Bessel Functions and Cylindrical Geometry

Steady state temperature distribution in a semi-infinite cylinder. The energy balance in cylindrical coordinates: $\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=0$

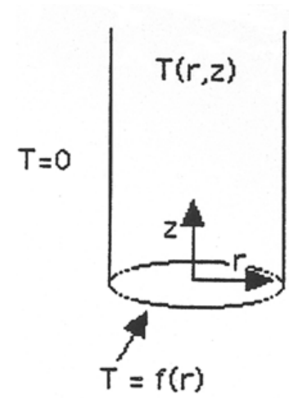
<u>Boundary Conditions</u>:

$T(1,z)=0$

$T(0,z)$ finite

$T(r,0)=f(r)$

$T(r,z)\Rightarrow 0$ as $z\to\infty$



Assume a separation of variables solution exists:
(can be shown using boundary conditions & Sturm-Liouville Thm)

$$T(r,z) = R(r)\, Z(z)$$

hence

$$Z\frac{d^2R}{dr^2}+\frac{Z}{r}\frac{dR}{dr}+R\frac{d^2T}{dz^2}=0$$

Divide by RZ to get (primes denote differentials)

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-\frac{Z''}{Z}=-\lambda^2$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-\frac{Z''}{Z}=-\lambda^2$$ (chosen to give exponentials in Z directions)

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\lambda^2R=0$$

or

$$r\frac{d^2R}{dr^2}+\frac{dR}{dr}+r\lambda^2R=0$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right)+r\lambda^2R=0$$

Remember *S-L* Equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-s(x)\,y+\lambda^2r(x)\,y=0$$

Clearly our equation is a *SL* equation:

$p(x) = r$

$s(x) = 0$

$r(x) = r \quad \Leftarrow$ weighting function

Remember that if the B.C.'s are appropriate, the solutions of this equation will be orthogonal eigenfunctions w.r.t the weight function ***r***

$$\int_0^1 R_n(\lambda_n r) R_m(\lambda_m r) \underset{\downarrow}{\overset{\downarrow \text{wt. function}}{\boldsymbol{r}}} \, dr = \delta_{mn} \quad \int_0^1 R_{mn}^2(\lambda_m r)\, r dr$$

We can also show that $R(r)$ is the well-known Bessel function:

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + r^2 \lambda^2 R = 0$$

Let $x = \lambda r \Rightarrow x/\lambda = r$

$dr = dx/\lambda$

to get

$$\left(\frac{x}{\lambda}\right)^2 \frac{d^2 R}{d(x/\lambda)^2} + \frac{x}{\lambda}\frac{dR}{d(x/\lambda)} + \left(\frac{x}{\lambda}\right)^2 \lambda^2 R = 0$$

$$x^2 \frac{d^2 R}{dx^2} + x\frac{dR}{dx} + x^2 R = 0$$

This is Bessel equation of order 0 and the solution is

$$R = a\,\underset{\uparrow 1^{\text{st}}\text{ kind}}{J_0(x)} + B\,\underset{\uparrow 2^{\text{nd}}\text{ kind}}{Y_0(x)} \text{ or}$$

$$R = a\,J_0(\lambda r) + B\,Y_0(\lambda r)$$

$$T = (r,z) = \left(A e^{-\lambda z} + B e^{+\lambda z}\right)\left(C\,J_0(\lambda r) + D\,Y_0(\lambda r)\right)$$

$T(0,z)$ finite $\Rightarrow D = 0$ because $Y_0(0) \to \infty$

$T(r, z \to \infty)$ finite $\Rightarrow B = 0$

$T(r,z) = A' e^{-\lambda z}\, J_0(\lambda \mathrm{r})$

$T(r,z) = 0 \Rightarrow J_0(\lambda) = 0$

Let λ_n be the nth root of J_0

then

$$T(r,z) = \sum_{n=1}^{\infty} A_n' e^{-\lambda_n z}\, J_0(\lambda_n \mathrm{r})$$

$$T(r,0) = f(r) = \sum_{n=1}^{\infty} A_n'\, J_0(\lambda_n \mathrm{r})$$

to solve for the $A_n^{'}$, we use the fact that the $J_0\left(\lambda_n \mathrm{r}\right)$ is orthogonal. Multiply both sides by $r\, J_0\left(\lambda_m \mathrm{r}\right) dr$ and integrate from 0 to 1:

$$\int_0^1 f\left(r\right) J_0\left(\lambda_m r\right) r\, dr = \sum_{n=1}^{\infty} A_n^{'} \int_0^1 J_0\left(\lambda_n r\right) J_0\left(\lambda_m r\right) r\, dr$$

$$= \sum_{n=1}^{\infty} \frac{A_n^{'}}{2} J_1^2\left(\lambda_m\right) \delta_{mn}$$

$$= \frac{A_m^{'}}{2} J_1^2\left(\lambda_m\right) \text{ by the properties of Bessel Functions}$$

and the solution is

$$T\left(r,z\right) = 2\sum_{n=1}^{\infty} \frac{\int_0^1 f\left(r\right) J_0\left(\lambda_n r\right) r\, dr}{J_1^2\left(\lambda_n\right)} J_0\left(\lambda_n r\right) e^{-\lambda_n z}$$

The first four values of λ_n are 2.404, 5.5.20, 8.654, 11.792.