Entanglement of Random Walks

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Abstract

This paper explores the relationship between the variance of random walks and the Finite-Time Braiding Exponent (FTBE), a measure of entanglement. By simulating Brownian motion for multiple trajectories and encoding their crossings using braid group mathematics, the study provides insights into the complexity of particle movements, with applications to modeling ocean currents. Utilizing Braidlab for computations, results indicate that while FTBE increases rapidly with low variance, it stabilizes for higher variances, suggesting a plateau in entanglement. These findings challenge intuitive hypotheses about entanglement and variance, offering new perspectives for analyzing particle dynamics in fluid systems.

1 Introduction

To build a comprehensive vector field for ocean currents would take a lot of data. Braids provide a method to make sense of ocean current patterns using sparse data. This data can be collected from tracking the trajectories of buoys in the ocean to find patterns in currents. From these intertwining trajectories a braid can be defined. Properties like FTBE (Finite Time Braiding Exponent) can be computed for a braid which measure the entanglement of these buoys. Given the measure of entanglement it can be known how long it takes these buoys to leave a certain ocean region like the Labrador sea as described in [2]. The goal is to relate the variance of random walks (simulated buoy data) with FTBE.

2 Random Walks and Brownian Motion

Simulate the N trajectories using standard Brownian motion with the following equation for each strand:

$$x_j = x_{j-1} + \sigma \cdot \sqrt{\mathrm{dt}} \cdot I_2 \cdot r \tag{1}$$

where $x_j \in \mathbb{R}^2$, σ is variance, dt is the change in time, I_2 is the 2 dimensional identity matrix, and r is a 2 by 1 matrix of random numbers chosen from a normal distribution. The next step, x_j, is determined by the previous step, x_{j-1}

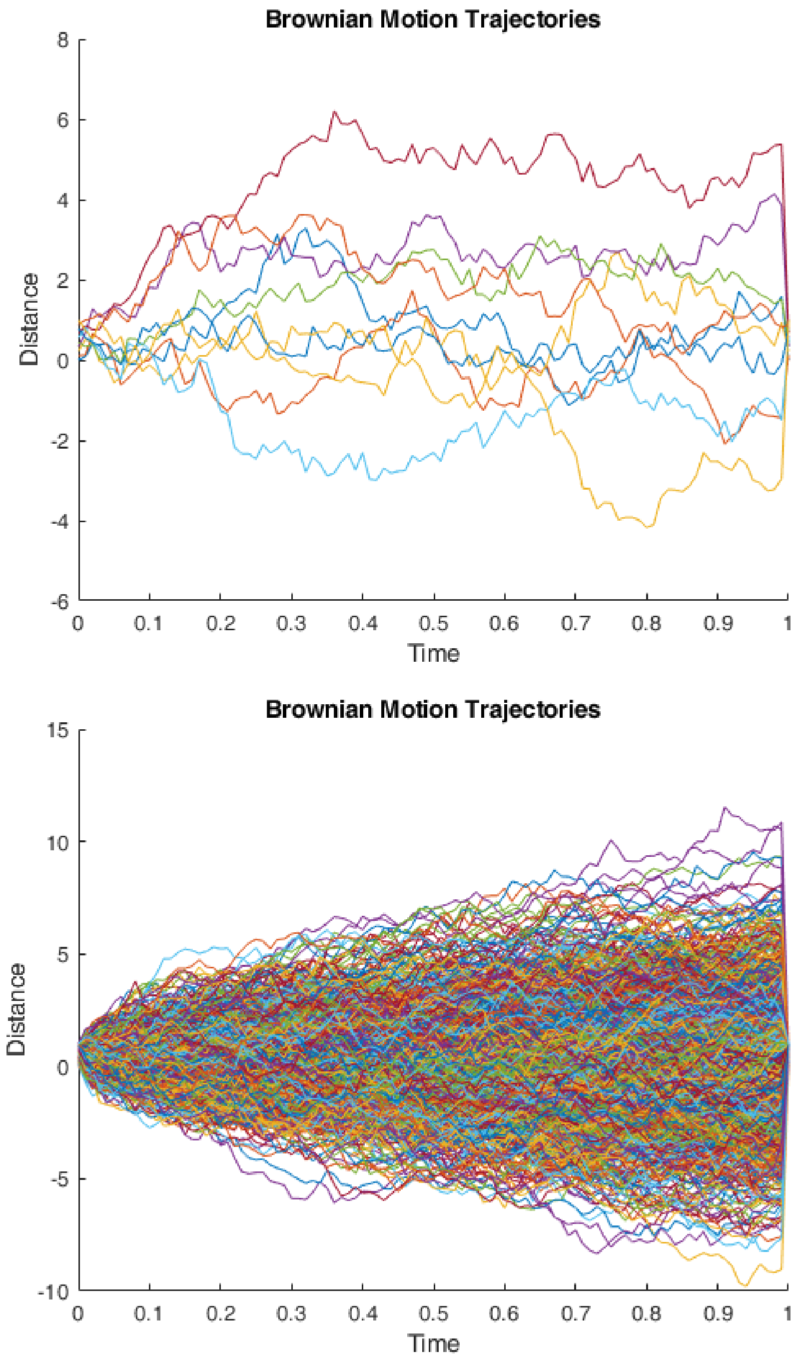

Figure 1: Random Walks of N=10 and N=1000 Strands.

with a random move dependent on the variance. This equation will generate a random walk for each strand. In Figure 1 there are graphs of trajectories with $N = 10$ and $N = 1000$. For this experiment, the variance is varied for $N \in \{10, 20, 30\}$.

3 Braid Group

As N particle trajectories (strands) cross each other time, their crossings can be encoded using the mathematics of the braid group. A left strand over the right strand is a positive generator, σ. Inversely, the right strand over the left strand is a negative generator, σ^{-1} (to make sense of left and right, the images in Figure 1 have to be reoriented). The generator σ_i encodes the crossing at the i and $i+1$ strands, while keeping the other strands fixed. Also, the index i does not denote the global position of the strand, but the localized position of the strand. If there are N strands, then there are $N-1$ possible generators. By concatenating a generator with its inverse generator i.e. $\sigma_i\sigma_i^{-1}$ the resulting braid is the identity braid corresponding to the identity element of the braid group. The set of all braids on N strands created by concatenation of generators forms the braid group B_N. The braid group B_N is generated by the $N-1$ generators $\{\sigma_1, ..., \sigma_{N-1}\}$ with the concatenation relation as the group operation. This is an infinite group which is finitely generated. There are two relations in this group (which stem from properties of a physical braid), the criss cross property and the commutator. The criss cross property can be written as $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ where i and j are adjacent strands i.e. $|i-j| = 1$. Generators may be commuted as long as they do not share a common strand i.e. $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$. Concisely the braid group on N strands may be written as

$$B_N = \langle \sigma_1, ..., \sigma_{N-1} | \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \rangle \tag{2}$$

For more detail on the braid group and braid diagrams see [2].

4 FTBE

The Finite-Time Braiding exponent is a measure of entropy or complexity of particle trajectories. The Finite-Time Braiding Exponent of the braid $\mathbf{b}$ corresponding to N trajectories over the time interval T is given by

$$\text{FTBE}(\mathbf{b}) = \frac{1}{T}\log\frac{|\mathbf{b}l_E|}{|l_E|} \tag{3}$$

where l_E is the loop representing a generating set of the non-oriented fundamental group on the N-punctured disk, and $|\cdot|$ is the number of intersections of the loop with the horizontal axis. The braid $\mathbf{b}$ acts on the loop in a natural way through Dynnikov coordinates which encodes the movement of the braid

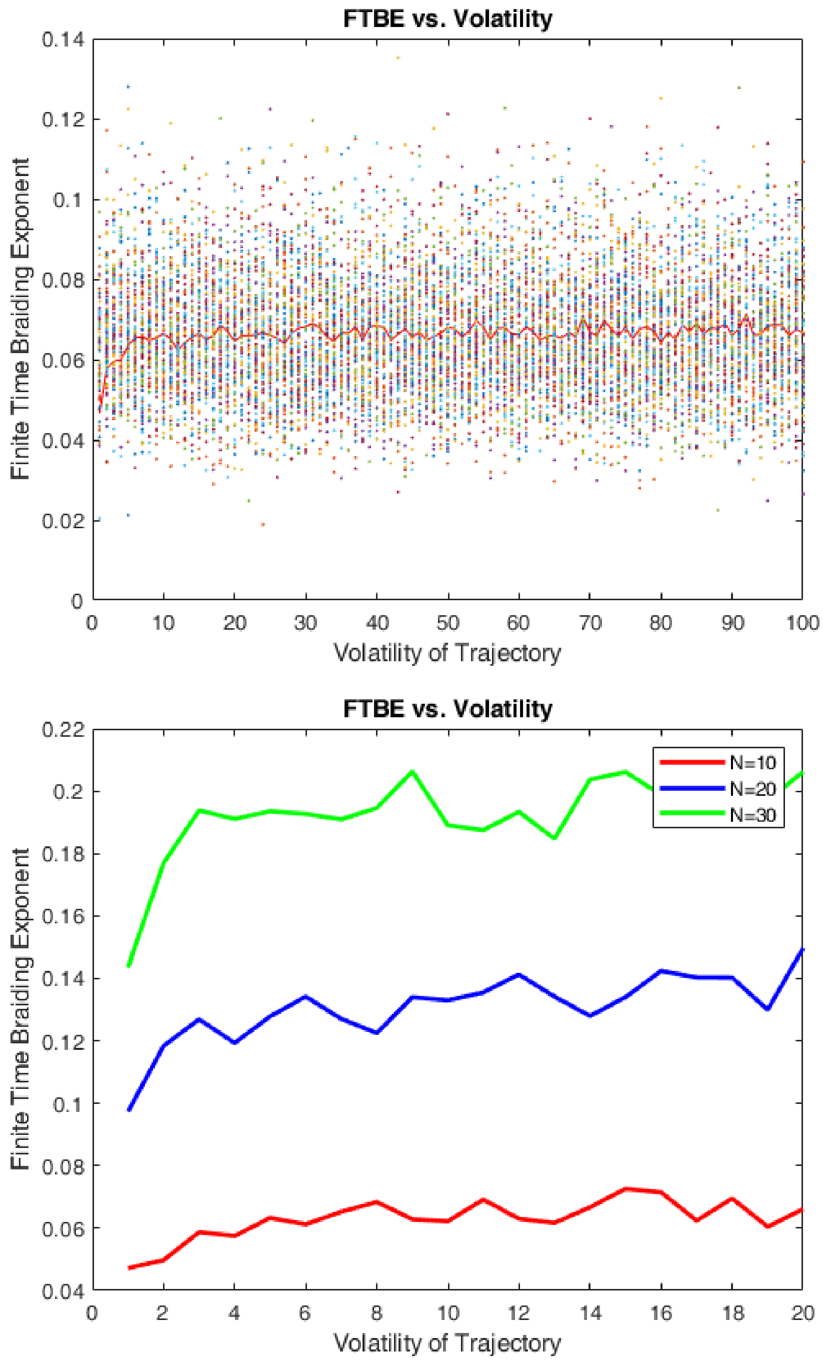

Figure 2: The top image is Volatility of Random Walk vs. FTBE with the average line highlighted in red. The below image is the same type of graph for various N.

itself. The statement about l_E requires lots of justification and can be found in [1] and [3]. Intuitively, the loop is wrapped around the N punctures and the braid determines the movement of these punctures (the punctures exchange places) which in turn entangles the loop. You can think of this like a taffy puller machine (which is a valid application found in [3]).

5 Results

Braidlab packages were used to encode the crossings of random walks, perform braid group simplifications, and compute the FTBE [4]. The goal is to examine the behavior of how the variance of random walks influences the entanglement of the loop acted on by the braid. Naively, the more volatile the trajectories are, the more entangled the loop becomes and the bigger the FTBE is. However this is not what Figure 2 shows. In the top image of Figure 2, $\sigma \in [0, 100]$ and it is averaged over several trials. If $\sigma \in \{0, 5\}$ then the FTBE is increasing rapidly. For $\sigma > 5$, the FTBE levels out and stays around the same value of 0.065. For different N values the FTBE value increases as N grows. This suggests that the relationship between variance and FTBE is constant for large σ values which goes against the intuition hypothesized. Thus a non-trivial relationship between variance of random walks and measure of entanglement is shown.

References

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[3] Jean-Luc Thiffeault. *Braids and Dynamics*. Springer, 2022.

[4] Jean-Luc Thiffeault and Marko Budišić. Braidlab: A software package for braids and loops, 2013–2021.