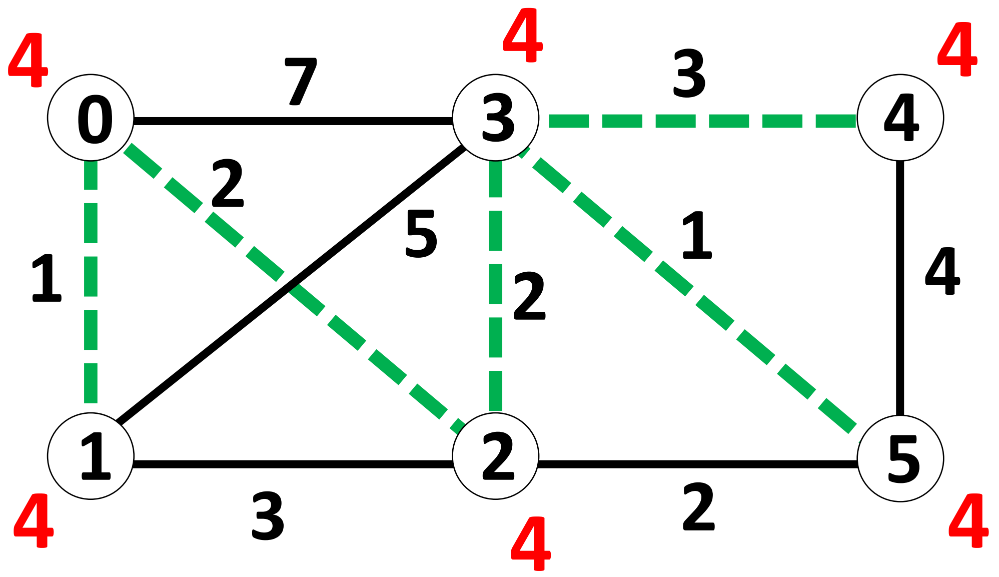
Cycle Finding



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        Edge edge = edgeQueue.poll();
        if (connectedComponentMarker[edge.getVertices()[0]]
                   != connectedComponentMarker[edge.getVertices()[1]]) {
            mst.add(edge);
            int newMarker = connectedComponentMarker[edge.getVertices()[0]];
            int oldMarker = connectedComponentMarker[edge.getVertices()[1]];



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    HashSet<Edge> mst = new HashSet<>();

    int[] connectedComponentMarker = new int[graph.getNumVertices()];
    for (int i = 0; i < connectedComponentMarker.length; i++) {
        connectedComponentMarker[i] = i;
    }

    PriorityQueue<Edge> edgeQueue = new PriorityQueue<>();
    for (Edge edge : graph.getEdges()) {
        edgeQueue.add(edge);
    }

    while (!edgeQueue.isEmpty()) {
        Edge edge = edgeQueue.poll();
        if (connectedComponentMarker[edge.getVertices()[0]]
                   != connectedComponentMarker[edge.getVertices()[1]]) {
            mst.add(edge);
            int newMarker = connectedComponentMarker[edge.getVertices()[0]];
            int oldMarker = connectedComponentMarker[edge.getVertices()[1]];
            for (int i = 0; i < connectedComponentMarker.length; i++) {



}   }   }   }
```
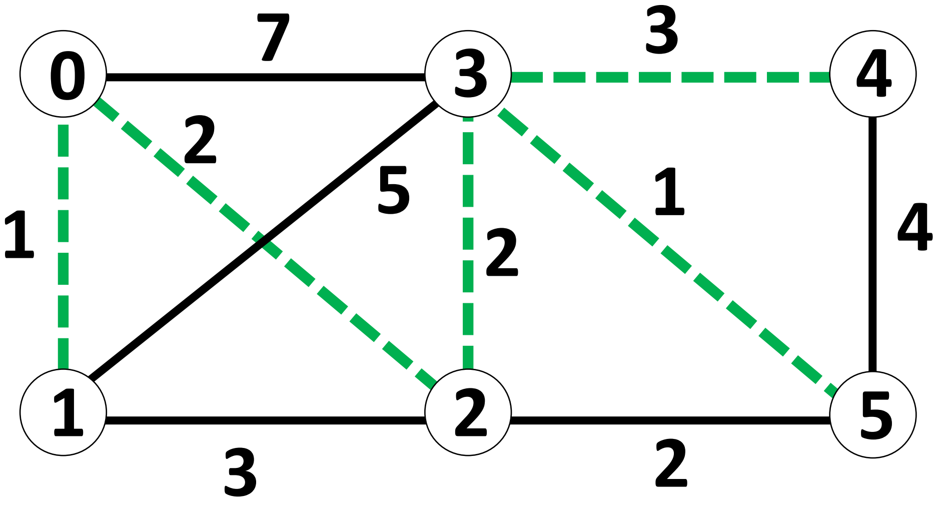

```
public MinimumSpanningTree(EdgeWeightedGraph graph) {
    HashSet<Edge> mst = new HashSet<>();

    int[] connectedComponentMarker = new int[graph.getNumVertices()];
    for (int i = 0; i < connectedComponentMarker.length; i++) {
        connectedComponentMarker[i] = i;
    }

    PriorityQueue<Edge> edgeQueue = new PriorityQueue<>();
    for (Edge edge : graph.getEdges()) {
        edgeQueue.add(edge);
    }

    while (!edgeQueue.isEmpty()) {
        Edge edge = edgeQueue.poll();
        if (connectedComponentMarker[edge.getVertices()[0]]
                   != connectedComponentMarker[edge.getVertices()[1]]) {
            mst.add(edge);
            int newMarker = connectedComponentMarker[edge.getVertices()[0]];
            int oldMarker = connectedComponentMarker[edge.getVertices()[1]];
            for (int i = 0; i < connectedComponentMarker.length; i++) {
                if (connectedComponentMarker[i] == oldMarker) {
                    connectedComponentMarker[i] = newMarker;
}   }   }   }   }
```

Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.



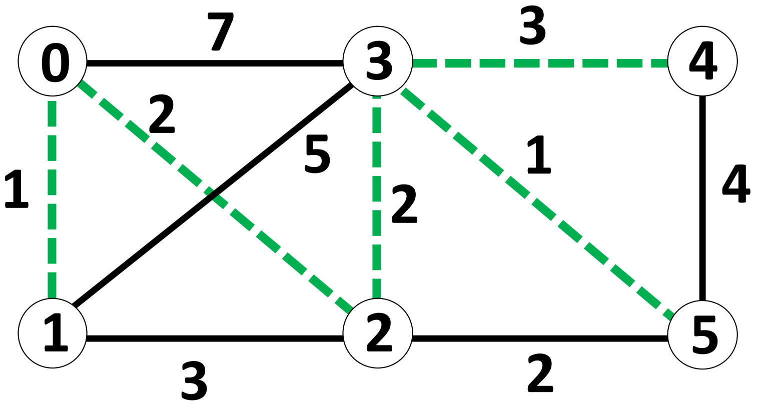
Is the solution valid (i.e., is the output a spanning tree)?

Is the solution optimal (i.e., is the output a <u>minimum</u> spanning tree)?

Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.

Proof of validity: Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.



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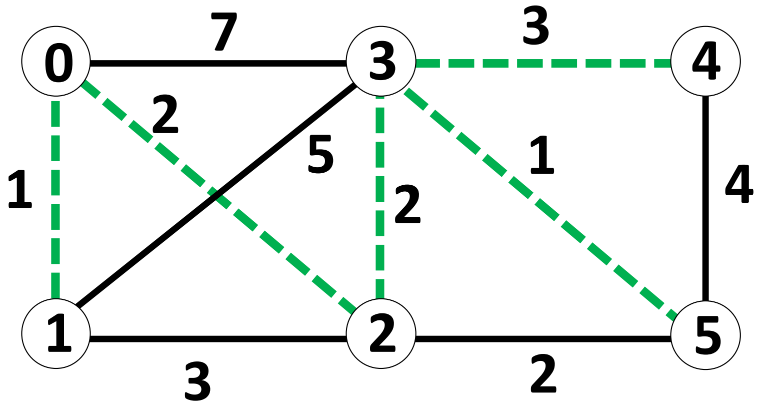
At each iteration, add smallest weight edge that doesn’t create a cycle.

<u>Proof of validity:</u> Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

T is a tree because…?

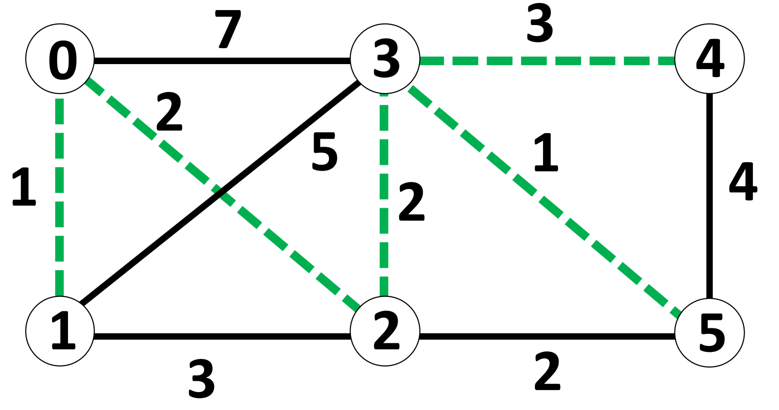


Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.

Proof of validity: Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

T is a tree because it is connected (otherwise we could have added more edges without creating cycles) and there are no cycles.



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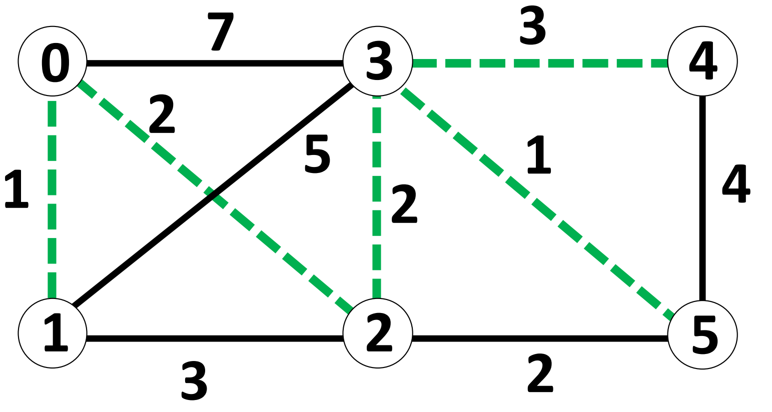
T spans G because…?



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At each iteration, add smallest weight edge that doesn’t create a cycle.

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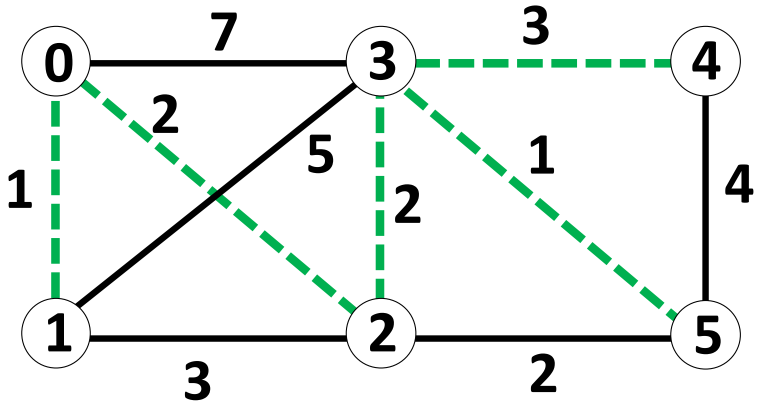
T is a tree because it is connected (otherwise we could have added more edges without creating cycles) and there are no cycles.

T spans G because if it did not, we could have added more edges to connected unreached nodes without creating cycles.



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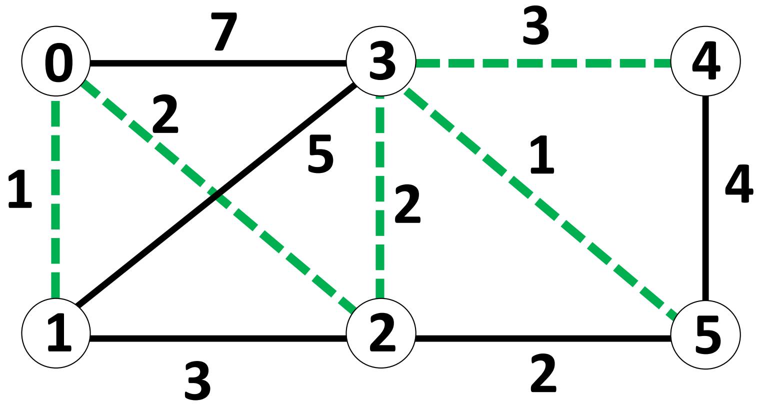
$\therefore T$ is a spanning tree of G



Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.

Proof of optimality: Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

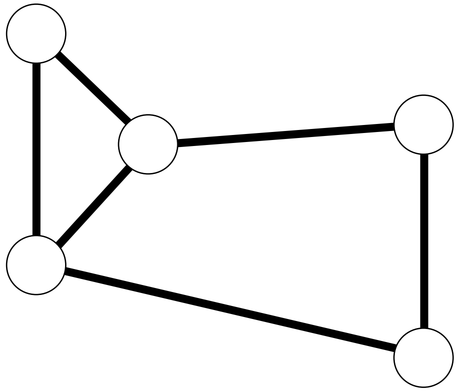
T is an MST because???

MST Cut Property

Assume unique edge costs.

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V \backslash S$ is part of every MST.

Proof:



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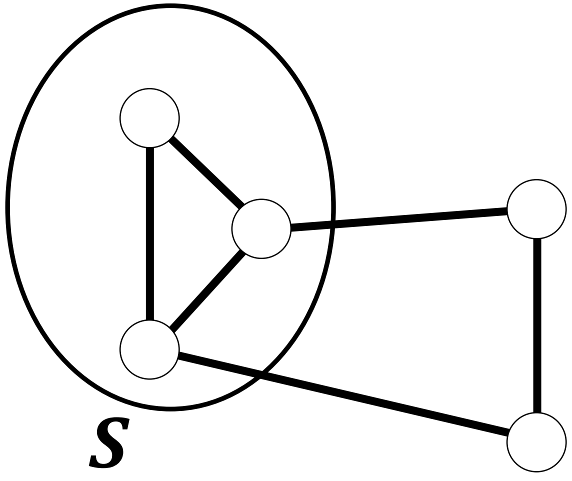
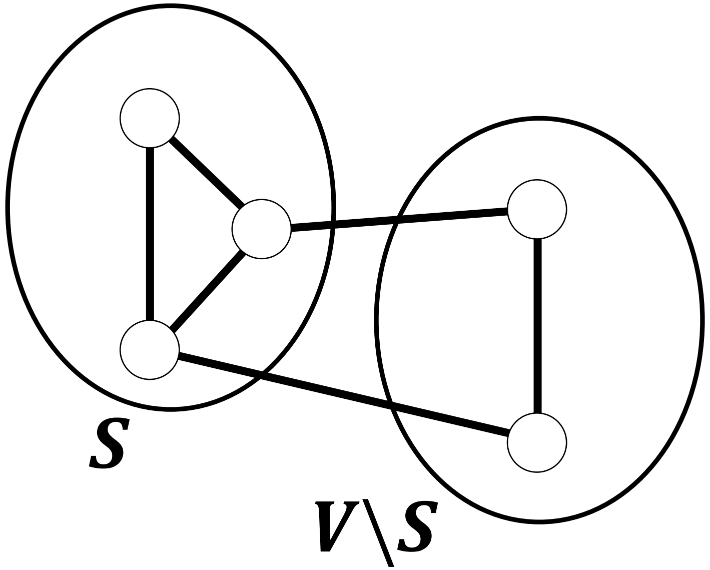


MST Cut Property

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V\setminus S$ is part of every MST.

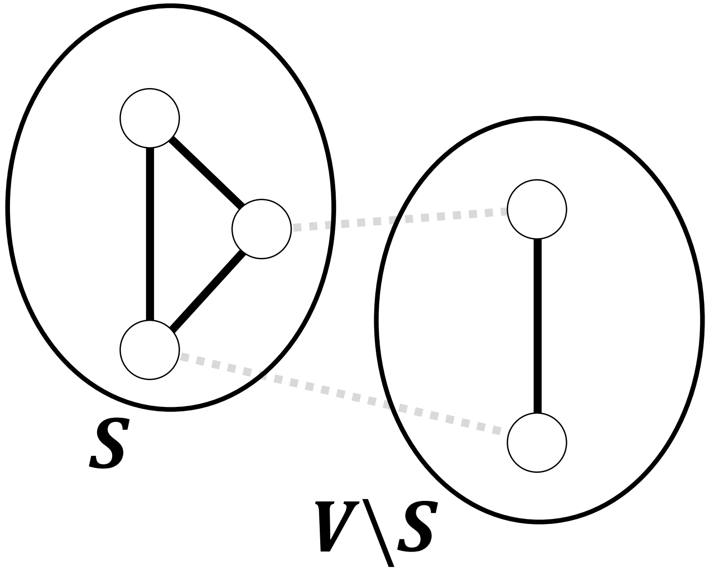
Proof:



MST Cut Property

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V\backslash S$ is part of every MST.

Proof: Any MST of G must include some edge between S and $V\backslash S$ (otherwise it would not be a tree).



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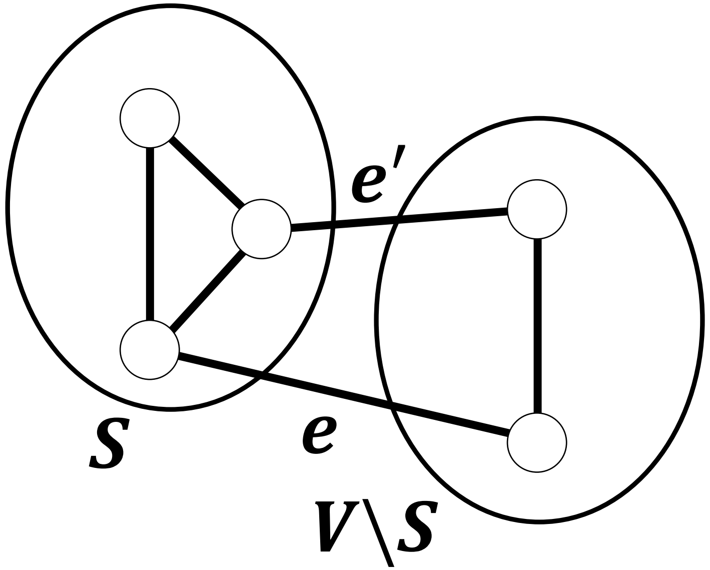

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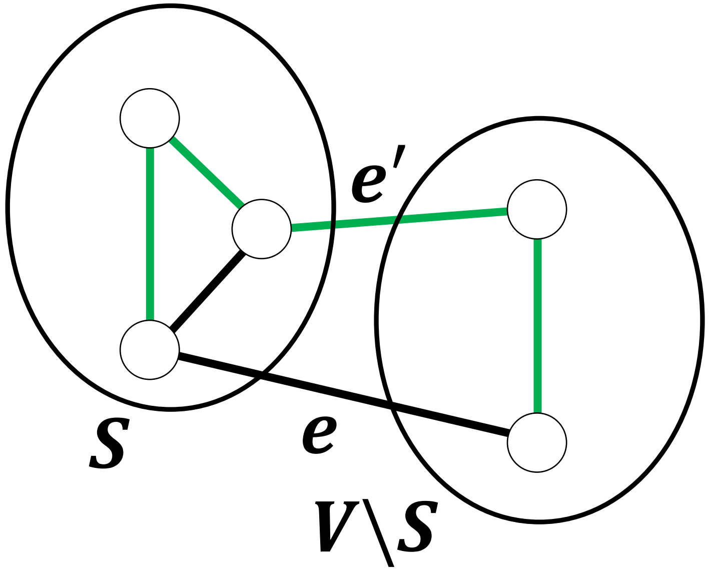

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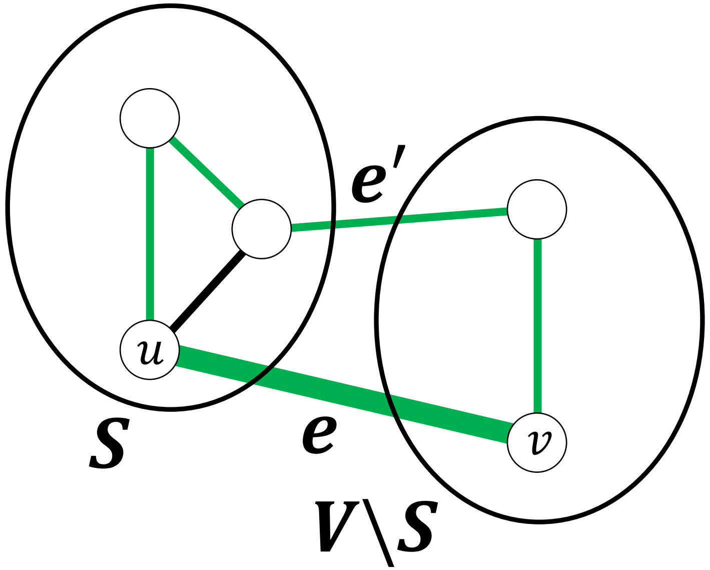
Suppose T is a ST that does not include e. Then:

1. $T \cup \{e\}$ must have a cycle.
2. That cycle must have another edge e' between S and $V\backslash S$.



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Let e be the cheapest edge between S and $V\backslash S$.

Suppose T is a ST that does not include e. Then:

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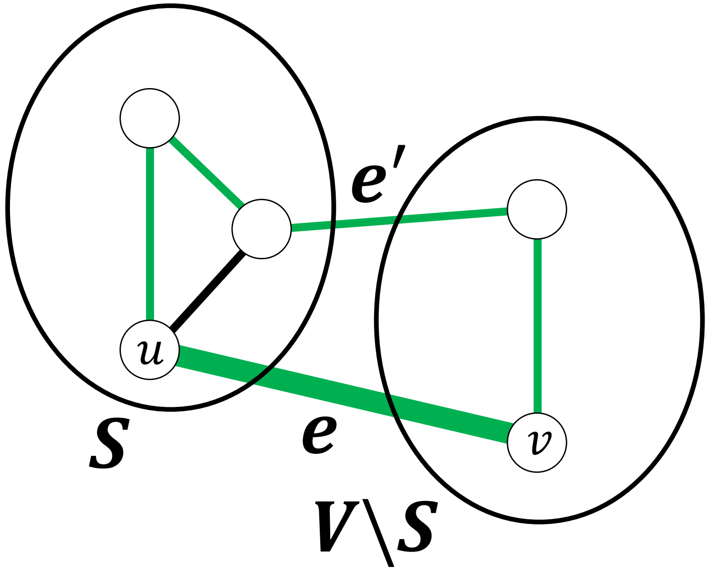
2. That cycle must have another edge e' between S and $V\backslash S$.



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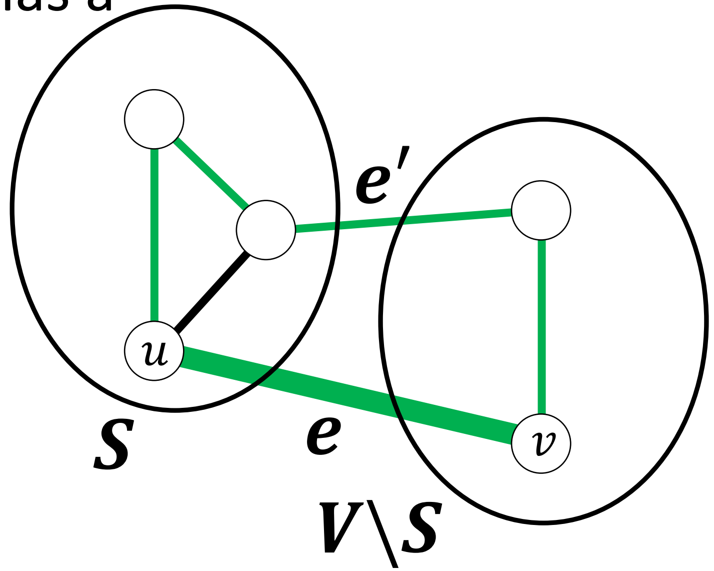
Let e be the cheapest edge between S and $V\backslash S$.

Suppose T is a ST that does not include e. Then:

1. $T \cup \{e\}$ must have a cycle. (Since spanning tree T already has a path between u and v, so adding e will create a cycle)
2. That cycle must have another edge e' between S and $V\backslash S$. Because?



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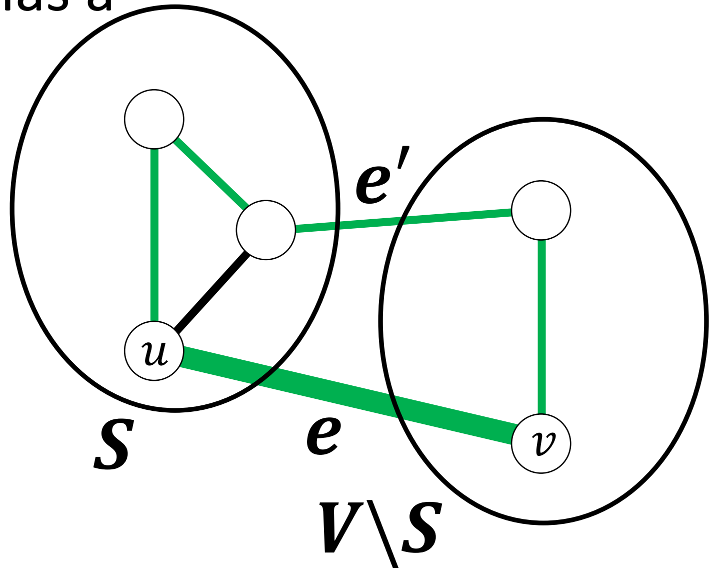

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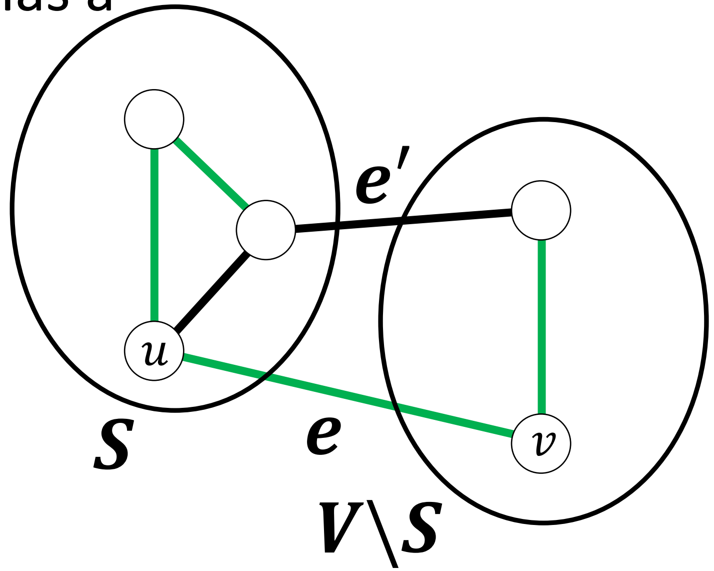

Thus, removing e' and including e results in a cheaper spanning tree!

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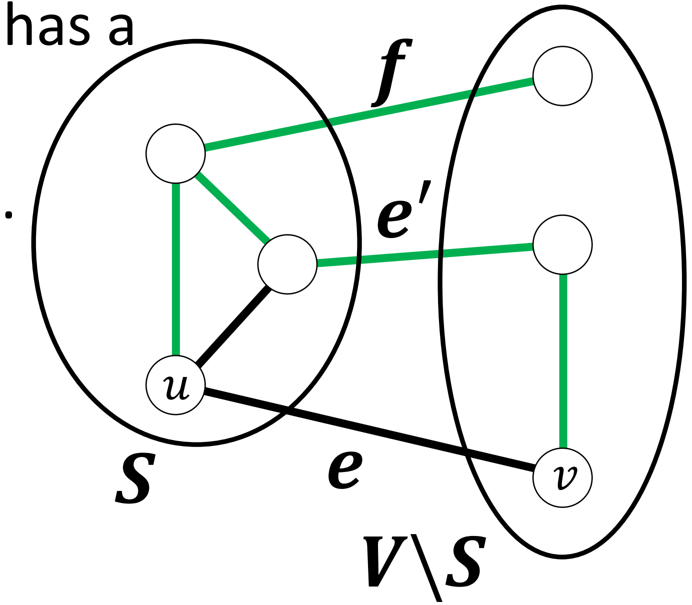

Need to make sure we pick the edge between S and $V\backslash S$ on the cycle!

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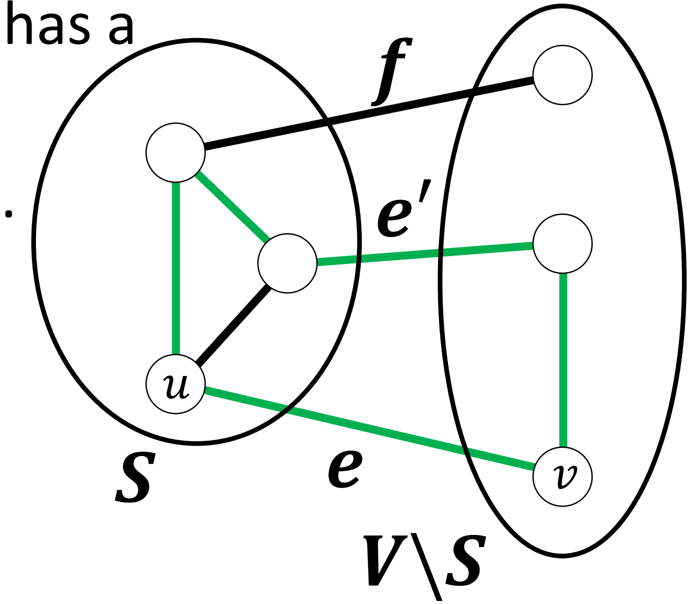

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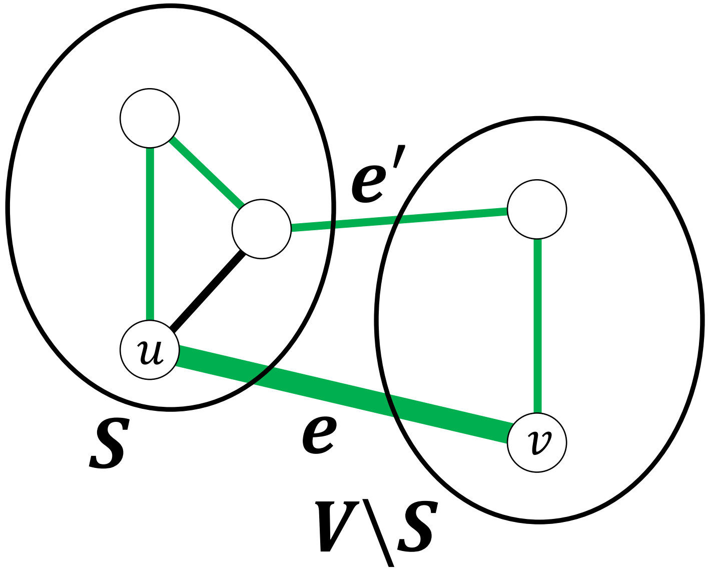

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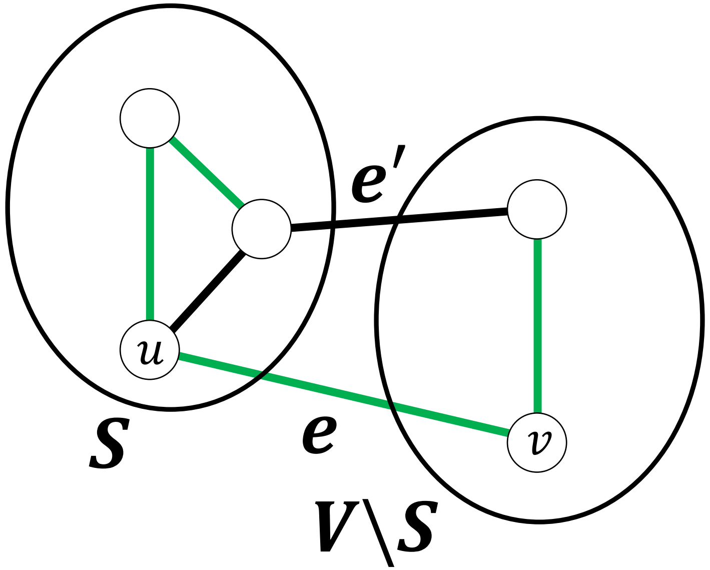
Let e be the cheapest edge between S and $V\backslash S$.

Suppose T is a ST that does not include e. $T \cup \{e\}$ must have a cycle and that cycle must have another edge e' between S and $V\backslash S$.

Remove e' to form $T' = T \cup \{e\}\backslash\{e'\}$.



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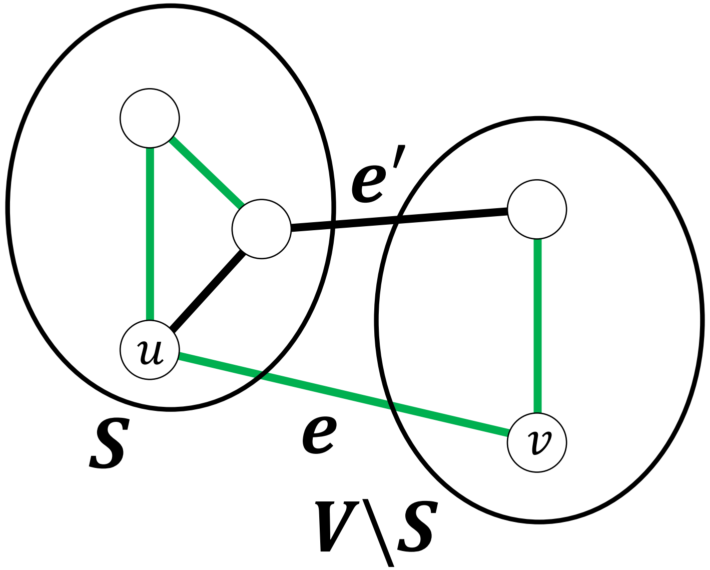
T' is a tree (removing edge from cycle cannot disconnect graph)



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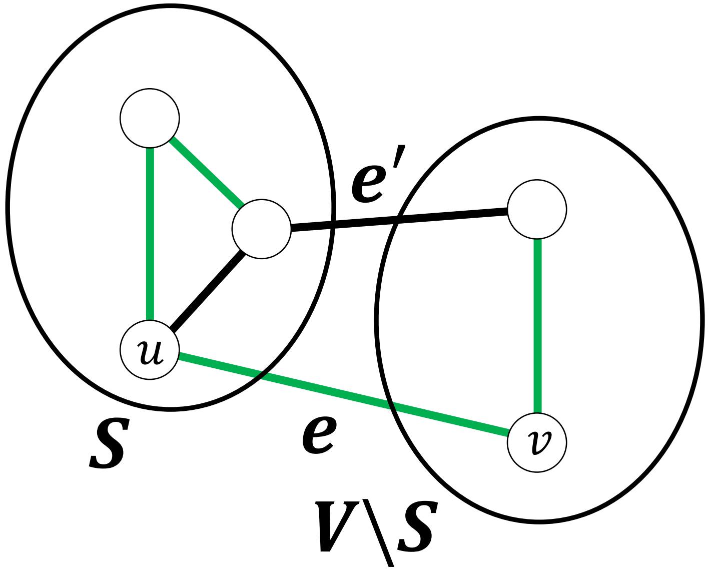
Remove e' to form $T' = T \cup \{e\} \backslash \{e'\}$.

T' is a tree (removing edge from cycle cannot disconnect graph)
T' spans V (same number of edges as ST T)



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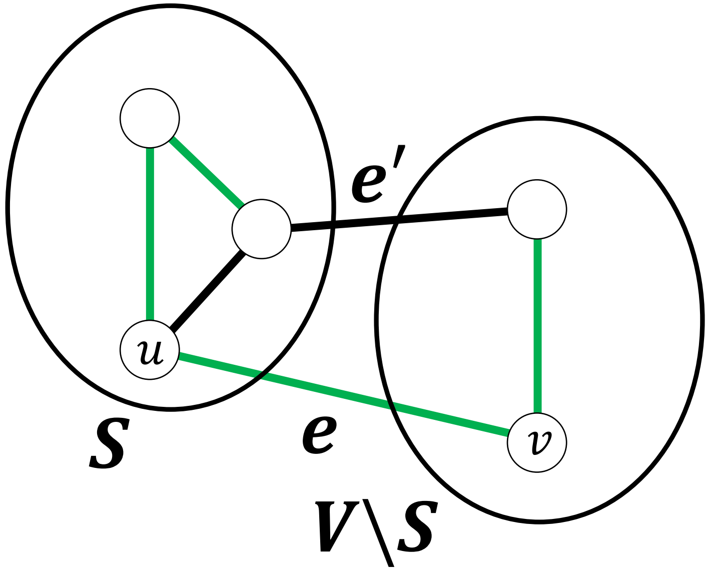
Remove e' to form $T' = T \cup \{e\}\backslash\{e'\}$.

T' is a tree (removing edge from cycle cannot disconnect graph)
T' spans V (same number of edges as ST T)
weight(T') = weight(T) + weight(e) – weight(e').



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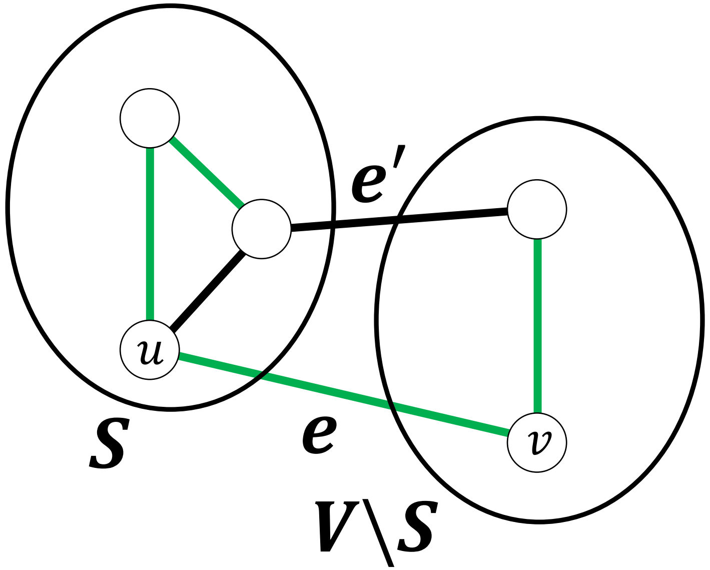
⇒ weight(T') < weight(T), since weight(e) < weight(e').



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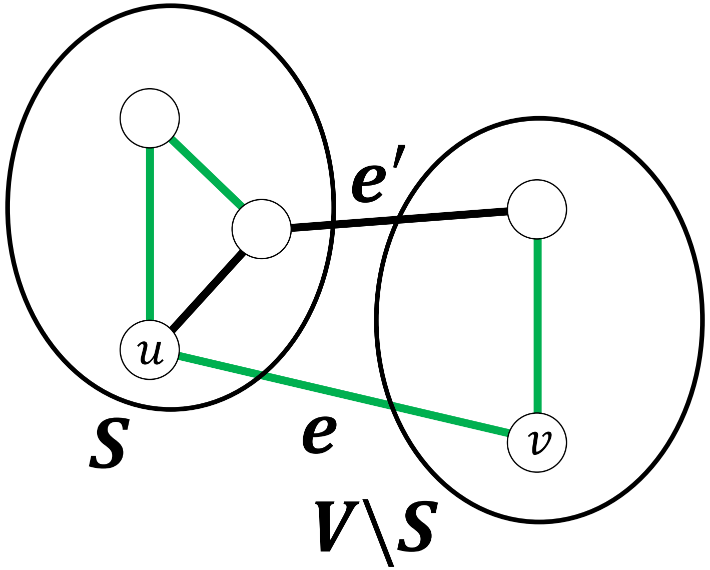
weight(T') = weight(T) + weight(e) – weight(e').
⇒ weight(T') < weight(T), since weight(e) < weight(e').
⇒ T' is a cheaper ST.



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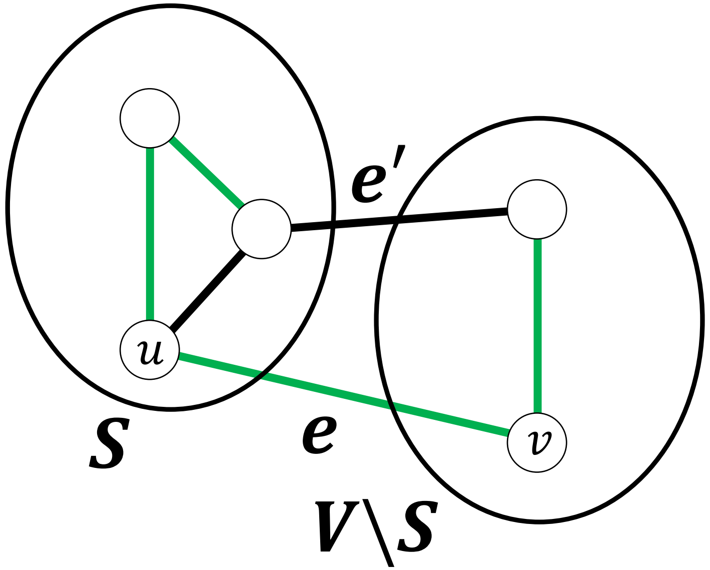
weight(T') = weight(T) + weight(e) – weight(e').
  $\Rightarrow$ weight(T') < weight(T), since weight(e) < weight(e').
  $\Rightarrow$ T' is a cheaper ST.

So, e is part of every MST.

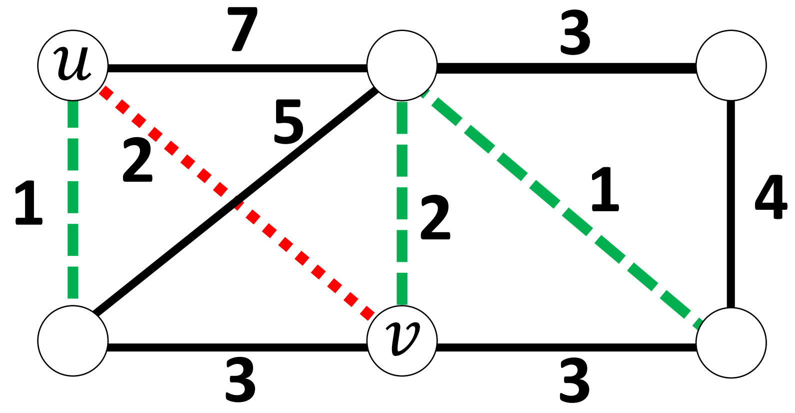


Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.

Proof of optimality: Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

Consider the iteration that edge $e = (u, v)$ is added by Kruskal’s algorithm.



Lemma: The cheapest edge between $S \subseteq V$ and $V \backslash S$ is part of every MST.

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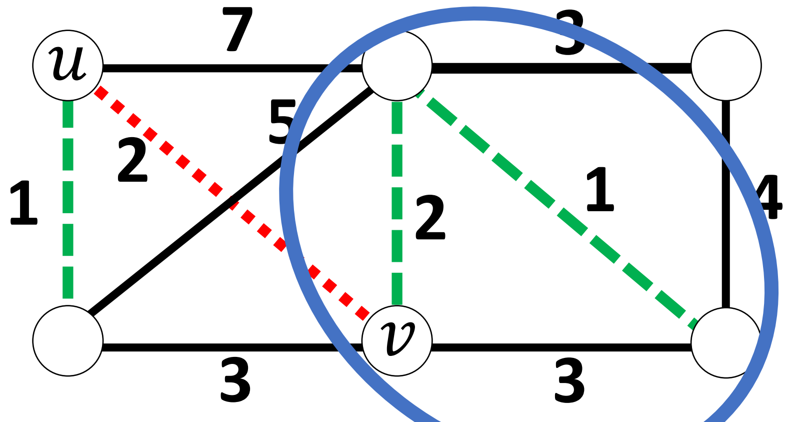
Consider the iteration that edge $e = (u, v)$ is added by Kruskal’s algorithm. Let S be the set of v and nodes connected to v. (or u and all nodes connected to u)



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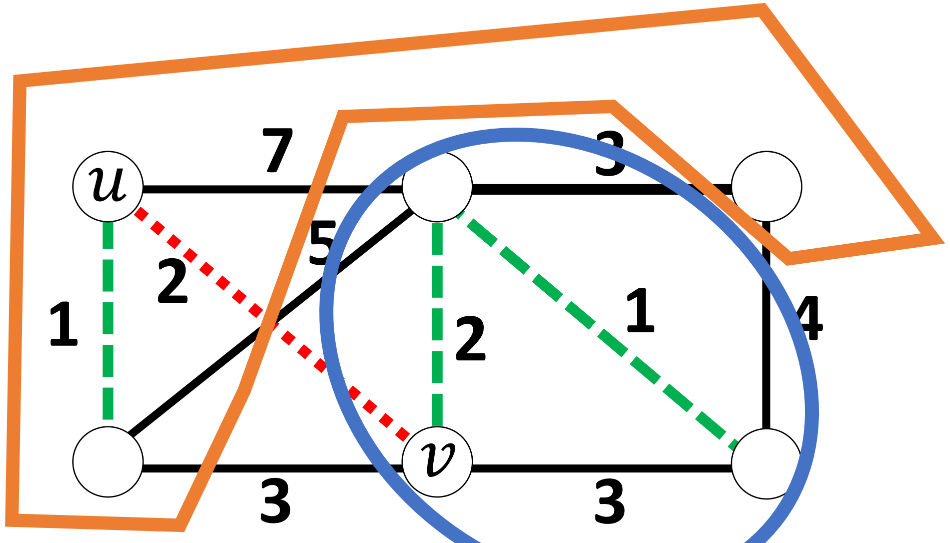
Kruskal’s MST Algorithm

At each iteration, add smallest weight edge that doesn’t create a cycle.

Proof of optimality: Let $G = (V, E)$ be connected, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

Consider the iteration that edge $e = (u, v)$ is added by Kruskal’s algorithm. Let S be the set of v and nodes connected to v. Clearly $v \in S$ and $u \in V \backslash S$ (otherwise adding e would have created a cycle).



Lemma: The cheapest edge between $S \subseteq V$ and $V \backslash S$ is part of every MST.

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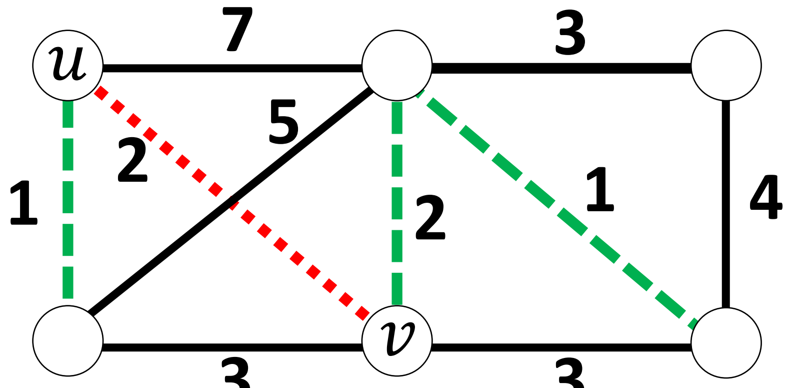

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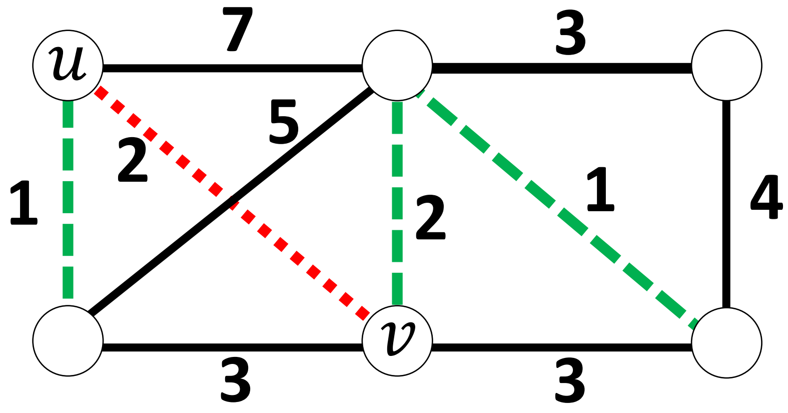

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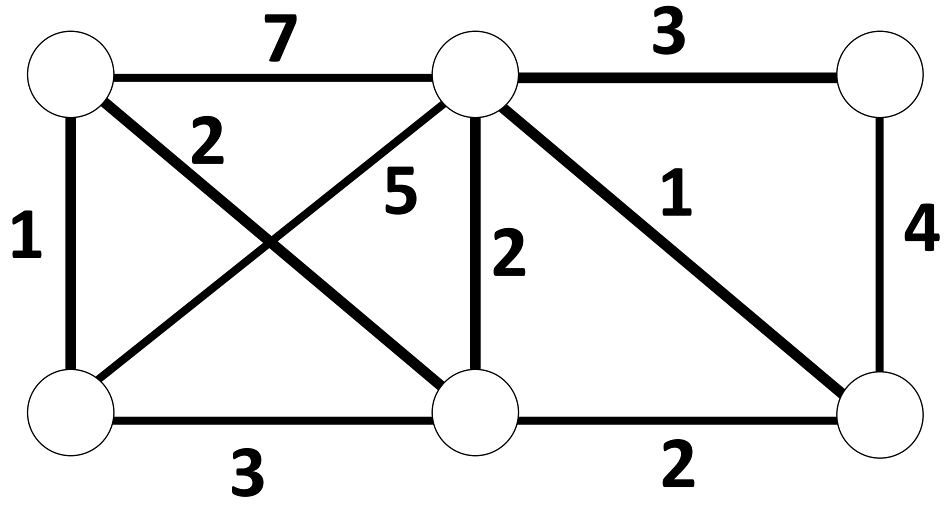
Prim’s MST Algorithm

Beginning at any node, add the node that can be connected as cheaply as possible to the tree we are building.



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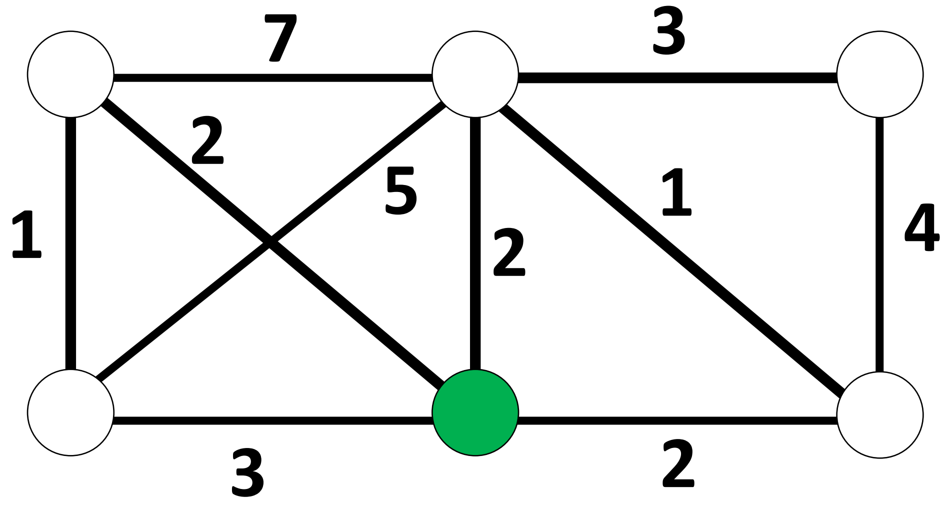

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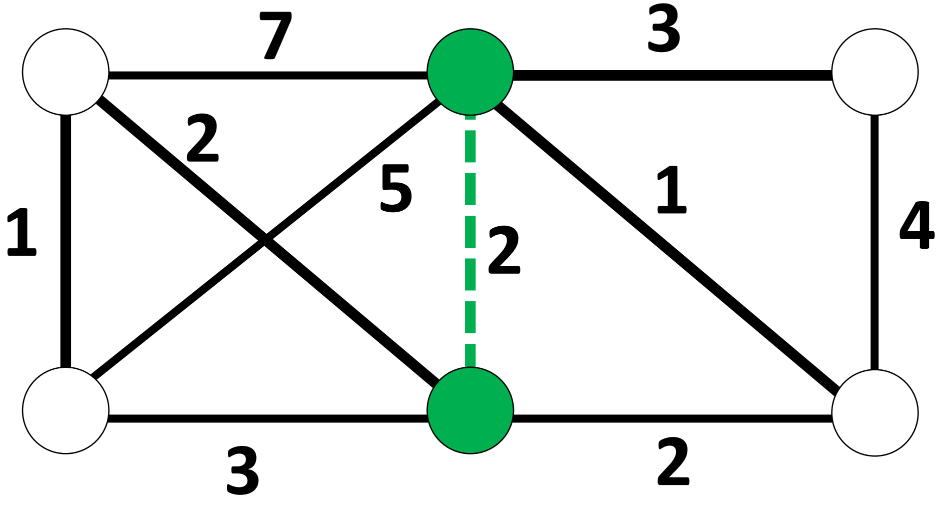

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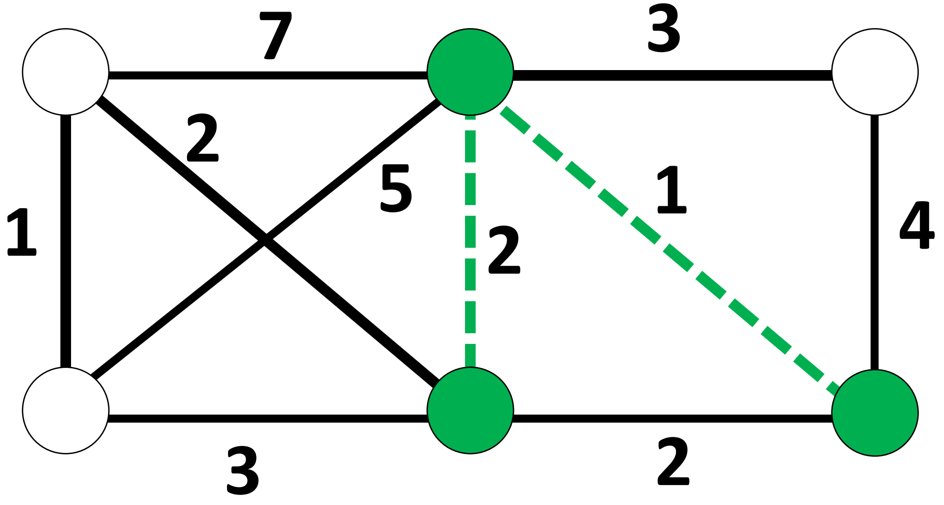

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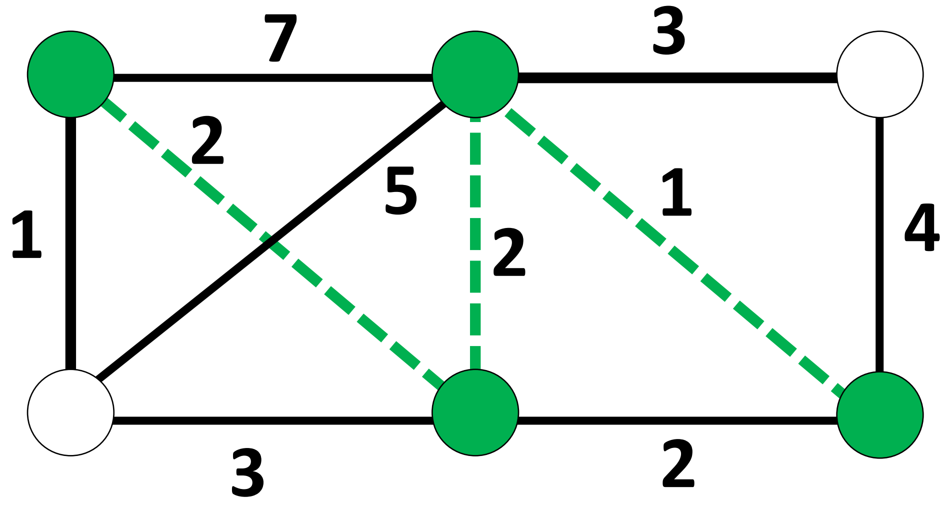

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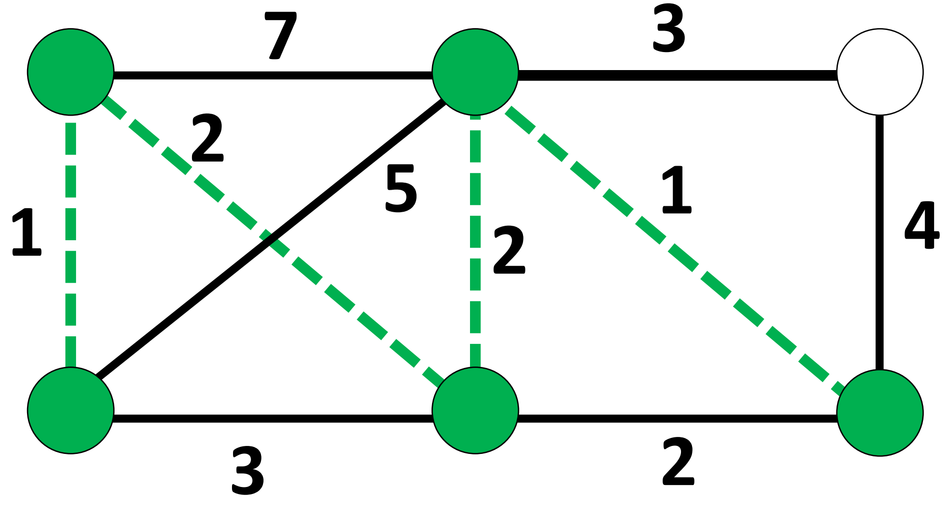

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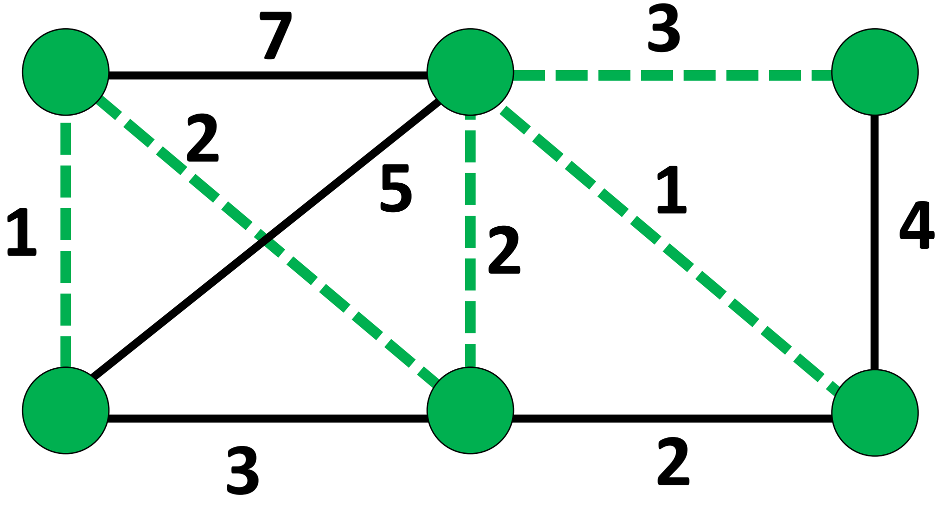
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