Physics Letters B 800 (2020) 135077

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

5d superconformal field theories and graphs

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ARTICLE INFO

Article history:
Received 24 September 2019
Received in revised form 25 October 2019
Accepted 31 October 2019
Available online 6 November 2019
Editor: M. Cvetič

ABSTRACT

We propose graphs, the Combined Fiber Diagrams (CFD), to characterize all 5d superconformal field theories (SCFTs) that arise as S^1-reductions of 6d SCFTs. Transitions between CFDs encode mass deformations that trigger RG-flows between SCFTs. They provide a combinatorial classification of all such 5d SCFTs and encode physical information about the strongly coupled theories, like the superconformal flavor symmetry and BPS states. We consistently reproduce known results, but more importantly predict new theories and strong coupling effects in 5d SCFTs.

1. Introduction

5d $\mathcal{N}=1$ SCFTs are intrinsically non-perturbative quantum field theories. At low energies these can have effective descriptions in terms of weakly coupled gauge theories, however to interpolate between the infrared (IR) and ultraviolet (UV) fixed points requires methods beyond ordinary field theory, motivating a string theoretic approach. 5d theories have been engineered in string theory by (p,q)-fivebrane webs [1], or M-theory on non-compact Calabi–Yau threefolds with canonical singularities [2,3]. In the latter approach, there is a particularly elegant correspondence between geometry and physics, whereby the resolution of the singularity may be identified with a renormalization group (RG)-flow from the UV to an effective IR description.

In this letter, we show how this approach comprehensively surveys 5d SCFTs and their salient physical properties. In essence, singularities in the M-theory realization, where complex surfaces have collapsed to points, correspond to SCFTs. In the smooth phase, when these surfaces have finite volume, their geometry determines the low-energy gauge theory descriptions for the SCFT, if one exists. Complex curves inside these surfaces determine the spectrum of matter hypermultiplets, as well as additional non-perturbative states, all of which become part of the BPS spectrum in the SCFT limit, where the surfaces collapse.

Recent progress in identifying M-theory geometries related to 5d SCFTs has been made in [4–10]. The approach in this letter is fundamentally different, as it intrinsically captures some of the strongly coupled physics and gives an efficient way of mapping out the landscape of 5d SCFTs.

We define for each SCFT a graph, the *combined fiber diagram* (CFD), which encodes key properties of the geometry. Each such graph corresponds to an equivalence class of surface configurations inside a Calabi–Yau threefold, whose singular limit defines the same SCFT. The vertices of each graph correspond to curves contained within the surfaces, and give rise to BPS states in the UV.

Transitions between CFDs encode flows between SCFTs. These reflect geometric transitions that modify the curve configuration on the surfaces, such that their collapse generates a different singularity. The graph theoretic description gives an efficient method to map out all SCFTs obtained by mass deformations from a given, starting point SCFT.

An intrinsically strongly coupled characteristic of a 5d SCFT is its flavor symmetry G_{F}, which generally is larger than that of its low-energy description [11]. Determining this flavor enhancement is notoriously difficult. While techniques such as the superconformal index require an effective gauge description [12], these approaches are inapplicable for examples without such a description. However, the CFD manifestly encodes the Dynkin diagram of G_{F} in terms of a marked subgraph. The CFD-transitions correspond to precise rules how vertices are removed and unmarked. Finally we can compute the representations of BPS states under G_{F}, knowing the CFD.

Our approach is rooted in the duality between M- and F-theory on a singular, elliptically fibered Calabi–Yau threefold, Y. F-theory

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https://doi.org/10.1016/j.physletb.2019.135077

on Y determines a 6d $\mathcal{N} = (1,0)$ SCFT, with flavor symmetry $G_{\text{F}}^{(6\text{d})}$, whose S^1-reduction with holonomies in the 6d flavor symmetry yields 5d SCFTs realized as M-theory on different geometric limits of Y. In these limits, we can manifestly track the unbroken subgroup of $G_{\text{F}}^{(6\text{d})}$ that constitutes the flavor symmetry [7] and the BPS spectrum [13] in 5d. We develop the geometric foundation of this approach in the companion paper [14]. In a second companion paper [15], the focus is the gauge description on the Coulomb branch of 5d SCFTs, using the methods developed in [16], complementing the CFD approach in cases lacking a gauge description.

2. SCFTs from graphs

A collection of compact complex surfaces inside a Calabi–Yau threefold defines, under suitable assumptions [3,4,6,17], a 5d $\mathcal{N} = 1$ SCFT. While determining the IR descriptions requires precise knowledge of the surface geometries, the SCFT limit is insensitive to many of the geometric details. We encode this reduced set of properties, upon which the, rank N, SCFT depends, in the *Combined Fiber Diagram* (CFD): the graph's vertices are complex curves, C_i, inside the collection of surfaces, $\mathcal{S} = \bigcup_{k=1}^{N} S_k$, and the number of edges connecting two vertices C_i and C_j is the intersection number $m_{i,j} = C_i \cdot C_j$. Each vertex has labels (n_i, g_i), the self-intersection number $n_i = C_i \cdot C_i$ of C_i inside $\mathcal{S}$ and the genus of C_i (if $g_i = 0$ the label is omitted). A detailed geometric derivation of the CFDs appears in [14].

Vertices with $(n_i, g_i) = (-2, 0)$ are marked (colored) and define a subgraph, corresponding to the Dynkin diagram of the non-abelian part of the flavor group of the 5d SCFT, G_{F}.[1] The rank of G_{F} is known, as discussed shortly, and thus one can determine the abelian factors in G_{F}. Vertices with $(n_i, g_i) = (-1, 0)$ encode mass deformations.

Given a CFD a new, *descendant CFD*, and thereby 5d SCFT, can be constructed by a *(CFD-)transition*: remove a vertex C_i with $(n_i, g_i) = (-1, 0)$ and update the CFD data:

$$\begin{aligned} n'_j &= n_j + m_{i,j}^2 \\ g'_j &= g_j + \frac{m_{i,j}^2 - m_{i,j}}{2} \\ m'_{j,k} &= m_{j,k} + m_{i,j} m_{i,k} \,, \end{aligned} \tag{1}$$

for $j, k \neq i$. A marked vertex for which n_j changes becomes unmarked after the transition. Geometrically, a transition is the collapse of a curve C_i in $\mathcal{S}$. In the SCFT, this corresponds to a mass deformation and subsequent RG-flow to the descendant SCFT. Such a transition is irreversible, reflecting the nature of RG-flows; one cannot flow "backwards" without knowing the decoupled degrees of freedom.

There are natural candidate starting points to construct descendant SCFTs, the so-called marginal theories, whose UV fixed points are 6d $(1,0)$ SCFTs. We define associated *marginal CFDs*, which have marked vertices forming *affine* Dynkin diagrams. Starting from such theories and their CFDs, our transition rules (1) generate all descendant CFDs/SCFTs. Note that this generates only 'irreducible' SCFTs that are not products of SCFTs.

For marginal theories, the rank of the flavor symmetry is $1 + \text{rank}(G_{\text{F}}^{(6\text{d})})$. With each transition, i.e., mass deformation, the flavor rank drops by one, thus the superconformal flavor symmetry algebra is fully determined.

[1] We discuss here only the simply-laced case and defer the more general case to [14].

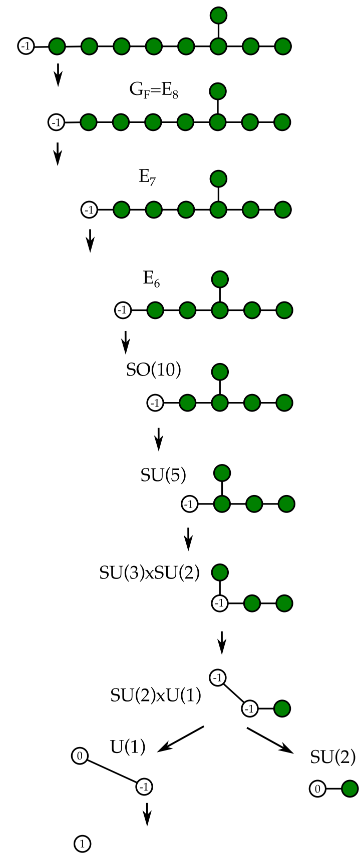
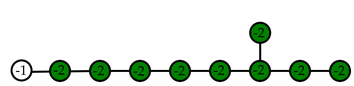


Fig. 1. CFD-transition tree for rank one 5d SCFTs including the superconformal flavor symmetries G_{F}.

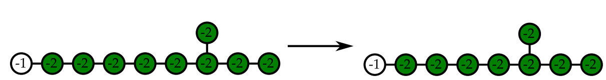
In the present letter, we consider marginal theories originating from 6d conformal matter (CM) theories [18]. The marginal CFD contains the affine Dynkin diagram of $\widehat{G_{\text{F}}^{(6\text{d})}}$ as a marked subgraph, in addition to unmarked vertices with $(n_i, g_i) = (-1, 0)$.

3. Rank one theories

We now show how CFD-transitions provide an alternative derivation of all rank one 5d SCFTs [2,11]. The marginal theory is associated to the rank one E-string theory and has CFD, where the green nodes are the marked $(-2, 0)$ vertices,


. (2)

Applying a transition to this marginal CFD describes the theory that is related by mass deformation and RG-flow. The first transition yields


, (3)

which is a CFD for a 5d SCFT with E_8 flavor symmetry—the UV fixed point for $SU(2)$ with $N_F = 7$ fundamental flavors. The complete tree of descendant CFDs is comprised of ten rank one 5d SCFTs with $G_{\text{F}} = E_{N_F+1}$, as shown in Fig. 1, in agreement with [2,11], capturing also the "E_0 theory", which lacks a gauge description.

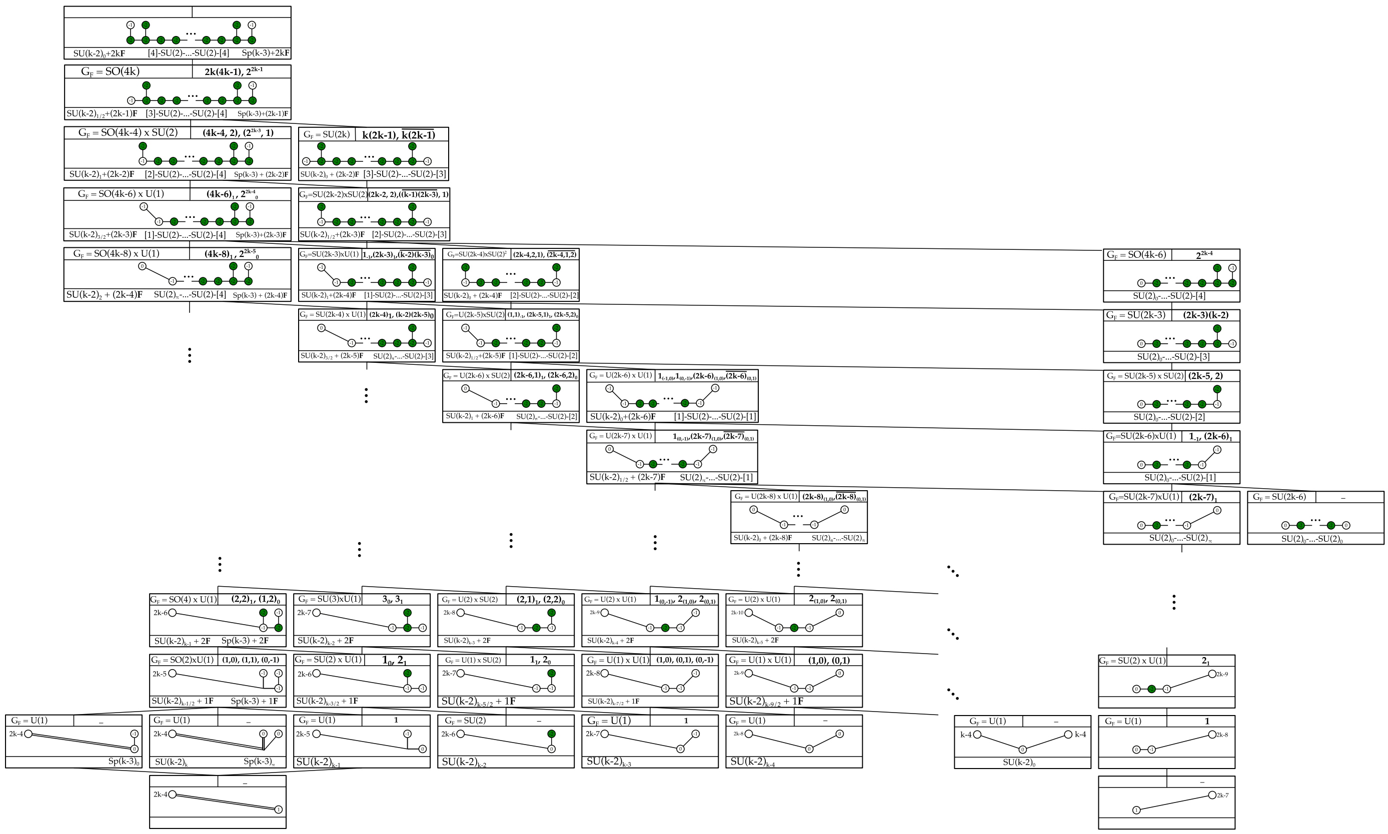
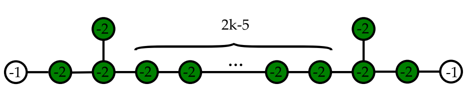


Fig. 2. CFDs for all 5d SCFTs descending from 6d (D_k, D_k) CM. Each box contains the 5d superconformal flavor symmetry, G_{F}, and the G_{F} representations of the spin 0 BPS states (right upper corner). In cases when there is a weakly coupled gauge theory description, this is noted at the bottom of each box. Connecting lines between boxes indicate transitions.

4. 5d SCFTs from (D_k, D_k) CM

Next we consider theories of arbitrary rank, descending from 6d (D_k, D_k) minimal CM, whose marginal CFD is



. (4)

The marked $(-2, 0)$-vertices form a $\widehat{D}_{2k}$ affine Dynkin diagram and $G_{\mathrm{F}}^{(6\mathrm{d})} = D_{2k}$. There are $(k+2)^2 - 3$ descendant CFDs/SCFTs, shown in Fig. 2, including the superconformal flavor symmetry. In the supplementary material we explicitly determine all descendants for (D_9, D_9).

Three dual gauge descriptions for the marginal theory are known

$$SU(k-2)_0 + 2k\mathbf{F}\,, \quad Sp(k-3) + 2k\mathbf{F}\,, \qquad [4\mathbf{F}] - SU(2)^{k-3} - [4\mathbf{F}]\,, \tag{5}$$

where $SU(2)^{k-3}$ is the linear quiver with $(k-3)$ $SU(2)$ gauge nodes connected by bifundamental hypermultiplets; the factors without flavors have $\theta = 0$ [19,20]. Giving mass to the flavors populates subtrees in Fig. 2.

Any of the $SU(k-2)$ gauge descriptions are specified by the number, m, of fundamental hypermultiplets and the Chern–Simons level, κ. Decoupling a flavor hypermultiplet shifts κ by $\pm\frac{1}{2}$ [3]. Moreover, $SU(k-2)_\kappa$ is dual to $SU(k-2)_{-\kappa}$. Overall, there are $k(k+2)$ 5d SCFTs with this weakly coupled gauge description.

The CFDs predict the following flavor enhancement for theories with an $SU(k-2)_\kappa + m\mathbf{F}$ description:

$$\begin{array}{rl} \kappa & \text{SCFT Flavor Symmetry } G_{\mathrm{F}} \\ k - \dfrac{m}{2}: & \left\{ \begin{array}{rl} SO(4k) & m = 2k-1 \\ SO(4k-4) \times SU(2) & m = 2k-2 \\ SO(2m) \times U(1) & m = 0, \ldots, 2k-3 \end{array} \right. \\ k - 1 - \dfrac{m}{2}: & \left\{ \begin{array}{rl} SU(2k) & m = 2k-2 \\ SU(2k-2) \times SU(2) & m = 2k-3 \\ SU(m+1) \times U(1) & m = 0, \ldots, 2k-4 \end{array} \right. \\ k - 2 - \dfrac{m}{2}: & \left\{ \begin{array}{rl} SU(2k-4) \times SU(2)^2 & m = 2k-4 \\ U(m) \times SU(2) & m = 0, \ldots, 2k-5 \end{array} \right. \end{array} \tag{6}$$

These flavor symmetries agree with those recently obtained by independent methods in [20–22].

By decoupling stepwise the $2k$ fundamental hypermultiplets from the marginal $Sp(k-3)$ theory in (5), we get $(2k+1)$ descendants, where the lowest two are $Sp(k-3)_0$ or $Sp(k-3)_\pi$; $2k$ have a dual $SU(k-2)$ gauge description. There is a unique theory with only an $Sp(k-3)_0$ gauge description, whose classical and superconformal flavor symmetry is $U(1)$.

For any k, there are six SCFTs, which have only an effective gauge description via the quivers

$$\begin{aligned} & SU(2)_0^{k-4} - SU(2) - [m\mathbf{F}], \quad m = 1, \ldots, 4 \\ & SU(2)_0^{k-4} - SU(2)_\theta\,, \qquad \theta = 0, \pi\,. \end{aligned} \tag{7}$$

The superconformal flavor symmetries are

$$
\begin{aligned}
m=4: &\quad SO(4k-6)\\
m=3: &\quad SU(2k-3)\\
m=2: &\quad SU(2k-5)\times SU(2)\\
m=1: &\quad SU(2k-6)\times U(1)\\
m=0,\ \theta=0: &\quad SU(2k-6)\\
m=0,\ \theta=\pi: &\quad SU(2k-7)\times U(1)\,.
\end{aligned}
\tag{8}
$$

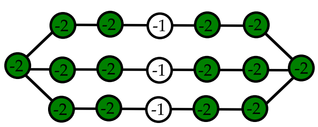
Our approach using CFDs not only determines these flavor symmetries much more efficiently and purely combinatorially than approaches using a gauge description, we can even determine the flavor symmetry in cases when such a weakly coupled description is absent. In the present case, there are $2k-6$ SCFTs that do not have any known gauge description, but we determine their superconformal flavor symmetry to be

$$
U(2k-7-i)\,,\quad i=0,\cdots,2k-7\,. \tag{9}
$$

These CFDs and their associated geometries [14] are evidence that such non-trivial 5d UV fixed points exist; these have been observed for rank two, $k=5$, in [6,14,23].

5. 5d SCFTs from (E_6, E_6) CM

Another class of higher rank theories, that have thus far not been studied in generality, are the rank five SCFTs descending from (E_6, E_6) minimal CM. The marginal CFD is



. (10)

CFD-transitions applied to this yield 93 descendant CFDs/SCFTs, included in the supplemental material. This predicts a large class of new 5d SCFTs. The only known weakly coupled description of the marginal theory is the quiver [18]

$$
\begin{array}{c}
\quad\quad\quad\quad [2]\\
\quad\quad\quad\quad |\\
\quad\quad\quad\quad SU(2)\\
\quad\quad\quad\quad |\\
[2]-SU(2)-SU(3)_0-SU(2)-[2]\,.
\end{array}
\tag{11}
$$

Decoupling the flavor hypermultiplets of each $SU(2)$, step-by-step, yields descendants with quiver descriptions. Denote these by a triple (q_1, q_2, q_3), where the q_i is either the number of fundamentals under, or the theta angle of, each of the three $SU(2)$ factors. For these quivers we find the following superconformal flavor symmetries:

$$
\begin{aligned}
(1\mathbf{F},2\mathbf{F},2\mathbf{F}) &: E_6\times E_6\\
(0,2\mathbf{F},2\mathbf{F}),\ (\pi,2\mathbf{F},2\mathbf{F}) &: E_6\times SU(6)\\
(1\mathbf{F},1\mathbf{F},2\mathbf{F}) &: SO(10)^2\times U(1)\\
(0,1\mathbf{F},2\mathbf{F}),\ (\pi,1\mathbf{F},2\mathbf{F}) &: SO(10)\times SU(5)\times U(1)\\
(1\mathbf{F},1\mathbf{F},1\mathbf{F}) &: SO(8)^2\times U(1)^2\\
(0,0,2\mathbf{F}),\ (\pi,\pi,2\mathbf{F}) &: SO(10)\times SU(4)\times U(1)\\
(0,\pi,2\mathbf{F}) &: SU(5)^2\times U(1)\\
(0,1\mathbf{F},1\mathbf{F}),\ (\pi,1\mathbf{F},1\mathbf{F}) &: SO(8)\times SU(4)\times U(1)^2\\
(0,0,1\mathbf{F}),\ (\pi,\pi,1\mathbf{F}) &: SO(8)\times SU(3)\times U(1)^2\\
(0,\pi,1\mathbf{F}) &: SU(4)^2\times U(1)^2\\
(0,0,0),(\pi,\pi,\pi) &: SO(8)\times SU(2)\times U(1)^2\\
(0,0,\pi),\ (\pi,\pi,0) &: SU(4)\times SU(3)\times U(1)^2\,.
\end{aligned}
\tag{12}
$$

This populates only a small subtree of twelve elements in the CFD tree. Note that the CFDs are sensitive to the number of independent discrete parameters; they capture dualities between theories with different theta angles [24,25]. It would be interesting to determine the gauge theory descriptions, where they exist, for the remaining 81 CFD/SCFTs.

6. BPS states

BPS states, Φ_C, arise in M-theory from wrapped M2-branes on holomorphic curves C in $\mathcal{S}$. In CFD-terms, $(n, g) = (-1, 0)$-vertices for instance correspond to spin 0 states under the 5d massive little group $SO(4)$ [26,27]. More generally, C can be a non-negative linear combination of vertices in the CFD, $C=\sum_i q_i C_i$, $q_i \geq 0$, where the q_i constrained by the decorations (n, g), which are recursively determined

$$
\begin{aligned}
n &= (C_1+C_2)^2 = C_1\cdot C_1 + C_2\cdot C_2 + 2C_1\cdot C_2\,,\\
g(C_1+C_2) &= g(C_1)+g(C_2)+C_1\cdot C_2 - 1\,.
\end{aligned}
\tag{13}
$$

Each C is associated to a weight of a representation of $G_{\rm F}$, where the highest weights under the non-abelian subalgebra, $G_{\rm F,na}$, are determined through the intersection numbers between C and the marked curves, F_i in the CFD

$$
C\cdot F_i \geq 0\,,\ (i=1,\ldots,\mathrm{rk}(G_{\rm F,na}))\,. \tag{14}
$$

Charges under the abelian subalgebra are determined through intersection with specific combinations of unmarked vertices orthogonal to $G_{\rm F,na}$, the $U(1)$ generators. Applying this to rank one theories reproduces the spin 0 BPS states in [28]. For the (D_k, D_k) descendants, Fig. 2 contains the predictions for spin 0 BPS states in these 5d strongly coupled SCFTs.

Acknowledgements

We thank J. Distler, J. J. Heckman, N. Mekareeya, A. Tomasiello, G. Zafrir and in particular M. Weidner for discussions. The work of FA, SSN, YNW is supported by the ERC Consolidator Grant 682608 “Higgs bundles: Supersymmetric Gauge Theories and Geometry (HIGGSBNDL)”. CL is supported by NSF CAREER grant PHY-1756996; LL is supported by DOE Award DE-SC0013528Y. FA and CL thank the 2019 Pollica summer workshop, where part of this work was completed. YNW thanks the Aspen Center for Physics, which is supported by National Science Foundation grant PHY-1607611, where part of this work was finished. SSN thanks the Mainz Institute for Theoretical Physics, Milano Bicocca University and Kavli IPMU hospitality during the completion of this work. YNW was also partially supported by a grant from the Simons Foundation at the Aspen Center for Physics. The authors thank the 2019 String-Phenomenology Conference and CERN for hospitality during the final stages of this work.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.physletb.2019.135077.

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