Quantum Field Theory and Differential Geometry

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Abstract

We introduce the historical development and physical idea behind topological Yang-Mills theory and explain how a physical framework describing subatomic physics can be used as a tool to study differential geometry. Further, we emphasize that this phenomenon demonstrates that the interrelation between physics and mathematics have come into a new stage.

1   Introduction

In the past 20th century the interrelation between mathematics and theoretical high energy physics had gone through an unprecedented revolution. The traditional viewpoint that mathematics is a language tool of describing a physical law and a physical problem stimulates the development in mathematics has been changed considerably. Mathematics, especially one of its branches, differential geometry, has begun to merge together with the theoretical framework describing subatomic physics. The effect of theoretical physics on the advancement in mathematics is no longer indirect and unilateral. It is well known that Einstein's theory of general relativity had ever greatly promoted the study on Riemannian geometry. But now it is not the case any more, and it is rather than a physical principle is becoming a tool of studying mathematics. A remarkable phenomenon is during the past twenty years a number of Fields medalists' works are relevant to theoretical physics. Especially, in 1990 a leading theoretical physicist, Professor Edward Witten at the Institute of Advanced Study in Princeton, was awarded the Fields Medal for his pioneering work using a relativistic quantum field theory to study differential topology of low-dimensional manifold. This event made a great sensation among both physicists and mathematicians at that time. These facts imply that the relation between mathematics and theoretical high energy physics has come to a new stage and fundamental principles in physics are becoming one of the necessities in pushing mathematics forward.

Naturally people may be puzzled with this phenomenon: mathematics and physics are two distinct subjects. Mathematics lays stress on rigor and logic, and the development in each step needs a rigorous proof to support; While theoretical physics is based on a scientific hypothesis and the subsequent experimental test. How can these two distinct disciplines merge together. The aim of this article is using topological quantum field theory to introduce and explain why and how this phenomenon has happened.

2   Symmetry in Quantum Field Theory and Group Representation

The interrelation between mathematics and theoretical high energy physics is an inevitable outcome of modern physics. It is well known that modern physics started at the beginning of 20th century and originated from the discovery on the theory of special relativity and quantum mechanics. The theory of special relativity is a fundamental physical principle for the matter

at extremely high energy, and it puts space and time on the same footing. This description has greatly modified the traditional space-time version described by Newtonian mechanics. In fact, the space-time theory described by Newton's mechanics is only a low-energy approximation to the version depicted by the theory of special relativity. Correspondingly, in mathematics, a mathematician Hermann Minkowski defined a highly symmetric pseudo-Euclidean space, which is now called the Minkowski space. Then the theory of special relativity is just realized as an isometry symmetry in this space, i.e., the distance of the Minkowski space is invariant under the Lorentz transformation. It is an elegant geometric description to the theory of special relativity. On the other hand, the quantum theory is another fundamental principle for the matter at microscopic scale (atomic and subatomic size). The atom presents wave behavior and obeys quantum principle different from that described by classical physics. The classical physical principle is just the macroscopic limit of quantum theory. The geometric description on quantum theory is the Hilbert space, named after the great German mathematician David Hilbert. A physical state and a physical observable are a vector and a self-adjoint operator in the Hilbert space, respectively. Further, the combination of the theory of special relativity and quantum mechanics had led to the birth of a most powerful theoretical framework describing the interaction among elementary particles — quantum field theory.

The characteristics of quantum field theory determines that it not only serves as a theoretical framework describing particle interactions, but also as a tool for realizing group representation and exploring the topology of differential manifold. A quantum field theory can be constructed from a classical field theory through a so-called quantization procedure. The basic elements constituting a classical field theory are field functions. However, they are not simply functions of space-time coordinate, and they have to constitute certain representations of the Lorentz group (or its covering group). Thus field functions are classified into scalar, spinor, vector and higher order tensor according to irreducible representations of the Lorentz group. In addition, the field functions may carry other indices other than the space-time ones, so they can form representations of certain other Lie groups. Further, the eminent mathematician Hermann Weyl proposed the celebrated gauge principle for the Abelian $U(1)$ group, and it was later generalized to the non-Abelian $SU(2)$ group by theoretical physicists Chen-Ning Yang and Robert Mills [1]. The gauge principle states that if we localize the group representation realized on a field function, i.e., the representation matrix of the Lie group realized on a field function being a function of space-time coordinate, then the group representation index will become a dynamical degree of freedom and a certain new vector field must be introduced to preserve the symmetry represented by the Lie group. In physics, the newly introduced vector field plays the role of mediating the interactions between elementary particles. Therefore, gauge principle is a fundamental principle of constructing a theory describing the interaction among elementary particles.

The application of quantum field theory in group representation theory is attributed to the female mathematician Emmy Nöether. She discovered a direct relation between the symmetry of a field theory and its conservative quantity, and proposed the celebrated Nöether theorem. This theorem shows that the conservative quantity in a field theory is just a representation for the generator of the continuous symmetry group. A classical field theory is described by a functional composed of field functions and their space-time derivatives not more than second order, which is called a classical action. The symmetry of a field theory is the invariance of the classical action under the transformation of the field function. In principle, a quantum field

theory can be obtained through a standard procedure — either canonical quantization or path integral quantization. Similar to its classical counterpart, the quantum field theory is also described by a functional composed of quantum field variables and their derivatives with respect to space-time coordinate. This functional is usually called a quantum effective action, which consists of the classical action plus quantum corrections. The quantum effective action is actually the sum of inner products between two vectors in the Hilbert space of quantum field theory, and these products are termed as the Green functions, which take into account the casualty of physical processes. However, due to the highly non-linear couplings among field functions in the classical action, we usually have no way to obtain the precise form of the quantum effective action. In the case that the coupling is weak, we can use the perturbative iteration method to calculate the quantum correction to a certain order of coupling constant. Moreover, one has to use the regularization and renormalization techniques invented by theoretical physicists to define the quantum effective action. In some cases, the quantum effective action may fail to present certain symmetries that the classical action possesses, this means certain symmetries become anomalous. If the quantum effective action still has various symmetries presented in the classical theory, i.e., no anomaly arises, the Green functions of a quantum field theory must satisfy certain relations dominated by symmetries, and these relations are called the Ward identities in gauge theories. Furthermore, if the vacuum state or ground state (the state with the lowest energy) of the theory provides a trivial representation to the symmetry group, which is called no spontaneous symmetry breaking in physics, then the Hilbert space is a natural platform providing various representations for the symmetry group. The quantum states, i.e., vectors in the Hilbert space, are classified into various irreducible representations of the symmetry group. In physics the vectors sharing the same irreducible representation is called constituting a multiplet. The Ward identities among various Green functions are just reflections of the symmetry in the field theory system. On the other hand, these identities have imposed very strong constraints on the quantum effective action and provide selection rules to the occurrences of physical processes. Overall, quantum field theory is a natural physical framework to study group representation due to its own specifical features, and the physical phenomena it describes is actually a reflection of symmetry in nature.

3 Fibre Bundle, Gauge Theory, Instanton Moduli Space and Instanton Tunneling Effect

The connection between gauge theory and the geometry of fibre bundle is very dramatic. The non-Abelian gauge theory was proposed by theoretical physicists Chen-Ning Yang and Robert Mills in the early middle of 1950s [1]. At that time, the fibre bundle theory had already developed ripely in differential geometry, but physicists almost knew nothing about it. It was until early 1960s that a theoretical physicist Elihu Lubkin realized that the classical Yang-Mills gauge theory and the affine geometry of fibre bundle are identical [2]. A gauge field is actually the pull-back of the connection to the base manifold of a certain principle fibre bundle and the gauge field strength is the pull-back of the curvature of principle bundle, while the gauge group is the structure group of principle bundle. Further, matter fields can be considered as the sections of some associated bundles of the principle fibre bundle, and the gauge transformation is the action of structure group on the section of associated bundle. However, these facts had not been taken

seriously by physicists. The extensive application of fibre bundle geometry in gauge theory was caused by an article on the global structure of electromagnetic field written by Tai-Tsun Wu and Chen-Ning Yang in 1975 [3]. In this article, they defined the electromagnetic potential on two half spheres S^2 with overlapping and avoided the singular string problem in the magnetic potential produced by the Dirac monopole [4]. In the overlapping region of two spheres, the magnetic potentials are related by a $U(1)$ gauge transformation. The geometry of this physical system is precisely a principle $U(1)$-bundle with base manifold S^2 [3]. In the the same year as Wu and Yang published their paper, A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Yu.S. Tyupkin from the former Soviet Union found a solution to the classical $SU(2)$ Yang-Mills theory with finite action in the Euclidean space [5], which is now called the BPST instanton (due to the physical effect it produces which will be mentioned later). This solution describes a gauge field configuration with (anti-)self-dual field strength, and its geometrical description is the principle $SU(2)$-bundle on the base manifold S^4. Further, it had been realized that this solution has topological meaning, and it is characterized by a topological index, which is precisely the Chern number of the second Chern Class in fibre bundle theory [6]. The BPST instanton is actually a classical solution to the Euclidean $SU(2)$ Yang-Mills theory with the Chern number equal to one. In the following years, theoretical physicists including Edward Witten [7], E. Corrigan and D.B. Fairlie [8], Roman Jackiw, C. Nohl and Claudio Rebbi [9] found the multiple (anti-)instanton solution with Chern number $|k| > 1$. Especially, Jackiw and Rebbi showed that the classical Yang-Mills theory has not only non-Abelian gauge symmetry and the Poincaré space-time symmetry, but also a larger conformal space-time symmetry [10], which consists of the Poincaré symmetry composed of translational and Lorentz rotational invariance, dilatational invariance and special conformal symmetry. As we know, if a field theory has a certain symmetry, then the symmetry transformation on a space-time dependent classical solution should lead to another solution to the classical equation of motion. This means that there exists a family of solutions to a field theory with symmetries. The inequivalent solutions under symmetry transformations constitute a finite-dimensional space, which is called the moduli space of classical solution. Concretely speaking, if a space-time dependent solution to the classical equation of motion cannot manifest a certain symmetry of the field theory explicitly, then there must have some parameter in the solutions to specify the symmetry. The number of independent parameters in the solution is the dimension of the moduli space of the classical solution. At the late stage of 1970s, both theoretical physicists and mathematicians employed various distinct methods to determine the dimension of instanton moduli space. A.S. Schwartz [11], Michael Atiyah, Nigel Hitchin and Isadore Singer [12] used the celebrated Atiyah-Singer index theorem in algebraic geometry to have identified the dimension of $SU(2)$ instanton moduli space as $8|k| - 3$. At the same time, Jackiw and Rebbi [13], Lowell Brown, Robert Carlitz and Choon-Kyu Lee [14] determined the dimension by analyzing directly degrees of freedom in the instanton solution and calculating the number of fermionic zero-modes of the Dirac operator in the instanton background, respectively. Moreover, the number of independent parameters for the instanton solution to the Yang-Mills theory with a general gauge group was worked out by theoretical physicists Claude Bernard, Norman Christ, Alan Guth and Erick Weinberg [15] as well as mathematician Atiyah, Hitchin and Singer [16]. Further, mathematicians [17] solved the problem how to construct an instanton solution for an arbitrarily given Chern number by using Roger Penrose's twistor description [18] to Yang-Mills theory. This construction showed the power of algebraic geometry in gauge theory [17]. Finally, mathematicians proved that a finite action solution to the Yang-Mills theory on a compactified Euclidean space in four dimensions must be an instanton solution [19]. At the end

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of 1970s, all the puzzles on instanton solution and the instanton moduli space had been cleared. To summarize, the joint efforts made by both physicists and mathematicians at the late of 1970s had laid the foundation for the breakthrough made in 1980s in understanding differential topological structure of a simply-connected smooth four-manifold.

On the other hand, in the same time theoretical physicists started investigating physical effects induced by instanton. In the middle of 1970s, Jackiw and Rebbi [20], Curtis Callan, Roger Dashen and David Gross [21] found the vacuum structure of a non-Abelian gauge theory is highly nontrivial: the Yang-Mills theory with gauge group $SU(2)$ has vacua with an infinite number of degeneracies. These vacua are classified into distinct homotopy classes by the mapping from S^3 to $SU(2)$ and characterized by topological indices. People usually thought these topological vacua should be absolutely stable since topological numbers should prevent the vacuum from decaying. However, the existence of instanton breaks this naive physical pattern. Gerard 't Hooft first studied quantum gauge theory in the instanton background [22]. He found that an instanton can cause the transition between two topological vacua if the difference of their topological indices equals to the Chern number carried by the instanton. This is the famous tunneling effects produced by instanton in a quantum gauge theory. A clear physical interpretation on the tunneling effect in quantum chromodynamics was further given by Callan, Dashen and Gross [23]. The tunneling phenomenon has lifted the degeneracy of topological vacua and the true vacuum state is the so-called θ-vacuum, the superposition of topological vacua. However, if the theory has massless fermions, then the tunneling effect produced by instanton disappears. The reason for this phenomenon is that a massless fermion carries not only usual fermionic charge (electric charge, lepton number or baryon number, depending on physical objects fermionic fields represent), but also a fermionic charge which changes sign under mirror (parity) transformation. In physics, it is called that the massless fermionic field theory has chiral symmetry (or equivalently axial vector $U_A(1)$ symmetry). However, in the presence of instanton configuration, this symmetry is violated by quantum correction. Thus the fermionic charge that flips a sign under the mirror reflection transformation is not conserved and acquires a contribution proportional to the instanton number, which is induced by quantum correction. This phenomenon in physics is called that $U_A(1)$ symmetry suffers from chiral anomaly [24]. At the late stage of 1970s, it was realized that chiral anomaly is independent of perturbative calculation of quantum field theory and has a topological origin. The violation of the axial fermionic charge is equal to the difference of the left- and right-handed fermionic zero modes of the Dirac operator in the instanton background [25]. This means that the Dirac operator acting on the massless fermionic fields in the instanton background must have non-paired zero modes. Since a fermionic field is represented by a Grassmann quantity, so the existence of non-paired zero modes leads to vanishing integration over fermionic fields and hence the tunneling effect is suppressed by massless fermions. ' t Hooft used the integration property of the Grassmann quantity to have realized that this phenomenon is actually a topological section rule for the physical process. Some gauge invariant quantities carrying the fermonic charges which change sign under parity transformation can absorb the fermionic zero modes and present non-vanishing expectation values, and hence contribute to the 't Hooft quantum effective action. 't Hooft used this idea to have solved the notorious $U_A(1)$ problem in particle physics.

The instanton background can also cause zero modes for the operators acting on bosonic fields such as scalar and vector fields. However, the origins of these bosonic zero modes are

completely different from the fermionic ones for the Dirac operator. As mentioned before, the instanton solution must contain some parameters to manifest gauge and space-time symmetries of the theory. The variations of these parameters do not alter the action or potential energy of the theory. Therefore, the bosonic operators have zero modes along the directions represented by these parameters. In physics one can consider these bosonic modes as the Goldstone modes corresponding to the breaking of certain global symmetries in the directions represented by the parameters. Viewed from the geometry of instanton moduli space, these bosonic zero modes are actually tangent vectors to the moduli space and hence the number of these zero modes is the dimension of the instanton moduli space, since the dimension of a tangent space to a manifold is equal to the dimension of the manifold. One usually makes use of a so-called collective coordinate method to handle the bosonic zero modes in quantum field theory. In the path integral description of quantum field theory, the essence of collective coordinate method is separating the integrations over zero modes from those non-zero modes. After the non-zero modes have been integrated out, the path integration over the bosonic fields reduces to an integration on the instanton moduli space and the key point in this process is how to define the integration measure on the instanton moduli space. Note that this is actually the physical idea used by Witten to describe the Donaldson invariant with a quantum gauge theory. It should emphasize that the above mentioned 't Hooft's instanton calculus is a pioneer work in non-perturbative calculation. During these past thirty years, most of important developments in non-perturbative quantum field theory such as instanton calculus in supersymmetric gauge theory [26, 27, 28, 29], topological quantum field theory [30] and the Seiberg-Witten duality [31] are somehow the prolongs of 't Hooft's instanton calculus. 't Hooft converted the complicated calculation in the instanton background into a central potential problem in quantum mechanics, and it is still not an easy task to repeat his calculations for a beginner despite that more than thirty years have passed.

4 Supersymmetry, Supersymmetric Gauge Theory, R-symmetry and Super-instanton Calculus

Supersymmetry plays a vital role in constructing topological Yang-Mills theory. It entered physics in the middle of 1970s and brought about a relativistic quantum field theory with supersymmetry [32]. It is well known that all the elementary particles are classified into two types, bosons with integer spin and fermions of half-integral spin, and they are described by Lorentz tensors (scalar, vector etc.) and spinors, respectively. It should be emphasized that supersymmetry is a fermionic-type space-time symmetry, i.e., its generator is a conservative charge with spin 1/2. Therefore, supersymmetry can collect bosons and fermions into one multiplet. The supersymmetry transformation turns a boson into a fermion and *vice versa*. Supersymmetry is the only generalization of the Poincaré group permitted by physical requirements. All the bosonic symmetries other than those in the Poincaré group impose too much restrictions on a relativistic quantum field theory [33] so that the scattering amplitude of interacting particles contradicts with experimental observation.

Just like the Lorentz symmetry is an isometry symmetry of the Minkowski space, with regard to supersymmetry one can also introduce fermionic type coordinates to define a superspace. Supersymmetry and the Poincaré symmetry then constitute an isometric symmetry of superspace

[34]. For the simple $N = 1$ supersymmery, the bosonic and fermionic fields in one supermultiplet form a superfield defined in the superspace with the isometry group $OSp(1, 3|4)$. The superfields are classified into chiral or vector superfield, depending on the field content sharing a supermultiplet. The classical action of a supersymmetric field theory in superspace can always divide into two terms: the Kähler potential and superpotential. It should emphasize that the superpotential has an elegant feature: it is an analytical (or anti-analytical) functional of chiral superfields, and this feature is called holomorphy (or anti-holomorphy). Since fermionic fields are represented by the Grassmann variables, so bosonic and fermionic fields in one supermultiplet contribute opposite quantum corrections. As a consequence of supersymmetric Ward identities, the perturbative quantum correction can be partially or fully canceled. This fact results in the nonrenormalization theorem, which states the superpotential in a supersymmetric quantum field theory receives no quantum correction. If formulated in superfield, this theorem means that the superpotential at quantum level keeps its holomorphic form. Therefore, a supersymmetric field is much more easily solvable than a non-supersymmetric theory.

A supersymmetric field theory has a rich mathematical structure. According to supersymmetry algebra, two consecutive supersymmetry transformations lead to a space-time coordinate translation, thus the generators of a supersymmetric transformation is directly related to the Hamiltonian of the theory. This implies that the ground state of a supersymmetric theory must be a zero-energy state. In particular, if a supersymmetric theory has no spontaneous breaking, its zero-energy vacuum state must exist. Therefore, the supersymmetry generator is a nilpotent operator when acting on a ground state of a supersymmetric field theory. This feature is similar to the exterior differential operator acting on differential forms in differential geometry, and consequently, the Hamiltonian of the theory corresponds to the Laplacian operator acting the harmonic differential forms. Therefore, if we can can construct a special supersymmetric quantum field theory whose Hilbert space consists only of the vacuum states of the theory, then its physical states are homological class of the supercharge. Further, if a correspondence between the homological class of the supercharge and the homology (or cohomology) class of the space-time manifold on which the theory is defined can be established, one can use a supersymmetric quantum field theory to study differential topology of a smooth manifold.

We should specify R-symmetry in a supersymmetric gauge theory since it plays a crucial role in reproducing the Donaldson polynomial invariants in terms of topological Yang-Mills theory. The generator of R-symmetry and the supercharge together with those generators for the Poincaré symmetry constitute the whole supersymmetry algebra. R-symmetry is an internal-like symmetry, so its generator(s) has (have) only non-trivial commutation relations with supersymmetry generators. It is a chiral symmetry and implements the automorphic chiral rotations among supercharges if they are represented by the Weyl spinors (or axial symmetry if the supersymmetry generators represented by four-component Majorana spinors). For an N-extended supersymmetry, if the supersymmetry algebra has central extension, the R-symmetry group is usually $U(N)$; While in the presence of central charge, it is $USp(N)$, the compact version of the symplectic group $Sp(N)$. Accompanying the supersymmetry, the R-symmetry has a natural representation in a supersymmetric field theory. The field functions in a supermultiplet carry the representations of R-symmetry according to commutative relations between the supercharge and the generator of R-symmetry. For $N = 1, 2$ supersymmetric Yang-Mills theories, their R-symmetries are $U_R(1)$ and $U(2) = U_R(1) \times SU(2)$, respectively, and each step of supersymmetry

transformation on a field function in a supermultiplet increases its $U_R(1)$-charge by one. The R-symmetry of $N = 4$ supersymmetric Yang-Mills theory is described by the non-Abelian group $SU(4)$. In a classical supersymmetric gauge theory, according to the supersymmetry algebra, the R-symmetry current, the energy-momentum tensor and the supersymmetry current constitute a superconformal current supermultiplet. At quantum level, the $U_R(1)$ symmetry usually becomes anomalous due to its chiral feature. Especially, the chiral anomaly of $U_R(1)$ current, the trace anomaly of the energy-momentum tensor and the gamma-trace anomaly of the supersymmetry current share a superconformal anomaly supermultiplet, and the common anomaly coefficient is proportional to the beta function of supersymmetric Yang-Mills theory. This feature is very useful for determining the perturbative part of quantum effective action for the $N = 2$ supersymmetric Yang-Mills theory since its perturbation theory is one-loop exhausted.

In the remained part of this section, we shall introduce the instanton calculus in a supersymmetric gauge theory. Witten actually employed the supersymmetric instanton calculus to reproduce the Donaldson invariants in terms of the observables of topological Yang-Mills theory [30]. In comparison with the instanton in the usual Yang-Mills theory, a supersymmetric instanton have new features. In the following we illustrate these features using an $N = 1$ supersymmetric gauge theory. First, the instanton has a fermionic partner with definite chirality and it is automatically a fermionc zero mode of the Dirac operator in the instanton background [27]. Moreover, the fermionic partner of (anti-)self-dual instanton must be described a (right-)left-handed Weyl spinor. Second, just like the classical Yang-Mills theory has a conformal symmetry, a classical supersymmetric Yang-Mills theory has a superconformal symmetry consisting of the usual conformal space-time symmetry, Poincaré supersymmetry and conformal supersymmetry [27]. The conformal supersymmetry transformation parameters are space-time coordinate dependent, and the conformal supersymmetry transformation on the instanton solution yields the conformal supersymmetric partner of the instanton. Naturally it is also a fermionic zero mode of the Dirac operator in the instanton background, but it has opposite chirality with the Poincaré supersymmetric partner. The reason is that the generators for the Poincaré supersymmetry and conformal supesymmetry have opposite chiralities. Finally, it should be emphasized that the instanton configuration breaks the conformal supersymmetry and preserves only the Poincaré supersymmetry. This is the reason why there exist the fermionic zero modes: it is just the action of conformal supersymmetry generators on the instanton solution that leads to the fermionic zero modes. In the supersymmetric instanton calculus, there are two approaches to proceed. The first one is using the original instanton calculus invented by 't Hooft and one deals with the fermionic zero modes in the same way as manipulating the usual fermionic zero modes in the instanton background, the only difference is that the present fermionic zero modes are in the adjoint representation of gauge group rather than in the fundamental representation. The other way, which has been used in most of literatures, is considering the instanton and the fermionic zero-modes as an instanton supermultiplet, i.e., superinstanton [27]. In this way, the instanton moduli space of a supersymmetric gauge theory is described by the parameters in the instanton solution and the Grassmann parameters in the fermionic zero modes, which reflect the breaking of conformal supersymmetry. The super-instanton moduli space is thus a supermanifold. Therefore, the integration measure in the instanton moduli space in a supersymmetric gauge theory consists of not only the integration over the parameters denoting the center and the size of instanton, but also the integration over the Grassmann parameters representing the conformal supersymmetry in the fermionic zero modes. Because the fermionic zero modes have definite

chiralities, they carry the $U(1)$ R-symmetry charges. So the moduli space of super-instanton can be thought as a "differential form" fibre bundle space with the usual Yang-Mills instanton moduli space as the base manifold, and the degrees of "differential forms" are just R-charges. This is the key idea that Witten used to reproduce the Donadlson invariants in terms of topological Yang-Mills theory.

For a supersymmetric gauge theory with extended supersymmetries, i.e., $N = 2$ or $N = 4$ case, the theory has scalar fields, which are components of the extended supermultiplet [29]. This makes the instanton calculus delicate. The reason is that the classical action of an extended supersymmetric gauge theory contains additional scalar potential and the Yukawa coupling term among scalar and fermionic fields. So if we have a careful look at the classical equation of motion, it seems that the fermionic and scalar zero modes arising at the lowest order of gauge coupling become non-zero modes. In this case the instanton solution is called a quasi-instanton and the parameters defining the instanton moduli space are called quasi-collective coordinates. However, as long as the theory has chiral $U(1)$ R-symmetry like $N = 1, 2$ supresymmetric Yang-Mills theories, it must suffer from chiral anomaly. Then according to the topological origin of chiral anomaly described by the Atiyah-Singer index theorem in algebraic geometry, there must exist fermionic zero modes for the Dirac operator and the scalar field zero modes for the Laplacian operator in the instanton background. But $N = 4$ supersymmetric Yang-Mills theory has only $SU(4)$ non-Abelian chiral R-symmetry, the situation is different. Another problem is that the scalar potential causes spontaneous breaking of gauge symmetry, i.e., the scalar field has non-vanishing vacuum expectation value. Rigorously speaking, the theory has no instanton solution in this case, and we must use the notion of constrained instanton proposed by Ian Affleck [35]. This kind of instanton behaves similarly as the Yang-Mills instanton at short-distance (or equivalently, the vacuum expectation value of scalar field is very small), and presents the exponential decay at long distance. This is actually a good phenomenon for instanton calculus since the infrared divergence caused by the large size instantons can be cured. Witten used the instanton calculus in a twisted $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory to reproduce the Donaldosn invariant, where it does not matter whether the scalar field has vacuum expectation value or not. As it will be introduced later, if the $SU(2)$ gauge symmetry breaks spontaneously to $U(1)$, the Donaldson invariants can be calculated from the magnetic dual theory of $N = 2$ supersymmetric Yang-Mills theory and the problem even becomes much more simple [36].

5 Differential Structure of Four-dimensional Manifold and Mathematical Construction of Donaldson Invariant

It was one of the most perplexed and subtle problems in differential geometry to distinguish differential structures of a differentiable manifold. Two manifolds are called homeomorphic if there exists a continuous one-to-one mapping between them. All the manifolds with same dimensions can be classified into equivalent classes according to the homeomorphic mapping. Further, if the homeomorphic mapping between two differential manifolds is differentiable, then these two manifolds are said to be diffeomorphic. This means that the manifolds sharing the same homeomorphic class can be more elaborately distinguished in terms of the diffeomorphism mapping. The manifolds presenting the same topology may have distinct differential structures. Roughly speaking, the notion of differential structure on a manifold describes how a coordinate atlas

on the manifold is assigned and how the open coverings of the manifold are "glued" together. It reflects the smoothness of a differential manifold. The notorious examples of illustrating differential structure are seven-dimensional spheres: S^7 and the Milnor exotic spheres $\widetilde{S}^7$ are homeomorphic but not diffeomorphic. The $\widetilde{S}^7$ can be created from S^7 by cutting it along the equator, transforming the boundary of one hemisphere and then "gluing" the two halves back. Mathematicians proved that low-dimensional manifolds (dimension less than 4) have unique differential structure, but higher dimensional manifolds usually have a variety of differential structures, and the most delicate case is the four-dimensional manifold, the physical space-time.

The Donaldson polynomial invariants distinguish differential topological structure of a simple-connected smooth manifold in four dimensions [37]. Before its invention, the invariant characterizing the topological structure of a four-dimensional differential manifold is the characteristic class on the cotangent bundle of the manifold constructed by the Russian mathematician Lev Semenovich Pontrjagin. This characteristic class is invariant under the homeomorphism transformation on the manifold, but the topological number obtained from the integration of the Pontrjagin class over the manifold can only describe the topology of a differentiable manifold and has little power to distinguish the delicate differential topological structure. At the early stage of 1980s, a mathematician Michael Freedman realized that the differential topology of a simply connected smooth four-dimensional manifold M is related to the intersection form defined on the second integer cohomology class $H^2(M, Z)$ over M [38], which is a generalization of the Pontrjagin class on a four-dimensional manifold. In mathematical terminology, an intersection form $\Omega_M(\alpha, \beta)$ is a mapping from the $H^2(M, Z)$ space to an integer set Z. It is defined by taking the wedge (or exterior) product $\alpha \wedge \beta$ of two elements α, β in $H^2(M, Z)$ and integrating over the top homology class $H_4(M, Z)$ on the manifold M. If a set of orthonormal basis in the $H^2(M, Z)$ space is chosen, then the intersection form can be written as a $n \times n$ matrix of integer-valued element, where n is the dimension of $H^2(M, Z)$ and it is an even number. Further, since the structure group of the cotangent bundle on a four-dimensional orientable manifold is $SO(4)$ and the fourth rank antisymmetric tensor $\epsilon_{\mu\nu\lambda\rho}$ is an $SO(4)$ invariant, all the elements in $H^2(M, Z)$ can be classified into the orthogonal self-dual and anti-self-dual sectors through a projection of $\epsilon_{\mu\nu\lambda\rho}$ on the second-rank antisymmetric tensors. Consequently, the intersection matrix decomposes two $n/2 \times n/2$ blocks and they are the matrix representation of an intersection form realized on the self- and anti-self-dual basis of $H^2(M, Z)$. The reason why an intersection form can describe the differential topological structure of a smooth four-manifold is that the homeomorphism transformation induced a homeomorphism transformation on $H^2(M, Z)$ and correspondingly, the intersection matrix undergoes a similar transformation. As a consequence, the intersection matrices are classified into equivalent classes. In this way, the equivalent classes of the intersection forms are related to the classification of differential topology of the manifold. At the beginning of 1980s, Freedman proved that depending on whether the diagonal elements of an intersection matrix are even or odd numbers, the classification of the simply-connected smooth four-manifolds under the homeomorphism can almost be uniquely determined by the equivalent classes of the intersection matrices [38]. In 1983, Simon Donaldson proved a crucial theorem that a positive-(or negative) definite intersection matrix of a simply-connected smooth four-manifold can always be converted into a positive (or negative) unit matrix [37]. This theorem has provided a criteria for the smoothness (infinite differentiability) of a simply connected four-manifold. In proving this theorem, Donaldson used the notion of instanton moduli space in the Yang-Mills theory, and the number of the eigenvalues equal to $+1$ in an intersection matrix is

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actually the number of singularities on the instanton moduli space. Further, Donaldson defined a mapping from the second integer homology group $H_2(M, Z)$ on a simply connected smooth four-manifold M to the second cohomology group $H^2(\mathcal{M}_k)$ over the instnaton moduli space $\mathcal{M}_k$. This is actually a "differential form " on the Yang-Mills instanton moduli space defined by the Donaldson mapping. Then Donaldson took the wedge product of d differential forms defined on $H_2(M, Z)$ but taking values in $H^2(\mathcal{M}_k)$ and performed integration over the compactified instanton moduli space $\mathcal{M}_k^c$. In this way, he finally constructed the powerful integer-valued polynomial invariants of order d to distinguish the delicate differential topological structure of a simply-connected smooth four-manifold.

6 Donaldson Invariant as Physical Observable of Witten's Topological Yang-Mills Theory

The differential topological invariants constructed by Donaldson using the Yang-Mills instanton moduli space had been re-derived from a relativistic quantum field theory by Witten in a physical way [30]. In the following we shall explain the physical ideas behind this field theory reproduction. First, as a topological invariants depicting differential topology of a manifold, it must be independent of the metric of the manifold. In physics, the Einstein equation tells that the energy-momentum tensor of matter field is the source of resulting in the variation of the metric on space-time manifold. So if the physical observable of a relativistic quantum field theory has nothing to do with the metric of the space-time manifold on which the theory is defined, the vacuum expectation value of energy-momentum tensor must equal to zero. Further, the Hamiltonian of the field theory is a component of the energy-momentum tensor and moreover, by definition, the zero mode of the Hamiltonian operator is a vacuum state of quantum field theory, thus the local quantum states in the Hilbert space of the anticipated theory should be only the vacuum states (or equivalently the contribution to the physical observable from local quantum excited states should be suppressed). This fact reminds us that the anticipated quantum field theory should have supersymmetry or a certain symmetry similar to supersymmetry since it can make the quantum corrections coming from local bosonic and fermionic excited states canceled. In addition, as discussed before, the notion of instanton moduli space and the cohomology group on it had been used to construct the Donaldson invariant. From the introduction to the super-instanton moduli space in a supersymmetric gauge theory, we know that the instanton moduli space of $N = 2$ supersymmetric Yang-Mills theory is a "differential form" fibre bundle over the usual Yang-Mills instanton moduli space and the degree of "the differential form" is the chiral $U_R(1)$-charge. However, one cannot use the standard $N = 2$ supersymmetric Yang-Mills theory to get the Donaldson invariant. The physical reason is that $N = 2$ supersymmetry is not powerful enough to eliminate all the quantum corrections contributed from local quantum excited states. The other reason comes from mathematical side. As we know from the knowledge in algebraic topology, the homology group of a manifold straightforwardly describes its topological structure. A geometrical or physical object must have a one-to-one correspondence with an element in the homology group to reflect the topology of the manifold. For example, the reason why the de Rahm cohomology group can describe the topology of a certain manifold is because it establishes a correspondence with the homology group through the Stokes theorem. Therefore, to construct a quantum field theory description to the Donaldson invariant, the generator of the supersymmetry-like symmetry must be nilpotent like the boundary operator (or

exterior differential operator) and the physical states must form a homology class with respect to the symmetry generator. The nilpotency of the symmetry generator provides a possibility that establishes a correspondence between a physical observable of quantum field theory and an element in the homology (or cohomology) group on a differential manifold. As mentioned before, the supersymmetry generator is nilpotent only when acting on vacuum states. The problem is how to achieve its nilpotency and in the meantime to generate the symmetry of the full theory.

In fact, in a quantum gauge theory, we have such a symmetry at hand. As it is well known, a gauge field theory has redundant degrees of freedom due to gauge symmetry, so when performing path integral quantization of a non-Abelian gauge field theory, one must first fix the gauge to make the path integral measure well defined. This problem was beautifully solved by Russian mathematical physicists Ludvig Faddeev and Victor Popov [39]. Consequently, the explicitly gauge-invariant action is replaced by an effective action containing ghost fields and the local gauge symmetry manifests itself as the BRST symmetry named after its discovers C. Becchi, A. Rouet, R. Stora [40] and I.V. Tyupin [41]. The generator of this symmetry is nilpotent and it is a fermionic-like quantity. The ghost fields are non-physical field functions since they are scalars with respect to space-time symmetry but obey fermionic statistics. In addition, the gauge-fixed effective action also has a $U(1)$ ghost number symmetry, which are charged only with ghost fields. The usual convention is that the BRST charge should carry a ghost number -1 and then each BRST transformation on a field function increases the ghost number by one. It turned out that the BRST transformation and the ghost field also admit elegant geometric descriptions. As we know, the classical action of a gauge field theory is a functional on the gauge field configuration space $\mathcal{U} = \{A\}$, which is an infinite dimensional affine space with trivial topology. However, due to the gauge symmetry, the physical field configuration space is actually the infinite dimensional gauge orbit space $\mathcal{U}/\mathcal{G}$, here $\mathcal{G} = \{g(x)\}$ is the gauge transformation group, which consists of all of the gauge transformations $g(x)$ and is an infinite dimensional group due to the space-time coordinate dependence of the group element [42, 43, 44, 45, 46]. Further, $\mathcal{U}$ can be naturally considered as a fiber bundle space with the base manifold $\mathcal{U}/\mathcal{G}$ and the fibre $\mathcal{G}$. Each point $a(x) = a_\mu dx^\mu$ on $\mathcal{U}/\mathcal{G}$ is a 1-form connection on the principle G-bundle $P(M,G)$ over the base manifold M and each point $A(x)$ on $\mathcal{U}(\mathcal{U}/\mathcal{G}, \mathcal{G})$ just differs a gauge transformation with $a(x)$ implemented by an element of $\mathcal{G}$, i.e., $A = g^{-1}ag + g^{-1}dg$. Like $P(M,G)$, one can introduce a connection $\mathcal{A}$ on $\mathcal{U}(\mathcal{U}/\mathcal{G}, \mathcal{G})$ and use it to decompose a cotangent vector δA at the point $A \in \mathcal{U}$ into vertical and horizontal components. Then the ghost field is the " Maurer-Cartan form " $v(x) \equiv g^{-1}(x)\delta g(x)$ in the connection $\mathcal{A}$, and the BRST transformation δA on the gauge field $A(x)$ is the variation of the cotangent vector $\widetilde{\delta}A$ along a fibre of $\mathcal{G}$ [42, 43, 44, 45, 46]. Naturally the BRST transformation preserves $a(x)$ but produces a infinitesimal gauge variation on $A(x)$: $\delta a = 0$ and $\delta A = Dv$, where D denotes the gauge covariant derivative operator defined with respect to $A(x)$.

The above geometric descriptions to the BRST transformation and the ghost field are quite mathematical. We prefer to a more physical way and re-interpret it in an extended principle fibre bundle theory [47, 48]. First, a principle G-bundle $P(M,G)$ over the base M is a fibre bundle whose fibre space and the structure group are identical, both being a Lie group manifold G. The connection one-form on the principle G-bundle in the cotangent space approach is $\omega(x) = g^{-1}(x)A(x)g(x) + g^{-1}(x)dg(x)$ [49] and now $g = g(x)$ is an element of Lie group fibre located at the point $x \in M$. The projection of the first term $g^{-1}Ag$ on the base manifold

M yields the gauge field $A = A_\mu^a(x)T^a dx^\mu$ and the second one $g^{-1}dg = \phi^a(x)T^a$ is the vertical Maurer-Cartan form with one-form ϕ^a satisfying the Maurer-Cartan structure equation. Now we extend the base manifold M to $M \times G$ and define an extended principle G-bundle $\mathcal{P}$ over the base manifold $M \times G$ with the same Lie group fibre G, then the coordinate of the fibre space space G becomes $g = g(x,\theta)$. Near the unit element it can be locally written as $g(x,\theta) = \exp[\theta^a(x)T^a]$, where θ^a $(a = 1, 2, \cdots, \dim G)$ are the parameters of Lie group G. As a consequence, the connection one-form on the extended principle G-bundle $\mathcal{P}$ reads $\widehat{\omega}(x,\theta) = g^{-1}(x,\theta)A(x)g(x,\theta) + g^{-1}(x,\theta)(d+\delta)g(x,\theta) \equiv \omega(x,\theta) + v(x)$ and $\delta \equiv d\theta^a \otimes \partial/\partial\theta^a$ is an exterior differential operator on the Lie group manifold. Then the ghost field $c^a(x)$ in the gauge-fixed Yang-Mills theory plays precisely the role of the Maurer-Cartan one-form in the group space, $v(x) = g^{-1}(x,\theta)\delta g(x,\theta) \equiv c^a(x)T^a$, and the BRST transformation is the exterior differentiation (i.e., the action of δ) on the Lie group manifold [47, 48, 50, 51]. Since the geometric counterpart of the BRST charge is the exterior differential operator δ and the ghost fields are sections of a certain exterior algebra bundle, the geometric description thus explains why the BRST charge is nilpotent and the ghost fields present the anticommuting character despite that they are scalar fields with respect to the space-time symmetry. Later we shall show that the geometric description to the BRST transformation plays an essential role in deriving the celebrated Stora-Zumino descent equations [48, 50, 51, 52], which is another key element used by Witten to construct the physical observables for reproducing the Donaldson polynomial invariants in terms of quantum field theory.

In physics, based on the BRST symmetry, physicists developed a canonical quantization method starting from the gauge-fixed effective action [53], and it is called the BRST quantization, in which all the physical states belong to holomogical classes of the BRST charge, i.e., they are BRST invariant modulo those BRST trivial states which have zero norms.

According to the above gathered materials, the anticipated quantum field theory should be a Yang-Mills gauge theory with a global symmetry presenting the features of both supersymmetry and BRST symmetry. In 1988 Witten constructed such a theory by twisting $N = 2$ supersymmetric Euclidean Yang-Mills theory, and this theory is now called the topological Yang-Mills theory [30]. The twisting operation means choosing a $SU(2)_X$ subgroup of the space-time Euclidean Lorentz group $SO(4) \cong SU(2)_X \times SU(2)_Y$ and another $SU(2)_R$ subgroup from the R-symmetry group $U(2) = SU(2)_R \times U(1)_R$ and taking the diagonal $SU(2)$ subgroup of the direct product $SU(2)_X \times SU(2)_R$ (i.e., decomposing the representation of direct product group $SU(2)_X \times SU(2)_R$ in terms of the representation of its subgroup group $SU(2)$), and then using this $SU(2)$ group and the remained group $SU(2)_Y$ and $U(1)_R$ to form a new space-time symmetry group $SU(2) \times SU(2) \times U(1) \cong SO(4) \times U(1)$. Consequently, all the field functions and supercharges of $N = 2$ supersymmetric Yang-Mills theory are re-classified according to new space-time symmetry group $SU(2) \times SU(2) \times U(1)$ and they become twisted field variables and symmetry generators. Then we rewrite the classical action of $N = 2$ supersymmetric Yang-Mills theory in terms of twisted field functions and the twisted theory presents a BRST symmetry. The BRST charge comes from the twisting of the right-handed supercharge carrying R-charge, and it is a scalar with respect to the new space-time symmetry group $SO(4)$ and carries R-charge -1 inherited from the $N = 2$ supersymmetric Yang-Mills theory before the twisting. In comparison with the counterpart in the BRST quantization, the $U(1)$ R-symmetry is the ghost number symmetry and the R-charge is identical to the ghost number. It is a miracle that both the

twisted classical action and the energy-momentum tensor is BRST trivial as expected. With the idea of BRST quantization of gauge theory, the theory has no local quantum excited states and its physical observables constitute a homology class of the BRST charge. The physical process described by the theory is only the transition among topological vacua caused by instanton tunneling effect. The transition amplitudes can be reduced to integrals over the instanton moduli space since all the quantum corrections contributed from local excited states cancel, attributing to the powerful BRST symmetry [30].

Now the confronting problem is how to construct appropriate physical observables in the topological Yang-Mills theory to reproduce the Donaldson polynomial invariants. According to the mathematical construction on the Donaldson polynomial invariants introduced before, the to-be-constructed physical observables, in addition that they are gauge invariant, should satisfy two requirements. First, as emphasized previously, the expected physical observable must have a correspondence with an element in a homology (or cohomology) class on the space-time manifold so that it can indeed represent a certain topological invariant. Second, the Donaldson polynomial invariant was constructed mathematically with a mapping from the integer homology group on a differential manifold in four dimensions to the cohomological group on the instanton moduli space of Yang-Mills theory, thus the anticipated physical observable must also reflect the correspondence between the cohomological class over the instanton moduli space and the homological (or cohomological) class on the space-time manifold. On the other hand, in topological Yang-Mills theory the field functions are graded by ghost number (or R-charge called in the untwisted $N = 2$ supersymmetric Yang-Mills theory), and the BRST transformation on a field function increases the ghost number by one, just behaving as an exterior differential operator acting on a differential form. Based on these analogues, Witten ingeniously made use of the descent equations in an anomalous gauge theory and successfully fulfilled the above two geometrical requirements [30].

The chain of descent equations plays crucial role in attaining the physical observables for the Donaldson polynomial invariants. It was found by Raymond Stora, Bruno Zumino and Juan Mañes [50, 51, 52], Bonora and Cotta-Ramusino [48] in evaluating the non-Abelian chiral anomaly through observing the Wess-Zumino consistency condition [54]. One chain of descent equations is precisely the local version of the Wess-Zumino consistency condition [54] and hence provide a physically legitimate basis of using solely differential geometry to find the non-Abelian chiral anomaly [55] (or the Bardeen anomaly in the covariant case [56]). In the following we use a physical and geometrical hybrid way to introduce the descent equations. In an anomalous gauge theory consisting of chiral fermions coupled with an external chiral gauge field $A_\mu = A_\mu^a T^a$, the anomalous Ward identity can be formally expressed as the form that the anomaly $G^a[A]$ comes from the action of gauge transformation generator X^a on the quantum effective action $W[A]$, which is is a functional of the external gauge field A_μ obtained by integrating out the chiral fermions in the path integral quantization. Since the gauge transformation generator X^a is a local functional differential operator representation to the Lie algebra generator T^a of the gauge group, Julius Wess and Bruno Zumino realized that there should naturally imposes a consistent condition on the non-Abelian chiral anomaly $G^a[A]$. Reversely, the non-trivial solution to this consistent condition should yield the chiral anomaly $G^a[A]$. If one defines a top-rank differential form $Q_{2n}^1 \equiv c^a(x) G^a[A(x)] d^{2n}x$ with the ghost number one on a $2n$-dimensional space-time manifold M^{2n} by multiplying the ghost field $c^a(x)$ with the non-Abelian anomaly $G^a[A(x)]$, the

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Wess-Zumino consistent condition can elegantly expressed as $\delta \int_{M^{2n}} Q_{2n}^1 = 0$, i.e., the BRST invariance of the integration of the top-rank differential form Q_{2n}^1 over the space-time manifold. This means locally the BRST transformation of the above top-rank differential form with ghost number one can be written as the exterior differentiation of a differential form with one degree less and the ghost number two, i.e., $\delta Q_{2n}^1 = -dQ_{2n-1}^2$.

It turned out that the above local version of the Wess-Zumino condition is only one chain of the descent equations and they can be derived from the geometric description to the BRST transformation and the ghost field, described in either the connection bundle $\mathcal{U}(\mathcal{U}/\mathcal{G}, \mathcal{G})$ or the extended principle G-bundle $\mathcal{P}(M \times G, G)$ introduced before [42, 43]. The context of the first option is mathematically rigorous but complicated [42, 43, 46]. One must define a direct product bundle $P(M, G) \times \mathcal{U}(\mathcal{U}/\mathcal{G}, \mathcal{G})$, and then introduce the total connection $A + \mathcal{A}$ and the corresponding curvature, $\mathcal{F} = (d + \widetilde{\delta})(A + \mathcal{A}) + (A + \mathcal{A}) \wedge (A + \mathcal{A})$. It can be proved that $\mathcal{F}$ has only non-vanishing horizonal lift in the product bundle space (i.e, the lift along $M \times \mathcal{U}/\mathcal{G}$ direction). Therefore, along the fibre of the connection bundle $\mathcal{U}(\mathcal{U}/\mathcal{G}, \mathcal{G})$, $\mathcal{F} = F$, the "differential operator" $\widetilde{\delta}$ on the connection space $\mathcal{U}$ reduces to the BRST operator δ and the connection $\mathcal{A}$ becomes the ghost field $v = g^{-1}\delta g$. Since the exterior differential operators d, δ and $d + \delta$ are all nilpotent, we can construct the Chern characteristic class $C_n(\mathcal{F}) = \text{Tr}(\mathcal{F}^n)$ on $P \times \mathcal{U}$. Because the Chern class is closed, and hence according to the Poincaré lemma, it can be locally written as an exact differential form. Further, there exists $\mathcal{F} = F$ along the fiber of $\mathcal{U}$, so we have $\text{Tr}(\mathcal{F}^n) = (d + \delta)Q_{2n-1}(A + v) = \text{Tr}(F^n) = dQ_{2n-1}(A)$. Expanding the above equation in powers of v and comparing the differential forms with the same degrees, we obtain the chain of descent equations.

The descent equations can also derived from the extended principle G-bundle $\mathcal{P}(M \times G, G)$ in a more easily accessible way [48, 50, 51]. First, we define the extended exterior differential operator $\Delta \equiv d + \delta$ on the extended base manifold $M \times G$. It is a formal direct sum of the exterior differential operator d on the space-time manifold and the differential operator δ on the Lie group space, and is thus nilpotent, $\Delta^2 = 0$; Second, we construct an extended Chern characteristic class $C_n[\widehat{\Omega}(x, \theta)] = \text{Tr}(\widehat{\Omega}^n)$ on the extended principle G-bundle using the curvature $\widehat{\Omega}(x, \theta) = \Delta\widehat{\omega}(x, \theta) + \widehat{\omega}(x, \theta) \wedge \widehat{\omega}(x, \theta)$. Third, there exists a natural result that the curvature of the extended principle G-bundle $\mathcal{P}(M \times G, G)$ equals to the curvature of the original principle G-bundle $P(M, G)$: $\widehat{\Omega} = \Delta\widehat{\omega} + \widehat{\omega} \wedge \widehat{\omega} = d\omega + \omega \wedge \omega = \Omega = g^{-1}(dA + A \wedge A)g = g^{-1}Fg$. Therefore, the extended Chern class $C_n[\widehat{\Omega}]$ should equal to the Chern class $C_n[\Omega]$ and also the Chern class $C_n[A]$ defined in terms of the pull-back A of the principle G-bundle curvature ω to the base manifold M: $C_n[\widehat{\Omega}] = C_n[\Omega]$. Finally, the closure and local exactness of the Chern class means $C_n[\Omega] = dQ_{2n-1}[\omega, \Omega]$. In the local form $\Delta Q_{2n-1}[\widehat{\omega}, \widehat{\Omega}] = (d+\delta)Q_{2n-1}[\omega + v, \Omega] = dQ_{2n-1}[\omega, \Omega]$, we expand the Chern-Simons form in powers of v: $Q_{2n-1}[\omega + v, \Omega] = \sum_{p=0}^{2n-1} Q_{2n-1-p}^p[v^p, \omega, \Omega]$. Then comparing the differential forms of the same degrees on the base manifold M^{2n} and choosing $g = e$, the unit element of the Lie group G, we immediately obtain the Stora-Zumino chain of descent equations: $\delta Q_{2n-1-p}^p[v^p, A, F] = -dQ_{2n-p-2}^{p+1}[v^{p+1}, A, F]$ and $\delta Q_0^{2n-1} = 0$ $(0 \leq p \leq 2n-1)$.

The chain of descent equations relates the physical property of quantum field configuration to space-time geometry. It shows the topological obstructions at various ranks for some physical

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variables to have global definitions on a space-time manifold. It was invented by theoretical physicists and provided a physical accessible tool of studying the global aspects of quantum field theory but described in terms of the infinitesimal local gauge transformation. This is the reason why it plays such a decisive role in investigating physical phenomena originating from the topologically non-trivial field configuration of a quantum gauge theory. The chain of descent equations establishes a correspondence between the BRST cohomology in field configuration space (or Hilbert space) of quantum gauge theory and the de Rahm cohomology class over the space-time manifold. Consequently, it relates the ghost number directly to the degree of a certain differential form on the space-time manifold. Thus it is no wonder that Witten naturally employed an analogue of these equations to construct physical observables to reproduce the Donaldson polynomial invariants in topological Yang-Mills theory. Substituting the descent equations into certain relevant Green functions, we can obtain various anomalous Ward identities among the Green functions. The further topological meaning embodied in the descent equations had been tapped by Michel Dubois-Violette, Michel Talon and Claude-Michel Viallet [57], and it implies the existence of $H^p_{2n-1-p}(\delta/d)$'s, the so-called δ-cohomology modulo d. This means those Q^p_{2n-1-p}'s entering the descent equations must take away the terms of the form $Q^p_{2n-1-p} = \delta\theta^{p-1}_{2n-p} + d\eta^{p-1}_{2n-p-2}$. The physical meaning that $H^p_{2n-1-p}(\delta/d)$'s are various non-trivial anomalous terms was clarified by Mañes and Zumino [58]. For examples, $Q^1_{2n-2}(v, A, F)$, the chain term to the order v in the expansion of the Chern-Simons form $Q_{2n-1}(A+v, F)$, describes the consistent non-Abelian anomaly [55] and its δ-trivial form can be canceled by introducing local counterterm into the quantum effective action [58]; Q^2_{2n-3}, the chain term to the order v^2, represents the Schwinger term appearing in an equal-time commutator of the Gauss law operators in the Yang-Mills theory [59]; Q^3_{2n-3}, the chain term to the order v^3, reflects the failure of the Bianchi identity in the presence of magnetic monopole [60, 61]. The quantum anomalous effects relevant to the chain terms of higher order v have not been found yet up to now.

Since one chain of the descent equations is just the local version of the Wess-Zumino consistency condition imposed on the non-Abelian consistent anomaly, based on this observation, Zumino, Yong-Shi Wu and Anthony Zee [55] invented the pure differential geometrical approach to evaluate the non-Abelian chiral anomaly with only simple calculations on the Feynman diagram to determine the normalization constant, i.e., starting from the Chern characteristic class and observing its infinitesimal gauge transformations, and then taking into account the Wess-Zumino consistent condition to derive the anomaly. It is an elegant and universal method, working not only for the non-Abelian chiral anomaly in any even dimensions, but also for the pure gravitational anomaly in $4k + 2$ dimensions [62]. Especially, it is extremely useful in calculating anomaly in higher dimensional space-time and has great advantage over other methods of evaluating anomaly such as calculating directly the Feynman diagram and observing the non-invariance of path integral measure proposed by Kazuo Fujikawa [63]. Actually, this approach provided a basis for Michael Green and John Schwarz to find the celebrated anomaly cancelation mechanism in superstring theory [64] and hence to some extent promoted the first superstring revolution.

The application of the descent equations for constructing the Donaldson polynomial in terms of topological Yang-Mills theory works like " killing two birds with one stone ". The chain of descent equations, $\delta Q^p_{2n-p-1} = -dQ^{p+1}_{2n-p-2}$, shows that δQ^p_{2n-p-1} $(0 \leq p \leq 2n - 1)$, the BRST

transformation of a certain $(2n - p - 1)$-degree differential form with ghost number p on a $2n$-dimensional manifold M^{2n}, equals to dQ_{2n-p-2}^{p+1}, the exterior differential of the one-degree-less differential form Q_{2n-p-2}^{p+1} with ghost number p. Therefore, in topological Yang-Mills theory, if we can find a set of Q_{2n-p}^{p-1}'s composed of the field variables and choose their integrals over $(2n - p)$-homological cycles Σ_{2n-p} on the space-time manifold as physical observables, then these BRST invariant physical observables not only describe a direct correspondence between the BRST cohomological class in the Hilbert space of the field theory and the homological class on the space-time manifold, but also reflect that the BRST cohomology can reduce to a cohomology over the Yang-Mills instanton moduli space once the non-zero modes of field variables have been integrated out. Witten chose the second Chern class $\mathrm{Tr}(F \wedge F)$ as the top differential form and constructed other three differential forms with the application of the BRST transformations [30]. These four differential forms composed of the field functions constitute a chain of descent equations. The integrals of these four differential forms over homological cycles Σ_p ($0 \leq p \leq 3$) on the space-time manifold are BRST invariants and depend only on the homological class Σ_p, and hence constitute a set of fundamental physical observables to produce topological invariants on the space-time manifold. According to the topological selection rule on the physical process induced by the instanton tunneling effect [22], one can use the product of above four fundamental observables to construct a set of physical observables whose ghost numbers equal to the dimension of instanton moduli space $d(\mathcal{M})$ [30]. Then the expectation values of these physical observables, after integrating out the non-zero modes, indeed yield the Donaldson polynomial invariants constructed through the mapping from the homology group on the space-time manifold to the homological group over the compactified Yang-Mills instanton moduli space.

Witten's topological Yang-Mills theory has not only provided a physical approach to reproduce the Donaldson invariant, but also initiated a new type of quantum field theory [30]. Usual quantum field theories have an infinite number of degrees of freedoms and its Hilbert space is infinite dimensional. But local quantum states in the Hilbert space of a topological field theory are only vacuum states, and it is exactly solvable to some extent. In physics topological quantum field theory describes a special phase of field theory in which there exist only topological (or no-local) quantum excitations. It might be helpful for understanding quantum gravity since a graviton can arise as the Goldstone boson corresponding to the spontaneous breaking of general covariance in topological quantum field theory, despite this symmetry breaking is hard to realize. From the end of 1980s to the early stage of 1990s, the study on topological quantum field theory is a hot topic for both theoretical physicists and mathematicians. However, it should emphasize that the topological Yang-Mills theory approach is not much helpful for the practical calculation on the Donaldson invariant. This can be easily understood from physical consideration. First, $N = 2$ supersymmetric Yang-Mills theory is strongly coupled at low-energy and there is no a feasible way to solve the theory; Second, the integration over the instanton moduli space suffers from infrared divergence due to the large size of instanton (or large volume of instanton moduli space), and this brings obstacles on the explicit calculation on the Donaldson invariant. This stagnant situation lasted until the early middle of 1990s.

7 Electric-magnetic Duality, Seiberg-Witten Monopole Equation and Donaldson Invariant

In 1994 Nathan Seiberg and Edward Witten made a breakthrough in understanding $N = 2$ non-perturbative supersymmetric Yang-Mills theory in four dimensions [31]. They synthesized a number of nonperturbative methods of solving supersymmetric gauge theory and physical phenomena occurred in the supersymmetric gauge theory including supersymmetric instanton calculus, spontaneous breaking of gauge symmetry, the holomorphy imposed by supersymmetric Ward identities and quantum anomaly of superconformal symmetry. Especially, they made use of electric-magnetic duality conjecture, which is a notion that had left out in the cold for a long time. With the accumulative application of these methods, Seiberg and Witten successfully determined the structure of vacuum moduli space for $N = 2$ supersymmetric Yang-Mills theory in the Coulomb phase and the corresponding quantum Wilsonian effective action including all of the perturbative and non-perturbative quantum corrections. They further gave a quantitative explanation to the mechanisms for confinement and chiral symmetry breaking. This is the first time in the history of quantum field theory that an exact solution for a strongly coupled quantum field theory in four dimensions has been obtained. The electric-magnetic duality conjecture plays a crucial role in attaining the solution. The electric-magnetic dual transformation in the vacuum moduli space of the Coulomb phase of $N = 2$ supersymmetric Yang-Mills theory is represented as a subgroup of the discrete group $SL(2, Z)$, which is generated by transformation matrices (called monodromy) around the singularities in the vacuum moduli space. In the following we give a brief introduction to the history of electric-magnetic duality in order to understand Seiberg and Witten's work.

Electric-magnetic duality conjecture has a long history and it came out not long after the birth of modern physics. As early as the beginning of 1930s, Dirac realized the asymmetry of electric and magnetic field sources in the Maxwell equations and hence proposed the notion of magnetic monopole [4], which is a particle carrying magnetic charge, just like a charged particle carrying electric charge. The Maxwell equations with magnetic charge sources present an elegant electric-magnetic duality: when the electric and magnetic charges change their roles with each other, the electric and magnetic fields also exchange (up to a sign). Dirac further found that when one considers quantum mechanics of an electrically charged particle moving in the magnetic field produced by a monopole, in order to achieve single-valuedness of the wave function, the product of the electric charge of the particle with the magnetic charge of the monopole must be a multiple of 2π, this is the celebrated Dirac quantization condition. In electrodynamics the electric charge is directly related to electromagnetic coupling, so the Dirac quantization condition implies that electric-magnetic duality can exchange a strongly coupled electric theory with a weakly coupled magnetic theory and *vice versa*. However, a quantum field theory with magnetic monopole had not been found until 40 years later. In 1974, Gerard 't Hooft [65] and A.M. Polyakov [66] independently found a $SU(2)$ gauge field theory that mediates scalar field interaction and contains a self-interacting scalar potential, which breaks the $SU(2)$ gauge symmetry break to $U(1)$ symmetry spontaneously, can have a topological soliton solution with finite energy. This solution presents the feature of a magnetic monopole and now is called the 't Hooft-Polyakov monopole. At large distance, the 't Hooft-Polyakov monopole behaves exactly as the Dirac monopole. By the way, the gauge field theory with monopole solution was actually the bosonic part of a field model proposed by Howard Georgi and Sheldon

Glashow [67], which describes the suppression of the physical process involving leptonic neutral currents, and hence is usually called the Georgi-Glashow model. When the energy of the theory takes a minimal finite value and the self-interaction of scalar field is very weak, the explicit analytic form of the 't Hooft-Polyakov monopole solution can be worked out. This is the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopole [68]. The mass of the BPS monopole is proportional to its magnetic charge. On the other hand, since the self-interacting scalar potential causes the spontaneous breaking of $SU(2)$ symmetry to $U(1)$, two electrically charged components of gauge field acquire mass through the Higgs mechanism and the masses proportional to electric charges they carry. Naturally, both the masses of BPS monopole and gauge field measured in terms of the vacuum expectation value of the scalar field since it is the only parameter with mass dimension. Several years after the discovery of 't Hooft-Polyakov monopole, Claus Montonen and David Olive observed the symmetry between the mass spectrums of gauge particle and BPS monopole, which are the fundamental quantum excitation and topological soliton of the George-Glashow model, respectively. Hence they boldly proposed that the Georgi-Glashow model should have a physically equivalent magnetic dual description, in which the BPS monopole should play the role of a gauge field and the massive vector boson in the broken phase of the Georgi-Glashow model becomes an "electric" monopole [69]. In particular, when the Dirac quantization condition is taken into account, the gauge couplings of these two dual theories should be opposite. Later, Witten [70] considered the strong charge-conjugate and parity symmetry violation effect in the George-Glashow model and introduced a topological θ-term proportional to the instanton number. This term causes the integer-valued magnetic charge to have a shift proportional to the θ parameter consistent with the Dirac quantization condition [70]. Consequently, the duality operation extends from Z_2 to $SL(2, Z)$ transformation.

However, the Montonen-Olive duality conjecture for the Georgi-Glashow model has several fatal problems. First, the vector bosons and magnetic monopole as dual particles have different spins; Second, quantum correction can usually modify the classical potential [71], and hence corrects the mass of gauge particles acquired from the spontaneous breaking of gauge symmetry happened at classical level. Third, the gauge coupling in a quantum field theory changes along with the energy scale. Finally, since electric-magnetic duality involves two theories with opposite couplings, it is almost not possible to verify this duality since usually there is no way to solve a strongly coupled theory. Two years after Montonen and Olive proposed the conjecture, Hugo Osborn from the Cambridge University found that these drawbacks can be refrained in the $N = 4$ supersymmetric Yang-Mills theory [72]. Thus electric-magnetic duality should exist in a gauge theory with maximal global supersymmetry. After Osborn's work, the notion of electric-magnetic duality had been laid aside and almost neglected in field theory.

This situation lasted until early 1990s, the Indian string theorist Ashoke Sen found a duality existing in the heterotic string theory compactified on a six-dimensional torus [73]. The resultant theory after compactification is a four-dimensional string theory with $N = 4$ supersymmetry, and its low-energy effective theory is $N = 4$ supergravity coupled with $N = 4$ supersymmetric Yang-Mills theory. Therefore, this duality is the Montonen-Olive-Osborn duality in heterotic string theory. In the latter half of 1994, Seiberg and Witten found that $N = 2$ supersymmtric Yang-Mills theory in the Coulomb phase at low-energy also presents an electric-magnetic duality [31], with which an exact Wilsonian effective action can be determined, and led to a great stir among theoretical physicists. This work had also rung the bell for the second superstring

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revolution. However, it should be emphasized that the duality found by Seiberg and Witten is different from the Montonen-Olive-Osborn duality in $N = 4$ supersymmetric Yang-Mills theory. The latter is somehow a self-duality, the magnetic monopole lies in the $N = 4$ vector supermultiplet, and the classical action of magnetic dual theory has the same form as the original "electric" theory. But for the Seiberg-Witten duality, the magnetic monopole lies in $N = 2$ matter supermultiplet (hypermultiplet), that is, the monopole belongs to a matter superfield. This electric-magnetic duality is actually close to the original duality conjecture proposed by Dirac. To summarize, according to Seiberg and Witten's work, the low-energy Wilsonian effective action of $N = 2$ supersymmetric Yang-Mills theory in the Coulomb phase, composed of perturbative quantum correction and non-perturbative contribution from instanton tunneling effect, is dual to a $N = 2$ supersymmeytric $U(1)$ gauge theory weakly coupled with a $N = 2$ magnetic monopole hypermultiplet. These two dual theories describe the same physics.

Shortly after the discovery of the Seiberg-Witten duality in $N = 2$ supersymmetric Yang-Mills theory, Witten applied it to the calculation on the Donaldson invariants [36]. As introduced before, the Donaldson invariant is a physical observable of the twisted $N = 2$ supersymmetric Yang-Mills theory, which can be expressed as an integration over the Yang-Mills instanton moduli space. Viewed from physics side, they are physical amplitudes of fermionic zero modes in the instanton background. According to electric-magnetic duality, these physical observables can be equivalently calculated in the dual magnetic theory. The calculation on the physical observables becomes much more tractable since the dual theory is a $N = 2$ Abelian supersymmetric gauge theory. Especially, the physical obervable in the twisted dual magnetic theory can only come from the integration over the moduli space of magnetic monopole solution. The weak-coupling of the magnetic dual theory means that it is sufficient to achieve the Donaldson invariant by considering only the moduli space of classical magnetic monopole, which is now called the Seiberg-Witten monopole equation. Therefore, the calculation on the Donaldson invariant in the magnetic dual theory converts into counting the number of independent solutions to the Seiberg-Witten monopole equation. Overall, the study on the differential topology of a simply connected four-manifold in terms of a relativistic quantum field theory had finally achieved a successful outcome.

8 Other Applications of Quantum Field Theory in Differential Geometry

There are more examples about the application of quantum field theory in the study on topology, differential geometry and algebraic geometry. Some of them are list as the following.

- The Morse inequalities in the Morse theory, which is an alternative and powerful approach to investigate topological structure of a manifold by studying the critical points of a function defined on the manifold, was derived through the tunneling effect in $0 + 1$-dimensional supersymmetric nonlinear sigma model [74].

- The Atiyah-Singer index theorem, which states the analytical index of an elliptic differential operator (i.e., the number of solutions to a differential equation) should equal to the topological index of the manifold on which the operator is defined, was verified by

calculating the Witten index [75] in $0 + 1$-dimensional supersymmetric nonlinear sigma model [76, 77].

- The elliptic cohomology was proposed as a generalization of K-theory in algebraic geometry and correspondingly, the elliptic genus should be related to the elliptic cohomology in the same way as the Dirac index is related to K-theory. Witten suggested that the role in elliptic cohomology analogous to the Dirac operator in K-theory should be played by the supercharge of an $N = (1, 1)$ supersymmetric non-linear sigma model in $1 + 1$ dimensions [78], provided that the left-moving fermion is assigned to the Neveu-Schwarz boundary condition and the right-moving fermion to a Ramond-Ramond boundary condition. The elliptic genus is precisely the index of the supercharge operator, just like the $\widehat{A}$-genus being the index of the Dirac operator.

- The Jones polynomials which classify knots and links topologically in three-dimensional space are identical to quantum Wilson loops of Chern-Simons topological gauge theory in three dimensions[79].

- The Thurston conjecture which can give a topological classification on three-dimensional manifolds could be studied by a certain three-dimensional gravity theory [80].

Since the middle of 1980s there has arisen an upsurge in studying superstring theory since in physics it can give a consistent theoretical description on quantum gravity and possibly provide a framework for unifying four fundamental interactions. The elementary particles, which are represented by quantum field in gauge field theory, now appear as various vibrational modes on open string and closed string, which are termed according to their geometrical shapes looking like a segment or a circle of thread, respectively. The dynamics of string theory is described by certain two-dimensional $N = 1$ supersymmetric nonlinear sigma models but with conformal symmetry. The rising of superstring theory has brought about many new mathematical tools to apply in physics. For example, the dynamical symmetry of string theory on world-sheet is conformal symmetry in two dimensions, which is described by infinite dimensional Virasovo and Kac-Moody algebras [81]; The calculation on multi-loop amplitude of string theory needs the knowledge about the Riemann-Roch theorem on the Riemannian surface and Teichmüller space etc. [82]. The conformal symmetry of quantum string theory requires the background space-time must be ten-dimensional. To get to the anticipated physics in four-dimensional space-time, we need to perform a compactification from ten to four dimensions, and the compactified six-dimensional space must be a Calabi-Yau manifold [83, 84]. In the middle of 1990s the discovery on the string soliton – D-brane led to a breakthrough in understanding non-perturbative string theory. The formerly discovered five string theories can be connected together by dualities and unified into a so-called M-theory living in an eleven-dimensional space-time. Even now people has not understood much about M-theory such as the fundamental principle behind it and its fundamental degrees of freedom. The only confirmed fact is that its low-energy limit should be the $N = 1$ supergravity in eleven dimensions. D-brane plays a crucial role in establishing string dualities and it carries topological charges with respect to the antisymmetric fields produced by quantum excitations of the world-sheet fermion in closed string theory under the periodic boundary condition. Witten pointed out that the topological charge carried by D-brane can be classified by the K-theory in fibre bundle theory [85]. String theory has occupied so many advanced mathematics, and reversely, it can also be considered as a tool to study differential

geometry and algebraic geometry. For instance, the mirror symmetry between two Calabi-Yau manifolds is a pure geometrical property and string theory greatly facilitates the study on mirror symmetry. At the beginning of 1990s, Witten constructed two types of topological strings (called A- and B-models) by " twisting " two-dimensional $N = 2$ supersymmetric non-linear sigma model in two different ways, and the search on a pair of mirror manifolds converts into calculating and comparing physical observables of two topological strings [86]. In recent years, theoretical physicists and mathematicians are interested in topological string theory in the background of D-instanton, this may helpful for exploring the mirror symmetry between a pair of the generalized Calabi-Yau manifolds [87]. In a colloquium on the prospects of mathematics in the 21st century, the leading mathematician in geometric analysis and mathematical physics and the Field medalist, Professor Shing-Tung Yau at the Harvard University, pointed out that " the developments in string theory have successfully unified some important sectors of differential geometry, algebraic geometry, group representation theory, number theory and topology " and he predicted that " the grand unification in mathematics could be bred from and born out of the grand unified field theory in physics " [88]. These facts indicate that string theory has much greater impacts on the development in mathematics.

9 Summary

We have introduced the physical idea how a relativistic quantum field theory has been used to derive the Donaldson invariants in differential geometry. The aim is to use this fact as an example to illustrate how quantum gauge theory, which is a theoretical framework describing the interaction among elementary particles, can be applied to study differential topology. It shows that once a physical notion has a mathematical counterpart, a physical framework can be used as a tool to study mathematics. Both quantum field theory and string theory have deep roots in differential geometry, algebraic geometry and group theory, and most of notions in these two theoretical frameworks are just another appearances (or physical representations) of certain mathematical objects. Thus it is no wonder that they are becoming an alternative way to develop new mathematics. Therefore, the interrelation between mathematics and physics are stepping into a new era. Mathematics and physics are two oldest disciplines in the scientific civilization of mankind, both born out of the cognition on natural world. They have finally melted into with each other after more than two hundred years' relatively independent developments, despite that there were some intersections in the past times. Nowadays more and more mathematicians and theoretical physicists are learning from each other to propose mathematical conjectures and prove new theorems.

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