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Introduction to Classical Field Theory

Charles G. Torre
Department of Physics, Utah State University, charles.torre@usu.edu

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C. G. Torre
Department of Physics
Utah State University

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About this text

This is a quick and informal introduction to the basic ideas and mathematical methods of classical relativistic field theory. Scalar fields, spinor fields, gauge fields, and gravitational fields are treated. The material is based upon lecture notes for a course I teach from time to time at Utah State University on Classical Field Theory.

The following is version 1.3 of the text. It is roughly the same as version 1.2. The update to 1.3 includes:

- numerous small improvements in exposition;

- fixes for a number of typographical errors and various bugs;

- a few new homework problems.

Contents

Preface

This document was created to support a course in classical field theory which gets taught from time to time here at Utah State University. In this course, hopefully, you acquire information and skills that can be used in a variety of places in theoretical physics, principally in quantum field theory, particle physics, electromagnetic theory, fluid mechanics and general relativity. As you may know, it is usually in courses on such subjects that the techniques, tools, and results we shall develop here are introduced – if only in bits and pieces as needed. As explained below, it seems better to give a unified, systematic development of the tools of classical field theory in one place. If you want to be a theoretical/mathematical physicist you must see this stuff at least once. If you are just interested in getting a deeper look at some fundamental/foundational physics and applied mathematics ideas, this is a good place to do it.

The traditional physics curriculum supports a number of classical[1] field theories. In particular, there is (i) the "Newtonian theory of gravity", based upon the Poisson equation for the gravitational potential and Newton's laws, and (ii) electromagnetic theory, based upon Maxwell's equations and the Lorentz force law. Both of these field theories appear in introductory physics courses as well as in upper level courses. Einstein provided us with another important classical field theory – a relativistic gravitational theory – via his general theory of relativity. This subject takes some investment in geometrical technology to adequately explain. It does, however, also typically get a course of its own. These courses (Newtonian gravity, electrodynamics, general relativity) are traditionally used to cover a lot of the concepts

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[1]Here, and in all that follows, the term "classical" is to mean "not quantum", *e.g.,* as in "the classical limit". Sometimes people use "classical" to also mean non-relativistic; we shall definitely not being doing that here. Indeed, every field theory we shall consider is "relativistic".

and methodology of classical field theory. The other field theories that are important (*e.g.,* Dirac, Yang-Mills, Klein-Gordon) typically arise, physically speaking, not as classical field theories but as *quantum* field theories, and it is usually in a course in quantum field theory that these other field theories are described. So, in a typical physics curriculum, it is through such courses that a student normally gets exposed to the tools and results of classical field theory. This book reflects an alternative approach to learning classical field theory, which I will now try to justify.

The traditional organization of material just described, while natural in some ways, overlooks the fact that field theories have many fundamental features in common – features which are most easily understood in the classical limit – and one can get a really good feel for what is going on in "the big picture" by exploring these features in a general, systematic way. Indeed, many of the basic structures appearing in classical field theory (Lagrangians, field equations, symmetries, conservation laws, gauge transformations, *etc.*) are of interest in their own right, and one should try to master them in general, not just in the context of a particular example (e.g., electromagnetism), and not just in passing as one studies the quantum field. Moreover, once one has mastered a sufficiently large set of such field theoretic tools, one is in a good position to discern how this or that field theory differs from its cousins in some key structural way. This serves to highlight physical and mathematical ingredients that make each theory special.

From a somewhat more pragmatic point of view, let me point out that most quantum field theory texts rely quite heavily upon one's facility with the techniques of classical field theory. This is, in part, because many quantum mechanical structures have analogs in a classical approximation to the theory. By understanding the "lay of the land" in the classical theory through a course such as this one, one gets a lot of insight into the associated quantum field theories. It is hard enough to learn quantum field theory without having to also assimilate at the same time concepts that are already present in the much simpler setting of classical field theory. So, if you are hoping to learn quantum field theory some day, this class should help out quite a bit.

A final motivation for the creation and teaching of this course is to support the research activities of a number of faculty and students here at Utah State University. The geometric underpinnings of classical field theory feature in a wide variety of research projects here. If you want to find out what is going on in this research – or even participate – you need to speak the language.

I have provided a number of problems you can use to facilitate your

learning the material. They are presented first within the text in order to amplify the text and to give you a contextual hint about what is needed to solve the problem. At the end of each chapter the problems which have appeared are summarized for your convenience.

I would like to thank the numerous students who have endured the rough set of notes from which this document originated and who contributed numerous corrections. I also would like to thank Joseph Romano for his (unfortunately rather lengthy) list of corrections. Finally, I would like to acknowledge Ian Anderson for his influence on my geometric point of view regarding Lagrangians and differential equations.

Chapter 1

What is a classical field theory?

In physical terms, a classical field is a dynamical system with an infinite number of degrees of freedom labeled by spatial location. In mathematical terms, a classical field is a section of some fiber bundle which obeys some PDEs. Of course, there is much, much more to the story, but this is at least a good first pass at a definition. By contrast, a *mechanical system* is a dynamical system with a finite number of degrees of freedom that is described by ODEs.

One place where classical fields naturally occur in physics is when non-rigid extended bodies such as bodies of water, elastic solids, strings under tension, portions of the atmosphere, and so forth, are described using a classical (as opposed to quantum) continuum approximation to their structure. Another place where classical fields arise – and this place is where we will focus most of our attention – is in a more or less fundamental description of matter and its interactions. Here the ultimate description is via quantum field theory, but the classical approximation sometimes has widespread macroscopic validity (*e.g.,* Maxwell theory) or the classical approximation can be very useful for understanding the structure of the theory. Here are some simple examples of classical fields in action, just to whet your appetite. Some of these theories will be explored in detail later.

1.1   Example: waves in an elastic medium

Let us denote by $u(\mathbf{r}, t)$ the displacement of some observable characteristic of an elastic medium at the position $\mathbf{r}$ at time t from its equilibrium value.

For example, if the medium is the air surrounding you, u could represent its compression/rarefaction relative to some standard value. For small displacements, the medium can often be well-modeled by supposing that the *displacement field u* satisfies the *wave equation*

$$\frac{1}{c^2}u_{,tt} - \nabla^2 u = 0, \tag{1.1}$$

where the comma notation indicates partial derivatives, ∇^2 is the Laplacian, and c is a parameter representing the speed of sound in the medium. We say that u is the *field variable* and that the wave equation is the *field equation*.

1.2 Example: Newtonian gravitational field

This is the original field theory. Here the field variable is a function $\phi(\mathbf{r}, t)$, the *gravitational potential*. The gravitational force $\mathbf{F}(\mathbf{r}, t)$ exerted on a (test) mass m at spacetime position $(\mathbf{r}, t)$ is given by

$$\mathbf{F}(\mathbf{r}, t) = -m\nabla\phi(\mathbf{r}, t). \tag{1.2}$$

The gravitational potential is determined by the mass distribution that is present via the field equation

$$\nabla^2\phi = -4\pi G\rho, \tag{1.3}$$

where $\rho = \rho(\mathbf{r}, t)$ is the mass density function and G is Newton's constant. Equation (1.3) is just Poisson's equation, of course.

Equations (1.2) and (1.3) embody a pattern in nature which persists even in the most sophisticated physical theories. The idea is as follows. Matter interacts via fields. The fields are produced by the matter, and the matter is acted upon by the fields. In the present case, the gravitational field, represented by the "scalar field" ϕ, determines the motion of mass via (1.2) and Newton's second law. Conversely, mass determines the gravitational field via (1.3). The next example also exhibits this fundamental pattern of nature.

1.3 Example: Maxwell's equations

Here the electromagnetic interaction of particles is mediated by a pair of vector fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r},t)$, the electric and magnetic fields. The force

on a test particle with electric charge q at spacetime event $(\mathbf{r}, t)$ is given by

$$\mathbf{F}(\mathbf{r}, t) = q\mathbf{E}(\mathbf{r}, t) + \frac{q}{c}\left[\mathbf{v}(t) \times \mathbf{B}(\mathbf{r}, t)\right], \tag{1.4}$$

where $\mathbf{v}(t)$ is the particle's velocity at time t and c is the speed of light in vacuum.[1] The electromagnetic field is determined from the charge distribution that is present via the Maxwell equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{1.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.6}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \tag{1.7}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.8}$$

where ρ is the electric charge density and $\mathbf{j}$ is the electric current density.

As we shall see later, the electromagnetic field is also fruitfully described using potentials. They are defined via

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1.9}$$

In each of these examples, the field variable(s) are u, ϕ, $(\mathbf{E}, \mathbf{B})$ – or $(\phi, \mathbf{A})$, and they are determined by PDEs. The "infinite number of degrees of freedom" idea is that, roughly speaking, the general solution to the field equations (wave, Poisson, or Maxwell) involves arbitrary functions. Thus the space of solutions to the PDEs – physically, the set of field configurations permitted by the laws of physics – is infinite dimensional.

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[1]We are using Gaussian units for the electromagnetic field.

Chapter 2

Klein-Gordon field

The simplest relativistic classical field is the Klein-Gordon field. It and its various generalizations are used throughout theoretical physics.

2.1 The Klein-Gordon equation

To begin with we can think of the Klein-Gordon field as simply a function on spacetime, also known as a *scalar field*, $\varphi\colon \mathbf{R}^4 \to \mathbf{R}$. Introduce coordinates $x^\alpha = (t, x, y, z)$ on $\mathbf{R}^4$. The field equation – known as the *Klein-Gordon equation* – is given by

$$\Box\varphi - m^2\varphi = 0, \tag{2.1}$$

where $\Box$ is the wave operator (or "d'Alembertian"),

$$
\begin{aligned}
\Box\varphi &= -\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \\
&= -\frac{\partial^2\varphi}{\partial t^2} + \nabla^2\varphi \\
&= -\partial_t^2\varphi + \partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi \\
&= -\varphi_{,tt} + \varphi_{,xx} + \varphi_{,yy} + \varphi_{,zz},
\end{aligned}
\tag{2.2}
$$

and m is a parameter known as the *mass* of the Klein-Gordon field.[1] (Here I have taken the liberty to show you most of the notations for derivatives that

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[1]Here we are using units in which $\hbar = c = 1$. It is a nice exercise to put things in terms of SI units.

you will see in this text.) You can see that the Klein-Gordon (KG) equation is just a simple generalization of the wave equation, which it reduces to when $m = 0$. In quantum field theory, quantum states of the Klein-Gordon field can be characterized in terms of "particles" with rest mass m and no other structure (*e.g.,* no spin, no electric charge, *etc.*) So the Klein-Gordon field is physically (and mathematically, too) the simplest of the relativistic fields that one can study.

If you like, you can view the Klein-Gordon equation as a "toy model" for the Maxwell equations which describe the electromagnetic field. The quantum electromagnetic field is characterized by "photons" which have vanishing rest mass, no electric charge, but they do carry intrinsic "spin". Coherent states of the quantum electromagnetic field contain many, many photons and are well approximated using "classical" electromagnetic fields satisfying the Maxwell equations. Likewise, you can imagine that coherent states involving many "Klein-Gordon particles" (sometimes called scalar mesons) are well described by a classical scalar field satisfying the Klein-Gordon equation.

The KG equation originally arose in an attempt to give a relativistic generalization of the Schrödinger equation. The idea was to let φ be the complex-valued wave function describing a spinless particle of mass m. But this idea didn't quite work out as expected (see below for a hint as to what goes wrong). Later, when it was realized that a more viable way to do quantum theory in a relativistic setting was via quantum field theory, the KG equation came back as a field equation for a quantum field whose classical limit is the KG equation above. The role of the KG equation as a sort of relativistic Schrödinger equation does survive the quantum field theoretic picture, however. The story is too long to go into in this course, but we will give a hint as to what this means in a moment.

2.2   Solving the KG equation

Let us have a closer look at the KG equation and its solutions. The KG equation is a linear PDE with constant coefficients. One standard strategy for solving such equations is via Fourier analysis. To this end, let us suppose that φ is sufficiently well behaved[2] so that we have the following Fourier

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[2] "Sufficiently well behaved" could mean for example that, for each t, $\varphi \in L^1(\mathbf{R}^3)$ or $\varphi \in L^2(\mathbf{R}^3)$.

expansion of φ at any given value of t:

$$\varphi(t, \mathbf{r}) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, \hat{\varphi}_{\mathbf{k}}(t) \, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.3}$$

where $\mathbf{k} = (k_x, k_y, k_z) \in \mathbf{R}^3$, and the complex-valued Fourier transform satisfies

$$\hat{\varphi}_{-\mathbf{k}} = \hat{\varphi}_{\mathbf{k}}^*, \tag{2.4}$$

since φ is a real-valued function. The KG equation implies the following equation for the Fourier transform $\hat{\varphi}_{\mathbf{k}}(t)$ of φ:

$$\ddot{\hat{\varphi}}_{\mathbf{k}} + (k^2 + m^2)\hat{\varphi}_{\mathbf{k}} = 0. \tag{2.5}$$

This equation is, of course, quite easy to solve. We have

$$\hat{\varphi}_{\mathbf{k}}(t) = a_{\mathbf{k}} e^{-i\omega_k t} + b_{\mathbf{k}} e^{i\omega_k t}, \tag{2.6}$$

where

$$\omega_k = \sqrt{k^2 + m^2}, \tag{2.7}$$

and $a_{\mathbf{k}}$, $b_{\mathbf{k}}$ are complex constants for each $\mathbf{k}$. The reality condition (2.4) on $\hat{\varphi}_{\mathbf{k}}$ implies

$$b_{-\mathbf{k}} = a_{\mathbf{k}}^*, \quad \forall \, \mathbf{k}, \tag{2.8}$$

so that φ solves the KG equation if and only if it takes the form

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, \left(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t} + a_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\right). \tag{2.9}$$

PROBLEM: *Verify (2.4)–(2.9).*

Let us pause and notice something familiar here. Granted a little Fourier analysis, the KG equation is, via (2.5), really just an infinite collection of uncoupled "harmonic oscillator equations" for the real and imaginary parts of $\hat{\varphi}_k(t)$ with "natural frequency" ω_k. Thus we can see quite explicitly how the KG field is akin to a dynamical system with an infinite number of degrees of freedom. Indeed, if we label the degrees of freedom with the Fourier wave vector $\mathbf{k}$ then each degree of freedom is a harmonic oscillator. It is this interpretation which is used to make the (non-interacting) quantum Klein-Gordon field: each harmonic oscillator is given the usual quantum mechanical treatment.

2.3   A small digression: one particle wave functions

From our form (2.9) for the general solution of the KG equation we can take a superficial glimpse at how this equation is used as a sort of Schrödinger equation in the quantum field theory description. To begin, we note that $\varphi = \varphi^+ + \varphi^-$, where

$$\varphi^+ := \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_k t}, \tag{2.10}$$

and

$$\varphi^- := \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, a_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t} \tag{2.11}$$

are each complex-valued solutions to the KG equation. These are called the *positive frequency* and *negative frequency* solutions of the KG equation, respectively. Let us focus on the positive frequency solutions. They satisfy the KG equation, but they also satisfy the stronger equation

$$i\frac{\partial\varphi^+}{\partial t} = \sqrt{-\nabla^2 + m^2}\,\varphi^+, \tag{2.12}$$

where the square root operator is defined via Fourier analysis to be[3]

$$\sqrt{-\nabla^2 + m^2}\,\varphi^+ := \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, \omega_k a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_k t}. \tag{2.13}$$

Thus the positive frequency solutions of the KG equation satisfy a Schrödinger equation with Hamiltonian

$$H = \sqrt{-\nabla^2 + m^2}. \tag{2.14}$$

This Hamiltonian can be interpreted as the kinetic energy of a relativistic particle (in the reference frame labeled by spacetime coordinates (t, x, y, z)). It is possible to give a relativistically invariant normalization condition on the positive frequency wave functions so that one can use them to compute probabilities for the outcome of measurements made on the particle. Thus

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[3]The domain of this operator will necessarily be limited to a subspace of functions.

the positive frequency solutions are sometimes called the "one-particle wave functions".

In a quantum field theoretic treatment, the (normalizable) positive frequency solutions represent the wave functions of KG particles. What about the negative frequency solutions? There are difficulties in using the negative frequency solutions to describe KG particles. For example, you can easily check that they satisfy a Schrödinger equation with a *negative* kinetic energy, which is unphysical. Moreover, the relativistic inner product with respect to which one can normalize the positive frequency solutions leads to a *negative* norm for the negative frequency solutions. This means that the negative frequency part of the solution to the KG equation cannot be used to describe the quantum mechanics of a single particle. In quantum field theory (as opposed to quantum mechanics), the negative frequency solutions are interpreted in terms of the possibility for destruction of particles. Quantum field theory, you see, allows for creation and destruction of particles. But now we are going too far afield. . .

2.4 Variational principle

In physics, fundamental theories always arise from a variational principle. Ultimately this stems from their roots as quantum mechanical systems, as can be seen from Feynman's path integral formalism. For us, the presence of a variational principle is a very powerful tool for organizing the information in a field theory. Presumably you have seen a variational principle or two in your life. I will not assume you are particularly proficient with the calculus of variations – one of the goals of this course is to make you better at it – but I will assume that you are familiar with the basic strategy of a variational principle.

So, consider *any* map $\varphi \colon \mathbf{R}^4 \to \mathbf{R}$ (not necessarily satisfying any field equations) and consider the following integral

$$S[\varphi] = \int_{\mathcal{R} \subset \mathbf{R}^4} d^4x \, \frac{1}{2} \left(\varphi_{,t}^2 - (\nabla \varphi)^2 - m^2 \varphi^2 \right). \tag{2.15}$$

The region $\mathcal{R}$ can be anything you like at this point, but typically we assume

$$\mathcal{R} = \{(t, x, y, z) \,|\, t_1 \leq t \leq t_2, \ -\infty < x, y, z < \infty\}. \tag{2.16}$$

We restrict attention to fields such that $S[\varphi]$ exists. For example, we can assume that φ is always a smooth function of compact spatial support. The value of the integral, of course, depends upon which function φ we choose, so the formula (2.15) assigns a real number to each function φ. We say that $S = S[\varphi]$ is a *functional* of φ. We will use this functional to obtain the KG equation by a variational principle. When a functional can be used in this manner it is called an *action functional* for the field theory.

The variational principle goes as follows. Consider any family of functions, labeled by a parameter λ, which includes some given function φ_0 at $\lambda = 0$. We say we have a *one-parameter family of fields*, φ_λ. As a random example, we might have

$$\varphi_\lambda = \cos(\lambda(t + x))\, e^{-(x^2+y^2+z^2)}. \tag{2.17}$$

You can think of φ_λ as defining a curve in the "space of fields" which passes through the point φ_0.[4] We can evaluate the functional S along this curve; the value of the functional defines an ordinary function of λ, again denoted S:

$$S(\lambda) := S[\varphi_\lambda]. \tag{2.18}$$

Note that different choices of curve φ_λ will determine different functions $S(\lambda)$. We now define a *critical point* φ_0 of the action functional $S[\varphi]$ to be a "point in the space of fields", that is, a field $\varphi = \varphi_0(x)$, which defines a critical point of the function $S(\lambda)$ for *any* curve passing through φ_0.

This way of defining a critical point is a natural generalization to the space of fields of the usual notion of critical point from multi-variable calculus. Recall that in ordinary calculus a critical point (x_0, y_0, z_0) of a function f is a point where all the first derivatives of f vanish,

$$\partial_x f(x_0, y_0, z_0) = \partial_y f(x_0, y_0, z_0) = \partial_z f(x_0, y_0, z_0) = 0. \tag{2.19}$$

This is equivalent to the vanishing of the rate of change of f along any curve through (x_0, y_0, z_0). To see this, define the parametric form of a curve $\vec{x}(\lambda)$ passing through $\vec{x}(0) = (x_0, y_0, z_0)$ via

$$\vec{x}(\lambda) = (x(\lambda), y(\lambda), z(\lambda)) \tag{2.20}$$

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[4]The space of fields is the set of all allowed functions on spacetime. It can be endowed with enough structure to view it as a smooth manifold whose points are the allowed functions. The curve in equation (2.17) passes through the point $\varphi = e^{-(x^2+y^2+z^2)}$ at $\lambda = 0$.

with $\lambda = 0$ corresponding to the point $\vec{x}_0$ on the curve,

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0). \tag{2.21}$$

The tangent vector $T(\lambda)$ to the curve at the point $\vec{x}(\lambda)$ has Cartesian components

$$\vec{T}(\lambda) = (x'(\lambda), y'(\lambda), z'(\lambda)), \tag{2.22}$$

and the rate of change of a function $f = f(x, y, z)$ along the curve at the point $\vec{x}(\lambda)$ is given by the directional derivative along $\vec{T}$:

$$\vec{T}(\lambda) \cdot \nabla f \Big|_{\vec{x} = \vec{x}(\lambda)} = x'(\lambda) \partial_x f(\vec{x}(\lambda)) + y'(\lambda) \partial_y f(\vec{x}(\lambda)) + z'(\lambda) \partial_z f(\vec{x}(\lambda)). \tag{2.23}$$

It should be apparent from this equation that the vanishing of the rate of change of f along *any* curve passing through the point $\vec{x}_0$ is equivalent to the vanishing of the gradient of f at $\vec{x}_0$, which is the same as the vanishing of all the first derivatives of f at $\vec{x}_0$. It is this interpretation of "critical point" in terms of vanishing rate of change along any curve that we generalize to the infinite-dimensional space of fields.

We shall show that the critical points of the functional $S[\varphi]$ correspond to functions on spacetime which solve the KG equation. To this end, let us consider a curve that passes through a putative critical point φ at, say, $\lambda = 0.$[5] This is easy to arrange. For example, let $\hat{\varphi}_\lambda$ be any curve in the space of fields. Define φ_λ via

$$\varphi_\lambda = \hat{\varphi}_\lambda - \hat{\varphi}_0 + \varphi. \tag{2.24}$$

If $\varphi_{\lambda=0} = \varphi$ is a critical point, then

$$\delta S := \left(\frac{dS(\lambda)}{d\lambda} \right)_{\lambda=0} = 0. \tag{2.25}$$

We call δS the *first variation of the action*; its vanishing for all curves through φ is the condition for a critical point. We can compute δS explicitly by applying (2.25) to $S[\varphi_\lambda]$; we find

$$\delta S = \int_{\mathcal{R}} d^4x \left(\varphi_{,t} \delta \varphi_{,t} - \nabla \varphi \cdot \nabla \delta \varphi - m^2 \varphi \delta \varphi \right), \tag{2.26}$$

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[5]We now drop the distracting subscript 0.

where the function $\delta\varphi$ – the *variation of* φ – is defined by

$$\delta\varphi := \left(\frac{d\varphi(\lambda)}{d\lambda}\right)_{\lambda=0}. \tag{2.27}$$

The critical point condition means that $\delta S = 0$ for *all* variations of φ, and we want to see what that implies about the critical point φ.

To this end we observe that $\delta\varphi$ is a completely arbitrary function (aside from regularity and boundary conditions to be discussed below). To see this, let ψ be any function you like and consider the curve

$$\varphi_\lambda = \varphi + \lambda\psi, \tag{2.28}$$

so that

$$\delta\varphi = \psi. \tag{2.29}$$

To make use of the requirement that $\delta S = 0$ must hold for arbitrary $\delta\varphi$, we "integrate by parts" in δS via the divergence theorem:

$$\delta S = \int_{\mathcal{R}} d^4x \left(-\varphi_{,tt} + \nabla^2\varphi - m^2\varphi \right)\delta\varphi$$
$$+ \left[\int_{\mathbf{R}^3} d^3x\, \varphi_{,t}\delta\varphi \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \int_{r\to\infty} d^2A\, \mathbf{n}\cdot\nabla\varphi\delta\varphi \tag{2.30}$$

Here the last two terms are the boundary contributions from $\partial\mathcal{R}$. For concreteness, I have assumed that

$$\mathcal{R} = [t_1, t_2] \times \mathbf{R}^3. \tag{2.31}$$

The last integral is over the "sphere at infinity" in $\mathbf{R}^3$, with $\mathbf{n}$ being the outward unit normal to that sphere.

If you need a little help seeing where (2.30) came from, the key is to write

$$\varphi_{,t}\delta\varphi_{,t} - \nabla\varphi\cdot\nabla\delta\varphi = \partial_t(\varphi_{,t}\delta\varphi) - \nabla\cdot(\nabla\varphi\delta\varphi) - \varphi_{,tt}\delta\varphi + (\nabla^2\varphi)\delta\varphi. \tag{2.32}$$

The first term's time integral is easy to perform, and the second term's spatial integral can be evaluated using the divergence theorem.

To continue with our analysis of (2.30), we make two assumptions regarding the boundary conditions to be placed on our various fields. First, we

note that φ, and φ_λ, and hence $\delta\varphi$, must vanish at spatial infinity ($r \to \infty$ at fixed t) in order for the action integral to converge. Further, we assume that φ and $\delta\varphi$ vanish as $r \to \infty$ fast enough so that in δS the boundary integral over the sphere at infinity vanishes. One way to do this systematically is to assume that all fields have "compact support" in space, that is, at each time t they all vanish outside of some bounded region in $\mathbf{R}^3$. Other asymptotic conditions are possible, but since the area element (dA) in the integral over the sphere grows like r^2 the integrand should fall off faster than $1/r^2$ as $r \to \infty$ for this boundary term to vanish.

Secondly, we hold fixed the initial and final values of the fields – a step which should be familiar to you from the variational formulation of classical mechanics. To this end we fix two functions

$$\phi_1, \phi_2 : \mathbf{R}^3 \to \mathbf{R} \tag{2.33}$$

and we assume that at t_1 and t_2, for any allowed φ (not just the critical point),

$$\varphi|_{t_1} = \phi_1, \quad \varphi|_{t_2} = \phi_2. \tag{2.34}$$

The functions ϕ_1 and ϕ_2 are fixed but arbitrary, subject to the asymptotic conditions as $r \to \infty$. Now, for our one parameter family of fields we also demand

$$\varphi_\lambda|_{t_1} = \phi_1, \quad \varphi_\lambda|_{t_2} = \phi_2, \tag{2.35}$$

which forces

$$\delta\varphi|_{t_1} = 0 = \delta\varphi|_{t_2}. \tag{2.36}$$

This forces the vanishing of the first term in the boundary contribution to δS in (2.30).

With these boundary conditions, we see that the assumption that φ is a (smooth) critical point implies that

$$0 = \int_{\mathcal{R}} d^4x \, \left(-\varphi_{,tt} + \nabla^2\varphi - m^2\varphi\right) \delta\varphi \tag{2.37}$$

for any function $\delta\varphi$ subject to (2.36) and the asymptotic conditions just described. Now, it is a standard theorem in calculus[6] that this implies

$$- \varphi_{,tt} + \nabla^2\varphi - m^2\varphi = 0 \tag{2.38}$$

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[6]The proof goes by choosing $\delta\varphi$ to be a localized "bump function" at any chosen point in $\mathcal{R}$.

everywhere in the region $\mathcal{R}$.

This, then, is the variational principle for the KG equation. The critical points of the KG action, subject to the two types of boundary conditions we described (asymptotic conditions at spatial infinity and initial/final boundary conditions), are the solutions of the KG equation.

I would understand if, at this point, you are thinking: "Why would I want to replace a relatively simple PDE with all this complicated variational stuff?" The payoff for this investment in variational technology turns out to be quite large, as I hope you will see by the end of this course.

2.5   Getting used to δ. Functional derivatives.

I have already introduced in passing the notation δ, known colloquially as a "variation". For any quantity $W[\varphi]$ built from the field φ, and for any one parameter family of fields φ_λ, we define

$$\delta W := \left(\frac{d}{d\lambda} W[\varphi_\lambda] \right)_{\lambda=0}. \tag{2.39}$$

Evidently, δW is the change in W given by letting λ be displaced an infinitesimal amount $d\lambda$ from $\lambda = 0$. A couple of important properties of δ are: (i) it is a linear operation obeying the Leibniz product rule (it is a derivation); (ii) variations commute with differentiation, *e.g.*,

$$\delta(\varphi_{,\alpha}) = \partial_\alpha(\delta\varphi) \equiv \delta\varphi_{,\alpha}. \tag{2.40}$$

In the last section we computed the *first variation* of the KG action:

$$\delta S[\varphi] = \int_{\mathcal{R}} d^4x \left(\varphi_{,t}\delta\varphi_{,t} - \nabla\varphi \cdot \nabla\delta\varphi - m^2\varphi\delta\varphi \right). \tag{2.41}$$

By definition, if the first variation of a functional $S[\varphi]$ can be expressed as

$$\delta S[\varphi] = \int_{\mathcal{R}} d^4x \, F(x) \, \delta\varphi(x) \tag{2.42}$$

then we say that S is *differentiable* and that

$$F(x) \equiv \frac{\delta S}{\delta\varphi(x)} \tag{2.43}$$

is the *functional derivative* of the action with respect to φ. For a differentiable functional $S[\varphi]$, then, we write

$$\delta S[\varphi] = \int_{\mathcal{R}} d^4x \, \frac{\delta S}{\delta \varphi} \delta \varphi. \tag{2.44}$$

We have seen that, with our choice of boundary conditions, the KG action is differentiable and that

$$\frac{\delta S}{\delta \varphi} = -\varphi_{,tt} + \nabla^2 \varphi - m^2 \varphi. \tag{2.45}$$

In general, the idea of variational principles is to encode the field equations into an action $S[\varphi]$ so that they arise as the equations

$$\frac{\delta S}{\delta \varphi} = 0. \tag{2.46}$$

2.6   The Lagrangian

Following the pattern of classical mechanics, I will now introduce the notion of a *Lagrangian L* for the KG equation, which is defined as an integral at fixed time t:

$$L = \int_{\mathbf{R}^3} d^3x \, \frac{1}{2} \left(\varphi_{,t}^2 - (\nabla \varphi)^2 - m^2 \varphi^2 \right), \tag{2.47}$$

so that

$$S[\varphi] = \int_{t_1}^{t_2} dt \, L. \tag{2.48}$$

In classical mechanics, the Lagrangian is a function of the independent variables, the dependent variables (the "degrees of freedom"), and the derivatives of the dependent variables. In field theory we have, in effect, degrees of freedom labeled by spatial points. We then have the possibility of expressing the Lagrangian as an integral over space (a sum over the degrees of freedom). In the KG theory we have that

$$L = \int_{\mathbf{R}^3} d^3x \, \mathcal{L}, \tag{2.49}$$

where

$$\mathcal{L} = \frac{1}{2} \left((\varphi_{,t})^2 - (\nabla \varphi)^2 - m^2 \varphi^2 \right) \tag{2.50}$$

is called the *Lagrangian density.* At a point (t, x, y, z), the Lagrangian density for the KG field depends on the values of the field φ and its first derivatives at (t, x, y, z). We say that $\mathcal{L}$ is a *local function of the field.*[7] Theories like the KG theory which admit an action which is a spacetime integral of a local Lagrangian density are called *local field theories.*

Finally, notice that the Lagrangian for the KG theory can be viewed as having the same structure as that for a finite dimensional dynamical system in non-relativistic Newtonian mechanics, namely, $L = T - U$, where T can be viewed as a kinetic energy for the field,

$$T = \int_{\mathbf{R}^3} d^3x \, \frac{1}{2}\varphi_{,t}^2 \tag{2.51}$$

and U plays the role of potential energy:

$$U := \int_{\mathbf{R}^3} d^3x \, \frac{1}{2} \left((\nabla\varphi)^2 + m^2\varphi^2\right). \tag{2.52}$$

Evidently, we can view $\frac{1}{2}\varphi_{,t}^2$ as the kinetic energy density, and view $(\nabla\varphi)^2 + m^2\varphi^2$ as the potential energy density.

2.7 The Euler-Lagrange equations

We have seen that the Lagrangian density $\mathcal{L}$ of the KG theory is a local function of the KG field. We can express the functional derivative of the KG action purely in terms of the Lagrangian density. To see how this is done, we note that

$$\delta\mathcal{L} = \varphi_{,t}\delta\varphi_{,t} - \nabla\varphi \cdot \nabla\delta\varphi - m^2\varphi\delta\varphi$$
$$= \left(-\varphi_{,tt} + \nabla^2\varphi - m^2\varphi\right)\delta\varphi + \frac{\partial}{\partial x^\alpha}V^\alpha, \tag{2.53}$$

where $x^\alpha = (t, x, y, z)$, $\alpha = 0, 1, 2, 3$, and we are using the Einstein summation convention,

$$\frac{\partial}{\partial x^\alpha}V^\alpha \equiv \sum_\alpha \frac{\partial}{\partial x^\alpha}V^\alpha. \tag{2.54}$$

---

[7]A non-local function of the field would depend upon the field and/or its derivatives at other points as well.

Here

$$V^0 = \varphi_{,t}\delta\varphi, \quad V^i = -(\nabla\varphi)^i\delta\varphi. \tag{2.55}$$

The term involving V^α is a four-dimensional *divergence* and leads to the boundary contributions to the variation of the action via the divergence theorem. Assuming the boundary conditions are such that these terms vanish, we see that the functional derivative of the action is computed by (1) varying the Lagrangian density, and (2) rearranging terms to move all derivatives of the field variations into divergences and (3) throwing away the divergences.

We now give a slightly more general way to think about this last computation, which is very handy for certain purposes. This point of view is developed more formally in the next section.

First, we view the formula giving the definition of the Lagrangian as a function of 9 variables

$$\mathcal{L} = \mathcal{L}(x, \varphi, \varphi_\alpha), \tag{2.56}$$

where now, formally, φ and φ_α are just a set of 5 variables upon which the Lagrangian density depends.[8] (The KG Lagrangian density does not actually depend upon x^α except through the field, so in this example $\mathcal{L} = \mathcal{L}(\varphi, \varphi_\alpha)$, but it is useful to allow for this possibility in the future.) This 9 dimensional space is called the *first jet space* for the scalar field. From this point of view, the field φ does not depend upon x^α and neither does φ_α. The fields are recovered as follows. For each function $f(x)$ there is a field obtained as a graph in the 5-dimensional space of (x^α, φ), specified by $\varphi = f(x)$. Similarly, in this setting we do not view φ_α as the derivatives of φ; given a function $f(x)$ we can extend the graph into the 9-dimensional space $(x^\alpha, \varphi, \varphi_\alpha)$ via $(\varphi = f(x), \varphi_\alpha = \partial_\alpha f(x))$. We can keep going like this. For example, we could work on a space parametrized by $(x^\alpha, \varphi, \varphi_\alpha, \varphi_{\alpha\beta})$, where $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}$ parametrizes the values of the second derivatives. This space is the *second jet space*; it has dimension 19 (exercise)! Given a field $\varphi = f(x)$ we have a graph in this 19 dimensional space given by $(x^\alpha, f(x), \partial_\alpha f(x), \partial_\alpha\partial_\beta f(x))$.

Next, for any formula $F(x, \varphi, \varphi_\alpha)$ built from the coordinates, the fields, and the first derivatives of the fields, introduce the *total derivative*

$$D_\alpha F(x, \varphi, \varphi_\alpha) = \frac{\partial F}{\partial x^\alpha} + \frac{\partial F}{\partial \varphi}\varphi_\alpha + \frac{\partial F}{\partial \varphi_\beta}\varphi_{\alpha\beta}. \tag{2.57}$$

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[8]Notice that we temporarily drop the comma in the notation for the derivative of φ. This is just to visually enforce our new point of view. You can mentally replace the comma if you like. We shall eventually put it back to conform to standard physics notation.

The total derivative just implements in this new setting the calculation of spacetime derivatives of F via the chain rule. In particular, if we imagine substituting a specific field, $\varphi = f(x)$ into the formula F, then F becomes a function $\mathcal{F}$ of x only:

$$\mathcal{F}(x) = F(x, f(x), \partial_\alpha f(x)). \tag{2.58}$$

The total derivative of F, when restricted to $\varphi = f(x)$, is the same as the derivative of $\mathcal{F}$:

$$D_\alpha F(x, \varphi, \varphi_\alpha)\Big|_{\varphi = f(x)} = \partial_\alpha \mathcal{F}(x). \tag{2.59}$$

We can extend this apparatus to include the jets of field variations $\delta\varphi$ and $\delta\varphi_\alpha$. The variation of the Lagrangian density is then defined as

$$\delta\mathcal{L} := \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\delta\varphi_\alpha. \tag{2.60}$$

As a nice exercise you should verify that $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\right)\delta\varphi + D_\alpha V^\alpha, \tag{2.61}$$

where

$$V^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\delta\varphi. \tag{2.62}$$

We define the *Euler-Lagrange derivative* of (or *Euler-Lagrange expression* for) the Lagrangian density via

$$\mathcal{E}(\mathcal{L}) := \frac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\frac{\partial\mathcal{L}}{\partial\varphi_\alpha}. \tag{2.63}$$

Evidently, with our boundary conditions the functional derivative of the KG action is the same as the EL derivative of the Lagrangian density (evaluated on a field $(\varphi = \varphi(x), \varphi_\alpha = \partial_\alpha\varphi(x), \varphi_{\alpha\beta} = \partial_\alpha\partial_\beta\varphi(x))$). We have

$$\frac{\delta S}{\delta\varphi} = \mathcal{E}(\mathcal{L})\Big|_{\varphi = \varphi(x)} = -\varphi_{,tt} + \nabla^2\varphi - m^2\varphi, \tag{2.64}$$

and the KG field equation is the *Euler-Lagrange equation* of the Lagrangian density $\mathcal{L}$:

$$\mathcal{E}(\mathcal{L}) = 0. \tag{2.65}$$

The reason I introduce you to all this jet space formalism is that often in field theory we want to manipulate a formula such as $\mathcal{L}(x, \varphi, \partial\varphi)$ using the ordinary rules of multivariable calculus, *e.g.,* calculate $\frac{\partial\mathcal{L}}{\partial x^\alpha}$, so that we are viewing the Lagrangian as a function of 9 variables. For example, it will be useful to know when a Lagrangian density does not depend explicitly on the spacetime coordinates. In the jet space formalism this is easy: $\frac{\partial\mathcal{L}}{\partial x^\alpha} = 0$. Indeed, the KG Lagrangian density has this property. If we view the Lagrangian density as a function of $\varphi(x)$, it his hard to usefully interpret such a condition – except that the Lagrangian is somehow a trivial constant. A more detailed discussion of jet spaces occurs in the next section. For the most part in this book, I will not use the fancy jet space formalism unless I really need it. Instead I will follow the usual physicist way of doing things where one always imagines that objects like Lagrangian densities are evaluated on fields $\varphi = \varphi(x)$ and the partial derivative is really the total derivative as in (2.59).

PROBLEM: Compute the Euler-Lagrange derivative of the KG Lagrangian density (2.50) and explicitly verify that the Euler-Lagrange equation is indeed the KG equation.

PROBLEM: Consider a Lagrangian density that is a divergence:

$$\mathcal{L} = D_\alpha W^\alpha, \tag{2.66}$$

where

$$W^\alpha = W^\alpha(\varphi). \tag{2.67}$$

Show that

$$\mathcal{E}(\mathcal{L}) \equiv 0. \tag{2.68}$$

so the EL equations are trivial $(0 = 0)$.

2.8   Jet Space

Here I will sketch some of the elements of a general jet space description of a field theory. To begin, we generalize to the case where the Lagrangian is a (local) function of the spacetime location, the fields, and the derivatives of the fields to any finite order. We write

$$\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi, \partial^2\varphi, \dots, \partial^k\varphi). \tag{2.69}$$

Viewed this way, the Lagrangian is a function on a large but finite-dimensional space called the kth *jet space* for the field theory. We denote this space by $\mathcal{J}^k$. Remarkably, if we vary $\mathcal{L}$ we can always rearrange things so that all derivatives of $\delta\varphi$ appear inside a total divergence. We have the *Euler-Lagrange identity*:

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha, \tag{2.70}$$

where the general form for the Euler-Lagrange derivative is given by

$$\mathcal{E}(\mathcal{L}) := \frac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}} + D_\alpha D_\beta\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha\beta}} - \cdots + (-1)^k D_{\alpha_1}\cdots D_{\alpha_k}\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha_1\cdots\alpha_k}}, \tag{2.71}$$

and where the general form of the total derivative operator on a function

$$F = F(x, \varphi, \partial\varphi, \partial^2\varphi, \ldots, \partial^k\varphi), \tag{2.72}$$

is given by

$$D_\alpha F = \frac{\partial F}{\partial x^\alpha} + \frac{\partial F}{\partial\varphi}\varphi_{,\alpha} + \frac{\partial F}{\partial\varphi_{,\beta}}\varphi_{,\alpha\beta} + \cdots + \frac{\partial F}{\partial\varphi_{,\alpha_1\cdots\alpha_k}}\varphi_{,\alpha\alpha_1\cdots\alpha_k} \tag{2.73}$$

Here we use the comma notation for (would-be) derivatives in conformation with standard notation in physics. Notice that the total derivative of a function on $\mathcal{J}^k$ is a function on $\mathcal{J}^{k+1}$.

From the total derivative formula, it follows that divergences have trivial Euler-Lagrange derivatives

$$\mathcal{E}(D_\alpha V^\alpha) = 0. \tag{2.74}$$

This reflects the fact that the Euler-Lagrange derivative corresponds to the functional derivative of the action integral in the case that the action functional is differentiable. In particular, the Euler-Lagrange derivative ignores all terms on the boundary of the domain of integration of the action integral.

In order to make contact between jet space and the usual calculus of variations, one evaluates jet space formulas on a specific function $\varphi = \varphi(x)$ via

$$\varphi = \varphi(x), \quad \varphi_{,\alpha} = \frac{\partial\varphi(x)}{\partial x^\alpha}, \quad \varphi_{,\alpha\beta} = \frac{\partial^2\varphi(x)}{\partial x^\alpha\partial x^\beta}, \quad \ldots. \tag{2.75}$$

In this way formulas defined as functions on jet space become formulas involving only the spacetime. A good framework for doing all this is to view jet

space as a fiber bundle over spacetime. A particular KG field defines a cross section of that fiber bundle which can be used to pull back various structures to the base space.

PROBLEM: Consider the following Lagrangian density, viewed as a function on $\mathcal{J}^2$:

$$\mathcal{L} = \frac{1}{2}\varphi(\Box - m^2)\varphi. \tag{2.76}$$

Compute the Euler-Lagrange equation of this Lagrangian and show that it yields the KG equation. Show that this Lagrangian density differs from our original Lagrangian density for the KG equation by a divergence.

PROBLEM: Obtain a formula for the vector field V^α appearing in the boundary term in the Euler-Lagrange identity (2.70).

2.9 Miscellaneous generalizations

There are a number of possible ways that one can generalize the KG field theory. Here I briefly mention a few generalizations that often arise in physical applications. The easiest way to describe them is in terms of modifications of the Lagrangian density.

2.9.1 External "sources"

It is useful for some purposes to consider an inhomogeneous version of the KG equation. This is done by adding a term to the Lagrangian representing the interaction of the KG field with a prescribed "source", which mathematically is a given function $\sigma(t, x, y, z)$. We have

$$\mathcal{L} = \frac{1}{2}(\varphi_{,t}^2 - (\nabla\varphi)^2 - m^2\varphi^2) - \sigma\varphi. \tag{2.77}$$

The Euler-Lagrange (EL) equations are then

$$0 = \mathcal{E}(\mathcal{L}) = (\Box - m^2)\varphi - \sigma. \tag{2.78}$$

The slickest way to solve this KG equation with sources is via Green's functions.

This is a "toy model" of the Maxwell equations for the electromagnetic field in the presence of sources (electric charges and currents). Note that we have here an instance of a Lagrangian which, viewed as a function on jet space, depends upon the spacetime point (t, x, y, z) via $\sigma = \sigma(t, x, y, z)$. In quantum field theory, the presence of a source will lead to particle production/annihilation via a transfer of energy-momentum (to be defined soon) between the field and the source.

2.9.2   Self-interacting field

The KG equation is linear; physically this corresponds to a "non-interacting" field. One way to see this is to recall the interpretation of the KG field as a collection of (many!) oscillators. System of coupled oscillators obey linear equations. When quantized the oscillator states can be interpreted in terms of particle observables. In such an interpretation the particles propagate freely. This is why the KG field we have been studied is often called the "free" or "non-interacting" scalar field.

One can of course modify the classical KG theory in a variety of ways to make it non-linear, that is, one can introduce "self-interactions". The simplest way to do this is to add a "potential term" to the Lagrangian so that we have

$$\mathcal{L} = \frac{1}{2}(\varphi_{,t}^2 - (\nabla\varphi)^2 - m^2\varphi^2) - V(\varphi), \tag{2.79}$$

where V is a differentiable function of one variable. From the oscillator point of view, such a term represents anharmonic contributions to the potential energy. The EL field equations are

$$0 = \mathcal{E}(\mathcal{L}) = (\Box - m^2)\varphi - V'(\varphi) = 0. \tag{2.80}$$

Provided V is not just a quadratic function, this field equation is non-linear.[9] Physically, this corresponds to a "self-interacting" field. Of course, one can also add a source to the self-interacting theory.

An interesting example of a scalar field potential is the "double well potential", which is given by

$$V(\varphi) = -\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4. \tag{2.81}$$

---

[9]Can you guess the interpretation of a quadratic potential?

We shall explore the physical utility of this potential a bit later.

PROBLEM: Consider a self-interacting scalar field with the potential (2.81.) Characterize the set of solutions of the form $\varphi = constant$ in terms of the values of the parameters m, a and b.

2.9.3 KG in arbitrary coordinates

We have presented the KG equation, action, Lagrangian, *etc.* using what we will call *inertial Cartesian coordinates* $x^\alpha = (t, x, y, z)$ on spacetime. Of course, we may use any coordinates we like to describe the theory. It is a straightforward – if perhaps painful – exercise to transform the KG equation into any desired coordinate system. As a physicist you would view the result as a new – but equivalent – representation of the KG field equation. Note however that, mathematically speaking, when you change coordinates you generally do get a *different* differential equation as the "new" KG equation. For example, in inertial Cartesian coordinates the KG equation is linear with constant coefficients. If you adopt spherical polar coordinates for space, thereby defining *inertial spherical coordinates*, then the new version of the KG equation will still be linear but it will have variable coefficients. While this distinction may appear to be largely a pedantic one, there are real consequences to the fact that the form of the equation changes when you change coordinate systems. We shall discuss this more later. For now, let us just note that (like most PDEs in physics) one needs to know what coordinate system one is going to use in order to come up with the correct version of the KG equation.

It is possible to give an elegant geometric prescription for computing the Lagrangian and field equations in any coordinate system. To do this will require a little familiarity with the basics of tensor analysis.

Introduce the *spacetime metric g*, which is a symmetric tensor field of type $\binom{0}{2}$ on Minkowski spacetime. Alternatively, you can view the metric as an assignment of an inner product on vectors at each point of spacetime. Using this latter point of view, if $V^\alpha = (V^t, V^x, V^y, V^z)$ and $W^\alpha = (W^t, W^x, W^y, W^z)$ are components of two vector fields $\vec{V}$ and $\vec{W}$ at a given point, their inner product at that point is given by

$$g(\vec{V}, \vec{W}) = g_{\alpha\beta}V^\alpha W^\beta = -V^t W^t + V^x W^x + V^y W^y + V^z W^z. \qquad (2.82)$$

Here we have defined the components of the metric in the $x^\alpha = (t, x, y, z)$ coordinates:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.83}$$

The metric has an inverse g^{-1} which is a symmetric tensor of type $\binom{2}{0}$ and which defines an inner product at each point on 1-forms (dual vector fields) at that point. If $A_\alpha = (A_t, A_x A_y, A_z)$ and $B_\alpha = (B_t, B_x, B_y, B_z)$ are components of 1-forms $\tilde{A}$ and $\tilde{B}$ at a point in spacetime, their inner product is

$$g^{-1}(\tilde{A}, \tilde{B}) = g^{\alpha\beta} A_\alpha B_\beta = -A_t B_t + A_x B_x + A_y B_y + A_z B_z, \tag{2.84}$$

where the components of the inverse metric in the inertial Cartesian coordinates happens to be the same as the components of the metric:

$$g^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.85}$$

The matrices $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are symmetric and they are each other's inverse:

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma. \tag{2.86}$$

Let me take a moment to spell out how the metric behaves under a change of coordinates. Call the old coordinates $x^\alpha = (t, x, y, z)$. Call the new coordinates $\hat{x}^\alpha$. Of course we will have an invertible transformation between the two coordinate systems. With the usual abuse of notation we will write

$$x^\alpha = x^\alpha(\hat{x}), \quad \hat{x}^\alpha = \hat{x}^\alpha(x). \tag{2.87}$$

The metric in the new coordinates has components

$$\hat{g}_{\alpha\beta}(\hat{x}) = \frac{\partial x^\gamma}{\partial \hat{x}^\alpha} \frac{\partial x^\delta}{\partial \hat{x}^\beta} g_{\gamma\delta}(x(\hat{x})). \tag{2.88}$$

Note that, while the original metric components formed a diagonal array of constants, the new metric components will, in general, form some 4×4

symmetric array of functions. One should now compute the inverse metric components,

$$\hat{g}^{\alpha\beta}(\hat{x}) = \frac{\partial \hat{x}^\gamma}{\partial x^\alpha} \frac{\partial \hat{x}^\delta}{\partial x^\beta} g^{\gamma\delta}(x(\hat{x})). \tag{2.89}$$

Equivalently, one can compute $\hat{g}^{\alpha\beta}(\hat{x})$ by finding the matrix inverse to $\hat{g}_{\alpha\beta}(\hat{x})$.

Although I won't prove it here, it is a fundamental result from the calculus of variations that the formula (2.63) for computing the EL equations does not depend upon the choice of coordinates. In particular, it is possible to show that, after a coordinate transformation, the EL equations of the transformed Lagrangian density are the transformed EL equations. (This is not obvious, but must be proved!) Consequently, the easiest way to find the transformed field equation is to transform the Lagrangian and then compute the field equations.

The KG Lagrangian density can be expressed in terms of the metric by

$$\mathcal{L} = -\frac{1}{2}\sqrt{-\det(g)}\left(g^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right)$$
$$= -\frac{1}{2}\sqrt{-\det(g)}\left(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + m^2\varphi^2\right). \tag{2.90}$$

There are two ingredients to this formula which should be explained. First, recall that if φ is a function on spacetime then $d\varphi$ – the differential of φ – is a 1-form with components $\varphi_{,\alpha}$. We have made a scalar from $d\varphi$ using the inner product defined by the inverse metric. Second, we have introduced an overall factor of $\sqrt{-\det(g)}$. You can easily check that this factor is unity when inertial Cartesian coordinates are used. Under a change of coordinates the determinant changes as

$$x^\alpha \to \hat{x}^\alpha, \quad \det(\hat{g}) = \det(\frac{\partial x}{\partial \hat{x}})^2 \det(g). \tag{2.91}$$

This compensates the change in the coordinate volume element in the action integral:

$$d^4\tilde{x}\sqrt{-\det(\hat{g})} = \left(\det(\frac{\partial \hat{x}}{\partial x})d^4x\right)\left(\det(\frac{\partial x}{\partial \hat{x}})\sqrt{-\det(g)}\right) = d^4x\sqrt{-\det(g)}, \tag{2.92}$$

so that the same formula (2.90) can be used in any coordinates *provided you use the metric appropriate to that coordinate system.* Notice that while the Lagrangian does not explicitly depend upon the coordinates when using an

inertial Cartesian coordinate system it may depend upon the coordinates
in general. Indeed, just switching (x, y, z) to spherical polar coordinates will
introduce explicit coordinate dependence in the Lagrangian, as you can easily
verify.

It is now a straightforward exercise to show that the EL equations of the
KG Lagrangian (2.90) take the form:

$$\partial_\alpha \left(\sqrt{-\det(g)} g^{\alpha\beta} \partial_\beta \varphi(x) \right) - m^2 \sqrt{-\det(g)} \varphi(x) = 0. \qquad (2.93)$$

Under a change of coordinates $x^\alpha \to \hat{x}^\alpha$ the transformed Euler-Lagrange
equations can be computed using (2.93) provided the metric $\hat{g}_{\alpha\beta}$ appropriate
to the $\hat{x}^\alpha$ coordinate system is used.

PROBLEM: Using (2.90), calculate the KG Lagrangian density in inertial
cylindrical coordinates. Compute the Euler-Lagrange equations for this La-
grangian and verify they are equivalent to the Euler-Lagrange equations in
inertial Cartesian coordinates.

At this point I want to try to nip some possible confusion in the bud.
While we have a geometric prescription for computing the KG Lagrangian in
any coordinates using the metric and so forth, it is not a good idea to think
that there is but one KG Lagrangian for all coordinate systems. Strictly
speaking, different coordinate systems will, in general, lead to different La-
grangians. This comment is supposed to be completely analogous to the
previously mentioned fact that, while we can compute "the KG equation"
in any coordinate system, each coordinate system leads, in general, to a
different PDE. Likewise, we have different functions $\mathcal{L}(x, \varphi, \partial\varphi)$ in different
coordinates. For example, in Cartesian coordinates $\mathcal{L}$ is in fact indepen-
dent of x^α, which need not be true in other coordinates, *e.g.,* spherical polar
coordinates.

Coordinates are convenient for computations, but they are more or less
arbitrary, so it should be possible – and is usually advantageous – to have
a formulation of the KG theory which is manifestly coordinate-free. Let me
just sketch this so you can get a flavor of how it goes.

We consider $\mathbf{R}^4$ equipped with a flat metric g of Lorentz signature.[10] We
introduce a scalar field $\varphi \colon \mathbf{R}^4 \to \mathbf{R}$ and a parameter m. Let $\epsilon(g)$ be the

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[10]This means that the eigenvalues of the metric component matrix have the signs (-
+++). If this is true in one coordinate system it will be true in any coordinate system.

volume 4-form defined by the metric. We define the KG Lagrangian as a 4-form via

$$\mathcal{L} = -\frac{1}{2} \left(g^{-1}(d\varphi, d\varphi) + m^2 \varphi^2 \right) \epsilon(g). \tag{2.94}$$

The Lagrangian is to be viewed as a function on the jet space $\mathcal{J}^1$ of the KG field. The metric must be specified to construct this function. Our discussion concerning the fact that different coordinates imply different Lagrangians (and EL equations) can be stated in coordinate free language as follows. Consider a diffeomorphism

$$f \colon \mathbf{R}^4 \to \mathbf{R}^4. \tag{2.95}$$

The diffeomorphism defines a new metric $\hat{g}$ by pull-back:

$$\hat{g} = f^* g. \tag{2.96}$$

We can define a new Lagrangian using this new metric.

$$\hat{\mathcal{L}} = -\frac{1}{2} \left(\hat{g}^{-1}(d\varphi, d\varphi) + m^2 \varphi^2 \right) \epsilon(\hat{g}). \tag{2.97}$$

This metric is flat and can equally well be used to build the KG theory. The EL equations arising from $\hat{\mathcal{L}}$ or $\mathcal{L}$ are the KG equations defined using $\hat{g}$ or g, respectively. The relation between the solution spaces of these two equations is that there is a bijection between the spaces of smooth solutions to each equation. The bijection between solutions φ to the EL equations of $\mathcal{L}$ and the solutions $\hat{\varphi}$ to the EL equations of $\hat{\mathcal{L}}$ is simply

$$\hat{\varphi} = f^* \varphi. \tag{2.98}$$

On the other hand, inasmuch as $\hat{g} \neq g$, the two Lagrangians and corresponding field equations are different as functions on jet space, strictly speaking.

2.9.4 KG on any spacetime

A *spacetime* is defined as a pair (M, g) where M is a manifold and g is a Lorentzian metric.[11] We will always assume that everything in sight is

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[11]Also called a "Lorentzian manifold", this means that at each point $x \in M$, there exists a basis e_α for $T_x M$ such that $g(e_\alpha, e_\beta) = \mathrm{diag}(-1, 1, 1, 1, \cdots, 1)$. By contrast, a *Riemannian manifold* uses a positive-definite metric so that the components of the metric at a point can be rendered as $\mathrm{diag}(1, 1, 1, 1, \cdots, 1)$.

smooth, unless otherwise indicated. Physically, we should take M to be four-dimensional, but this is not required mathematically and we shall not do it here. It is possible to generalize the KG field theory to any spacetime, but the generalization is not unique. Here I just mention two of the most popular possibilities.

First, we have the *minimally coupled KG theory*, defined by the Lagrangian

$$\mathcal{L}_1 = -\frac{1}{2} \left(g^{-1}(d\varphi, d\varphi) + m^2 \varphi^2 \right) \epsilon(g). \qquad (2.99)$$

Of course, this is formally the same as our flat spacetime Lagrangian. The term "minimal coupling" has a precise technical definition, but we will not bother to discuss it. It amounts to making the most straightforward generalization from flat spacetime to curved spacetime as you can see here.

A second possibility is the *curvature-coupled KG theory*, defined by the Lagrangian

$$\mathcal{L}_2 = -\frac{1}{2} \left[g^{-1}(d\varphi, d\varphi) + (m^2 + \xi R(g))\varphi^2 \right] \epsilon(g), \qquad (2.100)$$

where $R(g)$ is the scalar curvature of the metric and ξ is a parameter. The resulting theory is usually described with the terminology "non-minimally coupled".

If the spacetime is $(\mathbf{R}^4, g)$, with g a flat metric, then both of these Lagrangians reduce to one of the possible Lagrangians in flat spacetime that we discussed in the previous subsection. So, both (2.99) and (2.100) can be considered possible generalizations to curved spacetimes.

Finally, I emphasize that all of these Lagrangians require the specification of a metric for their definition. If you change the metric then, strictly speaking, you are using a different Lagrangian (viewed, say, as a function on jet space). This is why, in a precise technical sense, one does *not* use the adjectives "generally covariant" or "diffeomorphism invariant" to describe the KG field theories introduced above. If these Lagrangians had this property, then under a redefinition of the KG field via a diffeomorphism $f \colon M \to M$,

$$\varphi \to f^*\varphi, \qquad (2.101)$$

the Lagrangian should not change. Of course, in order for the Lagrangian to stay unchanged (say, as a function on jet space) one must also redefine the metric by the diffeomorphism,

$$g \to f^*g. \qquad (2.102)$$

But, as we already agreed, the Lagrangian changes when you use a different metric. The point is that the metric is not one of the fields in the KG field theory and you have no business treating it as such. Now, if the metric itself is treated as a field (not some background structure), subject to variation, EL equations, etc., then the Lagrangians we have written *are* generally covariant. Of course, we no longer are studying the KG theory, but something much more complex, *e.g.,* there are now 11 coupled non-linear field equations instead of 1 linear field equation. We will return to this issue again when we discuss what is meant by a symmetry.

2.10 PROBLEMS

1. Verify (2.4)–(2.9).

2. Compute the Euler-Lagrange derivative of the KG Lagrangian density (2.50) and explicitly verify that the Euler-Lagrange equation is indeed the KG equation.

3. Consider a Lagrangian density that is a divergence:

$$\mathcal{L} = D_\alpha W^\alpha, \tag{2.103}$$

where

$$W^\alpha = W^\alpha(\varphi). \tag{2.104}$$

Show that

$$\mathcal{E}(\mathcal{L}) \equiv 0. \tag{2.105}$$

4. Consider the Lagrangian density, viewed as a function on $\mathcal{J}^2$:

$$\mathcal{L} = \frac{1}{2}\varphi(\Box - m^2)\varphi. \tag{2.106}$$

Compute the Euler-Lagrange equation of this Lagrangian density and show that it yields the KG equation. Show that this Lagrangian density differs from our original Lagrangian density for the KG equation by a divergence.

5. Obtain a formula for the vector field V^α appearing in the boundary term in the Euler-Lagrange identity (2.70).

6. Consider a self-interacting scalar field with the potential (2.81). Characterize the set of solutions of the form $\varphi = constant$ in terms of the values of the parameters m, a and b.

7. Using (2.90), calculate the KG Lagrangian density in inertial cylindrical coordinates. Compute the Euler-Lagrange equations for this Lagrangian and verify they are equivalent to the Euler-Lagrange equations in inertial Cartesian coordinates.

Chapter 3

Symmetries and conservation laws

In physics, conservation laws are of undisputed importance. They are the foundation for every fundamental theory of nature. They also provide valuable physical information about the complicated behavior of non-linear dynamical systems. From the mathematical point of view, when analyzing the mathematical structure of differential equations and their solutions the existence of conservation laws (and their attendant symmetries via Noether's theorem) are also very important. We will now spend some time studying conservation laws for the KG equation. Later we will introduce the notion of symmetry and then describe a version of the famous Noether theorem relating symmetries and conservation laws. As usual, we begin by defining everything in terms of the example at hand: the KG field theory. It will then not be hard to see how the idea of conservation laws works in general.

3.1 Conserved currents

We say that a vector field on spacetime, constructed as a local function,

$$j^\alpha = j^\alpha(x, \varphi, \partial\varphi, \ldots, \partial^k\varphi), \tag{3.1}$$

is a *conserved current* or defines a *conservation law* if the divergence of j^α vanishes whenever φ satisfies its field equations (the KG equation). We write

$$D_\alpha j^\alpha = 0, \quad \text{when } (\Box - m^2)\varphi = 0. \tag{3.2}$$

It is understood that the current is divergence-free provided all relations between derivatives defined by the field equations and all the subsequent relations which can be obtained by differentiating the field equations to any order are imposed. Note that we are using the total derivative notation, which is handy when viewing j^α as a function on jet space, that is, as being built via some function of finitely many variables. The idea of the conservation law is that it provides a formula for taking any given solution the field equations

$$\varphi = \varphi(x), \quad (\Box - m^2)\varphi(x) = 0, \tag{3.3}$$

and then building a vector field on spacetime – also called j^α by a standard abuse of notation –

$$j^\alpha(x) := j^\alpha(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \ldots, \frac{\partial^k\varphi(x)}{\partial x^k}) \tag{3.4}$$

such that

$$\frac{\partial}{\partial x^\alpha} j^\alpha(x) = 0. \tag{3.5}$$

You can easily see in inertial Cartesian coordinates that our definition of a conserved current $j^\alpha = (j^0, j^1, j^2, j^3)$ simply says that the field equations imply a continuity equation for the density $\rho \equiv j^0$ (a function on spacetime) and the current density $\vec{j} = (j^1, j^2, j^3)$ (a time dependent vector field on space) associated with any solution $\varphi(x)$ of the field equations:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \vec{j} = 0, \tag{3.6}$$

where

$$\rho(x) = j^0(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \ldots, \frac{\partial^k\varphi(x)}{\partial x^k}), \tag{3.7}$$

and

$$(\vec{j}(x))^i = j^i(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \ldots, \frac{\partial^k\varphi(x)}{\partial x^k}), \quad i = 1, 2, 3. \tag{3.8}$$

The utility of the continuity equation is as follows. Define the *total charge contained in the region V* of space at a given time t to be

$$Q_V(t) := \int_V d^3x \, \rho(t, \vec{x}). \tag{3.9}$$

Note that the total charge is a functional of the field, that is, its value depends upon which field you choose. The integral over V of the continuity equation (3.6) implies that

$$\frac{d}{dt} Q_V(t) = -\int_{\partial V} \vec{j} \cdot d\vec{S} \ . \tag{3.10}$$

Here ∂V is the boundary of V and $d\vec{S}$ is its oriented surface element according to the divergence theorem. Keep in mind that this relation is meant to be valid when the field is a solution to the field equation.

PROBLEM: Derive (3.10) from the continuity equation.

We call the right hand side of (3.10) the *net flux into V*. We say the charge Q_V is conserved since we can account for its time rate of change purely in terms of the flux into or out of the region V. In this sense there is no "creation" or "destruction" of the charge, although the charge can move from place to place.

    With suitable boundary conditions, one can choose V such that charge cannot enter or leave the region and so the total charge is constant in time. In this case we speak of a *constant of the motion*. For example, we have seen that a reasonable set of boundary conditions to put on the KG field (motivated, say, by the variational principle) is to assume that the KG field vanishes at spatial infinity. Let us then consider the region V to be all of space, that is, $V = \mathbf{R}^3$. If the fields vanish at spatial infinity fast enough,[1] the flux will vanish asymptotically and we will have

$$\frac{dQ_V}{dt} = 0. \tag{3.11}$$

3.2   Conservation of energy

Let us look at an example of a conservation law for the KG equation. Consider the following spacetime vector field locally built from the KG field and its first derivatives – we give its components in an inertial Cartesian reference frame:

$$j^0 = \frac{1}{2}\left(\varphi_{,t}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right), \tag{3.12}$$

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[1]Since the area element of the sphere of radius r grows like r^2, the fields should fall off at infinity such that $\hat{n} \cdot \vec{j}$ falls off faster than $1/r^2$ as $r \to \infty$.

$$j^i = -\varphi_{,t}(\nabla\varphi)^i. \tag{3.13}$$

Let us see how this defines a conserved current. We compute[2]

$$\partial_0 j^0 = \varphi_{,t}\varphi_{,tt} + \nabla\varphi \cdot \nabla\varphi_{,t} + m^2\varphi\varphi_{,t}, \tag{3.14}$$

and

$$\partial_i j^i = -(\nabla\varphi_{,t}) \cdot (\nabla\varphi) - \varphi_{,t}\nabla^2\varphi. \tag{3.15}$$

All together, we get

$$\begin{aligned} \partial_\alpha j^\alpha &= \varphi_{,t}\left(\varphi_{,tt} - \nabla^2\varphi + m^2\varphi\right) \\ &= -\varphi_{,t}(\Box\varphi - m^2\varphi). \end{aligned} \tag{3.16}$$

It follows from the identity (3.16) that if $\varphi(x)$ is a solution to the Klein-Gordon equation then the resulting vector field $j^\alpha(x)$ defined in (3.12), (3.13) will satisfy $\partial_\alpha j^\alpha = 0$.

The conserved charge Q_V associated with this conservation law is called the *energy of the KG field* in the region V and is denoted by E_V:

$$E_V = \int_V d^3x \, \frac{1}{2}\left(\varphi_{,t}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right). \tag{3.17}$$

There are various reasons why we view this as an energy. First of all, if you put in physically appropriate units, you will find that E_V has the dimensions of energy. The best reason comes from Noether's theorem, which we shall discuss later. For now, let us recall that the Lagrangian has the form

$$L = T - U, \tag{3.18}$$

where the "kinetic energy" is given by

$$T = \int_V d^3x \, \frac{1}{2}\varphi_{,t}^2, \tag{3.19}$$

and the "potential energy" is given by

$$U = \int_V d^3x \frac{1}{2}\left((\nabla\varphi)^2 + m^2\varphi^2\right). \tag{3.20}$$

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[2] Here I have moved away from the total derivative notation back to the physicist's notation in which one imagines evaluating all the relevant formulas on a given field $\varphi = \varphi(x)$.

Naturally, then, the conserved charge that arises as

$$E_V = T + U \tag{3.21}$$

is called the total energy (in the region V).

The net flux of energy into V is given by

$$-\int_{\partial V} \vec{j} \cdot \mathbf{dS} = \int_{\partial V} \varphi_{,t} \nabla\varphi \cdot \mathbf{dS}. \tag{3.22}$$

Evidently, if the solution to the KG equation is chosen to be static, $\varphi_{,t} = 0$, or such that the component of its spatial gradient along the normal to the boundary vanishes, then the flux into the volume vanishes and the energy is a constant of the motion. If we choose $V = \mathbf{R}^3$, then the total energy of the KG field – in the whole universe – is independent of time if the product of the time rate of change of φ and the radial derivative of φ vanish as $r \to \infty$ faster than $\frac{1}{r^2}$. Of course, for the total energy to be defined in this case the integral of the *energy density* $j^0(x)$ must exist and this also imposes asymptotic decay conditions on the solutions $\varphi(x)$ to the KG equation. Indeed, we must have that $\varphi(x)$, its time derivative, and the magnitude of its spatial gradient should decay "at infinity" faster than $\frac{1}{r^{3/2}}$. This will guarantee that the net flux into $\mathbf{R}^3$ vanishes.

PROBLEM: What becomes of conservation of energy for the KG field when an external source is present, as in (2.77)? How do you physically interpret this state of affairs?

3.3 Conservation of momentum

Let us look at another conservation law for the KG equation known as the *conservation of momentum*. This arises as a triplet of conservation laws in which

$$\rho_{(i)} = \varphi_{,t}\varphi_{,i}, \quad i = 1, 2, 3 \tag{3.23}$$

$$(\vec{j}_{(i)})^l = -(\nabla\varphi)^l \varphi_{,i} + \frac{1}{2}\delta_i^l \left[(\nabla\varphi)^2 - (\varphi_{,t})^2 + m^2\varphi^2\right]. \tag{3.24}$$

PROBLEM: Verify that the currents (3.23), (3.24) are conserved. (If you like, you can just fix a value for i, say, $i = 1$ and check that j_1^α is conserved.)

The origin of the name "momentum" of these conservation laws can be understood on the basis of units: the conserved charges

$$P_{(i)} = \int_V d^3x \, \varphi_{,t} \varphi_{,i},$$
(3.25)

have the dimensions of momentum (if one takes account of the various dimensionful constants that we have set to unity). The name can also be understood from the fact that the each of the three charge densities $\rho_{(i)}$ corresponds to a component of the current densities for the energy conservation law. Roughly speaking, you can think of this quantity as getting the name "momentum" since it defines the "flow of energy". In a little while we will get an even better explanation from Noether's theorem. Finally, recall that the total momentum of a system is represented as a vector in $\mathbf{R}^3$. The Cartesian components of this vector in the case of a KG field are the $P_{(i)}$.

3.4   Energy-momentum tensor

The conservation of energy and conservation of momentum can be given a unified treatment by introducing a $\binom{0}{2}$ tensor field on spacetime known as the *energy-momentum tensor* (also known as the "stress-energy-momentum tensor", the "stress-energy tensor", and the "stress tensor"). Given a function on spacetime $\varphi \colon \mathbf{R}^4 \to \mathbf{R}$ (not necessarily satisfying any field equations), the energy-momentum tensor is defined as

$$T = d\varphi \otimes d\varphi - \frac{1}{2} \left\{ g^{-1}(d\varphi, d\varphi) - m^2\varphi^2 \right\} g,$$
(3.26)

where g is the metric tensor of spacetime. The components of the energy-momentum tensor take the form

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\varphi_{,\gamma}\varphi_{,\delta} - \frac{1}{2}m^2\varphi^2 g_{\alpha\beta}.$$
(3.27)

Our conservation laws were defined for the KG field on flat spacetime and the formulas were given in inertial Cartesian coordinates $x^\alpha = (t, x^i)$ such that the metric takes the form

$$g = g_{\alpha\beta}dx^\alpha \otimes dx^\beta,$$
(3.28)

with

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1). \tag{3.29}$$

The formulas (3.26) or (3.27) are in fact correct on any spacetime provided the metric in the chosen coordinates is specified. Note that the energy-momentum tensor is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}. \tag{3.30}$$

If desired, one can view the formula for T as defining a collection of functions on jet space representing a formula for a tensor field on spacetime. More precisely, we can view T as a mapping from $\mathcal{J}^1$ into the $\binom{0}{2}$ tensor fields on spacetime.

Using inertial coordinates $x^\alpha = (t, \vec{x})$ on flat spacetime you can check that the conserved energy current has components given by

$$j^\alpha_{energy} = -T^\alpha_t \equiv -g^{\alpha\beta}T_{t\beta}. \tag{3.31}$$

In particular the energy density is T^{tt}. Likewise, the components of the conserved momentum currents are given by

$$j^\alpha_{momentum} = -T^\alpha_i \equiv -g^{\alpha\beta}T_{i\beta}, \quad i = 1, 2, 3, \tag{3.32}$$

so that, in particular, the momentum density in the direction labeled i is given by $-T^{ti}$. The conservation of energy and momentum are encoded in the important identity:

$$g^{\beta\gamma}D_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi, \tag{3.33}$$

where I remind you that we have defined

$$\Box\varphi = g^{\alpha\beta}\varphi_{,\alpha\beta}. \tag{3.34}$$

This relation shows that when evaluated on a function satisfying the KG equations the resulting energy-momentum tensor field on spacetime has vanishing divergence.

Although we are not emphasizing relativistic considerations in our discussions, it is perhaps worth mentioning that there is no absolute distinction between energy and momentum. A change of reference frame will mix up these quantities. One therefore usually speaks of the "conservation of energy-momentum", or the "conservation of four-momentum", represented by the currents $j_{(\alpha)}$, $\alpha = 0, 1, 2, 3$ with components given by

$$j^\beta_{(\alpha)} = -T^\beta_\alpha = -g^{\beta\gamma}T_{\alpha\gamma}. \tag{3.35}$$

3.5   Conservation of angular momentum

Finally, let me introduce 6 more conservation laws, known as the conservation laws of *relativistic angular momentum*. They are given by 6 currents $M^{(\mu)(\nu)}$ with components:

$$M^{\alpha(\mu)(\nu)} = T^{\alpha\mu}x^{\nu} - T^{\alpha\nu}x^{\mu}. \tag{3.36}$$

Note that

$$M^{\alpha(\mu)(\nu)} = -M^{\alpha(\nu)(\mu)}, \tag{3.37}$$

which is why there are only 6 independent currents.

PROBLEM: Show that the six currents (3.36) are conserved. (*Hint: Don't panic! This is actually the easiest one to prove so far, since you can use*

$$g^{\beta\gamma}D_{\gamma}T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi, \tag{3.38}$$

which we have already established.)

In a given inertial reference frame labeled by coordinates $x^{\alpha} = (t, x^i) = (t, x, y, z)$ the relativistic angular momentum naturally decomposes into two pieces in which (α, β) take the values (i, j) and $(0, i)$. Let us look at the charge densities; we have

$$\rho^{(i)(j)} := M^{0(i)(j)} = T^{0i}x^j - T^{0j}x^i, \tag{3.39}$$

$$\rho^{(0)(i)} := M^{0(0)(i)} = T^{00}x^i - T^{0i}t. \tag{3.40}$$

The first charge density represents the usual notion of (density of) angular momentum. Indeed, you can see that it has the familiar *position $\times$ momentum* form. The second charge density, $\rho^{(0)(i)}$, when integrated over a region V yields a conserved charge which can be interpreted, roughly, as the "center of mass-energy at $t = 0$" in that region. Just as energy and momentum are two facets of a single, relativistic energy-momentum, you can think of these two conserved quantities as forming a single relativistic form of angular momentum.

Let us note that while the energy-momentum conserved currents are (in Cartesian coordinates) local functions of the fields and their first derivatives, the angular momentum conserved currents are also explicit functions of the spacetime coordinates. Thus we see that conservation laws may be, in general, functions on the full jet space $(x, \varphi, \partial\varphi, \partial^2\varphi, \ldots)$.

3.6 Variational symmetries

Let us now (apparently) change the subject to consider the notion of *symmetry* in the context of the KG theory. We shall see that this is not really a change in subject when we come to Noether's theorem relating symmetries and conservation laws.

A slogan for the definition of symmetry in the style of the late John Wheeler might be something like: "change without change". When we speak of an object admitting a "symmetry", we usually have in mind some kind of transformation of that object that leaves some specified aspects of that object unchanged. We can partition transformations into two types: *discrete* and *continuous*. The continuous transformations depend continuously on one or more parameters. For example, the group of rotations of $\mathbf{R}^3$ about the z-axis defines a continuous transformation parametrized by the angle of rotation. The "inversion" transformation

$$(x, y, z) \to (-x, -y, -z) \tag{3.41}$$

is an example of a discrete transformation. We will be focusing primarily on continuous transformations in what follows.

For a field theory such as the KG theory, let us define a *one parameter family of transformations* – also called a *continuous transformation* – to be a rule for building from any given field φ a family of KG fields (not necessarily satisfying any field equations), denoted by φ_λ. We always assume that the transformation starts at $\lambda = 0$ in the sense that

$$\varphi_{\lambda=0} = \varphi. \tag{3.42}$$

You are familiar with such "curves in field space" from our discussion of the variational calculus. There we considered a single (though arbitrary) curve through a critical point. Here we make a stronger assumption and assume that the transformation defines a curve through each point in the space of fields via a formula which is a local function of the field and its derivatives.[3] We view this family of curves as defining a transformation of any field φ, where the transformation varies continuously with the parameter λ, and such that $\lambda = 0$ is the identity transformation.

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[3] The tangents to all these curves defines a vector field in jet space.

A simple example of a one parameter family of transformations is the *scaling transformation:*

$$\varphi(t,x,y,z) \longrightarrow \varphi_\lambda(t,x,y,z) := e^\lambda \varphi(t,x,y,z). \qquad (3.43)$$

As another example, the following is the transformation induced on φ due to a *time translation*:

$$\varphi(t,x,y,z) \longrightarrow \varphi_\lambda(t,x,y,z) := \varphi(t+\lambda,x,y,z). \qquad (3.44)$$

We say that a continuous transformation is a *continuous variational symmetry* for the KG theory if it leaves the Lagrangian invariant in the sense that, for any KG field $\varphi = \varphi(x)$,

$$\mathcal{L}(x,\varphi_\lambda(x),\frac{\partial\varphi_\lambda(x)}{\partial x}) = \mathcal{L}(x,\varphi(x),\frac{\partial\varphi(x)}{\partial x}). \qquad (3.45)$$

Explicitly, we want

$$-\frac{1}{2}\left(g^{\alpha\beta}\partial_\alpha\varphi_\lambda(x)\partial_\beta\varphi_\lambda(x) + m^2\varphi_\lambda^2\right) = -\frac{1}{2}\left(g^{\alpha\beta}\partial_\alpha\varphi(x)\partial_\beta\varphi(x) + m^2\varphi^2\right). \qquad (3.46)$$

An equivalent way to express this is that

$$\frac{\partial}{\partial\lambda}\mathcal{L}(x,\varphi_\lambda(x),\frac{\partial\varphi_\lambda(x)}{\partial x}) = 0. \qquad (3.47)$$

I think you can see why this is called a "symmetry". While the KG field is certainly changed by the symmetry transformation, from the point of view of the Lagrangian nothing is changed by the transformation.

Our definition of variational symmetries did not rely in any essential way upon the continuous nature of the transformation. For example, you can easily see that the discrete transformation

$$\varphi \to -\varphi \qquad (3.48)$$

leaves the KG Lagrangian unchanged and so would be called a *discrete variational symmetry*. Any transformation of the KG field that leaves the Lagrangian unchanged will be called simply a *variational symmetry*. Noether's theorem, which is our goal, involves continuous variational symmetries.

There is a notion of "symmetry" for field equations, too. Briefly, a transformation is a symmetry of the field equations if it maps solutions of the field

equations to other solutions of the (same) field equations. A fundamental result of the variational calculus is that variational symmetries are always symmetries of the EL equations. (The converse is not true!)

The following problem provides a simple illustration.

PROBLEM: Consider the real scalar field with the double-well self-interaction potential (2.81). Show that $\varphi \to \hat{\varphi} = -\varphi$ is a variational symmetry. Consider the 3 constant solutions to the field equation (which you found in the problem just after (2.81)) and check that this symmetry maps these solutions to solutions.

This problem asks you to prove the general result.

PROBLEM: Let $\varphi \to \hat{\varphi} = F(\varphi)$ be a variational symmetry of a Lagrangian. Show that it maps solutions of the Euler-Lagrange equations to new solutions, that is, if φ is a solution, so is $\hat{\varphi}$. If you like, you can restrict your attention to Lagrangians $\mathcal{L}(x, \varphi, \partial \varphi)$.

3.7 Infinitesimal symmetries

Let us now restrict our attention to continuous symmetries. A fundamental observation going back to Lie is that, when considering aspects of problems involving continuous transformations, it is always best to formulate the problems in terms of *infinitesimal transformations*. Roughly speaking, the technical advantage provided by an infinitesimal description is that many non-linear equations that arise become linear. The idea of an infinitesimal transformation is that we consider continuous transformations $\varphi \to \varphi_\lambda$ for "very small" values of the parameter λ. More precisely, we define the infinitesimal change of the field in much the same way as we do a field variation,

$$\delta\varphi = \left(\frac{\partial \varphi_\lambda}{\partial \lambda}\right)_{\lambda=0}, \tag{3.49}$$

which justifies the use of the same notation, I think. Still it is important to see how these two notions of $\delta\varphi$ are the same and are different. A field variation in a variational principle involves studying curves in field space passing through a specific point (a critical point) so that, for each curve, $\delta\varphi$

is a single function on spacetime. An infinitesimal transformation $\delta\varphi$ will be a spacetime function which will depend upon the field φ being transformed, and it is this dependence which is the principal object of study. From a more geometric point of view, field variations in the calculus of variations represent tangent vectors at a single point in the space of fields. An infinitesimal transformation is a vector *field* on the space of fields – a continuous assignment of a vector to each point in field space. Just as one can restrict a vector field to a given point and get a vector there, one can restrict an infinitesimal transformation to a particular field and get a particular field variation there.

An example is in order. For the scaling transformation

$$\varphi_\lambda = e^\lambda \varphi, \tag{3.50}$$

we get

$$\delta\varphi = \varphi, \tag{3.51}$$

which shows quite clearly that $\delta\varphi$ is built from φ so that, while it is a function on spacetime for a given field $\varphi = \varphi(x)$, this spacetime function varies from point to point in the space of fields. Likewise for time translations:

$$\varphi_\lambda(t, x, y, z) = \varphi(t + \lambda, x, y, z) \tag{3.52}$$

$$\delta\varphi = \varphi_{,t}. \tag{3.53}$$

Of course, just as it is possible to have a constant vector field, it is possible to have a continuous transformation whose infinitesimal form happens to be independent of φ. For example, given some function $f = f(x)$ the transformation

$$\varphi_\lambda = \varphi + \lambda f \tag{3.54}$$

has the infinitesimal form

$$\delta\varphi = f. \tag{3.55}$$

This transformation is sometimes called a *field translation.*

The infinitesimal transformation gives a formula for the "first-order" change of the field under the indicated continuous transformation. This first order information is enough to completely characterize the transformation. The idea is that a finite continuous transformation can be viewed as being built by composition of "many" infinitesimal transformations. Indeed, if you think of a continuous transformation as a family of curves foliating field space, then an infinitesimal transformation is the vector field defined by

the tangents to those curves at each point. As you may know, it is enough to specify the vector field to determine the foliation of curves (via the "flow of the vector field"). If this bit of mathematics is obscure to you, then you may be happier by recalling that, say, the "electric field lines" are completely determined by specifying the electric vector field, and conversely.

For a continuous transformation to be a variational symmetry it is necessary and sufficient that its infinitesimal form defines an (infinitesimal) variational symmetry. By this I mean that the variation induced in $\mathcal{L}$ by the infinitesimal transformation vanishes for all fields.[4] That this condition is necessary is clear from our earlier observation that a continuous symmetry satisfies:

$$\frac{\partial}{\partial \lambda} \mathcal{L}(x, \varphi_\lambda(x), \frac{\partial \varphi_\lambda(x)}{\partial x}) = 0. \tag{3.56}$$

Clearly, this implies that

$$\begin{aligned} 0 &= \frac{\partial}{\partial \lambda} \mathcal{L}(x, \varphi_\lambda(x), \frac{\partial \varphi_\lambda(x)}{\partial x})\Big|_{\lambda=0} \\ &= \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \delta\varphi_\alpha \\ &= \delta L. \end{aligned} \tag{3.57}$$

That this condition is sufficient follows from the fact that it must hold at all points in the space of fields, so that the derivative with respect to λ vanishes everywhere on the space of fields. Thus one often checks whether a continuous transformation is a variational symmetry by just checking its infinitesimal condition (3.57).

3.8 Divergence symmetries

We have defined a variational symmetry as a transformation that leaves the Lagrangian unchanged. This is a reasonable state of affairs since the Lagrangian determines the field equations. However, we have also seen that any two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differing by a divergence

$$\mathcal{L}' = \mathcal{L} + D_\alpha V^\alpha, \tag{3.58}$$

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[4] Contrast this with idea of a critical point. An infinitesimal variational symmetry is a *particular* family of field variations that does not change the Lagrangian for *any* choice of the field. The critical point is a *particular field* such that the action does not change for *any* field variation.

where $V^\alpha = V^\alpha(x, \varphi, \partial\varphi, \dots)$, will define the same EL equations since

$$\mathcal{E}(D_\alpha V^\alpha) = 0 \tag{3.59}$$

so that

$$\mathcal{E}(\mathcal{L}') = \mathcal{E}(\mathcal{L}) + \mathcal{E}(D_\alpha V^\alpha) = \mathcal{E}(\mathcal{L}). \tag{3.60}$$

Therefore, it is reasonable – and we shall see quite useful – to consider generalizing our notion of symmetry. We say that a transformation is a *divergence symmetry* if the Lagrangian only changes by the addition of a divergence. In infinitesimal form, a divergence symmetry satisfies

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{3.61}$$

for some spacetime vector field W^α, built locally from the scalar field and its derivatives, $W^\alpha = W^\alpha(x, \varphi, \partial\varphi, \dots)$. Of course, a variational symmetry is just a special case of a divergence symmetry arising when $W^\alpha = 0$.

PROBLEM: Show that the scaling transformation (3.50) is neither a variational symmetry nor a divergence symmetry for the KG Lagrangian.

Time translation,

$$\delta\varphi = \varphi_{,t}, \tag{3.62}$$

defines a divergence symmetry of the KG Lagrangian. Let's see how this works. We begin by writing the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + m^2\varphi^2\right), \tag{3.63}$$

where $g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. We then have

$$\begin{aligned}
\delta\mathcal{L} &= -\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta t} + m^2\varphi\varphi_{,t}\right) \\
&= D_t\left[-\frac{1}{2}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + m^2\varphi^2\right)\right] \\
&= D_t\mathcal{L} \\
&= D_\alpha\left(\delta_t^\alpha\mathcal{L}\right),
\end{aligned} \tag{3.64}$$

so that we can choose

$$W^\alpha = \delta_t^\alpha\mathcal{L}. \tag{3.65}$$

Physically, the presence of this symmetry reflects the fact that there is no preferred instant of time in the KG theory. A shift in the origin of time $t \to t + \ constant$ does not change the field equations.

3.9 A first look at Noether's theorem

We now have enough technology to have a first, somewhat informal look at Noether's theorem relating symmetries and conservation laws. I will use the jet space formalism to ensure maximum precision. The idea is as follows. Consider a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi). \tag{3.66}$$

Of course, the KG Lagrangian is of this form. Suppose that $\delta\varphi$ is an infinitesimal variational symmetry. Then the induced change in the Lagrangian vanishes,

$$\delta\mathcal{L} = 0, \tag{3.67}$$

everywhere in the space of fields. But, at any given point in field space, we always have the identity, valid for any kind of variation, which defines the Euler-Lagrange expression:

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha, \tag{3.68}$$

where

$$\mathcal{E}(\mathcal{L}) = \frac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\right), \tag{3.69}$$

and[5]

$$V^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi. \tag{3.70}$$

This identity holds for *any field variation*. By hypothesis, our field variation $\delta\varphi$ is some expression built from x, φ, and its derivatives that has the property that $\delta\mathcal{L} = 0$. Consequently, for the infinitesimal symmetry transformation $\delta\varphi$ we have the relation

$$0 = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha \iff D_\alpha V^\alpha = -\mathcal{E}(\mathcal{L})\delta\varphi. \tag{3.71}$$

This is exactly the type of relationship that defines a conserved current V^α, since it says that the divergence of V^α will vanish if V^α is built from a KG field φ that satisfies the EL-equation (the KG equation). Note that the specific form of V^α as a function of φ (and its derivatives) depends upon

---

[5]There is an ambiguity in the definition of V^α here which we shall ignore for now to keep things simple. We will confront it when we study conservation laws in electromagnetism.

the specific form of the Lagrangian via $\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}$ and on the specific form of the transformation via $\delta\varphi$.

More generally, suppose that the infinitesimal transformation $\delta\varphi$ defines a divergence symmetry, that is, there exists a vector field W^α built from φ such that

$$\delta\mathcal{L} = D_\alpha W^\alpha. \tag{3.72}$$

We still get a conservation law since the variational identity applied to a divergence symmetry now yields

$$D_\alpha W^\alpha = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha, \tag{3.73}$$

which implies

$$D_\alpha(V^\alpha - W^\alpha) = -\mathcal{E}(\mathcal{L})\delta\varphi, \tag{3.74}$$

so that the conserved current is now $V^\alpha - W^\alpha$.

To summarize, if $\delta\varphi(x, \varphi, \partial\varphi, \dots)$ is a divergence symmetry of $\mathcal{L}(x, \varphi, \partial\varphi)$,

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{3.75}$$

then there is a conserved current given by

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}\delta\varphi - W^\alpha. \tag{3.76}$$

This is a version of "Noether's first theorem".

3.10   Time translation symmetry and conservation of energy

Using Noether's first theorem we can see how the conserved current defining conservation of energy arises via the time translation symmetry. (I will continue to use the jet space formalism – the total derivative D, in particular.) Recall that time translation symmetry is a divergence symmetry:

$$\delta\varphi = D_t\varphi = \varphi_{,t} \implies \delta\mathcal{L} = D_\alpha(\delta_t^\alpha \mathcal{L}). \tag{3.77}$$

With

$$\delta\varphi = D_t\varphi = \varphi_{,t}, \quad W^\alpha = \delta_t^\alpha \mathcal{L}, \quad \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} = -g^{\alpha\beta}\varphi_{,\beta}, \tag{3.78}$$

we can apply the results of the previous section to obtain a conserved current:

$$\begin{aligned} j^\alpha &= -g^{\alpha\beta}\varphi_{,\beta}\varphi_{,t} - \delta^\alpha_t \mathcal{L} \\ &= -T^\alpha_t, \end{aligned} \tag{3.79}$$

which is our expression of the conserved energy current in terms of the energy-momentum tensor.

It is worth pointing out that the existence of the time translation symmetry, and hence conservation of energy, is solely due to the fact that the KG Lagrangian has no explicit t dependence; no other structural features of the Lagrangian play a role. To see this, consider any Lagrangian whatsoever

$$\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi, \ldots) \tag{3.80}$$

satisfying

$$\frac{\partial}{\partial t}\mathcal{L}(x, \varphi, \partial\varphi, \ldots) = 0. \tag{3.81}$$

Here I should emphasize that this partial derivative is really a partial derivative – it only applies to the explicit coordinate dependence of the Lagrangian. So, for example, in the jet space context of the present discussion we have

$$\partial_t(t\varphi) = \varphi, \quad D_t(t\varphi) = \varphi + t\varphi_{,t}. \tag{3.82}$$

From the identity

$$D_t \mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,t} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}\varphi_{,t\alpha} \tag{3.83}$$

we have – provided $\partial_t \mathcal{L} = 0$:

$$\frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,t} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}\varphi_{,t\alpha} = D_t \mathcal{L}, \tag{3.84}$$

which can be interpreted as saying the time translation $\delta\varphi = \varphi_{,t}$ yields a divergence symmetry (3.77), leading to conservation of energy. One says that the conserved current for energy is the *Noether current* associated to time translational symmetry.

3.11 Space translation symmetry and conservation of momentum

We can use spatial translation symmetry to obtain conservation of momentum as follows. Let $\hat{n}$ be a constant unit vector field in space. With

$(t, x, y, z) \equiv (t, \vec{x})$, the continuous transformation corresponding to a spatial translation along the direction specified by $\hat{n}$ is given by

$$\varphi(t, \vec{x}) \rightarrow \varphi_\lambda(t, \vec{x}) = \varphi(t, \vec{x} + \lambda \hat{n}), \tag{3.85}$$

Infinitesimally, we have

$$\delta\varphi = \hat{n} \cdot \nabla\varphi = n^i \varphi_{,i}. \tag{3.86}$$

To check that this is a symmetry of the KG Lagrangian we compute

$$\begin{aligned}
\delta\mathcal{L} &= \varphi_{,t}(\hat{n} \cdot \nabla\varphi)_{,t} - \nabla\varphi \cdot \nabla(\hat{n} \cdot \nabla\varphi) - m^2\varphi(\hat{n} \cdot \nabla\varphi) \\
&= \hat{n} \cdot \nabla\mathcal{L} \\
&= \nabla \cdot (\hat{n}\mathcal{L}) \\
&= D_\alpha W^\alpha,
\end{aligned} \tag{3.87}$$

where

$$W^\alpha = (0, n^i\mathcal{L}), \tag{3.88}$$

and D is the total derivative.

As before, it is not hard to see that this result is a sole consequence of the fact that the Lagrangian has no dependence on the spatial (Cartesian) coordinates. In particular,

$$n^i \frac{\partial}{\partial x^i} \mathcal{L}(x, \varphi, \partial\varphi, \dots) = 0, \tag{3.89}$$

so that

$$n^i \frac{\partial\mathcal{L}}{\partial\varphi}\varphi_{,i} + n^i \frac{\partial\mathcal{L}}{\partial\varphi_{,j}}\varphi_{,ij} = n^i D_i \mathcal{L} \tag{3.90}$$

Thus we have the conservation law

$$j^\alpha = (\rho, j^i), \tag{3.91}$$

with

$$\rho = \varphi_{,t}\hat{n} \cdot \nabla\varphi, \tag{3.92}$$

and

$$j^i = -\varphi^{,i}\hat{n} \cdot \nabla\varphi + \frac{1}{2}n^i \left((\nabla\varphi)^2 - \varphi_{,t}^2 + m^2\varphi^2\right). \tag{3.93}$$

Since the direction $\hat{n}$ is arbitrary, it is easy to see that we really have three independent conservation laws corresponding to 3 linearly independent choices

for $\hat{n}$. These three conservation laws correspond to the conservation laws for momentum that we had before. The relation between ρ and j^i here and $\rho_{(i)}$ and $\vec{j}_{(i)}$ there is given by

$$\rho = n^k \rho_{(k)}, \quad j^i = n^k (\vec{j}_{(k)})^i. \tag{3.94}$$

You can see that the translational symmetry in the spatial direction defined by $\hat{n}$ leads to a conservation law for the component of momentum along $\hat{n}$. Thus the three conserved momentum currents are the Noether currents associated with spatial translation symmetry.

3.12 Conservation of energy-momentum revisited

Here we revisit the conservation of energy-momentum from a more relativistic point of view, bringing into play the energy-momentum tensor. Let a^α be any constant vector field on spacetime. Consider the following continuous transformation, which is a *spacetime translation*:

$$\varphi_\lambda(x^\alpha) = \varphi(x^\alpha + \lambda a^\alpha), \quad \delta\varphi = a^\alpha \varphi_{,\alpha}. \tag{3.95}$$

As a nice exercise you should check that we then have

$$\delta\mathcal{L} = D_\alpha(a^\alpha \mathcal{L}) \tag{3.96}$$

so that from Noether's theorem we have the following conserved current:

$$j^\alpha = -g^{\alpha\beta}\varphi_{,\beta}(a^\gamma\varphi_{,\gamma}) - a^\alpha\mathcal{L}. \tag{3.97}$$

Here we are using $g_{\alpha\beta} = g^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. By choosing a^α to define a time or space translation we get the corresponding conservation of energy or momentum, as explored in the previous sections.

Since the components a^α are arbitrary constants, it is easy to see that for each value of γ, the current

$$j^\alpha_{(\gamma)} = -g^{\alpha\beta}\varphi_{,\beta}\varphi_{,\gamma} - \delta^\alpha_\gamma\mathcal{L} \tag{3.98}$$

is conserved, corresponding to the four independent conservation laws of energy and momentum. Substituting for the KG Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + m^2\varphi^2\right), \tag{3.99}$$

we get that

$$j^\alpha_{(\gamma)} = -T^\alpha_\gamma \equiv -g^{\alpha\beta}T_{\beta\gamma}. \qquad (3.100)$$

Thus the energy-momentum tensor can be viewed as the set of Noether currents associated with spacetime translational symmetry.

PROBLEM: Verify that the Noether currents associated with a spacetime translation do yield the energy-momentum tensor.

3.13 Angular momentum revisited

We have seen the correspondence between spacetime translation symmetry and conservation of energy-momentum. What symmetry is responsible for conservation of angular momentum? It is *Lorentz symmetry*. Recall that the Lorentz group consists of "boosts" and spatial rotations. Mathematically, a Lorentz transformation is a linear transformation on the spacetime $\mathbf{R}^4$,

$$x^\alpha \longrightarrow S^\alpha_\beta x^\beta, \qquad (3.101)$$

that leaves invariant the quadratic form

$$g_{\alpha\beta}x^\alpha x^\beta = -t^2 + x^2 + y^2 + z^2. \qquad (3.102)$$

We have then

$$S^\alpha_\gamma S^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}. \qquad (3.103)$$

Consider a 1-parameter family of such transformations, $S(\lambda)$, so that

$$S^\alpha_\beta(0) = \delta^\alpha_\beta, \quad \left(\frac{\partial S^\alpha_\beta}{\partial \lambda}\right)_{\lambda=0} =: \omega^\alpha_\beta \qquad (3.104)$$

Using these transformations in (3.103) and differentiating with respect to λ we obtain

$$\omega^\alpha_\gamma g_{\alpha\delta} + \omega^\beta_\delta g_{\gamma\beta} = 0. \qquad (3.105)$$

This is the infinitesimal version of (3.103). Defining

$$\omega_{\alpha\beta} = g_{\beta\gamma}\omega^\gamma_\alpha \qquad (3.106)$$

we see that a Lorentz transformation is "generated" by ω if and only if the array $\omega_{\alpha\beta}$ is anti-symmetric:

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{3.107}$$

Infinitesimal Lorentz transformations are thus in one-to-one correspondence with antisymmetric arrays $\omega_{\alpha\beta}$. In a given inertial reference frame with inertial Cartesian coordinates $x^\alpha = (t, x^i)$, the spatial rotations are generated by the infinitesimal transformations provided by ω_{ij}, while the boosts are generated by the infinitesimal transformations provided by ω_{0i}.

Let us now determine the infinitesimal transformation induced on the scalar field by the infinitesimal Lorentz transformation generated by $\omega_{\alpha\beta}$. Consider the following transformation

$$\varphi_\lambda(x^\alpha) = \varphi(S_\beta^\alpha(\lambda)x^\beta), \tag{3.108}$$

Differentiate both sides with respect to λ and set $\lambda = 0$ to find the infinitesimal transformation:

$$\delta\varphi = (\omega_\beta^\alpha x^\beta)\varphi_{,\alpha}, \tag{3.109}$$

with an antisymmetric $\omega_{\alpha\beta}$ as above. It is now a short computation to check that, for the KG Lagrangian,

$$\delta\mathcal{L} = D_\alpha \left(\omega_\beta^\alpha x^\beta \mathcal{L} \right). \tag{3.110}$$

(Here I am still using the jet space notation with the total derivative D.) Thus the Lorentz transformations define a divergence symmetry for the KG Lagrangian. The resulting Noether current is given by

$$\begin{aligned} j^\alpha &= -g^{\alpha\beta}\varphi_{,\beta}(\omega_\delta^\gamma x^\delta)\varphi_{,\gamma} - \omega_\beta^\alpha x^\beta \mathcal{L} \\ &= \omega_{\gamma\delta} M^{\alpha(\gamma)(\delta)}, \end{aligned} \tag{3.111}$$

where $M^{\alpha(\gamma)(\delta)}$ are the conserved currents associated with relativistic angular momentum.

PROBLEM: Derive (3.110).

3.14   Spacetime symmetries in general

The symmetry transformations we have been studying involve spacetime translations:

$$x^\alpha \longrightarrow x^\alpha + \lambda a^\alpha, \qquad (3.112)$$

where $a^\alpha = const.$ and Lorentz transformations,

$$x^\alpha \longrightarrow S^\alpha_\beta(\lambda)x^\beta, \qquad (3.113)$$

where

$$S^\alpha_\beta S^\gamma_\delta \eta_{\alpha\gamma} = \eta_{\beta\delta}. \qquad (3.114)$$

These symmetries are, naturally enough, called *spacetime symmetries* since they involve transformations in spacetime. These symmetry transformations have a nice geometric interpretation which goes as follows.

   Given a spacetime (M, g) we can consider the group of diffeomorphisms, which are smooth mappings of M to itself with smooth inverses. Given a diffeomorphism

$$f \colon M \to M, \qquad (3.115)$$

there is associated to the metric g a new metric f^*g via the *pull-back*. In coordinates x^α on M the diffeomorphism f is given as

$$x^\alpha \to f^\alpha(x), \qquad (3.116)$$

and the pullback metric has components related to the components of g via

$$(f^*g)_{\alpha\beta}(x) = \frac{\partial f^\gamma}{\partial x^\alpha} \frac{\partial f^\delta}{\partial x^\beta} g_{\gamma\delta}(f(x)). \qquad (3.117)$$

We say that f is an *isometry* if

$$f^*g = g. \qquad (3.118)$$

The idea of an isometry is that it is a symmetry of the metric – the spacetime points have been moved around, but the metric can't tell it happened.

   Consider a 1-parameter family of diffeomorphisms f_λ such that $f_0 =$ *identity*. As the parameter varies, each point in M traces out a curve. The tangent vectors to all these curves constitute a vector field, which I will denote by X. The *Lie derivative* of the metric along X is defined as

$$L_X g := \left(\frac{d}{d\lambda} f^*_\lambda g \right)_{\lambda=0}. \qquad (3.119)$$

If f_λ is a 1-parameter family of isometries then we have that

$$L_X g = 0. \tag{3.120}$$

This is the infinitesimal version of (3.118).

It is not too hard to verify that the spacetime translations and the Lorentz translations define isometries of the Minkowski metric

$$g = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta. \tag{3.121}$$

In fact, it can be shown that all continuous isometries of flat spacetime are contained in the *Poincaré group*, which is the group of diffeomorphisms built from spacetime translations and Lorentz transformations. As an exercise you can convince yourself that this group is labeled by 10 parameters.

The KG Lagrangian depends upon a choice of spacetime for its definition. Recall that a spacetime involves specifying two structures: a manifold M and a metric g on M. Isometries are symmetries of that structure: they are diffeomorphisms – symmetries of M – that also preserve the metric. It is not too surprising, then, that the Lagrangian symmetries we have been studying are symmetries of the spacetime, since that is the only structure used to construct the Lagrangian. The existence of conservation laws of energy, momentum and angular momentum is contingent upon the existence of suitable spacetime symmetries.

3.15 Internal symmetries

Besides the spacetime symmetries, there is another class of symmetries in field theory that is very important since, for example, it is the source of myriad other conservation laws besides energy, momentum and angular momentum. This class of symmetries is known as *internal symmetries* since they do not involve transformations of spacetime, but only transformations on the space of fields.

A simple example of an internal Lagrangian symmetry (albeit a discrete symmetry) for the KG theory is given by $\varphi \to -\varphi$, as you can easily verify by inspection. This symmetry extends to self-interacting scalar theories with potentials which are an even function of φ, *e.g.,* the double-well potential.

For our purposes, there are no particularly interesting continuous internal symmetries of the KG theory unless one sets the rest mass to zero. Then we have the following situation.

PROBLEM: Consider the KG theory with $m = 0$. Show that the transformation

$$\varphi_\lambda = \varphi + \lambda \qquad (3.122)$$

is a variational symmetry. Use Noether's theorem to find the conserved current and conserved charge.

There is an important generalization of the KG theory which admits a fundamental internal symmetry, and this is our next topic.

3.16   The charged KG field and its internal symmetry

An important generalization of the Klein-Gordon field is the *charged Klein-Gordon field*. The charged KG field can be viewed as a mapping

$$\varphi \colon M \to \mathbf{C}, \qquad (3.123)$$

so that there are actually two real-valued functions in this theory. The Lagrangian for the charged KG field is

$$\mathcal{L} = -(g^{\alpha\beta}\varphi_{,\alpha}\varphi^*_{,\beta} + m^2|\varphi|^2), \quad g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1). \qquad (3.124)$$

PROBLEM: Show that this Lagrangian is the sum of the Lagrangians for two (real-valued) KG fields φ_1 and φ_2 with $m_1 = m_2$ and with the identification

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2). \qquad (3.125)$$

Recall from Lagrangian mechanics that two dynamical systems, described respectively by Lagrangians L_1 and L_2, can be combined into a total system (with no interactions) by using the Lagrangian $L_1 + L_2$. Thus the Lagrangian for the charged KG field represents that of two non-interacting KG fields. From this problem you can also surmise that the field equations for the charged KG field consist of two identical KG equations for the real and imaginary parts of φ. In terms of the complex-valued function φ you can

check that the field equations – computed as Euler-Lagrange equations or via the critical points of the action – are simply

$$\mathcal{E}_\varphi(\mathcal{L}) = (\Box - m^2)\varphi^* = 0, \tag{3.126}$$

$$\mathcal{E}_{\varphi^*}(\mathcal{L}) = (\Box - m^2)\varphi = 0. \tag{3.127}$$

Note that one can do field-theoretic computations such as deriving these Euler-Lagrange equations either using the real functions φ_1 and φ_2 or using the familiar trick of using "complex coordinates" on the space of fields, that is, treating φ and φ^* as independent variables. In the former case there are 2 EL equations:

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} - D_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{1,\mu}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \varphi_2} - D_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{2,\mu}} = 0. \tag{3.128}$$

In the latter case, the field equations — equivalent to those above — can be computed via

$$\frac{\partial \mathcal{L}}{\partial \varphi} - D_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \varphi^*} - D_\mu \frac{\partial \mathcal{L}}{\partial \varphi^*_{,\mu}} = 0. \tag{3.129}$$

PROBLEM: Using $\varphi = \varphi_1 + i\varphi_2$, show that the field equations (3.128) and (3.129) are equivalent. You should be able to do this for an arbitrary Lagrangian density, but you should at least do it for the one given in (3.124).

In any case, one has doubled the size of the field space. As we shall see, the new "degrees of freedom" that have been introduced allow for a notion of conserved electric charge.[6]

It is easy to see that the Lagrangian (3.124) for the complex KG field admits the continuous symmetry

$$\varphi_\lambda = e^{i\lambda}\varphi, \quad \varphi^*_\lambda = e^{-i\lambda}\varphi^*. \tag{3.130}$$

This continuous variational symmetry is given various names. Sometimes it is called a "phase transformation" for obvious reasons. Because the set of

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[6] In addition, in the corresponding quantum field theory, the doubling of the degrees of freedom leads to anti-particles which are distinct from the particles.

unitary linear transformations of the vector space of complex numbers, denoted $U(1)$, is precisely the multiplicative group of phases $e^{i\lambda}$, sometimes the symmetry transformation (3.130) is called a "rigid $U(1)$ transformation", or a "global $U(1)$ transformation", or just a "$U(1)$ transformation". For various reasons related to Noether's second theorem (as we shall see), sometimes this transformation is called a "gauge transformation of the first kind". You will also find various mixtures of these terms in the literature. Whatever the name, you can see that the transformation is simply a rotation in the vector space of values of the fields φ_1 and φ_2 which were defined in the last problem. The Lagrangian is rotationally invariant in field space, hence the symmetry.

It is straightforward to compute the conserved current associated with the $U(1)$ symmetry using Noether's (first) theorem. The only novel feature here is that we have more than one field. I will therefore give the gory details. The infinitesimal transformation is given by

$$\delta\varphi = i\varphi, \quad \delta\varphi^* = -i\varphi^*. \tag{3.131}$$

The variation of the Lagrangian is, in general, given by

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi_{,\alpha} + \frac{\partial\mathcal{L}}{\partial\varphi^*}\delta\varphi^* + \frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}}\delta\varphi^*_{,\alpha} \\
&= \mathcal{E}_\varphi(\mathcal{L})\delta\varphi + \mathcal{E}_{\varphi^*}(\mathcal{L})\delta\varphi^* + D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}}\delta\varphi^*\right).
\end{aligned}
\tag{3.132}
$$

From the phase symmetry we know that when we set $\delta\varphi = i\varphi$ it follows that $\delta\mathcal{L} = 0$, so we have

$$0 = \mathcal{E}_\varphi(\mathcal{L})i\varphi - \mathcal{E}_{\varphi^*}(\mathcal{L})i\varphi^* + D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}i\varphi - \frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}}i\varphi^*\right). \tag{3.133}$$

Using

$$\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}} = g^{\alpha\beta}\varphi^*_{,\beta}, \tag{3.134}$$

$$\frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}} = g^{\alpha\beta}\varphi_{,\beta}, \tag{3.135}$$

we get a conserved current

$$j^\alpha = -ig^{\alpha\beta}\left(\varphi^*\varphi_{,\beta} - \varphi\varphi^*_{,\beta}\right). \tag{3.136}$$

PROBLEM: Verify directly from the above formula for j^α that

$$D_\alpha j^\alpha = 0, \tag{3.137}$$

when the field equations for φ and φ^* are satisfied.

The total "$U(1)$ charge" contained in a spatial volume V at $t = const.$ is given by

$$Q = i \int_V d^3x \left(\varphi^* \varphi_{,t} - \varphi \varphi^*_{,t} \right). \tag{3.138}$$

Note that the sign of this charge is indefinite: the charged KG field contains both positive and negative charges. This charge can be used to model electric charge in electrodynamics. It can also be used to model the charge which interacts via neutral currents in electroweak theory.

PROBLEM: The double well potential can be generalized to the $U(1)$-symmetric charged scalar field by choosing $V(\varphi) = -a^2|\varphi|^2 + b^2|\varphi|^4$. Check that the Lagrangian with this self-interaction potential still admits the $U(1)$ symmetry. (Hint: this is really easy.) Plot the graph of V as a function of the real and imaginary parts of φ. (Hint: you should see why this is often called the "Mexican hat potential" for an appropriate choice for m and a.) Find all solutions of the field equations of the form $\varphi = constant$. How do these solutions transform under the $U(1)$ symmetry?

3.17 More generally...

We can generalize our previous discussion as follows. Recall that, given a group G, a (linear) *representation of G* is a pair (r, V) where V is a vector space and $r\colon G \to GL(V)$ is a group homomorphism, that is, r is an identification of linear transformations $r(g)$ on V with elements $g \in G$ such that

$$r(g_1 g_2) = r(g_1) r(g_2). \tag{3.139}$$

This way of viewing things applies to the $U(1)$-symmetric charged Klein-Gordon theory as follows. For the charged scalar field the group $G = U(1)$ is the set of phases $e^{i\lambda}$, labeled by λ, and with group multiplication being ordinary multiplication of complex numbers:

$$e^{i\lambda_1} e^{i\lambda_2} = e^{i(\lambda_1 + \lambda_2)}. \tag{3.140}$$

The vector space is $V = \mathbf{C}$, and the representation r is via ordinary multiplication of elements $z \in \mathbf{C}$ by the phase $z \to r(\lambda)z = e^{i\lambda}z$.

In this $U(1)$ case, the internal symmetry transformation is multiplication of the complex-valued field by a phase. So we can view this situation as coming from the facts that (1) we can view φ as a map from spacetime into the representation vector space $\mathbf{C}$, and (2) the group $U(1)$ acts on $\mathbf{C}$, so that the composition of the maps, $r(\lambda) \circ \varphi$ defines the symmetry transformation.

There is a pretty straightforward generalization of this to a general group. Given a group G one picks a representation (r, V). One introduces fields that are maps from spacetime into V; we write

$$\varphi \colon M \to V. \tag{3.141}$$

Each element $g \in G$ defines a field transformation via

$$\varphi \longrightarrow r(g)\varphi. \tag{3.142}$$

While it is possible to generalize still further, this construction captures almost all instances of (finite-dimensional) internal symmetries in field theory. Of course, for the transformation just described to be a (divergence) symmetry, it is necessary that the Lagrangian be suitably invariant under the action of $r(g)$. One can examine this issue quite generally, but we will be content with exhibiting another important example.

3.18 SU(2) symmetry

The group SU(2) can be defined as the group of unitary, unimodular transformations of the vector space $\mathbf{C}^2$, equipped with its standard inner-product and volume element.[7] In terms of the Hermitian conjugate (complex-conjugate-transpose) $\dagger$, the unitarity condition on a linear transformation U is

$$U^\dagger = U^{-1}, \tag{3.143}$$

which is equivalent to saying that the linear transformation preserves the standard Hermitian inner product. The unimodularity condition is

$$\det U = 1, \tag{3.144}$$

---

[7]This way of defining SU(2) in terms of a representation provides the "defining representation".

which is equivalent to saying that the linear transformation preserves the standard volume form on $\mathbf{C}^2$.

Let us focus on the "defining representation" of SU(2) as just stated. Then the representation vector space is $\mathbf{C}^2$ and each element of SU(2) can be represented by a matrix of the form

$$r(g) = U(\theta, \mathbf{n}) = \cos(\frac{\theta}{2}) I + i \sin(\frac{\theta}{2}) n^j \sigma_j, \qquad (3.145)$$

where

$$\mathbf{n} = (n^1, n^2, n^3), \quad (n^1)^2 + (n^2)^2 + (n^3)^2 = 1, \qquad (3.146)$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3.147)$$

are the *Pauli matrices*. Note that group elements are uniquely labeled by the values of three parameters; one corresponding to θ and two corresponding to the free parameters defining the unit vector $\mathbf{n}$.[8]

We can use this group representation to define a continuous transformation group of a field theory using the general strategy we outlined earlier. The fields are defined to be mappings

$$\varphi \colon M \to \mathbf{C}^2, \qquad (3.148)$$

so we now have two charged KG fields or, equivalently, four real KG fields. You can think of φ as a 2-component column vector whose entries are complex functions on spacetime. Let $U(\lambda)$ be any one parameter family of SU(2) transformations, as described above. We assume that

$$U(0) = I. \qquad (3.149)$$

We define

$$\varphi_\lambda = U(\lambda)\varphi. \qquad (3.150)$$

The infinitesimal form of this transformation is

$$\delta\varphi = i\tau\varphi, \qquad (3.151)$$

---

[8]The elements of SU(2) can be parametrized by a unit vector and an angle, just as are elements of the rotation group $SO(3)$. This is related to the fact that SU(2) provides a *spinor representation* of the group of rotations.

where τ is a Hermitian, traceless 2×2 matrix defined by

$$\tau = \frac{1}{i} \left(\frac{dU}{d\lambda} \right)_{\lambda=0}. \tag{3.152}$$

Note that

$$\delta\varphi^\dagger = -i\varphi^\dagger \tau^\dagger = -i\varphi^\dagger \tau. \tag{3.153}$$

By the way, you can see that τ is traceless and Hermitian by considering our formula for $U(\theta, n)$ above, or by simply noting that $U(\lambda)$ satisfies

$$U^\dagger(\lambda)U(\lambda) = I, \quad \det(U(\lambda)) = 1 \tag{3.154}$$

for all values of λ. Differentiation of each of these relations and evaluation at $\lambda = 0$ yields the Hermitian ($\tau^\dagger = \tau$) and trace-free conditions, respectively. It is not hard to see that every Hermitian trace-free matrix is a linear combination of the Pauli matrices:

$$\tau = a^i \sigma_i, \tag{3.155}$$

where $a^{i*} = a^i$. Thus the SU(2) transformations can also be parametrized by the three numbers a^i.

On $\mathbf{C}^2$ we have the usual Hermitian inner product $(\cdot, \cdot)$ that is invariant with respect to the SU(2) transformation. With

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{3.156}$$

the is inner product is given by

$$\langle u, v \rangle = u^\dagger v = u_1^* v_1 + u_2^* v_2. \tag{3.157}$$

We can use it to define an inner product on the values of the fields φ:

$$\langle \varphi_1, \varphi_2 \rangle = \varphi_1^\dagger \varphi_2. \tag{3.158}$$

Here is how to see that this inner product is SU(2)-invariant:

$$\begin{aligned}
\langle U\varphi_1, U\varphi_2 \rangle &= (U\varphi_1)^\dagger (U\varphi_2) \\
&= \varphi_1^\dagger U^\dagger U \varphi_2 \\
&= \varphi_1^\dagger \varphi_2 \\
&= \langle \varphi_1, \varphi_2 \rangle.
\end{aligned} \tag{3.159}$$

This allows us to build a Lagrangian that has the SU(2) transformation as an internal variational symmetry:

$$\mathcal{L} = -\left[g^{\alpha\beta}\langle\varphi_{,\alpha},\varphi_{,\beta}\rangle + m^2(\varphi,\varphi)\right], \quad g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1). \quad (3.160)$$

This Lagrangian describes a pair of charged KG fields (or a quartet of real KG fields) with mass m. To see this, we write

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad (3.161)$$

and then

$$\mathcal{L} = -\left[g^{\alpha\beta}(\varphi^{1*}_{,\alpha}\varphi^1_{,\beta} + \varphi^{2*}_{,\alpha}\varphi^2_{,\beta}) + m^2(\varphi^{1*}\varphi^1 + \varphi^{2*}\varphi^2)\right]. \quad (3.162)$$

Representing the components of φ as φ^a, $a = 1,2$, we have the Euler-Lagrange equations

$$\mathcal{E}_a = (\Box - m^2)\varphi^a = 0, \quad (3.163)$$

which are equivalent to

$$\mathcal{E} = (\Box - m^2)\varphi = 0, \quad (3.164)$$

in our matrix notation. Of course, the complex (or Hermitian) conjugates of these equations are also field equations.

Just as before, we can use Noether's theorem to find the current that is conserved by virtue of the SU(2) symmetry.

PROBLEM: Show that for the symmetry $\delta\varphi = i\tau\varphi$ the associated conserved current is given by

$$j^\alpha = ig^{\alpha\beta}(\varphi^\dagger_{,\beta}\tau\varphi - \varphi^\dagger\tau\varphi_{,\beta}). \quad (3.165)$$

Using (3.165) we can build three independent conserved currents corresponding to the three independent symmetry transformations. To see them explicitly, one can pick τ to be each of the Pauli matrices. The 3 conserved charges associated with the SU(2) symmetry are often called "isospin", stemming from their use in nuclear and particle physics.

3.19   A general version of Noether's theorem

Let me briefly indicate, without proof, a more general version of Noether's theorem. (This is sometimes called "Noether's first theorem".) Given all of our examples, this theorem should not be very hard to understand.

Consider a field theory described by a set of functions φ^a, $a = 1, 2, \ldots, m$, and a Lagrangian, viewed as a function on the jet space $\mathcal{J}^k$:

$$\mathcal{L} = \mathcal{L}(x, \varphi^a, \partial\varphi^a, \ldots, \partial^k\varphi^a). \tag{3.166}$$

The Euler-Lagrange equations arise via the identity

$$\delta\mathcal{L} = \mathcal{E}_a\delta\varphi^a + D_\alpha\eta^\alpha(\delta\varphi), \tag{3.167}$$

where $\eta^\alpha(\delta\varphi)$ is a linear differential operator on $\delta\varphi^a$ constructed from the fields φ^a and their derivatives via the usual integration by parts procedure.[9] Suppose that there is an infinitesimal transformation,

$$\delta\varphi^b = F^b(x, \varphi^a, \partial\varphi^a, \ldots, \partial^l\varphi^a) \tag{3.168}$$

that is a divergence symmetry:

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{3.169}$$

for some W^α locally constructed from x, φ^a, $\varphi^a_{,\alpha}$, *etc.* Then the following is a conserved current:

$$j^\alpha = \eta^\alpha(F) - W^\alpha. \tag{3.170}$$

Noether's theorem, as it is conventionally stated – more or less as above, shows that symmetries of the Lagrangian beget conservation laws. But the scope of this theorem is actually significantly larger. It is possible to prove a converse to the result shown above, to the effect that to each conservation law for a system of Euler-Lagrange equations there is a corresponding symmetry of the Lagrangian. Indeed, Noether's theorem establishes a one-to-one correspondence between conservation laws and symmetries of the Lagrangian for a wide class of field theories (including the KG field and its variants that have been discussed up until now). There is even more than this! But it is time to move on. . . .

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[9]There is an ambiguity in the definition of η^α here which we shall ignore for now to keep things simple. We will confront it when we study conservation laws in electromagnetism.

3.20 "Trivial" conservation laws

For any field theory there are two ways to construct conserved currents that are in some sense "trivial". The first is to suppose that we have a conserved current that happens to vanish when the field equations hold. For example, in the KG theory we could use

$$j^\alpha = (\Box - m^2)\partial^\alpha \varphi. \tag{3.171}$$

It is easy to check that this current is conserved. It is even easier to check that this current is completely uninteresting since it vanishes for any solution of the field equations. The triviality of such conservation laws also can be seen by constructing the conserved charge in a region by integrating j^0 over a volume. Of course, when you try to substitute a solution of the equations of motion into j^0 so as to perform the integral you get zero. Thus you end up with the trivial statement that zero is conserved.

The second kind of "trivial" conservation law arises as follows. Suppose we create an antisymmetric, $\binom{2}{0}$ tensor field locally from the fields and their derivatives:

$$S^{\alpha\beta} = -S^{\beta\alpha}. \tag{3.172}$$

For example, in KG theory we could use

$$S^{\alpha\beta} = k^\alpha \varphi^{,\beta} - k^\beta \varphi^{,\alpha}, \tag{3.173}$$

where $k^\alpha = k^\alpha(x)$ is any given vector field on spacetime. Now make a current via

$$j^\alpha = D_\beta S^{\alpha\beta}. \tag{3.174}$$

It is easy to check that such currents are always conserved, irrespective of field equations, because the order of differentiation is immaterial:

$$D_\alpha j^\alpha = D_\alpha D_\beta S^{\alpha\beta} = D_\beta D_\alpha S^{\alpha\beta} = -D_\beta D_\alpha S^{\beta\alpha} = -D_\alpha j^\alpha \implies D_\alpha j^\alpha = 0. \tag{3.175}$$

These sorts of conservation laws are "trivial" because they do not really reflect properties of the field equations but rather simple derivative identities analogous to the fact that the divergence of the curl is zero, or that the curl of the gradient is zero. Indeed, the current above is divergence free for any function $\varphi(x)$, whether or not it satisfies any field equations.

It is also possible to understand this second kind of triviality from the point of view of the conserved charge

$$Q_V = \int_V d^n x \, j^0. \tag{3.176}$$

I am setting the dimension of spacetime to be n. Here, in order to perform the integral, it is understood that the current has been evaluated on some field configuration, *i.e.*, $\varphi = \varphi(x)$. Keep this in mind as we proceed. For a trivial conservation law arising as the divergence of an antisymmetric tensor we can integrate by parts, *i.e.*, use the divergence theorem, to express Q_V as an area integral over the boundary B of V:

$$Q_V = \int_V d^n x \, \partial_\beta S^{0\beta} = \int_B d^{n-1} A \, n_i S^{0i}. \tag{3.177}$$

Here $d^{n-1}A$ is the area element of the boundary, n is the covariant unit normal to the boundary, and $i = 1, 2, \ldots, n$. From the continuity equation, the time rate of change of Q_V arises from the flux through B:

$$\frac{d}{dt} Q_V = - \int_B d^{n-1} A \, n_i j^i. \tag{3.178}$$

But because this continuity equation is an identity (rather than holding by virtue of field equations) this relationship is tautological. To see this, we write:

$$-\int_B d^{n-1} A \, n_i j^i = -\int_B d^{n-1} A \, n_i \left(S^{i0}{}_{,0} + S^{ij}{}_{,j} \right) = \frac{d}{dt} \int_B d^{n-1} A \, n_i S^{0i} = \frac{d}{dt} Q_V, \tag{3.179}$$

where I used (1) the divergence theorem, and (2) a straightforward application of Stokes theorem in conjunction with the fact that $\partial B = \partial \partial V = \emptyset$ to get

$$\int_B d^{n-1} A \, n_i S^{ij}{}_{,j} = 0. \tag{3.180}$$

Thus the conservation law is really just saying that $\frac{dQ_V}{dt} = \frac{dQ_V}{dt}$.

Another way to view this kind of trivial conservation law is to note that, from (3.177), the conserved charge is really just a function of the values of the field on the boundary of the region V and has nothing to do with the state of the field in the interior of V. Again, the charge is conserved whether or not any field equations are satisfied.

PROBLEM: Let S be a two dimensional surface in Euclidean space with unit normal $\vec{n}$ and boundary curve C with tangent $\vec{dl}$. Show that

$$\int_S d^2S \, n_i S^{ij}{}_{,j} = \frac{1}{2} \int_C \vec{V} \cdot \vec{dl}, \qquad (3.181)$$

where

$$V^i = \frac{1}{2}\epsilon^{ijk} S_{jk}. \qquad (3.182)$$

We have seen there are two kinds of conservation laws that are in some sense trivial. We can combine these two kinds of triviality. So, for example, the current

$$j^\alpha = D_\beta(k^{[\beta}(x)D^{\alpha]}\varphi) + (\Box\varphi - m^2\varphi)D^\alpha\varphi \qquad (3.183)$$

is trivial.

We can summarize our discussion with a formal definition. We say that a conservation law j^α is *trivial* if there exists a skew-symmetric tensor field $S^{\alpha\beta}$ – locally constructed from the fields and their derivatives – such that

$$j^\alpha = D_\beta S^{\alpha\beta}, \quad \text{modulo the field equations.} \qquad (3.184)$$

Given a conservation law j^α (trivial or non-trivial) we see that we have the possibility to redefine it by adding a trivial conservation law. Thus given one conservation law there are infinitely many others "trivially" related to it. In particular, this means that, without some other criteria to choose among these conservation laws, there is no unique notion of "charge density" $\rho = j^0$ since one can change the form of this quantity quite a bit by adding in a trivial conservation law. Furthermore, without some specific boundary conditions, there is no unique choice of the *total charge* contained in a region. Usually, in a physical application of these ideas, there are additional criteria and specific boundary conditions that largely – if not completely – determine the choice of charge density and charge in a region. (If not, then the physicist should find these criteria!) Mathematically speaking, the optimal way to view conservation laws is really in terms of equivalence classes, with two conservation laws being equivalent if they differ by a trivial conservation law.

3.21   Conservation laws in terms of differential forms

Let me briefly describe a particularly nice way to think about conservation laws, which is in terms of differential forms. If you don't know about differential forms, you could give this section a miss. But consider spending a little time learning about differential forms when you have a chance.

For simplicity, I will stick to four spacetime dimensions. On our four dimensional spacetime, the vector field j^α can be converted to a 1-form $\omega = \omega_\alpha dx^\alpha$ using the metric:

$$\omega_\alpha = g_{\alpha\beta} j^\beta. \tag{3.185}$$

This 1-form can be converted to a 3-form,

$$*\omega = \frac{1}{3!}(*\omega)_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \tag{3.186}$$

using the Hodge dual

$$(*\omega)_{\alpha\beta\gamma} = g^{\mu\delta}\epsilon_{\alpha\beta\gamma\delta}\omega_\mu. \tag{3.187}$$

If the vector field on spacetime j is divergence free modulo the field equations, this is equivalent to $*\omega$ being *closed* modulo the field equations:

$$d(*\omega) = 0, \quad \text{modulo the field equations.} \tag{3.188}$$

Keep in mind that $*\omega$ is really a 3-form locally constructed from the field and its derivatives, that is, it is a 3-form-valued function on the jet space for the theory. The exterior derivative in (3.188) is a total derivative. As you know, an *exact* 3-form is of the form $d\beta$ for some 2-form β. If there is a 2-form β *locally constructed from the fields* such that

$$*\,\omega = d\beta \quad \text{modulo the field equations,} \tag{3.189}$$

then clearly $*\omega$ is closed modulo the field equations. This is just the differential form version of a trivial conservation law. Indeed, the anti-symmetric tensor field that is the "potential" for the conserved current is given by

$$S^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\beta_{\gamma\delta}. \tag{3.190}$$

Let me mention and dispose of a common point of confusion concerning trivial conservation laws. This point of confusion is why I felt compelled to

occasionally stick in the phrase "locally constructed from the field" in the discussion above. For simplicity, I will use the flat metric and Cartesian coordinates on the spacetime manifold $M = \mathbf{R}^4$ in what follows. To expose the potential point of confusion, let me remind you of the following standard result from multivariable calculus. Let V^α be a vector field on Minkowski space, expressed in the usual inertial Cartesian coordinates. V^α is not to be viewed as locally constructed from the field, except in the trivial sense that it does not depend upon the fields at all, only the spacetime point, $V^\alpha = V^\alpha(x)$. If V^α is divergence free,

$$\partial_\alpha V^\alpha = 0, \tag{3.191}$$

then there exists an antisymmetric tensor field $S^{\alpha\beta}$ such that

$$V^\alpha = \partial_\beta S^{\alpha\beta}. \tag{3.192}$$

This is just the dual statement to the well-known fact that all closed 3-forms (indeed, all closed forms of degree higher than 0) on $\mathbf{R}^4$ are exact *i.e.*, the De Rham cohomology of $\mathbf{R}^4$ is trivial. This result might tempt you to conclude that all conservation laws are trivial! Unlike the case in real life, you should not give in to temptation here. There are two reasons. First, a conservation law should not be viewed as just a single divergence-free vector field on the spacetime manifold M. A conservation law is a formula which assigns a divergence-free vector field to each solution of the field equations. Each field configuration will, in principle, define a different conserved current. Second, as we have been saying, each of the (infinite number of) divergence-free vector fields is to be *locally constructed from the fields, i.e.,* are functions on jet space (rather than just x space). Put differently, the conserved current at a point x is required to depend upon the values of the fields and their derivatives at the point x. The correct notion of triviality is that a conserved current j^α is trivial if for each field configuration it is (modulo the field equations) always a divergence of a skew tensor field $S^{\alpha\beta}$ *that is itself locally constructed from the fields.* If we take a conservation law and evaluate it on a particular solution to the field equations, then we end up with a divergence-free vector field on M (or a closed 3-form on M, if you prefer). If $M = \mathbf{R}^4$ we can certainly write this vector field as the divergence of an antisymmetric tensor on M (or as the exterior derivative of a 2-form on M). But the point is that for non-trivial conservation laws there is no way to construct all the antisymmetric tensors (2-forms) for all possible field configurations using a local formula in terms of the fields and their derivatives. So, while conservation laws are in many ways

like de Rham cohomology (closed modulo exact forms on M), they actually represent a rather different kind of cohomology.[10] Sometimes this kind of cohomology is called "local cohomology" or "Euler-Lagrange cohomology".

Finally, as I have mentioned without explanation here and there via footnotes, there is some ambiguity in the definition of the vector field which appears in the divergence term in the variation of the Lagrangian density (see *e.g.,* V in (3.68)). In light of our definition of trivial conservation laws, I think you can easily see what that ambiguity is. Namely, the variational identity only determines this vector field up to addition of the divergence of a skew tensor (locally constructed from the fields and field variations). This ambiguity affects the Noether theorem formula (3.170) for the conserved current as follows. Using two different choices for the vector field in the divergence term in (3.167) one finds that the two resulting conservation laws, each constructed via (3.170), differ by a trivial conservation law.

PROBLEM: Consider two possible choices, η_1 and η_2, for η defined in (3.167). They are related by

$$\eta_2^\alpha = \eta_1^\alpha + D_\beta S^{\alpha\beta}, \qquad\qquad (3.193)$$

for some skew tensor S locally constructed from the fields and field variations. Suppose there is a divergence symmetry of the Lagrangian. Show that the conserved currents j_1 and j_2, constructed via (3.170) using η_1 and η_2, respectively, differ by a trivial conservation law.

3.22   PROBLEMS

1. Derive (3.10) from the continuity equation.

2. What becomes of conservation of energy for the KG field when an external source is present, as in (2.77)? How do you physically interpret this state of affairs?

3. Verify that the currents (3.23), (3.24) are conserved. (If you like, you can just fix a value for i, say, $i = 1$ and check that j_1^α is conserved.)

4. Show that the six currents (3.36) are conserved. (*Hint: Don't panic! This is actually the easiest one to prove so far, since you can use*

$$g^{\beta\gamma}D_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi,$$

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[10]One says the conservation laws are "horizontally" closed forms on the jet space.

which we have already established.)

5. Consider the real scalar field with the double well self-interaction potential (2.81). Show that $\hat{\varphi} = -\varphi$ is a symmetry. Consider the 3 constant solutions to the field equations (which you found in the problem just after (2.81)) and check that this symmetry maps these solutions to solutions.

6. Let $\hat{\varphi} = F(\varphi)$ be a variational symmetry of a Lagrangian. Show that it maps solutions of the Euler-Lagrange equations to new solutions, that is, if φ is a solution, so is $\hat{\varphi}$. If you like, you can restrict your attentions to Lagrangians $\mathcal{L}(x, \varphi, \partial\varphi)$.

7. Show that the scaling transformation (3.50) is neither a variational symmetry nor a divergence symmetry for the KG Lagrangian.

8. Verify that the Noether currents associated with a spacetime translation do yield the energy-momentum tensor.

9. Derive (3.110).

10. Consider the KG theory with $m = 0$. Show that the transformation

$$\varphi_\lambda = \varphi + \lambda$$

is a variational symmetry. Use Noether's theorem to find the conserved current and conserved charge.

11. Show that the Lagrangian (3.124) is the sum of the Lagrangians for two (real-valued) KG fields φ_1 and φ_2 with $m_1 = m_2$ and with the identification

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2).$$

12. Using $\varphi = \varphi_1 + i\varphi_2$, show that the field equations (3.128) and (3.129) are equivalent. You should be able to do this for an arbitrary Lagrangian density, but you should at least do it for the one given in (3.124).

13. Verify directly from (3.136) that

$$D_\alpha j^\alpha = 0,$$

when the field equations for φ and φ^* are satisfied.

14. The double well potential can be generalized to the $U(1)$-symmetric charged scalar field by choosing $V(\varphi) = -a^2|\varphi|^2 + b^2|\varphi|^4$. Check that the Lagrangian with this self-interaction potential still admits the $U(1)$ symmetry. (Hint: this is really easy.) Plot the graph of V as a function of the real and imaginary parts of φ. (Hint: you should see why this is often called the "Mexican hat potential" for an appropriate choice for m and a.) Find all solutions of the field equations of the form $\varphi = constant$. How do these solutions transform under the $U(1)$ symmetry?

15. Show that the SU(2) symmetry $\delta\varphi = i\tau\varphi$ of (3.160) has an associated conserved current given by

$$j^\alpha = ig^{\alpha\beta}(\varphi^\dagger_{,\beta}\tau\varphi - \varphi^\dagger\tau\varphi_{,\beta}).$$

16. Let S be a two dimensional surface in Euclidean space with unit normal $\vec{n}$ and boundary curve C with tangent $\vec{dl}$. Show that

$$\int_S d^2S\, n_i S^{ij}{}_{,j} = \int_C \vec{V}\cdot\vec{dl}, \tag{3.194}$$

where

$$V^i = \frac{1}{2}\epsilon^{ijk}S_{jk}. \tag{3.195}$$

17. Consider two possible choices, η_1 and η_2, for η defined in (3.167). They are related by

$$\eta_2^\alpha = \eta_1^\alpha + D_\beta S^{\alpha\beta}, \tag{3.196}$$

for some skew tensor S locally constructed from the fields and field variations. Suppose there is a divergence symmetry of the Lagrangian. Show that the conserved currents j_1 and j_2, constructed via (3.170) using η_1 and η_2, respectively, differ by a trivial conservation law.

Chapter 4

The Hamiltonian formulation

The Hamiltonian formulation of dynamics is in many ways the most elegant, powerful, and geometric approach. For example, the relation between symmetries and conservation laws becomes an identity in the Hamiltonian formalism. Another important motivation comes from quantum theory: one uses elements of the Hamiltonian formalism to construct quantum theories that have the original dynamical system as a classical limit. Such features provide ample motivation for a brief foray into the application of Hamiltonian techniques in field theory. To begin, it is worth reminding you how we do classical mechanics from the Hamiltonian point of view.

4.1 Review of the Hamiltonian formulation of mechanics

Probably you are familiar with the Hamiltonian formulation of mechanics in the following form. There is a set of "canonical coordinates and momenta", (q^i, p_i), $i = 1, 2, \ldots, n$, where n is the number of "degrees of freedom" of the mechanical system. The space parametrized by (q, p) is the "phase space" of the system. There is a function on phase space (and maybe time), $H = H(q, p, t)$, known as the "Hamiltonian", which defines the equations of motion via curves in the phase space according to a set of $2n$ first-order ODEs:

$$\dot{q}^i(t) = \left(\frac{\partial H}{\partial p_i} \right)_{\substack{q=q(t) \\ p=p(t)}}, \quad \dot{p}_i(t) = -\left(\frac{\partial H}{\partial q^i} \right)_{\substack{q=q(t) \\ p=p(t)}}. \tag{4.1}$$

These are "Hamilton's equations of motion".

The Hamiltonian form of the equations of motion is related to the Lagrangian version as follows. Given a Lagrangian $L = L(q, \dot{q}, t)$, the canonical momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{4.2}$$

One solves this equation for the velocities,

$$\dot{q}^i = \dot{q}^i(q, p, t), \tag{4.3}$$

and defines the Hamiltonian via

$$H(q, p, t) = p_i \dot{q}^i(q, p, t) - L(q, \dot{q}(q, p, t), t). \tag{4.4}$$

The Hamilton equations can then be shown to be equivalent to the EL equations when the relation (4.2) between the momenta and velocities are taken into account. Notice that the EL equations will be a system of n second-order ODEs, while the Hamilton equations will be a system of $2n$ first-order ODEs.

I would now like to introduce you to an elegant, more geometric version of all this. This fancier presentation has a number of advantages when it is applied to relativistic field theory, *e.g.* it allows for "manifest spacetime covariance".

In classical mechanics a Hamiltonian system has 2 ingredients: (1) a phase space, and (2) a function on phase space – the Hamiltonian – defining time evolution. The phase space is an even-dimensional manifold Γ equipped with a non-degenerate, closed 2-form Ω. In coordinates z^i on Γ, the 2-form can be viewed as a tensor field defined by components $\Omega_{ij} = \Omega_{ij}(q, p) = -\Omega_{ji}$ which form an antisymmetric invertible matrix:

$$\Omega = 2\Omega_{ij} dz^i \otimes dz^j = \Omega_{ij}(dz^i \otimes dz^j - dz^j \otimes dz^i) \equiv \Omega_{ij} dz^i \wedge dz^j. \tag{4.5}$$

The invertibility of the array Ω_{ij} is equivalent to the non-degeneracy requirement. To say that Ω is "closed" means that its exterior derivative vanishes, $d\Omega = 0$. In terms of its components this means

$$\partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega_{ij} = 0. \tag{4.6}$$

The 2-form is called the *symplectic* form, and the pair (Γ, Ω) is known mathematically as a *symplectic manifold*. Physicists call it the *phase space* or *state*

space for the mechanical system. The dimensionality of Γ is $2n$, where the integer n is the number of *degrees of freedom* of the system.

The second ingredient in a Hamiltonian system is a function, $H: \Gamma \to \mathbf{R}$, called the *Hamiltonian*. The Hamiltonian defines how states of the system evolve in time. Given coordinates z^i for Γ, the dynamical evolution of the mechanical system is a curve $z^i = z^i(t)$ in the phase space defined by the ordinary differential equations:

$$\dot{z}^i(t) = \Omega^{ij} \frac{\partial H}{\partial z^j} \bigg|_{z=z(t)}, \tag{4.7}$$

where Ω^{ij} are the components of the inverse symplectic form. The ODEs (4.7) are known as *Hamilton's equations* of motion. More generally, given any function $G: \Gamma \to \mathbf{R}$, there is an associated foliation of Γ by curves $z^i = z^i(s)$ defined by

$$\dot{z}^i(s) = \Omega^{ij} \frac{\partial G}{\partial z^j} \bigg|_{z=z(s)}. \tag{4.8}$$

The probably more familiar textbook presentation of the Hamiltonian formalism emerges via the *Darboux theorem*. This theorem asserts that, given a closed, non-degenerate 2-form such as Ω, there exist coordinates, $z^i = (q^a, p_b)$, $a, b = 1, 2, \ldots, n$, such that Ω can be put into a standard form:

$$\Omega = dp_a \otimes dq^a - dq^a \otimes dp_a \equiv dp_a \wedge dq^a. \tag{4.9}$$

You can check that this form of Ω is closed and non-degenerate. The coordinates (q, p) are called canonical coordinates (or "canonical coordinates and momenta"). The existence of such coordinates is a direct consequence of the non-degeneracy of Ω and the condition (4.6). There are infinitely many canonical coordinate systems.[1] As a nice exercise, you should determine the components Ω_{ij} of the symplectic form in canonical coordinates. In canonical coordinates the Hamilton equations take the familiar textbook form

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}. \tag{4.10}$$

PROBLEM: Derive (4.10) from (4.7) and (4.9).

---

[1]They are all related by "canonical transformations".

Given an action principle and/or a non-degenerate Lagrangian (see below for the meaning of "non-degenerate"), there is a canonical construction of an associated Hamiltonian system. It is worth seeing how this goes.

Let the configuration variables and velocities be denoted by $(q^a, \dot{q}^a)$. For simplicity, we only exhibit this construction in the familiar situation where $L = L(q, \dot{q}, t)$. Recall the variational identity:

$$\delta L = \mathcal{E}_a \delta q^a + \frac{d}{dt}\theta, \tag{4.11}$$

where $\mathcal{E}_a$ is the Euler-Lagrange formula,

$$\mathcal{E}_a = \frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a}, \tag{4.12}$$

and

$$\theta = \frac{\partial L}{\partial \dot{q}^a}\delta q^a. \tag{4.13}$$

represents the "boundary term" which arises via integration by parts in the associated variational principle. The quantity θ is a function of q^a, $\dot{q}^a$ and depends linearly upon δq^a:

We will construct the Hamiltonian formalism from these ingredients. This means we will use these data to define the phase space (Γ, Ω) and a Hamiltonian $H \colon \Gamma \to \mathbf{R}$. We do this as follows.

Define Γ to be the set of solutions to the EL equations $\mathcal{E}_a = 0$. In our setting, assuming $L = L(q, \dot{q}, t)$ and making the non-degeneracy assumption

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}\right) \neq 0, \tag{4.14}$$

the EL equations are a quasi-linear system of second order ODEs and we can expect that the solution space is a $2n$-dimensional manifold parametrized by, *e.g.,* the initial conditions. The points of Γ are thus solutions $q^a(t)$. Tangent vectors to Γ at a point defined by the solution $q^a(t)$ are then variations $\delta q^a(t)$, which are solutions to the linear system of equations obtained by linearizing the EL equations about the solution $q^a(t)$.

PROBLEM: Given $L = L(q, \dot{q}, t)$, give a formula for the EL equations linearized about a solution $q^a = q^a(t)$.

The symplectic form on Γ is constructed from θ using the following geometric point of view. Think of θ as the value of a 1-form Θ on a tangent vector $\delta q^a(t)$ to Γ,

$$\theta = \Theta(\delta q). \tag{4.15}$$

The symplectic 2-form is then constructed via

$$\Omega = d\Theta. \tag{4.16}$$

In formulas, this prescription says that the value of the symplectic 2-form on a pair of tangent vectors $\delta_1 q^a(t)$ and $\delta_2 q^a(t)$ at the point of Γ specified by $q^a(t)$ is given by

$$\Omega(\delta_1 q, \delta_2 q) = \frac{\partial^2 L}{\partial \dot{q}^a \partial q^b}(\delta_1 q^b \delta_2 q^a - \delta_1 q^a \delta_2 q^b) + \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}(\delta_1 \dot{q}^b \delta_2 q^a - \delta_1 q^a \delta_2 \dot{q}^b), \tag{4.17}$$

where it is understood that all expressions involving $q(t)$ and $\dot{q}(t)$ are evaluated at the given solution, and that all instances of $\delta q(t)$ are evaluated at solutions of the equations of motion linearized about $q(t)$.

The formula (4.17) appears to depend upon the time at which the solutions and the linearized solutions are evaluated. But it can be shown that Ω does not depend upon t by virtue of the EL equations satisfied by $q^a(t)$ and their linearization satisfied by $\delta q^a(t)$.

PROBLEM: Show that the symplectic form given in (4.17) does not depend upon time. (*Hint:* Differentiate (4.17) with respect to time and then use the EL equations and their linearization.)

Finally, we need to define a Hamiltonian and recover Hamilton's equations. Given a function H on Γ, Hamilton's equations specify a family of curves on Γ by specifying their tangent vector field X using the relation (4.7), which can be written as:

$$X = \Omega^{-1}(\cdot, dH). \tag{4.18}$$

Since we are currently viewing the phase space as the set of solutions to the EL equations, the Hamiltonian should define one parameter families of solutions corresponding to translations in time of those solutions. For simplicity in what follows, I must now suppose we are working with a system whose

Lagrangian is not explicitly dependent upon time, $L = L(q, \dot{q})$.[2] I will now show how the canonical energy function,

$$E = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L, \tag{4.19}$$

will implement (4.18) for time translations.

To begin, it is a little easier to rewrite (4.18) as

$$\Omega(\delta q, X) = dE(\delta q), \tag{4.20}$$

where δq is any solution to the linearized equations and X defines the linearized solution defined by infinitesimal time translations[3]:

$$X = \dot{q}^a. \tag{4.21}$$

Using (4.17), we compute:

$$\Omega(\delta q, X) = \frac{\partial^2 L}{\partial \dot{q}^a \partial q^b}(\dot{q}^a \delta q^b - \dot{q}^b \delta q^a) + \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}(\delta \dot{q}^b \dot{q}^a - \delta q^a \ddot{q}^b). \tag{4.22}$$

Let us compare this with (after some simplifications!)

$$dE(\delta q) = \frac{\partial^2 L}{\partial \dot{q}^a \partial q^b}\dot{q}^a \delta q^b + \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}\dot{q}^a \delta \dot{q}^b - \frac{\partial L}{\partial q^a}\delta q^a. \tag{4.23}$$

The first two terms appear as the first and third terms in (4.22). Since the EL equations are satisfied by $q^a(t)$, the third term is

$$-\frac{\partial L}{\partial q^a}\delta q^a = -\delta q^a \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a}$$
$$= -\delta q^a \left(\frac{\partial^2 L}{\partial \dot{q}^a \partial q^b}\dot{q}^b + \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}\ddot{q}^b \right) \tag{4.24}$$

This matches the second and fourth terms in (4.22), and (4.20) is verified.

---

[2]Otherwise, time translations will not map solutions to solutions, and a more sophisticated approach must be used.

[3]Notice that if the Lagrangian depends explicitly upon time, $L = L(q, \dot{q}, t)$ then $\dot{q}^a$ will not in general define a solution to the linearized equations and thus does not define a tangent vector to phase space. To deal with this this case takes us too far afield.

PROBLEM: A harmonic oscillator is defined by the Lagrangian and EL equations

$$L = \frac{1}{2}m\dot{q} - \frac{1}{2}m\omega^2 q^2, \quad \ddot{q}(t) = -\omega^2 q(t). \tag{4.25}$$

What does the symplectic structure look like when the space of solutions to the EL equations is parametrized according to the following formulas?

- $q(t) = A\cos(\omega t) + B\sin(\omega t)$

- $q(t) = ae^{-i\omega t} + a^* e^{i\omega t}$

4.2 Hamiltonian formulation of the scalar field

As we saw earlier, the Lagrangian density for the scalar field with self-interaction potential V is given by

$$\mathcal{L} = \frac{1}{2}\left(\varphi_{,t}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right) - V(\varphi), \tag{4.26}$$

and the Lagrangian is

$$L = \int_{\mathbf{R}^3} d^3x\,\mathcal{L}. \tag{4.27}$$

As shown earlier, the variational identity is

$$\delta\mathcal{L} = \left(-\varphi_{,tt} + \nabla^2\varphi - m^2\varphi - V'(\varphi)\right)\delta\varphi + D_\alpha W^\alpha, \tag{4.28}$$

where we can choose

$$W^0 = \varphi_{,t}\delta\varphi, \quad W^i = -\varphi_{,}^{\ i}\delta\varphi. \tag{4.29}$$

We can now mimic the construction of the Hamiltonian formalism from the previous section.

We let the phase space Γ consist of a suitable[4] function space of solutions φ to the EL equations,

$$\Box\varphi - m^2\varphi - V'(\varphi) = 0. \tag{4.30}$$

---

[4] "Suitable" could be, for example, smooth solutions to the field equations with compactly supported initial data.

Assuming the solutions vanish at spatial infinity, the variational identity for the Lagrangian evaluated on Γ reads:

$$\delta L = \frac{d}{dt} \int_{\mathbf{R}^3} d^3x\, W^0. \tag{4.31}$$

We define a 1-form Θ by its linear action on $\delta\varphi$, a tangent vector to Γ:

$$\Theta(\delta\varphi) = \int_{\mathbf{R}^3} d^3x\, W^0 = \int_{\mathbf{R}^3} d^3x\, \varphi_{,t}\delta\varphi. \tag{4.32}$$

From (4.16), the symplectic 2-form evaluated on a pair of tangent vectors $\delta_1\varphi$, $\delta_2\varphi$ at a solution φ to (4.30) is given by

$$\Omega(\delta_1\varphi, \delta_2\varphi) = \int_{\mathbf{R}^3} d^3x\, (\delta_2\varphi_{,t}\delta_1\varphi - \delta_1\varphi_{,t}\delta_2\varphi)\,. \tag{4.33}$$

To calculate the symplectic structure using (4.33) we have to pick a time t at which to perform the volume integral. Indeed, it would appear that we actually have defined many 2-forms – one for each value of t. Let us show that the symplectic form given in (4.33) does not in fact depend upon which time one chooses to perform the volume integral. Take a time derivative to get

$$
\begin{aligned}
\frac{d}{dt}\Omega(\delta_1\varphi, \delta_2\varphi) &= \int_{\mathbf{R}^3} d^3x\, (\delta_2\varphi_{,tt}\delta_1\varphi - \delta_1\varphi_{,tt}\delta_2\varphi + \delta_2\varphi_{,t}\delta_1\varphi_{,t} - \delta_1\varphi_{,t}\delta_2\varphi_{,t}) \\
&= \int_{\mathbf{R}^3} d^3x\, (\delta_2\varphi_{,tt}\delta_1\varphi - \delta_1\varphi_{,tt}\delta_2\varphi)\,.
\end{aligned}
$$
$$\tag{4.34}$$

The tangent vectors to Γ each satisfy the linearization of equation (4.30) about a solution φ:

$$\Box\delta\varphi - m^2\delta\varphi - V''(\varphi)\delta\varphi = 0. \tag{4.35}$$

This means, in particular, that

$$\delta\varphi_{,tt} = \nabla^2\delta\varphi - (m^2 + V''(\varphi))\delta\varphi \tag{4.36}$$

Using this relation for each linearized solution in (4.34) we get

$$
\begin{aligned}
\frac{d}{dt}\Omega(\delta_1\varphi, \delta_2\varphi) &= \int_{\mathbf{R}^3} d^3x\, \left[(\nabla^2\delta_2\varphi)\delta_1\varphi - (\nabla^2\delta_1\varphi)\delta_2\varphi\right] \\
&= \int_{\mathbf{R}^3} d^3x\, (\nabla\delta_1\varphi \cdot \nabla\delta_2\varphi - \nabla\delta_2\varphi \cdot \nabla\delta_1\varphi) \\
&= 0. \tag{4.37}
\end{aligned}
$$

To get this result I used the divergence theorem and the requirement that the solutions and their linearization vanish asymptotically.

Up to this point the phase space has been defined implicitly inasmuch as we have not given an explicit parametrization of the set of solutions to the field equations. There are various ways to parametrize the space of solutions depending upon how much analytic control you have over this space. Let us use the most traditional parametrization, which relies upon the existence of a well-posed initial value problem: for every pair of functions on $\mathbf{R}^3$, $(\phi(\vec{x}), \pi(\vec{x}))$ there is uniquely determined a solution $\varphi(x)$ of the field equations such that at a given initial time $t = t_0$:

$$\varphi(t_0, \vec{x}) = \phi(\vec{x}), \quad \partial_t \varphi(t_0, \vec{x}) = \pi(\vec{x}). \tag{4.38}$$

We can thus view a point in Γ as just a pair of functions (ϕ, π) on $\mathbf{R}^3$. Using this parametrization of solutions, from (4.32) and (4.33):

$$\Theta(\delta\phi, \delta\pi) = \int_{\mathbf{R}^3} d^3x \, \pi \, \delta\phi. \tag{4.39}$$

$$\Omega(\{\delta\phi_1, \delta\pi_1\}, \{\delta\phi_2, \delta\pi_2\}) = \int_{\mathbf{R}^3} d^3x \, (\delta_2\pi \, \delta_1\phi - \delta_1\pi \, \delta_2\phi). \tag{4.40}$$

Hopefully you recognize the pattern familiar from particle mechanics where $\Theta = p_i dq^i$ and $\Omega = dp_i \wedge dq^i$. Indeed, if one views the field as just a mechanical system with an infinite number of degrees of freedom labeled by the spatial point $\vec{x}$, then one can view the integrations over $\vec{x}$ as the generalizations of the various sums over degrees of freedom which occur in particle mechanics. For this reason people often call $\phi(\vec{x})$ the "coordinate" and $\pi(\vec{x})$ the "momentum" for the scalar field. Although we shall not worry too much about precisely what function spaces ϕ and π live in, it is useful to note that both of these functions must vanish at spatial infinity if the symplectic structure is to be defined.

The formula (4.40) makes it easy to check that the symplectic form is non-degenerate as it should be. Can you construct a proof?

PROBLEM: Show that the symplectic structure (4.40) is non-degenerate. (*Hint:* A 2-form Ω is non-degenerate if and only if $\Omega(u, v) = 0$, $\forall v$ implies $u = 0$.)

PROBLEM: Express the symplectic structure on the space of solutions of

the KG equation, $(\Box - m^2)\varphi = 0$ in terms of the Fourier parametrization of solutions which we found earlier:

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \, \left(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_k t} + a_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t}\right).$$

Hamilton's equations should define curves in Γ, that is, one parameter families $(\phi(\vec{x}, t), \pi(\vec{x}, t))$, corresponding to the time evolution dictated by the field equation (4.30). You can easily verify as an exercise that the following equations are equivalent to the original field equation (4.30) once we make the correspondence $\varphi(x) = \phi(\vec{x}, t)$.

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = \pi(\vec{x}, t) \tag{4.41}$$

$$\frac{\partial \pi(\vec{x}, t)}{\partial t} = \nabla^2 \phi(\vec{x}, t) - m^2 \phi(\vec{x}, t) - V'(\vec{x}, t). \tag{4.42}$$

I will now show that these equations can be put into Hamiltonian form,

$$\frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta \pi}$$
$$\frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta \phi}, \tag{4.43}$$

where the Hamiltonian H generating time evolution along the foliation of Minkowski spacetime by hypersurfaces $t = const.$ is given by the energy as defined in this inertial reference frame. In terms of our parametrization of Γ using initial data:

$$H[\phi, \pi] = \int_{\mathbf{R}^3} d^3x \, \left(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right), \tag{4.44}$$

where ∇ denotes the usual spatial gradient. In preparation for calculating the Hamilton equations (4.43), let us consider the functional derivatives of H. To do this we vary ϕ and π:

$$\delta H = \int_{\mathbf{R}^3} d^3x \, \left(\pi \, \delta\pi + \nabla\phi \cdot \nabla\delta\phi + m^2\phi \, \delta\phi + V' \, \delta\phi\right). \tag{4.45}$$

Integrating by parts in the term with the gradients, and using the fact that ϕ vanishes at infinity to eliminate the boundary terms, we get

$$\delta H = \int_{\mathbf{R}^3} d^3x \left[\pi\, \delta\pi + \left(-\nabla^2\phi + m^2\phi + V' \right) \delta\phi \right]. \qquad (4.46)$$

This means

$$\frac{\delta H}{\delta\pi(\vec{x})} = \pi(\vec{x}), \qquad (4.47)$$

and

$$\frac{\delta H}{\delta\phi(\vec{x})} = -\nabla^2\phi(\vec{x}) + m^2\phi(\vec{x}) + V'(\phi(\vec{x})). \qquad (4.48)$$

You can now see that the Hamilton equations (4.43) are equivalent to the field equations (4.30).

Finally, using standard techniques of classical mechanics it is possible to deduce the Hamiltonian from the Lagrangian. Recall that the KG Lagrangian is (in a particular inertial reference frame $(t, \vec{x})$)

$$L = \int_{\mathbf{R}^3} d^3x\, \mathcal{L} = \int_{\mathbf{R}^3} d^3x \left\{ \frac{1}{2} \left(\varphi_{,t}^2 - (\nabla\varphi)^2 - m^2\varphi^2 \right) - V(\varphi) \right\}. \qquad (4.49)$$

At any fixed time t, view $\phi \equiv \varphi(t, \vec{x})$ and $\dot\phi \equiv \varphi_{,t}(t, \vec{x})$ as independent fields on $\mathbf{R}^3$. Define the canonical momentum as the functional derivative of the Lagrangian with respect to the velocity:

$$\pi = \frac{\delta L}{\delta\dot\phi} = \dot\phi. \qquad (4.50)$$

The Lagrangian now assumes the canonical form:

$$L = \int_{\mathbf{R}^3} d^3x\, \pi\dot\phi - \int_{\mathbf{R}^3} d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + V(\phi) \right), \qquad (4.51)$$

so that the Hamiltonian is given by the familiar Legendre transformation:

$$H = \int_{\mathbf{R}^3} d^3x\, \pi\dot\phi - L = \int_{\mathbf{R}^3} d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + V(\phi) \right). \qquad (4.52)$$

4.3   Poisson brackets

Associated to any Hamiltonian system there is a fundamental algebraic structure known as the *Poisson bracket*. This bracket is a useful way to store information about the Hamiltonian system, and its algebraic properties correspond to those of the corresponding operator algebras in quantum theory. Here I will introduce the Poisson bracket for the scalar field theory primarily so we can use it in the next section to give an elegant formulation of the connection between symmetries and conservation laws. Let me begin with a brief summary of the Poisson bracket in mechanics.

Recall that in classical mechanics the Poisson bracket associates to any 2 functions on phase space, say A and B, a third function, denoted $[A, B]$ via:

$$[A, B] = \Omega^{ij} \partial_i A \, \partial_j B. \tag{4.53}$$

In terms of canonical coordinates and momenta (q^a, p_a), in which

$$\Omega = dp_a \wedge dq^a, \quad \Omega^{-1} = \frac{\partial}{\partial q^a} \wedge \frac{\partial}{\partial p_a}, \tag{4.54}$$

we have

$$[A, B] = \frac{\partial A}{\partial q^a} \frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a} \frac{\partial B}{\partial q^a}. \tag{4.55}$$

In particular, we have the *canonical Poisson bracket relations*,

$$[q^a, p_b] = \delta^a_b. \tag{4.56}$$

Notice that the Poisson bracket is antisymmetric,

$$[A, B] = -[B, A]; \tag{4.57}$$

it is bilinear,

$$[A, cB] = c[A, B], \quad [A, B + C] = [A, B] + [A, C], \quad c = const.; \tag{4.58}$$

and satisfies the Jacobi identity,

$$[[A, B], C], +[[B, C], A] + [[C, A], B] = 0. \tag{4.59}$$

This endows the vector space of functions on phase space with the structure of a Lie algebra called the *Poisson algebra* of functions. Finally, the Hamilton equations can be written[5]

$$\dot{q}^a = [q^a, H], \quad \dot{p}_a = [p_a, H]. \tag{4.60}$$

Returning to field theory, using the parametrization of the scalar field phase space in terms of initial data (ϕ, π), it is not too hard to see how (4.55) generalizes to functions on the scalar field phase space. Let $A = A[\phi, \pi]$ and $B = B[\phi, \pi]$ be differentiable functionals on phase space. The Poisson bracket is defined by

$$[A, B] = \int_{\mathbf{R}^3} d^3x \left(\frac{\delta A}{\delta \phi(\vec{x})} \frac{\delta B}{\delta \pi(\vec{x})} - \frac{\delta A}{\delta \pi(\vec{x})} \frac{\delta B}{\delta \phi(\vec{x})} \right). \tag{4.61}$$

Two important applications of this formula are the canonical Poisson bracket relations,

$$[\phi(\vec{x}), \pi(\vec{y})] = \delta(\vec{x} - \vec{y}), \tag{4.62}$$

and the Hamilton equations:

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = [\phi(\vec{x}), H], \quad \frac{\partial \pi(\vec{x}, t)}{\partial t} = [\pi(\vec{x}), H]. \tag{4.63}$$

PROBLEM: Verify (4.62) follows from (4.61) and that (4.63) agrees with (4.41), (4.42).

4.4 Symmetries and conservation laws

One of the most beautiful aspects of the Hamiltonian formalism is the way it implements the relation between symmetries and conservation laws. To see how this works, let us start by reviewing the relevant results from classical mechanics. Recall that the time evolution of any function C on phase space is given in terms of the Poisson bracket by:

$$\dot{C} = [C, H]. \tag{4.64}$$

---

[5]In quantum theory, the canonical Poisson bracket relations correspond to the canonical commutation relations, and the Hamilton equations correspond to the Heisenberg equations.

More generally, given any function G on phase space there is defined a 1-parameter family of canonical transformations whose infinitesimal form is given by

$$\delta q^a = [q^a, G], \quad \delta p_a = [p_a, G]. \tag{4.65}$$

G is called the *infinitesimal generator* of the transformation. Given the transformation generated by G, the infinitesimal change in any function C on phase space is given by

$$\delta C = [C, G]. \tag{4.66}$$

Given a Hamiltonian system with Hamiltonian function H, we say that G defines a *symmetry* of the Hamiltonian system if the canonical transformation it generates preserves H, that is,

$$\delta H = [H, G] = 0. \tag{4.67}$$

Since $[H, G] = -[G, H]$, we see that if G defines a symmetry, then

$$\dot{G} = [G, H] = -[H, G] = 0, \tag{4.68}$$

so that G is also a conserved quantity. Thus we have the ultimate statement of the connection between symmetries and conservation laws: a function on phase space generates a symmetry if and only if it is conserved.

Let us see how some of these results look in the case of a scalar field by way of a couple of examples. The idea will be to show that symmetry generators $G[\phi, \pi]$ are also constants of the motion. The key will be to compute the Poisson bracket $[G, H] = 0$. Of course, if we pick $G = H$ then this condition is satisfied just by the anti-symmetry of the Poisson bracket. Thus the total energy is conserved by virtue of time translation symmetry. A less trivial example is provided by the total momentum of the field. Recall that the component of the field momentum along a fixed direction $\vec{v}$ (with constant Cartesian components) is given by

$$P(\vec{v}) = \int_{\mathbf{R}^3} d^3x \, \varphi_{,t}\varphi_{,i}v^i, \tag{4.69}$$

The field momentum is a constant of the motion for solutions of the scalar field equation (4.30). We showed this earlier in the special case of the Klein-Gordon field ($V(\varphi) = 0$), but it is not hard to see that this result generalizes to the case of equation (4.30), as it must because the Lagrangian density still has the spatial translation symmetry.

PROBLEM: Show that $P(\vec{v})$ is a constant of the motion for solutions of the scalar field equation (4.30).

In terms of our parametrization of the phase space using initial data, the field momentum along $\vec{v}$ is given by

$$P(\vec{v}) = \int_{\mathbf{R}^3} d^3x \, \pi \phi_{,i} v^i, \tag{4.70}$$

Let us compute the Poisson bracket of the field momentum with the Hamiltonian. There are a few ways to organize this computation, but let us emphasize the role of P as an infinitesimal generator of translations. Our strategy is to use (4.45) in conjunction with the infinitesimal change in (ϕ, π) under the canonical transformation generated by $P(\vec{v})$. We have (try it!)

$$\delta\phi(\vec{x}) \equiv [\phi(\vec{x}), P(\vec{v})] = v^i \phi_{,i} \equiv \vec{v} \cdot \nabla\phi, \tag{4.71}$$

and

$$\delta\pi(\vec{x}) \equiv [\pi(\vec{x}), P(\vec{v})] = v^i \pi_{,i} \equiv \vec{v} \cdot \nabla\pi. \tag{4.72}$$

This is indeed what one expects to be the change in a function under an infinitesimal translation along $\vec{v}$. The change in the KG Hamiltonian (4.44) for any changes in the fields is given by (4.45). Using (4.71) and (4.72) we get

$$\delta H = \int_{\mathbf{R}^3} d^3x \left(\pi \, \vec{v} \cdot \nabla\pi + \nabla\phi \cdot \nabla(\vec{v} \cdot \nabla\phi) + m^2 \phi \, \vec{v} \cdot \nabla\phi + V' \vec{v} \cdot \nabla\phi \right)$$

$$= \int_{\mathbf{R}^3} d^3x \, \nabla \cdot \left[\vec{v} \left(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 + V(\phi) \right) \right]. \tag{4.73}$$

Next, we use the divergence theorem to convert this integral to a surface integral "at infinity". The asymptotic decay of the fields needed to make the energy finite in the first place then implies that the fields decay fast enough such that this integral vanishes. Thus the Hamiltonian is invariant under spatial translations. As already noted, this is equivalent to the fact that the momentum $P(\vec{v})$ is conserved. Of course, we already knew this from our discussion of Noether's theorem.

Here is an example for you to try.

PROBLEM: Consider the massless, free scalar field. Show that

$$Q = \int_{\mathbf{R}^3} d^3x \, \pi(\vec{x}) \tag{4.74}$$

has vanishing Poisson bracket with the Hamiltonian so that Q is conserved. What is the symmetry generated by Q? (For simplicity, you can assume that the fields (ϕ, π) vanish outside of a compact set.)

4.5   PROBLEMS

1. Derive (4.10) from (4.7) and (4.9).

2. Given $L = L(q, \dot{q}, t)$, give a formula for the EL equations linearized about a solution $q^a = q^a(t)$.

3. Show that the symplectic form given in (4.17) does not depend upon time. (*Hint:* Differentiate (4.17) with respect to time and then use the EL equations and their linearization.)

4. A harmonic oscillator is defined by the Lagrangian and EL equations

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2, \quad \ddot{q}(t) = -\omega^2 q(t). \tag{4.75}$$

What does the symplectic structure look like when the space of solutions to the EL equations is parametrized according to the following formulas?

- $q(t) = A\cos(\omega t) + B\sin(\omega t)$

- $q(t) = ae^{-i\omega t} + a^* e^{i\omega t}$?

5. Show that the symplectic structure (4.40) is non-degenerate. (*Hint:* A 2-form Ω is non-degenerate if and only if $\Omega(u, v) = 0$, $\forall v$ implies $u = 0$.)

6. Express the symplectic structure on the space of solutions of the KG equation, $(\Box - m^2)\varphi = 0$ in terms of the Fourier parametrization of solutions which we found earlier:

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} d^3k \left(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_k t} + a_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega_k t}\right).$$

7. Verify (4.62) follows from (4.61) and that (4.63) agrees with (4.41), (4.42).

8. Show that $P(\vec{v})$ in (4.69) is a constant of the motion for solutions of the scalar field equation (4.30).

9. Consider the massless, free scalar field. Show that

$$Q = \int_{\mathbf{R}^3} d^3x\, \pi(\vec{x}) \tag{4.76}$$

has vanishing Poisson bracket with the Hamiltonian so that Q is conserved. What is the symmetry generated by Q? (For simplicity, you can assume that the fields (ϕ, π) vanish outside of a compact set.)

Chapter 5

Electromagnetic field theory

Presumably you've had a course or two treating electrodynamics. Our focus here will be on the salient features of electrodynamics from a field-theoretic perspective. In particular, we will be exploring what it means for electrodynamics to be a "gauge theory". We shall see that certain structural features familiar from KG theory appear also for electromagnetic theory and that new structural features appear as well.

5.1   Review of Maxwell's equations

We begin with a quick review of Maxwell's equations. Hopefully you've seen some of this before.

PROBLEM: In units where $c = 1$, Maxwell's equations for the electric and magnetic field $(\vec{E}, \vec{B})$ associated to charge density and current density $(\rho, \vec{j})$ are given by

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{5.1}$$

$$\nabla \cdot \vec{B} = 0, \tag{5.2}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\sigma, \tag{5.3}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{5.4}$$

Show that for any function $\phi(\vec{x}, t)$ and vector field $\vec{A}(\vec{x}, t)$ the electric and magnetic fields defined by

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A} \tag{5.5}$$

satisfy (5.2) and (5.4).

PROBLEM: Define the anti-symmetric array $F_{\mu\nu}$ in inertial Cartesian coordinates $x^\alpha = (t, x, y, z), \alpha = 0, 1, 2, 3$ via

$$F_{0i} = -E_i, \quad F_{ij} = \epsilon_{ijk}B^k, \quad i, j = 1, 2, 3. \tag{5.6}$$

Under a change of inertial reference frame corresponding to a boost along the x axis with speed v,

$$t' = \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}, \tag{5.7}$$

the electric and magnetic fields change $(\vec{E}, \vec{B}) \to (\vec{E}', \vec{B}')$, where

$$E^{x\prime} = E^x, \quad E^{y\prime} = \gamma(E^y - vB^z), \quad E^{z\prime} = \gamma(E^z + vB^y) \tag{5.8}$$

$$B^{x\prime} = B^x, \quad B^{y\prime} = \gamma(B^y + vE^z), \quad B^{z\prime} = \gamma(B^z - vE^y). \tag{5.9}$$

Show that this is equivalent to saying that $F_{\mu\nu}$ are the components of a spacetime tensor of type $\binom{0}{2}$. Show that the two quantities $\vec{E}\cdot\vec{B}$ and $E^2 - B^2$ do not change under the boost.

PROBLEM: Define the 4-current

$$j^\alpha = (\rho, j^i), \quad i = 1, 2, 3. \tag{5.10}$$

Show that the Maxwell equations take the form

$$F^{\alpha\beta}{}_{,\beta} = 4\pi j^\alpha, \quad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \tag{5.11}$$

where indices are raised and lowered with the usual Minkowski metric.

PROBLEM: Show that the scalar and vector potentials, when assembled into the 4-potential

$$A_\mu = (-\phi, A_i), \quad i = 1, 2, 3, \tag{5.12}$$

are related to the electromagnetic tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.13}$$

Show that this formula for $F_{\mu\nu}$ solves the homogeneous Maxwell equations $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$.

5.2   Electromagnetic Lagrangian

While the electromagnetic field can be described in terms of the field tensor F in Maxwell's equations, if we wish to use a variational principle to describe this field theory we will have to use potentials.[1] So, we will describe electromagnetic theory using the scalar and vector potentials, which can be viewed as a spacetime *1-form*

$$A = A_\alpha(x)dx^\alpha. \tag{5.14}$$

Depending upon your tastes, you can think of this 1-form as (1) a cross-section of the cotangent bundle of the spacetime manifold M; (2) tensor field of type $\binom{0}{1}$; (3) a connection on a $U(1)$ fiber bundle; (4) a collection of 4 functions, $A_\alpha(x)$ defined in a given coordinate system x^α and such that in any other coordinate system $x^{\alpha'}$

$$A'_\alpha(x') = \frac{\partial x^\beta}{\partial x^{\alpha'}} A_\beta(x(x')). \tag{5.15}$$

In any case, A is called the "Maxwell field", the "electromagnetic field", the "electromagnetic potential", the "gauge field", the "4-vector potential", the "U(1) connection", and some other names as well, along with various mixtures of these.

As always, having specified the geometric nature of the field, the field theory can be defined by choosing a Lagrangian. To define the Lagrangian we introduce the *field strength tensor F*, also known as the "Faraday tensor", or as the "curvature" of the gauge field A. We write

$$F = F_{\alpha\beta}(x)dx^\alpha \otimes dx^\beta, \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{5.16}$$

The field strength is in fact a two-form (an anti-symmetric $\binom{0}{2}$ tensor field):

$$F_{\alpha\beta} = -F_{\beta\alpha}, \tag{5.17}$$

and we can write

$$F = \frac{1}{2}F_{\alpha\beta}(dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha) = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta, \tag{5.18}$$

---

[1]It can be shown using techniques from the *inverse problem of the calculus of variations* that there is no variational principle for Maxwell's equations built solely from $(\vec{E}, \vec{B})$ (equivalently from $F_{\alpha\beta}$) and their derivatives.

where the anti-symmetric tensor product is known as the "wedge product", denoted with $\wedge$. In terms of differential forms, the field strength is the exterior derivative of the Maxwell field:

$$F = dA. \tag{5.19}$$

This guarantees that $dF = 0$, which is equivalent to the homogeneous Maxwell equations $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$. So, the use of potentials solves half the Maxwell equations; the only remaining Maxwell equations to be considered are $F^{\alpha\beta}{}_{,\beta} = 4\pi j^\alpha$.

The 6 independent components of F in an inertial Cartesian coordinate chart (t, x, y, z) define the electric and magnetic fields as perceived in that reference frame. Note, however, that all of the definitions given above are in fact valid on an arbitrary spacetime manifold in an arbitrary system of coordinates.

The Lagrangian for electromagnetic theory – on an arbitrary spacetime (M, g) – can be defined by the n-form (where $n = dim(M)$),

$$\mathbf{L} = -\frac{1}{4} F \wedge *F = \mathcal{L}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \tag{5.20}$$

where $*F$ is the Hodge dual defined by the spacetime metric g. In terms of components in a coordinate chart we have

$$(*F)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}, \tag{5.21}$$

and the Lagrangian density given by

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}F^{\alpha\beta}F_{\alpha\beta}, \tag{5.22}$$

where

$$F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}. \tag{5.23}$$

Of course, we can – and usually will – restrict attention to the flat spacetime in the standard Cartesian coordinates for explicit computations. It is always understood that F is built from A in what follows.

Let us compute the Euler-Lagrange derivative of $\mathcal{L}$. For simplicity we will work on flat spacetime in inertial Cartesian coordinates so that

$$M = \mathbf{R}^4, \quad g = g_{\alpha\beta}dx^\alpha \otimes dx^\beta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{5.24}$$

We have

$$
\begin{aligned}
\delta \mathcal{L} &= -\frac{1}{2} F^{\alpha\beta} \delta F_{\alpha\beta} \\
&= -\frac{1}{2} F^{\alpha\beta} (\delta A_{\beta,\alpha} - \delta A_{\alpha,\beta}) \\
&= -F^{\alpha\beta} \delta A_{\beta,\alpha} \\
&= F^{\alpha\beta}{}_{,\alpha} \delta A_\beta + D_\alpha \left(-F^{\alpha\beta} \delta A_\beta \right).
\end{aligned} \tag{5.25}
$$

From this identity the Euler-Lagrange expression is given by

$$
\mathcal{E}^\beta(\mathcal{L}) = F^{\alpha\beta}{}_{,\alpha}, \tag{5.26}
$$

and the *source-free Maxwell equations are*

$$
F^{\alpha\beta}{}_{,\alpha} = 0. \tag{5.27}
$$

There are some equivalent expressions of the field equations that are worth knowing about. First of all, we have that

$$
F^{\alpha\beta}{}_{,\alpha} = 0 \iff g^{\alpha\gamma} F_{\alpha\beta,\gamma} \equiv F_{\alpha\beta,}{}^\alpha = 0, \tag{5.28}
$$

so that the field equations can be expressed as

$$
g^{\alpha\gamma}(A_{\beta,\alpha\gamma} - A_{\alpha,\beta\gamma}) = 0. \tag{5.29}
$$

We write this using the wave operator $\square$ (which acts component-wise on the 1-form A) and the operator

$$
\operatorname{div} A = g^{\alpha\beta} A_{\alpha,\beta} = A_{\alpha,}{}^\alpha = A^\alpha{}_{,\alpha}. \tag{5.30}
$$

via

$$
\square A_\beta - (\operatorname{div} A)_{,\beta} = 0. \tag{5.31}
$$

You can see that this is a modified wave equation.

A more sophisticated expression of the field equations, which is manifestly valid on any spacetime, uses the technology of differential forms. Recall that on a spacetime one has the *Hodge dual*, which identifies the space of p-forms with the space of $n - p$ forms. This mapping is denoted by

$$
\alpha \to *\alpha. \tag{5.32}
$$

If F is a 2-form, then $*F$ is an $(n-2)$-form. The components are related by (5.21). The source-free field equations (5.27) are equivalent to the vanishing of a 1-form:

$$*d*F = 0, \tag{5.33}$$

where d is the exterior derivative. This equation is valid on any spacetime (M, g) and is equivalent to the EL equations for the Maxwell Lagrangian as defined above on any spacetime.

Let me show you how this formula works in our flat 4-d spacetime. The components of $*F$ are given by

$$*F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}. \tag{5.34}$$

The exterior derivative maps the 2-forms $*F$ to a 3-form $d*F$ via

$$(d*F)_{\alpha\beta\gamma} = 3\partial_{[\alpha}*F_{\beta\gamma]} = \frac{3}{2}\partial_{[\alpha}\epsilon_{\beta\gamma]\mu\nu}F^{\mu\nu}. \tag{5.35}$$

The Hodge dual maps the 3-form $d*F$ to a 1-form $*d*F$ via

$$\begin{aligned}
(*d*F)\sigma &= \frac{1}{6}\epsilon_{\sigma\alpha\beta\gamma}(d*F)^{\alpha\beta\gamma} \\
&= \frac{1}{4}\epsilon_{\sigma\alpha\beta\gamma}\epsilon^{\beta\gamma\mu\nu}\partial^{\alpha}F_{\mu\nu} \\
&= -\delta_{\sigma}^{[\mu}\delta_{\alpha}^{\nu]}\partial^{\alpha}F_{\mu\nu} \\
&= F_{\alpha\sigma},^{\alpha}.
\end{aligned} \tag{5.36}$$

So that

$$*d*F = 0 \iff F_{\alpha\beta},^{\alpha} = 0. \tag{5.37}$$

The following problems establish some key structural features of electromagnetic theory.

PROBLEM: Show that the EL derivative of the Maxwell Lagrangian satisfies the differential identity

$$D_{\beta}\mathcal{E}^{\beta}(\mathcal{L}) = 0. \tag{5.38}$$

PROBLEM: Restrict attention to flat spacetime in Cartesian coordinates, as usual. Fix a vector field on spacetime, $j^{\alpha} = j^{\alpha}(x)$. (1) Show that the Lagrangian

$$\mathcal{L}_j = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + 4\pi j^{\alpha}A_{\alpha} \tag{5.39}$$

gives the field equations

$$F^{\alpha\beta}{}_{,\alpha} = -4\pi j^{\beta}. \tag{5.40}$$

These are the Maxwell equations with prescribed *electric sources* – a charge density ρ and current density $\vec{j}$, where

$$j^{\alpha} = (\rho, \vec{j}). \tag{5.41}$$

(2) Use the results from the preceding problem to show that the Maxwell equations with sources have no solution unless the vector field representing the sources is divergence-free:

$$\partial_{\alpha} j^{\alpha} = 0. \tag{5.42}$$

(3) Show that this condition is in fact the usual continuity equation representing conservation of electric charge.

PROBLEM: Show that the Lagrangian density for source-free electromagnetism can be written in terms of the electric and magnetic fields (in any given inertial frame) by $\mathcal{L} = \frac{1}{2}(E^2 - B^2)$. This is one of the 2 relativistic invariants that can be made algebraically from $\vec{E}$ and $\vec{B}$.

PROBLEM: Show that $\vec{E} \cdot \vec{B}$ is relativistically invariant (unchanged by a Lorentz transformation). Express it in terms of potentials and show that it is just a divergence, with vanishing Euler-Lagrange expression.

5.3 Gauge symmetry

Probably the most significant new aspect of electromagnetic theory, field theoretically speaking, is that it admits an infinite-dimensional group of variational symmetries known as *gauge symmetries*. Their appearance stems from the fact that the electromagnetic Lagrangian only depends upon the vector potential through the field strength tensor via

$$F = dA, \tag{5.43}$$

so that if we redefine the 4-vector potential by a *gauge transformation*

$$A \to A' = A + d\Lambda, \tag{5.44}$$

where $\Lambda\colon M \to \mathbf{R}$, then

$$F' = dA' = d(A + d\Lambda) = dA = F, \tag{5.45}$$

where we used the fact that, on any differential form, $d^2 = 0$. It is easy to check all this explicitly in terms of components.

PROBLEM: For any function $\Lambda = \Lambda(x)$, define

$$A'_\alpha = A_\alpha + \partial_\alpha \Lambda. \tag{5.46}$$

Show that

$$A'_{\alpha,\beta} - A'_{\beta,\alpha} = A_{\alpha,\beta} - A_{\beta,\alpha}. \tag{5.47}$$

Show that, in terms of the scalar and vector potentials, this gauge transformation is equivalent to

$$\phi \to \phi' = \phi - \partial_t \Lambda, \quad \vec{A} \to \vec{A}' = \vec{A} + \nabla \Lambda. \tag{5.48}$$

This shows, then, that under the transformation

$$A \to A' \tag{5.49}$$

we have

$$F \to F. \tag{5.50}$$

Evidently, the Lagrangian – which contains A only through F – is invariant under this transformation of A. We say that the Lagrangian is *gauge invariant.*

The gauge transformations constitute a very large set of variational symmetries. Up to boundary conditions, one can use *any* function Λ to define a new set of potentials. Mathematically, the gauge transformations form an infinite-dimensional Abelian group.

Insofar as classical electrodynamics can be formulated in terms of the field strength tensor, the gauge transformation symmetry has no physical content in the sense that one always identifies physical situations described by gauge-equivalent Maxwell fields. Thus the Maxwell fields A provide a redundant description of the physics. On the other hand, while the potential does not have direct classical physical significance, it does have a physical role to play: the need to use the potential A can be understood from the desire to have a

local variational principle – which is crucial for quantum theory. Indeed, in the quantum context the potential plays a more important role.

The gauge symmetry is responsible for the fact that the Maxwell equations for the potential

$$\Box A - d(div\, A) = -4\pi j \tag{5.51}$$

are not hyperbolic. (For comparison, the KG equation *is* hyperbolic.) Indeed, hyperbolic equations will have a Cauchy problem with unique solutions for given initial data. It is clear that, because the function Λ is arbitrary, one can never have unique solutions to the field equations for A associated to given Cauchy data. To see this, let A be any solution for prescribed Cauchy data on a hypersurface $t = const$. Let A' be any other solution obtained by a gauge transformation:

$$A' = A + d\Lambda. \tag{5.52}$$

It is easy to see that A' also solves the field equations. This follows from a number of points of view. For example, the field equations are conditions on the field strength F, which is invariant under the gauge transformation. Alternatively, the field equations are invariant under the field equations because the Lagrangian is. Finally, you can check directly that $d\Lambda$ solves the source-free field equations:

$$\left[(\Box - d\,div)d\Lambda\right]_\alpha = \partial^\beta \partial_\beta (\partial_\alpha \Lambda) - \partial_\alpha (\partial^\beta \partial_\beta \Lambda) = 0. \tag{5.53}$$

Since Λ is an arbitrary smooth function, we can choose the first two derivatives of Λ to vanish on the initial hypersurface so that A' and A are distinct solutions with the same initial data.

To uniquely determine the potential A from Cauchy data and the Maxwell equations one has to add additional conditions on the potential beyond the field equations. This is possible since one can adjust the form of A via gauge transformations. It is not too hard to show that one can gauge transform any given potential into one which satisfies the *Lorenz gauge* condition:

$$\partial^\alpha A_\alpha = 0. \tag{5.54}$$

To see this, take any potential, say, $\tilde{A}$ and gauge transform it to a potential $A = \tilde{A} + d\Lambda$ such that the Lorenz gauge holds; this means

$$\partial^\alpha \partial_\alpha \Lambda = -\partial^\alpha \tilde{A}_\alpha. \tag{5.55}$$

Viewing the right-hand side of this equation as given, we see that to find such a gauge transformation amounts to solving the wave equation with a given source, which can always be done.

In the Lorenz gauge the Maxwell equations are just the usual, hyperbolic wave equation for each inertial-Cartesian component of the 4-vector potential,

$$\Box A_\alpha = -4\pi j_\alpha. \tag{5.56}$$

It is therefore tempting to suppose that one can thus identify the EM field theory with 4 copies of a massless KG field theory. Things are a little more interesting than that: the 4-potential still must satisfy (5.54), and even in the Lorenz gauge the potentials are not uniquely determined! Pick any solution Λ of the wave equation $\Box\Lambda = 0$ and use Λ to make a gauge transformation. You can easily check that the transformed potentials still satisfy the condition (5.54). It is possible to show that the Lorenz condition along with this *residual gauge freedom* ultimately permits the elimination of two functions worth of information from A.[2] One says that the electromagnetic field has "two degrees of freedom per spatial point"; this is intimately related to the two helicity states of photons in the associated quantum theory.

5.4   Noether's second theorem in electromagnetic theory

We have seen that a variational (or divergence) symmetry leads to a conserved current. The gauge transformation defines a variational symmetry for electromagnetic theory. Actually, there are *many* gauge symmetries: because each function on spacetime (modulo an additive constant) defines a gauge transformation the set of gauge transformations is infinite dimensional! Let us consider our Noether type of analysis for these symmetries. We will see that the analysis that led to Noether's (first) theorem can be taken a little further when the symmetry involves arbitrary functions.

Consider a 1-parameter family of gauge transformations:

$$A' = A + d\Lambda_s, \tag{5.57}$$

characterized by a 1-parameter family of functions Λ_s where

$$\Lambda_0 = 0. \tag{5.58}$$

---

[2]In detail, one can choose Λ such that $A_0 = 0$ and, therefore, so that $\nabla \cdot \vec{A} = 0$.

Infinitesimally, we have[3]

$$\delta A = d\sigma, \tag{5.59}$$

where

$$\sigma = \left(\frac{\partial \Lambda_s}{\partial s}\right)_{s=0}. \tag{5.60}$$

It is easy to see that the function σ can be chosen arbitrarily just as we had for field variations in the usual calculus of variations analysis. The Lagrangian is invariant under the gauge transformation; therefore it is invariant under its infinitesimal version. Let us check this explicitly. For any variation we have

$$\delta \mathcal{L} = -\frac{1}{2}F^{\alpha\beta}\delta F_{\alpha\beta}, \tag{5.61}$$

and under a variation defined by an infinitesimal gauge transformation

$$\begin{aligned}
\delta F_{\alpha\beta} &= \partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha \\
&= \partial_\alpha(\partial_\beta \sigma) - \partial_\beta(\partial_\alpha \sigma) \\
&= 0,
\end{aligned} \tag{5.62}$$

so that $\delta \mathcal{L} = 0$.

For any variation, the first variational identity is

$$\delta \mathcal{L} = \mathcal{E}^\beta \delta A_\beta + D_\alpha\left(-F^{\alpha\beta}\delta A_\beta\right), \tag{5.63}$$

where

$$\mathcal{E}^\beta = F^{\alpha\beta}{}_{,\alpha}. \tag{5.64}$$

For a variation induced by an infinitesimal gauge transformation we know that $\delta \mathcal{L} = 0$, so the variational identity tells us that

$$0 = \mathcal{E}^\beta \partial_\beta \sigma + D_\alpha\left(-F^{\alpha\beta}\partial_\beta \sigma\right), \tag{5.65}$$

which is valid for any function σ. Now we take account of the fact that the function σ is arbitrary. We rearrange the derivatives of σ to get them inside a divergence:

$$0 = -D_\beta \mathcal{E}^\beta \sigma + D_\alpha\left(-F^{\alpha\beta}\partial_\beta \sigma + F^{\alpha\beta}{}_{,\beta}\,\sigma\right) \tag{5.66}$$

---

[3]Notice that the infinitesimal gauge transformation has the same form as a finite gauge transformation. This is due to the fact that the gauge transformation is an affine (as opposed to non-linear) transformation.

Restrict this equation to a potential $A = A(x)$, then integrate the result over a spacetime region $\mathcal{R}$:

$$0 = -\int_{\mathcal{R}} F^{\alpha\beta}{}_{,\alpha\beta}\, \sigma + \int_{\partial\mathcal{R}} \left(-F^{\alpha\beta}\partial_\beta\sigma + F^{\alpha\beta}{}_{,\beta}\,\sigma\right)\, d\Sigma_\alpha. \qquad (5.67)$$

This must hold for any function σ; we can use the fundamental theorem of variational calculus to conclude that the Euler-Lagrange equations must satisfy the differential identity

$$D_\beta \mathcal{E}^\beta = 0, \qquad (5.68)$$

which you proved directly in a previous homework problem. Note that this says the Euler-Lagrange expression is divergence-free, and that this holds whether or not the field equations are satisfied – it is an *identity* arising due to the gauge symmetry of the Lagrangian.

Compare our results above to Noether's first theorem. We have seen that the gauge symmetry – being a continuous variational symmetry – leads to a divergence-free vector field, as it must by Noether's first theorem. But we now have a new ingredient: the gauge symmetry is built from an arbitrary function of all the independent variables x^α so that the gauge transformation can be localized to an arbitrary location in spacetime. This leads to the vector field being divergence-free *identically*, independent of the field equations. Indeed, the divergence relation is an identity satisfied *by* the field equations. All this is an example of *Noether's second theorem*, and the resulting identity is sometimes called the "Noether identity" associated to the gauge symmetry.

PROBLEM: Consider the electromagnetic field coupled to sources with the Lagrangian density

$$\mathcal{L}_j = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + 4\pi j^\alpha A_\alpha \qquad (5.69)$$

Show that this Lagrangian is gauge invariant (up to a divergence) if and only if the spacetime vector field j^α is chosen to be divergence-free. What is the Noether identity in this case?

5.5   Noether's second theorem

Let us spend a moment having a look at Noether's second theorem from a more general point of view.

Consider a system of fields φ^a, $a = 1, 2, \ldots$, on a manifold labeled by coordinates x^α, described by a Lagrangian density $\mathcal{L}$ and field equations $\mathcal{E}_a(\mathcal{L}) = 0$ defined by the Euler-Lagrange identity

$$\delta\mathcal{L} = \mathcal{E}_a(\mathcal{L})\delta\varphi^a + D_\alpha\eta^\alpha, \tag{5.70}$$

where η^α is a linear differential operator acting on $\delta\varphi^a$. As usual, all these quantities are functions on jet space.

Let us define an *infinitesimal gauge transformation* to be an infinitesimal transformation

$$\delta\varphi^a = \delta\varphi^a(\Lambda), \tag{5.71}$$

that is constructed as a linear differential operator $\mathcal{D}$ (constructed locally from the fields and their derivatives) on arbitrary functions $\Lambda^A = \Lambda^A(x)$, $A = 1, 2, \ldots$:

$$\delta\varphi^a(\Lambda) = [\mathcal{D}(\Lambda)]^a. \tag{5.72}$$

The gauge transformation is an *infinitesimal gauge symmetry* if it leaves the Lagrangian invariant up to a divergence of a vector field W^α,

$$\delta\mathcal{L} = D_\alpha W^\alpha(\Lambda), \tag{5.73}$$

where $W^\alpha(\Lambda)$ is a linear differential operator acting on the functions Λ^A, with the linear operator being locally constructed from the fields φ^a and their derivatives.

Noether's second theorem now asserts that the existence of a gauge symmetry implies differential identities satisfied by the field equations. To see this, we simply use the fact that, for any functions Λ^A,

$$0 = \delta\mathcal{L} - D_\alpha W^\alpha = \mathcal{E}_a(\mathcal{L})[\mathcal{D}(\Lambda)]^a + D_\alpha(\eta^\alpha - W^\alpha), \tag{5.74}$$

where both η and W are built as linear differential operators acting Λ^A. As before, we integrate this identity over a region and choose the functions Λ^A to vanish in a neighborhood of the boundary so that the divergence terms can be neglected. We then have that, for all functions Λ^A,

$$\int_{\mathcal{R}} \mathcal{E}_a(\mathcal{L})[\mathcal{D}(\Lambda)]^a = 0. \tag{5.75}$$

Now imagine integrating by parts each term in the linear operator $[\mathcal{D}(\Lambda)]^a$ so that all derivatives of Λ are removed. The boundary terms that arise vanish

with our boundary conditions on Λ^A. This process defines the *formal adjoint* $\mathcal{D}^*$ of the linear differential operator $\mathcal{D}$:

$$\int_{\mathcal{R}} \mathcal{E}_a [\mathcal{D}(\Lambda)]^a = \int_{\mathcal{R}} \Lambda^A [\mathcal{D}^*(\mathcal{E})]_A. \tag{5.76}$$

The gauge symmetry condition is now

$$\int_{\mathcal{R}} \Lambda^A [\mathcal{D}^*(\mathcal{E})]_A = 0, \tag{5.77}$$

for all functions Λ^A (vanishing in the neighborhood of the boundary). The fundamental theorem of variational calculus then tells us that the Euler-Lagrange expressions must obey the differential identities:

$$[\mathcal{D}^*(\mathcal{E})]_A = 0. \tag{5.78}$$

You can easily check out this argument via our Maxwell example. The gauge transformation is defined by the exterior derivative on functions:

$$[\mathcal{D}(\Lambda)]_\alpha = \partial_\alpha \Lambda. \tag{5.79}$$

The infinitesimal transformation

$$\delta A_\alpha = [\mathcal{D}(\Lambda)]_\alpha \tag{5.80}$$

is a symmetry of the Lagrangian with

$$W^\alpha = 0. \tag{5.81}$$

The adjoint of the exterior derivative is given by (minus) the divergence operation:

$$V^\alpha \partial_\alpha \Lambda = -\Lambda \partial_\alpha V^\alpha + \partial_\alpha (\Lambda V^\alpha) \tag{5.82}$$

so that

$$[\mathcal{D}^*(\mathcal{E})] = D_\alpha \mathcal{E}^\alpha, \tag{5.83}$$

which leads to the Noether identity

$$D_\alpha \mathcal{E}^\alpha = 0 \tag{5.84}$$

for any field equations coming from a gauge invariant Lagrangian.

Let me emphasize that the above considerations only work because Λ can be chosen to be any function of all the independent variables x^α. This is needed for various integrations by parts, and it is needed to use the fundamental theorem of the variational calculus. The bottom line here is that you have a gauge symmetry (as opposed to an ordinary symmetry) if and only if the *support in spacetime* of the transformation can be freely specified.

5.6 The canonical energy-momentum tensor

In §3.12 we saw how the translational symmetries of Minkowski spacetime led to the construction of the energy-momentum tensor for the KG field. We now apply similar reasoning to the electromagnetic field.

Consider a 1-parameter family of translations, say,

$$x^\alpha \to \overline{x}^\alpha = x^\alpha + a^\alpha, \quad \text{where} \quad a^\alpha = \lambda b^\alpha. \tag{5.85}$$

The corresponding ("pullback") transformation of the vector potential is given by

$$A_\alpha \to \overline{A}_\alpha = \frac{\partial x^\beta}{\partial x^\alpha} A_\beta(x + \lambda b) = A_\alpha(x + \lambda b). \tag{5.86}$$

We have then[4]

$$\delta A_\alpha = b^\beta A_{\alpha,\beta}. \tag{5.87}$$

This implies that

$$\delta F_{\mu\nu} = b^\alpha F_{\mu\nu,\alpha} \tag{5.88}$$

and hence the translations define a divergence symmetry:

$$\delta \mathcal{L} = -\frac{1}{2} b^\gamma F^{\alpha\beta} F_{\alpha\beta,\gamma} = D_\gamma \left(-\frac{1}{4} b^\gamma F^{\alpha\beta} F_{\alpha\beta} \right). \tag{5.89}$$

Recalling the variational identity:

$$\delta \mathcal{L} = F^{\alpha\beta}{}_{,\alpha} \delta A_\beta + D_\alpha \left(-F^{\alpha\beta} \delta A_\beta \right), \tag{5.90}$$

this leads to the conserved current

$$j^\alpha = -b^\gamma \left(F^{\alpha\beta} A_{\beta,\gamma} - \frac{1}{4} \delta^\alpha_\gamma F^{\mu\nu} F_{\mu\nu} \right). \tag{5.91}$$

You can easily check with a direct computation that j^α is conserved, that is,

$$D_\alpha j^\alpha = 0, \tag{5.92}$$

when the field equations hold.

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[4]It is worth noting that this formula is not gauge-invariant; it really only defines the change in the fields due to a translation modulo a gauge transformation. We will address this issue soon.

PROBLEM: Verify equations (5.87)–(5.92).

Since this conservation law exists for each constant vector b^α, we can summarize these conservation laws using the *canonical energy-momentum tensor*

$$\mathcal{T}_\gamma^\alpha = \left(F^{\alpha\beta} A_{\beta,\gamma} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu} \right), \tag{5.93}$$

which satisfies

$$D_\alpha \mathcal{T}_\beta^\alpha = 0, \tag{5.94}$$

modulo the field equations. We view the energy-momentum tensor as a collection of conserved currents labeled by the index γ.

5.7 Improved Maxwell energy-momentum tensor

There is one glaring defect in the structure of the canonical energy-momentum tensor: it is not gauge invariant. Indeed, under a gauge transformation

$$A \longrightarrow A + d\Lambda \tag{5.95}$$

we have

$$\mathcal{T}_\beta^\alpha \longrightarrow \mathcal{T}_\beta^\alpha + F^{\alpha\mu} \partial_\mu \partial_\beta \Lambda. \tag{5.96}$$

In order to see what to do about this, we need to use some of the flexibility we have in defining conserved currents. This is our next task.

It is possible to show that all local and gauge invariant expressions must depend on the vector potential only through the field strength. Consequently, the currents are not gauge invariant because of the explicit presence of the potentials A. With that in mind we write

$$\mathcal{T}_\gamma^\alpha = \left(F^{\alpha\beta} F_{\gamma\beta} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu} \right) - F^{\alpha\beta} A_{\gamma,\beta}$$

$$= \left(F^{\alpha\beta} F_{\gamma\beta} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu} \right) - D_\beta (F^{\alpha\beta} A_\gamma) + A_\gamma D_\beta F^{\alpha\beta}. \tag{5.97}$$

According to §3.20, the last two terms are trivial conservation laws. So, modulo a set of trivial conservation laws, the canonical energy-momentum tensor takes the gauge-invariant form

$$T_\gamma^\alpha = F^{\alpha\beta} F_{\gamma\beta} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu}. \tag{5.98}$$

This tensor is called the "gauge-invariant energy-momentum tensor" or the "improved energy-momentum tensor" or the "general relativistic energy-momentum tensor". The latter term arises since this energy-momentum tensor serves as the source of the gravitational field in general relativity and can be derived using the variational principle of that theory.

The improved energy-momentum tensor has another valuable feature relative to the canonical energy-momentum tensor (besides gauge invariance). The canonical energy-momentum tensor,

$$\mathcal{T}_\gamma^\alpha = \left(F^{\alpha\beta} A_{\beta,\gamma} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu} \right), \tag{5.99}$$

is not a symmetric tensor. If we define

$$\mathcal{T}_{\alpha\beta} = g_{\alpha\gamma} \mathcal{T}_\beta^\gamma \tag{5.100}$$

then you can see that

$$\mathcal{T}_{[\alpha\beta]} = F_{[\alpha}{}^\gamma \partial_{\beta]} A_\gamma. \tag{5.101}$$

Here we used the notation

$$\mathcal{T}_{[\alpha\beta]} \equiv \frac{1}{2} (\mathcal{T}_{\alpha\beta} - \mathcal{T}_{\beta\alpha}). \tag{5.102}$$

On the other hand the improved energy-momentum tensor,

$$T_\gamma^\alpha = F^{\alpha\beta} F_{\beta\gamma} - \frac{1}{4} \delta_\gamma^\alpha F^{\mu\nu} F_{\mu\nu}. \tag{5.103}$$

is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}. \tag{5.104}$$

Why is all this important? Well, think back to the KG equation. There, you will recall, the conservation of angular momentum, which stems from the symmetry of the Lagrangian with respect to the Lorentz group, comes from the currents

$$M^{\alpha(\mu)(\nu)} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu} = 2T^{\alpha[\mu} x^{\nu]}. \tag{5.105}$$

These 6 currents were conserved since (1) $T^{\alpha\beta}$ is divergence free (modulo the equations of motion) and (2) $T^{[\alpha\beta]} = 0$. This result will generalize to give conservation of angular momentum in electromagnetic theory using the

improved energy-momentum tensor. So the improved tensor in electromagnetic theory plays the same role relative to angular momentum as does the energy-momentum tensor of KG theory.

Why did we have to "improve" the canonical energy-momentum tensor? Indeed, we have a paradox: the Lagrangian is gauge invariant, so why didn't Noether's theorem automatically give us the gauge invariant energy-momentum tensor? As with most paradoxes, the devil is in the details. Noether's theorem involves using the variational identity in the form

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L}) + D_\alpha\eta^\alpha \tag{5.106}$$

to construct the conserved current from η^α and any divergence $D_\alpha W^\alpha$ which arises in the symmetry transformation of $\mathcal{L}$. To construct the canonical energy-momentum tensor from Noether's first theorem we used

$$W^\alpha = -\frac{1}{4}b^\alpha F_{\mu\nu}F^{\mu\nu}, \tag{5.107}$$

along with

$$\eta^\alpha = -F^{\alpha\beta}\delta A_\beta, \quad \text{and} \quad \delta A_\beta = b^\gamma A_{\beta,\gamma} \implies \eta^\alpha = -b^\gamma F^{\alpha\beta}A_{\beta,\gamma}. \tag{5.108}$$

The lack of gauge invariance snuck into the calculation via the formula for the change in A under an infinitesimal translation, $\delta A_\beta = b^\gamma A_{\beta,\gamma}$. We destroyed gauge invariance with this formula since its right hand side is not gauge invariant. A gauge invariant formula for the change of A under an infinitesimal translation can be gotten by accompanying the translation with a gauge transformation:

$$\delta A_\beta = b^\gamma A_{\beta,\gamma} - D_\beta(b^\gamma A_\gamma) = b^\gamma F_{\gamma\beta}. \tag{5.109}$$

With this improved symmetry transformation we get the improved energy-momentum tensor from Noether's theorem. Notice also that this additional gauge transformation is a symmetry and corresponds to the additional trivial conservation law needed to improve the energy-momentum tensor in the first place.

5.8   The Hamiltonian formulation of Electromagnetism.

The Hamiltonian formulation of the electromagnetic field offers some significant new features beyond what we found for the scalar field. These new

features are associated with gauge invariance. Mathematically speaking, the new features stem from the failure of the non-degeneracy condition (4.14). The novelties which appear here also appear (in a more elaborate form) in Yang-Mills theories and in generally covariant theories, *e.g.,* of gravitation.

5.8.1   Phase space

We start with the source-free EM Lagrangian in Minkowski spacetime using inertial-Cartesian coordinates:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}, \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{5.110}$$

The phase space is the vector space of solutions to the source-free Maxwell equations,

$$\partial_\beta F^{\alpha\beta} = 0, \tag{5.111}$$

equipped with a symplectic structure which is constructed as follows.

The variation of the Lagrangian is given by:

$$\delta L = \int_{\mathbf{R}^3} d^3x \left\{ F^{\alpha\beta}{}_{,\alpha}\delta A_\beta + \partial_\alpha \left(-F^{\alpha\beta}\delta A_\beta \right) \right\} \tag{5.112}$$

$$= \int_{\mathbf{R}^3} d^3x \left\{ F^{\alpha\beta}{}_{,\alpha}\delta A_\beta + \partial_0 \left(-F^{0\beta}\delta A_\beta \right) \right\}, \tag{5.113}$$

where the integral takes place at some chosen value for $x^0 \equiv t$ and we used the spatial divergence theorem along with boundary conditions at spatial infinity to get the second equality. The 1-form Θ which is normally used to construct the symplectic 2-form is then defined by

$$\Theta(\delta A) = \int_{\mathbf{R}^3} d^3x \, F^{\beta 0}\delta A_\beta = \int_{\mathbf{R}^3} d^3x \, F^{i0}\delta A_i, \tag{5.114}$$

where we shall use Latin letters to denote spatial components, $i = 1, 2, 3$. Using this naive definition of the symplectic form $\Omega = d\Theta$, we get

$$\Omega(\delta_1 A, \delta_2 A) = \int_{\mathbf{R}^3} d^3x \left\{ (\partial_t \delta_1 A^i - \partial^i \delta_1 A_t)\delta_2 A_i - (\partial_t \delta_2 A^i - \partial^i \delta_2 A_t)\delta_1 A_i \right\}. \tag{5.115}$$

PROBLEM: Show that the 2-form (5.115) does not depend upon the time at which it is evaluated.

The new feature that appears here is that the 2-form Ω is in fact degenerate. This means that there exists a vector $\vec{v}$ such that $\Omega(\vec{u}, \vec{v}) = 0$ for all $\vec{u}$. Let us see how this happens in detail.

Consider using in (5.115) a field variation $\delta_2 A$ which is an infinitesimal gauge transformation:

$$\delta_2 A_t = \partial_t \Lambda, \quad \delta_2 A_i = \partial_i \Lambda, \tag{5.116}$$

where $\Lambda \colon \mathbf{R}^3 \to \mathbf{R}$ is any function of compact support. Substituting this into (5.115) it is easy to see that the second group of terms vanish by the commutativity of partial derivatives. Let us examine the first group of terms. Upon substitution of (5.116) we get:

$$\int_{\mathbf{R}^3} d^3x \, (\partial_t \delta_1 A^i - \partial^i \delta_1 A_t) \delta_2 A_i = \int_{\mathbf{R}^3} d^3x \, (\partial_t \delta_1 A^i - \partial^i \delta_1 A_t) \partial_i \Lambda$$
$$= -\int_{\mathbf{R}^3} d^3x \, \Lambda \, \partial_i (\partial_t \delta_1 A^i - \partial^i \delta_1 A_t), \tag{5.117}$$

where integration by parts and the divergence theorem were used to get the last equality. The boundary term "at infinity" vanishes since Λ has compact support. Next, recall that the field variations δA represent tangent vectors to the space of solutions of the Maxwell equations (5.111) and so are solutions to the linearized equations. Because the Maxwell equations are linear, their linearization is mathematically the same:

$$\partial^\beta (\partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha) = 0. \tag{5.118}$$

Setting $\alpha = 0$ we get

$$\partial^i (\partial_t \delta A_i - \partial_i \delta A_t) = 0. \tag{5.119}$$

This means that (5.117) vanishes and we conclude that the "pure gauge" tangent vectors (5.116) are degeneracy directions for the 2-form Ω.

Degenerate 2-forms are often called "pre-symplectic" because there is a canonical procedure for extracting a unique symplectic form on a smaller space from a pre-symplectic form. We will not develop this elegant geometric result here. Instead, we will proceed in a useful if more roundabout route by examining the Hamiltonian formulation of the theory that arises when we parametrize the space of solutions to the field equations (5.111) with initial data.

5.8.2 Equations of motion

Returning to (5.114), we see from its "$p\,dq$" form that we can identify the canonical coordinates with the initial data for the spatial part of the vector potential and the conjugate momentum is then given by minus the electric field. We define

$$Q_i(\vec{x}) = A_i(t = 0, \vec{x}), \quad P^i(\vec{x}) = F^{i0}(t = 0, \vec{x}). \tag{5.120}$$

Because of the definition of the field strength tensor in terms of the vector potential, we have

$$P_i = \partial_t A_i - \partial_i A_t, \tag{5.121}$$

so we can write

$$\dot{Q}_i = P_i + \partial_i A_t. \tag{5.122}$$

Next, consider the equations of motion with time and space (as given in an inertial reference frame) explicitly separated:

$$\partial_t F^{\alpha t} + \partial_i F^{\alpha i} = 0, \tag{5.123}$$

which yields four equations:

$$\partial_t F^{jt} + \partial_i F^{ji} = 0 \tag{5.124}$$

$$\partial_i F^{ti} = 0, \tag{5.125}$$

These equations allow us to view the Maxwell equations as determining a curve $(Q_i = Q_i(t, \vec{x}), P^i = P^i(t, \vec{x}))$ in the space of initial data along with a *constraint* on the canonical variables. Equations (5.122) and (5.124) are equivalent to the evolution equations:

$$\dot{Q}_i = \delta_{ij} P^j + \partial_i A_t, \tag{5.126}$$
$$\dot{P}^j = \partial_j F^{ij}, \tag{5.127}$$

which correspond to Ampere's law. Equation (5.125) corresponds to the constraint

$$\partial_j P^j = 0. \tag{5.128}$$

The constraint equation does not specify time evolution but restricts the canonical variables at any given instant of time. You can easily check that this

is the differential version of Gauss' law, in the case of vanishing charge density. One can also interpret the constraint as a restriction on admissible initial conditions for the evolution equations (5.127), since we have the following result.

PROBLEM: Show that if the constraint (5.128) holds at one time and the canonical variables evolve in time according to (5.126), (5.127) then (5.128) will hold at any other time. (*Hint:* Consider the time derivative of (5.128).)

Notice that the equations (5.126), (5.127) are evolution equations for (Q_i, P^i) only. The time component A_t is not determined by these equations, it simply defines a gauge transformation of the Q_i as time evolves.

Equations (5.126), (5.127) and (5.128) determine solutions to the Maxwell equations as follows. Specify 6 functions on $\mathbf{R}^3$, namely $(q_i(\vec{x}), p^i(\vec{x}))$, where p^i is divergence-free, $\partial_i p^i = 0$. Pick a function $A_t(t, \vec{x})$ any way you like. Solve the evolution equations (5.126), (5.127) subject to the initial conditions $(Q_i(0) = q^i, P_i(0) = p_i)$ to get $(Q_i(t), P^i(t))$. In the given inertial reference frame define

$$F_t^i = P^i, \quad F_{ij} = \partial_i Q_j - \partial_j Q_i. \tag{5.129}$$

The resulting field strength tensor $F_{\alpha\beta}$ satisfies the (source-free) Maxwell equations, as you can easily verify.

5.8.3 The electromagnetic Hamiltonian and gauge transformations

Following the same strategy I mentioned when studying the KG field, we can compute the electromagnetic Hamiltonian from the Lagrangian using Legendre transformation. An examination of this Hamiltonian will give us useful new perspectives on the electromagnetic phase space, on the Hamilton equations, and on the role of gauge transformations.

To this end we use an inertial reference frame

$$x^\alpha = (x^0, x^i) = (t, \vec{x}) \tag{5.130}$$

and decompose the electromagnetic Lagrangian density with respect to it.

$$\mathcal{L} = -\frac{1}{2}\left[-F_{0i}F_0{}^i + \frac{1}{2}F_{ij}F^{ij}\right] = \frac{1}{2}\left[(\partial_t A_i - \partial_i A_t)(\partial_t A^i - \partial^i A_t) - \frac{1}{2}F_{ij}F^{ij}\right] \tag{5.131}$$

I will choose $\phi \equiv -A_t(\vec{x})$ and $Q_i \equiv A_i(\vec{x}))$ as the canonical coordinates, the momentum conjugate to Q_i is

$$P^i = \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = (\partial_t A_i - \partial_i A_t) = (\partial_t Q_i + \partial_i \phi), \qquad (5.132)$$

as we found earlier. Notice that there are no time derivatives of A_t in the Lagrangian – its conjugate momentum vanishes:

$$\frac{\partial \mathcal{L}}{\partial(\partial_t A_t)} = 0. \qquad (5.133)$$

Viewing the spacetime fields as $Q_i(t, \vec{x}), P^i(t, \vec{x}), \phi(t, \vec{x})$, the Lagrangian can be viewed as a functional of (Q_i, P^j, ϕ) and can be written in the Hamiltonian "$p\dot{q} - H$" form (exercise):

$$L[Q, P, \phi] = \int_{\mathbf{R}^3} d^3x \left\{ P^i \dot{Q}_i - \frac{1}{2}(P_i P^i + \frac{1}{2}F_{ij}F^{ij}) + P^i \partial_i \phi \right\}$$

$$= \int_{\mathbf{R}^3} d^3x \left\{ P^i \dot{Q}_i - \left[\frac{1}{2}(P_i P^i + \frac{1}{2}F_{ij}F^{ij}) + \phi \partial_i P^i \right] \right\} \qquad (5.134)$$

To get the second equality I integrated by parts and used the divergence theorem on the last term. At each time t, we will assume that (Q_i, P^i) vanish sufficiently rapidly at infinity so that boundary term vanishes. As usual, the EL equations (using functional derivatives) for P^i reproduce the definition (5.132) so that the EL equations for (A_t, Q_i) then yield the Maxwell equations. Notice that ϕ enters as a Lagrange multiplier enforcing the (Gauss law) constraint on the canonical variables,

$$\partial_i P^i = 0, \qquad (5.135)$$

which is how we shall treat it in all that follows.

PROBLEM: Show that the EL equations defined by $L = L[Q, P, \phi]$,

$$\frac{\delta L}{\delta P^i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{P}^i} = 0, \quad \frac{\delta L}{\delta Q_i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{Q}_i} = 0, \quad \frac{\delta L}{\delta \phi} = 0, \qquad (5.136)$$

are equivalent to the Maxwell equations.

From (5.134) the Hamiltonian is given by

$$H[Q, P, \phi] = \int_{\mathbf{R}^3} d^3x \left[\frac{1}{2}(P_i P^i + \frac{1}{2} F_{ij} F^{ij}) + \phi \, \partial_i P^i \right]. \qquad (5.137)$$

The first two terms are what you might expect: they define the energy of the electromagnetic field (once you recognize that $\frac{1}{4} F_{ij} F^{ij} = \frac{1}{2} B^2$ is the magnetic energy density). You can see that the first term (the electric energy) is akin to the kinetic energy of a particle, while the second term (the magnetic energy) is akin to the potential energy of a particle. The term we want to focus on is the third term – what's that doing there? Well, first of all note that this term does not affect the value of the Hamiltonian provided the canonical variables satisfy the constraint (5.135). Secondly, let us consider Hamilton's equations:

$$\dot{Q}_i = \frac{\delta H}{\delta P^i} = P_i - \partial_i \phi, \qquad (5.138)$$

$$\dot{P}^i = -\frac{\delta H}{\delta Q_i} = \partial_i F^{ij}. \qquad (5.139)$$

From (5.138), which is secretly the relation between the electric field and the potentials, you can see that the last term in the Hamiltonian is precisely what is needed to generate the gauge transformation term we already found in (5.126).

PROBLEM: Using the Poisson brackets

$$[M, N] = \int_{\mathbf{R}^3} d^3x \left\{ \frac{\delta M}{\delta Q_i(\vec{x})} \frac{\delta N}{\delta P^i(\vec{x})} - \frac{\delta M}{\delta P^i(\vec{x})} \frac{\delta N}{\delta Q_i(\vec{x})} \right\}, \qquad (5.140)$$

show that

$$G = -\int_{\mathbf{R}^3} d^3x \, \Lambda(\vec{x}) \, \partial_i P^i \qquad (5.141)$$

is the generating function for gauge transformations

$$\delta Q_i = [Q_i, G] = \partial_i \Lambda, \quad \delta P^i = [P^i, G] = 0. \qquad (5.142)$$

Thus, the Maxwell equations can be viewed as a *constrained Hamiltonian system* with canonical coordinates and momentum given by the vector potential and electric field, with the Hamiltonian given by (5.137), and with the constraint (5.135). The Hamiltonian generates the time evolution of (Q_i, P^i) with the constraint term contributing a gauge transformation with gauge function ϕ. The Hamiltonian structures of non-Abelian gauge theory and of generally covariant gravitation theories follows a similar pattern.

5.9 PROBLEMS

1. Maxwell's equations for the electric and magnetic field $(\vec{E}, \vec{B})$ associated to charge density and current density $(\rho, \vec{j})$ are given by

$$\nabla \cdot \vec{E} = 4\pi\rho,$$

$$\nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 4\pi\sigma,$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0.$$

Show that for any function $\phi(\vec{x}, t)$ and vector field $\vec{A}(\vec{x}, t)$ the electric and magnetic fields defined by

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}$$

satisfy (5.2) and (5.4).

2. Define the anti-symmetric array $F_{\mu\nu}$ in inertial Cartesian coordinates $x^\alpha = (t, x, y, z), \alpha = 0, 1, 2, 3$ via

$$F_{ti} = -E_i, \quad F_{ij} = \epsilon_{ijk}B^k, \quad i, j = 1, 2, 3.$$

Under a change of inertial reference frame corresponding to a boost along the x axis with speed v the electric and magnetic fields change $(\vec{E}, \vec{B}) \rightarrow (\vec{E}', \vec{B}')$, where

$$E^{x\prime} = E^x, \quad E^{y\prime} = \gamma(E^y - vB^z), \quad E^{z\prime} = \gamma(E^z + vB^y)$$

$$B^{x'} = B^x, \quad B^{y'} = \gamma(B^y + vE^z), \quad B^{z'} = \gamma(B^z - vE^y).$$

(Here $\gamma = 1/\sqrt{1 - v^2}$.) Show that this is equivalent to saying that $F_{\mu\nu}$ are the components of a spacetime tensor of type $\binom{0}{2}$. Show that the two quantities $\vec{E} \cdot \vec{B}$ and $E^2 - B^2$ do not change under the boost.

3. Define the 4-current

$$j^\alpha = (\rho, j^i), \quad i = 1, 2, 3.$$

Show that the Maxwell equations take the form

$$F^{\alpha\beta}{}_{,\beta} = 4\pi j^\alpha, \quad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0,$$

where indices are raised and lowered with the usual Minkowski metric.

4. Show that the scalar and vector potentials, when assembled into the 4-potential

$$A_\mu = (-\phi, A_i), \quad i = 1, 2, 3, \tag{5.143}$$

are related to the electromagnetic tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.144}$$

Show that this formula for $F_{\mu\nu}$ solves the homogeneous Maxwell equations $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$.

5. Show that the EL derivative of the Maxwell Lagrangian satisfies the differential identity

$$D_\beta \mathcal{E}^\beta(\mathcal{L}) = 0.$$

6. Restrict attention to flat spacetime in Cartesian coordinates, as usual. Fix a vector field on spacetime, $j^\alpha = j^\alpha(x)$. Show that the Lagrangian

$$\mathcal{L}_j = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + 4\pi j^\alpha A_\alpha$$

gives the field equations

$$F^{\alpha\beta}{}_{,\alpha} = -4\pi j^\beta.$$

These are the Maxwell equations with prescribed *electric sources* having a charge density ρ and current density $\vec{j}$, where

$$j^\alpha = (\rho, \vec{j}).$$

Use the results from the preceding problem to show that the Maxwell equations with sources have no solution unless the vector field representing the sources is divergence-free:

$$\partial_\alpha j^\alpha = 0.$$

Show that this condition is in fact the usual continuity equations representing conservation of electric charge.

7. Show that the Lagrangian density for source-free electromagnetism can be written in terms of the electric and magnetic fields (in any given inertial frame) by $\mathcal{L} = \frac{1}{2}(E^2 - B^2)$. This is one of the 2 relativistic invariants that can be made algebraically from $\vec{E}$ and $\vec{B}$.

8. Show that $\vec{E} \cdot \vec{B}$ is relativistically invariant (unchanged by a Lorentz transformation). Express it in terms of potentials and show that it is just a divergence, with vanishing Euler-Lagrange expression.

9. For any function $\Lambda = \Lambda(x)$, define

$$A'_\alpha = A_\alpha + \partial_\alpha \Lambda.$$

Show that

$$A'_{\alpha,\beta} - A'_{\beta,\alpha} = A_{\alpha,\beta} - A_{\beta,\alpha}.$$

Show that, in terms of the scalar and vector potentials, this gauge transformation is equivalent to

$$\phi \to \phi' = \phi - \partial_t \Lambda, \quad \vec{A} \to \vec{A}' = \vec{A} + \nabla \Lambda.$$

10. Consider the electromagnetic field coupled to sources with the Lagrangian density

$$\mathcal{L}_j = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + 4\pi j^\alpha A_\alpha$$

Show that this Lagrangian is gauge invariant (up to a divergence) if and only if the spacetime vector field j^α is chosen to be divergence-free. What is the Noether identity in this case?

11. Verify equations (5.87)–(5.92).

12. Show that the 2-form (5.115) does not depend upon the time at which it is evaluated.

13. Show that if the constraint (5.128) holds at one time and the canonical variables evolve in time according to (5.126), (5.127) then (5.128) will hold at any other time. (*Hint:* Consider the time derivative of (5.128).)

14. Show that the EL equations defined by $L = L[Q, P, \phi]$,

$$\frac{\delta L}{\delta P^i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{P}^i} = 0, \quad \frac{\delta L}{\delta Q_i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{Q}_i} = 0, \quad \frac{\delta L}{\delta \phi} = 0, \tag{5.145}$$

are equivalent to the Maxwell equations.

15. Using the Poisson brackets

$$[M, N] = \int_{\mathbf{R}^3} d^3x \left\{ \frac{\delta M}{\delta Q_i(\vec{x})} \frac{\delta N}{\delta P^i(\vec{x})} - \frac{\delta M}{\delta P^i(\vec{x})} \frac{\delta N}{\delta Q_i(\vec{x})} \right\},$$

show that

$$G = -\int_{\mathbf{R}^3} d^3x \, \Lambda(\vec{x}) \, \partial_i P^i$$

is the generating function for gauge transformations

$$\delta Q_i = [Q_i, G] = \partial_i \Lambda, \quad \delta P^i = [P^i, G] = 0.$$

Chapter 6

Scalar Electrodynamics

Let us have an introductory look at the field theory called *scalar electrodynamics*, in which one considers a coupled system of Maxwell and charged KG fields. There are an infinite number of ways one could try to couple these fields. There is one particularly interesting way, physically speaking, and this is the one we shall be exploring. Mathematically, too, this particular coupling has many interesting features which we shall explore. To understand the motivation for the postulated form of scalar electrodynamics, it is easiest to proceed via Lagrangians. For simplicity we will restrict attention to flat spacetime in inertial Cartesian coordinates, but our treatment is easily generalized to an arbitrary spacetime.

6.1   Electromagnetic field with scalar sources

Let us return to the electromagnetic theory, but now with electrically charged sources. Recall that if $j^\alpha(x)$ is some *given* divergence-free vector field on spacetime, representing some externally specified charge-current distribution, then the behavior of the electromagnetic field interacting with the given source is dictated by the Lagrangian

$$\mathcal{L}_j = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + 4\pi j^\alpha A_\alpha. \tag{6.1}$$

Incidentally, given the explicit appearance of A_α, one might worry about the gauge symmetry of this Lagrangian. But it is easily seen that the gauge transformation is a divergence symmetry of this Lagrangian. Indeed, under

a gauge transformation of the offending term we have

$$
\begin{aligned}
j^\alpha A_\alpha \longrightarrow & j^\alpha(A_\alpha + \partial_\alpha \Lambda) \\
= & j^\alpha A_\alpha + D_\alpha(\Lambda j^\alpha),
\end{aligned} \tag{6.2}
$$

where we had to use the fact

$$
\partial_\alpha j^\alpha(x) = 0. \tag{6.3}
$$

The idea now is that we don't want to specify the sources in advance, we want the theory to tell us how they behave. In other words, we want to include the sources as part of the dynamical variables of our theory. In all known instances the correct way to do this always follows the same pattern: the gauge fields affect the "motion" of the sources, and the sources affect the form of the gauge field. Here we will use the electromagnetic field as the gauge field and the charged ($U(1)$ symmetric) KG field as the source. The reasoning for this latter choice goes as follows.

We are going to use fields to model things like electrically charged matter, so we insist upon a model for the charged sources built from a classical field.[1] So, we need a classical field theory that admits a conserved current that we can interpret as an electric 4-current. The KG field admitted 10 conserved currents corresponding to conserved energy, momentum and angular momentum. But we know that the electromagnetic field is not driven by such quantities, so we need another kind of current. To find such a current we turn to the charged KG field. In the absence of any other interactions, this field admits the conserved current

$$
j^\alpha = -ig^{\alpha\beta}\left(\varphi^* \varphi_{,\beta} - \varphi \varphi^*_{,\beta}\right). \tag{6.4}
$$

The simplest thing to try is to build a theory in which this is the current that drives the electromagnetic field. This is the correct idea, but the most naive attempt to implement this strategy falls short of perfection. To see this, imagine a Lagrangian of the form

$$
\mathcal{L}_{wrong} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - (\partial^\alpha \varphi^* \partial_\alpha \varphi + m^2 |\varphi|^2) - iA_\alpha g^{\alpha\beta}\left(\varphi^* \varphi_{,\beta} - \varphi \varphi^*_{,\beta}\right). \tag{6.5}
$$

This Lagrangian was obtained by simply taking (6.1), substituting (6.4) for the current, and then adding the KG Lagrangian for the scalar field.[2] The

---

[1] After all, this *is* a course in classical field theory.

[2] Recall from Lagrangian mechanics that one can find a Lagrangian for the combination of mechanical systems by adding their respective Lagrangians.

idea is that the EL equations for A will give the Maxwell equations with the KG current as the source. The EL equations for the scalar field will now involve A, but that is reasonable since we expect the presence of the electromagnetic field to affect the sources. But there is one big problem with this Lagrangian: it is no longer gauge invariant! Recall that the gauge invariance of the Maxwell Lagrangian with prescribed sources made use of the fact that the current was divergence-free. But now the current is only divergence-free when the field equations hold. The key to escaping this difficulty is to let the KG field participate in the gauge symmetry. This forces us to modify the Lagrangian as we shall now discuss.

6.2 Minimal coupling: the gauge covariant derivative

The physically correct way to get a gauge invariant Lagrangian for the coupled Maxwell-KG theory, which still gives the $j^\alpha A_\alpha$ kind of coupling is rather subtle and clever. Let me begin by just stating the answer. Then I will try to show how it works and how one might even be able to derive it from some new, profound ideas.

The answer is to modify the KG Lagrangian via "minimal coupling", in which one replaces

$$\partial_\alpha \varphi \to \mathcal{D}_\alpha \varphi := (\partial_\alpha + iqA_\alpha)\varphi, \tag{6.6}$$

and

$$\partial_\alpha \varphi^* \to \mathcal{D}_\alpha \varphi^* := (\partial_\alpha - iqA_\alpha)\varphi^*. \tag{6.7}$$

Here q is a parameter reflecting the coupling strength between the charged field φ and the gauge field. It is a *coupling constant.* In the quantum field theory description, q is the bare electric charge of a particle excitation of the quantum field φ. The effect of the electromagnetic field, described by A_α, upon the KG field is then described by the Lagrangian

$$\mathcal{L}_{KG} = -\mathcal{D}^\alpha \varphi^* \mathcal{D}_\alpha \varphi - m^2 |\varphi|^2. \tag{6.8}$$

This Lagrangian yields field equations which involve the wave operator modified by terms built from the electromagnetic potential. These additional terms represent the effect of the electromagnetic field on the charged scalar field.

PROBLEM: Compute the EL equations of $\mathcal{L}_{KG}$ in (6.8).

The Lagrangian (6.8) still admits the $U(1)$ phase symmetry of the charged KG theory, but because this Lagrangian depends explicitly upon A it will not be gauge invariant unless we include a corresponding transformation of φ. We therefore extend the gauge transformation to be:

$$A_\alpha \longrightarrow A_\alpha + \partial_\alpha \Lambda, \tag{6.9}$$

$$\varphi \longrightarrow e^{-iq\Lambda}\varphi, \quad \varphi^* \longrightarrow e^{iq\Lambda}\varphi^*. \tag{6.10}$$

You should now verify that under a gauge transformation we have the fundamental relation (which justifies the minimal coupling prescription)

$$\mathcal{D}_\alpha \varphi \longrightarrow e^{-iq\Lambda}\mathcal{D}_\alpha\varphi, \tag{6.11}$$

$$\mathcal{D}_\alpha \varphi^* \longrightarrow e^{iq\Lambda}\mathcal{D}_\alpha\varphi^*. \tag{6.12}$$

For this reason $\mathcal{D}_\alpha$ is sometimes called the *gauge covariant derivative*. There is a nice geometric interpretation of this covariant derivative, which we shall discuss later. For now, because of this "covariance" property of $\mathcal{D}$, it follows that the Lagrangian (6.8) is gauge invariant with respect to the transformations (6.9), (6.10).

The Lagrangian for scalar electrodynamics is taken to be

$$\mathcal{L}_{SED} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - m^2|\varphi|^2. \tag{6.13}$$

We now discuss some important structural features of this Lagrangian.

If we expand the gauge covariant derivatives we see that

$$\mathcal{L}_{SED} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \partial^\alpha\varphi^*\partial_\alpha\varphi - m^2|\varphi|^2 + iqA_\alpha\left(\varphi^*\partial^\alpha\varphi - \varphi\partial^\alpha\varphi^* + iqA^\alpha|\varphi|^2\right). \tag{6.14}$$

This Lagrangian is the sum of the electromagnetic Lagrangian, the free charged KG Lagrangian, and a $j \cdot A$ "interaction term". The vector field that is contracted with A is almost the conserved current (6.4), but is modified by the last term involving the square of the gauge field, which is needed for invariance under the gauge transformation (6.9), (6.10) and for the current to be conserved when the new form of the field equations are satisfied. The EL equations for the Maxwell field are of the desired form:

$$\partial_\beta F^{\alpha\beta} = -4\pi J^\alpha, \tag{6.15}$$

where the current is defined using the covariant derivative instead of the ordinary derivative:

$$J^\alpha = -\frac{iq}{4\pi} \left(\varphi^* \mathcal{D}^\alpha \varphi - \varphi \mathcal{D}^\alpha \varphi^* \right). \tag{6.16}$$

As you will verify in the problem below, this current can be derived from Noether's first theorem applied to the $U(1)$ phase symmetry of the Lagrangian (6.13). Thus we have solved the gauge invariance problem *and* obtained a consistent version of the Maxwell equations with conserved sources using the minimal coupling prescription.

PROBLEM: Derive (6.15), (6.16) from (6.13).

PROBLEM: Verify that (6.16) is the Noether current coming from the $U(1)$ symmetry of the Lagrangian and that it is indeed conserved when the field equations for φ hold.

One more interesting feature to ponder: the charged current (6.16) serving as the source for the Maxwell equations is built from the KG field *and the Maxwell field.* Physically this means that one cannot say the charge "exists" only in the KG field. In an interacting system the division between source fields and fields mediating interactions is somewhat artificial and arbitrary. This is physically reasonable, if perhaps a little unsettling. Mathematically, this feature stems from the demand of gauge invariance. Just like the vector potential, the KG field is no longer uniquely defined - it is subject to a gauge transformation as well! In the presence of interaction, the computation of the electric charge involves a gauge invariant combination of the KG and electromagnetic field. To compute, say, the electric charge contained in a volume V one should take a solution (A, φ) of the coupled Maxwell-KG equations and substitute it into

$$Q_V = \frac{1}{4\pi} \int_V d^3x \, iq \left(\varphi^* \mathcal{D}^0 \varphi - \varphi \mathcal{D}^0 \varphi^* \right). \tag{6.17}$$

This charge is conserved and gauge invariant.

6.3 Global and Local Symmetries

The key step in constructing the Lagrangian for scalar electrodynamics was to introduce the coupling between the Maxwell field A and the charged KG field

by replacing in the KG Lagrangian the ordinary derivative with the gauge covariant derivative. With this replacement, the coupled KG-Maxwell theory is defined by adding the modified KG Lagrangian to the electromagnetic Lagrangian. There is a rather deep way of viewing this construction which we shall now explore.

Let us return to the free, charged KG theory, described by the Lagrangian

$$\mathcal{L}_{KG}^0 = -(\varphi^{*},^{\alpha}\, \varphi_{,\alpha} + m^2|\varphi|^2). \tag{6.18}$$

This field theory admits a conserved current

$$j^{\alpha} = -iq(\varphi^{*}\varphi,^{\alpha} - \varphi\varphi^{*},^{\alpha}), \tag{6.19}$$

which we want to interpret as corresponding to a conserved electric charge "stored" in the field. Of course, the presence of electric charge in the universe only manifests itself by virtue of its electromagnetic interactions. How should the conserved charge in the KG field be interacting? Well, we followed one rather *ad hoc* path to introducing this interaction in the last lecture. Let us revisit the construction with a focus upon symmetry considerations, which will lead to a very profound way of interpreting and systematizing the construction.

The current j^{α} is conserved because of the *global $U(1)$* phase symmetry. For any $\alpha \in \mathbf{R}$ this symmetry transformation is

$$\varphi \to e^{-iq\alpha}\varphi, \quad \varphi^{*} \to e^{iq\alpha}\varphi^{*}, \tag{6.20}$$

where q is a parameter which would ultimately be fixed by experimental considerations. This transformation shifts the phase of the scalar field by the same amount α everywhere in space and for all time. This is why the transformation is called "global". The phase of the scalar field is defined by an angle – a point on a circle – and the global $U(1)$ symmetry can be interpreted as saying that the Lagrangian does not depend on the choice of origin for that phase. This is analogous to the translational symmetry of special relativistic theories; the Lagrangian does not depend upon the choice of origin in a given inertial reference frame.

The presence of the electromagnetic interaction can be seen as a "localizing" or "gauging" of this global symmetry so that one is free to redefine the phase of the field independently at each spacetime event (albeit smoothly).

This "general relativity" of phase is accomplished by demanding that the theory be modified so that one has the symmetry

$$\varphi \to e^{-iq\alpha(x)}\varphi, \quad \varphi^* \to e^{iq\alpha(x)}\varphi^*, \tag{6.21}$$

where $\alpha(x)$ is any function on the spacetime manifold M. Of course, the original Lagrangian $\mathcal{L}^0_{KG}$ fails to have this *local* $U(1)$ transformation as a symmetry since

$$\partial_\mu(e^{-iq\alpha(x)}\varphi) = e^{-iq\alpha(x)}\varphi_{,\mu} - iqe^{-iq\alpha(x)}\varphi\alpha_{,\mu}. \tag{6.22}$$

However, we can introduce a *gauge field* A_μ, which is affected by the local $U(1)$ transformation via the gauge transformation:

$$A_\mu \longrightarrow A_\mu + \partial_\mu\alpha, \tag{6.23}$$

and then introduce the *covariant derivative*

$$\mathcal{D}_\alpha\varphi := (\partial_\alpha + iqA_\alpha)\varphi, \tag{6.24}$$

and

$$\mathcal{D}_\alpha\varphi^* := (\partial_\alpha - iqA_\alpha)\varphi^*, \tag{6.25}$$

which satisfies

$$\mathcal{D}_\mu(e^{-iq\alpha(x)}\varphi) = e^{-iq\alpha(x)}\mathcal{D}_\mu\varphi. \tag{6.26}$$

Then with the Lagrangian modified via

$$\partial_\mu\varphi \to \mathcal{D}_\mu\varphi, \tag{6.27}$$

so that

$$\mathcal{L}_{KG} = -\mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - m^2|\varphi|^2, \tag{6.28}$$

we get the local $U(1)$ symmetry, as shown previously. Thus the minimal coupling rule that we invented earlier can be seen as a way of turning the global $U(1)$ symmetry into a local $U(1)$ gauge symmetry. One also obtains the satisfying mental picture that the electromagnetic interaction of charges is the principal manifestation of this local phase symmetry in nature. Thus the electromagnetic interaction is introduced via the *principle of local gauge invariance.*

The interaction of the KG field with the electromagnetic field is described mathematically by the $\partial_\alpha \to \mathcal{D}_\alpha$ prescription described above. But the story

is not complete since we have not given a complete description of the electromagnetic field itself. Indeed, even in the absence of the KG field the EM field has a description in terms of the Lagrangian (5.22). We need to incorporate the electromagnetic Lagrangian into the total Lagrangian for the system. But first, how should we think about the electromagnetic Lagrangian from the point of view the principle of local gauge invariance? The electromagnetic Lagrangian is the simplest non-trivial relativistic invariant that can be made from the field strength tensor. The field strength tensor itself can be viewed as the "curvature" of the gauge covariant derivative, computed via the commutator:

$$\left(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu\right)\varphi = iqF_{\mu\nu}\varphi. \tag{6.29}$$

From this relation it follows immediately that F is gauge invariant; of course we already knew that F was gauge invariant.

PROBLEM: Verify the result (6.29).

Thus the electromagnetic Lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{6.30}$$

admits the local $U(1)$ symmetry and can be added to the locally invariant KG Lagrangian to get the final Lagrangian for the theory

$$\mathcal{L}_{SED} = \mathcal{L}_{KG} + \mathcal{L}_{EM}. \tag{6.31}$$

In this way we have an interacting theory designed by local $U(1)$ gauge symmetry. The parameter q, which appears via the gauge covariant derivative, is a "coupling constant" and characterizes the strength with which the electromagnetic field couples to the charged aspect of the KG field. In the limit in which $q \to 0$ the theory becomes a decoupled juxtaposition of the non-interacting (or "free") charged KG field theory and the non-interacting (free) Maxwell field theory. In principle, the parameter q is determined by suitable experiments.

Scalar electrodynamics still admits the global $U(1)$ symmetry, with $\alpha = const.$

$$\varphi \to e^{-iq\alpha}\varphi, \quad \varphi^* \to e^{iq\alpha}\varphi^*, \tag{6.32}$$

$$A_\mu \to A_\mu, \tag{6.33}$$

as a variational symmetry. Infinitesimally, we can write this transformation as

$$\delta\varphi = -iq\alpha\varphi, \quad \delta\varphi^* = iq\alpha\varphi^*, \tag{6.34}$$

$$\delta A_\mu = 0. \tag{6.35}$$

From Noether's first theorem this leads to the identity

$$D_\mu J^\mu - iq(\varphi\mathcal{E}_\varphi - \varphi^*\mathcal{E}_{\varphi^*}) = 0, \tag{6.36}$$

where

$$J^\alpha = -\frac{iq}{4\pi}\left(\varphi^*\mathcal{D}^\alpha\varphi - \varphi\mathcal{D}^\alpha\varphi^*\right). \tag{6.37}$$

Evidently, J is divergence-free when the scalar field equations of motion hold. J is the conserved Noether current J corresponding to the electric charge carried by the scalar field. This is the current that serves as source for the Maxwell field. The presence of the gauge field renders the Noether current suitably "gauge invariant", that is, insensitive to the local $U(1)$ transformation. It also reflects the fact that the equations of motion for φ, which must be satisfied in order for the current to be conserved, depend upon the Maxwell field as is appropriate since the electromagnetic field affects the motion of its charged sources.

By construction, the theory of scalar electrodynamics admits the local $U(1)$ gauge symmetry. With $\alpha(x)$ being any function, the symmetry is

$$\varphi \to e^{-iq\alpha(x)}\varphi, \quad \varphi^* \to e^{iq\alpha(x)}\varphi^*, \tag{6.38}$$

$$A_\mu \to A_\mu + \partial_\mu\alpha(x). \tag{6.39}$$

There is a corresponding Noether identity (Noether's second theorem, remember?). To compute it we consider an infinitesimal gauge transformation:

$$\delta\varphi = -iq\alpha(x)\varphi, \quad \delta\varphi^* = iq\alpha(x)\varphi^*, \tag{6.40}$$

$$\delta A_\mu = \partial_\mu\alpha(x). \tag{6.41}$$

Since the Lagrangian is gauge invariant we have the identity

$$0 = \delta\mathcal{L} = \mathcal{E}_\varphi(-iq\alpha(x)\varphi) + \mathcal{E}_{\varphi^*}(iq\alpha(x)\varphi^*) + \mathcal{E}^\mu(\partial_\mu\alpha(x)) + \text{divergence}, \tag{6.42}$$

where $\mathcal{E}_\varphi$ and $\mathcal{E}_{\varphi^*}$ are the scalar field EL expressions and $\mathcal{E}^\mu$ is the gauge field EL expression. If we integrate this relation over a compact region and

choose $\alpha(x)$ to vanish at the boundary of this region, then the divergence term vanishes. We can integrate by parts in the third term to get the Noether identity

$$D_\mu \mathcal{E}^\mu + iq(\varphi \mathcal{E}_\varphi - \varphi^* \mathcal{E}_{\varphi^*}) = 0. \tag{6.43}$$

(Notice that this identity does *not* follow if the gauge transformation includes a global part.) The terms involving the EL expressions for the KG field are the same as arise in the identity (6.36). Thus the Noether identity (6.43) can also be written as

$$D_\alpha D_\beta F^{\alpha\beta} = 0. \tag{6.44}$$

Thanks to this Noether identity we can obtain – again! – the conservation law of electric charge. We have the electromagnetic field equation

$$\mathcal{E}^\beta \equiv D_\alpha F^{\alpha\beta} - 4\pi J^\beta = 0. \tag{6.45}$$

Taking the divergence of this equation and using (6.44) we get $D_\beta J^\beta = 0$ when $E^\beta = 0$.

6.4   A lower-degree conservation law

There is an interesting alternative point of view on the conservation of electric charge which I would like to mention. Let me set the stage. The conserved current for scalar electrodynamics,

$$J^\alpha = -\frac{iq}{4\pi} \left(\varphi^* \mathcal{D}^\alpha \varphi - \varphi \mathcal{D}^\alpha \varphi^* \right), \tag{6.46}$$

features in the Maxwell equations via the EL equation $\mathcal{E}^\alpha = 0$ for the gauge field A_α, where

$$\mathcal{E}^\beta = F^{\alpha\beta}_{,\alpha} - 4\pi J^\beta. \tag{6.47}$$

Rearranging this formula,

$$J^\beta = \frac{1}{4\pi} F^{\alpha\beta}_{,\alpha} - \frac{1}{4\pi} \mathcal{E}^\beta, \tag{6.48}$$

and recalling the discussion of §3.20, you will see that J^β is a "trivial" conservation law! Mathematically, this result is intimately related to the fact that the conservation law arises from a symmetry, which contains both a gauge part and a global part. Physically, it reflects the fact that the electric

charge contained in a given spatial region can be computed using just *electromagnetic* data on the surface bounding that region. Indeed, the conserved electric charge in a 3-dimensional spacelike region V at some time $x^0 = const.$ is given by

$$Q_V = \int_V dV \, J^0 = \frac{1}{4\pi} \int_V dV \, F^{i0}_{,i} = \frac{1}{4\pi} \int_S dS \, \hat{n} \cdot \vec{E}, \tag{6.49}$$

where $S = \partial V$ and $E^i = F^{0i}$ is the electric field in the inertial frame with time $t = x^0$. This is of course the integral form of the Gauss law.

A more elegant and geometric way to characterize this relationship is via differential forms and Stokes' theorem. Recall that if ω is a differential p-form and V is a $(p+1)$-dimensional region with p-dimensional boundary S (*e.g.*, V is the interior of a 2-sphere and S is the 2-sphere), then Stokes' theorem says

$$\int_V d\omega = \int_S \omega. \tag{6.50}$$

This generalizes the purely vectorial version of the Stokes' and divergence theorems you learned in multi-variable calculus in Euclidean space to manifolds of any dimension. As I have mentioned before, we can view the electromagnetic tensor as a 2-form F via

$$F = F_{\alpha\beta} dx^\alpha \otimes dx^\beta = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta. \tag{6.51}$$

Using the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$, we can construct the *Hodge dual* $\star F$, defined by

$$\star F = \frac{1}{2} (\star F)_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{6.52}$$

where

$$(\star F)_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}. \tag{6.53}$$

In terms of F and $\star F$ the Maxwell equations read

$$dF = 0, \quad d\star F = 4\pi \mathcal{J}, \tag{6.54}$$

where $\mathcal{J}$ is the Hodge dual of the electric current:

$$\mathcal{J} = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} J^\delta dx^\alpha \wedge dx^\beta \wedge dx^\gamma. \tag{6.55}$$

The conservation of electric current is normally expressed in terms of a divergence-free vector field J, but this is equivalent to having a closed 3-form:

$$d\mathcal{J} = 0, \quad \text{modulo the field equations.} \tag{6.56}$$

Indeed, applying the exterior derivative to both sides of the field equations (6.54) and using the identity $d^2 = 0$ you see that (6.56) must hold. (This is just an elegant repackaging of the result at the end of the last section.) In terms of differential forms, the electric charge in a three-dimensional region V at a fixed time t and with boundary S is the integral of $\mathcal{J}$ over that region:

$$Q_V = \int_V \mathcal{J}. \tag{6.57}$$

When the field equations hold we have

$$Q_V = \int_V \mathcal{J} = \frac{1}{4\pi} \int_V d \star F = \frac{1}{4\pi} \int_S \star F, \tag{6.58}$$

recovering Gauss' law in differential form form.

An important application of Stokes' theorem we will need goes as follows. Let χ be the integral of the p-form ω over a closed[3] p-dimensional space S:

$$\chi = \int_S \omega. \tag{6.59}$$

Consider any continuous deformation of S into a new (closed) surface S' and let χ' be the integral of ω over that space:

$$\chi' = \int_{S'} \omega. \tag{6.60}$$

The relation between these 2 quantities can be obtained using Stokes theorem:

$$\chi' - \chi = \int_{S'} \omega - \int_S \omega = \int_{S'-S} \omega = \int_V d\omega, \tag{6.61}$$

where $\partial V = S' - S$. In particular, if ω is a *closed p-form*, that is, $d\omega = 0$, then $\chi = \chi'$ and the integral χ is independent of the choice of the space S in the sense that χ is unchanged by any continuous deformation of S. A simple illustration of this result is the following.

---

[3]To say that S is "closed" means that it has no boundary, $\partial S = \emptyset$, *e.g.,* a 2-sphere.

PROBLEM: Consider two concentric circles of radii a and b in the "x-y plane". Let V be the area between the circles so that its boundary S consists of the concentric circles. In Cartesian coordinates with origin at the center of the circles, let $\vec{A}$ be a vector field defined in V by

$$\vec{A} = \frac{x\hat{y} - y\hat{x}}{x^2 + y^2}. \tag{6.62}$$

Show that the curl of $\vec{A}$ vanishes in V. Show that the line integral of $\vec{A}$ around either of the circles gives the same result, independent of a and b.

Consider a charge distribution which is localized in some compact region. Because the 3-form $\mathcal{J}$ is closed when the field equations hold, we can apply the preceding result to conclude that the integral of $\mathcal{J}$ over a spacelike hypersurface (say, at a fixed time) will be independent of the choice of the hypersurface. This is just a fancy way of saying the total charge is conserved. But from (6.58) we have an alternative way to compute the conserved charge in a compact volume: integrate $\star F$ over a surface enclosing that volume. As long as the surface is computed outside of the charge distribution the form $\star F$ is closed there and the resulting integral is independent of a continuous deformation of that surface. This deformation could just be redrawing the surface at a fixed time, or it could involve evaluating the surface at a different time. Thus the conservation of electric charge can be expressed in term of a closed 2-form.

The existence of conservation laws of the traditional sort – divergence-free currents or closed 3-forms – is tied to the existence of symmetries of an underlying Lagrangian via Noether's theorem. It is natural to ask if there is any symmetry-based origin to conserved 2-forms such as we have in source-free electrodynamics with $\star F$. The answer is yes. Details would take us too far afield, but let me just mention that closed 2-forms (in 4 dimensions) arise in a field theory when (1) the theory admits a gauge symmetry, and (2) every solution of the field equations admits a gauge transformation which does not change that solution. Now consider pure electromagnetism in a region of spacetime with no sources. Of course, criterion (1) is satisfied. To see that criterion (2) is satisfied consider the gauge transformation by a constant function.[4] It is this symmetry structure which corresponds to the conservation law associated to $\star F$ in source-free regions. [5]

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[4]Notice that this is the "global" version of the $U(1)$ symmetry transformation.

[5]To read more about such things, have a look at `arxiv.org/abs/hep-th/9706092` .

6.5   Scalar electrodynamics and fiber bundles

There is a beautiful geometric interpretation of SED in terms of a famous mathematical structure called a fiber bundle. I debated with myself for a long time whether or not to try and describe this to you. I decided that I could not resist mentioning it, so that those of you who are so-inclined can get exposed to it. On the other hand, a complete presentation would take us too far afield and not everybody who is studying this material is going to be properly prepared (or interested enough!) for a full-blown treatment. So, if you don't mind, I will just give a quick and dirty summary of the salient points. Those who are not ready for this material can just skip it. A more complete – indeed, a more correct – treatment can be found in many advanced texts. Of course, the problem is that to use these advanced texts takes a considerable investment in acquiring prerequisites. The idea of our brief discussion is to provide a first introductory step in that direction. A technical point for those who have some background in this stuff: for simplicity in what follows we shall not emphasize the role of the gauge field as a connection on a *principal bundle*, but rather its role as defining a connection on associated vector bundles.

Recall that our charged KG field can be viewed as a mapping

$$\varphi\colon M \to \mathbf{C}. \tag{6.63}$$

We can view φ as a *cross section* of a fiber bundle

$$\pi\colon E \to M \tag{6.64}$$

where

$$\pi^{-1}(x) = \mathbf{C}, \quad x \in M. \tag{6.65}$$

M is called the *base space*. The space $\pi^{-1}(x) \approx \mathbf{C}$ is the *fiber* over x. Since $\mathbf{C}$ is a vector space, this type of fiber bundle is called a *vector bundle*. For us, $M = \mathbf{R}^4$ and it can be shown that for a contractible base space such as $\mathbf{R}^n$ there is always a (non-unique) diffeomorphism that makes possible the global identification:

$$E \approx M \times \mathbf{C}. \tag{6.66}$$

For a general base space M, such an identification will only be valid locally.

Next, recall that a cross section of E (often just called a "section") is a map (or graph)

$$\sigma\colon M \to E \tag{6.67}$$

satisfying

$$\pi \circ \sigma = id_M. \tag{6.68}$$

Using coordinates (x, z) adapted to (6.66), we can identify a KG field $\varphi(x)$ with the cross section

$$\sigma(x) = (x, \varphi(x)). \tag{6.69}$$

Thus, given the identification $E \approx M \times \mathbf{C}$, we see that the bundle point of view just describes the geometric setting of our theory: complex valued functions on $\mathbf{R}^4$. For the purposes of this discussion the most interesting issue is that this identification is far from unique. Let us use coordinates (x^α, z) for E, where $x^\alpha \in \mathbf{R}^4$ and $z \in \mathbf{C}$. Each set of such coordinates provides an identification of E with $M \times \mathbf{C}$. Since we use a fixed (flat) metric on M, one can restrict attention to inertial Cartesian coordinates on M, in which case one can only redefine x^α by a Poincaré transformation. What is more interesting for us in this discussion is the freedom to redefine the way that the complex numbers are "glued" to each spacetime event.

Recall that to build the Lagrangian for the charged KG field we also had to pick an inner product on the vector space $\mathbf{C}$; of course we just used the standard one

$$\langle z, w \rangle = z^* w. \tag{6.70}$$

We can therefore restrict attention to linear changes of our coordinates on $\mathbf{C}$ which preserve this inner product. Thus the allowed changes of fiber coordinates are just the phase transformations

$$z \to e^{-i\alpha} z, \quad \alpha \in \mathbf{R}. \tag{6.71}$$

We can make this change of coordinates on $\mathbf{C}$ for each fiber so that on $\pi^{-1}(x)$ we make the transformation

$$z \to e^{-i\alpha(x)} z. \tag{6.72}$$

There is no intrinsic way to compare points on different fibers, and this fact reflects itself in the freedom to redefine our labeling of those points in a way that can vary from fiber to fiber. We have seen this already; the change of fiber coordinates $z \to e^{-i\alpha(x)} z$ corresponds to the gauge transformation of the charged KG field:

$$\varphi(x) \to e^{-i\alpha(x)} \varphi(x). \tag{6.73}$$

When building a field theory of the charged KG field we need to take derivatives. Now, to take a derivative means to compare the value of φ at

two neighboring points on M. From our fiber bundle point of view, this means comparing points on two different fibers. Because this comparison is not defined *a priori*, there is no natural way to take derivatives of a section of a fiber bundle. This is closely related to the fact that the ordinary derivative of the KG field does not transform homogeneously under a gauge transformation. Thus, for example, to say that a KG field is a constant, $\partial_\alpha \varphi = 0$, is not an intrinsic statement since a change in the bundle coordinates will negate it.

A definition of derivatives of sections of the fiber bundle requires the introduction of additional structure beyond the bundle and the metric. (One often introduces this structure implicitly!) This additional structure is called a *connection* and the resulting notion of derivative is called the *covariant derivative* defined by the connection. A connection can be viewed as a definition of how to compare points on neighboring fibers. If you are differentiating in a given direction, the derivative will need to associate to that direction a linear transformation (actually, a phase transformation) which "aligns" the vector spaces/fibers and allows us to compare them. Since the derivative involves an infinitesimal motion in M, it turns out that this fiber transformation is an infinitesimal phase transformation, which involves multiplication by a pure imaginary number (think: $e^{i\alpha} = 1 + i\alpha + \dots$). So, at each point $x \in M$, a connection assigns an imaginary number to every direction in M. It can be specified by an imaginary-valued 1-form on M, which we write as $A = iqA_\alpha(x)dx^\alpha$. The covariant derivative is then

$$\mathcal{D}_\alpha \varphi = (\partial_\alpha + iqA_\alpha)\varphi. \tag{6.74}$$

The role of the connection A is to define the rate of change of a section by adjusting the correspondence between fibers relative to that provided by the given choice of coordinates.

As we have seen, if we make a redefinition of the coordinates on each fiber by a local gauge transformation, then we must correspondingly redefine the 1-form via

$$z \to e^{-i\alpha(x)}z, \quad A_\alpha \to A_\mu + \partial_\mu \alpha. \tag{6.75}$$

This guarantees that the covariant derivative transforms homogeneously under a gauge transformation in the same way that the KG field itself does, which means it represents an invariant tensorial quantity on E. In particular, if a KG field is constant with the given choice of connection then this remains true in any (fiber) coordinates.

The connection A must be specified to define the charged KG Lagrangian (6.8). A different choice of connection, even just in a different gauge, changes the Lagrangian (6.8) and is not a symmetry of the KG field in a given electromagnetic field. By contrast, in scalar electrodynamics we view the connection as one of the dependent variables of the theory and we therefore have, as we saw, the full gauge invariance since now we can let the connection be transformed along with the KG field.

As you may know from differential geometry, when using a covariant derivative one will, in general, lose the commutativity of the derivative operation. The commutator of two covariant derivatives defines the *curvature* of the connection. We have already seen that this curvature is precisely the electromagnetic field strength:

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu] \varphi = iqF_{\mu\nu}\varphi. \tag{6.76}$$

To continue the analogy with differential geometry a bit further, you see that the field φ is playing the role of a vector, with its vector aspect being the fact that it takes values in the vector space **C** and transforms homogeneously under the change of fiber coordinates, that is, the gauge transformation. The complex conjugate can be viewed as living in the dual space to **C**, so that it is a "covector". Quantities like the Lagrangian density, or the conserved electric current are "scalars" from this point of view – they are gauge invariant. In particular, the current

$$J^\mu = -\frac{iq}{4\pi}(\varphi^* D^\mu \varphi - \varphi \mathcal{D}^\mu \varphi^*) \tag{6.77}$$

is divergence free with respect to the ordinary derivative, which is the correct covariant derivative on "scalars".

Do we really need all this fancy mathematics? Perhaps not. But, since all the apparatus of gauge symmetry, covariant derivatives, *etc.,* which show up repeatedly in field theory, arises so naturally from this geometric structure, it is clear that this is the right way to be thinking about gauge theories. Moreover, there are certain results that would, I think, be very hard to come by without using the fiber bundle point of view. I have in mind certain important topological structures that can arise via global effects in classical and quantum field theory. These topological structures are, via the physics literature, appearing in the guise of "monopoles" and "instantons". Such structures would play a very nice role in a second semester for this course, if there were one.

6.6   PROBLEMS

1. Compute the EL equations of $\mathcal{L}_{KG}$ in (6.8).

2. Derive (6.15), (6.16) from (6.13).

3. Verify that (6.16) is the Noether current coming from the $U(1)$ symmetry of the Lagrangian and that it is indeed conserved when the field equations for φ hold.

4. Verify the result (6.29).

5. Consider two concentric circles of radii a and b in the "x-y plane". Let V be the area between the circles so that its boundary S consists of the concentric circles. In Cartesian coordinates with origin at the center of the circles, let $\vec{A}$ be a vector field defined in V by

$$\vec{A} = \frac{x\hat{y} - y\hat{x}}{x^2 + y^2}. \tag{6.78}$$

Show that the curl of $\vec{A}$ vanishes in V. Show that the line integral of $\vec{A}$ around either of the circles gives the same result, independent of a and b.

Chapter 7

Spontaneous symmetry breaking

We now will take a quick look at some of the classical field theoretic underpinnings of "spontaneous symmetry breaking" (SSB) in quantum field theory. Quite generally, SSB can be a very useful way of thinking about phase transitions in physics. In particle physics, SSB is used, in collaboration with the "Higgs mechanism", to give masses to gauge bosons (and other elementary particles) without destroying gauge invariance.

7.1 Symmetry of laws versus symmetry of states

To begin to understand spontaneous symmetry breaking in field theory we need to refine our understanding of "symmetry", which is the goal of this section. The idea will be that there are two related kinds of symmetry one can consider: symmetry of the "laws" governing the field (*i.e.*, the field equations), and symmetries of the "states" of the field (*i.e.*, the solutions to the field equations).

So far we have been studying "symmetry" in terms of transformations of a field which preserve the Lagrangian, possibly up to a divergence. For our present aims, it is good to think of this as a "symmetry of the laws of physics" in the following sense. The Lagrangian determines the "laws of motion" of the field via the Euler-Lagrange equations. As was pointed out in Chapter 3, symmetries of a Lagrangian are also symmetries of the equations

of motion. This means that if φ is a solution to the equations of motion and if $\tilde{\varphi}$ is obtained from φ via a symmetry transformation, then $\tilde{\varphi}$ also satisfies the *same* equations of motion. Just to make sure this is clear, let me exhibit a very elementary example.

Consider the massless KG field described by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi. \tag{7.1}$$

It is easy to see that this Lagrangian admits the symmetry

$$\tilde{\varphi} = \varphi + const. \tag{7.2}$$

You can also easily see that the field equations

$$\partial^\alpha\partial_\alpha\varphi = 0 \tag{7.3}$$

admit this symmetry in the sense that if φ is solution then so is $\varphi + const.$ Thus a symmetry of a Lagrangian is also a symmetry of the field equations and we will sometimes call it a symmetry of the law governing the field.

PROBLEM: While every symmetry of a Lagrangian is a symmetry of its EL equations, it is not true that every symmetry of the field equations is a symmetry of the Lagrangian. Consider the massless KG field. Show that the scaling transformation $\tilde{\varphi} = (const.)\varphi$ is a symmetry of the field equations but is *not* a symmetry of the Lagrangian.

If the Lagrangian and its field equations represent the "laws", then the solutions of the field equations are the "states" of the field that are allowed by the laws. The function $\varphi(x)$ is an allowed state of the field when it solves the field equations. A symmetry of a given "state", $\varphi_0(x)$ say, is then defined to be a transformation of the fields, $\varphi \to \tilde{\varphi}[\varphi]$, which preserves the given solution

$$\tilde{\varphi}[\varphi_0(x)] = \varphi_0(x). \tag{7.4}$$

Since symmetry transformations form a group, such solutions to the field equations are sometimes called "group-invariant solutions".

Let us consider an elementary example of group-invariant solutions. Consider the KG field with mass m. Use inertial Cartesian coordinates. We have seen that the spatial translations, $x^i \to x^i + const.$, $i = 1, 2, 3$, form a group of symmetries of the theory. Functions which are invariant under the group

of spatial translations will depend upon t only: $\varphi = \varphi(t)$. It is easy to see that the corresponding group invariant solutions to the field equations are of the form:

$$\varphi = A\cos(mt) + B\sin(mt), \tag{7.5}$$

where A and B are constants. Another very familiar type of example of group-invariant solutions you will have seen by now occurs whenever you are finding rotationally invariant solutions of PDEs.

PROBLEM: Derive the result (7.5).

An important result from the geometric theory of differential equations which relates symmetries of laws to symmetries of states goes as follows. Suppose G is a group of symmetries of a system of differential equations $\Delta = 0$ for fields φ on a manifold M, (*e.g.*, G is the Poincaré group). Let $K \subset G$ be a subgroup (*e.g.*, spatial rotations). Suppose we are looking for solutions to $\Delta = 0$ which are invariant under K. Then the field equations $\Delta = 0$ reduce to a system of differential equations $\hat{\Delta} = 0$ for K-invariant fields $\hat{\varphi}$ on the reduced space M/K.[1]

As a simple and familiar example, consider the Laplace equation for functions on $\mathbf{R}^3$,

$$\partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi = 0. \tag{7.6}$$

The Laplace equation is invariant under the whole Euclidean group G consisting of translations and rotations. Consider the subgroup $K = SO(3)$ consisting of rotations. The rotationally invariant functions are of the form[2]

$$\varphi(x, y, z) = f(r), \quad r = \sqrt{x^2 + y^2 + z^2}. \tag{7.7}$$

Rotationally invariant solutions to the Laplace equation are characterized by a reduced field f satisfying a reduced differential equation on the reduced space $\mathbf{R}^+ = \mathbf{R}^3/SO(3)$ given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = 0. \tag{7.8}$$

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[1]The quotient space M/K is the set of orbits of K in M. Equivalently it is the set of equivalence classes of points in M where two points are equivalent if they can be related by an element of the transformation group K.

[2]Some boundary conditions have to be imposed at $r = 0$, but we will not worry about that here.

This is the principal reason one usually makes a "symmetry ansatz" for solutions to field equations which involves fields invariant under a subgroup K of the symmetry group G of the equations. It is not illegal to make other kinds of ansatzes, of course, but most will lead to inconsistent equations or equations with trivial solutions.

Having said all this, I should point out that just because you ask for group invariant solutions according to the above scheme it doesn't mean you will find any! There are two reasons for this. First of all, it may be that there are no (non-trivial) fields invariant with respect to the symmetry group you are trying to impose on the state. For example, consider the symmetry group $\varphi \to \varphi + const.$ we mentioned earlier for the massless KG equation. You can easily see that there are no functions which are invariant under that transformation group. Secondly, the reduced differential equation may have no (or only trivial) solutions, indicating that no (interesting) solutions exist with that symmetry. Finally, I should mention that not all states have symmetry - indeed the generic states are completely asymmetric. States with symmetry are special, physically simpler states than what you expect generically.

To summarize, field theories may have two types of symmetry. There may be a group G of symmetries of its laws – the symmetry group of the Lagrangian (and field equations). There can be symmetries of states, that is, there may be a transformation group (usually a subgroup of G) which preserves certain solutions to the field equations.

7.2   The "Mexican hat" potential

Let us now turn to a class of examples which serve to illustrate the preceding remarks and which we shall use to understand spontaneous symmetry breaking. We have actually seen these examples before.

We start by considering the real KG field with the double-well potential:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - (-\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4). \tag{7.9}$$

As usual, we are working in Minkowski space with inertial Cartesian coordinates. This Lagrangian admits the Poincaré group as a symmetry group. It also admits the symmetry $\varphi \to -\varphi$, which forms a 2 element discrete subgroup Z_2 of the symmetry group of the Lagrangian. In the Problems

in Chapter 3 we identified 3 simple solutions to the field equations for this Lagrangian:

$$\varphi = 0, \pm \frac{a}{b}, \tag{7.10}$$

where a and b are constants. These solutions are highly symmetric: they admit the whole Poincaré group of symmetries, as you can easily verify. Because Z_2 is a symmetry of the Lagrangian it must be a symmetry of the field equations, mapping solutions to solutions. You can verify that this is the case for the solutions (7.10). Thus the group consisting of the Poincaré group and Z_2 form a symmetry of the law governing the field. The 3 solutions in (7.10) represent 3 (of the infinite number of) possible solutions to the field equations – they are possible states of the field. The states represented by $\varphi = \pm \frac{a}{b}$ have Poincaré symmetry, but not Z_2 symmetry. In fact the Z_2 transformation maps between the solutions $\varphi = \pm \frac{a}{b}$. The state represented by $\varphi = 0$ has both the Poincaré and the Z_2 symmetry. The solution $\varphi = 0$ thus has more symmetry than the states $\varphi = \pm \frac{a}{b}$.

Let us consider the energetics of these highly symmetric solutions $\varphi = $ *const.* In an inertial reference frame with coordinates $x^\alpha = (t, x^i)$, the conserved energy in a spatial volume V for this non-linear field is easily seen (from Noether's theorem) to be

$$E = \int_V dV \left(\frac{1}{2}\varphi_{,t}^2 + \frac{1}{2}\varphi_{,i}\varphi_{,}^i - \frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4 \right). \tag{7.11}$$

You can easily check that the solutions given by $\varphi = 0, \pm \frac{a}{b}$ are critical points of this energy functional.

PROBLEM: Compute the first variation δE of the functional (7.11). Show that it vanishes when evaluated on fields $\varphi = 0, \pm \frac{a}{b}$.

Given the double-well shape of the potential, it might be intuitively clear that these solutions ought to represent global minima at $\varphi = \pm \frac{a}{b}$ and a local maximum at $\varphi = 0$. But let us see how one might try to prove it. Consider the change in the energy to quadratic-order in a displacement $u = u(t, x, y, z)$ from equilibrium in each case. The idea is that a displacement from a local minimum can only increase the energy and a displacement from a local maximum can only decrease the energy. To look into this, we assume that u has compact support for simplicity. We write $\varphi = \varphi_0 + u$ where φ_0 is

a constant and expand E to quadratic order in u. We get

$$E = -\frac{V}{4}\frac{a^4}{b^2} + \int_V dV \left\{ \frac{1}{2}(u_{,t}^2 + u_{,i}u_{,}^i) + a^2 u^2 \right\} + \mathcal{O}(u^3), \quad \text{when } \varphi_0 = \pm\frac{a}{b}$$

(7.12)

and

$$E = \int_V dV \left\{ \frac{1}{2}(u_{,t}^2 + u_{,i}u_{,}^i) - \frac{1}{2}a^2 u^2 \right\} + \mathcal{O}(u^3), \quad \text{when } \varphi_0 = 0. \quad (7.13)$$

Evidently, as we move away from $\varphi = \pm\frac{a}{b}$ the energy increases so that the critical points $\varphi = \pm\frac{a}{b}$ represent local minima. The situation near $\varphi = 0$ is less obvious. One thing is for sure: by choosing functions u which are suitably "slowly varying", one can ensure that the energy becomes negative in the vicinity of the solution $\varphi = 0$ so that $\varphi = 0$ is a saddle point if not a local maximum. We conclude that the state $\varphi = 0$ – the state of highest symmetry – is unstable and will not be seen "in the real world". On the other hand we expect the critical points $\varphi = \pm\frac{a}{b}$ to be stable. They are in fact the states of lowest energy and represent the possible ground states of the classical field. Evidently, the lowest energy is doubly degenerate.

Because the stable ground states have less symmetry than possible, one says that the ground state has "spontaneously broken" the (maximal) symmetry group $Z_2 \times$ Poincaré to just the Poincaré group. This terminology is useful, but can be misleading. The theory retains the full symmetry group as a symmetry of its laws, of course. Compare this with, say, the ordinary Klein-Gordon theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2. \tag{7.14}$$

You can easily check that the solution $\varphi = 0$ is the global minimum energy state of the theory and that it admits the full symmetry group $Z_2 \times$ Poincaré. There is evidently no spontaneous symmetry breaking here.

Let us now generalize this example by allowing the scalar field to become complex, $\varphi \colon M \to \mathbf{C}$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi^* - \left(-\frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4\right). \tag{7.15}$$

We assume $a \geq 0$, $b \geq 0$. This Lagrangian still admits the Poincaré symmetry, but the discrete Z_2 symmetry has been enhanced into a continuous $U(1)$

symmetry. Indeed, it is pretty obvious that the transformation

$$\varphi \to e^{i\alpha}\varphi, \quad \alpha \in \mathbf{R} \tag{7.16}$$

is a symmetry of $\mathcal{L}$. If you graph this potential in Cartesian coordinates (x, y, z) with $x = \Re(\varphi)$, $y = \Im(\varphi)$ and $z = V$, then you will see that the graph of the double well potential has been extended into a surface of revolution about z with the resulting shape being of the famous "Mexican hat" form. From this graphical point of view, the $U(1)$ phase symmetry of the Lagrangian corresponds to symmetry of the graph of the potential with respect to rotations in the x-y plane.

Let us again consider the simplest possible states of the field, namely, the ones which admit the whole Poincaré group as a symmetry group. These field configurations are necessarily constants, and you can easily check that in order for a constant field to solve the EL equations the constant must be a critical point of the potential viewed as a function in the complex plane. So, $\varphi = const.$ is a solution to the field equations if and only if

$$\frac{b^2}{4}\varphi|\varphi|^2 - \frac{1}{2}a^2\varphi = 0. \tag{7.17}$$

There is an isolated solution $\varphi = 0$, and (now assuming $b > 0$) a continuous family of solutions characterized by

$$|\varphi| = \frac{a}{b}. \tag{7.18}$$

The solution $\varphi = 0$ "sits" at the local maximum of the potential at the top of the "hat". The solutions (7.18) sit at the circular set of global minima of the potential. As you might expect, the transformation (7.16) maps the solutions (7.18) among themselves. To see this explicitly, write the general form of φ satisfying (7.18) as

$$\varphi = \frac{a}{b}e^{i\theta}, \quad \theta \in \mathbf{R}. \tag{7.19}$$

The $U(1)$ symmetry transformation (7.16) then corresponds to $\theta \to \theta + \alpha$. The $U(1)$ transformation is a symmetry of the state $\varphi = 0$. Thus the solution $\varphi = 0$ has more symmetry than the family of solutions characterized by (7.18).

The stability analysis of these highly symmetric states of the complex scalar field generalizes from the double well example as follows. (I will

spare you most of the details of the computations, but you might try to fill them in as a nice exercise.) In an inertial reference frame with coordinates $x^\alpha = (t, x^i)$, the conserved energy for this non-linear field is easily seen (from Noether's theorem) to be

$$E = \int d^3x \left(\frac{1}{2}|\varphi_{,t}|^2 + \frac{1}{2}\varphi_{,i}\,\varphi^{*i}_{,} - \frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4 \right). \tag{7.20}$$

You can easily check that the solutions given by $|\varphi| = 0, a/b$ are critical points of this energy functional. As before, the maximally symmetric state $\varphi = 0$ is unstable. The circle's worth of states (7.18) are quasi-stable in the following sense. Any displacement in field space yields a non-negative change in energy. To see this, write

$$\varphi = \rho e^{i\Theta}, \tag{7.21}$$

where ρ and Θ are spacetime functions. The energy takes the form

$$E = \int d^3x \left(\frac{1}{2}\rho_{,t}^2 + \frac{1}{2}\rho_{,i}\,\rho^{,i} + \frac{1}{2}\rho^2(\Theta_{,t}^2 + \Theta_{,i}\,\Theta^{,i}) - \frac{1}{2}a^2\rho^2 + \frac{1}{4}b^2\rho^4 \right). \tag{7.22}$$

The critical points of interest lie at $\rho = \frac{a}{b}$, $\Theta = const$. Expanding the energy in displacements $(\delta\rho, \delta\Theta)$ from equilibrium yields

$$E = -\frac{1}{4}\frac{a^4}{b^2} + \int d^3x \left(\frac{1}{2}\delta\rho_{,t}^2 + \frac{1}{2}\delta\rho_{,i}\,\delta\rho^{,i} + \frac{1}{2}\left(\frac{a}{b}\right)^2(\delta\Theta_{,t}^2 + \delta\Theta_{,i}\,\delta\Theta^{,i}) + a^2\delta\rho^2 \right) \tag{7.23}$$

Evidently, all displacements *except $\delta\rho = 0$, $\delta\Theta = const$.* increase the energy. The displacements $\delta\rho = 0$, $\delta\Theta = const$. do not change the energy, as you might have guessed, since they correspond to displacements along the circular locus of minima of the potential energy function. The states (7.18) are the lowest energy states – the ground states. Thus the lowest energy is infinitely degenerate – the set of ground states (7.18) is topologically a circle. That these stable states form a continuous family *and* have less symmetry than the unstable state will have some physical ramifications which we will unravel after we take a little detour.

7.3   Dynamics near equilibrium and Goldstone's theorem

A significant victory for classical mechanics is the complete characterization of motion near stable equilibrium in terms of normal modes and character-

istic frequencies of vibration. It is possible to establish analogous results in classical field theory via the process of *linearization.* This is even useful when one considers the associated quantum field theory: one can interpret the linearization of the classical field equations as characterizing particle states in the Fock space built on the vacuum state whose classical limit corresponds to the ground state about which one linearizes. If this seems a little too much to digest, that's ok – the point of this section is to at least make it easier to swallow.

Let us begin again with our simplest example: the real KG field with the double-well potential. Suppose that φ_0 is a given solution to the field equations. Any "nearby" solution we will denote by φ and we define the difference to be $\delta\varphi$.

$$\delta\varphi = \varphi - \varphi_0. \tag{7.24}$$

The field equation is the non-linear PDE:

$$\Box\varphi + a^2\varphi - b^2\varphi^3 = 0. \tag{7.25}$$

Using (7.24) we substitute $\varphi = \varphi_0 + \delta\varphi$. We then do 2 things: (1) we use the fact that φ_0 is a solution to the field equations; (2) we assume that $\delta\varphi$ is in some sense "small" so we can approximate the field equations in the vicinity of the given solution φ_0 by ignoring quadratic and cubic terms in $\delta\varphi$. We thus get the field equation linearized about the solution φ_0:

$$\Box\delta\varphi + (a^2 - 3b^2\varphi_0^2)\delta\varphi = 0. \tag{7.26}$$

This result can be obtained directly from the variational principle.

PROBLEM: Using (7.24) expand the action functional for (7.25) (see (7.9)) to quadratic order in $\delta\varphi$. Show that this approximate action, viewed as an action functional for the displacement field $\delta\varphi$, has (7.26) as its Euler-Lagrange field equation.

Evidently, the linearized equation (7.26) is a linear PDE for the displacement field $\delta\varphi$. In general, this linear PDE has variable coefficients due to the presence of φ_0. But if the given solution φ_0 is a constant in spacetime, the linearized PDE is mathematically identical to a Klein-Gordon equation for $\delta\varphi$ with mass given by $(-a^2 + 3b^2\varphi_0^2)$. The mass at the minima, $\varphi_0 = \pm a/b$, is $2a^2$. The mass at the maximum, $\varphi_0 = 0$, is $-a^2$. The negative mass-squared is a symptom of the instability of this state of the field.

From the way the linearized equation is derived, you can easily see that any displacement field $\delta\varphi$ constructed as an infinitesimal symmetry of the field equations will automatically satisfy the linearized equations when the fields being used to build $\delta\varphi$ satisfy the field equations. Indeed, this fact is the defining property of an infinitesimal symmetry of the field equations. Here is a simple example.

PROBLEM: Consider time translations $\varphi(t, x, y, z) \to \tilde{\varphi} = \varphi(t+\lambda, x, y, z)$. Compute the infinitesimal form $\delta\varphi$ of this transformation as a function of φ and its derivatives. Show that if φ solves the field equation coming from (7.9) then $\delta\varphi$ solves the linearized equation (7.26).

All the preceding results easily generalize to the $U(1)$-invariant complex scalar field case, but a new and important feature emerges which leads to an instance of (the classical limit of) a famous result known as "Goldstone's theorem". Let us go through it carefully.

Things will be most transparent in the polar coordinates (7.21). The Lagrangian density takes the form

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\rho\,\partial^\alpha\rho - \frac{1}{2}\rho^2\partial_\alpha\Theta\,\partial^\alpha\Theta + \frac{1}{2}a^2\rho^2 - \frac{1}{4}b^2\rho^4. \qquad (7.27)$$

The EL equations take the form

$$\Box\rho - \rho\partial_\alpha\Theta\,\partial^\alpha\Theta + a^2\rho - b^2\rho^3 = 0, \qquad (7.28)$$

$$\partial_\alpha(\rho^2\partial^\alpha\Theta) = 0. \qquad (7.29)$$

There are two things to notice here. First, the symmetry under $\Theta \to \Theta + const.$ is manifest – only derivatives of Θ appear. Second, the associated conservation law is the content of (7.29).

Let us consider the linearization of these field equations about the circle's worth of equilibria $\rho = a/b$, $\Theta = const.$ We could proceed precisely as before, of course. But it will be instructive to perform the linearization in the Lagrangian. To this end we write

$$\rho = \frac{a}{b} + \delta\rho, \quad \Theta = \theta + \delta\Theta, \qquad (7.30)$$

where $\theta = const.$, and we expand the Lagrangian to quadratic order in $\delta\rho$, $\delta\Theta$:

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\delta\rho\,\partial^\alpha\delta\rho - \frac{1}{2}\left(\frac{a}{b}\right)^2\partial_\alpha\delta\Theta\,\partial^\alpha\delta\Theta - \frac{1}{4}\frac{a^4}{b^2} - a^2\delta\rho^2 + \mathcal{O}(\delta\varphi^3). \quad (7.31)$$

Evidently, in a suitably small neighborhood of equilibrium the complex scalar field can be viewed as 2 real scalar fields $(\delta\rho, \delta\Theta)$; one of the fields $(\delta\rho)$ has a mass $m = a$ and the other field $(\delta\Theta)$ is massless.

To get a feel for what just happened, let us consider a very similar $U(1)$ symmetric theory, just differing in the sign of the quadratic potential term in the Lagrangian. The Lagrangian density is

$$\mathcal{L}' = -\frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi^* - \mu^2|\varphi|^2 - \frac{1}{4}b^2|\varphi|^4. \tag{7.32}$$

There is only a single Poincaré invariant critical point, $\varphi = 0$, which is a global minimum of the energy and which is also $U(1)$ invariant, so the $U(1)$ symmetry is *not* spontaneously broken in the ground state. In the vicinity of the ground state the linearized Lagrangian takes the simple form[3]

$$\mathcal{L}' = -\frac{1}{2}\partial_\alpha\delta\varphi\,\partial^\alpha\delta\varphi^* - \mu^2|\delta\varphi|^2 + \mathcal{O}(\delta\varphi^3). \tag{7.33}$$

Here of course we have the Lagrangian of a complex-valued KG field $\delta\varphi$ with mass μ; equivalently, we have two real scalar fields with mass μ.

To summarize thus far: With a complex KG field described by a potential such that the ground state shares all the symmetries of the Lagrangian, the physics of the theory near the ground state is that of a pair of real, massive KG fields. Using instead the Mexican hat potential, the ground state of the complex scalar field does not share all the symmetries of the Lagrangian – there is spontaneous symmetry breaking – and the physics of the field theory near equilibrium is that of a pair of scalar fields, one with mass and one which is massless.

To some extent, it is not too hard to understand *a priori* how these results occur. In particular, we can see why a massless field emerged from the spontaneous symmetry breaking. For Poincaré invariant solutions – which are constant functions in spacetime – the linearization of the field equations about a Poincaré invariant solution involves:

1. the derivative terms in the Lagrangian, which are quadratic in the fields and, since the Poincaré invariant state is constant, are the same in the linearization as for the full Lagrangian;

2. the Taylor expansion of the potential $V(\varphi)$ to second order about the constant equilibrium solution φ_0.

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[3]We do not use polar coordinates which are ill-defined at the origin.

Because of (1), the mass terms comes solely from the expansion of the potential in (2). Because of the $U(1)$ symmetry of the potential, through each point in the set of field values there will be a curve (with tangent vector given by the infinitesimal symmetry) along which the potential will not change. Because the symmetry is broken, this curve connects all the ground states of the theory. Taylor expansion about the ground state in this symmetry direction can yield only vanishing contributions because the potential has vanishing derivatives in that direction. Thus the broken symmetry direction(s) defines direction(s) in field space which correspond to massless fields in an expansion about equilibrium. This is the essence of the (classical limit of the) Goldstone theorem: to each broken continuous symmetry generator there is a massless field.

7.4 The Abelian Higgs model

The Goldstone result in conjunction with minimal coupling to an electromagnetic field yields a very important new behavior known as the "Higgs phenomenon". This results from the interplay of the spontaneously $U(1)$ symmetry and the local gauge symmetry. We start with a charged self-interacting scalar field coupled to the electromagnetic field; the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - V(\varphi). \tag{7.34}$$

We will again choose the potential so that spontaneous symmetry breaking occurs:

$$V(\varphi) = -\frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4. \tag{7.35}$$

To see what happens in detail, we return to the polar coordinates (7.21). The Lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{2}\partial_\alpha\rho\partial^\alpha\rho - \frac{1}{2}\rho^2(\partial_\alpha\Theta + qA_\alpha)(\partial^\alpha\Theta + qA^\alpha) + \frac{1}{2}a^2\rho^2 - \frac{1}{4}b^2\rho^4. \tag{7.36}$$

The Poincaré invariant ground state(s) can be determined as follows. As we have observed, a Poincaré invariant function φ is necessarily a constant. Likewise, it is too not hard to see that the only Poincaré invariant (co)vector is the the zero (co)vector $A_\alpha = 0$. Consequently, the Poincaré invariant

ground state, as before, is specified by

$$\rho = \frac{a}{b}, \quad \Theta = const. \quad A_\alpha = 0. \tag{7.37}$$

As before, the $U(1)$ symmetry of the theory is not a symmetry of this state, but instead maps these states among themselves. As before, we want to expanding to quadratic order about the ground state. To this end we write

$$A_\alpha = 0 + \delta A_\alpha, \quad \rho = \frac{a}{b} + \delta\rho, \quad \Theta = \Theta_0 + \delta\Theta, \tag{7.38}$$

where Θ_0 is a constant. We also define

$$B_\alpha = \delta A_\alpha + \frac{1}{q}\partial_\alpha \delta\Theta. \tag{7.39}$$

Ignoring terms of cubic and higher order in the displacements $(\delta A, \delta\rho, \delta\Theta)$ we then get

$$\mathcal{L} \approx -\frac{1}{4}(\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial^\alpha B^\beta - \partial^\beta B^\alpha) - \frac{1}{2}\left(\frac{aq}{b}\right)^2 B_\alpha B^\alpha$$
$$- \frac{1}{2}\partial_\alpha \delta\rho\, \partial^\alpha \delta\rho - \frac{1}{2}a^2 \delta\rho^2 \tag{7.40}$$

PROBLEM: Starting from (7.34) derive the results (7.36) and (7.40).

As you can see, excitations of ρ around the ground state are those of a scalar field with mass a, as before. To understand the rest of the Lagrangian (7.40) we need to understand the *Proca Lagrangian*:

$$\mathcal{L}_{Proca} = -\frac{1}{4}(\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial^\alpha B^\beta - \partial^\beta B^\alpha) - \frac{1}{2}\kappa^2 B_\alpha B^\alpha. \tag{7.41}$$

For $\kappa = 0$ this is just the usual electromagnetic Lagrangian. Otherwise...

PROBLEM: Assuming $\kappa \neq 0$, show that the Euler-Lagrange equations for B_α defined by (7.41) are equivalent to

$$(\Box - \kappa^2)B_\alpha = 0, \quad \partial^\alpha B_\alpha = 0. \tag{7.42}$$

Each component of B_α behaves as a Klein-Gordon field with mass κ. The divergence condition means that there are only 3 independent, positive energy scalar fields in play, and it can be shown that these field equations define an irreducible representation of the Poincaré group corresponding to a *massive* spin-1 field. Inasmuch as the Lagrangian reduces to the electromagnetic Lagrangian in the limit $\kappa \to 0$, one can interpret the Proca field theory as the classical limit of a theory of massive photons.[4] Notice that the Proca theory does not admit the gauge symmetry of electromagnetism because of the "mass term" in its Lagrangian. Gauge symmetry is a special feature of field theories describing massless particles.

The punchline here is that spontaneous symmetry breaking coupled with gauge symmetry leads to a field theory whose dynamics *near the ground state* is that of a massive scalar and a massive vector field. This is the simplest instance of the "Higgs phenomenon".

It is interesting to ask what became of the gauge symmetry and what became of the massless "Goldstone boson" which appeared when we studied spontaneous symmetry breaking of the scalar field without the gauge field. First of all, the $U(1)$ gauge symmetry of the theory – the symmetry of the law – is intact, but hidden in the linearization about a non-gauge invariant ground state. The direction in field space (the Θ direction) which would have yielded the massless Goldstone field is now a direction corresponding to the $U(1)$ gauge symmetry. The change of variables (7.39) which we used to put the (linearized) Lagrangian into the Proca form amounts to modifying A by a gauge transformation. The effect of this is that the field Θ provides a "longitudinal" mode to the field B, corresponding to its acquisition of a mass.

Finally, we point out that the Higgs phenomenon can be generalized considerably. We do not have time to go into it here, but the idea is as follows. Consider a system of matter fields with symmetry group of its Lagrangian being a continuous group G and with a ground state which breaks that symmetry to some subgroup of G. Couple these matter fields to gauge fields, the latter with gauge group which includes G. Excitations of the theory near the ground state will have the gauge fields corresponding to G acquiring a mass. This is precisely how the W and Z bosons of the weak interaction acquire their effective masses at (relatively) low energies.

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[4]The experimental upper limit on κ is very, very small – but it isn't zero!

7.5 PROBLEMS

1. While every symmetry of a Lagrangian is a symmetry of its EL equations, it is not true that every symmetry of the field equations is a symmetry of the Lagrangian. Consider the massless KG field. Show that the scaling transformation $\tilde{\varphi} = (const.)\varphi$ is a symmetry of the field equations but is *not* a symmetry of the Lagrangian.

2. Derive the result (7.5).

3. Compute the first variation δE of the functional (7.11). Show that it vanishes when evaluated on fields $\varphi = 0, \pm\frac{a}{b}$.

4. Using (7.24) expand the action functional to quadratic order in $\delta\varphi$. Show that this approximate action, viewed as an action functional for the displacement field $\delta\varphi$, has (7.26) as its Euler-Lagrange field equation.

5. Consider time translations $\varphi(t, x, y, z) \rightarrow \tilde{\varphi} = \varphi(t + \lambda, x, y, z)$. Compute the infinitesimal form $\delta\varphi$ of this transformation as a function of φ and its derivatives. Show that if φ solves the field equation coming from (7.9) then $\delta\varphi$ solves the linearized equation (7.26).

6. Starting from (7.34) derive the results (7.36) and (7.40).

7. Assuming $\kappa \neq 0$, show that the Euler-Lagrange equations for B_α defined by (7.41) are equivalent to

$$(\Box - \kappa^2)B_\alpha = 0, \quad \partial^\alpha B_\alpha = 0. \tag{7.43}$$

Chapter 8

The Dirac field

Recall that *spin* is an "internal degree of freedom", that is, an internal bit of structure for the particle excitations of a quantum field whose classical approximation we have been studying. The spin gets its name since it contributes to the angular momentum conservation law. In quantum field theory, the KG equation is an equation useful for describing relativistic particles with spin 0; the Maxwell and Proca equations are equations used to describe particles with spin 1. One of the many great achievements of Dirac was to devise a system of differential equations that is perfectly suited for studying the quantum theory of relativistic particles with "spin $\frac{1}{2}$". This is the *Dirac equation* and it used in quantum field theory to describe electrons, neutrinos, quarks, *etc.* Originally, this equation was obtained from trying to find a relativistic analog of the Schrödinger equation. The idea is that one wants a PDE that is first order in time, unlike the KG or the 4-potential form of Maxwell equations. Here we shall simply define the Dirac equation without trying to give details regarding its historical derivation. Then we will explore some of the simple field theoretic properties associated with the equation.

8.1   The Dirac equation

Given our flat Minkowski spacetime,

$$(M, g) = (\mathbf{R}^4, \eta), \tag{8.1}$$

the classical Dirac field can be viewed as a mapping

$$\psi \colon M \to \mathbf{C}^4. \tag{8.2}$$

To define the field equations, we introduce four linear transformations on $\mathbf{C}^4$,

$$\gamma_\mu \colon \mathbf{C}^4 \to \mathbf{C}^4, \quad \mu = 0, 1, 2, 3 \tag{8.3}$$

satisfying the anti-commutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I. \tag{8.4}$$

Such linear transformations define a representation of what is known as a *Clifford algebra*. We shall see why such transformations are relevant when we discuss how the Dirac fields provide a representation of the Poincaré group. The matrix forms of the γ_μ are intimately related to the Pauli matrices that we considered earlier. Recall that these matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.5}$$

We can define

$$\gamma_0 = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \tag{8.6}$$

and

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3. \tag{8.7}$$

Here we are using a 2×2 block matrix notation.

The γ-matrices are often called, well, just the "gamma matrices". They are also called the "Dirac matrices". The Dirac matrices satisfy

$$\gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i, \tag{8.8}$$

and

$$\gamma_0^\dagger = -\gamma_0^{-1}, \quad \gamma_i^\dagger = \gamma_i^{-1} \tag{8.9}$$

The Dirac matrices are not uniquely determined by their anti-commutation relations. It is possible to find other, equivalent representations. However, Pauli showed that if two sets of matrices γ_μ and γ_μ' satisfy the anti-commutation relations and the Hermiticity relations of the gamma matrices as shown above, then there is a unitary transformation U on $\mathbf{C}^4$ such that

$$\gamma_\mu' = U^{-1} \gamma_\mu U. \tag{8.10}$$

So, given the freedom to perform unitary transformations, *i.e.*, changes of basis preserving the inner product on $\mathbf{C}^4$, we have all the Dirac matrices.

With all this structure in place, we can now define the Dirac equation. Using matrix multiplication, it is given by

$$\gamma^\mu \partial_\mu \psi + m\psi = 0. \tag{8.11}$$

Here we raise the "spacetime index" on γ_μ with the flat spacetime metric:

$$\gamma^\mu = \eta^{\mu\nu}\gamma_\nu. \tag{8.12}$$

Also, the derivative operation is understood to be applied to each component of the Dirac field. Thus, if

$$\psi = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}, \tag{8.13}$$

then

$$\partial_\mu \psi = \begin{pmatrix} \partial_\mu \alpha \\ \partial_\mu \beta \\ \partial_\mu \gamma \\ \partial_\mu \delta \end{pmatrix}, \tag{8.14}$$

The Dirac equation is a coupled system of 4 complex (or 8 real) first-order, linear PDEs with constant coefficients (in Cartesian coordinates).

In the quantum description of the Dirac field the parameter m plays the role of the mass of the particle excitations of the field. Indeed, it is easy to see that if ψ is a solution of the Dirac equation, then each component of the Dirac field satisfies a KG-type of equation with mass m. To see this simply apply the *Dirac operator*

$$\gamma^\nu \partial_\nu + mI \tag{8.15}$$

to the left hand side of the Dirac equation (here I is the identity transformation on $\mathbf{C}^4$.). You get

$$(\gamma^\nu \partial_\nu + mI)(\gamma^\mu \partial_\mu \psi + m\psi) = \Box\psi + 2m\gamma^\mu \partial_\mu \psi + m^2\psi. \tag{8.16}$$

Now suppose that ψ is a solution of the Dirac equation so that

$$\gamma^\mu \partial_\mu \psi = -m\psi, \tag{8.17}$$

then we have that

$$0 = (\gamma^\nu \partial_\nu + mI)(\gamma^\mu \partial_\mu \psi + m\psi)$$
$$= (\square - m^2)\psi, \qquad\qquad (8.18)$$

where $\square$ and m^2 are operating on ψ component-wise, that is

$$\square \equiv \square I, \quad m^2 \equiv m^2 I. \qquad\qquad (8.19)$$

Note: That each component of the Dirac field satisfies the KG equation is necessary for ψ to satisfy the Dirac equation, but it is not sufficient.

The preceding computation allows us to view the Dirac operator as a sort of "square root" of the KG operator. This is one way to understand the utility of the Clifford algebra.

I have no choice but to assign you the following.

PROBLEM: Verify that the Dirac γ matrices shown in (8.6), (8.7) satisfy

$$\{\gamma_\mu, \gamma_\nu\} := \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I. \qquad\qquad (8.20)$$

(Although it is easy to check this by hand, I would recommend using a computer for this since it is good to learn how to make the computer do these boring tasks.)

8.2   Representations of the Poincaré group

The Dirac equation was simply postulated in the previous section. It is worth spending a little time trying to understand why it is the way it is and why one might find it useful. These questions can be answered on a number of levels, but probably the most outstanding aspect of the Dirac theory is that it provides a new *projective, unitary representation of the Poincaré group* called a *spinor representation.* To understand this statement, first recall the discussion in §2.3 where I sketched how solutions of a field equation can be used to build a Hilbert space of one particle states in quantum theory. Symmetry transformations of the theory, such as the Poincaré group, will map solutions of the field equation to other solutions. Physically, this represents the change in state which occurs when one views a physical system from a different inertial reference frame. The Poincaré group is thus represented by

unitary transformations in the quantum theory. The reason why this is important is that irreducible projective unitary representations of the Poincaré group are identified with elementary particles in quantum field theory. The Dirac equation, in particular, is used to describe elementary spin $1/2$ particles like the electron and quarks. These fermionic representations in the quantum theory can be seen already at the level of the classical field theory, and this is what we want to describe here. But first it will be instructive to spend a little time systematically thinking about the Poincaré group, its Lie algebra, and its representations.

To begin, we'll review the definition of the Poincaré group. Recall that the Poincaré group can be viewed as the transformation group $x \to x'$ which leaves the flat spacetime metric η invariant:

$$\eta_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}. \tag{8.21}$$

This group of transformations then takes the form

$$x'^{\alpha} = L^{\alpha}_{\beta} x^{\beta} + a^{\alpha}, \tag{8.22}$$

where a^{α} is any constant vector field (defining a spacetime translation) and L is an element of the Lorentz group (also called O(3,1)).

$$L^{\alpha}_{\gamma} L^{\beta}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}. \tag{8.23}$$

Elements of the Poincaré group are thus labeled by pairs (L, a).[1] The group composition law for two elements $g_1 = (L_1, a_1)$, $g_2 = (L_2, a_2)$ is defined by

$$(g_2 g_1 \cdot x)^{\alpha} = L^{\alpha}_{2\beta}(L^{\beta}_{1\gamma} x^{\gamma} + a^{\beta}_1) + a^{\beta}_2 = (L^{\alpha}_{2\beta} L^{\beta}_{1\gamma}) x^{\gamma} + L^{\alpha}_{2\beta} a^{\beta}_1 + a^{\alpha}_2, \tag{8.24}$$

so that

$$g_2 g_1 = (L_2 L_1, L_2 \cdot a_1 + a_2). \tag{8.25}$$

As mentioned earlier, in quantum theory it is the "projective unitary representations" of symmetry groups which are relevant. Recall from §3.17 that a representation of a group G is an assignment of a linear transformation on a vector space V (a matrix, if you like) to each element of G such that the linear transformations "behave" like the group. More precisely, we have a mapping $r\colon G \to GL(V)$ which satisfies the group homomorphism property:

$$r(g_1 \cdot g_2) = r(g_1) r(g_2). \tag{8.26}$$

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[1]As an exercise: In terms of this characterization of group elements, what is the identity and inverse transformation?

A *projective* representation relaxes this requirement to be

$$r(g_1 \cdot g_2) = c_{g_1, g_2}\, r(g_1) r(g_2), \tag{8.27}$$

where $c_{g,h}$ are constants which depend upon a pair of group elements. A representation is now the special case $c_{g,h} = 1$, $\forall\, g, h$. The projective representations are of interest because of quantum theory. Recall that the set of states of a physical system are identified with elements of a vector space (in fact a Hilbert space) with the proviso that any two linearly dependent vectors – two vectors which differ by a scalar multiple – define the same state. When considering the transformation of states by symmetries, one thus considers linear transformations, but the group homomorphism property is only required up to a constant rescaling, as in (8.27).

It can be shown that all irreducible unitary, projective representations of the Poincaré group are characterized by two parameters, the *spin* and the mass. Nature only seems to take advantage of the representations in which the mass is real and positive and the spin is a non-negative integer or half integer.[2] In quantum field theory, the spin of the representation is principally controlled by the geometrical type of the field. The principal role of the field equations is then to pick out the mass of the representation and ensure its irreducibility. So far we have considered a scalar field and a 1-form (or 4-vector field), corresponding to spin-0 and spin-1. The Dirac field is a new type of geometric object: a *spinor field*, corresponding to spin-$\frac{1}{2}$.

To better understand this last statement, we now want to see how the Poincaré group is represented as a transformation group of the classical fields. A *field representation* of the Poincaré group G of transformations assigns to each group element $g \in G$ a linear transformation Ψ_g of the fields,

$$\varphi \to \varphi' = \Psi_g(\varphi), \tag{8.28}$$

satisfying the representation property:

$$\Psi_{g_2}(\Psi_{g_1}(\varphi)) = \Psi_{g_2 g_1}(\varphi), \quad g_1, g_2 \in G. \tag{8.29}$$

To understand this point of view, consider our previously studied fields. The Klein-Gordon field was defined as a function or scalar field, $\varphi \colon \mathbf{R}^4 \to \mathbf{R}$, so

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[2]The other allowed representations have imaginary mass, or vanishing mass and continuous spin. Both mathematical possibilities appear to have unphysical properties.

that we have the representation of the Poincaré group on the space of KG fields given by:

$$\varphi(x) \to \varphi'(x) = \varphi(L^{-1} \cdot (x - a)). \tag{8.30}$$

It is easy to check that the space of KG fields, equipped with this transformation rule provides a field representation of the Poincaré group. This representation is called the spin-0 representation. The electromagnetic field was defined in terms of a 1-form A,

$$A \colon \mathbf{R}^4 \to T^*M. \tag{8.31}$$

The Poincaré group acts upon

$$A = A_\alpha dx^\alpha \tag{8.32}$$

via

$$A_\alpha(x) \to A'_\alpha(x) = L_\alpha^\beta A_\beta(L^{-1} \cdot (x - a)). \tag{8.33}$$

This transformation law leads to the spin-1 representation of the Poincaré group.

PROBLEM: Verify that the transformation laws (8.30) and (8.33) define representations of the Poincaré group.

We can build other representations of G via tensor products. Our goal now is to introduce the spinor representation.

8.3 The spinor representation

The Dirac field actually exemplifies a generalized notion of representation – called a "projective representation" – of the Poincaré group. To build this representation and to understand this "projective" business, it is easiest to work at the infinitesimal, Lie algebraic level.

Consider infinitesimal Poincaré transformations. See §3.13. Denote a Poincaré transformation by the pair (L, a) and let $(L(s), a(s))$ be a 1-parameter subgroup such that $s = 0$ is the identity transformation:

$$L_\beta^\alpha(0) = \delta_\beta^\alpha, \quad a^\alpha(0) = 0. \tag{8.34}$$

For any fixed vector b^α we can write

$$a^\alpha(s) = s\, b^\alpha. \tag{8.35}$$

Infinitesimally, $L(s)$ is characterized by a skew symmetric tensor,

$$\omega_{\alpha\beta} \equiv \eta_{\alpha\gamma}\omega^{\gamma}{}_{\beta} = -\omega_{\beta\alpha}, \tag{8.36}$$

such that

$$L^{\alpha}_{\beta}(s) = \exp(s\omega^{\alpha}{}_{\beta}) \approx \delta^{\alpha}_{\beta} + s\,\omega^{\alpha}{}_{\beta} + \cdots. \tag{8.37}$$

Thus ω and b label the infinitesimal generators of the 1-parameter family of Poincaré transformations.

The formal definition of the Lie algebra goes as follows.[3] The underlying vector space is the set of pairs $h = (\omega, b)$ with addition given by

$$h_1 + h_2 = (\omega_1 + \omega_2, b_1 + b_2). \tag{8.38}$$

Scalar multiplication of (ω, b) a real number λ is given by

$$\lambda \cdot (\omega, b) = (\lambda\omega, \lambda b). \tag{8.39}$$

The Lie bracket is the commutator of infinitesimal transformations:

$$[h_1, h_2] = h_3, \tag{8.40}$$

where

$$(\omega_3)^{\beta}_{\alpha} = [\omega_1, \omega_2]^{\beta}_{\alpha} = \omega^{\gamma}_{1\alpha}\omega^{\beta}_{2\gamma} - \omega^{\gamma}_{2\alpha}\omega^{\beta}_{1\gamma}, \tag{8.41}$$

$$b^{\alpha}_3 = \omega^{\alpha}_{1\beta}b^{\beta}_2 - \omega^{\alpha}_{2\beta}b^{\beta}_1 + b^{\alpha}_1 - b^{\alpha}_2. \tag{8.42}$$

Recall that a Lie group is (among other things) a differentiable manifold. Its component containing the identity is completely characterized by its infinitesimal generators via the commutation relations of the associated Lie algebra. What this means in the present example is that, up to discrete transformations like time reversal or spatial inversion, the group elements can be defined by pairs $(b^{\alpha}, \omega_{\alpha\beta})$ and the group multiplication law is characterized by the commutators among the infinitesimal generators. We shall therefore focus on the infinitesimal generators for most of our discussion.

We have seen that the action of the Poincaré group on a scalar field or a 1-form involves a transformation of the argument of the field along with

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[3]Recall that a Lie algebra is a vector space V equipped with a skew-symmetric bilinear mapping $[\cdot, \cdot] \colon V \times V \to V$ – the *Lie bracket* – which satisfies the Jacobi identity.

a transformation of the scalar or vector value of the field by the Lorentz transformation.[4] Spinor fields also follow this pattern. So we begin just by seeing how the value of a spinor field transforms under a Lorentz transformation. We are interested in representations of the group of transformations on the spinor space $\mathbf{C}^4$ associated to the matrices L^α_β. Infinitesimally, we are interested in representations on $\mathbf{C}^4$ of the Lie algebra of the anti-symmetric matrices $\omega^\alpha{}_\beta$, that is, we need to assign a linear transformation on $\mathbf{C}^4$ to each antisymmetric ω. We will insist that these linear transformations on $\mathbf{C}^4$ should obey the bracket relations (8.41) and (8.42) of the Lie algebra,[5] and they should also define symmetries of the Dirac equation. We will check the latter requirement a little later. As for the former, we will do this using the Dirac gamma matrices in what follows.

Define 6 independent linear transformations on $\mathbf{C}^4$ via

$$S_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]. \tag{8.43}$$

Given an infinitesimal Lorentz transformation specified by $\omega^\alpha{}_\beta$, we define its representative on $\mathbf{C}^4$ by

$$S(\omega) = \frac{1}{2}\omega^{\mu\nu} S_{\mu\nu}. \tag{8.44}$$

Let us compute the commutator of these infinitesimal generators on $\mathbf{C}^4$. We have

$$[S(\omega), S(\chi)] = \frac{1}{4}\omega^{\mu\nu}\chi^{\alpha\beta}[S_{\mu\nu}, S_{\alpha\beta}]. \tag{8.45}$$

PROBLEM: Using

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}I, \tag{8.46}$$

show that

$$[S(\omega), S(\chi)] = \frac{1}{4}\omega^{\mu\nu}\chi^{\alpha\beta}[S_{\mu\nu}, S_{\alpha\beta}] = \frac{1}{2}[\omega, \chi]^{\alpha\beta} S_{\alpha\beta} = S([\omega, \chi]). \tag{8.47}$$

This shows that the linear transformations $S(\omega)$ on the space $\mathbf{C}^4$ represent the Lie algebra of the Lorentz group. This representation is called the (infinitesimal) *spinor representation* of the (infinitesimal) Lorentz group. The

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[4]The transformation of the value of a scalar field is just the identity transformation.

[5]We shall see that this representation of the Lie algebra will lead to a projective representation of the corresponding group.

Lorentz transformation $L(\omega)$ defined by the matrix $\omega^{\alpha}{}_{\beta}$ is represented on vectors in $\mathbf{C}^4$ – "spinors" – by the matrix exponential $R(\omega)$:

$$R(\omega) = \exp\left\{S(\omega)\right\}. \tag{8.48}$$

The fact that $S(\omega)$ represents the Lie algebra ensures that the exponential will (projectively) represent the Lorentz group. As with scalar and vector fields, the translational part of the Poincaré group is represented trivially on the values of the spinor.

We have considered how $\mathbf{C}^4$ – the values of taken by a Dirac field – can provide a representation of the Lie algebra of the Poincaré group. We now extend this representation to a field representation (of the connected component of the identity of the Poincaré group) by adding in the Lorentz and translational transformations to the argument of the field and exponentiating. We specify a Poincaré transformation[6] by using a skew tensor ω, and a constant vector b. We define the transformation $\psi \to \psi'$ by

$$\psi'(x) = e^{S(\omega)}\psi(e^{-\omega} \cdot (x - b)). \tag{8.49}$$

Under an infinitesimal Poincaré transformation the Dirac field transforms according to

$$\delta\psi(x) = S(\omega)\psi(x) - \omega^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\psi - b^{\alpha}\partial_{\alpha}\psi. \tag{8.50}$$

To see that this provides a representation of the Lie algebra we consider the commutator of two successive infinitesimal transformations to find:

$$\delta_1\delta_2\psi - \delta_2\delta_1\psi = \delta_3\psi, \tag{8.51}$$

where

$$\omega^{\beta}_{3\,\alpha} = [\omega_1, \omega_2]^{\beta}{}_{\alpha}, \quad b^{\alpha}_3 = \omega^{\alpha}_{1\beta}b^{\beta}_2 - \omega^{\alpha}_{2\beta}b^{\beta}_1 + b^{\alpha}_1 - b^{\alpha}_2. \tag{8.52}$$

Comparing with (8.41) and (8.42) we see that the infinitesimal transformations (8.50) do indeed represent the Lie algebra of the Poincaré group.

As I have mentioned in the preceding discussion, the "spinor representation" of the Poincaré group we have been describing is not quite a true representation of the group, but rather a *projective* representation. (As we just saw, we *do* have a true representation of the Lie algebra.) To see why

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[6]We are restricting to the component connected to the identity transformation this way. For simplicity I will suppress the interesting discussion of how to represent the remaining transformations (time reversal and spatial reflection).

I say this, consider a Poincaré transformation consisting of a rotation by 2π about, say, the z-axis. You can easily check that this is generated by an infinitesimal transformation with $b^\alpha = 0$ and

$$\omega_{01} = \omega_{02} = \omega_{03} = \omega_{13} = \omega_{23} = 0, \quad \omega_{12} = 2\pi, \tag{8.53}$$

so that the matrix $\omega^\alpha{}_\beta$ is given by

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\pi & 0 \\ 0 & -2\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.54}$$

The corresponding Lorentz transformation matrix is given by

$$L^\alpha_\beta = [e^\omega]^\alpha_\beta = \delta^\alpha_\beta, \tag{8.55}$$

which is just showing that a rotation by 2π is the same as the identity transformation. (Check this calculation as a nice exercise.) The corresponding transformation $\psi \to \psi'$ of the spinor field is determined by

$$S(\omega) = \frac{1}{4}\omega^{\alpha\beta}[\gamma_\alpha, \gamma_\beta] = \pi[\gamma_1, \gamma_2] = i\pi \operatorname{diag}(1, 1, -1, -1). \tag{8.56}$$

So that

$$e^{S(\omega)} = -I, \tag{8.57}$$

and

$$\psi'(x) = -\psi(x). \tag{8.58}$$

PROBLEM: Verify (8.56) through (8.58).

The minus sign is not a mistake. The representative on spinors of a rotation by 2π is minus the identity. This means that we cannot have a true representation of the Poincaré group. The reason I say this is that the principal requirement for a representation, given in equation (8.29), is not satisfied. For example, consider two rotations about the z-axis by π. The composition of these two rotations is, for a true representation, the representative of the 2π rotation which is the identity. For our spinor "representation" we get minus the identity; the identity only appears after a rotation by 4π. It can be shown that the spinor "representation" we have constructed only differs

by a sign from a true representation in the manner just illustrated. One thus speaks of a "representation up to a sign" or a "projective representation". If the spinor field were intended as a directly measurable quantity, this projective representation would be a disaster since physical observables must be unchanged after a spatial rotation by 2π. However, the way in which the spinor field is used in physics – stemming from quantum theory in which states are rays in Hilbert space – is such that only quantities that provide true representations of the Poincaré group are deemed physically measurable. For example, the energy density of the Dirac field will be exhibited below; this quantity is quadratic in the Dirac field so it behaves properly under a 2π rotation.

One might wonder if the spinor "representation" of the Poincaré group constructed here is not the right thing to use. Maybe some other representation is needed. As we shall see a little later, the spinor representation is precisely what is needed for Poincaré symmetry of the Dirac Lagrangian and Dirac equation. Moreover, while the spinor field itself cannot be physically observable directly, the fact that it changes sign under a 2π rotation does lead to physical effects! For example, neutron interferometry experiments have been performed in which a beam of neutrons is split with one half of the beam passing through a magnetic field that rotates the spin state while the other half propagates with no rotation. The neutron interference pattern depends upon the magnetic field intensity in a fashion that only agrees with the theory if the Dirac field changes sign under a 2π rotation.[7]

8.4   Dirac Lagrangian

Let me now present a Lagrangian for the Dirac equation. To begin, we have the identity

$$(\gamma^0 \gamma^\mu)^\dagger = \gamma^0 \gamma^\mu. \tag{8.59}$$

It is then convenient to define

$$\bar{\psi} = i\psi^\dagger \gamma^0, \tag{8.60}$$

and the Lagrangian density takes the form

$$\mathcal{L} = -\frac{1}{2}\left\{ \bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi \right\} - m\bar{\psi}\psi. \tag{8.61}$$

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[7]See, for example, `arxiv.org/pdf/1601.07053.pdf` and references therein.

We view the Lagrangian as a function of ψ, $\bar{\psi}$ and their derivatives. As an exercise you should verify that $\mathcal{L}$ is real.

To compute the field equations we vary the fields and put the result in the standard Euler-Lagrange form:

$$\delta\mathcal{L} = -\delta\bar{\psi}(\gamma^\mu\partial_\mu\psi + m\psi) - (-\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi})\delta\psi - \frac{1}{2}\partial_\mu(\bar{\psi}\gamma^\mu\delta\psi - \delta\bar{\psi}\gamma^\mu\psi). \quad (8.62)$$

Evidently, the EL expression for $\bar{\psi}$ is

$$\mathcal{E}_{\bar{\psi}} = -(\gamma^\mu\partial_\mu\psi + m\psi), \quad (8.63)$$

so the EL equations coming from varying $\bar{\psi}$, namely $\mathcal{E}_{\bar{\psi}} = 0$, yields the Dirac equation for ψ The EL equations coming from varying ψ are determined by

$$\mathcal{E}_\psi = \partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi}. \quad (8.64)$$

PROBLEM: Show that the equations $\mathcal{E}_\psi = 0$ are equivalent to the Dirac equation for ψ.

8.5 Poincaré symmetry

Let us briefly check the Poincaré symmetry of the Lagrangian, at least infinitesimally. This will largely explain the specific form of the Dirac equation and it will justify the use of the spinor representation. Since the Lagrangian has no explicit dependence upon the coordinates x^α it is easy to see that under an infinitesimal spacetime translation,

$$x^\alpha \to x^\alpha + a^\alpha, \quad \delta\psi = -a^\alpha\psi_{,\alpha} \quad (8.65)$$

the Lagrangian changes by a divergence (as usual)

$$\delta\mathcal{L} = D_\mu(-a^\mu\mathcal{L}). \quad (8.66)$$

More interesting perhaps is the way in which the spinor representation makes the Lagrangian Lorentz-invariant. We need to consider how the Lagrangian changes when we make a transformation

$$\delta\psi = S(\omega)\psi - \omega^\alpha{}_\beta x^\beta\psi_{,\alpha}. \quad (8.67)$$

We have

$$\delta\mathcal{L} = -\frac{1}{2}[\delta\bar{\psi}\gamma^\mu\psi_{,\mu} + \bar{\psi}\gamma^\mu(D_\mu\delta\psi) - (D_\mu\delta\bar{\psi})\gamma^\mu\psi - \bar{\psi}_{,\mu}\gamma^\mu\delta\psi] - m(\delta\bar{\psi}\psi + \bar{\psi}\delta\psi).$$
(8.68)

To see how to proceed, we need a few small results. First, we have that

$$\gamma^\mu S(\omega) = S(\omega)\gamma^\mu + \omega^\mu{}_\alpha\gamma^\alpha.$$
(8.69)

Next, if $\delta\psi$ is as given in (8.67), then using (8.4) and (8.8) it follows that

$$\delta\bar{\psi} = -\bar{\psi}S(\omega) - \omega^\alpha{}_\beta x^\beta \bar{\psi}_{,\alpha}$$
(8.70)

Using these facts we can compute the change in the Lagrangian under an infinitesimal Lorentz transformation to be

$$\delta\mathcal{L} = D_\mu(-\omega^\mu{}_\nu x^\nu \mathcal{L}).$$
(8.71)

Thus the Lorentz transformations – at least those in the component connected to the identity in the Lorentz group – are a divergence symmetry of the Dirac Lagrangian.

PROBLEM: Prove (8.71).

8.6   Energy

It is instructive to have a look at the conserved energy for the Dirac field, both to illustrate previous technology and to motivate the use of Grassmann-valued fields.

From (8.66) the time translation symmetry is a divergence symmetry with

$$\delta\mathcal{L} = D_\mu(-\delta_0^\mu\mathcal{L}).$$
(8.72)

According to Noether's theorem, then, we have the conserved current

$$j^\mu = \frac{1}{2}[\bar{\psi}\gamma^\mu\psi_{,0} - \bar{\psi}_{,0}\gamma^\mu\psi] - \delta_0^\mu\mathcal{L}.$$
(8.73)

Next we take note of the following.

PROBLEM: Derive (8.73). Show that $\mathcal{L} = 0$ when the field equations hold. Therefore, modulo a trivial conservation law and a scaling by (-1), we can define the conservation of energy via

$$j^\mu = -\frac{1}{2}[\bar\psi\gamma^\mu\psi_{,0} - \bar\psi_{,0}\gamma^\mu\psi]. \tag{8.74}$$

PROBLEM: Check that the current (8.74) is conserved when the Dirac equation holds.

The energy density is given by

$$\rho \equiv j^0 = -\frac{1}{2}(\bar\psi\gamma^0\partial_0\psi - \partial_0\bar\psi\gamma^0\psi) = \frac{i}{2}(\partial_0\psi^\dagger\psi - \psi^\dagger\partial_0\psi). \tag{8.75}$$

Let us now expose a basic difficulty with the classical Dirac field theory: the energy density is not bounded from below (or above). Contrast this with the energy density of the KG field, (3.12), or with the energy density of the EM field. To reveal this problem for the classical Dirac field, let us write down an elementary solution to the Dirac equation and evaluate its energy density.

Consider a Dirac field that only depends upon the time $x^0 = t$, that is, we consider a Dirac field which has spatial translation symmetry:

$$\partial_i\psi = 0. \tag{8.76}$$

The Dirac equation for $\psi = \psi(t)$ is now

$$\gamma^0\partial_t\psi + m\psi = 0. \tag{8.77}$$

Let us write

$$\psi = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \\ d(t) \end{pmatrix}. \tag{8.78}$$

Since $\gamma_0 = \text{diag}(i, i, -i, -i)$ we have that $\gamma^0 = \text{diag}(-i, -i, i, i)$ so that the Dirac equation reduces to the decoupled system

$$-i\dot a + ma = 0$$
$$-i\dot b + mb = 0$$
$$i\dot c + mc = 0$$
$$i\dot d + md = 0. \tag{8.79}$$

The solution is then

$$\psi = \begin{pmatrix} a_0 e^{-imt} \\ b_0 e^{-imt} \\ c_0 e^{imt} \\ d_0 e^{imt} \end{pmatrix}, \tag{8.80}$$

where (a_0, b_0, c_0, d_0) are constants representing the value of ψ at $t = 0$.

Let us compute the energy density of this solution. We have

$$\begin{aligned} \rho &= \frac{i}{2}(\psi^\dagger \partial_0 \psi - \partial_0 \psi^\dagger \psi) \\ &= m(|a_0|^2 + |b_0|^2 - |c_0|^2 - |d_0|^2). \end{aligned} \tag{8.81}$$

This energy is clearly not bounded from below. By choosing initial conditions with c_0 and d_0 sufficiently large in magnitude we can make the energy as negative as we wish. Physically, this is a disaster since it means that one can extract an infinite amount of energy from the Dirac field by coupling it to other dynamical systems. Note that, while we are free to consider redefining the energy density by a change of sign, this won't help because ρ is not bounded from above either. One could try to avoid this problem by simply decreeing that one must only consider solutions in which $c = d = 0$. This would work if the only thing in the universe were the Dirac field. But when the field interacts with its environment – which it must do to be observable! – it can be shown that as time evolves one eventually gets solutions with $c \neq 0$, $d \neq 0$, so this escape route fails. This difficulty reflects the limited physical domain of the purely classical field theory for fermions. (Contrast this with electrodynamics, where there exist a multitude of quantum coherent states which exhibit classical behavior and provide a wide physical domain of validity of the classical field theory).

8.7   Anti-commuting fields

There is a novel way to redefine the mathematical structure of the classical Dirac field theory so that difficulties such as were highlighted above do not arise. This is done by appealing to the underlying quantum field theory. In quantum field theory, roughly speaking, one replaces the set-up where fields take values in a commutative algebra (*e.g.,* real numbers, complex numbers, *etc.*) with a set-up where fields take values in an operator algebra

on a Hilbert space.[8] Because operator algebras need not be commutative, this means that, in particular, one must keep track of the ordering of the operators within products. For example, for a KG quantum field,

$$\varphi(x)\varphi(x') \neq \varphi(x')\varphi(x). \tag{8.82}$$

Up until now, we have always assumed that fields could be built out of functions which take values in a commutative algebra (*e.g.*, real numbers). Insofar as the classical field theory is the "classical limit" of a quantum field theory, it turns out that this commutative algebra assumption is reasonable for bosonic fields (integer spin). But for fermionic fields, *e.g.,* spin $1/2$ fields like the Dirac field, one can show that the "classical limit" of the operator algebra leads to fields which *anti-commute*. This leads one to formulate a classical Dirac field theory as a theory of *Grassmann-valued spinors*. In the following I will briefly outline how this goes.

A finite-dimensional Grassmann algebra $\mathcal{A}$ is a (real or complex) vector space V with a basis χ_α, $\alpha = 1, 2, \ldots, n$, and equipped with a product such that

$$\chi_\alpha \chi_\beta = -\chi_\beta \chi_\alpha. \tag{8.83}$$

Notice that this implies

$$(\chi_\alpha)^2 = 0. \tag{8.84}$$

The algebra is, as usual, built up by sums and products of the χ_α. Every element $\Omega \in \mathcal{A}$ is of the form

$$\Omega = \omega_0 + \omega^\alpha \chi_\alpha + \omega^{\alpha\beta} \chi_\alpha \chi_\beta + \cdots + \omega^{\alpha_1 \cdots \alpha_n} \chi_{\alpha_1} \cdots \chi_{\alpha_n}, \tag{8.85}$$

where ω_0 can be identified with the field (real or complex) of the vector space, the coefficients $\omega^{\alpha_1 \cdots}$ are totally antisymmetric. The terms of odd degree in the χ_α are "anti-commuting" while the terms of even degree are "commuting". If the vector space is complex, then there is a notion of complex conjugation in which

$$(\alpha\beta)^* = \beta^* \alpha^*, \quad \alpha, \beta \in \mathcal{A}. \tag{8.86}$$

Note that if α and β are real and anti-commuting,

$$\alpha^* = \alpha, \quad \beta^* = \beta, \tag{8.87}$$

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[8]More precisely: fields are viewed as "unbounded, self-adjoint operator-valued distributions on spacetime".

then their product is pure imaginary in the sense that

$$(\alpha\beta)^* = \beta^*\alpha^* = \beta\alpha = -\alpha\beta. \tag{8.88}$$

The usual exterior algebra of forms over a vector space is an example of a Grassmann algebra.

In the Dirac theory with anti-commuting fields, one builds $\mathcal{A}$ from $V = \mathbf{C}^4$. The Dirac fields are then considered as maps from spacetime into V. Of course, one cannot directly measure a "Grassmann number", but the "commuting" part of the algebra can be interpreted in terms of real numbers, which is how one interprets observable aspects of the classical field theory.

Let us return to our simple example involving the energy density associated to spatially homogeneous solutions to the Dirac equation. Given the anti-commuting nature of the values of the Dirac field, it follows that the solution to the field equations we wrote down above has a, b, c, d now being interpreted as anti-commuting Grassmann numbers. It then becomes an issue as to what order to put the factors in the definition of various quantities, say, the energy density. To answer this question it is best to appeal to the underlying quantum field theory, from which it turns out that the correct version of (8.81) is

$$\rho = (a^*a + b^*b - cc^* - dd^*) = (a^*a + b^*b + c^*c + d^*d). \tag{8.89}$$

It turns out that in the classical limit of the quantum theory each of the quantities a^*a, b^*b, c^*c, d^*d correspond to positive real numbers whence the energy density is now positive.

8.8   Coupling to the electromagnetic field

One of the most successful theories in all of physics is the quantum field theory describing the interactions of electrons, positrons, and photons. This is quantum electrodynamics (QED). While QED is fundamentally a quantum theory, and really should be studied as such, we have enough field-theoretic tools to write down the classical Lagrangian using the principal of local gauge invariance. Let us have a quick look. You may want to review Chapter 6 to remind yourself of the strategy.

The Dirac field admits the transformation group $U(1)$ in much the same way as the complex KG field. We have for any $\psi \in \mathbf{C}^4$

$$\psi \to e^{i\alpha}\psi, \quad \alpha \in \mathbf{R}, \tag{8.90}$$

or, infinitesimally,

$$\delta\psi = i\psi. \tag{8.91}$$

You can easily check that this transformation is a symmetry of the Dirac Lagrangian density (8.61). Noether's first theorem then provides a conserved current.

PROBLEM: Show that the conserved current associated to the variational symmetry (8.90) by Noether's theorem is given by

$$j^{\alpha} = i\overline{\psi}\gamma^{\alpha}\psi. \tag{8.92}$$

Verify that this vector field is divergence-free when the Dirac equation is satisfied.

This current can be interpreted as the electric 4-current of the field in the absence of interaction with an electromagnetic field. To introduce that interaction we "gauge" the global $U(1)$ symmetry to make it "local". We do this by introducing the electromagnetic potential $A = A_{\alpha}dx^{\alpha}$ and defining a "covariant derivative",

$$\mathcal{D}_{\mu}\psi = \partial_{\mu}\psi - iqA_{\mu}\psi. \tag{8.93}$$

Here q represents the electromagnetic "coupling constant". The local $U(1)$ gauge transformation is defined as

$$\psi \to \tilde{\psi} = e^{iq\alpha(x)}\psi, \quad A_{\mu} \to \tilde{A}_{\mu} = A_{\mu} + \partial_{\mu}\alpha(x), \tag{8.94}$$

where α is any function on spacetime. Under a gauge transformation the covariant derivative transforms homogeneously:

$$\tilde{\mathcal{D}}_{\mu}\tilde{\psi} = e^{iq\alpha(x)}\mathcal{D}_{\mu}\psi. \tag{8.95}$$

Consequently, if we replace ordinary derivatives with covariant derivatives in the Dirac Lagrangian density the resulting Lagrangian density will admit the local $U(1)$ gauge symmetry and we thereby introduce the coupling of the Dirac field to the electromagnetic field. Adding this Lagrangian density to the usual electromagnetic Lagrangian we get the Lagrangian density for the classical limit of QED:

$$\mathcal{L} = -\frac{1}{2}\left\{\bar{\psi}\gamma^{\mu}\mathcal{D}_{\mu}\psi - (\mathcal{D}_{\mu}\bar{\psi})\gamma^{\mu}\psi\right\} - m\bar{\psi}\psi - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}. \tag{8.96}$$

Here we have defined
$$\mathcal{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} + iqA_\mu \bar{\psi}. \tag{8.97}$$

PROBLEM: Calculate the field equations for ψ and A from this Lagrangian density. Notice that, unlike the case with scalar electrodynamics, the electric 4-current that acts as a source for the electromagnetic field does not depend upon the electromagnetic field.

8.9   PROBLEMS

1. Verify that the Dirac γ matrices shown in (8.6), (8.7) satisfy
$$\{\gamma_\mu, \gamma_\nu\} := \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I.$$

(Although it is easy to check this by hand, I would recommend using a computer for this since it is good to learn how to make the computer do these boring tasks.)

2. Verify that the transformation laws (8.30) and (8.33) define representations of the Poincaré group.

3. Using
$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I,$$
show that
$$\frac{1}{4}\omega^{\mu\nu}\chi^{\alpha\beta}[S_{\mu\nu}, S_{\alpha\beta}] = \frac{1}{2}[\omega, \chi]^{\alpha\beta} S_{\alpha\beta} = S([\omega, \chi]).$$

4. Verify (8.56) through (8.58).

5. Show that the equations $\mathcal{E}_\psi = 0$ (see (8.64)) are equivalent to the Dirac equation for ψ.

6. Prove (8.71).

7. Derive (8.73). Show that $\mathcal{L} = 0$ when the field equations hold. Therefore, modulo a trivial conservation law, we can define the conservation of energy via
$$j^\mu = -\frac{1}{2}[\bar{\psi}\gamma^\mu \psi_{,0} - \bar{\psi}_{,0}\gamma^\mu \psi].$$

8. Check that the current (8.74) is conserved when the Dirac equation holds.

9. Show that the conserved current associated to the variational symmetry (8.90) by Noether's theorem is given by

$$j^\alpha = i\overline{\psi}\gamma^\alpha\psi.$$

Verify that this vector field is divergence-free when the Dirac equation is satisfied.

10. Calculate the field equations for ψ and A from the Lagrangian density (8.96). Notice that, unlike the case with scalar electrodynamics, the electric 4-current that acts as a source for the electromagnetic field does not depend upon the electromagnetic field.

Chapter 9

Non-Abelian gauge theory

We were introduced to some of the underpinnings of what is now called "gauge theory" when we studied the electromagnetic field. Let us now consider another gauge theory, often called "Yang-Mills theory" after its inventors. It is also sometimes called "non-Abelian gauge theory" since the gauge transformations are coming as a non-Abelian generalization of the $U(1)$ type of gauge transformations of electromagnetic theory. The Yang-Mills (YM) theory was originally conceived (in 1954) as way of formulating interactions among protons and neutrons. This approach turned out not to bear much fruit (as far as I know), but the structure of the theory was studied for its intrinsic field theoretic interest by a relatively small number of physicists through the 1960's. During this time, theories of the Yang-Mills type were used to slowly devise a scheme for describing a theory of electromagnetic and weak interactions. This work was performed by Glashow, then Weinberg and also Salam. All three eventually won the Nobel prize for this work. These theories became truly viable when it was shown by 't Hooft that the non-Abelian gauge theories were well-behaved from the point of view of perturbative quantum field theory. It wasn't long after that that people found how to describe the strong interactions using a non-Abelian gauge theory. It is even possible to think of the gravitational interactions as described by Einstein's general relativity as a sort of non-Abelian gauge theory – although this requires some generalization of the term "gauge theory". Certainly one can think of Maxwell theory as a very special case of a non-Abelian gauge theory. So, one can take the point of view that all the interactions that are observed in nature can be viewed as an instance of a gauge theory. This is certainly ample motivation for spending some additional time studying them.

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The non-Abelian gauge theory substitutes a non-Abelian Lie group[1] for the group $U(1)$ that featured in electrodynamics. Although, when among friends, one often uses the terms "non-Abelian gauge theory" or "Yang-Mills theory" or just "gauge theory" interchangeably, one properly distinguishes the Yang-Mills theory as the specialization of non-Abelian gauge theory to the gauge group built from SU(2), as Yang and Mills originally did. We shall try to make this distinction as well.

Non-Abelian gauge theory is a theory of interactions among matter just as electrodynamics is. Following Weyl, we were able to view electromagnetic interactions between electrically charged matter as a manifestation of a "local $U(1)$ symmetry" of matter. Recall that the charged KG field acquired its conserved charge by virtue of the $U(1)$ phase symmetry. This symmetry acted "globally" in the sense that the symmetry transformation shifted the phase of the complex KG field by the same amount throughout spacetime. The coupling of the charged KG field to the EM field can be viewed as corresponding to the requirement that this phase change could be made independently (albeit smoothly) at each spacetime event. Making the simplest "minimal" generalization of the charged KG Lagrangian to incorporate this "local" gauge symmetry involved introducing the Maxwell field to define a connection, *i.e.,* a gauge covariant derivative. This gives the correct Lagrangian for the charged KG field in the presence of an electromagnetic field. The dynamics of the EM field itself could then be incorporated by adjoining to this Lagrangian the simplest gauge invariant Lagrangian for the Maxwell field. This leads to the theory of scalar electrodynamics. Similar constructions in §8 led to a classical version of QED.

We have already discussed another kind of "charged KG" theory in which one has not a single conserved current but three of them, corresponding to a global SU(2) symmetry group. What happens if we try to make that symmetry local? This is one way to "invent" the YM theory.

9.1   SU(2) doublet of KG fields, revisited

I will use the idea of "localizing" a symmetry to generate an interaction. So, we begin by reviewing the field theory of two complex KG fields admitting an SU(2) symmetry. See §3.18.

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[1]For physical reasons pertaining to the quantum field theory, the gauge group is usually built on a compact, semi-simple Lie group.

The fields are

$$\varphi \colon M \to \mathbf{C}^2. \tag{9.1}$$

You can think of φ as a column vector with two complex-valued functions for its components. SU(2) is the group of linear transformations on $\mathbf{C}^2$ preserving the inner product:

$$\langle \alpha, \beta \rangle = \alpha^\dagger \beta, \quad \alpha, \beta \in \mathbf{C}^2, \tag{9.2}$$

$$\langle U\alpha, U\beta \rangle = \langle \alpha, \beta \rangle \quad \Longleftrightarrow \quad U^\dagger = U^{-1}, \tag{9.3}$$

and with unit determinant

$$\det U = 1. \tag{9.4}$$

We can express these linear transformations as matrices:

$$U = U(\theta, n) = \cos\theta \, I + i \sin\theta \, n^i \sigma_i, \tag{9.5}$$

where

$$\mathbf{n} = (n^1, n^2, n^3), \quad (n^1)^2 + (n^2)^2 + (n^3)^2 = 1, \tag{9.6}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.7}$$

are the *Pauli matrices.* Note that there are three free parameters in this group, corresponding to θ and the two free parameters defining n^i.

The group SU(2), as represented on $\mathbf{C}^2$, acts on the fields in the obvious way:

$$\varphi(x) \to U\varphi(x). \tag{9.8}$$

One parameter subgroups $U(\lambda)$ are defined by any curve in the 3-d parameter space associated with θ and $\mathbf{n}$. As an example, set $\mathbf{n} = (1, 0, 0)$ and $\theta = \lambda$ to get

$$U(\lambda) = \begin{pmatrix} \cos\lambda & i\sin\lambda \\ i\sin\lambda & \cos\lambda \end{pmatrix}. \tag{9.9}$$

The infinitesimal form of the SU(2) transformation is

$$\delta\varphi = \tau\varphi, \tag{9.10}$$

where τ is an anti-Hermitian, traceless matrix obtained from a 1-parameter group $U(\lambda)$ by

$$\tau = \left(\frac{dU}{d\lambda} \right)_{\lambda=0}. \tag{9.11}$$

Note that

$$\delta\varphi^\dagger = -\varphi^\dagger\tau. \tag{9.12}$$

We can write

$$\tau = -ia^j\sigma_j, \tag{9.13}$$

where $a^i \in \mathbf{R}^3$. Thus infinitesimal transformations can be identified with a three-dimensional vector space with a basis

$$\delta_j\varphi = i\sigma_j\varphi. \tag{9.14}$$

The infinitesimal generators of the transformation group SU(2) are given by the anti-Hermitian matrices $\tau = -ia^i\sigma_i$. The commutator algebra of these matrices defines a representation of the Lie algebra of SU(2), which we shall denote by su(2). A basis e_k for this matrix representation of su(2) is provided by $\frac{1}{2i}$ times the Pauli matrices:

$$e_k = -\frac{i}{2}\sigma_k, \quad \tau = a^k e_k, \ a^k \in \mathbf{R}^3. \tag{9.15}$$

It is easy to check that in this basis the structure constants of su(2) are given by

$$[e_i, e_j] = \epsilon_{ij}{}^k e_k, \tag{9.16}$$

where ϵ_{ijk} is the three-dimensional Levi-Civita symbol and indices are raised and lowered using the Kronecker delta.

PROBLEM: Verify that the e_k in (9.15) satisfy the su(2) Lie algebra (9.16) as advertised.

An SU(2)-invariant Lagrangian which generalizes that of the KG and charged KG theories is given by (in flat spacetime, with metric η)

$$\mathcal{L} = -\eta^{\alpha\beta}\langle\varphi_{,\alpha}, \varphi_{,\beta}\rangle - m^2\langle\varphi, \varphi\rangle, \tag{9.17}$$

You can see quite easily that the transformation $\varphi \to U\varphi$ yields $\mathcal{L} \to \mathcal{L}$; this is the global SU(2) symmetry. This symmetry is responsible for three independent conservation laws, which arise via Noether's theorem applied to the variational symmetries

$$\delta_j\varphi = i\sigma_j\varphi. \tag{9.18}$$

The 3 conserved currents are

$$j_k^\beta = i(\varphi^{\dagger,\beta}\sigma_k\varphi - \varphi^\dagger\sigma_k\varphi^{,\beta}), \quad k = 1, 2, 3. \tag{9.19}$$

These currents "carry" the SU(2) charge possessed by the scalar fields.

PROBLEM: (a) show that the Lagrangian (9.17) is invariant with respect to the transformation (9.8); (b) compute its Euler-Lagrange equations; (c) derive the conserved currents (9.19) from Noether's theorem; (d) verify that the currents (9.19) are divergence-free when the Euler-Lagrange equations are satisfied.

PROBLEM: Show that if any one of the currents (9.19) is conserved, then the other 2 are automatically conserved. (*Hint:* Consider the behavior of the currents under SU(2) transformations of the fields.)

9.2 Local SU(2) symmetry

The SU(2) symmetry group studied in the previous section tacitly uses a prescription for comparing the values of the fields at each point of spacetime. To be sure, at each spacetime event the field takes its value in a copy of $\mathbf{C}^2$, but there are infinitely many ways to put all these different $\mathbf{C}^2$ spaces in correspondence – as many ways as there are linear isomorphisms of this vector space. You can think of a choice of correspondence between all these complex vector spaces as a sort of preferred choice of internal reference frame for the fields φ.

In analogy with the case of scalar electrodynamics, one can introduce an interaction between conserved charges by insisting that there is no fixed, *a priori* rule for comparing SU(2) phases from event to event in spacetime. To do this we add the "rule" as a new variable in the theory, which gets interpreted as the gauge field mediating the interaction between charges. Thus we are, in effect, following the path of Einstein in his general theory of relativity. There, by insisting that there is no privileged spacetime reference frame at each point he was able to correctly describe the gravitational interaction. In YM theory we use a similar relativity principle, but now it is regarding the "internal frame" on $\mathbf{C}^2$. We do this in two steps: (1) introduce a fixed but arbitrary connection A or covariant derivative $\mathcal{D}$, which makes explicit how

we are comparing values of φ at infinitesimally separated points, (2) treat the connection as a new field – a gauge field – and give it its own Lagrangian. The resulting field equations determine the "matter fields" φ and the "gauge fields" A. The result is an interacting theory of charges and gauge fields dictated by a sort of "general relativity of SU(2) phases".

So, the upshot of the preceding discussion is that we seek to build a theory in which the "global" symmetry transformation

$$\varphi(x) \to U\varphi(x), \tag{9.20}$$

$$U \colon \mathbf{C}^2 \to \mathbf{C}^2, \quad U^\dagger = U^{-1}, \quad \det U = 1, \tag{9.21}$$

becomes a "local" symmetry:

$$\varphi(x) \to U(x)\varphi(x), \tag{9.22}$$

$$U(x) \colon M \to SU(2), \quad U^\dagger(x) = U^{-1}(x), \quad \det U(x) = 1. \tag{9.23}$$

It is easy to see that the mass term in our Lagrangian (9.17) for φ allows this transformation as a symmetry. As in the case of SED, the derivative terms do not allow this transformation to be a symmetry. This reflects the fact that the derivatives are defined using a fixed notion of how to compare the values of φ at two neighboring points. The local SU(2) transformation can be viewed as redefining the method of comparison by redefining the basis of $\mathbf{C}^2$ differently at each point of M; obviously the partial derivatives will respond to this redefinition.

To generalize the derivative to allow for an arbitrary method of comparison of values of φ from point to point of M we introduce the "gauge field" or "connection", or "Yang-Mills (YM) field" A, which is a *Lie algebra-valued 1-form*. What this means is that, at each point of the spacetime M, A is a linear mapping from the tangent space at that point to the representation of the Lie algebra su(2) discussed above. This way, one can use A to define an infinitesimal SU(2) transformation at any point which tells how to compare the values of the SU(2) phase of φ as one moves in any given direction. Since A is defined as a linear mapping on vectors, we can write it in terms of 1-forms,

$$A = A_\mu dx^\mu, \tag{9.24}$$

where, for each value of μ, A_μ is a map from M to su(2). We write:

$$A_\mu = A_\mu^i e_i, \tag{9.25}$$

where e_i are the basis for su(2) defined in (9.15). Thus, if you like, you can think of the gauge field as $4 \times 3 = 12$ real fields $A_\mu^k(x)$ labeled according to their spacetime (index $\mu = 0, 1, 2, 3$) and "internal" su(2) structure (index $k = 1, 2, 3$).

The *gauge covariant derivative* of φ is now defined as[2]

$$\mathcal{D}_\mu \varphi = \partial_\mu \varphi + A_\mu \varphi. \tag{9.26}$$

A fixed, given gauge field A_μ just defines another way to compare phases of fields at different points. Nothing is gained symmetry-wise by its introduction. Indeed, under a gauge transformation

$$\varphi(x) \to U(x)\varphi(x) \tag{9.27}$$

we have to redefine A_μ via

$$A_\mu \to U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x). \tag{9.28}$$

This is designed to give

$$\begin{aligned}
\mathcal{D}_\mu \varphi &\to \left\{ \partial_\mu + \left[U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x) \right] \right\} (U(x)\varphi) \\
&= U(x) \left(\partial_\mu + A_\mu \right) \varphi \\
&= U(x)\mathcal{D}_\mu \varphi.
\end{aligned} \tag{9.29}$$

Granted this set-up, we can then define the Lagrangian for φ using the "minimal coupling prescription

$$\partial_\mu \to \mathcal{D}_\mu, \tag{9.30}$$

so that we have

$$\mathcal{L} = -\eta^{\alpha\beta} \langle \mathcal{D}_\alpha \varphi, \mathcal{D}_\beta \varphi \rangle - m^2 \langle \varphi, \varphi \rangle. \tag{9.31}$$

Since, at the moment, we view the gauge field as fixed/prescribed, strictly speaking, this Lagrangian is no more gauge invariant than our original Lagrangian (9.17),[3] which it includes as the special case $A = 0$, but it has the virtue of allowing for any choice of connection between $\mathbf{C}^2$ at neighboring points. Physically, this Lagrangian defines the dynamics of the fields φ under the influence of the "force" due to any given gauge field A.

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[2]For simplicity we absorb a coupling constant into the definition of A.

[3]This is completely analogous to our discussion of the behavior of the KG Lagrangian under a coordinate transformation, back in §2.9.3.

I think you can imagine that we could play the preceding game whenever one is given (1) some fields $\varphi \colon M \to V$, where V is some vector space, and (2) given a representation of SU(2) on this vector space, (3) given a globally SU(2)-invariant Lagrangian. This is all we really used here. We will see an example of some of this in the following section. One can in fact generalize this construction further to the case where we are given the representation of any Lie group G on a vector space V and given a G-invariant Lagrangian for mappings $\varphi \colon M \to V$.

9.3   Infinitesimal gauge transformations

Let us consider the infinitesimal form of the gauge transformation (9.28) on the YM field. We consider a 1-parameter family of gauge transformations $U_\lambda(x)$ with

$$U_0(x) = I, \tag{9.32}$$

and

$$\delta U(x) := \left(\frac{dU_\lambda(x)}{d\lambda} \right)_{\lambda=0} \equiv \xi(x). \tag{9.33}$$

The infinitesimal generator, $\xi(x)$, is a Lie algebra-valued function on M, that is, we have

$$\xi = \xi^i e_i. \tag{9.34}$$

Note that

$$\delta(U^{-1}(x)) = -\xi(x). \tag{9.35}$$

If we then consider

$$A_\mu(\lambda) = U_\lambda(x) A_\mu U_\lambda^{-1}(x) - (\partial_\mu U_\lambda(x)) U_\lambda^{-1}(x) \tag{9.36}$$

we can compute

$$\delta A_\mu = \left(\frac{dA_\mu(x)}{d\lambda} \right)_{\lambda=0} = -\left(\partial_\mu \xi + [A_\mu, \xi] \right). \tag{9.37}$$

We can interpret this last result as follows. The Lie algebra su(2), like any Lie algebra, is a vector space. You can easily check that the set of 2×2 anti-Hermitian trace-free matrices is a real three-dimensional vector space using the usual notion of addition of matrices and multiplication of matrices

by scalars. This three dimensional vector space provides a representation of the group SU(2) called the *adjoint representation*. This representation is defined by

$$\tau \to U\tau U^{-1}. \tag{9.38}$$

You can check that this transformation provides an isomorphism of the algebra onto itself:

$$[U\tau U^{-1}, U\tau'U^{-1}] = U[\tau, \tau']U^{-1}. \tag{9.39}$$

Anyway, at any given point, $\xi \equiv \delta U$ is a field taking values in this vector space and we have a representation of SU(2) acting on this vector space, so we can let SU(2) act on ξ via

$$\xi \to U\xi U^{-1}. \tag{9.40}$$

This linear representation can be made into an explicit matrix representation on the matrix elements of ξ (viewed as entries in a column vector) but we will not need to do this explicitly. The infinitesimal adjoint action of SU(2) on the vector space su(2) is via the commutator. To see this, simply consider a 1-parameter family of SU(2) transformations U_λ with

$$U_{\lambda=0} = I, \quad \left(\frac{dU_\lambda}{d\lambda}\right)_{\lambda=0} = \tau, \tag{9.41}$$

where τ is trace-free and anti-Hermitian. We have the infinitesimal adjoint action of SU(2) on su(2) given by

$$\delta\xi = [\tau, \xi] \tag{9.42}$$

PROBLEM: Show that the adjoint representation (9.40) has the infinitesimal form (9.42).

The point of these observations is to show that we can now extend our definition of the covariant derivative to ξ via the adjoint representation. We can define a covariant derivative of ξ as we did for φ by replacing the representation of infinitesimal SU(2) on φ,

$$\delta\varphi = \tau\varphi, \tag{9.43}$$

with its action on ξ:

$$\delta\xi = [\tau, \xi]. \tag{9.44}$$

We define

$$\mathcal{D}_\mu \xi = \partial_\mu \xi + [A_\mu, \xi]. \tag{9.45}$$

It then follows that we can interpret the infinitesimal gauge transformation of the gauge field in terms of the gauge covariant derivative of ξ:

$$\delta A_\mu = -\mathcal{D}_\mu \xi. \tag{9.46}$$

It is instructive to note that the preceding set of results can be viewed as a generalization of the construction of the $U(1)$ gauge theory in §6. To see how this works, take account of the fact that $U(1)$ is Abelian, so commutators of Lie-algebra valued elements vanish and the adjoint representation is trivial. Thus, in particular, the infinitesimal transformation (9.46) becomes (6.41) with $\xi \to -\alpha$.

9.4   Geometrical interpretation: parallel propagation

I have made various vague references to the fact that the gauge field provides a rule for "comparing the values of φ at different spacetime points". Let us try, very briefly, to be a little more precise about this.

Consider an infinitesimal displacement from a point x along a vector v. To each such displacement we can associate an element $\tau \in su(2)$ (in our matrix representation) via

$$\tau = A_\mu v^\mu. \tag{9.47}$$

The idea is that this infinitesimal transformation is used to define the linear transformation that, by definition, "lines up" the bases for $\mathbf{C}^2$ at infinitesimally neighboring points x and $x + v$. Let us explore this a little. Suppose that we have a curve $x = \gamma(s) \colon [0, 1] \to M$ starting at x_1, and ending at x_2

$$\gamma(0) = x_1, \quad \gamma(1) = x_2. \tag{9.48}$$

We can associate a group element $g[\gamma] \in SU(2)$ to each point on this curve by, in effect, exponentiating this infinitesimal transformation. More precisely, we define a group transformation at each point along the curve by solving the differential equation

$$\frac{d}{ds} g(s) = -\dot{\gamma}^\mu(s) A_\mu(\gamma(s)) g(s), \tag{9.49}$$

subject to the initial condition

$$g(0) = I. \tag{9.50}$$

We say that φ has been *parallelly propagated from x_1 to x_2 along the curve γ if*[4]

$$\varphi(x_2) = g(1)\varphi(x_1). \tag{9.51}$$

The idea is that, given the curve, the group transformation $g(1)$ is used to define the relationship between the spaces $\mathbf{C}^2$ at x_1 and x_2. Equivalently, we can define the parallel propagation of φ along the curve γ to be defined by solving the equation

$$\frac{d}{ds}\varphi(\gamma(s)) = -\dot{\gamma}^\mu(s)A_\mu(\gamma(s))\varphi(\gamma(s)). \tag{9.52}$$

Thus the gauge field A defines what it means to have elements of $\mathbf{C}^2$ to be "parallel", that is, to stay "unchanged" as we move from point to point in M. In particular, we say that the field φ is not changing (relative to A) or is parallelly propagated along the curve if

$$\begin{aligned}
0 &= \frac{d}{ds}\varphi(\gamma(s)) + \dot{\gamma}^\mu(s)A_\mu(\gamma(s))\varphi(\gamma(s)) \\
&= \dot{\gamma}^\mu\left(\varphi_{,\mu}(\gamma) + A_\mu(\gamma)\varphi(\gamma)\right) \\
&= \dot{\gamma}^\mu \mathcal{D}_\mu\varphi(\gamma). \tag{9.53}
\end{aligned}$$

You can see how the covariant derivative determines the rate of change of φ along the curve. So how a field is changing from point to point is determined by the choice of gauge field or "connection".

Let me emphasize that the parallel propagation of a field (relative to a given connection) from one point to another is, in general, path-dependent, *i.e.*, depends upon the choice of curve connecting the points. This is because the form of the ODE shown above will depend upon γ. We will say a little more about this below.

It is possible to give a formal series solution to the differential equation of parallel propagation. Indeed, if you have studied quantum mechanics, you will perhaps note the similarity of the equation

$$\frac{d}{ds}g(s) = -\dot{\gamma}^\mu(s)A_\mu(\gamma(s))g(s). \tag{9.54}$$

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[4] "Parallel propagation" is also called "parallel transport".

to the Schrödinger equation for the time evolution operator $U(t)$ of a time-dependent Hamiltonian $H(t)$:

$$i\hbar \frac{\partial U(t)}{\partial t} = H(t)U(t). \tag{9.55}$$

In quantum mechanics one solves the Schrödinger equation using the "time-ordered exponential"; here we can use the analogous quantity, the "path-ordered exponential":

$$g(s) = \sum_{n=0}^{\infty} (-1)^n \int_0^1 ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \dot{\gamma}^\mu(s_n) A_\mu(\gamma(s_n)) \cdots \dot{\gamma}^\mu(s_1) A_\mu(\gamma(s_1))$$

$$\equiv P \exp \left\{ - \int_0^1 ds \, \dot{\gamma}^\mu A_\mu \right\}. $$

$$\tag{9.56}$$

The ordering of the factors of A is crucial since they do not commute.

It may happen that for a particular choice of A there is a field satisfying

$$\mathcal{D}_\mu \varphi = 0, \tag{9.57}$$

such fields are called *covariantly constant*, or just "constant", or just "parallel". Covariantly constant fields have the property that their value at any point is parallel with its value at any other point – relative to the definition of "parallel" provided by the connection A and independently of any choice of curve connecting the points. As we shall see below, the existence of parallel fields requires a special choice of connection.

It is instructive to point out that everything we have done for SU(2) gauge theory could be also done with the group $U(1)$ in SED. Now the connections can be viewed as $i\times$ ordinary, real-valued 1-forms ω_μ, which commute. In this case the path-ordered exponential becomes the ordinary exponential:

$$P \exp \left\{ - \int_0^1 ds \, \dot{\gamma}^\mu A_\mu \right\} = \exp \left\{ -i \int_0^1 ds \, \dot{\gamma}^\mu \omega_\mu \right\}. \tag{9.58}$$

9.5   Geometrical interpretation: curvature

Given a covariant derivative, that is, a notion of parallel transport, one obtains also a notion of *curvature*. The simplest way to define curvature is as we

did for SED, namely, via the commutation relations for covariant derivatives. A straightforward computation reveals

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi = F_{\mu\nu}\varphi, \tag{9.59}$$

where $F_{\mu\nu}$ defines a Lie algebra-valued 2-form, the curvature of the gauge field,

$$F_{\mu\nu} = F^i_{\mu\nu} e_i, \tag{9.60}$$

given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{9.61}$$

and

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^i{}_{jk} A^j_\mu A^k_\nu. \tag{9.62}$$

PROBLEM: Verify the results (9.59) – (9.62) on the YM curvature.

We noted earlier that the parallel propagation of φ from one point to another depends upon what curve is used to connect the points. Infinitesimally, the parallel transport in the direction x^μ is defined by the covariant derivative $\mathcal{D}_\mu$. Since F arises as the commutator of covariant derivatives, you can interpret F as an infinitesimal measure of the path dependence of parallel transport. Indeed, the parallel transport is path independent (in a region) if and only if the connection is flat (in that region), $F = 0$.

The curvature can also be viewed as the obstruction to the existence of parallel fields φ. Since these satisfy

$$\mathcal{D}_\mu\varphi = 0, \tag{9.63}$$

the integrability condition

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi = 0 \tag{9.64}$$

arises, yielding

$$0 = F_{\mu\nu}\varphi = F^i_{\mu\nu}\, e_i\, \varphi. \tag{9.65}$$

It is not hard to check that, for SU(2),

$$\det(v^i e_i) = \delta_{ij} v^i v^j I. \tag{9.66}$$

Therefore, assuming $\varphi \neq 0$,

$$F^i_{\mu\nu}\, e_i\, \varphi = 0 \implies F_{\mu\nu} = 0. \tag{9.67}$$

So, if there exists non-vanishing parallel fields φ then

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0, \tag{9.68}$$

This condition, which says that the connection is "flat", is also the integrability condition for the existence of SU(2)-valued functions U satisfying

$$\partial_\mu U + A_\mu U = 0. \tag{9.69}$$

Thus, if the connection is flat then there exists $U(x)\colon M \to SU(2)$ such that

$$A_\mu = -(\partial_\mu U(x))U^{-1}(x). \tag{9.70}$$

This means that the gauge field is just a gauge transformation of the trivial connection $A = 0$. Thus we see that, at least locally,[5] *all flat connections are gauge-equivalent to the zero connection.*

Finally, let us consider the behavior of the curvature under a gauge transformation.

PROBLEM: Let $U(x)\colon M \to SU(2)$ define a gauge transformation, for which

$$A_\mu \to U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x). \tag{9.71}$$

Show that

$$F_{\mu\nu} \to U(x)F_{\mu\nu}U^{-1}(x). \tag{9.72}$$

Thus while the gauge field transforms inhomogeneously under a gauge transformation, the curvature transforms homogeneously. The curvature is transforming according to the adjoint representation of SU(2) on su(2) in which, with

$$\tau \in su(2), \quad U \in SU(2), \tag{9.73}$$

we have

$$\tau \to U\tau U^{-1}. \tag{9.74}$$

Infinitesimally, if we have a gauge transformation defined by

$$\delta U(x) = \xi(x), \tag{9.75}$$

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[5]I say "locally" because arguments based upon integrability conditions only guarantee local solutions to the differential equations.

then it is easy to see that

$$\delta F_{\mu\nu} = [\xi(x), F_{\mu\nu}]. \tag{9.76}$$

PROBLEM: Verify (9.76).

The curvature of the YM field differs from the curvature of the Maxwell field in a few key ways. First, the YM curvature is really a trio of 2-forms, while the Maxwell curvature is a single 2-form. Second, the YM curvature is a non-linear function of the the gauge field in contrast to the linear relation $F = dA$ arising in Maxwell theory. Finally, the YM curvature transforms homogeneously under a gauge transformation, while the Maxwell curvature is gauge invariant. You can now interpret this last result as coming from the fact that the adjoint representation of an Abelian group like $U(1)$ is trivial (exercise).

9.6 Lagrangian for the YM field

Let us recall the Lagrangian that represents the coupling of φ to A:

$$\mathcal{L}_{\varphi} = -\eta^{\mu\nu}\langle \mathcal{D}_\mu\varphi, \mathcal{D}_\nu\varphi\rangle - m^2\langle\varphi, \varphi\rangle. \tag{9.77}$$

If we view A as fixed, *i.e.,* given, then it is not a field variable but instead provides explicit functions of x^α which appear in $\mathcal{L}_\varphi = \mathcal{L}_\varphi(x, \varphi, \partial\varphi)$. Gauge-related connections A will give the same Lagrangian if we transform φ as well, but this has no immediate field theoretic consequence, *e.g.,* it doesn't provide a symmetry for Noether's theorem. The Lagrangian in this case is viewed as describing the dynamics of KG fields interacting with a prescribed YM field. The full, gauge invariant theory demands that A also be one of the dynamical fields. Following the examples of the electrically charged KG field and Dirac field, we need to adjoin to this Lagrangian a term providing dynamics for the YM field.[6] This is our next consideration.

We can build a Lagrangian for the YM field A by a simple generalization of the Maxwell Lagrangian. Our guiding principle is gauge invariance.

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[6]Simply using (9.77) to describe the dynamics of the scalars *and* the YM field is unsatisfactory since the Euler-Lagrange equations for the gauge field imply that φ is covariantly constant and hence that A is flat.

Now, unlike the Maxwell curvature, the YM curvature is not gauge invariant, rather, it is gauge "covariant", transforming homogeneously under a gauge transformation via the adjoint representation of SU(2) on su(2). However, it is easy to see that the trace of a product of su(2) elements is invariant under this action of SU(2) on su(2):

$$\text{tr}\left\{(U\tau_1 U^{-1})(U\tau_2 U^{-1})\right\} = \text{tr}\left\{\tau_1\tau_2\right\}. \tag{9.78}$$

This implies that the following Lagrangian density is gauge invariant:

$$\mathcal{L}_{YM} = \frac{1}{2}\text{tr}\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{9.79}$$

Writing

$$F_{\mu\nu} = F_{\mu\nu}^i\, e_i, \tag{9.80}$$

and using

$$\text{tr}(e_i e_j) = -\frac{1}{2}\delta_{ij}, \tag{9.81}$$

we get

$$\mathcal{L}_{YM} = -\frac{1}{4}\delta_{ij}F^{\mu\nu\,i}F_{\mu\nu}^j. \tag{9.82}$$

So, the YM Lagrangian is arising really as a sum of 3 Maxwell-type Lagrangians. However there is a very significant difference between the Maxwell and YM Lagrangians: the Maxwell Lagrangian is quadratic in the gauge field while the YM Lagrangian includes terms cubic and quartic in the gauge field. This means that, unlike the source-free Maxwell equations, the source-free YM equations will be non-linear. Physically this means that the YM fields are "self-interacting".

9.7   The source-free Yang-Mills equations

We have built the Yang-Mills Lagrangian as a generalization of the Maxwell Lagrangian. Let us for a moment forget about the scalar fields φ and derive the source-free YM equations as the Euler-Lagrange equations of

$$\mathcal{L}_{YM} = \frac{1}{2}\text{tr}\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{9.83}$$

We proceed as usual: we vary the Lagrangian and assemble the derivatives of the variations into a divergence term. This calculation will not involve

any explicit x^α dependence; to avoid confusion with the gauge covariant derivative, I will use the symbol ∂_μ to denote the total derivative with respect to x^μ. This total derivative is used when building the covariant derivative.

The variation of the Lagrangian density is

$$\delta\mathcal{L}_{YM} = \text{tr}\left(F^{\mu\nu}\delta F_{\mu\nu}\right), \tag{9.84}$$

where

$$\delta F_{\mu\nu} = \mathcal{D}_\mu\delta A_\nu - \mathcal{D}_\nu\delta A_\mu, \tag{9.85}$$

and

$$\mathcal{D}_\mu\delta A_\nu = \partial_\mu\delta A_\nu + [A_\mu, \delta A_\nu]. \tag{9.86}$$

PROBLEM: Verify equations (9.84) and (9.85).

Note that (9.86) is appropriate since δA_μ can be viewed as a Lie-algebra valued field transforming according to the adjoint representation of SU(2). To see this, start from the gauge transformation rule

$$A_\mu \to U A_\mu U^{-1} - (\partial_\mu U)U^{-1}, \tag{9.87}$$

and apply it to a 1-parameter family of gauge fields. From there it is easy to compute the transformation of the variation of the gauge field to get (exercise):

$$\delta A_\mu \to U\delta A_\mu U^{-1}, \tag{9.88}$$

giving us that adjoint transformation rule.

Returning now to our computation of the EL equations, we have

$$\begin{aligned}
\delta\mathcal{L}_{YM} &= 2\text{tr}\left(F^{\mu\nu}\mathcal{D}_\mu\delta A_\nu\right) \\
&= 2\text{tr}\left(F^{\mu\nu}\{\partial_\mu\delta A_\nu + [A_\mu, \delta A_\nu]\}\right) \\
&= -2\text{tr}\left\{(\partial_\mu F^{\mu\nu})\delta A_\nu - F^{\mu\nu}[A_\mu, \delta A_\nu]\right\} + \partial_\mu(2\text{tr}F^{\mu\nu}\delta A_\nu) \\
&= -2\text{tr}\left\{(\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}])\delta A_\nu\right\} + \partial_\mu(2\text{tr}F^{\mu\nu}\delta A_\nu) \\
&= -2\text{tr}\left\{\mathcal{D}_\mu F^{\mu\nu}\delta A_\nu\right\} + \partial_\mu(2\text{tr}F^{\mu\nu}\delta A_\nu) \\
&= \mathcal{D}_\mu F^{\mu\nu i}\delta A_\nu^j\delta_{ij} + \partial_\mu(-F^{\mu\nu i}\delta A_\nu^j\delta_{ij}), \tag{9.89}
\end{aligned}$$

where we have defined the covariant derivative of the curvature via

$$\mathcal{D}_\mu F^{\alpha\beta} = \partial_\mu F^{\alpha\beta} + [A_\mu, F^{\alpha\beta}], \tag{9.90}$$

which is appropriate given the fact that (like δA) F transforms according to the adjoint representation of SU(2).

From this computation we see that the EL derivative of $\mathcal{L}_{YM}$ is

$$\mathcal{E}_i^\nu(\mathcal{L}_{YM}) = \mathcal{D}_\mu F_i^{\mu\nu}, \qquad (9.91)$$

where the Latin (su(2) component) indices are raised and lowered with δ_{ij}. The *source-free YM equations* are then

$$\mathcal{D}_\mu F^{\mu\nu} = 0. \qquad (9.92)$$

Of course, this compact geometric notation hides a lot of stuff. The PDEs for the gauge field A are, more explicitly,

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]) + [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu] + [A_\mu, [A^\mu, A^\nu]] = 0. \quad (9.93)$$

Thus we get $3 \times 4 = 12$ equations providing a non-linear generalization of the Maxwell equations.

Despite their complexity, a good number of solutions of the Yang-Mills equations are known. It would take us too far afield to discuss them, but I will show you a very famous solution, the "Wu-Yang monopole."

PROBLEM: Consider the Yang-Mills field whose non-zero components, A_α^i, $\alpha = (t, x, y, z)$, $i = (1, 2, 3)$, relative to an inertial Cartesian coordinate system and the basis e_i are given by

$$A_x^2 = -\frac{z}{r^2}, \quad A_x^3 = \frac{y}{r^2}, \quad A_y^1 = \frac{z}{r^2}, \quad A_y^3 = -\frac{x}{r^2}, \quad A_z^1 = -\frac{y}{r^2}, \quad A_z^2 = \frac{x}{r^2}. \tag{9.94}$$

Show that this YM field solves the YM equations.

9.8 Yang-Mills with sources

Let us now consider the YM field coupled to its "source" φ.[7] We will compute the EL equations associated to

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_\varphi, \qquad (9.95)$$

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[7]While calling φ the "source" is quite all right, it should be kept in mind that, owing to the non-linearity of the source-free YM field equations, one can consider the YM field as its own source!

where

$$\mathcal{L}_{\varphi} = - \left(\langle \mathcal{D}_{\mu}\varphi, \mathcal{D}^{\mu}\varphi \rangle + m^2 \langle \varphi, \varphi \rangle \right). \tag{9.96}$$

It is not hard to check that the EL equations for φ are given by

$$\mathcal{D}^{\mu}\mathcal{D}_{\mu}\varphi - m^2\varphi = 0, \tag{9.97}$$

while the EL equations for A^i_{ν} are given by

$$\mathcal{D}_{\mu}F^{\mu\nu}_i + \left(\varphi^{\dagger} e_i \mathcal{D}^{\nu}\varphi - \mathcal{D}^{\nu}\varphi^{\dagger} e_i \varphi \right) = 0. \tag{9.98}$$

We can write the latter as a Lie algebra-valued expression:

$$\mathcal{D}_{\mu}F^{\mu\nu} - j^{\nu} = 0. \tag{9.99}$$

You can see that the fields φ are coupled to the gauge field through a sort of covariant version of the KG equation. The scalar field acts as a source of the YM field via the *current*

$$j^{\mu}_i = \left(\mathcal{D}^{\mu}\varphi^{\dagger} e_i \varphi - \varphi^{\dagger} e_i \mathcal{D}^{\mu}\varphi \right). \tag{9.100}$$

Note that, just as in SED, the definition of the current involves the gauge field itself.

PROBLEM: Derive equations (9.97), (9.98), (9.99).

9.9 Noether theorems

Let us briefly mention some considerations arising from applying Noether's theorems to the SU(2) transformations arising in YM theory. There are some significant differences compared to Maxwell theory. First of all, we can try to consider the global SU(2) symmetry for which

$$\varphi \to U\varphi, \quad A \to UAU^{-1}. \tag{9.101}$$

But this kind of symmetry is not that meaningful (in general) since there is no gauge invariant way of demanding that U be constant except via the covariant derivative, and this would force the curvature to vanish. The availability of "global" or "rigid" gauge symmetries is a special feature permitted by Abelian gauge groups. It does not immediately generalize to YM theory.

However, we can still consider Noether's second theorem as applied to the gauge symmetry

$$\varphi \to U(x)\varphi, \quad A_\mu \to U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x). \qquad (9.102)$$

To begin with, the source-free theory described by $\mathcal{L}_{YM}$ also has the gauge symmetry, and this implies the identity (which you can easily check directly)

$$\mathcal{D}_\nu \mathcal{D}_\mu F^{\mu\nu} = 0. \qquad (9.103)$$

PROBLEM: Verify by direct computation

$$\mathcal{D}_\nu \mathcal{D}_\mu F^{\mu\nu} = 0. \qquad (9.104)$$

(Note that this is not quite as trivial as in the $U(1)$ case since covariant derivatives appear.)

For the YM field coupled to its source described by $\mathcal{L}_\varphi$, the differential identity is

$$\{\mathcal{D}_\nu \left[\mathcal{D}_\mu F_i^{\mu\nu} - j_i^\nu\right]\} + 2Re\left[\mathcal{E}_\varphi e_i\varphi\right] = 0. \qquad (9.105)$$

We conclude then that the scalar current (9.100) satisfies a *covariant* conservation equation when the field equations hold:

$$\mathcal{D}_\mu j^\mu \equiv \partial_\mu j^\mu + [A_\mu, j^\mu] = 0, \quad \text{modulo field equations.} \qquad (9.106)$$

PROBLEM: Derive (9.105) and (9.106).

This last result is significant: a covariant divergence appears in (9.106) rather than an ordinary divergence. This means that one does not have a continuity equation for j^μ! Physically, one interprets this by saying that the "charge" described by j^μ is not all the charge in the system. Indeed, one says that the YM field itself carries some charge! As we have mentioned, this is why the YM equations are non-linear: the YM field can serve as its own source. As it turns out, without making some kind of special restrictions (see below), there is no gauge-invariant way to localize the charge and current densities, so there is no meaningful conserved current in general! With appropriate asymptotic conditions it can be shown that there *is* a well-defined notion of total charge in the system, but that is another story. All these

features have analogs in the gravitational field within the general theory of relativity.

PROBLEM: Suppose the gauge field A takes the form

$$A = A^1 e_1, \quad A^1 = \alpha_\mu dx^\mu. \tag{9.107}$$

Show that there exist fields $\phi = \phi^i e_i$ transforming according to the adjoint representation of SU(2), which satisfy

$$\mathcal{D}_\mu \phi = 0. \tag{9.108}$$

PROBLEM: Suppose that the gauge field A is such that there exists a covariantly constant field $\phi = \phi^i e_i$ (*e.g.*, as in the previous problem),

$$\mathcal{D}_\mu \phi = 0. \tag{9.109}$$

Show that $J^\mu := \phi^i j_i^\mu$ (see (9.100)) defines a *bona fide* conserved current.

9.10 Non-Abelian gauge theory in general

There are a number of generalizations of our results for YM theory. For example, here I show how to generalize YM theory to any gauge group associated to a matrix Lie group G. Let G be a Lie group represented as a set of matrices acting on a real (for simplicity) vector space V. Let $\mathfrak{g}$ be its Lie algebra, also represented as a set of matrices acting on V. Let

$$\varphi \colon M \to V, \tag{9.110}$$

and let A be a $\mathfrak{g}$-valued 1-form on M. Define a covariant derivative

$$\mathcal{D}_\mu \varphi = \partial_\mu \varphi + A_\mu \varphi. \tag{9.111}$$

Define the curvature of A via

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] \varphi = F_{\mu\nu} \varphi, \tag{9.112}$$

so that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{9.113}$$

The curvature is a $\mathfrak{g}$-valued 2-form. The behavior of the gauge field and curvature under a gauge transformations is the same as when the gauge group was SU(2):

$$A_\mu \to U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x), \qquad (9.114)$$

$$F_{\mu\nu} \to U(x)F_{\mu\nu}U^{-1}(x). \qquad (9.115)$$

To build a Lagrangian we need two bi-linear forms. Let

$$\alpha \colon V \times V \to \mathbf{R} \qquad (9.116)$$

be a G-invariant, non-degenerate bilinear form on V. (Recall that when we chose $V = \mathbf{C}^2$ and $G = SU(2)$ we used the standard $U(2)$ invariant Hermitian inner product for α.) "G-invariance" means that for any $U \in G$

$$\alpha(Uv, Uw) = \alpha(v, w), \quad v, w \in V. \qquad (9.117)$$

"Non-degenerate" means that

$$\alpha(v, w) = 0 \quad \forall\, w \quad \implies \quad v = 0. \qquad (9.118)$$

Next, let

$$\beta \colon \mathfrak{g} \times \mathfrak{g} \to \mathbf{R} \qquad (9.119)$$

be a non-degenerate bi-linear form on the Lie algebra, invariant with respect to the adjoint representation of G on $\mathfrak{g}$. If $U \in G$ the adjoint representation, $Ad_U \colon \mathfrak{g} \to \mathfrak{g}$, is given by

$$Ad_U \cdot \tau = U\tau U^{-1}, \quad \tau \in \mathfrak{g}. \qquad (9.120)$$

The invariance condition on β is

$$\beta(Ad_U\tau_1, Ad_U\tau_2) \equiv \beta(U\tau_1 U^{-1}, U\tau_2 U^{-1}) = \beta(\tau_1, \tau_2). \qquad (9.121)$$

(When we used the 2×2 matrix representation of $G = SU(2)$ we used (minus) the trace of a product of matrices for β. This is equivalent to using the Killing form for β.)

The Lagrangian for the theory can now be constructed via

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\varphi, \qquad (9.122)$$

where

$$\mathcal{L}_A = \beta(F_{\mu\nu}, F^{\mu\nu}), \tag{9.123}$$

and

$$\mathcal{L}_\varphi = -\alpha(\mathcal{D}_\mu\varphi, \mathcal{D}^\mu\varphi) - m^2\alpha(\varphi, \varphi). \tag{9.124}$$

I think you can see that, given the the transformation properties of the covariant derivative and the curvature, and given the invariance properties of the bi-linear forms α and β, the Lagrangian $\mathcal{L}$ is gauge invariant.

9.11 Chern-Simons Theory

There is a significant new option for gauge theory when the underlying manifold of independent variables has three dimensions (as opposed to the four we have been using for the most part so far). This gauge theory is known as *Chern-Simons theory*. Here I will just give a brief introduction within the general setting of the previous section. The goals will be to define the Lagrangian, compute the field equations, and examine some symmetries. I will begin by summarizing the technology we will need.

Fix a basis e_i, $i = 1, 2, \ldots, m$, for a chosen m-dimensional Lie algebra, so that

$$A = \left(A^i_\mu dx^\mu\right) e_i, \quad [e_i, e_j] = C^k_{ij}e_k. \tag{9.125}$$

(In this section Greek letters denote coordinates x^μ, $\mu = 1, 2, 3$.) The invariant bilinear form on the Lie algebra is then

$$\beta = \beta_{ij}\omega^i \otimes \omega^j, \tag{9.126}$$

where ω^i is the basis of 1-forms dual to the basis e_i. This means that, with

$$v = v^i e_i, \quad w = w^i e_i, \tag{9.127}$$

we have

$$\beta(v, w) = \beta_{ij}v^i w^j. \tag{9.128}$$

The adjoint-invariance of β implies that

$$C^c_{ka}\beta_{cb} + C^c_{kb}\beta_{ac} = 0. \tag{9.129}$$

Along with the usual anti-symmetry of the structure constants, $C^c_{ab} = -C^c_{ba}$, this implies

$$\beta_{ca}C^c_{bd} = \beta_{c[a}C^c_{bd]}. \tag{9.130}$$

The next ingredient we will need to build the Lagrangian is the 3-dimensional *permutation symbol*.[8] This object is denoted $\eta^{\alpha\beta\gamma}$, $\alpha, \beta, \gamma = 1, 2, 3$, and is defined to have the following properties: (i) it is totally anti-symmetric (changes sign under any permutation of its components), and (ii) it has components consisting of $0, \pm 1$ where, *in any coordinate system* (with the same orientation),

$$\eta^{123} = 1. \tag{9.131}$$

The permutation symbol plays a fundamental role in defining the determinant of a matrix. From the Leibniz formula for the determinant of a 3×3 matrix B_β^α we have

$$(\det B)\, \eta^{\mu\nu\sigma} = \eta^{\alpha\beta\gamma} B_\alpha^\mu B_\beta^\nu B_\gamma^\sigma. \tag{9.132}$$

Using this identity, the permutation symbol can be interpreted as a rank-3, contravariant *tensor density of weight one*. By definition, under a coordinate transformation $x^\alpha \to \overline{x}^\alpha$, the components $W^{\alpha\beta\gamma}$ of a rank-3, contravariant tensor density of weight one transform as

$$\overline{W}^{\alpha\beta\gamma} = \left[\det\left(\frac{\partial \overline{x}}{\partial x}\right)\right]^{-1} \frac{\partial \overline{x}^\alpha}{\partial x^\mu} \frac{\partial \overline{x}^\beta}{\partial x^\nu} \frac{\partial \overline{x}^\gamma}{\partial x^\sigma} W^{\mu\nu\sigma} \tag{9.133}$$

Applying this rule to $\eta^{\alpha\beta\gamma}$ we get

$$\overline{\eta}^{\alpha\beta\gamma} = \left[\det\left(\frac{\partial \overline{x}}{\partial x}\right)\right]^{-1} \frac{\partial \overline{x}^\alpha}{\partial x^\mu} \frac{\partial \overline{x}^\beta}{\partial x^\nu} \frac{\partial \overline{x}^\gamma}{\partial x^\sigma} \eta^{\mu\nu\sigma} = \eta^{\alpha\beta\gamma}. \tag{9.134}$$

Thus if (9.131) holds in any one coordinate system it will hold in all coordinate systems, consistent with the definition of the permutation symbol. An important point to make in this context is that nowhere did we need to use a metric tensor to define the permutation symbol; such an object is always available as a rank-n tensor density on any n-dimensional manifold.

PROBLEM: Show that one can likewise define a permutation symbol $\eta_{\alpha\beta\gamma}$, which is a *covariant* tensor density of weight *minus* one. (Do not introduce a metric to lower the indices!)

We now have enough machinery to build the Lagrangian density for Chern-Simons theory. I will simply postulate it; you can consult, *e.g.*,

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[8]Also known as the Levi-Civita tensor density.

the *nLab* page `ncatlab.org/nlab/show/Chern-Simons+theory` for background, motivations, additional results, details, and references. The Chern-Simons Lagrangian density is

$$\mathcal{L} = \eta^{\alpha\beta\gamma}\beta_{ij}\left(2A_\alpha^i\partial_\beta A_\gamma^j + \frac{2}{3}C_{kl}^j A_\alpha^i A_\beta^k A_\gamma^l\right). \tag{9.135}$$

Let's calculate the Chern-Simons field equations using the Euler-Lagrange formula. We have

$$\frac{\partial\mathcal{L}}{\partial A_\mu^r} = \eta^{\mu\beta\gamma}\left(2\beta_{jr}\partial_\beta A_\gamma^j + 2\beta_{j[r}C_{kl]}^j A_\beta^k A_\gamma^l\right), \quad \frac{\partial\mathcal{L}}{\partial A_{\mu,\nu}^r} = -2\eta^{\mu\nu\gamma}\beta_{jr}A_\gamma^j, \tag{9.136}$$

Consequently, the Euler-Lagrange expression $\mathcal{E}_r^\mu(\mathcal{L})$ for the Chern-Simons Lagrangian is

$$\begin{aligned}
\mathcal{E}_r^\mu(\mathcal{L}) &= \eta^{\mu\beta\gamma}\left(2\beta_{jr}\partial_\beta A_\gamma^j + 2\beta_{j[r}C_{kl]}^j A_\beta^k A_\gamma^l\right) + 2\eta^{\mu\nu\gamma}\beta_{jr}\partial_\nu A_\gamma^j \\
&= 2\eta^{\mu\beta\gamma}\left(2\beta_{jr}\partial_\beta A_\gamma^j + \beta_{j[r}C_{kl]}^j A_\beta^k A_\gamma^l\right) \\
&= 2\eta^{\mu\beta\gamma}\left(2\beta_{jr}\partial_\beta A_\gamma^j + \beta_{jr}C_{kl}^j A_\beta^k A_\gamma^l\right) \\
&= 2\eta^{\mu\beta\gamma}\beta_{jr}\left(2\partial_{[\beta}A_{\gamma]}^j + C_{kl}^j A_\beta^k A_\gamma^l\right)
\end{aligned} \tag{9.137}$$

To get the last equality I used (9.130). The curvature formula (9.113) expressed in the basis (9.125) is

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + C_{jk}^i A_\mu^j A_\nu^k. \tag{9.138}$$

We thus get

$$\mathcal{E}_r^\mu(\mathcal{L}) = 2\eta^{\mu\beta\gamma}\beta_{jr}F_{\beta\gamma}^j. \tag{9.139}$$

Because of the identities

$$\eta^{\mu\beta\gamma}\eta_{\mu\rho\sigma} = 2\delta_{[\rho}^\beta\delta_{\sigma]}^\beta, \quad \beta_{jr}\beta^{rs} = \delta_j^s, \tag{9.140}$$

we can conclude

$$\mathcal{E}_r^\mu(\mathcal{L}) = 0 \iff F_{\mu\nu}^i = 0. \tag{9.141}$$

The EL equations for the Chern-Simons Lagrangian say that the gauge field (or connection) is flat!

I will conclude this brief introduction to Chern-Simons theory by exposing the somewhat some remarkable symmetries of the Lagrangian. To begin, it

is reasonable to inquire whether there is a gauge symmetry. At a first glance, existence of such a symmetry would appear doubtful since the Lagrangian is not built in an invariant way from the curvature. Still, it is instructive to look more closely. Use (9.138) to write the Lagrangian as

$$\mathcal{L} = \eta^{\alpha\beta\gamma}\beta_{ij}\left(A_\alpha^i F_{\beta\gamma}^j - \frac{1}{3}C_{kl}^j A_\alpha^i A_\beta^k A_\gamma^l\right)$$

$$= \eta^{\alpha\beta\gamma}\left(\beta(A_\alpha, F_{\beta\gamma}) - \frac{1}{3}\beta(A_\alpha, [A_\beta, A_\gamma])\right)$$

$$= \eta^{\alpha\beta\gamma}\left(\beta(A_\alpha, F_{\beta\gamma}) - \frac{2}{3}\beta(A_\alpha, A_\beta\, A_\gamma)\right). \qquad (9.142)$$

Define

$$\tilde{A}_\alpha = UA_\alpha U^{-1} - \partial_\alpha UU^{-1}, \quad \tilde{F}_{\beta\gamma} = UF_{\beta\gamma}U^{-1}. \qquad (9.143)$$

Use the gauge transformation formulas (9.114) and (9.115) and the invariance conditions (9.121) and (9.130) to obtain

$$\eta^{\alpha\beta\gamma}\beta(\tilde{A}_\alpha, \tilde{F}_{\beta\gamma}) = \eta^{\alpha\beta\gamma}\beta(A_\alpha, F_{\beta\gamma}) - \eta^{\alpha\beta\gamma}\beta(U^{-1}\partial_\alpha U, F_{\beta\gamma})$$

$$= \eta^{\alpha\beta\gamma}\left\{\beta(A_\alpha, F_{\beta\gamma}) - 2\beta(U^{-1}\partial_\alpha U, \partial_\beta A_\gamma) - 2\beta(U^{-1}\partial_\alpha U, A_\beta A_\gamma)\right\}$$
$$(9.144)$$

and

$$\beta(\tilde{A}_\alpha, \tilde{A}_\beta\, \tilde{A}_\gamma) = \eta^{\alpha\beta\gamma}\big\{\beta(A_\alpha, A_\beta\, A_\gamma) - \beta(\partial_\alpha UU^{-1}, \partial_\beta UU^{-1}\partial_\gamma UU^{-1})$$
$$+ 3\beta(U^{-1}\partial_\alpha UU^{-1}\partial_\beta U, A_\gamma) - 3\beta(U^{-1}\partial_\alpha U, A_\beta A_\gamma)\big\}$$
$$(9.145)$$

We now have

$$\tilde{\mathcal{L}} \equiv \eta^{\alpha\beta\gamma}\beta(\tilde{A}_\alpha, \tilde{F}_{\beta\gamma}) - \frac{2}{3}\beta(\tilde{A}_\alpha, \tilde{A}_\beta\, \tilde{A}_\gamma)$$

$$= \mathcal{L} + \frac{2}{3}\eta^{\alpha\beta\gamma}\beta(\partial_\alpha UU^{-1}, \partial_\beta UU^{-1}\partial_\gamma UU^{-1})$$

$$- 2\eta^{\alpha\beta\gamma}\left\{\beta(U^{-1}\partial_\alpha U, \partial_\beta A_\gamma) + \beta(U^{-1}\partial_\alpha UU^{-1}\partial_\beta U, A_\gamma)\right\}. \qquad (9.146)$$

From $U^{-1}U = 1$ we can deduce

$$U^{-1}\partial_\alpha UU^{-1} = -\partial_\alpha U^{-1}, \qquad (9.147)$$

so that

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{2}{3}\eta^{\alpha\beta\gamma}\beta(\partial_\alpha UU^{-1}, \partial_\beta UU^{-1}\partial_\gamma UU^{-1}) + \partial_\alpha \left\{\eta^{\alpha\beta\gamma}\beta(2U^{-1}\partial\beta U, A_\gamma)\right\}.$$
(9.148)

The last term is a divergence; what about the second term? This term is interesting. If the gauge transformation can be obtained by iterating infinitesimal transformations, that is, if it can be written as

$$U = \exp(\lambda^i(x)e_i),$$
(9.149)

for some functions $\lambda(x)$ then

$$\partial_\alpha UU^{-1} = \partial_\alpha\lambda^i e_i, \quad \Longrightarrow \quad \partial_{[\alpha}(\partial_{\beta]}UU^{-1}) = 0.$$
(9.150)

You can check that in this case the term of interest in (9.148) can also be written as a divergence. Thus infinitesimal gauge transformations will be divergence symmetries of the Lagrangian. Noether's second theorem then applies implying the existence of differential identities satisfied by the Euler-Lagrange equations. In this case these are equivalent to the "Bianchi identities":

$$\mathcal{D}_{[\alpha}F_{\beta\gamma]} = 0.$$
(9.151)

PROBLEM: Show by direct computation just using the definition (9.113) and the definition of covariant derivative that

$$\mathcal{D}_{[\alpha}F_{\beta\gamma]} = 0.$$
(9.152)

It can happen that there are gauge transformations not obtained by iterating infinitesimal transformations as in (9.149). To explain this in detail will take us too far afield, so let me just provide you with an example.

PROBLEM: Consider a Chern-Simons theory where $M = S^3$ and $G = SU(2)$, and $\beta_{ij} = -2\delta_{ij}$. Using the Euler angle coordinates $x^\mu = (\gamma, \beta, \alpha)$, $(0 < \alpha < 2\pi, 0 < \beta < 2\pi, 0 < \gamma < \pi)$ for S^3, define $U: M \to G$ by

$$U = \begin{pmatrix} e^{i(\alpha+\beta)}\cos(\gamma/2) & e^{-i(\alpha-\beta)}\sin(\gamma/2) \\ -e^{i(\alpha-\beta)}\sin(\gamma/2) & e^{-i(\alpha+\beta)}\cos(\gamma/2) \end{pmatrix}$$
(9.153)

Show that

$$\int_M \eta^{\alpha\beta\gamma}\beta(\partial_\alpha U U^{-1}, \partial_\beta U U^{-1}\partial_\gamma U U^{-1}) = 48\pi^2. \qquad (9.154)$$

This means that the integrand can't be written as a divergence since in such a case the integral would have to vanish by the divergence theorem. Thus for these "large" gauge transformations the Lagrangian is not gauge invariant.[9]

Finally, let me point out a remarkable symmetry enjoyed by the Chern-Simons Lagrangian. Consider a *diffeomorphism* of M. This is a smooth 1-1 mapping of M onto itself whose inverse is also smooth. Diffeomorphisms are the "symmetries" of manifolds in the sense that one considers two manifolds related by a diffeomorphism to be the same abstract manifold. The diffeomorphisms form a group, which is the symmetry group of "generally covariant" theories like Einstein's general theory of relativity, as we shall discuss later. I want to show you that, unlike the usual Yang-Mills type of Lagrangian, the Chern-Simons Lagrangian is generally covariant in the sense that it admits the diffeomorphism symmetry. For brevity, I will just sketch the proof of invariance with respect to infinitesimal transformations.

Given a coordinate system x^μ on M, a diffeomorphism $\phi\colon M \to M$ looks like a change of coordinates

$$x^\mu \to \tilde{x}^\mu = \phi^\mu(x), \qquad (9.155)$$

where ϕ^μ will be smooth with a smooth inverse in the domain of the coordinates. An infinitesimal diffeomorphism comes from a 1-parameter family of diffeomorphisms ϕ_s in the usual way:

$$\phi_s^\mu(x) = x^\mu + sV^\mu(x) + \mathcal{O}(s^2). \qquad (9.156)$$

Here the V^μ define – and are defined by – the 1-parameter family of transformations. Geometrically, V^α is a vector field on M. So, for any infinitesimal diffeomorphism defined by a vector field V^α on M, define the infinitesimal transformation of the gauge field as[10]

$$\delta A_\alpha = V^\beta F_{\beta\alpha}. \qquad (9.157)$$

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[9]Interestingly and importantly, with a proper normalization the action integral changes by an additive integer under such gauge transformations. This allows gauge invariance in the associated quantum field theory.

[10]For those who know about these things, this is the "gauge covariant Lie derivative" of A.

PROBLEM: A "covariant" tensor like A_α transforms under (9.155) by

$$A_\mu(x) \to \frac{\partial \phi_\nu}{\partial x^\mu} A_\nu(\phi(x)). \tag{9.158}$$

Show that (9.157) corresponds to the infinitesimal form of this transformation law (using (9.156)) up to an infinitesimal gauge transformation generated by $V^\alpha A_\alpha$ (see (9.46)).

The transformation (9.157) induces the following change in the Lagrangian density:

$$\delta \mathcal{L} = 2\eta^{\alpha\beta\gamma} \beta_{ij} V^\sigma F^i_{\sigma\alpha} F^j_{\beta\gamma} + \partial_\beta \left(2\eta^{\alpha\beta\gamma} \beta_{ij} A^i_\alpha V^\sigma F^j_{\sigma\gamma} \right). \tag{9.159}$$

As a nice exercise you can check the identity

$$\eta^{\alpha\beta\gamma} \beta_{ij} F^i_{\sigma\alpha} F^j_{\beta\gamma} = 0. \tag{9.160}$$

Consequently, the infinitesimal transformation (9.157) defines a divergence symmetry of the Lagrangian.

PROBLEM: Apply Noether's second theorem to the infinitesimal gauge and diffeomorphism symmetries of the Chern-Simons Lagrangian. What are the corresponding differential identities satisfied by the Euler-Lagrange expression (9.139)?

9.12 PROBLEMS

1. Verify that the e_k in (9.15) satisfy the su(2) Lie algebra (9.16) as advertised.

2. (a) Show that the Lagrangian (9.17) is invariant with respect to the transformation (9.8); (b) compute its Euler-Lagrange equations; (c) derive the conserved currents (9.19) from Noether's theorem; (d) verify that the currents (9.19) are divergence-free when the Euler-Lagrange equations are satisfied.

3. Show that if any one of the currents (9.19) are conserved, then the other 2 are automatically conserved. (*Hint:* Consider the behavior of the currents under SU(2) transformations of the fields.)

4. Show that the adjoint representation (9.40) has the infinitesimal form (9.42).

5. Verify the results (9.59) – (9.62) on the YM curvature.

6. Let $U(x): M \to SU(2)$ define a gauge transformation, for which

$$A_\mu \to U(x)A_\mu U^{-1}(x) - (\partial_\mu U(x))U^{-1}(x). \tag{9.161}$$

Show that

$$F_{\mu\nu} \to U(x)F_{\mu\nu}U^{-1}(x). \tag{9.162}$$

7. Verify equation (9.76).

8. Verify equations (9.84) and (9.85).

9. Consider the Yang-Mills field whose non-zero components, A_α^i, $\alpha = (t, x, y, z)$, $i = (1, 2, 3)$, relative to an inertial Cartesian coordinate system and the basis e_i are given by

$$A_x^2 = -\frac{z}{r^2}, \quad A_x^3 = \frac{y}{r^2}, \quad A_y^1 = \frac{z}{r^2}, \quad A_y^3 = -\frac{x}{r^2}, \quad A_z^1 = -\frac{y}{r^2}, \quad A_z^2 = \frac{x}{r^2}. \tag{9.163}$$

Show that this YM field solves the YM equations.

10. Derive equations (9.97), (9.98), (9.99).

11. Verify by direct computation

$$\mathcal{D}_\nu \mathcal{D}_\mu F^{\mu\nu} = 0. \tag{9.164}$$

(Note that this is not quite as trivial as in the $U(1)$ case since covariant derivatives appear.)

12. Derive (9.105) and (9.106).

13. Suppose the gauge field takes the form

$$A = A^1 e_1, \quad A^1 = \alpha_\mu dx^\mu. \tag{9.165}$$

Show that there exist fields $\phi = \phi^i e_i$ transforming according to the adjoint representation of SU(2), which satisfy

$$\mathcal{D}_\mu \phi = 0. \tag{9.166}$$

14. Suppose that the gauge field A is such that there exists a covariantly constant field $\phi = \phi^i e_i$,

$$\mathcal{D}_\mu \phi = 0. \tag{9.167}$$

Show that $J^\mu := \phi^i j_i^\mu$ (see (9.100)) defines a *bona fide* conserved current.

15. Show that one can define a permutation symbol $\eta_{\alpha\beta\gamma}$, which is a *covariant* tensor density of weight *minus* one. (Do not introduce a metric to lower the indices!)

16. Show by direct computation just using the definition (9.113) and the definition of covariant derivative that

$$\mathcal{D}_{[\alpha} F_{\beta\gamma]} = 0. \tag{9.168}$$

17. A "covariant" tensor like A_α transforms under (9.155) by

$$A_\mu(x) \to \frac{\partial \phi_\nu}{\partial x^\mu} A_\nu(\phi(x)). \tag{9.169}$$

Show that (9.157) corresponds to the infinitesimal form of this transformation law (using (9.156)) up to an infinitesimal gauge transformation generated by $V^\alpha A_\alpha$ (see (9.46)).

18. Apply Noether's second theorem to the infinitesimal gauge and diffeomorphism symmetries of the Chern-Simons Lagrangian. What are the corresponding differential identities satisfied by the Euler-Lagrange expression (9.139)?

Chapter 10

Gravitational field theory

Currently, our best theory of gravitation is the field-theoretic description occurring within Einstein's geometrical model of spacetime as a Lorentzian manifold, which is the essence of his 1915 General Theory of Relativity. There are 3 principal ingredients to this theory. (1) *The geodesic hypothesis:* freely falling test particles move on geodesics of the spacetime geometry. (2) *The Einstein field equations:* the curvature of the geometry is specified by energy density, pressure, momentum and stress of matter. (3) *The principle of general covariance:* the only fixed structure needed to implement (1) and (2) is the manifold structure of spacetime. Everything else is determined dynamically.

The following sections are not intended to be an introduction to relativity, but rather an exploration of some of its field theoretical aspects. I am assuming that you have seen some relativistic physics in a previous class.

10.1   Spacetime geometry

We begin with a very brief review of spacetime geometry. The goal in this section is simply to refresh your memory regarding some of the things we will need and to establish notation. The set of all spacetime events is mathematically represented as a differentiable manifold M equipped with a spacetime metric g. The metric determines the spacetime interval between events via its *line element*. In coordinates x^μ on M, the metric tensor field takes the form

$$g := g_{\mu\nu}(x)\mathbf{dx}^\mu \otimes \mathbf{dx}^\nu, \quad g_{\mu\nu} = g_{\nu\mu}, \tag{10.1}$$

where $\mathbf{dx}^\mu$ are the coordinate basis dual vectors. The line element is

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{10.2}$$

We denote by $g^{\alpha\beta} = g^{\beta\alpha}$ the symmetric array which is inverse to the array of components $g_{\alpha\beta}$:

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma. \tag{10.3}$$

It is a basic result from linear algebra that any quadratic form Q can be put into canonical form by a change of basis (try googling "Sylvester's law of inertia"). What this means is that there always exists a basis for the vector space upon which Q is defined such that the matrix of components of the quadratic form takes the form

$$Q = \mathrm{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n), \tag{10.4}$$

where each $\epsilon = \pm 1, 0$ and $n = \dim(M)$. The number of times $+1$ occurs, -1 occurs, and 0 occurs cannot be modified by a change of basis and is an intrinsic feature – the only intrinsic feature – of Q. These numbers determine the *signature* of Q.

At each point of M the metric determines a quadratic form on the tangent space to M at that point. So we can apply the above math facts to the metric at any given point. First of all, a metric cannot have any $\epsilon = 0$ (since the metric should be non-degenerate). Secondly, if each $\epsilon = 1$, then the metric is called *Riemannian*. Otherwise, the metric is called *pseudo-Riemannian*. If a single $\epsilon = -1$ and the rest have $\epsilon = 1$ the pseudo-Riemannian metric is called *Lorentzian*.[1] All this analysis took place at one point, but since we require $\epsilon \neq 0$, the signature cannot change from point to point (assuming the metric components and hence the signature vary continuously). Thus the signature is actually a fixed feature of the whole metric tensor field.

Spacetime in general relativity is a Lorentzian manifold – a manifold equipped with a metric of Lorentz signature. Most of what we do here will not depend on the signature of the metric; only in a few places will we assume that the metric is Lorentzian. It is perhaps worth pointing out that not all manifolds admit a globally defined Lorentzian metric. For example, the manifold $\mathbf{R}^4$ admits a Lorentzian metric, but the four-sphere (S^4) does not.

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[1]There is a convention (which we shall not use) where one also calls "Lorentzian" the case where a single $\epsilon = 1$ and the rest are minus one.

On a Riemannian manifold (a manifold equipped with a Riemannian metric) the line element of the metric determines the infinitesimal distance ds between neighboring points x^μ and $x^\mu + dx^\mu$. On a Lorentzian manifold the line element of the metric represents the invariant spacetime interval and determines the infinitesimal proper time elapsed along nearby timelike separated events and the infinitesimal proper distance between nearby spacelike separated events.

The metric defines a notion of squared-"length" of vectors and a notion of "angles" between vectors. Consequently, one can use the metric to define the difference between two vectors (and other kinds of tensors) at neighboring spacetime points. Using the metric to define parallelism on the spacetime, one obtains a derivative operator (or "covariant derivative") ∇ which is defined as follows. The derivative maps functions to 1-forms using the exterior derivative:

$$\nabla_\mu \phi = \partial_\mu \phi. \tag{10.5}$$

The derivative maps vector fields to tensor fields of type $\binom{1}{1}$:

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta_{\gamma\alpha} v^\gamma, \tag{10.6}$$

where the *Christoffel symbols* are given by

$$\Gamma^\beta_{\gamma\alpha} = \frac{1}{2} g^{\beta\sigma} (\partial_\gamma g_{\alpha\sigma} + \partial_\alpha g_{\gamma\sigma} - \partial_\sigma g_{\alpha\gamma}). \tag{10.7}$$

The derivative maps 1-forms to tensor fields of type $\binom{0}{2}$:

$$\nabla_\alpha w_\beta = \partial_\alpha w_\beta - \Gamma^\gamma_{\beta\alpha} w_\gamma. \tag{10.8}$$

The derivative is extended to all other tensor fields as a derivation (linear, Leibniz product rule) on the algebra of tensor fields. In particular, the Christoffel symbols are defined so that the metric determines parallelism; we have:

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\sigma_{\beta\alpha} g_{\sigma\gamma} - \Gamma^\sigma_{\gamma\alpha} g_{\beta\sigma} = 0. \tag{10.9}$$

PROBLEM: Show that (10.7) enforces $\nabla_\alpha g_{\beta\gamma} = 0$.

Geodesics play an important role in Einstein's theory of gravity. Mathematically, geodesics are curves whose tangent vector T is parallel transported

along the curve, $T^\alpha \nabla_\alpha T^\beta = 0$. If the curve is given parametrically by functions q^μ, *i.e.*, $x^\mu = q^\mu(s)$, where s is proportional to the arc length along the curve[2], then the geodesics are solutions to

$$\frac{d^2 q^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dq^\alpha}{ds} \frac{dq^\beta}{ds} = 0, \tag{10.10}$$

$$g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds} = \kappa, \tag{10.11}$$

where κ is a given constant; $\kappa > 0$ for spacelike curves, $\kappa < 0$ for timelike curves and $\kappa = 0$ for lightlike curves.

PROBLEM: Show that $g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds}$ is a constant of motion for (10.10).

For essentially the same reasons as in Yang-Mills theory, the commutator of two covariant derivatives defines a tensor field called the Riemann *curvature tensor*:

$$2\nabla_{[\alpha} \nabla_{\beta]} w_\gamma = R_{\alpha\beta\gamma}{}^\delta w_\delta, \tag{10.12}$$

where

$$R_{\alpha\beta\gamma}{}^\delta = -2\partial_{[\alpha} \Gamma^\delta_{\beta]\gamma} + 2\Gamma^\epsilon_{\gamma[\alpha} \Gamma^\delta_{\beta]\epsilon}. \tag{10.13}$$

The curvature tensor satisfies the following identities:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}, \tag{10.14}$$

$$\nabla_{[\sigma} R_{\alpha\beta]\gamma\delta} = 0. \tag{10.15}$$

The latter identity is known as the *Bianchi identity*.

The *Ricci tensor* is defined to be

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}. \tag{10.16}$$

The scalar curvature, or *Ricci scalar*, is defined to be

$$R = R^\alpha_\alpha = R^{\gamma\alpha}{}_{\gamma\alpha} \tag{10.17}$$

The *Weyl tensor* is defined (in n dimensions) by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-2}(g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}) + \frac{2}{(n-1)(n-2)} R g_{\alpha[\gamma} g_{\delta]\beta}. \tag{10.18}$$

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[2]Or is an "affine parameter" if the curve is lightlike.

The curvature tensor is completely determined by the Weyl tensor, Ricci tensor, and Ricci scalar through the following formula:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{2}{n-2}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) - \frac{2}{(n-1)(n-2)}Rg_{\alpha[\gamma}g_{\delta]\beta}.$$
$$(10.19)$$

10.2 The Geodesic Hypothesis

Einstein's Big Idea is that gravitation is a manifestation of the curvature of spacetime. In particular, the motion of test particles in a gravitational field is defined to be geodesic motion in the curved spacetime. From this point of view, the principal manifestation of gravitation is a focusing/defocusing of families of geodesics. Let us briefly explore this.

For the geodesic hypothesis to work, it must be that any possible initial conditions for a particle can be evolved uniquely into one of the geodesic curves. This works because geodesics are solutions $x^\mu = q^\mu(s)$ to the geodesic equation (10.10). Like any consistent system of ODEs, the geodesic equations have a well-posed initial value problem. What this means is that at any given point, say x_0^μ, and for any any tangent vector v^μ at this point satisfying (for a given constant κ),

$$g_{\mu\nu}(x_0)v^\mu v^\nu = \kappa, \tag{10.20}$$

there exists a unique solution to the geodesic equations (10.10) such that

$$q^\mu(0) = x_0^\mu, \quad \frac{dq^\mu(0)}{ds} = v^\mu. \tag{10.21}$$

Here is a useful mathematical result with an important physical interpretation. Pick a spacetime event, *i.e.*, a point $p \in M$. Events q sufficiently close to p can be labeled using geodesics as follows. Find a geodesic starting at p which passes through q. This geodesic will be unique if q is in a sufficiently small neighborhood of p. Let u^μ be the components of the tangent vector at p which, along with p, provides the initial data for the geodesic which passes through q. If the geodesic in question is timelike, then normalize u^μ to have length $g_{\alpha\beta}(p)u^\alpha u^\beta = -1$. If the geodesic is spacelike, normalize u^μ to have length $+1$. If the geodesic is null, normalization is not an issue. The geodesic in question will pass through q when the affine parameter s takes some value, say, s_0. Assign to q the coordinates $x^\alpha = s_0 u^\alpha$. It can be shown that this

construction defines a coordinate chart called *geodesic normal coordinates*. This chart has the property that

$$g_{\alpha\beta}(p) = \eta_{\alpha\beta}, \quad \Gamma^{\gamma}_{\alpha\beta}(p) = 0. \tag{10.22}$$

These relations will not in general hold away from the origin p of the normal coordinates. Physically, the geodesics correspond to the local reference frame of a freely falling observer at p. For a sufficiently small spacetime region around p the observer sees spacetime geometry in accord with special relativity insofar as (10.22) is a good approximation for any measurements made. This is the mathematical manifestation of the equivalence principal of Einstein. Sometimes (10.22) is characterized by saying that spacetime is "locally flat", but you should be warned that this slogan is misleading – spacetime curvature can never be made to disappear in any reference frame.[3] Let us see what in fact is the physical meaning of spacetime curvature.

Nearby geodesics will converge/diverge from each other according to the curvature there. This "tidal acceleration" is the principal effect of gravitation and is controlled by the *geodesic deviation equation*, which is defined as follows. Consider a 1-parameter (λ) family of geodesics, $x^{\mu} = q^{\mu}(s, \lambda)$. This means that, for each λ there is a geodesic obeying

$$\frac{d^2 q^{\mu}(s, \lambda)}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dq^{\alpha}(s, \lambda)}{ds} \frac{dq^{\beta}(s, \lambda)}{ds} = 0, \tag{10.23}$$

$$g_{\mu\nu} \frac{dq^{\mu}(s, \lambda)}{ds} \frac{dq^{\nu}(s, \lambda)}{ds} = \kappa, \tag{10.24}$$

The *deviation vector field*

$$r^{\mu}(s, \lambda) = \frac{\partial q^{\mu}(s, \lambda)}{\partial \lambda} \tag{10.25}$$

defines the displacement of the geodesic labeled by $\lambda + d\lambda$ relative to the geodesic labeled by λ. The *relative velocity* $v^{\mu}(s, \lambda)$ of the geodesic labeled by $\lambda + d\lambda$ relative to the geodesic labeled by λ is the directional derivative of the deviation vector in the direction of the geodesic labeled by λ:

$$v^{\mu}(s, \lambda) = \frac{\partial r^{\mu}}{\partial s} + \Gamma^{\mu}_{\nu\sigma} r^{\nu} \frac{\partial q^{\sigma}}{\partial s}. \tag{10.26}$$

---

[3]Roughly speaking, at a given point the zeroth and first derivative content of the metric can be adjusted more or less arbitrarily by a choice of coordinates, but (some of the) second derivative information is immutable.

The *relative acceleration* $a^\mu(s, \lambda)$ of the neighboring geodesic labeled by $\lambda + d\lambda$ relative to the geodesic labeled by λ is the directional derivative of the relative velocity in the direction of the geodesic labeled by λ:

$$a^\mu(s, \lambda) = \frac{\partial v^\mu}{\partial s} + \Gamma^\mu_{\nu\sigma} v^\nu \frac{\partial q^\sigma}{\partial s}. \tag{10.27}$$

A nice exercise is to show that:

$$a^\mu(s, \lambda) = R^\mu{}_{\alpha\beta\gamma} v^\alpha v^\beta r^\gamma. \tag{10.28}$$

This is the precise sense in which spacetime curvature controls the relative acceleration of geodesics. Note that while one can erect a freely falling reference frame such that (10.22) holds at some event, it is not possible to make the curvature vanish at a point by a choice of reference frame. Thus the geodesic deviation (10.28) is an immutable feature of the gravitational field. One can say that gravity *is* geodesic deviation.

10.3 The Principle of General Covariance

The principle of general covariance is usually stated in physical terms as "the laws of physics take the same form in all reference frames". Since physical laws are typically defined using differential equations, and since reference frames in physics can be mathematically represented in terms of coordinates (possibly along with other structures) a pragmatic implementation of the principle is to require that the differential equations characterizing physical laws take the same form in all coordinate systems. A more elegant global form of this statement is that the field equations only depend upon the manifold structure of spacetime for their construction and hence must be unchanged by any transformations which fix that structure. The Einstein field equations (to be described soon) are, of course, the paradigm for such "generally covariant field equations". To better understand the principle of general covariance it is instructive to look at a simple and familiar example of a field equation which is *not* generally covariant.

Consider the wave equation (massless KG equation) for a scalar field φ on a given spacetime (M, g). The wave equation takes the form

$$\Box\varphi \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi = 0. \tag{10.29}$$

This definition can be used to compute the wave equation in any coordinate system, but the form of the equation will, in general, be different in different coordinate systems because the metric and Christoffel symbols – which must be specified in order to define the equation for φ – are different in different coordinate systems. For example, suppose that the spacetime is Minkowski spacetime (so the curvature tensor of g vanishes). Then one can introduce (inertial) coordinates $x^\alpha = (t, x, y, z)$ such that the metric is the usual Minkowski metric and the wave equation is

$$\Box\varphi = (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\varphi = 0. \tag{10.30}$$

Now suppose that we use spherical polar coordinates to label events at the same t. Labeling these coordinates as $x^\alpha = (t, r, \theta, \phi)$ we have

$$\Box\varphi = (-\partial_t^2 + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2 \sin^2 \theta}\partial_\phi^2)\varphi = 0. \tag{10.31}$$

Clearly these differential equations are not the same. The KG equation on a given background spacetime is not "generally covariant".

A mathematically precise way to implement the principle of general covariance in a theory is to demand that all equations defining the theory are constructed using only the underlying manifold structure. The KG equation as described above fails to obey this principle since one needs to specify additional structure – the metric – in order to define the equation for φ. Recall that two manifolds are equivalent if and only if they are related by a *diffeomorphism*, which is a smooth mapping $\phi\colon M \to M$ which has a smooth inverse ϕ^{-1}. The principle of general covariance specifies that the differential equations of interest will be insensitive to diffeomorphisms.

In coordinates, a diffeomorphism is a transformation $x^\alpha \to y^\alpha = \phi^\alpha(x)$ with inverse $x^\alpha = \phi^{-1\alpha}(y)$. Let us consider how the metric behaves under such a transformation. The metric components $\tilde{g}_{\alpha\beta}$, expressed in terms of y^α, are related to the components in terms of x^α by the "pullback" operation:[4]

$$(\phi^*g)_{\alpha\beta}(x) = \frac{\partial\phi^\mu}{\partial x^\alpha}\frac{\partial\phi^\nu}{\partial x^\beta}\tilde{g}_{\mu\nu}(\phi(x)). \tag{10.32}$$

More generally, given any tensor field we can relate its components $\phi^*T^{\alpha\cdots}_{\beta\cdots}(x)$

---

[4]One way to see this is to take the line element defined with y^α and express it in terms of x^α using the mapping ϕ.

in terms of x^μ and its components $\tilde{T}^{\alpha\cdots}_{\beta\cdots}(y)$ in terms of y^μ:

$$(\phi^* T)^{\alpha\cdots}_{\beta\cdots}(x) = \frac{\partial \phi^\nu}{\partial x^\beta} \cdots \left(\frac{\partial \phi^{-1\alpha}}{\partial y^\mu} \right)_{y=\phi(x)} \cdots \tilde{T}^{\mu\cdots}_{\nu\cdots}(\phi(x)). \tag{10.33}$$

Now we exhibit (without proof) a fundamental identity involving the curvature. The curvature formula (10.13) takes a metric[5] g in some coordinate system and produces the curvature tensor $R(g)$ in that same coordinate system. Given a diffeomorphism ϕ – an equivalent presentation of the manifold – the metric changes to $\phi^* g$. Applying (10.13) we get the important result:

$$R(\phi^* g) = \phi^* R(g). \tag{10.34}$$

What this formula says in English is: If you apply the curvature formula to the transformed metric, the result is the same as applying the formula to the untransformed metric and then transforming the result. In this sense the curvature formula is "the same" in all coordinate systems or, more elegantly, is defined from the metric only using the underlying manifold structure. We say that the curvature tensor is "naturally" defined in terms of the metric.[6] More generally, any tensor field obtained from the curvature using the metric and covariant derivatives with the metric-compatible connection will be "naturally" constructed from the metric. We will call such tensor fields *natural*.

The principle of general covariance is the requirement that the field equations for spacetime are naturally constructed from the metric (and any other matter fields which may be present). Just considering spacetime (with no matter), let the field equations be of the form $G = 0$, where $G = G(g)$ is a tensor constructed naturally from the metric g. Supposed g_0 is solution to the field equations, $G(g_0) = 0$. Then, thanks to the property (10.34), we have

$$G(\phi^* g_0) = \phi^* G(g_0) = 0. \tag{10.35}$$

This shows that generally covariant field theories on a manifold M have a very large group of symmetries – the diffeomorphism group of M.

---

[5] We could also phrase this in terms of a connection.

[6] Some people call such an object a "tensor" or "covariant". The former term is clearly ambiguous. We will use the modern terminology: "natural tensor" to denote things like the curvature which obey (10.34).

10.4   The Einstein-Hilbert Action

In this section we shall build the Einstein field equations for the gravitational field "in vacuum", that is, in regions of spacetime where there is no matter. We will obtain the field equations as the Euler-Lagrange equations associated to a famous variational principle obtained by Hilbert and Einstein. To implement the principle of general covariance, we need to use a Lagrangian which is constructed naturally from the metric. Since we aim to vary the metric to get the field equations, we must take care to identify all places where the metric is used.

A Lagrangian density is meant to be the integrand in an action integral. For an integral over a manifold of dimension n, the integrand is most properly viewed as a differential n-form — a completely antisymmetric tensor of type $\binom{0}{n}$. Given a coordinate chart, x^α, $\alpha = 0, 1, 2, 3$, the integral over some region B in that chart of a function $f(x)$ is

$$\int_B f(x)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \equiv \int_B f(x)\, dx^0 dx^1 dx^2 dx^3 \qquad (10.36)$$

This, for example, explains the use of Jacobian determinants to transform coordinates in an integral. Of course, the result of such an integral is as arbitrary as are the coordinates used to make the 4-form $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The metric provides an invariant notion of integration by defining a natural 4-form:

$$\epsilon = \epsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta, \quad \epsilon_{\alpha\beta\gamma\delta} = \frac{1}{24}\sqrt{|g|}\eta_{\alpha\beta\gamma\delta}. \qquad (10.37)$$

Here $\sqrt{|g|}$ is the square root of the absolute value of the determinant of the metric components $g_{\alpha\beta}$ (in the current basis of dual vectors), and $\eta_{\alpha\beta\gamma\delta}$ is the *alternating symbol* defined by the properties: (1) $\eta_{\alpha\beta\gamma\delta}$ is totally antisymmetric, (2) $\eta_{1234} = 1$. We call ϵ the *volume form* defined by the metric. Notice that all 4-forms in four dimensions are proportional to one another; here we have

$$\epsilon = \sqrt{|g|}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \qquad (10.38)$$

The principal property of the volume form (10.37) is contained in the following problem:

PROBLEM: Show that the volume form (10.37) is a natural tensor.

According to the rules for integrating forms, the integral of the volume form over a region B is the volume $V(B)$ of B defined by the metric. In coordinates x^a

$$\int_B \epsilon = \int_B \sqrt{|g|}\, dx^0 dx^1 dx^2 dx^3 = V(B).$$

(10.39)

While the volume form by itself is not suitable as a gravitational Lagrangian, it will be instructive and useful to see how to it varies with respect to variations of the metric. To this end, consider a 1-parameter family of metrics $g(\lambda)$ passing through a given metric $g \equiv g(0)$. As usual we define

$$\delta g_{\alpha\beta} = \left(\frac{\partial g_{\alpha\beta}(\lambda)}{\partial \lambda}\right)_{\lambda=0}.$$

(10.40)

For each λ there is an inverse metric, which satisfies

$$g^{\alpha\beta}(\lambda)g_{\beta\gamma}(\lambda) = \delta^\alpha_\gamma,$$

(10.41)

Differentiating both sides with respect to λ gives

$$\delta g^{\alpha\beta} g_{\beta\gamma} + g^{\alpha\beta}\delta g_{\beta\gamma} = 0,$$

(10.42)

so that

$$\delta g^{\alpha\beta} = -g^{\alpha\gamma}g^{\beta\delta}\delta g_{\gamma\delta}.$$

(10.43)

Henceforth we must remember to make this exception in the usual tensor notation for raising and lowering indices with the metric. Next, we recall a couple of results from linear algebra. Let $\mathbf{A}$ be a non-singular square matrix. We have the identity

$$\ln(\det(\mathbf{A})) = \text{tr}(\ln(\mathbf{A})).$$

(10.44)

Now let $\mathbf{A}(\lambda)$ be a non-singular square matrix depending upon a parameter λ. It follows from (10.44) that

$$\frac{d}{d\lambda}\det(\mathbf{A}(\lambda)) = \det(\mathbf{A}(\lambda))\text{tr}\left(\mathbf{A}^{-1}(\lambda)\frac{d}{d\lambda}\mathbf{A}(\lambda)\right).$$

(10.45)

PROBLEM: Prove (10.44) and (10.45).

From these results it follows that the variation of $g \equiv \det(g_{ab})$ is given by

$$\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta}.$$

(10.46)

Therefore, the variation of the volume form ϵ is given by

$$\delta\epsilon = \left(\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\right)\epsilon. \tag{10.47}$$

After the volume form, the next simplest natural Lagrangian density is the Einstein-Hilbert Lagrangian:

$$\mathcal{L}_{EH} = (const.)R\epsilon, \tag{10.48}$$

where R is the scalar curvature. Einstein's theory – including the cosmological constant Λ – arises from a Lagrangian density which is a combination of the Einstein-Hilbert Lagrangian density and the volume form:

$$\mathcal{L} = \frac{1}{2\kappa}(R - 2\Lambda)\epsilon, \tag{10.49}$$

where κ is given in terms of Newton's constant $\mathcal{G}$ and the speed of light c as

$$\kappa = \frac{8\pi\mathcal{G}}{c^4}$$

To compute the Euler-Lagrange expression for $\mathcal{L}$ we will need to know how to compute the variation of the scalar curvature. Here are the results we will need.[7]

$$\delta\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\nabla_{\alpha}\delta g_{\beta\delta} + \nabla_{\beta}\delta g_{\alpha\delta} - \nabla_{\delta}\delta g_{\alpha\beta}\right), \tag{10.50}$$

$$\delta R_{\alpha\beta\gamma}{}^{\delta} = \nabla_{\beta}\delta\Gamma^{\delta}_{\alpha\gamma} - \nabla_{\alpha}\delta\Gamma^{\delta}_{\beta\gamma}, \tag{10.51}$$

$$\delta R_{\alpha\gamma} = \delta R_{\alpha\beta\gamma}{}^{\beta} = \nabla_{\beta}\delta\Gamma^{\beta}_{\alpha\gamma} - \nabla_{\alpha}\delta\Gamma^{\beta}_{\beta\gamma}. \tag{10.52}$$

Finally,

$$\delta R = \delta g^{\alpha\gamma}R_{\alpha\gamma} + g^{\alpha\gamma}\delta R_{\alpha\gamma} = -R^{\alpha\gamma}\delta g_{\alpha\gamma} + \nabla_{\sigma}\left(g^{\alpha\gamma}\delta\Gamma^{\sigma}_{\alpha\gamma} - g^{\sigma\gamma}\delta\Gamma^{\beta}_{\beta\gamma}\right) \tag{10.53}$$

PROBLEM: Derive equations (10.50)–(10.53).

---

[7]All these formulas can be viewed as functions on the jet space of metrics. From this point of view, the covariant derivative should be viewed as a total derivative plus the Christoffel terms, e.g., $\nabla_{\alpha}\omega_{\beta} = D_{\alpha}\omega_{\beta} - \Gamma^{\gamma}_{\alpha\beta}\omega_{\gamma}$

The variation of the Lagrangian for Einstein's theory is thus given by

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{1}{2\kappa}\left\{(\delta R)\epsilon + (R - 2\Lambda)\delta\epsilon\right\} \\
&= \frac{1}{2\kappa}\left\{-(R^{\alpha\gamma} - \frac{1}{2}g^{\alpha\gamma}R + \Lambda g^{\alpha\gamma})\delta g_{\alpha\gamma} + \nabla_\sigma\left(g^{\alpha\gamma}\delta\Gamma^\sigma_{\alpha\gamma} - g^{\sigma\gamma}\delta\Gamma^\beta_{\beta\gamma}\right)\right\}\epsilon \\
&= \frac{1}{2\kappa}\left\{-(G^{\alpha\gamma} + \Lambda g^{\alpha\gamma})\delta g_{\alpha\gamma} + \nabla_\alpha\Theta^\alpha\right\}\epsilon,
\end{aligned}
\tag{10.54}
$$

where

$$
G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}
\tag{10.55}
$$

is the celebrated *Einstein tensor*, and the vector field featuring in the divergence is

$$
\Theta^\sigma = g^{\alpha\gamma}\delta\Gamma^\sigma_{\alpha\gamma} - g^{\sigma\gamma}\delta\Gamma^\beta_{\beta\gamma}.
\tag{10.56}
$$

The Einstein action for the gravitational field is given by

$$
S_{grav}[g] = \int_M \frac{1}{2\kappa}(R - 2\Lambda)\epsilon.
\tag{10.57}
$$

If we consider metric variations with compact support on a manifold M, the functional derivative of the action (also known as the Euler-Lagrange expression $\mathcal{E}$ for the Einstein Lagrangian) is given by

$$
\mathcal{E}^{\alpha\beta} \equiv \frac{\delta S_{grav}}{\delta g_{\alpha\beta}} = -\frac{1}{2\kappa}(G^{\alpha\gamma} + \Lambda g^{\alpha\gamma})\epsilon.
\tag{10.58}
$$

10.5 Vacuum spacetimes

In the absence of matter, and for a given value of the cosmological constant, the Einstein equations or the metric are

$$
G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 0.
\tag{10.59}
$$

These are usually called the "vacuum Einstein equations" (with a cosmological constant). Using the definition of the Einstein tensor, and contracting this equation with $g_{\alpha\beta}$ ("taking the trace"), we get (in 4-dimensions)

$$
R = 4\Lambda.
\tag{10.60}
$$

Consequently, the vacuum Einstein equations with a cosmological constant Λ are equivalent to (exercise)

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}. \tag{10.61}$$

The vacuum equations are 10 coupled non-linear PDEs for the 10 components of the metric, in any given coordinate system. Despite their complexity, many solutions are known. The majority of the known solutions have $\Lambda = 0$, but here is a pretty famous solution which includes Λ. It is called the "Kottler metric". It is also called the "Schwarzschild-de Sitter metric". In coordinates (t, r, θ, ϕ) it is corresponds to the line element given by

$$ds^2 = -(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)dt^2 + \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{10.62}$$

When $m = 0 = \Lambda$ this is the metric (in spherical coordinates) of the flat spacetime used in special relativity (and throughout this text!). When $\Lambda = 0$ this is the famous Schwarzschild solution. When $m = 0$ and $\Lambda > 0$ this is the de Sitter metric. When $m = 0$ and $\Lambda < 0$ this is the anti-de Sitter metric. The metric (10.62) can be physically interpreted as defining the exterior gravitational field of a spherical body embedded in a space which is (anti) de Sitter at large distances from the body.

PROBLEM: Use your favorite tensor analysis software (*e.g., Differential-Geometry* in Maple) to verify that (10.62) is, for any m, a solution to the vacuum Einstein equations (10.59) with cosmological constant Λ.

10.6   Diffeomorphism symmetry and the contracted Bianchi identity

The Euler-Lagrange expression computed from the gravitational Lagrangian satisfies an important identity by virtue of its "naturality" (also known as "general covariance" or "diffeomorphism invariance"). One way to see this is as follows.

To begin with, the Einstein-Hilbert action is invariant under diffeomorphisms of M in the following sense. Let $\phi\colon M \to M$ be a diffeomorphism. Compute the action

$$S[g] = \int_M \mathcal{L}(g) \tag{10.63}$$

for two different metrics, g and ϕ^*g. Because of naturality of the Lagrangian and the fact that manifolds are invariant under diffeomorphisms[8], we have

$$S[\phi^*g] = \int_M \mathcal{L}(\phi^*g) = \int_M \phi^*(\mathcal{L}(g)) = \int_{\phi(M)} \mathcal{L}(g) = \int_M \mathcal{L}(g) = S[g].$$
$$(10.64)$$

Thus the Einstein action has a diffeomorphism symmetry.

Next we need an important fact about the transformation of a metric by infinitesimal diffeomorphisms.[9] Consider a 1-parameter family of diffeomorphisms, ϕ_λ, with ϕ_0 being the identity. We specialize to infinitesimal diffeomorphisms, characterized by $\lambda << 1$. In coordinates x^α we have

$$\phi_\lambda^\alpha(x) = x^\alpha + \lambda v^\alpha(x) + \mathcal{O}(\lambda^2). \qquad (10.65)$$

It can be shown that ϕ_λ is completely determined by the vector field v^α, which is called the *infinitesimal generator* of the 1-parameter family of diffeomorphisms. A fundamental result of tensor analysis is that

$$\phi_\lambda^* g_{\alpha\beta} = g_{\alpha\beta} + \lambda \delta g_{\alpha\beta} + \mathcal{O}(\lambda^2), \qquad (10.66)$$

where the infinitesimal variation in the metric induced by the family of diffeomorphisms is given by the *Lie derivative L_v*:

$$\delta g_{\alpha\beta} = L_v g_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha. \qquad (10.67)$$

With that little mathematical result in hand, consider the variation of the Einstein action,

$$S_{grav} = \int_M \frac{1}{2\kappa}(R - 2\Lambda)\epsilon \qquad (10.68)$$

induced by an infinitesimal diffeomorphism. We already know that, for *any* variation $\delta g_{\alpha\beta}$ (with compact support away from the boundary of M),

$$\delta S_{grav} = -\frac{1}{2\kappa} \int_M \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta} \right) \delta g_{\alpha\beta}\, \epsilon \qquad (10.69)$$

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[8]Our discussion will also apply to the case where we have a manifold with boundary provided we restrict attention to diffeomorphisms which preserve the boundary

[9]This result holds for any tensor of type $\binom{0}{2}$, and it generalizes to tensor fields of any type via the Lie derivative.

For a variation (10.67) due to an infinitesimal diffeomorphism we have

$$\delta S_{grav} = -\frac{1}{2\kappa} \int_M \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) \left(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha\right) \epsilon$$

$$= -\frac{1}{\kappa} \int_M \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) \nabla_\alpha v_\beta \, \epsilon$$

$$= \frac{1}{\kappa} \int_M \left\{\nabla_\alpha \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) v_\beta - \nabla_\alpha k^\alpha\right\} \epsilon$$

$$= \frac{1}{\kappa} \int_M \nabla_\alpha \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) v_\beta \, \epsilon - \frac{1}{\kappa} \int_{\partial M} k \cdot \epsilon, \qquad (10.70)$$

where

$$(k \cdot \epsilon)_{\beta\gamma\delta} = k^\alpha \epsilon_{\alpha\beta\gamma\delta}, \qquad (10.71)$$

and

$$k^\alpha = \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) v_\beta \qquad (10.72)$$

The last equality in (10.70) follows from the divergence theorem, which can be expressed as:

$$\int_M \left(\nabla_\alpha k^\alpha\right) \epsilon = \int_{\partial M} k \cdot \epsilon. \qquad (10.73)$$

If the manifold M has no boundary, then the boundary term is absent. The boundary could be "at infinity", which can be the limit as $r \to \infty$ of a sphere of radius r. We assume the diffeomorphism is the identity transformation on the boundary, which means the vector field v^α vanishes there (at a fast enough rate, in the sphere at infinity case). Either way, the boundary term is absent and we have the important identity:

$$\delta S_{grav} = \frac{1}{\kappa} \int_M \nabla_\alpha \left(G^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) v_\beta \, \epsilon = \frac{1}{\kappa} \int_M \nabla_\alpha G^{\alpha\beta} v_\beta \, \epsilon, \qquad (10.74)$$

were we used $\nabla_\alpha g_{\beta\gamma} = 0$.

Now we finish the argument. We know *a priori* the action is invariant under diffeomorphisms. This means that for infinitesimal diffeomorphisms (which are trivial on the boundary of M)

$$\delta S_{grav} = 0. \qquad (10.75)$$

On the other hand, we have the identity (10.74), which must vanish for any choice of the vector field v^α thanks to (10.75). Evidently, using the usual calculus of variations reasoning we must have the identity

$$\nabla_\alpha G^{\alpha\beta} = 0. \qquad (10.76)$$

This is the contracted Bianchi identity; it can be derived from the Bianchi identity (10.15).

PROBLEM: Derive (10.76) from (10.15).

While (10.76) can be understood as a consequence of (10.15), we now know its origins can also be traced to the diffeomorphism invariance of the Einstein action. Indeed, from our preceding discussion it is easy to see that *any* natural Lagrangian will give rise to an Euler-Lagrange expression $\mathcal{E}^{\alpha\beta} = \mathcal{E}^{\beta\alpha}$ which satisfies the identity

$$\nabla_\alpha \mathcal{E}^{\alpha\beta} = 0. \qquad (10.77)$$

Finally, you might re-read §5.5. Then you will see that the discussion above is just an instance of Noether's second theorem.

10.7 Coupling to matter - scalar fields

In Einstein's theory of gravity "matter" is any material phenomena which admits an energy-momentum tensor. In the limit where matter's influence on gravitation can be ignored, and when modeling matter as made of "test particles", the geodesic hypothesis describes the motion of matter in a given gravitational field. But to understand where the gravitational field comes from in the first place one needs to go beyond the test particle approximation and consider the interacting system of gravity coupled to matter. In this setting, matter curves spacetime and spacetime affects the motion of the matter.[10] Here we will briefly outline how to use classical field theory techniques to obtain equations which model the curving of spacetime by matter and the propagation of matter in the curved spacetime.

The two principal phenomenological models of classical (non-quantum) matter which are used in gravitational physics to model macroscopic phenomena in the "real world" are fluids and electromagnetic fields. A scalar field also provides a sort of fluid source for gravitation – one with a "stiff equation of state" – and also provides a nice simple warm-up for incorporation of electromagnetic sources. Of course, we are very familiar with scalar

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[10]It is worth mentioning that a parallel set of comments can be made about electric charges and currents in electromagnetic theory. Think about it!

fields by now and, since this *is* a course in field theory, let us begin with this matter field.[11]

We will use a Klein-Gordon field φ. We begin by recalling the definition of its Lagrangian as a top-degree differential form on any curved spacetime (M, g):

$$\mathcal{L}_{KG} = -\frac{1}{2}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + m^2\varphi^2\right)\epsilon. \qquad (10.78)$$

The Euler-Lagrange equation for φ is a curved spacetime generalization of the original Klein-Gordon equation:

$$\nabla^\alpha\nabla_\alpha\varphi - m^2\varphi = 0 \qquad (10.79)$$

You can think of the appearance of the metric in this equation as bringing into play the effect of the gravitational field on the "motion" of the scalar field.

PROBLEM: The following line element represents a class of "big bang" cosmological models, characterized by a scale factor $a(t)$.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \qquad (10.80)$$

Compute the Klein-Gordon equation (10.79). Can you find a solution?

The Lagrangian (10.78) describes the effect of gravity on the field φ. To allow the field to serve as a source of gravity we add the gravitational Lagrangian (10.49) to get

$$\mathcal{L} = \left\{\frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{2}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + m^2\varphi^2\right)\right\}\epsilon. \qquad (10.81)$$

There are now EL equations for the metric and for the scalar field. The EL equation for the scalar field is (10.79), given above. The EL equations for the metric take the form

$$\frac{1}{2\kappa}\left(-G^{\alpha\gamma} + \Lambda g^{\alpha\gamma}\right) + \frac{1}{2}T^{\alpha\gamma} = 0, \qquad (10.82)$$

---

[11]The only fundamental scalar field known to actually exist is the Higgs field, and it should be treated quantum mechanically. Bound states of quarks called "mesons" exist which have spin zero and are represented via scalar fields, but this is principally in the quantum domain. So you should think of the following discussion in the context of a classical limit of a quantum theory or as a phenomenological study of matter with a simple relativistic fluid model built from a scalar field, or as a humble version of the electromagnetic field.

or

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = \kappa T^{\alpha\beta}, \tag{10.83}$$

where

$$T^{\alpha\gamma} = \nabla^\alpha \varphi \nabla^\gamma \varphi - \frac{1}{2} g^{\alpha\gamma} \nabla_\beta \varphi \nabla^\beta \varphi - \frac{1}{2} g^{\alpha\gamma} m^2 \varphi^2 \tag{10.84}$$

is the energy-momentum tensor of the scalar field. The equations (10.83) are the celebrated *Einstein field equations*.

PROBLEM: With

$$S[g, \varphi] = \int_M \left\{ \frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{2}\left(g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi + m^2 \varphi^2\right) \right\} \epsilon \tag{10.85}$$

compute the energy-momentum tensor (10.84) of the scalar field via

$$T^{\alpha\beta} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\alpha\beta}}. \tag{10.86}$$

Note that here the energy-momentum tensor is not defined via Noether's theorem but instead via the coupling of the scalar field to gravity. States of the combined system of matter (modeled as a scalar field) interacting with gravity satisfy the coupled system of 11 non-linear PDEs given by (10.79) and (10.83), known as the Einstein-Klein-Gordon (or Einstein-scalar) system of equations.

PROBLEM: Show that the following metric and scalar field, given in the coordinate chart (t, r, θ, ϕ) define a solution to the Einstein-scalar field equations for $m = 0$.

$$g = -r^2 dt \otimes dt + \frac{2}{1 - \frac{2}{3}\Lambda r^2} dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta \, d\phi \otimes d\phi), \tag{10.87}$$

$$\varphi = \pm \frac{1}{\sqrt{\kappa}} t + const. \tag{10.88}$$

10.8   The contracted Bianchi identity revisited

We saw earlier that the Einstein tensor satisfies the contracted Bianchi identity, $\nabla_\alpha G^{\alpha\beta} = 0$. Evidently, if one manages to find a metric and scalar field satisfying the Einstein field equations (10.83) then the energy-momentum tensor of the scalar field must be divergence free,

$$\nabla_\alpha T^{\alpha\beta} = 0, \quad \text{for solutions to the field equations.} \tag{10.89}$$

You can think of this as a necessary condition for a metric and scalar field to satisfy the Einstein-scalar field system. Let me show you how this condition on the energy-momentum tensor is guaranteed by the KG equation via the diffeomorphism invariance of the action for the metric and scalar field.[12]

The analysis is very much like the one we did for the vacuum theory. Return to the Lagrangian (10.81) and form the action integral over some region M:

$$S[g, \varphi] = \int_M \left\{ \frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{2}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + m^2\varphi^2\right) \right\} \epsilon. \tag{10.90}$$

As before, consider a diffeomorphism $\Psi \colon M \to M$ supported away from the boundary of M. The change in the metric and scalar field induced by a diffeomorphism is via the pullback operation, discussed earlier. Because the Lagrangian is natural, we have that diffeomorphisms are a symmetry group of the action:

$$S[\Psi^*g, \Psi^*\varphi] = S[g, \varphi]. \tag{10.91}$$

Given a 1-parameter family of diffeomorphisms, we can define an infinitesimal diffeomorphism via a vector field v^α on M. The infinitesimal changes in the metric and scalar field are given by the Lie derivatives:

$$\delta g_{\alpha\beta} = L_v g_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha, \quad \delta\varphi = L_v\varphi = v^\alpha \nabla_\alpha\varphi. \tag{10.92}$$

For any variations of compact support the change in the action is

$$\delta S = \int_M \left\{ \left[\frac{1}{2\kappa}\left\{-G^{\alpha\gamma} + \Lambda g^{\alpha\gamma}\right\} + \frac{1}{2}T^{\alpha\gamma} \right] \delta g_{\alpha\gamma} + \left[\nabla^\alpha\nabla_\alpha\varphi - m^2\varphi\right]\delta\varphi \right\} \epsilon. \tag{10.93}$$

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[12]You should compare this discussion with its analog in scalar electrodynamics in §6.3.

For variations corresponding to infinitesimal diffeomorphisms the action is unchanged by the argument we used earlier, so we have the identity

$$0 = \int_M \left\{ \left[\frac{1}{2\kappa} \{-G^{\alpha\gamma} + \Lambda g^{\alpha\gamma}\} + \frac{1}{2}T^{\alpha\gamma} \right] (\nabla_\alpha v_\gamma + \nabla_\gamma v_\alpha) \right. \tag{10.94}$$

$$\left. + \left[\nabla^\alpha \nabla_\alpha \varphi - m^2 \varphi \right] v^\alpha \nabla_\alpha \varphi \right\} \epsilon \tag{10.95}$$

This identity must hold for all v^α of compact support. Integrating by parts via the divergence theorem, with the boundary terms vanishing because v^α is of compact support, we can remove the derivatives of the vector field v^α to get

$$\int_M \left\{ -\nabla_\alpha \left[\frac{1}{\kappa} \{-G^{\alpha\gamma} + \Lambda g^{\alpha\gamma}\} + T^{\alpha\gamma} \right] v_\gamma + \left[\nabla^\alpha \nabla_\alpha \varphi - m^2 \varphi \right] v^\alpha \nabla_\alpha \varphi \right\} \epsilon = 0, \tag{10.96}$$

which can be simplified (using the contracted Bianchi identity and covariant constancy of the metric) to

$$\int_M \left[\nabla_\alpha T^\alpha_\beta - \left(\nabla^\alpha \nabla_\alpha \varphi - m^2 \varphi \right) \nabla_\beta \varphi \right] v^\beta \epsilon = 0. \tag{10.97}$$

Since v^α is arbitrary on the interior of M, it follows that

$$\nabla_\alpha T^\alpha_\beta - \left(\nabla^\alpha \nabla_\alpha \varphi - m^2 \varphi \right) \nabla_\beta \varphi = 0, . \tag{10.98}$$

PROBLEM: Show by direct calculation (differentiation of the energy-momentum tensor) that it satisfies (10.98).

From this identity it is easy to see that any solution (g, φ) to the KG equation (10.79) will obey the compatibility condition (10.89). Recall that (10.89) was a necessary condition implied by the Einstein field equations (10.83) and the contracted Bianchi identity. In this way the coupled Einstein-scalar field equations are compatible. But it is even more interesting to note that any solution (g, φ) to the Einstein field equations (10.83) alone, provided $\nabla\varphi \neq 0$, will automatically satisfy the KG equation (10.79)! The logic goes as follows. If (g, φ) define a solution to (10.83) then by the Bianchi identity the energy-momentum tensor built from the solution (g, φ) has vanishing covariant divergence. But then the identity (10.98) and the assumption that

φ is not constant means that (g, φ) must satisfy (10.79). So, if we opt to work on the space of non-constant functions φ only, which is reasonable since constant functions cannot satisfy the KG equation with $m \neq 0$, then the equations of motion for matter are already contained in the Einstein field equations!

The divergence condition $\nabla_a T^{ab} = 0$ is closely related to – but not quite the same as – conservation of energy and momentum. You will recall that in flat spacetime conservation of energy-momentum corresponds to the fact that the energy-momentum tensor is divergence free, which is interpreted as providing a divergence-free collection of currents. Here we do not have an ordinary divergence of a vector field, but rather a covariant divergence of a tensor field:

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\alpha_{\alpha\sigma} T^{\sigma\beta} + \Gamma^\beta_{\alpha\sigma} T^{\alpha\sigma}. \tag{10.99}$$

For this reason one cannot use the divergence theorem to convert (10.89) into a true conservation law.[13] Physically, this state of affairs reflects the fact that the matter field φ can exchange energy and momentum with the gravitational field; since the energy momentum tensor pertains only to the matter field it need not be conserved by itself. In a freely falling reference frame the effects of gravity disappear to some extent and one might expect at least an approximate conservation of energy-momentum of matter. This idea can be made mathematically precise by working in geodesic normal coordinates (defined earlier). At the origin of such coordinates the Christoffel symbols vanish and the coordinate basis vectors can be used to define divergence free currents at the origin. So, in an infinitesimally small region around the origin of such coordinate system – physically, in a suitably small freely falling reference frame – one can define an approximate set of conservation laws for the energy-momentum (and angular momentum) of matter. But because gravity *can* carry energy and momentum this conservation law cannot be extended to any finite region. The crux of the matter is that there is no useful way to define gravitational energy-momentum *densities* – there is no suitable energy-momentum current for gravity. Indeed, if you had such a current, you ought to be able to make it vanish at the origin of normal coordinates, whence it can't be a purely geometrical quantity or it should

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[13]There are actually two reasons for this. First of all, as just mentioned, the ordinary divergence is not zero! And, even if it were, there is no useful definition of the integral of a vector (namely, the divergence of T) over a volume in the absence of additional structures, such as preferred vector fields with which to take components.

vanish in any coordinates. In fact, it is possible to prove that any vector field locally constructed from the metric and its derivatives to any order which is divergence-free when the Einstein equations hold is a trivial conservation law in the sense of §3.20.[14]

The upshot of these considerations is that gravitational energy, momentum, and angular momentum will not occur via the usual mechanism of conserved currents. Thanks to the equivalence principle, gravitational energy, momentum, and angular momentum must be non-local in character. You may recall that an analogous situation arose for the sources of the Yang-Mills field. Despite our inability to localize gravitational energy and momentum, it *is* possible to define a notion of conserved *total* energy, momentum, and angular momentum for gravitational systems which are suitably *isolated* from their surroundings. So, for example, it is possible to compute these quantities for a star or galaxy if we ignore the rest of the universe. Based on our preceding discussion you will not be surprised to hear that such quantities are not constructed by volume integrals of densities. Perhaps a future version of this text will explain how all that works.

10.9 PROBLEMS

1. Show that (10.7) enforces $\nabla_\alpha g_{\beta\gamma} = 0$.

2. Show that $g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds}$ is a constant of motion for (10.10).

3. Show that the volume form (10.37) is a natural tensor.

4. Prove (10.44) and (10.45).

5. Derive equations (10.50)–(10.53).

6. Use your favorite tensor analysis software (*e.g., DifferentialGeometry* in Maple) to verify that (10.62) is, for any m, a solution to the vacuum Einstein equations (10.59) with cosmological constant Λ.

7. Derive (10.76) from (10.15).

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[14]I. Anderson and C. Torre, *Phys. Rev. Lett.* **70**, 3525 (1993).

8. The following line element represents a class of "big bang" cosmological models, characterized by a scale factor $a(t)$.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \tag{10.100}$$

Compute the Klein-Gordon equation (10.79). Can you find a solution?

9. With

$$S[g, \varphi] = \int_M \left\{ \frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{2}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + m^2\varphi^2\right) \right\} \epsilon \tag{10.101}$$

compute the energy-momentum tensor (10.84) of the scalar field via

$$T^{\alpha\beta} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\alpha\beta}}. \tag{10.102}$$

10. Show that the following metric and scalar field, given in the coordinate chart (t, r, θ, ϕ) define a solution to the Einstein-scalar field equations for $m = 0$.

$$g = -r^2 dt \otimes dt + \frac{2}{1 - \frac{2}{3}\Lambda r^2} dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi), \tag{10.103}$$

$$\varphi = \pm\frac{1}{\sqrt{\kappa}}t + const. \tag{10.104}$$

11. Repeat the analysis of §10.7 and §10.8 for the case where the matter field is the electromagnetic field defined by

$$\mathcal{L}_{EM} = -\frac{1}{4}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}\,\epsilon, \tag{10.105}$$

where

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{10.106}$$

12. Show by direct calculation (differentiation of the energy-momentum tensor) that it satisfies

$$\nabla_\alpha T^{\alpha\beta} = \nabla^\beta\varphi\left(\nabla^\alpha\nabla_\alpha\varphi - m^2\varphi\right). \tag{10.107}$$

Chapter 11

Goodbye

If you have stayed with me to the end of this exhilarating mess, I congratulate you. If you feel like there is a lot more you would like to learn about classical field theory, then I agree with you. I have only given a simple introduction to some of the possible topics. Slowly but surely I hope to add more to this text, correct errors, add problems, *etc.* so check back every so often to see if the version has been updated. Meanwhile, the next section has a list of resources which you might want to look at, depending upon your interests.

11.1   Suggestions for Further Reading

General Relativity, R. M. Wald, University of Chicago Press

Spacetime and Geometry, S. Carroll, Addison-Wesley

Tensors, Differential Forms, and Variational Principles, D. Lovelock and H. Rund, Dover

Lecture notes on classical fields, J. Binney,
    http://www-thphys.physics.ox.ac.uk/user/JamesBinney/classf.pdf

Classical Field Theory, D. Soper, Dover

Applications of Lie Groups to Differential Equations, D. Olver, Springer

Classical Theory of Fields, L. Landau and E. Lifshitz, Pergamon Press

Dynamical Theory of Groups and Fields, B. DeWitt, Routledge

Classical Field Theory and the Stress-Energy Tensor, S. Swanson, Morgan & Claypool Publishers

Quantum Field Theory, M. Srednicki,
    web.physics.ucsb.edu/∼mark/ms-qft-DRAFT.pdf