DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **25**, Number **4**, December **2009**

doi:10.3934/dcds.2009.25.1219

pp. **1219–1227**

RESONANCE AND NONRESONANCE FOR P-LAPLACIAN PROBLEMS WITH WEIGHTED EIGENVALUES CONDITIONS

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(Communicated by Nikolaos Papageorgiou)

Abstract. We study multiplicity of solutions for a quasilinear elliptic problem related to the p-Laplacian operator. Our assumptions rely on the first eigenvalue depending on a weight function. We treat both resonant and non-resonant cases.

1. **Introduction.** Let us consider the problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and Δ_p denotes the p-Laplace operator, i.e., $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type and satisfies $f(x,0) \equiv 0$. We are interested in solving (1) by doing assumptions on the asymptotic behavior of f, i.e., by considering

$$l_\pm(x) = \liminf_{t\to\pm 0} \frac{pF(x,t)}{|t|^p} \quad \text{and} \quad K_\pm(x) = \limsup_{t\to\pm\infty} \frac{pF(x,t)}{|t|^p},$$

where $F(x,s) = \int_0^s f(x,t)dt$. We allow $l_\pm, K_\pm$ to lie in a borderline space $L^r(\Omega)$ where

$$r = N/p \ \text{ if } \ 1 < p < N \ \text{ and } \ r = 1 \text{ if } p > N. \tag{2}$$

This class of problems, with more regularity on $l_\pm, K_\pm$, has been studied by many authors (see [1] and references therein). Here we study multiplicity of solutions for (1) in the resonant and nonresonant cases. To this aim we use variational methods and we obtain results related to [1, 4, 6, 8, 9].

The paper is organized as follows: In Section 2 we recall some basic results on the first weighted eigenvalue of the p-Laplacian. Section 3 deals with resonance and nonresonance around λ_1. In Section 4 and 5 we deal with strong resonance and near resonance at λ_1, respectively.

2000 *Mathematics Subject Classification.* Primary: 35J25; Secondary: 58E05.
Key words and phrases. p-Laplacian, indefinite weight, multiplicity of solution.
This work was supported by CNPq/Brazil - FNRS/Belgium.

The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|$. The positive and negative part of a function u are denoted by $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$. If A is a measurable set of $\mathbb{R}^N$, $|A|$ stands for its Lebesgue measure.

2. **The eigenvalue problem.** In this section we collect some results on the eigenvalue problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega) \tag{3}$$

where $m \in L^r(\Omega)$ is a weight such that $m^+ \not\equiv 0$ and r satisfies condition (2). This is a borderline condition in the sense that no less integrability on m can be required. In this case the imbedding $W_0^{1,p}(\Omega) \subset L^{rp'}(\Omega)$ is not compact, since $rp' = p^*$. This lack of compactness can be overcome by some weak continuity, as stated in the next proposition (see [11, Lemma 2.13]):

Proposition 1. *The mapping $u \longmapsto \int_\Omega m|u|^p dx$, $u \in W_0^{1,p}(\Omega)$ is weakly continuous.*

Therefore we can use the Rayleigh minimum formula to define the first positive eigenvalue of (3). We refer to [10, Theorem 4.1] and [7, Proposition 3.1] for the proof of the following proposition.

Proposition 2. *The problem (3) admits a first positive eigenvalue, given by*

$$\lambda_1(m) := \inf\left\{\int_\Omega |\nabla u|^p dx; u \in W_0^{1,p}(\Omega), \int_\Omega m|u|^p dx = 1\right\}.$$

Moreover, $\lambda_1(m)$ is simple and it is achieved by $\varphi_m > 0$ a.e. in Ω.

Let us remark (as observed in [7]) that under the assumptions on m we cannot ensure any regularity for solutions of (3). However, by Proposition 1.2 of [5], solutions of (3) are in $L^t(\Omega)$, for all $t \in [1, \infty)$.

By means of Picone's identity, we can show that if (u, λ) is an eigenpair of (3) with $\lambda > \lambda_1$ then u is sign-changing (see [2]). Furthermore we get an estimate on the measure of *supp* (u^-) and, as a consequence, the isolation of λ_1 in the spectrum of (3). For the proof of Corollary 1, we refer to [2].

Proposition 3. *Let (u, λ) be a solution of (3), with $\lambda > \lambda_1$. Then*

$$|supp\ (u^-)| \geq \ (C\lambda||m||_r)^{-\gamma},$$

where C and γ are positive constants depending only on N and p.

Proof. Let us set $\Omega^- := supp\ (u^-)$.
If $p < N$ we take u^- as test function in (3) and we obtain, for some $s \in]\frac{p^*}{p}, p^*[$:

$$\begin{aligned} \int_\Omega |\nabla u^-|^p dx &= \lambda \int_\Omega m(u^-)^p dx \leq \lambda ||m||_{\frac{N}{p}} ||u^-||_s^p |\Omega^-|^{1-\frac{p}{N}-\frac{1}{s}} && (4) \\ &\leq C\lambda ||m||_{\frac{N}{p}} ||u^-||_{p^*}^p |\Omega^-|^{1-\frac{p}{N}-\frac{1}{s}}, && (5) \end{aligned}$$

where we applied Hölder inequality and the imbedding $L^{p^*}(\Omega) \subset L^s(\Omega)$. Now, by Sobolev inequality, we have

$$\overline{C}||u^-||_{p^*}^p \leq \int_\Omega |\nabla u^-|^p dx$$

for some $\overline{C} = \overline{C}(N,p)$, which combined with (5) yields the estimate.
If $p > N$, from the imbedding $W_0^{1,p}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ we have

$$\int_\Omega |\nabla u^-|^p dx = \lambda \int_\Omega m|u^-|^p dx \leq \lambda ||m||_1 ||u^-||_\infty^p.$$

On the other hand, Morrey's theorem provides a constant $C_3 = C_3(N,p)$ such that

$$||u^-||_\infty \leq C_3 |\Omega^-|^{-\frac{1}{p}+\frac{1}{N}} ||\nabla u^-||_p.$$

□

Corollary 1. *$\lambda_1(m)$ is isolated in the spectrum of (3).*

3. **Non-resonance and resonance with respect to λ_1.** We aim to obtain multiplicity of solutions of (1) under assumptions on

$$l_\pm(x) = \liminf_{t\to\pm 0} \frac{pF(x,t)}{|t|^p} \quad \text{and} \quad K_\pm(x) = \limsup_{t\to\pm\infty} \frac{pF(x,t)}{|t|^p}. \tag{6}$$

We assume that the above limits are uniform in x, have nontrivial positive parts and belong to $L^r(\Omega)$, with r as in (2). Moreover, let f have a subcritical growth:

$$|f(x,t)| \leq c|t|^{q-1} + b(x), \quad \text{a.e in} \quad \Omega, \quad t \in \mathbb{R}, \tag{7}$$

where $p < q < p^*$ if $1 < p < N$ and $p < q < \infty$ if $N < p$, $b \in L^{\overline{q}'}(\Omega)$ where $\overline{q} = \frac{p^*}{s}$ with $p < s < p^*$, and c is a constant.

Theorem 3.1. *Assume that $\lambda_1(l_\pm) < 1 < \lambda_1(K_\pm)$. Then problem (1) admits two nontrivial solutions $u_+ \geq 0$ and $u_- \leq 0$.*

Proof. We are going to find a nontrivial solution u_+ of the problem

$$\begin{cases} -\Delta_p u = f_+(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t \geq 0, \\ 0, & t \leq 0. \end{cases}$$

and show that $u_+ \geq 0$, what proves that u_+ is a solution of (1). In a similar way we can find a solution $u_- \leq 0$.

We define

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F_+(x,u)dx, \quad u \in W_0^{1,p}(\Omega)$$

where $F_+(x,u) = \int_0^u f_+(x,t)dt$. We start by proving that Φ is coercive. By the uniformity on x of the limits in (6), for each $\varepsilon > 0$ there exists $c_\varepsilon \in L^r(\Omega)$ (we can assume c_ε positive) such that

$$pF_+(x,t) \leq (K_+(x) + \epsilon)|t|^p + c_\varepsilon(x) \quad \text{for} \quad t \in \mathbb{R},$$

so that

$$\begin{aligned} \Phi(u) \quad &\geq \quad \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega K_+(x)|u^+|^p dx - \frac{\varepsilon}{p}\int_\Omega |u^+|^p dx - C \\ &\geq \quad \frac{1}{p}\left(1 - \frac{1}{\lambda_1(K_+)} - \frac{\varepsilon}{\lambda_1(1)}\right)\int_\Omega |\nabla u^+|^p dx + \frac{1}{p}\int_\Omega |\nabla u^-|^p dx - C, \end{aligned}$$

where we used the variational characterization of $\lambda_1(K_+)$. Now, as $\lambda_1(K_+) > 1$ we can pick ε such that $\left(1 - \frac{1}{\lambda_1(K_+)} - \frac{\varepsilon}{\lambda_1(1)}\right) > 0$. So Φ is coercive and therefore every (PS) sequence for Φ is bounded. By the subcritical growth of f we have that every bounded (PS) sequence has a convergent subsequence. Hence Φ satisfies (PS). Moreover as Φ is bounded below, it achieves its infimum. Let (u_n) be a minimizing sequence for Φ. We apply Ekeland variational principle in order to get a minimizing (PS) sequence, which has, by the (PS) condition, a subsequence converging to a critical point u of Φ. In order to prove that u is nontrivial we show that 0 is not a local minimizer for Φ. From (6) we have that for $\varepsilon > 0$ there is $\delta > 0$ such that

$$pF_+(x,t) \geq l_+(x)t^p - \epsilon t^p, \quad \text{for} \quad 0 < t \leq \delta. \tag{9}$$

On the other hand, from the growth assumption on f we have that, for $t \geq \delta$,

$$\begin{aligned} F_+(x,t) &\geq -ct^q - \beta(x)t \\ &\geq -ct^q - d\beta(x)t^s - \varepsilon\frac{t^p}{p} + \frac{l_+(x)}{p}t^p - d\frac{|l_+(x)|}{p}t^s, \end{aligned} \tag{10}$$

for some $d > 0$ and $s > p$. By (9) we have that (10) holds for every $t > 0$.

Then

$$\begin{aligned} \Phi(t\varphi_{l_+}) &\leq \frac{t^p}{p}(\lambda_1(l_+) - 1 + \varepsilon||\varphi_{l_+}||_p^p) + ct^q||\varphi_{l_+}||_q^q + d\, t^s \int_\Omega \beta|\varphi_{l_+}|^s dx \\ &+ \frac{t^s}{p}\int_\Omega |l_+||\varphi_{l_+}|^s dx. \end{aligned}$$

Now, since $\lambda_1(l_+) < 1$ and $q, s > p$ we see that the right-hand side expression is negative if t and ε are small enough. We conclude the proof by observing that if we take u^- as test function in (8) we get $\int_\Omega |\nabla u^-|^p dx = \int_\Omega f_+(x,u)u^- dx = 0$, so $u \geq 0$ solves (1). $\square$

In order to deal with the resonant case, we introduce the L^r-functions $k_\pm, L_\pm$ defined by

$$k_\pm(x) = \liminf_{t\to\pm\infty} \frac{pF(x,t)}{|t|^p} \quad \textit{and} \quad L_\pm(x) = \limsup_{t\to 0^\pm} \frac{pF(x,t)}{|t|^p} \tag{11}$$

and assume that $k_\pm, L_\pm$ have nontrivial positive parts, and that the limits are uniform a.e. in Ω.

Theorem 3.2. *Under (7), assume that* $\max\{\lambda_1(k_\pm)\} = 1 < \lambda_1(L_\pm)$,

$$\lim_{|t|\to\infty} [tf(x,t) - pF(x,t)] = -\infty, \quad \text{and} \tag{12}$$

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = K(x) \in L^r. \tag{13}$$

Then problem (1) admits a nontrivial solution.

Proof. We apply the mountain-pass theorem in order to get a positive critical value for the functional

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,p}(\Omega).$$

By (7), (12) and (13) we can show that Φ satisfies the Cerami condition (we can follow the steps of the proof of [3, Lemma 2] and make use of Proposition 1). Let

us show that the expected geometry holds for Φ:
Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$pF(x,t) \leq \begin{cases} (L_+(x)+\varepsilon)|t|^p & \text{for} \quad 0<t<\delta \\ (L_-(x)+\varepsilon)|t|^p & \text{for} \quad 0<-t<\delta. \end{cases}$$

Moreover, by (7) we have, for s as in (7), that

$$F(x,t) \leq \frac{1}{p}L_{\pm}(x)|t|^p + \frac{\epsilon}{p}|t|^p + c|t|^q + cb(x)|t|^s \quad \forall\, t \in \mathbb{R}.$$

Hence

$$\begin{aligned} \Phi(u) \quad \geq \quad & \frac{1}{p}\left(1-\frac{1}{\lambda_1(L_+)}-\frac{\varepsilon}{\lambda_1(1)}\right)\|u^+\|^p + \frac{1}{p}\left(1-\frac{1}{\lambda_1(L_-)}-\frac{\varepsilon}{\lambda_1(1)}\right)\|u^-\|^p \\ & - C\|u\|^q - C\|u\|^s, \end{aligned}$$

and we can choose $\varepsilon > 0$ such that $(1-\frac{1}{\lambda_1(L_\pm)}-\frac{\varepsilon}{\lambda_1(1)}) > 0$ so that $\Phi(u) > 0$ if $\|u\| = \rho$ for some $\rho > 0$.

Now we prove that $\Phi(t\varphi_{k_+}) \to -\infty$ as $t \to \infty$. Let us assume that $\mu_1(k_+) = 1$.

Given $\epsilon > 0$ there is an L^r-function $c(x)$ (depending on ϵ) such that

$$pF(x,t) \geq (k_+(x)-\epsilon)|t|^p + c(x) \quad \text{for} \quad t \geq 0.$$

Then for $t \geq 0$, we have

$$\begin{aligned} \Phi(t\varphi_{k_+}) \quad &= \quad \frac{1}{p}\int_\Omega |t\nabla\varphi_{k_+}|^p dx - \int_\Omega F(x,t\varphi_{k_+})dx \\ &= \quad \frac{1}{p}\int_\Omega k_+(x)(t\varphi_{k_+})^p dx - \int_\Omega F(x,t\varphi_{k_+})dx \\ &= \quad \frac{1}{p}\int_\Omega H(x,t\varphi_{k_+})dx. \end{aligned}$$

Here $H(x,s) = k_+(x)|s|^p - pF(x,s)$ and by (12) we see that $H(x,s) \to -\infty$ as $s \to \infty$ uniformly a.e. in Ω (see for instance [4, Lemma 2]). Since φ_{k_+} is positive in Ω, we infer that

$$\limsup_{t\to\infty} \Phi(t\varphi_{k_+}) \leq \frac{1}{p}\int_\Omega \limsup_{t\to\infty} H(x,t\varphi_{k_+})dx = -\infty.$$

□

4. **Strong resonance at infinity.** Now let $a, b \in L^r(\Omega)$, with r given by (2). Consider the problem

$$\begin{cases} -\Delta_p u = a(x)(u^+)^{p-1} - b(x)(u^-)^{p-1} + f(x,u) & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where

$$|f(x,s)| \leq \alpha(x)|s|^{\sigma-1} + \beta(x) \tag{15}$$

for some $1 < \sigma < p$, $\alpha \in L^{(\frac{p^*}{\sigma})'}(\Omega)$, $\beta \in L^{(p^*)'}(\Omega)$. We assume that F satisfies

$$F(x,s) \to F_\pm(x) \text{ as } s \to \pm\infty \tag{16}$$

uniformly a.e. in Ω with $F_\pm \in L^1(\Omega)$ and $\int_\Omega F_\pm(x) \leq 0$. We set

$$J(u) = \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx - \int_\Omega [a(u^+)^p - b(u^-)^p]dx\right) - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,p}(\Omega). \tag{17}$$

Proposition 4. *Assume (15), (16) and that* $\min\{\lambda_1(a),\lambda_1(b)\}=1$. *Then* J *satisfies the* $(PS)_c$ *condition for all* $c<\min\left(-\int_\Omega F_+(x)dx, -\int_\Omega F_-(x)dx\right)$.

Proof. Let (u_n) be a $(PS)_c$ sequence and suppose that it is unbounded. We can assume that $||u_n||\to\infty$. By setting $v_n := \frac{u_n}{||u_n||}$ we have, passing to a subsequence if necessary, that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ and $v_n \to v$ in L^q for every $q<p^*$. By the $(PS)_c$ condition we have

$$\left|\frac{1}{p}\left(\int_\Omega |\nabla v_n|^p dx - \int_\Omega [a(x)(v_n^+)^p + b(x)(v_n^-)^p]dx\right) - \int_\Omega \frac{F(x,u_n)}{||u_n||^p}dx\right| \leq \frac{M}{||u_n||^p},$$

for some $M>0$. From (15) and (16) we have that $|F(x,s)|\leq \psi(x)$ for some $\psi\in L^1(\Omega)$. It follows that

$$\int_\Omega \frac{F(x,u_n)}{||u_n||^p}dx \to 0,$$

so $||v||^p = \int_\Omega [a(x)(v^+)^p + b(x)(v^-)^p]dx$, which implies that

$$\int_\Omega |\nabla v^+|^p dx \leq \int_\Omega a(x)(v^+)^p dx \quad \text{or} \quad \int_\Omega |\nabla v^-|^p dx \leq \int_\Omega b(x)(v^-)^p dx.$$

Thus, by the assumption on $\lambda_1(a)$ and $\lambda_1(b)$ we must have $v^+=\varphi_a$ or $v^-=\varphi_b$ or $v=0$. If $v=0$ then $||v_n||\to 0$, which contradicts $||v_n||=1$. We assume that $v^+=\varphi_a$ (the remaining case is analogous). Then $v=\varphi_a$ and so we have that $u_n\to\infty$ a.e. on Ω. Now, as (u_n) is a $(PS)_c$ sequence, for $\varepsilon>0$ given there exists n_0 such that $J(u_n)\leq c+\varepsilon$ for all $n\geq n_0$. From the assumption $\min\{\lambda_1(a),\lambda_1(b)\}=1$ we get

$$-\int_\Omega F_+(x,u_n(x))dx \leq c+\varepsilon$$

for every $n\geq n_0$, and so

$$-\int_\Omega F_+(x)dx \leq c,$$

which contradicts the assumption on c. □

In the next theorem we make assumptions on the asymptotic behaviour of the right-hand side of (14). We suppose that

$$l_\pm(x) = \liminf_{t\to\pm 0} \frac{pF(x,t)}{|t|^p} + \begin{cases} a(x), & t\to +0 \\ b(x), & t\to -0. \end{cases} \tag{18}$$

and

$$L_\pm(x) = \limsup_{t\to\pm 0} \frac{pF(x,t)}{|t|^p} + \begin{cases} a(x), & t\to +0 \\ b(x), & t\to -0. \end{cases} \tag{19}$$

are uniform in x, have nontrivial positive parts and belong to $L^r(\Omega)$, with r as in (2).

Theorem 4.1. *Under the assumptions of Proposition (4), assume further* $\lambda_1(l_\pm)<1$ *and* $F_\pm\leq 0$. *Then (14) admits two nontrivial solutions.*

Proof. Once again we consider

$$f_+(x,t) = \begin{cases} f(x,t), & t\geq 0, \\ 0, & t\leq 0. \end{cases}$$

and the problem

$$\begin{cases} -\Delta_p u = a(x)(u^+)^{p-1} + f_+(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

If we set $F_+(x,t) = \int_0^t f_+(x,s)ds$ and

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega a(u^+)^p dx - \int_\Omega F_+(x,u)dx, \quad u \in W_0^{1,p}(\Omega).$$

then

$$\lim_{s\to\pm\infty} F_+(x,s) = \begin{cases} F_+(x), & s \to +\infty, \\ 0, & s \to -\infty. \end{cases}$$

and $\Phi(u) = J(u)$ for $u \geq 0$.

By (16) we have that $|\int_\Omega F_+(x,s)dx| \leq C$ for a constant C, so that

$$\Phi(u) \geq \frac{1}{p}\left(1 - \frac{1}{\lambda_1(a)}\right)||u||^p - C,$$

viz, Φ is bounded below. From (18) for $\varepsilon > 0$ we can find $\delta > 0$ such that

$$pF_+(x,t) \geq (l_+(x) - a(x) - \varepsilon)|t|^p$$

for $0 < t < \delta$. On the other hand, from (7), for $t \geq \delta$ we get

$$\begin{aligned} F_+(x,t) &\geq -ct^q - \beta(x)t \\ &\geq -ct^q - d\beta(x)t^s - \varepsilon\frac{t^p}{p} + \frac{l_+(x)-a(x)}{p}t^p - d\frac{|l_+(x)-a(x)|}{p}t^s, \end{aligned} \tag{21}$$

for some $d > 0$ and $s > p$. So (21) holds for all $t > 0$ and

$$\begin{aligned} \Phi(t\varphi_{l_+}) \leq\ & \frac{t^p}{p}(\lambda_1(l_+) - 1 + \varepsilon||\varphi_{l_+}||_p^p) + ct^q||\varphi_{l_+}||_q^q + dt^s\int_\Omega \beta|\varphi_{l_+}|^s dx \\ & + \frac{t^s}{p}\int_\Omega |l_+ - a||\varphi_{l_+}|^s dx, \end{aligned}$$

which is negative if we choose $\varepsilon > 0$ and $t > 0$ small enough.

Then

$$\inf_{W_0^{1,p}(\Omega)} \Phi < 0 \leq -\int_\Omega F_+(x)dx$$

and by Proposition (4) we can obtain a critical point $u \neq 0$ for Φ. Taking u^- as test function in (20) we get $\int_\Omega |\nabla u^-|^p dx = \int_\Omega a(u^+)^{p-1}u^- dx - \int_\Omega f_+(x,u^-)u^- dx = 0$, so $u \geq 0$ is a solution of (14). ☐

5. **Near resonance with respect to λ_1.** Again let $a, b \in L^r(\Omega)$, with r given by (2). Consider the problem

$$\begin{cases} -\Delta_p u = \lambda a(x)(u^+)^{p-1} - \gamma b(x)(u^-)^{p-1} + f(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where $\lambda, \gamma \in \mathbb{R}$ and f satisfies (15).

We consider (22) under a nonresonant condition of the type 'below the first eigenvalue', i.e., we assume that λ and γ are sufficiently near to $\lambda_1(a)$ and $\lambda_1(b)$, respectively, from the left.

Theorem 5.1. *Under (15), assume further that* $\lim_{|s|\to\infty} F(x,s) = \infty$. *Then there exists* $\epsilon > 0$ *for which (14) admits at least three nontrivial solutions if* $\lambda_1(a) - \epsilon < \lambda < \lambda_1(a)$ *and* $\lambda_1(b) - \epsilon < \gamma < \lambda_1(b)$.

Proof. We use a standard minimization procedure to prove the existence of two local minima of J, defined by,

$$J(u) = \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx - \int_\Omega [\lambda a(u^+)^p - \gamma b(u^-)^p] dx\right) - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,p}(\Omega).$$

Thereafter we find a third critical point of J by a Mountain Pass argument. Let us start by showing that J is coercive:

$$\begin{aligned} J(u) \quad &\geq \quad \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1(a)}\right)||u^+||^p + \frac{1}{p}\left(1 - \frac{\gamma}{\lambda_1(b)}\right)||u^-||^p - \int_\Omega F(x,u)dx \\ &\geq \quad \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1(a)}\right)||u^+||^p + \frac{1}{p}\left(1 - \frac{\gamma}{\lambda_1(b)}\right)||u^-||^p \\ &\quad\quad -C||u||^\sigma - C||u||, \end{aligned}$$

where we used the variational characterization of $\lambda_1(a), \lambda_1(b)$ and the growth condition on $f(x,s)$. The coercivity follows from the assumption on σ. Now we split the proof in two cases, according to the dimension of $V := span\{\varphi_a, \varphi_b\}$.

Case 1: V is one-dimensional

In this case we have $W_0^{1,p}(\Omega) = V \oplus W$, where

$$\begin{aligned} W :=& \{u \in W_0^{1,p}(\Omega); \int_\Omega a(x)|\varphi_a|^{p-1} u \, dx = 0\} \\ =& \{u \in W_0^{1,p}(\Omega); \int_\Omega b(x)|\varphi_b|^{p-1} u \, dx = 0\}. \end{aligned}$$

We set $\theta := \inf_W J$.
From $\lim_{|s|\to\infty} F(x,s) = \infty$ we can choose $t^+ > 0$ such that $\int_\Omega F(x, t^+\varphi_a)dx > -\theta$, so that

$$J(t^+\varphi_a) = \frac{(t^+)^p}{p}(\lambda_1(a) - \lambda) - \int_\Omega F(x, t^+\varphi_a)dx < \frac{(t^+)^p}{p}(\lambda_1(a) - \lambda) + \theta.$$

Therefore, if λ is close enough to $\lambda_1(a)$ from the left, we have that $J(t^+\varphi_a) < \theta$. Similarly, we can pick $t^- < 0$ such that $J(t^-\varphi_b) < \theta$ for γ close enough to $\lambda_1(b)$ from the left. By setting $O_a := \{\ t\varphi_a + w;\ t > 0, w \in W\}$, we have that

$$\inf_{O_a} J < \theta.$$

The following step is the proof of the $(PS)_c$ condition of J in O_a for all $c < \theta$. Let (u_n) be a $(PS)_c$ sequence in O_a. From the coercivity of J, (u_n) is bounded and hence it has a convergent subsequence, say (u_n) itself. Since $\partial O_a = W$ and $\inf_W J = \theta$, u_n converges to a point $u \in O_a$. Now we apply Ekeland variational principle in $\overline{O}_a$ in order to obtain a critical point $u_a \in O_a$. A similar procedure yields a critical point u_b in $O_b := \{\ -s\varphi_b + w;\ t > 0, w \in W\}$. We can easily see that $O_a \cap O_b = \emptyset$, so that $u_a \neq u_b$.

Case 2: V is two-dimensional

Let U be a topological complement of V and put $W := U + span\{\varphi_a - \varphi_b\}$. Then O_a and O_b defined as above are open and disjoint, and the construction of u_a and u_b holds as in Case 1.

Once we have proved the existence of two local minima u_a and u_b, we may see that a mountain pass geometry holds. Indeed, in both cases we have that

$$J(u_a), J(u_b) < \theta < \max_{u \in \gamma[-1,1]} J(u),$$

for all $\gamma \in C([-1,1], W_0^{1,p}(\Omega))$ such that $\gamma(-1) = u_a$ and $\gamma(1) = u_b$. Finally we can easily show that J satisfies the (PS) condition.
It follows that

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} J(\gamma(t)),$$

where $\Gamma = \{\ \gamma \in C([-1,1], W_0^{1,p}(\Omega))\ ;\ \gamma(-1) = u_a \ \text{ and } \ \gamma(1) = u_b\}$, is a critical value of J. Furthemore we have that $d \geq \theta > J(u_a), J(u_b)$, so d is a third critical level. □

Acknowledgments. The authors would like to thank CNPq/Brazil and FNRS/Belgium for financial support. The first author wishes to thank the Université Libre de Bruxelles for the invitation and hospitality. The second author thanks IMECC-Unicamp for the invitation and the hospitality.

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Received December 2007; revised July 2009.

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