Exploring computational geometry of fundamental interactions in static limits

Article · September 2019

DOI: 10.4006/0836-1398-32.3.411]

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Exploring computational geometry of fundamental interactions in static limits

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(Received 7 May 2019; accepted 10 August 2019; published online 18 September 2019)

Abstract: We explore computational geometry, particularly the three-dimensional (3D) contours of the electrostatic potential due to electrical charges with similar and also opposite polarity. We also explore the 3D contours of the gravitational potential by computational geometry. The noticeable outcome of these exercises is that electrostatic interaction affects the geometry of the space as does the gravitational interaction but with the difference that electrostatic interaction affects the geometry of the space differently for charges with different polarity and at different distance scales.
 [http://dx.doi.org/10.4006/0836-1398-32.3.411]

Résumé: Nous explorons la géométrie informatique, en particulier les contours tridimensionnels (3-D) du potentiel électrostatique dus aux charges électriques de polarité à la fois similaire et opposée. Par le biais de la géométrie algorithmique, nous pourrons également explorer les contours tridimensionnels (3-D) du potentiel gravitationnel. Le résultat visible de ces exercices montre que l'interaction électrostatique affecte la géométrie de l'espace tout comme l'interaction gravitationnelle, mais avec la différence que l'interaction électrostatique affecte la géométrie de l'espace d'une manière différente pour des charges de polarité opposée et à des échelles de distance diverse.

Key words: Electrostatic Interaction; Gravitational Potential; Fabric of the Space; Equipotential Curves; Equipotential Surfaces; Contours of Potential; World Sheet.

I. INTRODUCTION

This discussion is a follow up of the recent explorations of computation of geometry due to electrostatic interactions.[1,2] Also, the present discussion aims to explore the computational geometry of fundamental interactions in wider perspective including that of gravitational interaction. We analyze the computational images of the electrostatic potentials with the specific expressions of electrostatics used by Aalok,[1] Mayes,[2] and Wangsness.[3] We also take note of exercises given by Gilat[4] and Landau *et al*.,[5] which present valuable concept demonstrations of electrostatic interactions, but do not explore anything further. The discussion on "electrostatic interactions between charged particles" by Lindgren *et al*.[6] is also significant progress in this direction, but, turns out to be more relevant in Chemistry. In our earlier discussion,[1] we reported having observed a few niceties in the geometry of electrostatic interactions resulting from the computations. We noticed that two-dimensional (2D) contours of these graphics represent equipotential curves and the three-dimensional (3D) contours represent equipotential surfaces.[1] We now raise a pertinent question: What does this computational geometry represent? Does it represent the changes in the geometry of the space due to electrostatic potential and electrostatic interaction? Can it be analyzed in the light of similar geometrical effects caused by gravitation?

We attempt to answer these questions in the present exercise. However, the polarity of the electric charges does affect the geometry of the plots of electrostatic potential due to similar charges, the computation of the potential due to dissimilar charges is a challenge to address that we take up in this exercise. The cylindrical world-sheets as arise from computation of the 3D contours of electric potential hold only for similar electrical charges, and do not sustain if we apply it for two dissimilar electrical charges. Thus, to add to the meaningful deliberations of the computational graphics, we employ the idea of flavor or polarity changing mechanism. In other words, we employ the idea of particles lying along the world-lines directed in forward and backward directions, respectively. It is pertinent to mention that by using the trick of generic mouse dragging the graphics in 3D could be rotated, reproduced, and presented in different perspectives.

II. THE COMPUTATIONAL INVESTIGATIONS

We again make use of the standard forms of electrostatic potential and field to explore the geometry of electrostatic interactions in various scenarios. We make use of standard

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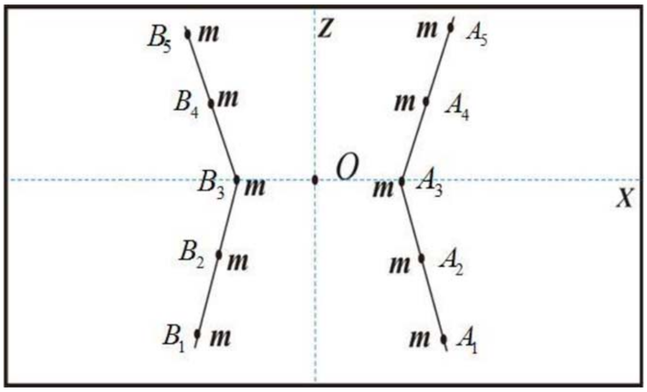

FIG. 1. (Color online) Set of stationary particles each with mass $m = 1.66 \times 10^{-27}$ kg and located along two world-lines A and B.

expressions of electrostatic and gravitational potential given in Eqs. (A1) and (A2) in the Appendix. We consider a number of cases to compute the geometry of electrostatic interaction in an extended scenario that was discussed in our earlier exercise.[1]

We investigate the computational geometry of electrostatic potential due to a number of stationary particles each of which bears electrical charge -1.6×10^{-19}C or $+1.6 \times 10^{-19}$C are located along a path which in a dynamical scenario one may call a world line passing through these points. In the same picture, we consider another set of electrical charges located along a path separated from the first one at a small distance.

The challenging part of this exercise lies in the case when we consider a set of opposite electrical charges along two paths. We also examine the computational geometry, particularly, the 3D contours of gravitational potential due to masses of the fundamental particles as we did for these particles in our earlier exercises. We consider the following case:

Case (i): Identical stationary particles each with mass $m = 1.66 \times 10^{-27}$ kg are located at points $A_1(+.082 \times 10^{-10},\ 0,\ -.070 \times 10^{-10})$, $A_2(+.068 \times 10^{-10},\ 0,\ -.060 \times 10^{-10})$, $A_3(+.01 \times 10^{-10},\ 0,\ 0)$, $A_4(+.068 \times 10^{-10},\ 0,\ +.060 \times 10^{-10})$, and $A_5(+.082 \times 10^{-10},\ 0,\ +.070 \times 10^{-10})$ along a world line A as shown in Fig. 1. Similarly, as shown in Fig. 1, identical stationary particles each with mass $m = 1.66 \times 10^{-27}$kg are located at points $B_1(-.082 \times 10^{-10},\ 0,\ -.070 \times 10^{-10})$, $B_2(-.068 \times 10^{-10},\ 0,\ -.060 \times 10^{-10})$, $B_3(-.01 \times 10^{-10},\ 0,\ 0)$, $B_4(-.068 \times 10^{-10},\ 0,\ +.060 \times 10^{-10})$, and $B_5(-.082 \times 10^{-10},\ 0,\ +.070 \times 10^{-10})$ along a world line B. We compute the geometry of the resultant gravitational potential by means of 3D contours in the range $-.082 \times 10^{-10}\,\mathrm{m} < x < +.082 \times 10^{-10}\,\mathrm{m}$, $-.14 \times 10^{-10}\,\mathrm{m} < y < +.15 \times 10^{-10}\,\mathrm{m}$, and $-.097 \times 10^{-10}\,\mathrm{m} < z < +.097 \times 10^{-10}\,\mathrm{m}$.

The MATHEMATICA command for this computation is given as:

```
ContourPlot3D[ (-6.66*10^-11)*(1.66*(10)^-
27)*(1/Sqrt[ (x+.0100*10^(-10))^2+y^2+z^2] +          1/Sqrt[ (x-
.0100*10^(-10))^2+y^2+z^2] +                  1/Sqrt[ (x+.068*10^(-
10))^2+y^2+(z+.060*10^(-10))^2] +            1/Sqrt[ (x+.082*10^(-
10))^2+y^2+(z+.070*10^(-10))^2] +             1/Sqrt[ (x-.068*10^(-
10))^2+y^2+(z-.060*10^(-10))^2] +             1/Sqrt[ (x-.082*10^(-
10))^2+y^2+(z-.070*10^(-10))^2] +            1/Sqrt[ (x+.068*10^(-
10))^2+y^2+(z-.060*10^(-10))^2] +            1/Sqrt[ (x+.082*10^(-
10))^2+y^2+(z-.070*10^(-10))^2] +             1/Sqrt[ (x-.068*10^(-
10))^2+y^2+(z+.060*10^(-10))^2] +             1/Sqrt[ (x-.082*10^(-
10))^2+y^2+(z+.070*10^(-10))^2]),{x,-.082*10^(-10), .082*10^(-
10)},{y,-.14*10^(-10),.15*10^(-10)},{z,-.097*10^(-10),
.097*10^(-10)}]
```

The resultant geometry appears as follows in Fig. 2.

Interestingly, the computed graphic of the gravitational potential too follows a particular pattern which seems to mimic the string theoretic picture, but, at low energies, in static limits, and at atomic distance scale.

We now revisit the computational geometry due to electrostatic potential presented earlier and look into various aspects of it. Earlier, we had computed the geometry of 3D contours due to two charges located at points (x_1, y_1, z_1, t_1) and (x'_1, y'_1, z'_1, t_1), and we found it to be cylindrical world-sheet which represents equipotential surface.[1] We then argued that if we compute the geometry of 3D contours due to similar charges located at points (x_2, y_2, z_2, t_2) and (x'_2, y'_2, z'_2, t_2) and for small enough time interval $\Delta t = t_2 - t_1$, the resultant geometry has plethora of information in store.[1] We thus look at the geometry of electrostatic potential, but, in a different perspective as follows:

Case (ii): Identical stationary particles each with electrical charge -1.6×10^{-19}C are located at points $A_1(+.082 \times 10^{-10}, 0, -.070 \times 10^{-10})$, $A_2(+.068 \times 10^{-10}, 0, -.060 \times 10^{-10})$, $A_3(+.01 \times 10^{-10},\ 0,\ 0)$, $A_4(+.068 \times 10^{-10}, 0, +.060 \times 10^{-10})$ and $A_5(+.082 \times 10^{-10}, 0, +.070 \times 10^{-10})$ along a world line A as shown in Fig. 3. Similarly, as shown in Fig. 3, identical stationary particles each with charge -1.6×10^{-19}C are located at points $B_1(-.082 \times 10^{-10},\ 0,\ -.070 \times 10^{-10})$, $B_2(-.068 \times 10^{-10},\ 0,\ -.060 \times 10^{-10})$, $B_3(-.01 \times 10^{-10},\ 0,\ 0)$, $B_4(-.068 \times 10^{-10},\ 0,\ +.060 \times 10^{-10})$, and $B_5(-.082 \times 10^{-10},\ 0,

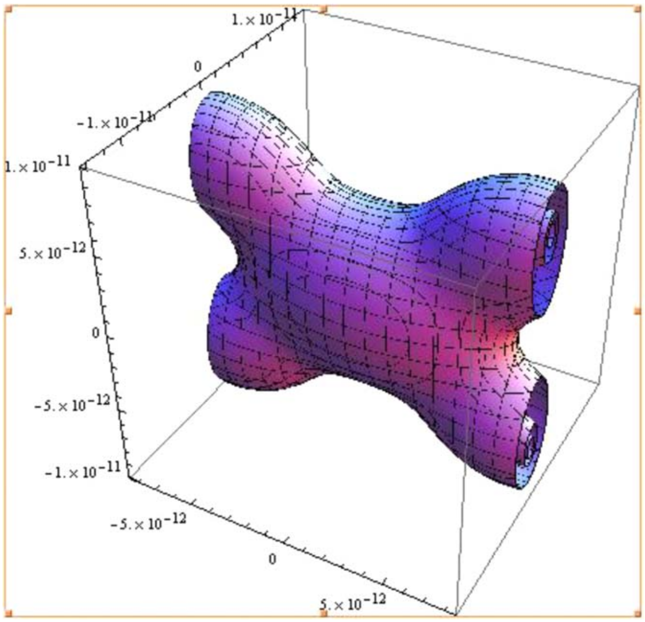

FIG. 2. (Color online) Geometry of the gravitational potential due to set of charges discussed in Fig. 1.

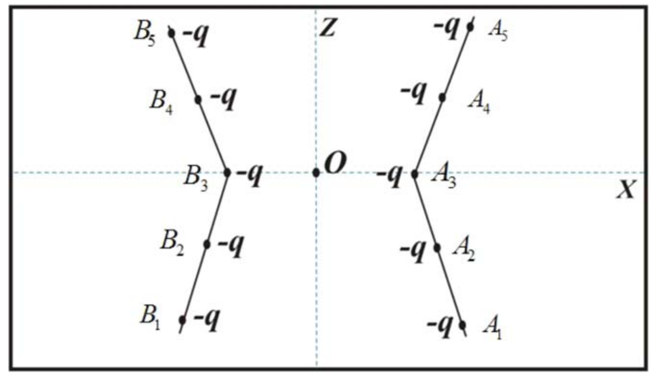

FIG. 3. (Color online) Set of stationary charged particles each with electrical charge $q = -1.6 \times 10^{-19}$C and located along two world-lines A and B.

$+.070 \times 10^{-10}$) along a world line B. We compute the geometry of the resultant electrostatic potential by means of 3D contours in the range $-.082 \times 10^{-10}$ m $< x <$ $+.082 \times 10^{-10}$ m, $-.14 \times 10^{-10}$m $< y < +.15 \times 10^{-10}$ m, and $-.097 \times 10^{-10}$ m $< z < +.097 \times 10^{-10}$ m.

We compute the geometry of the resultant electrostatic potential by means of 3D contours.

The MATHEMATICA command for this computation is given as:

```
ContourPlot3D[ (9*10^9)*(-1.6*(10)^-19)*(1/Sqrt[ (x+.0100*10^(-10))^2+y^2+z^2]
+1/Sqrt[ (x-.0100*10^(-10))^2+y^2+z^2]+
1/Sqrt[ (x+.068*10^(-10))^2+y^2+(z+.060*10^(-
10))^2]+1/Sqrt[ (x+.082*10^(-10))^2+y^2+(z+.070*10^(-
10))^2]+1/Sqrt[ (x-.068*10^(-10))^2+y^2+(z-.060*10^(-
10))^2]+1/Sqrt[ (x-.082*10^(-10))^2+y^2+(z-.070*10^(-
10))^2]+1/Sqrt[ (x+.068*10^(-10))^2+y^2+(z-.060*10^(-10))^2]+
1/Sqrt[ (x+.082*10^(-10))^2+y^2+(z-.070*10^(-10))^2]+1/Sqrt[ (x-
.068*10^(-10))^2+y^2+(z+.060*10^(-10))^2]+1/Sqrt[ (x-.082*10^(-10))^2+y
^2+(z+.070*10^(-10))^2]),{x,-.082*10^(-10),.082*10^(-
10)},{y,-.14*10^(-10),+.15*10^(-10)},{z,-.097*10^(-10),
.097*10^(-10)}]
```

And the resultant geometry appears as in Fig. 4.

As interpreted earlier, the 3D contour that appears to mimic the interaction picture predicted by Superstring theory represents the equipotential surface. We shall further elaborate various aspects of it.

Case (iii): Identical stationary particles each with charge $q = -1.6 \times 10^{-19}$C are located at points $A_1(+.006 \times 10^{-10}$, 0, $-.012 \times 10^{-10})$, $A_2(+.0009 \times 10^{-10}, 0, -.001 \times 10^{-10})$, $A_3(+.0001 \times 10^{-10},\ 0,\ 0)$, $A_4(+.0009 \times 10^{-10},\ 0, +.001 \times 10^{-10})$, and $A_5(+.006 \times 10^{-10}$, 0, $+.012 \times 10^{-10})$ along a world line A as shown in Fig. 5. Similarly, as shown in Fig. 5, identical stationary charged particles each with charge $q = +1.6 \times 10^{-19}$C are located at points $B_1(-.006 \times 10^{-10},\ 0, -.012 \times 10^{-10})$, $B_2(-.0009 \times 10^{-10},\ 0,\ -.001 \times 10^{-10})$, $B_3(-.0001 \times 10^{-10}, 0, 0)$, $B_4(-.0009 \times 10^{-10},\ 0, +.001 \times 10^{-10})$, and $B_5(-.006 \times 10^{-10}, 0, +.012 \times 10^{-10})$ along a world line B. We compute the geometry of the electrostatic potential by means of 3D contours in the range, $-.028 \times 10^{-10}$ m $< x < +.028 \times 10^{-10}$ m, $-.11 \times 10^{-10}$ m $<$ $y < +.12 \times 10^{-10}$ m, and $-.0099 \times 10^{-10}$ m $< z <$ $+.0099 \times 10^{-10}$ m.

We compute the geometry of the resultant electrostatic potential by means of 3D contours.

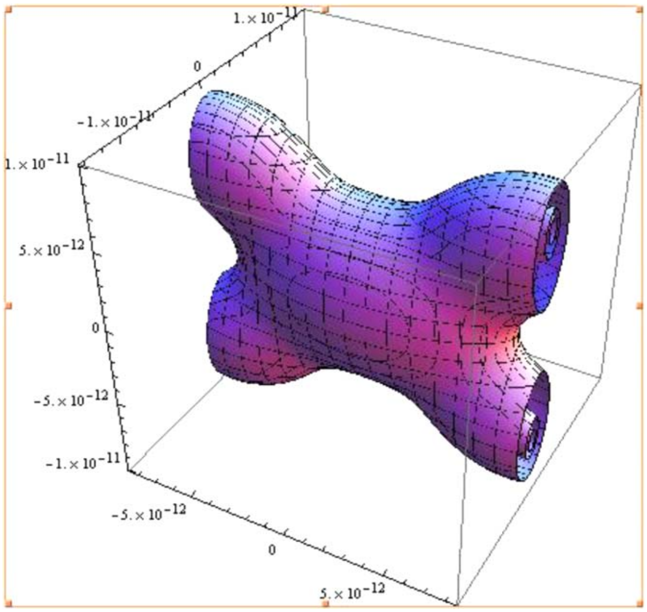

FIG. 4. (Color online) Geometry of the electrostatic potential due to charges discussed in Fig. 3.

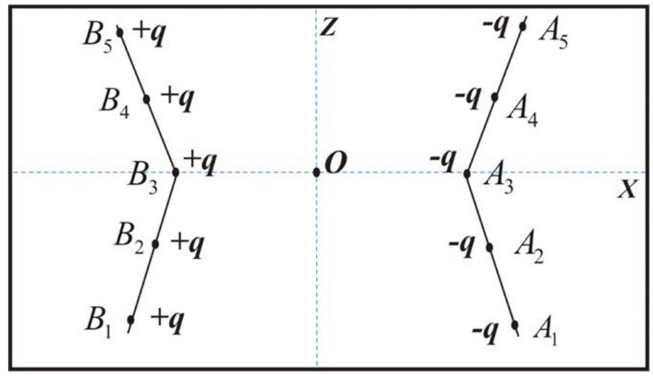

FIG. 5. (Color online) Stationary charged particles each with electrical charge $+1.6 \times 10^{-19}$C located along the world-line B, and each with electrical charge -1.6×10^{-19}C located along the world-line A.

The MATHEMATICA command for this computation is given as follows:

```
ContourPlot3D[ (9*10^9)*(1.6*(10)^-19)*(
-1/Sqrt[ (x-.006*10^(-10))^2+y^2+(z+.012*10^(-10))^2]
-1/Sqrt[ (x-.0009*10^(-10))^2+y^2+(z+.001*10^(-10))^2]
-1/Sqrt[ (x-.0009*10^(-10))^2+y^2+(z-.001*10^(-10))^2]
-1/Sqrt[ (x-.006*10^(-10))^2+y^2+(z-.012*10^(-10))^2]
-1/Sqrt[ (x-.0001*10^(-10))^2+y^2+z^2]
+1/Sqrt[ (x+.006*10^(-10))^2+y^2+(z+.012*10^(-10))^2]
+1/Sqrt[ (x+.0009*10^(-10))^2+y^2+(z+.001*10^(-10))^2]
+1/Sqrt[ (x+.0009*10^(-10))^2+y^2+(z-.001*10^(-10))^2]
+1/Sqrt[ (x+.006*10^(-10))^2+y^2+(z-.012*10^(-10))^2]
+1/Sqrt[ (x+.0001*10^(-10))^2+y^2+z^2]),
{x,-.028*10^(-10),.028*10^(-10)},{y,-.11*10^(-10),.12*10^(-
10)},{z,-.0099*10^(-10),.0099*10^(-10)}]
```

And the resultant geometry appears as in Fig. 6.

However, a plane sheet is sighted at the interface of the two cylindrical world-sheets while generic mouse dragging on this graphic. With whatever effort, it could not be eliminated. This gives us an impression of method of images often invoked in electro-statics. This aspect is open for further deliberations.

Inspired by these investigations, once again we explore the behavior of gravitational potential *vis-a-vis* the behavior of electrostatic potential for the same set of objects from different perspectives. We redo the computational exercise that we performed in the previous study[1] but for gravitational potential due to two electrons.

Case (iv): We evaluate the geometry of gravitational potential due to two similar charged particles say, electrons and masses $m_1 = 9.1 \times 10^{-31}$ kg and $m_2 = 9.1 \times 10^{-31}$ kg positioned in the $X-Y$ plane at points $(0.264 \times 10^{-10}, 0, 0)$ and $(-0.264 \times 10^{-10}, 0, 0)$ m, respectively. We explore the potential due to these particles at points in the $X-Y$ plane in the domain $-.2 \times 10^{-10} \le x \le .2 \times 10^{-10}$ and $-.2 \times 10^{-10} \le y \le .2 \times 10^{-10}$ m, respectively. This separation is of the order of Bohr's radius.

The MATLAB code for this graphic can be written as follows:

```
≫m1=9.1e-31; m2=9.1e-31; G=6.66e-11;
x=-0.2e-10:0.005e-10:0.2e-10; y=-0.2e-10:0.005e-10:0.2e-10;
[X,Y]=meshgrid(x,y);
r1=sqrt((X+0.264e-10).^2+Y.^2);
r2=sqrt((X-0.264e-10).^2+Y.^2);
V=-G*(m1./r1+m2./r2);
meshc(X,Y,V);xlabel('x(m)'); ylabel('y(m)'); zlabel('V(Nt-m/kg)')
```

The resultant geometry appears as in Fig. 7.

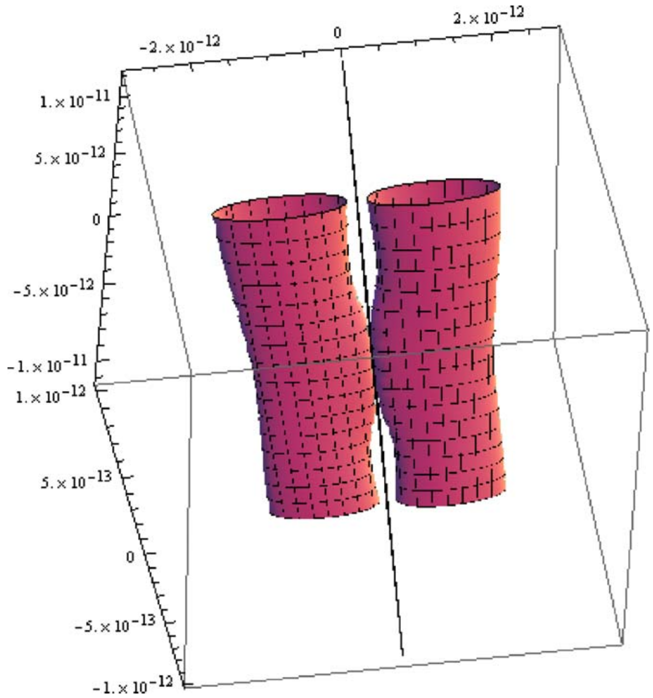

FIG. 6. (Color online) Geometry of the electrostatic potential due to charges discussed in Fig. 5.

It is pertinent to mention here that in the computed geometry the 2D contours on the horizontal plane represent curves corresponding to equipotential points on the surface. We now evaluate the cumulative effect on the computed geometry due to gravitational as well as electrostatic potential geometry and discuss the similarities and dissimilarities.

Case (v): Two particles have opposite charges $q_1 = 1.6 \times 10^{-19}$C and $q_2 = -1.6 \times 10^{-19}$C, respectively, and masses $m_1 = 1.66 \times 10^{-27}$ kg and $m_2 = 9.1 \times 10^{-31}$ kg positioned in the $X - Y$ plane at points $(0.264 \times 10^{-10}, 0, 0)$ and $(-0.264 \times 10^{-10}, 0, 0)$ m, respectively. We explore the geometry of the effective potential due to these particles at points in the $X - Y$ plane in the domain $-.264 \times 10^{-10} \leq x \leq .264 \times 10^{-10}$ and $-.264 \times 10^{-10} \leq y \leq .264 \times 10^{-10}$ m,

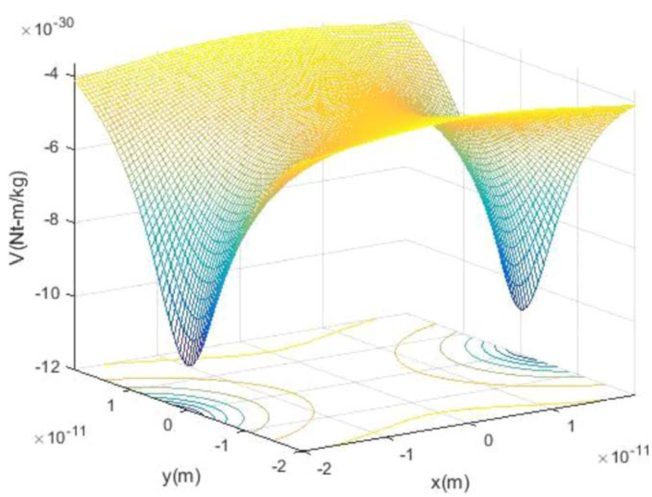

FIG. 7. (Color online) Geometry of the gravitational potential due to particles with similar masses.

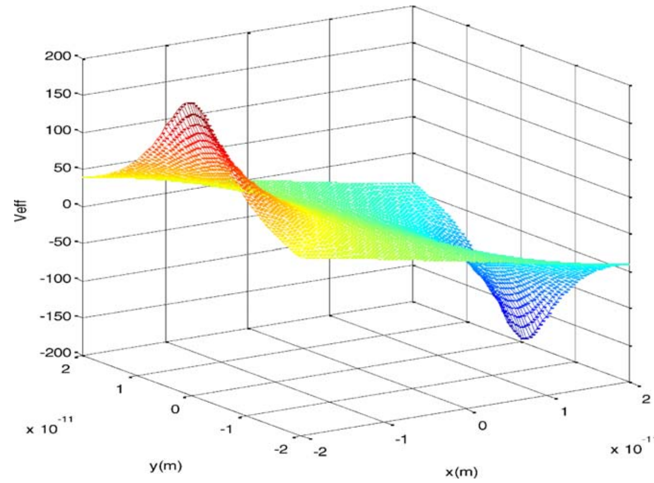

FIG. 8. (Color online) Geometry of gravitational and electrostatic potentials due to an electron and a proton.

respectively. The MATLAB code for this graphic can be written as follows:

```
≫ q1 = 1.6e-19; q2=-1.6e-19; eps = 8.85e-12; k = 1/(4*pi*eps);
m1 = 1.66e-27; m2 = 9.1e-31; G = 6.66e-11;
x=-0.2e-10:0.005e-10:0.2e-10;
y=-0.2e-10:0.005e-10:0.2e-10;
[X,Y] =meshgrid(x,y);
r1 = sqrt((X + 0.264e-10).^2+Y.^2);
r2 = sqrt((X-0.264e-10).^2+Y.^2);
Veff=k*(q1./r1+q2./r2)-G*(m1./r1+m2./r2);
mesh(X,Y,Veff); xlabel('x(m)'); ylabel('y(m)'); zlabel('Veff')
```

The resultant computed geometry appears as in Fig. 8.

The graphic reveals that space has been affected positively as well as negatively due to this combined potential. This is the case of electrostatic interaction, as the effects due to gravitational potential are not expected to be different for different polarity of electron and the proton. Thus, we may conclude that the effect of gravitational potential is submerged in that of the electrostatic potential. At large scale of distances, the charge neutrality reins and the electrostatic potential turns insignificant and only gravitational potential affects the space.

We now repeat the same exercise but at larger scale of distance that is to say—we explore geometry of the effective potential due to superposition of gravitational and the electrostatic potentials arising because of two particles having opposite charges at atomic distance scale.

Case (vi): We explore geometry of space due to gravitational potential arising because of two particles with masses $m_1 = 1.66 \times 10^{-27}$kg and $m_2 = 9.1 \times 10^{-31}$kg positioned in the $X - Y$ plane at points $(0.264 \times 10^{-10}, 0, 0)$ and $(-0.264 \times 10^{-10}, 0, 0)$ m, respectively. We explore the potential due to these particles at points in the $X - Y$ plane in the domain $-.264 \times 10^{-10} \leq x \leq .264 \times 10^{-10}$ and $-.264 \times 10^{-10} \leq y \leq .264 \times 10^{-10}$ m, respectively.

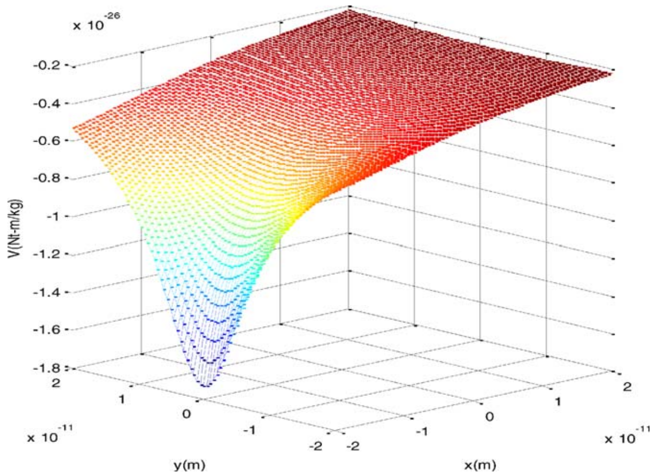

FIG. 9. (Color online) Geometry of the gravitational potential due to an electron and a proton.

The MATLAB code for this graphic can be written as:

```
>> m1 = 1.66e-27; m2 = 9.1e-31; G = 6.66e-11;
x=-0.2e-10:0.005e-10:0.2e-10;
y=-0.2e-10:0.005e-10:0.2e-10;
[X,Y] =meshgrid(x,y);
r1 = sqrt((X + 0.264e-10).^2+Y.^2);
r2 = sqrt((X-0.264e-10).^2+Y.^2);
V=-G*(m1./r1+m2./r2);
mesh(X,Y,V); xlabel( 'x(m)'); ylabel(
'y(m)'); zlabel('V(Nt-m/kg)')
```

The resultant geometry appears as in Fig. 9.

Apparently, the effect of gravitational potential due to electron on the space is submerged in the effect due to a proton and thus it is not visible in the graphic. Furthermore, we propose to explore geometry of the effective potential due to superposition of gravitational and the electrostatic potentials arising because of two particles having opposite charges at a smaller distance scale.

III. SUMMARY AND DISCUSSION

The most important lesson that emerges out of this discussion can be summarized as follows: Electrostatic interaction too affects the geometry of the space. The differences arise at different distance scales. As is evident from the graphics, electrostatic interactions also affect the geometry of the space as does the gravitational interaction but with a significant difference that electrostatic interaction affects the geometry of the space with both positive as well as negative polarity, whereas, gravitational interaction affects the geometry of the space in different magnitudes and curvature but remains indifferent with respect to variation in polarity of electrical charges.

We are posed with another question: Can 3D contours witnessed in this exercise which we refer as equipotential surfaces and that appear like cylindrical world-sheets have any analogy with superstring theory in electrostatic limits and how? We are amazed by the striking similarity of the computational outcome of our exercises with the interactions depicted in the Superstring theory. In the Superstring theory, the strings are extended objects that represent excitations of fundamental particles and the cylindrical world-sheet represents evolution of the string from one location to other. We feel that more could be explored on this aspect. In the Superstring theory, a closed string while propagating sweeps a 2D surface known as the world-sheet of the string.[7,8] It could be an indicator of the residual effect of these interactions for stationary objects, at low energy, at atomic distance scale, and in the form of equipotential surfaces! If the computational outcome of the contour plot of gravitational potential along with electrostatic potentials is any indicator, the issue really deserves further attention!

There are obvious limitations of the present discussion: (i) It explains static interactions and does not incorporate dynamic phenomena; (ii) It explains electrostatic interaction of charged particles only and does not say, anything about electrically neutral particles. Therefore, it is imperative that the present discussion which is confined to the electrostatic limits should be extended by dynamical simulations. We again clarify that the computed geometry in this paper cannot be reproduced at all distance scales, and, computation breaks down with different range of distances and at different distance scales.

APPENDIX: ELECTROSTATIC AND GRAVITATIONAL POTENTIAL

The electric potential at a point at the distance $\vec{r}$ from the origin due to charge q_i located at distance $\vec{r}_i$ from the origin is given by

$$V_q(r) = \frac{kq_i}{|\vec{r}-\vec{r}_i|} = \frac{kq_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}, \tag{A1}$$

and the gravitational potential at a point at the distance $\vec{r}$ from the origin due to mass m_i located at distance $\vec{r}_i$ from the origin is given by

$$V_g(r) = -\frac{Gm_i}{|\vec{r}-\vec{r}_i|} = -\frac{Gm_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}, \tag{A2}$$

where symbols have their usual meaning.

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