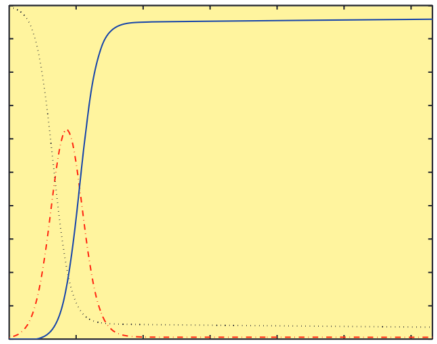
Chapter 21 / Case Study

More Models of Infection: It's Epidemic



In the case study of Chapter 19, we studied a model of the spread of an infection through a hospital ward. The ward was small enough that we could track each patient as an individual.

When the size of the population becomes large, it becomes impractical to use that kind of model, so in this case study we turn our attention to models which study the population as a whole.

As before, we divide the population into three groups: At day t, $I(t)$ is the proportion of the population that is infected and $S(t)$ is the proportion of the population that has never been infected. These quantities satisfy $0 \le I(t) \le 1$ and $0 \le S(t) \le 1$ for $t \ge 0$. The third part, $R(t)$, is the proportion of the population that was once infected but has now recovered, and it can be derived from these two: $R(t) = 1 - I(t) - S(t)$.

Models without Spatial Variation

In the models we studied before, the probability of an individual becoming infected depended on the status of the individual's neighbors. In the models in this section, we consider all individuals to be neighbors. This is equivalent to assuming a **well-mixed** model, in which all members of the population have contact with all others.

How might we model the three groups in the population? If the infection (or at least the contagious phase of the infection) lasts k days, then we might assume as an approximation that the rate of recovery is equal to the number infected divided by k. Thus, on average, $1/k$ of the infected individuals recover each day.

Let τ be the proportion of encounters between an infected individual and a susceptible individual that transmit the infection. Then the rate of new infections should increase as any of the parameters I, S, or τ increases, so we model this rate as $\tau I(t)S(t)$.

Next, we take the limit as the "timestep" Δt goes to zero, obtaining a system of ODEs. This gives us a simple but interesting Model 1:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - I(t)/k,$$
$$\frac{dS(t)}{dt} = -\tau I(t)S(t),$$

POINTER 21.1. Software.

The MATLAB function `ode23s` provides a good solver for the ODEs of Challenge 21.1. Most ODE software provides a mechanism for stopping the integration when some quantity goes to zero; in `ode23s` this is done by using the `Events` property in an option vector.

For Challenge 21.2, some ODE software, including `ode23s`, can be used to solve some DAEs; in MATLAB, this is done using the `Mass` property in the option vector.

In MATLAB, some DDEs can be solved using `dde23`.

$$\frac{dR(t)}{dt} = I(t)/k\,.$$

We start the model by assuming some proportion of infected individuals; for example, $I(0) = 0.005$, $S(0) = 1 - I(0)$, $R(0) = 0$.

CHALLENGE 21.1.

Solve Model 1 using `ode23s` for $k = 4$ and $\tau = .8$ for $t > 0$ until either $I(t)$ or $S(t)$ drops below 10^{-5}. Plot $I(t)$, $S(t)$, and $R(t)$ on a single graph. Report the proportion of the population that became infected and the maximum difference between $I(t)+S(t)+R(t)$ and 1.

Instead of using the equation $dR/dt = I/k$, we could have used the conservation principle

$$I(t)+S(t)+R(t) = 1$$

for all time. Substituting this for the dR/dt equation gives us an equivalent system of differential-algebraic equations (DAEs), studied in Section 20.4, and we call this Model 2.

CHALLENGE 21.2.

Redo Challenge 21.1 using Model 2 instead of Model 1. One way to do this is to differentiate the conservation principle and express the three equations of the model as $\boldsymbol{M}\boldsymbol{u}' = \boldsymbol{f}(t,\boldsymbol{u})$, where $\boldsymbol{M}$ is a 3×3 matrix and $\boldsymbol{u}$ is a function of t with three components. Another way is to use a DAE formulation.

There are many limitations in the model, but one of them is that the recovery rate is proportional to the current number of infections. This means that we are not very faithful to the hypothesis that each individual is infectious for k days. One way to model this more closely is to use a **delay differential equation** (DDE). We modify Model 1 by specifying that the rate of recovery at time t is equal to the rate of new infections at time $t-k$. This gives us a new model, Model 3:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \tau I(t-k)S(t-k)\,,$$

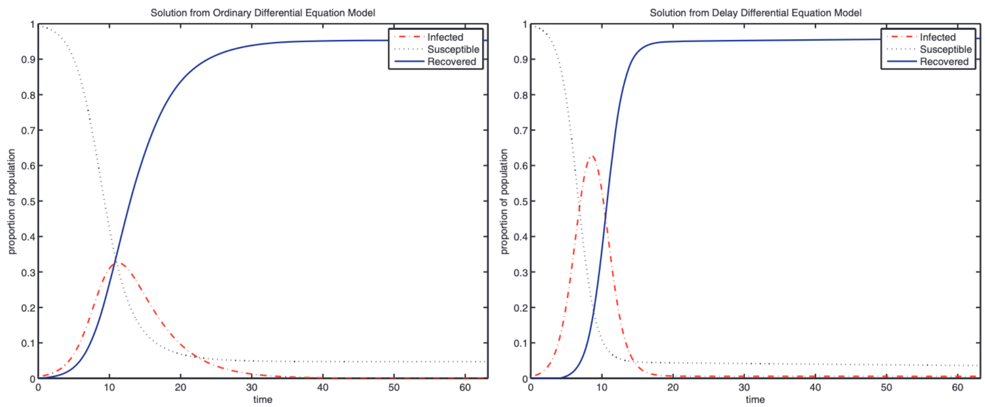


Figure 21.1. *Results of our models. (Left) Proportion of individuals infected by the epidemic from the* ODE *Model 1 or the* DAE *Model 2. (Right) Proportion of individuals infected by the epidemic from the* DDE *Model 3.*

$$\frac{dS(t)}{dt} = -\tau I(t)S(t),$$
$$\frac{dR(t)}{dt} = \tau I(t-k)S(t-k).$$

One disadvantage of this model is that we need to specify initial conditions not just at $t = 0$ but for $-k \le t \le 0$, so it requires a lot more information. A second disadvantage is that the functions I, S, and R are likely to have discontinuous derivatives (for example, at $t = 0$ and $t = k$, as we switch from dependence on the initial conditions to dependence only on the integration history). This causes solvers to do extra work at these points of discontinuity.

CHALLENGE 21.3.

Redo Challenge 21.1 with MATLAB's `dde23` using Model 3 instead of Model 1. For $t < 0$, use the initial conditions

$$I(t) = 0, \;\; S(t) = 1, \;\; R(t) = 0,$$

and let $I(0) = 0.005$, $S(0) = 1 - I(0)$, $R(0) = 0$. Note that these conditions match our previous ones, but have a jump at $t = 0$. Compare the results of the three models, as illustrated in Figure 21.1.

Models that Include Spatial Variation

Epidemics vary in space as well as time. They usually start in a single location and then spread, based on interaction of infected individuals with neighbors, as in the models of Chapter 19. The models of the previous section lose this characteristic. To recover it, we now let S, I, and R depend on a spatial coordinate (x, y) as well as t and see what such a model predicts.

Since people move in space, we introduce a **diffusion term** that allows infected individuals to affect susceptible individuals that are close to them in space. Diffusion adds a term $\delta((\partial^2 I)/(\partial x^2) + (\partial^2 I)/(\partial y^2))S$ to dI/dt, and subtracts the same term from dS/dt. This produces differential equations analogous to Model 1:

$$\begin{aligned}
\frac{\partial I(t,x,y)}{\partial t} &= \tau I(t,x,y)S(t,x,y) - I(t,x,y)/k \\
&\quad + \delta\left(\frac{\partial^2 I(t,x,y)}{\partial x^2} + \frac{\partial^2 I(t,x,y)}{\partial y^2}\right) S(t,x,y), \\
\frac{\partial S(t,x,y)}{\partial t} &= -\tau I(t,x,y)S(t,x,y) - \delta\left(\frac{\partial^2 I(t,x,y)}{\partial x^2} + \frac{\partial^2 I(t,x,y)}{\partial y^2}\right) S(t,x,y), \\
\frac{\partial R(t,x,y)}{\partial t} &= I(t,x,y)/k\,.
\end{aligned}$$

We assume that the initial values $I(0,x,y)$ and $S(0,x,y)$ are given, that we study the problem for $0 \le x \le 1$, $0 \le y \le 1$, and $t \ge 0$, and that there is no diffusion across the boundaries $x = 0$, $x = 1$, $y = 0$, and $y = 1$.

To solve this problem, Model 4, we **discretize** and approximate the solution at the points of a grid of size $n \times n$. Let $h = 1/(n-1)$ and let $x_i = ih$, $i = 0, \ldots, n-1$, and $y_j = jh$, $j = 0, \ldots, n-1$. Our variables are our approximations $I(t)_{ij} \approx I(t, x_i, y_j)$ and similarly for $S(t)_{ij}$ and $R(t)_{ij}$.

CHALLENGE 21.4.

(a) Use Taylor series expansions to show that we can approximate

$$\frac{\partial^2 I(t,x_i,y_j)}{\partial x^2} = \frac{I(t,x_{i-1},y_j) - 2I(t,x_i,y_j) + I(t,x_{i+1},y_j)}{h^2} + O(h^2).$$

A similar expression can be derived for $\partial^2 I(t,x_i,y_j)/\partial y^2$.

(b) Form a vector $\widehat{\boldsymbol{I}}(t)$ from the approximate values of $I(t)$ by ordering the unknowns as I_{00}, $I_{01}, \ldots,$ $I_{0,n-1}$, I_{10}, $I_{11}, \ldots, I_{1,n-1}, \ldots,$ $I_{n-1,0}$, $I_{n-1,1}, \ldots, I_{n-1,n-1}$, where $I_{ij}(t) = I(t,x_i,y_j)$. In the same way, form the vectors $\widehat{\boldsymbol{S}}(t)$ and $\widehat{\boldsymbol{R}}(t)$ and derive the matrix $\boldsymbol{A}$ so that our discretized equations become Model 4:

$$\begin{aligned}
\frac{\partial \widehat{\boldsymbol{I}}(t)}{\partial t} &= \tau \widehat{\boldsymbol{I}}(t).*\widehat{\boldsymbol{S}}(t) - \widehat{\boldsymbol{I}}(t)/k + \delta(\boldsymbol{A}\widehat{\boldsymbol{I}}(t)).*\widehat{\boldsymbol{S}}(t), \\
\frac{\partial \widehat{\boldsymbol{S}}(t)}{\partial t} &= -\tau \widehat{\boldsymbol{I}}(t).*\widehat{\boldsymbol{S}}(t) - \delta(\boldsymbol{A}\widehat{\boldsymbol{I}}(t)).*\widehat{\boldsymbol{S}}(t), \\
\frac{\partial \widehat{\boldsymbol{R}}(t)}{\partial t} &= \widehat{\boldsymbol{I}}(t)/k,
\end{aligned}$$

where the notation $\widehat{\boldsymbol{I}}(t).*\widehat{\boldsymbol{S}}(t)$ means the vector formed by the product of each component of $\widehat{\boldsymbol{I}}(t)$ with the corresponding component of $\widehat{\boldsymbol{S}}(t)$. To form the approximation near the boundary, assume that the (Neumann) boundary conditions imply that $I(t,-h,y) = I(t,h,y)$,

$I(t,1+h,y) = I(t,1-h,y)$ for $0 \le y \le 1$, and similarly for S and R. Make the same type of assumption at the two other boundaries.

There are two ways to use this model. First, suppose we fix the timestep Δt and use Euler's method to approximate the solution; this means we approximate the solution at $t+\Delta t$ by the solution at t, plus Δt times the derivative at t. This gives us an iteration

$$\begin{aligned}
\widehat{\boldsymbol{I}}(t+\Delta t) &= \widehat{\boldsymbol{I}}(t) + \Delta t(\tau \widehat{\boldsymbol{I}}(t). * \widehat{\boldsymbol{S}}(t) - \widehat{\boldsymbol{I}}(t)/k + \delta(\boldsymbol{A}\widehat{\boldsymbol{I}}(t)). * \widehat{\boldsymbol{S}}(t)),\\
\widehat{\boldsymbol{S}}(t+\Delta t) &= \widehat{\boldsymbol{S}}(t) + \Delta t(-\tau \widehat{\boldsymbol{I}}(t). * \widehat{\boldsymbol{S}}(t) - \delta(\boldsymbol{A}\widehat{\boldsymbol{I}}(t)). * \widehat{\boldsymbol{S}}(t)),\\
\widehat{\boldsymbol{R}}(t+\Delta t) &= \widehat{\boldsymbol{R}}(t) + \Delta t(\widehat{\boldsymbol{I}}(t)/k).
\end{aligned}$$

This model is very much in the spirit of the models we considered in the case study of Chapter 19, except that it is deterministic rather than stochastic.

Alternatively, we could apply a more accurate ODE solver to this model, and we investigate this in the next challenge.

CHALLENGE 21.5.

(a) Set $n = 11$ (so that $h = 0.1$), $k = 4$, $\tau = 0.8$ and $\delta = 0.2$ and use an ODE solver to solve Model 4. For initial conditions, set $S(0,x,y) = 1$ and $I(0,x,y) = R(0,x,y) = 0$ at each point (x,y), except that $S(0,0.5,0.5) = I(0,0.5,0.5) = .5$. (For simplicity, you need only use I and S in the model, and you may derive $\underline{R}(t)$ from these quantities.) Stop the simulation when either the average value of $\widehat{\boldsymbol{I}}(t)$ or $\widehat{\boldsymbol{S}}(t)$ drops below 10^{-5}. Form a plot similar to that of Challenge 21.1 by plotting the average value of $I(t)$, $S(t)$, and $R(t)$ vs time. Compare the results.

(b) Let's vaccinate the susceptible population at a rate

$$\frac{\nu S(t,x,y) I(t,x,y)}{I(t,x,y) + S(t,x,y)}.$$

This rate is the derivative of the vaccinated population $V(t,x,y)$ with respect to time, and this term is subtracted from $\partial S(t,x,y)/\partial t$. So now we model four segments of the population: susceptible $S(t)$, infected $I(t)$, recovered $R(t)$, and vaccinated $V(t)$. Your program can track three of these and derive the fourth from the conservation principle $S(t) + I(t) + R(t) + V(t) = 1$. Run this model with $\nu = 0.7$ and compare the results with those of Model 4.

POINTER 21.2. Further Reading.

Model 1 is the SIR model of Kermack and McKendrick, introduced in 1927. It is discussed, for example, by Britton [21].

DDEs such as those in Challenge 21.3 arise in many applications, including circuit analysis. To learn more about these problems, consult a textbook such as that by Bellman and Cooke [12] or by Hale and Lunel [67].

Stochastic differential equations are an active area of research. Higham [77] gives a good introduction to computational aspects and supplies references for further investigation.

The differential equations leading to Model 4 are presented, for example, by Callahan [24], following a model with one space dimension given in [87].

If you want to experiment further with Model 4, incorporate the delay recovery term in place of $-\widehat{\boldsymbol{I}}(t)/k$.

CHALLENGE 21.6. (Extra)

Include a delay in Model 4. Solve the resulting DDE model numerically and compare with the previous results.

In the models we used in the case study of Chapter 19, we incorporated some randomness to account for factors that were not explicitly modeled. We could also put randomness into our differential equation models, resulting in **stochastic differential equations**. See Pointer 21.2 for references on this subject.