Proposal of the Network Resonance Method for Estimating Eigenvalues of the Scaled Laplacian Matrix

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Proposal of the Network Resonance Method for Estimating Eigenvalues of the Scaled Laplacian Matrix

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***Abstract*—Eigenvalues of the Laplacian matrix play an important role in characterizing structural and dynamical properties of networks. In the procedure for calculating eigenvalues of the Laplacian matrix, we need to get the Laplacian matrix that represents structures of the network. Since the actual structure of networks and the strength of links are difficult to know, it is difficult to determine elements of the Laplacian matrix. To solve this problem, our previous study proposed a concept of the network resonance method, which is for estimating eigenvalues of the scaled Laplacian matrix using resonance of oscillation dynamics on networks. This method does not need a priori information about the network structure. In this research, we investigate feasibility of the network resonance method, and show that the method can estimate eigenvalues of the scaled Laplacian matrix of the entire network through observations of oscillation dynamics even if observable nodes are restricted to a part of network.**

I. Introduction

In the real world, there are many network systems, such as the Internet, the World Wide Web and social networks. For effectively utilizing networks, it is important to reveal the properties of networks.

When we analyze a network, we generally represent the network by using the adjacency matrix or the Laplacian matrix. The network, whose links are undirected, is generally represented by a symmetric matrix. On the other hand, the network, whose links are directed, is generally represented by an asymmetric matrix. Symmetric matrices are easier to analyze than asymmetric matrices, so matrix-based analysis is suitable for symmetric network.

In spectral graph theory, structural and dynamical properties of the network are studied through eigenvalues and eigenvectors of the Laplacian matrix which represents the structure of the corresponding network. Eigenvalues of the Laplacian matrix is powerful tools for describing properties of the network. Particularly, the second smallest eigenvalue of the Laplacian matrix is actively studied because it plays an important role in the synchronization on networks [2], [3]. Additionally, eigenvectors of the Laplacian matrix is also useful tools for describing properties of the network [4].

Now, for calculating a eigenvalue λ of a Laplacian matrix L, we need to solve the eigenvalue equation $\det(\lambda I - L) = 0$ where I is the identity matrix. This means that we need to know all the elements of the Laplacian matrix that represents structures of the network in advance for calculating eigenvalues of L. However, the actual structure of networks and the strength of links are difficult to know. So, it is difficult to determine elements of the Laplacian matrix. There are two reasons for that. The first reason is that we cannot necessarily observe the structure of the whole network. For example, it is hard to observe the whole network of cyber attackers. One other reason is that even if we can observe the topological structure of the network, we cannot observe the network structure considered the weight of links between adjacent nodes. For example, in social networks, we cannot observe the strength or significance of friendship between users.

To solve this problem, in [5], Aida *et al.* proposed a concept of the network resonance method, which is the method for estimating eigenvalues of the scaled Laplacian matrix without requiring a priori information about network structure. However, there is still room for consideration in its feasibility. In this paper, we discuss technical issues in the network resonance method and show the eigenvalues of the scaled Laplacian matrix can be estimated by using the network resonance method.

This paper is organized as follows. Section II presents the definition of the (scaled) Laplacian matrix, and the oscillation model on networks modeled by the (scaled) Laplacian matrix. Section III describes the network resonance method. Then, we discuss technical issues in the network resonance method, in Section IV. In Section V, we show that our method can estimate eigenvalues of the scaled Laplacian matrix. Section VI concludes the paper.

II. Preliminaries

A. Scaled Laplacian Matrix

Let $G = G(V, E)$ be an undirected graph, where $V = \{1, 2, \dots, n\}$ is the set of nodes and $E \subset V \times V$ is the set of links in G. The (weighted) adjacency matrix $A = [A_{ij}]$ is the $n \times n$ matrix defined as

$$A_{ij} \;:=\; \begin{cases} w_{ij} & ((i,j) \in E), \\ 0 & ((i,j) \notin E). \end{cases} \tag{1}$$

As the simplest case, if all link weights are 1, A is the ordinary adjacency matrix and describes topological structure of the network. The degree of each node is $d_i := \sum_{j \in \partial i} w_{ij}$ where ∂i is the set of nodes adjacent to node i. The degree matrix is defined as $D := \mathrm{diag}(d_1, d_2, \ldots, d_n)$, and the Laplacian matrix is defined as follows [6]:

$$L := D - A \tag{2}$$

Let us introduce a scaling factor $m_i > 0$ called mass as a characteristic of each node i and define the mass matrix as $M := \mathrm{diag}(m_1, m_2, \ldots, m_n)$. The scaled Laplacian matrix is defined as

$$S := M^{-1/2}\, L\, M^{-1/2}. \tag{3}$$

Note that S is symmetric. If the scaling factor m_i is chosen to be equal to the node degree d_i, the scaled Laplacian matrix comes to the well-known the normalized Laplacian matrix $N := D^{-1/2} L D^{-1/2}$. In addition, if the scaling factor m_i is homogeneous, i.e. $m_i = 1$ for all i, the scaled Laplacian matrix is reduced to be the Laplacian matrix.

We denote the eigenvalues of S as

$$0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}, \tag{4}$$

and the eigenvector corresponding to λ_μ as $\boldsymbol{v}_\mu = {}^t(\{v_\mu(i)\})$. We can choose the eigenvectors as the orthonormal eigenbasis which satisfy

$$S\, \boldsymbol{v}_\mu = \lambda_\mu\, \boldsymbol{v}_\mu, \quad \boldsymbol{v}_\mu \cdot \boldsymbol{v}_\nu = \delta_{\mu\nu}, \tag{5}$$

where $\delta_{\mu\nu}$ is the Kronecker delta.

S is a real symmetric matrix, so S can be factorized as

$$S = P\, \Lambda\, {}^t P, \tag{6}$$

where P is the orthogonal matrix defined as

$$P := (\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}), \tag{7}$$

and Λ is the diagonal matrix defined as

$$\Lambda := \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}). \tag{8}$$

This means that if we get the information of all eigenvalues and eigenvectors, we can know the network structure.

B. Oscillation Model on Networks

To describe the propagation of influence on networks, Aida *et al.* proposed oscillation model on networks in [5]. Let assume the system in which each node in the network gives influence to the adjacent nodes (Fig. 1) [5]. Although the figure shows a 1-dimensional network, it is extendable to general networks.

Let x_i of node i be the displacement from the equilibrium, and its restoring force be proportional to the difference of the displacements of the adjacent nodes. In addition, we define that the mass of node i is represented as the scaling factor m_i and the spring constant of the spring between node i and node j is represented as the link weight w_{ij}. Then, the Hamiltonian $\mathcal{H}$ of this coupled oscillator system is expressed as

$$\begin{aligned} \mathcal{H} &:= \sum_{i \in V} \frac{(p_i)^2}{2m} + \sum_{(i,j) \in E} \frac{w_{ij}}{2} (x_i - x_j)^2 \\ &= \sum_{i \in V} \frac{(p_i)^2}{2m} + \frac{1}{2} ({}^t\boldsymbol{x}\, L\, \boldsymbol{x}) \end{aligned} \tag{9}$$

where p_i is the conjugate momentum of the displacement x_i. We can derive the canonical equations of motion expressed as follows:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{n} L_{ij}\, x_j \tag{10}$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i} \tag{11}$$

By eliminating p_i from these equations, we derive the equation of motion in the form of the wave equation as

$$m_i\, \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = -\sum_{j=1}^{n} L_{ij}\, x_j, \tag{12}$$

or it can be in vector form as

$$M\, \frac{\mathrm{d}^2 \boldsymbol{x}(t)}{\mathrm{d}t^2} = -L\, \boldsymbol{x}(t). \tag{13}$$

By introducing the vector $\boldsymbol{y}(t) = M^{1/2} \boldsymbol{x}(t)$, (13) can be rewritten as

$$\frac{\mathrm{d}^2 \boldsymbol{y}(t)}{\mathrm{d}t^2} = -S\, \boldsymbol{y}(t). \tag{14}$$

In this regard, it is the important question whether this equation describes actually meaningful phenomena on networks. In [5] and [7], Aida, Takano, and Murata reported that the oscillation energy of each nodes reproduces well-known indices of node centrality, including degree centrality and betweenness

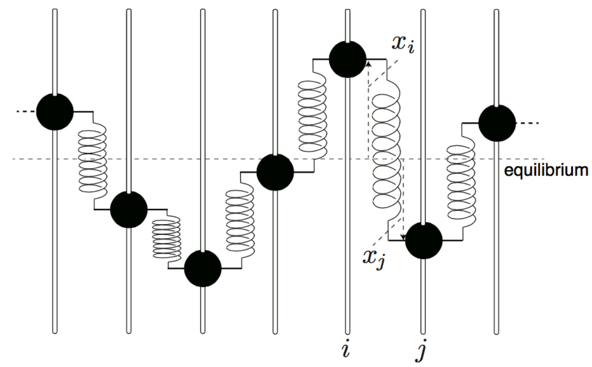

Fig. 1. Oscillation model on networks.

centrality, by applying appropriate weight to each links.

This result suggests that the oscillation on networks is related to the real world networks through the oscillation energy.

III. The network resonance method

In the procedure for calculating eigenvalues of Laplacian matrix, we need to get Laplacian matrix that represents structures of the corresponding networks. Unfortunately, since the actual structure of networks and the strength of links are difficult to know, it is difficult to determine elements of the Laplacian matrix.

In this section, we explain the network resonance method, which is the method for estimating eigenvalues of the scaled Laplacian matrix without requiring a priori information about network structure.

The procedure of the network resonance method is follows. First, we enter the periodic external force with frequency ω into a certain node j and observe its reaction at node i. In this process, topological structure of the entire network is assumed to be not necessarily known. Next, by analyzing the observed reaction, we calculate the estimate of eigenvalues of S. Figure 2 is the schematic diagram of this mechanism.

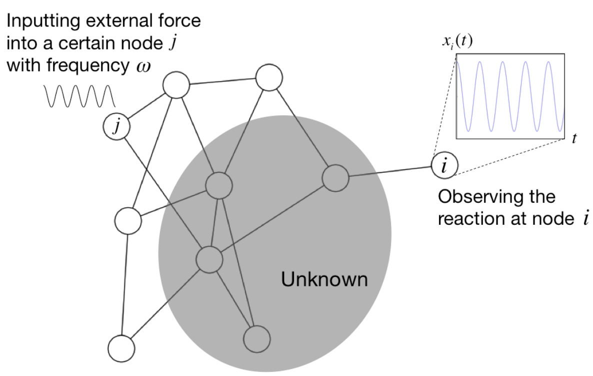

Fig. 2. The mechanism of the network resonance method.

To support this procedure, we consider the forced oscillation on networks. We suppose that the damped force is proportional to the velocity $\mathrm{d}x_i(t)/\mathrm{d}t$ of node i, and assume that inputting the periodic external force with frequency ω into a certain node j. The equation of motion of the forced oscillation on networks can be written by using the Laplacian matrix as follows:

$$M\,\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2}+M\,\gamma\,\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}+L\,\boldsymbol{x}(t)=(F\,\cos\omega\,t)\,\mathbf{1}_{\{j\}},\quad(15)$$

where γ and F are constants and $\mathbf{1}_{\{j\}}$ is the n-dimensional vector whose j-th component is 1 and all other components are 0. By using the vector $\boldsymbol{y}(t)=M^{1/2}\boldsymbol{x}(t)$, (15) can be rewritten as

$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2}+\gamma\,\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t}+S\,\boldsymbol{y}(t)=\frac{F\cos\omega\,t}{\sqrt{m_j}}\mathbf{1}_{\{j\}}.\quad(16)$$

The vectors $\boldsymbol{y}(t)$ and $\mathbf{1}_{\{j\}}$ can be expanded using eigenbasis $\boldsymbol{v}_\mu$ of S and eigenmodes $a_\mu(t)$ and b_μ [8], [9], that is,

$$\boldsymbol{y}(t)=\sum_{\mu=0}^{n-1}a_\mu(t)\,\boldsymbol{v}_\mu,\quad \mathbf{1}_{\{j\}}=\sum_{\mu=0}^{n-1}b_\mu\,\boldsymbol{v}_\mu.\quad(17)$$

By using orthogonality of $\boldsymbol{v}_\mu$, (16) can be decomposed into equations of motion for the eigenmode $a_\mu(t)$ expressed as follows:

$$\frac{\mathrm{d}^2a_\mu(t)}{\mathrm{d}t^2}+\gamma\,\frac{\mathrm{d}a_\mu(t)}{\mathrm{d}t}+\lambda_\mu\,a_\mu(t)=\frac{F\cos\omega\,t}{\sqrt{m_j}}v_\mu(j).\quad(18)$$

By substituting the stationary solution $a_\mu(t)$ of (18) into $\boldsymbol{x}(t)=M^{-1/2}\sum_{\mu=0}^{n-1}a_\mu(t)\,\boldsymbol{v}_\mu$, we derive the solution of the (15) as

$$\boldsymbol{x}(t)=M^{-1/2}\sum_{\mu=0}^{n-1}A_\mu(\omega)\cos\big(\omega\,t+\theta_\mu(\omega)\big)\boldsymbol{v}_\mu.\quad(19)$$

where $A_\mu(\omega)$ and $\theta_\mu(\omega)$ are the amplitude and the phase for eigenmode μ, respectively, which are expressed as

$$A_\mu(\omega)=\frac{F\,v_\mu(j)}{\sqrt{m_j}}\frac{1}{\sqrt{(\omega_\mu^2-\omega^2)^2+(\gamma\,\omega)^2}},\quad(20)$$

$$\theta_\mu(\omega)=\tan^{-1}\left(\frac{\gamma\,\omega}{\omega^2-\omega_\mu^2}\right).\quad(21)$$

where $\omega_\mu=\sqrt{\lambda_\mu}$. The amplitude $A_\mu(\omega)$ of (20) increases sharply around $\omega\simeq\omega_\mu$. This phenomenon is called resonance. The amplitude $A_\mu(\omega)$ takes the maximal value at

$$\omega_\mu^{\max}=\sqrt{\omega_\mu^2-\gamma^2/2}.\quad(22)$$

Next, to represent the sharpness of the amplitude $A_\mu(\omega)$ with respect to ω, we define the Q-factor. Let ω_μ^- and ω_μ^+ be frequencies that give $1/\sqrt{2}$ times the value of $A_\mu(\omega_\mu^{\max})$ ($\omega_\mu^-<\omega_\mu^+$). Namely,

$$A_\mu(\omega_\mu^-)=A_\mu(\omega_\mu^+)=\frac{1}{\sqrt{2}}A_\mu(\omega_\mu^{\max}).$$

The Q-factor is defined as

$$Q_\mu:=\frac{\sqrt{\omega_\mu^2-\gamma^2/2}}{\omega_\mu^+-\omega_\mu^-}.\quad(23)$$

We assume $\gamma\ll\omega_\mu$ and $\omega\simeq\omega_\mu$. By using $\omega^2-\omega_\mu^2\simeq 2\omega_\mu(\omega-\omega_\mu)$,

$$\begin{aligned}A_\mu(\omega)&\simeq\frac{F\,v_\mu(j)}{\sqrt{m_j}}\frac{1}{\sqrt{(2\omega_\mu(\omega-\omega_\mu))^2+(\gamma\,\omega_\mu)^2}}\\&=\frac{F\,v_\mu(j)}{\sqrt{m_j}\,\omega_\mu}\frac{1}{\sqrt{4(\omega-\omega_\mu)^2+\gamma^2}}.\end{aligned}\quad(24)$$

Therefore,

$$A_\mu(\omega_\mu)\simeq\frac{F\,v_\mu(j)}{\sqrt{m_j}\,\omega_\mu\,\gamma},\quad A_\mu(\omega_\mu\pm\gamma/2)\simeq\frac{1}{\sqrt{2}}A_\mu(\omega_\mu),$$

and we have

$$\omega_\mu^\pm \simeq \omega_\mu \pm \frac{\gamma}{2} \quad \text{(double-sign corresponds).} \tag{25}$$

From (22) and (25), the eigenfrequency ω_μ and the damping factor γ are approximately represented as

$$\omega_\mu \simeq 2\,\omega_\mu^\pm - \sqrt{2(\omega_\mu^\pm)^2 - (\omega_\mu^{\max})^2}, \tag{26}$$

$$\gamma \simeq \omega_\mu^+ - \omega_\mu^-. \tag{27}$$

ω_μ^-, ω_μ^+ and $\omega_\mu^{\max}$ are the values obtained by observing amplitude $A_\mu(\omega)$, so we substitute them into (26) and obtain the estimated value of the eigenfrequency ω_μ. From $\lambda_\mu = \omega_\mu^2$, we obtain the estimated value of the eigenvalue λ_μ.

ω_μ^-, ω_μ^+ and $\omega_\mu^{\max}$ are the values obtained by observing the amplitude $A_\mu(\omega)$ of eigenmode μ. However, we cannot actually observe the amplitude $A_\mu(\omega)$ for each μ alone. The amplitude we can observe at node i is

$$\begin{aligned} a_i(\omega) &= \max_t |x_i(t)| \\ &= \max_t \left| \frac{1}{\sqrt{m_i}} \sum_{\mu=0}^{n-1} A_\mu(\omega) \cos\big(\omega\, t + \theta_\mu(\omega)\big) v_\mu(i) \right|. \end{aligned} \tag{28}$$

However, we can expect that this is not the serious problem from the following reason.

From (20) and (28), it is assumed that the contribution of the amplitude with eigenmode μ is dominant to the amplitude $a_i(\omega)$ around $\omega \simeq \omega_\mu$. So, we expect that we can estimate the eigenvalues of S by obtaining the values corresponding to ω_μ^-, ω_μ^+ and $\omega_\mu^{\max}$ from each peaks of the amplitude $a_i(\omega)$ (Fig. 3).

Theoretically, if the damping factor γ is sufficiently small, this method can always estimate the all eigenvalues of S.

In practical applications, input of the forced oscillation corresponds to, for example, to update information of a popular blog, market information, and news, periodically with frequency ω. Note that the node centrality can be understood as the oscillation energy that is proportional to the square of the amplitude [7]. So, by observing the strength of node activity as the response of the forced input, we can expect to get the value of amplitude by calculating node centralities.

IV. Technical Issues in the Network Resonance Method

In Sec. III, we explained the concept of the network resonance method. However, there is still room for consideration in its feasibility. In this section, we discuss technical issues in the network resonance method and the measures for them. Note that, in this section, the displacement and the amplitude observed at node i is represented as $x_i(t;\, j)$ and $a_i(\omega;\, j) = \max_t |x_i(t;\, j)|$ if the external force is input into node j.

The network resonance method has two technical issues.

The first issue is the issue which the peak of the amplitude $a_i(\omega;\, j)$ is not necessarily to be formed around $\omega \simeq \omega_\mu$. In Sec. III, we assume that the contribution of the amplitude with eigenmode μ is dominant to the amplitude $a_i(\omega;\, j)$ around $\omega \simeq \omega_\mu$ because of resonance. However, there is the case where the peak of the amplitude $a_i(\omega;\, j)$ cannot be observed even if $\omega \simeq \omega_\mu$ (Fig. 4). We cannot derive the values w_μ^-, w_μ^+ and $w_\mu^{\max}$ of (26) if the peak cannot be observed. Therefore, we cannot estimate eigenvalues.

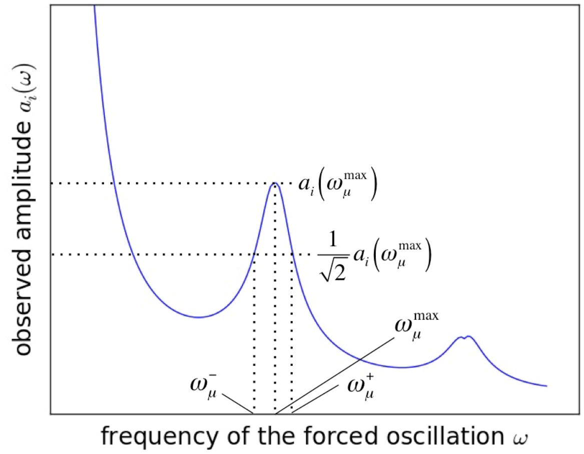

Fig. 3. Example of the observed amplitude at node i.

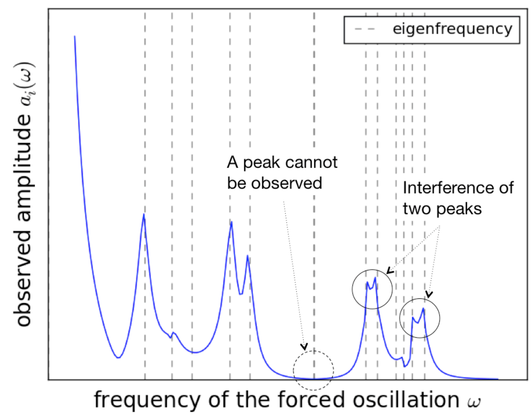

Fig. 4. Technical issues in the network resonance method: Dashed lines represent the position of each eigenfrequencies ω_μ.

The second issue is the issue which the peak of the amplitude $a_i(\omega;\, j)$ interfere at the narrow area of the eigenfrequency spacing. At the narrow area of the eigenfrequency spacing, there is the case where the peak of the amplitude $a_i(\omega;\, j)$ interfere and become skew (Fig. 4). The network resonance method supposes that each peaks of amplitude $a_i(\omega;\, j)$ are nearly symmetry. Thus, the eigenvalue estimated from the skew peak may be the poor precision.

The measures for these issues is to change the pair of the observing node i and the inputting node j. The amplitude $a_i(\omega;\, j)$ depends on i and j. Therefore, by changing the pair, the shape of the amplitude $a_i(\omega;\, j)$ also changes.

Finally, by repeating trial-and-error in this measures for selecting good node pairs and by finding the peak which was suitable to estimate, it is expected that all eigenvalues of S can be estimated.

Additionally, we discuss the number of shape patterns of $a_i(\omega; j)$, which we can observe, by changing the pair. If external force is input into node j, the displacement observed at node i is

$$\begin{aligned} & x_i(t;\, j) \\ & = \sum_{\mu=0}^{n-1} \frac{F\, v_\mu(j)\, v_\mu(i)}{\sqrt{m_i\, m_j\left((\omega_\mu^2 - \omega^2)^2 + (\gamma\, \omega)^2\right)}} \cos\left(\omega\, t + \theta_\mu(\omega)\right). \end{aligned}$$

Thus, $x_i(t;\, j) = x_j(t;\, i)$. This means that the displacement $x_i(t;\, j)$ and the amplitude $a_i(\omega;\, j)$ are symmetric with respect to the replacement of i and j. Therefore, in theory, we can observe $n(n+1)/2$ shape patterns to n nodes by changing the pair.

V. Numerical results

In this section, we demonstrate that the network resonance method can estimate eigenvalues of the scaled Laplacian matrix.

Figure 5 describes the network topology for the simulation. The link weight of the network is 1 for all $(i, j) \in E$, and the mass of each nodes is equal to the degree of each nodes, i.e. $m_i = d_i$ for all i. In the simulation, we set the parameter $F = 5$, relating to external force, and the damping factor $\gamma = 0.05$.

In addition to the above setting, we restrict both the input and observed nodes to a part of nodes. This is because the situation that we can input the external force into all nodes, and can observe reaction for all nodes, might be unrealistic assumption. We input the periodic external force with frequency ω into node 2 or 3 only and observed the reaction against the external force at node 2, 3, 9 or 10 only. Figure 6 shows the amplitudes observed at nodes 2, 3, 9 and 10, in the case we input the periodic external force into node 2. Dashed lines in Fig. 6 represent the position of each eigenfrequencies ω_μ.

Figure 7 shows the comparison between actual eigenvalue and the estimated eigenvalue for each eigenmodes μ. Note that, the minimum eigenvalue λ_0 of the scaled Laplacian matrix S is

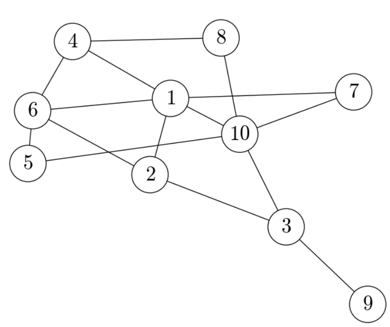

Fig. 5. Network topology: The link weight $w_{ij} = 1, \forall (i, j) \in E$. The node mass $m_i = d_i, \forall i \in V$

trivial, i.e. $\lambda_0 = 0$, so we omitted λ_0 from Fig. 7. From Fig. 7, we obtain the result that eigenvalues estimated by the network resonance method are very close to actual eigenvalues. The error ratio of estimated values to actual values are smaller than $\pm 1.5\%$ for each eigenmodes μ.

From the above, the network resonance method can estimate eigenvalues of the scaled Laplacian matrix with high precision even if observable nodes are restricted to a part of network.

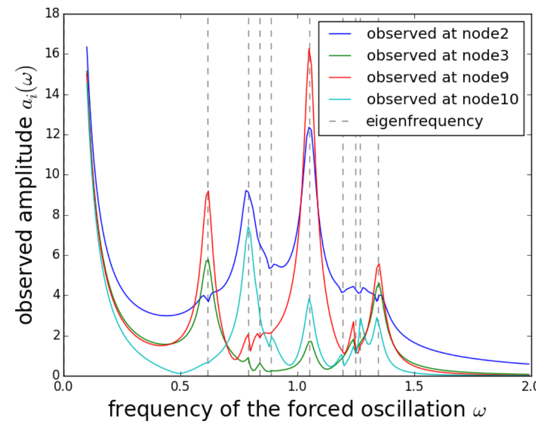

Fig. 6. Observed amplitudes (input node: 2).

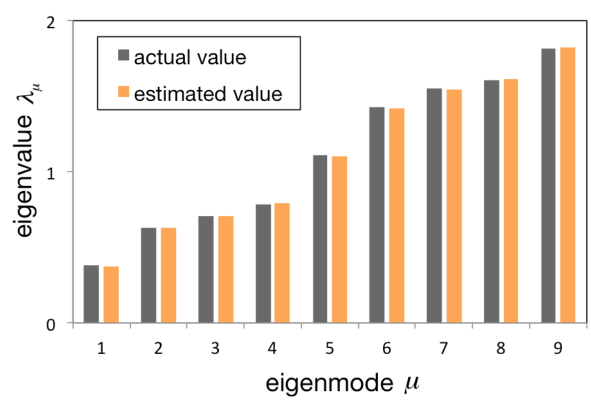

Fig. 7. Actual eigenvalues and estimated eigenvalues.

VI. Conclusions

In this research, we investigate feasibility of the network resonance method, which is the method for estimating eigenvalues of the scaled Laplacian matrix. As technical issues in the network resonance method, we discussed the issue which the peak of the amplitude $a_i(\omega)$ is not necessarily to be formed around $\omega \simeq \omega_\mu$ and the issue which the peaks of the amplitude $a_i(\omega)$ at the narrow area of the eigenfrequency spacing are mutually interfered. We explained these issues can be solved by changing the pair of the observing node and the inputting node.

In addition, by the simulation, we confirmed that the network resonance method can estimate eigenvalues of the scaled Laplacian matrix even if observable nodes are restricted in a part of network.

The significance of this study is that we showed the eigenvalues of the (scaled) Laplacian matrix, which is useful information for analyzing the properties of networks, can be estimated by using the network resonance method. This means that we can know the structural and dynamical properties of networks even if the actual structure of networks and the strength of links are difficult to know.

Incidentally, In Sec. II, we explained that if we get the information of all eigenvalues and eigenvectors, we can know the network structure. Namely, if we also estimate eigenvectors of the scaled Laplacian matrix by applying the network resonance method, we can estimate the strength of links between nodes in networks. For example, in the social network, it is expected that we can estimate the strength or significance of friendship between users which we cannot directly observe. To develop the method for estimating eigenvectors of the scaled Laplacian matrix is our future challenges.

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