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A geometric description for simple and damped harmonic oscillators

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Abstract: In this work we consider the Riemannian geometry associated with the differential equations of one dimensional simple and damped linear harmonic oscillators. We show that the sectional curvatures are completely determined by the oscillation frequency and the friction coefficient and these physical constants can be thought as obstructions for the manifold to be flat. Moreover, equations of simple and damped harmonic oscillators describe nonisomorphic solvable Lie groups with nonpositive scalar curvature.

Key words: Harmonic oscillator, Riemannian geometry, constant scalar curvature, oscillation frequency, damping force, metric Lie groups

1. Introduction

An oscillatory motion of a massive particle of one degree of freedom is governed by second-order linear ordinary differential equation (ODE)

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F(t), \tag{1.1}$$

here ω_0 is angular frequency, γ is damping coefficient and $F(t)$ is a time-dependent external force exerted on the system. The simple harmonic oscillator is the primary model describing motion of a massive particle accelerating with a linear spring restoring force and conserves its initial energy and is represented by $\gamma \equiv 0$ and $F(t) \equiv 0$. In the real world, an oscillatory physical system evolves in an actual environment and loses its energy in time due to the presence of damping γ or another external force $F(t)$. Such motions are called damped or forced harmonic motions. For an extensive discussion, we refer to [1]. Although these equations stand for the models for different physical systems, they are equivalent in the sense that they can be mapped to each other by a change of dependent and independent variables [15, 20, 27]. On the other hand, it is possible to distinguish them by considering such an equation as a free particle equation due to the curvature of Riemannian structure constructed by the coframe associated with given ODE and scalar curvature is completely interpreted in terms of angular frequency and damping coefficient.

In this work, our aim is to uncover the geometric meaning of physical constants angular frequency and damping coefficient. By this means, we consider the equations of harmonic motions in the framework of Riemannian geometry associated with second-order ODEs. This work can be seen as the continuation of the recent work of the authors dealing with a first order differential equation as a curved space in jet manifold in

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the context of Riemannian geometry [26]. Our consideration in this paper is based on constructing Riemannian metric and connection 1-form from the exterior differential system encoding a second-order ODE in the second-order jet space [2]. By this approach it is seen that 2-jet of a solution curve of a given equation defines a geodesic curve of the metric compatible connection. In other words, an integral curve of given equation in (t, x) plane is obtained by projecting a geodesic curve via bundle projection $J^2(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$. Structure equations for simple and damped harmonic motions are distinguished in terms of their associated curvatures of the metric connections constructed by a basis for the Maurer–Cartan forms on non-isomorphic Lie groups. Accordingly, Riemannian metrics constructed from equations of simple and damped oscillators define left invariant metrics on nonisomorphic solvable Lie groups which are not necessarily flat. We showed that presence of damping force is one of the obstructions for the connection to be flat. The other obstruction is due to the value of the oscillation frequency. In three dimensions, in terms of an orthonormal frame the scalar curvature is described in terms of sectional curvatures as $R = 2(R^1_{212} + R^1_{313} + R^2_{323})$. We showed that the scalar curvature of the Riemannian manifold corresponding to simple harmonic oscillator is completely determined by the angular frequency by the formula $2R = -(\omega_0^2 - 1)^2$ and hence curvature vanishes provided that the oscillation frequency is equal to one. Since $\omega_0 = \sqrt{\frac{k}{m}}$, where k is spring restoring force and m is the mass of particle, the curvature of the space is determined by the restoring force and vice versa. Accordingly, equation of a simple harmonic oscillator with $\omega_0^2 \leq 1$ is described as

$$\frac{d^2x}{dt^2} + (1 - \sqrt{-2R})x = 0. \tag{1.2}$$

On the other hand, the scalar curvature of the Riemannian manifold corresponding to damped harmonic oscillator is completely determined by the angular frequency and damping coefficient by the formula $2R = -[(\omega_0^2 - 1)^2 + 4\gamma^2]$. The latter formula tells us that the presence of damping force leads to nonflat space. In both cases we confront with a space of constant nonpositive scalar curvature. By this means, angular frequency and damping coefficient can be regarded as quantities which measure how much space is negatively curved. Moreover, it is deduced that the curvature of space is independent of time-dependent external force exerted on the system.

2. Riemannian structure associated with a second-order ODE

Geometric study of differential equations is a remarkable subject that has recently gained more interest and has been separated into a variety of branches, such as equivalence and rectification problems [7, 13, 15, 20, 27], associated theory of linear and nonlinear connections [8, 9], Finsler, projective and metric structures related to differential equations [10, 11, 14, 25], and metrizability of projective structures [4, 5, 12] and so on.

As it is considered in [2, 26], an ODE can be treated as a submanifold of an appropriate jet space and Riemannian structure may be given by the sum of squares of the elements of the exterior differential system encoding given differential equation. Our aim in this paper is to uncover the geometric role of the physical quantities oscillation frequency and damping coefficient in equations of harmonic motions. In order to deal with different types of oscillatory motions, we mainly follow [2] and consider the general second-order ordinary differential equation of the form

$$\frac{d^2x}{dt^2} = f(t, x, x'). \tag{2.1}$$

A second-order ODE is realized geometrically as a submanifold in the second-order jet bundle $J^2(\mathbb{R}, \mathbb{R})$ of maps

$\mathbb{R} \to \mathbb{R}$ by the zero set of the function $F(t,x,p,q) = q - f(t,x,p)$, where (t,x,p,q) standard local coordinates on $J^2(\mathbb{R},\mathbb{R})$. The second order jet bundle $J^2(\mathbb{R},\mathbb{R})$ of maps $\mathbb{R} \to \mathbb{R}$ is a smooth fibered manifold of all 2-jets of smooth sections of the trivial bundle $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. If we denote the submanifold corresponding to (2.1) by $\mathcal{S}$, then $\mathcal{S}$ is parametrized by $(t,x,p) \to (t,x,p,q = f(t,x,p))$. Accordingly, (t,x,p) defines local coordinates on $\mathcal{S}$. The 2-jet of a solution curve of differential equation (2.1) is a curve on $\mathcal{S}$ represented by 2-jet of a smooth section $(t,x(t))$ of the bundle $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, say $t \mapsto (t, x(t), x'(t), x''(t))$, on which the contact forms

$$\omega^2 = dx - pdt, \quad \omega^3 = dp - fdt \tag{2.2}$$

vanish. Namely, solutions of the exterior differential system

$$\omega^1 \neq 0, \quad \omega^2 = 0, \quad \omega^3 = 0, \tag{2.3}$$

are in one-to-one correspondence with the solutions of (2.1). Here $\omega^1 = dt$. For a detailed account of the geometric formulation of differential equations, we refer to [19, 21, 28].

Since ω^is are linearly independent at each point of a coordinate neighbourhood of $\mathcal{S}$, $(\omega^1, \omega^2, \omega^3)$ defines a local coframe which is dual to the frame of the vector fields

$$e_1 = \frac{\partial}{\partial t} + p\frac{\partial}{\partial x} + f\frac{\partial}{\partial p}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial p}. \tag{2.4}$$

Assume that Riemannian metric on a coordinate neighbourhood of $\mathcal{S}$ is given in a way that (e_1, e_2, e_3) forms an orthonormal frame:

$$ds^2 = \sum_i \omega^i \otimes \omega^i. \tag{2.5}$$

In terms of the local coordinates (t,x,p), Riemannian metric (2.5) is given by

$$ds^2 = (1 + p^2 + f^2)dt \otimes dt - p(dt \otimes dx + dx \otimes dt) - f(dt \otimes dp + dp \otimes dt) + dx \otimes dx + dp \otimes dp. \tag{2.6}$$

Note that a solution curve of (2.1) are obtained by the projection of an integral curve of e_1 to (t,x) plane.

Structure equations for the coframe $(\omega^1, \omega^2, \omega^3)$ are of the form

$$\begin{aligned} d\omega^1 &= 0 \\ d\omega^2 &= \omega^1 \wedge \omega^3 \\ d\omega^3 &= f_x\omega^1 \wedge \omega^2 + f_p\omega^1 \wedge \omega^3 \end{aligned} \tag{2.7}$$

$\mathfrak{o}(3,\mathbb{R})$-valued torsion free connection is constructed by solving the system of equations

$$d\omega^i = -\theta^i_j \wedge \omega^j, \quad \theta^i_j = -\theta^j_i, \tag{2.8}$$

Accordingly we have the following [2]:

Theorem 2.1 *For a given second-order ODE of the form* $\frac{d^2x}{dt^2} = f(t,x,x')$*, the* $\mathfrak{o}(3,\mathbb{R})$*-valued connection 1-form is determined by*

$$\theta = \begin{pmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & -\gamma \\ \beta & \gamma & 0 \end{pmatrix}, \tag{2.9}$$

where α, β*, and* γ *are defined respectively by* $\alpha = \frac{1}{2}(f_x + 1)\omega^3$*,* $\beta = \frac{1}{2}(f_x + 1)\omega^2 + f_p\omega^3$ *and* $\gamma = -\frac{1}{2}(f_x - 1)\omega^1$*.*

For all the computations below, see Appendix. Structure equations $\Omega^i_j = d\theta^i_j + \theta^i_k \wedge \theta^k_j$ determine the curvature 2-form associated to the connection 1-form θ. In terms of the coframe $(\omega^1, \omega^2, \omega^3)$ the components of Ω are found explicitly as

$$\begin{aligned}
\Omega^1_2 &= -\frac{1}{2}\left(\frac{3}{2}f_x^2 + f_x - \frac{1}{2}\right)\omega^1 \wedge \omega^2 - \frac{1}{2}f_{xx}\omega^2 \wedge \omega^3 \\
&- \frac{1}{2}\left((f_t + ff_p)_x + f_x f_p + pf_{xx}\right)\omega^1 \wedge \omega^3 \\
\Omega^1_3 &= -\frac{1}{2}\left((f_t + ff_p)_x + f_x f_p + pf_{xx}\right)\omega^1 \wedge \omega^2 - \frac{1}{2}f_{xp}\omega^2 \wedge \omega^3 \\
&- \frac{1}{2}\left(f_x - \frac{1}{2}f_x^2 + \frac{3}{2} + 2pf_{xp} + 2(f_t + ff_p)_p\right)\omega^1 \wedge \omega^3 \\
\Omega^2_3 &= -\frac{1}{2}f_{xx}\omega^1 \wedge \omega^2 - \frac{1}{2}f_{xp}\omega^1 \wedge \omega^3 + \frac{1}{2}\left(\frac{1}{2}(f_x + 1)^2\right)\omega^2 \wedge \omega^3.
\end{aligned}$$

Independent components of the Riemann curvature tensor $\Omega^i_j = \Sigma_{k<l} R^i_{jkl}\omega^k \wedge \omega^l$ are given by

$$\begin{aligned}
&R^1_{212} = -\frac{1}{2}\left(\frac{3}{2}f_x^2 + f_x - \frac{1}{2}\right), \quad R^1_{213} = -\frac{1}{2}\left((f_t + ff_p)_x + f_x f_p + pf_{xx}\right), \ R^1_{223} = -\frac{1}{2}f_{xx} \\
&R^1_{313} = -\frac{1}{2}\left(f_x - \frac{1}{2}f_x^2 + \frac{3}{2} + 2pf_{xp} + 2(f_t + ff_p)_p\right), R^1_{323} = -\frac{1}{2}f_{xp}, \ R^2_{323} = \frac{1}{4}(f_x + 1)^2.
\end{aligned}$$

At a given point, the sectional curvatures associated with the two dimensional subspaces of the tangent space spanned by the orthonormal pair of the vector fields (e_1, e_2), (e_1, e_3), and (e_2, e_3) are determined by R^1_{212}, R^1_{313}, and R^2_{323}, respectively. The scalar curvature is found by $R = 2(R^1_{212} + R^1_{313} + R^2_{323})$ as

$$2R = -\left(f_x^2 + 2f_x + 4\left(pf_{xp} + (f_t + ff_p)_p\right) + 1\right) \tag{2.10}$$

By using the definition (4.4) for the covariant differentiation we obtain

$$\nabla_{e_1}e_1 = 0, \ \nabla_{e_1}e_2 = -\frac{1}{2}(f_x - 1)e_3, \ \nabla_{e_1}e_3 = \frac{1}{2}(f_x - 1)e_2, \tag{2.11}$$

$$\nabla_{e_2}e_1 = \frac{1}{2}(f_x + 1)e_3, \nabla_{e_2}e_2 = 0, \ \nabla_{e_2}e_3 = -\frac{1}{2}(f_x + 1)e_1, \tag{2.12}$$

$$\nabla_{e_3}e_1 = \frac{1}{2}(f_x + 1)e_2 + f_p e_3, \ \nabla_{e_3}e_2 = -\frac{1}{2}(f_x + 1)e_1, \ \nabla_{e_3}e_3 = -f_p e_1. \tag{2.13}$$

From $\nabla_{e_1}e_1 = 0$ we see that once a second-order ODE is given one can constructed unique torsion free metric compatible connection on the tangent bundle of a submanifold by means of the exterior differential system (2.3) such that 2-jet of a solution curve defines a geodesic curve on the manifold corresponding to a second-order ODE. Also, $\nabla_{e_2}e_2 = 0$ implies that 2-jet of an integral curve of e_2 is also geodesic on $\mathcal{S}$.

Corollary 2.2 *The 2-jet of a solution curve of equation (2.1) is a geodesic curve on the Riemannian manifold $\mathcal{S}$.*

2.1. Lie group structure of harmonic motions

Since the structure functions for the equations for linear harmonic motions are all constant, they can be identified by the structure constants of a (local) group of transformations of the space $\mathcal{S}$ corresponding to the second-order ODE under consideration. In this case, the coframe $(\omega^1, \omega^2, \omega^3)$ defines a canonical coframe on $\mathcal{S}$ [23]:

Lemma 2.3 (Framing Lemma) *Let B be an N-dimensional manifold endowed with a coframing $\theta = (\theta^1, . . ., \theta^N)$. Then the (local) group G of diffeomorphisms of B that preserves this coframing is a finite-dimensional (local) Lie group of dimension at most N. The bound N is achieved if and only if*

$$d\theta^i = \sum c^i_{jk}\theta^j \wedge \theta^k,$$

with all of the c^i_{jk} constant. In this case c^i_{jk} are the structure constants of G, G acts freely and transitively on B, and the coframe can be identified with left-invariant one-forms on G.

By the framing lemma, Riemannian manifolds corresponding to the equations

$$x'' + \omega_0^2 x = 0 \tag{2.14}$$

and

$$x'' + \gamma x' + \omega_0^2 x = 0 \tag{2.15}$$

can be considered as (local) Lie groups, since the structure functions are all constant. By the structure equations (2.7), it follows that the equation for a simple harmonic motion leads to

$$\begin{aligned} d\omega^1 &= 0 \\ d\omega^2 &= \omega^1 \wedge \omega^3 \\ d\omega^3 &= -\omega_0^2 \omega^1 \wedge \omega^2. \end{aligned} \tag{2.16}$$

Also, for the equation of a damped motion, we have

$$\begin{aligned} d\omega^1 &= 0 \\ d\omega^2 &= \omega^1 \wedge \omega^3 \\ d\omega^3 &= -\omega_0^2 \omega^1 \wedge \omega^2 - \gamma\omega^1 \wedge \omega^3 \end{aligned} \tag{2.17}$$

Note that, for the equation of a simple harmonic motion we have $f = -\omega_0^2 x$ and for the equation of a damped harmonic motion we have $f = -\omega_0^2 x - \gamma p$. The Lie algebras determined by (2.16) and (2.17) are solvable. To see this, we write these equations in the vector form as $[e_i, e_j] = -c^i_{jk} e_k$ to obtain

$$[e_1, e_2] = \omega_0^2 e_3,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = 0 \tag{2.18}$$

and

$$[e_1, e_2] = \omega_0^2 e_3,\ [e_1, e_3] = -e_2 + \gamma e_3,\ [e_2, e_3] = 0 \tag{2.19}$$

respectively. A Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$, $\mathfrak{g}^{(3)} = [\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)}], \cdots$ vanishes for some $k > 0$, i.e. $\mathfrak{g}^{(k)} = 0$. Note that in both cases (2.18) and

(2.19) dimension of the first derived algebra $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is 2 and generated by e_2 and e_3. Since $[e_2, e_3] = 0$, the corresponding Lie algebras are clearly solvable.

Notice that the Lie algebras (2.18) and (2.19) are not isomorphic whenever $\gamma \neq 0$. If it is used $[e_j, e_k] = -c^i_{jk} e_i$ for (2.18) and $[\hat{e}_j, \hat{e}_k] = -\hat{c}^i_{jk} \hat{e}_i$ for (2.19), it can be seen that it is not possible to find an invertible matrix (a^i_j) satisfying the system of equations

$$\sum_{r,s} a^r_i a^s_j c^l_{rs} = \sum_k \hat{c}^k_{ij} a^l_k. \tag{2.20}$$

It is suitable to note that in three dimensional unimodular case (up to isomorphism), there are only two distinct solvable but not nilpotent Lie algebras one of which is the group of rigid motions in Euclidean plane $\mathbb{E}(2)$ and the other is group of rigid motions in two dimensional Minkowski space $\mathbb{E}(1,1)$ [22]. Also, since the trace of the linear transformation $\mathrm{ad}(e_1)(e_i) = [e_1, e_i]$ is nonzero, the Lie algebra (2.19) is nonunimodular in the presence of the damping force. Assume that $\gamma \neq 0$. If we introduce the basis

$$\begin{aligned} \hat{e}_1 &= \frac{2}{\gamma} e_1 + e_3 \\ \hat{e}_2 &= -e_2 + \gamma e_3 \\ \hat{e}_3 &= e_3, \end{aligned} \tag{2.21}$$

then we have

$$[\hat{e}_1, \hat{e}_2] = 2\hat{e}_2 - 2\frac{\omega_0^2}{\gamma}\hat{e}_3, \ [\hat{e}_1, \hat{e}_3] = \frac{2}{\gamma}\hat{e}_2, \ [\hat{e}_2, \hat{e}_3] = 0 \tag{2.22}$$

so that the trace of A is $a + d = 2$, where A is determined by the equations $[\hat{e}_1, \hat{e}_2] = a\hat{e}_2 + b\hat{e}_3$ and $[\hat{e}_1, \hat{e}_3] = c\hat{e}_2 + d\hat{e}_3$ together with $[\hat{e}_2, \hat{e}_3] = 0$. By Lemma 4.10. in [22], the determinant $D = ad - bc = 4\frac{\omega_0^2}{\gamma^2}$ is a complete isomorphism invariant. The value of the determinant has also a physical importance to determine whether the motion is underdamped, overdamped, or critically damped.

As a final remark, we should note that since $\omega^i(e_j) = \delta^i_j$, and $\mathfrak{g}$ is identified with left-invariant vector fields on a Lie group G, $\omega^i \in \mathfrak{g}^*$ defines a basis for the left-invariant 1-forms on G so called Maurer-Cartan forms and hence a Riemannian metric defined by (2.5) is left invariant. In the subsequent section we will investigate the curvature properties of the left invariant metrics constructed from the equations for simple oscillator and damped oscillator. This approach leads one to investigate the geometry of three dimensional Lie groups with certain left-invariant metrics by means of the certain second-order ODEs.

3. Curvature associated with the harmonic motions

3.1. Simple harmonic oscillator

Since $f(t, x, p) = -\omega_0^2 x$ holds for the equation $x'' + \omega_0^2 x = 0$ of simple harmonic oscillator, Riemannian metric (2.6) and the connection 1-form take the form

$$\begin{aligned} ds^2 &= (1 + p^2 + \lambda^2 x^2) dt \otimes dt - p(dt \otimes dx + dx \otimes dt) + \lambda x (dt \otimes dp + dp \otimes dt) \\ &+ dx \otimes dx + dp \otimes dp. \end{aligned} \tag{3.1}$$

and

$$\theta = \frac{1}{2}\begin{pmatrix} 0 & -(1-\lambda)\omega^3 & -(1-\lambda)\omega^2 \\ (1-\lambda)\omega^3 & 0 & -(1+\lambda)\omega^1 \\ (1-\lambda)\omega^2 & (1+\lambda)\omega^1 & 0 \end{pmatrix} \tag{3.2}$$

respectively. Here $\lambda = \omega_0^2$. Accordingly, components of the Riemannian curvature tensor are given by

$$R^1_{212} = -\frac{1}{2}\left(\frac{3}{2}\lambda^2 - \lambda - \frac{1}{2}\right), \; R^1_{313} = -\frac{1}{2}\left(-\frac{1}{2}\lambda^2 - \lambda + \frac{3}{2}\right), \; R^2_{323} = \frac{1}{4}(1-\lambda)^2 \tag{3.3}$$

and $R^1_{213} = R^1_{223} = R^1_{323} = 0$. Thus, the scalar curvature is found as

$$2R = -(\lambda - 1)^2. \tag{3.4}$$

We have reached the following conclusion:

Theorem 3.1 *Riemannian manifold corresponding to the equation*

$$x'' + \omega_0^2 x = 0 \tag{3.5}$$

has nonpositive constant scalar curvature for any $\omega_0^2 > 0$. *Curvature vanishes if and only if the oscillation frequency is equal to 1.*

3.2. Damped harmonic oscillator

By the similar argument, since $f(t,x,p) = -\omega_0^2 x - \gamma p$ holds for the equation $x'' + \gamma x' + \omega_0^2 x = 0$ of damped harmonic oscillator we obtain

$$\begin{aligned} ds^2 &= (1 + p^2 + (\lambda x + \gamma p)^2) dt \otimes dt - p(dt \otimes dx + dx \otimes dt) \\ &+ (\lambda x + \gamma p)(dt \otimes dp + dp \otimes dt) + dx \otimes dx + dp \otimes dp. \end{aligned} \tag{3.6}$$

and

$$\theta = \frac{1}{2}\begin{pmatrix} 0 & -(1-\lambda)\omega^3 & -(1-\lambda)\omega^2 + 2\gamma\omega^3 \\ (1-\lambda)\omega^3 & 0 & -(1+\lambda)\omega^1 \\ (1-\lambda)\omega^2 - 2\gamma\omega^3 & (1+\lambda)\omega^1 & 0 \end{pmatrix}. \tag{3.7}$$

Components of the Riemannian curvature tensor take the following form

$$R^1_{212} = -\frac{1}{2}\left(\frac{3}{2}\lambda^2 - \lambda - \frac{1}{2}\right), \; R^1_{313} = -\frac{1}{2}\left(-\frac{1}{2}\lambda^2 - \lambda + \frac{3}{2} + 2\gamma^2\right) \tag{3.8}$$

$$R^2_{323} = -\lambda\gamma, \; R^2_{323} = \frac{1}{4}(1-\lambda)^2 \tag{3.9}$$

and $R^1_{223} = R^1_{323} = 0$. The scalar curvature is found as

$$2R = -\left[(\lambda - 1)^2 + 4\gamma^2\right]. \tag{3.10}$$

Thus, we have the following:

Theorem 3.2 *Riemannian manifold corresponding to the equation*

$$x'' + \gamma x' + \omega_0^2 x = 0, \tag{3.11}$$

has nonpositive constant scalar curvature for any $\omega_0^2 > 0$ *and* $\gamma > 0$. *Curvature vanishes if and only if the* $\omega_0^2 = 1$ *and* $\gamma = 0$.

Corollary 3.3 *Equation (3.11) with* $\gamma > 0$ *cannot describe a scalar-flat Riemannian manifold.*

On the other hand, for the equation of a forced oscillator, we have $f(t,x,p) = -\omega_0^2 x - \gamma p + F(t)$. Thus, we obtain

$$\begin{aligned} ds^2 &= (1 + p^2 + (\lambda x + \gamma p + F(t))^2) dt \otimes dt - p(dt \otimes dx + dx \otimes dt) \\ &+ (\lambda x + \gamma p - F(t))(dt \otimes dp + dp \otimes dt) + dx \otimes dx + dp \otimes dp. \end{aligned} \tag{3.12}$$

However, the connection 1-form remains the same and scalar curvature is again determined by the equation (3.10). That is, whenever the external force F exerted on the system depends only on time t, sectional curvatures and hence the scalar curvature are not affected by the external force. This brings an interesting result that the geometry associated with a harmonic motion is independent from a time-dependent external force.

4. Concluding remarks

In this paper we deal with the Riemannian geometry associated with the equations for one dimensional linear harmonic motions. We see that the Riemannian metric, and hence the curvature, is determined by the oscillation frequency and the friction coefficient. In this sense, we have obtained a family of Riemannian metrics depending on two parameters. That is to say, apart from their physical meaning, they can also be thought as deformation parameters of the underlying Riemannian structure. Accordingly, equations for simple and damped harmonic motions are distinguished by means of their associated curvatures and these equations define nonisomorphic solvable Lie groups with certain left invariant metrics. Besides, these equations correspond to free particle equation on a manifold with certain curvature in a way that it is possible to define a Riemannian metric whose geodesics identified with integral curves of an equation of harmonic motion. On the other hand, the current work is ultimately related to study of Riemannian geometry of certain Lie groups in the sense that a certain class of second-order ODEs provides some Lie group structures and can readily be used to investigate the geometry of submanifolds in such spaces, see for example [3, 16–18, 22].

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APPENDIX: Basics of Riemannian geometry in an orthonormal frame

In this appendix part, we give some brief information on Riemannian geometry in an orthonormal frame in accordance with what we have used in this paper. We have assumed that all objects are smooth and all considerations are local. For details, see [6, 19, 24].

Let $(\mathcal{S}, g)$ be an n-dimensional Riemannian manifold and let $T\mathcal{S}$ and $T^*\mathcal{S}$ denote the tangent bundle and the cotangent bundle. Consider the coframe $(\omega^1, ..., \omega^n)$ dual to the orthonormal frame $(e_1, ..., e_n)$ of vector fields. Namely, for $i, j = 1, ..., n$ $\omega^i \in \Gamma(T^*\mathcal{S})$ and $e_j \in \Gamma(T\mathcal{S})$ such that $\omega^i(e_j) = \delta^i_j$ and $g(e_i, e_j) = \delta_{ij}$. Here $\Gamma(T\mathcal{S})$ and $\Gamma(T^*\mathcal{S})$ denote the set of sections of tangent bundle and cotangent bundle respectively. Orthonormality condition implies that the metric g is described as

$$ds^2 = \sum_i \omega^i \otimes \omega^i. \tag{4.1}$$

For a given metric, there exists unique $\mathfrak{o}(3, \mathbb{R})$-valued 1-form $\theta = (\theta^j_k)$ satisfying

$$d\omega^i = -\theta^i_j \wedge \omega^j, \quad \theta^i_j = -\theta^j_i. \tag{4.2}$$

$\mathfrak{o}(3, \mathbb{R})$-valued 1-form $\theta = (\theta^j_k)$ is called the connection 1-form of the Levi-Civita connection which is defined as

$$\nabla : \Gamma(T\mathcal{S}) \to \Gamma(T^*\mathcal{S} \otimes T\mathcal{S}) \tag{4.3}$$

such that

$$\nabla_X e_j = \theta^i_j(X) e_i, \qquad X \in T\mathcal{S}. \tag{4.4}$$

Here $\nabla_X e_j$ is called the covariant derivative of e_j relative to X. From (4.2) one has $\theta^i_j = \gamma^i_{jk}\omega^k$ for some functions γ^i_{jk}. The coefficients γ^i_{jk} are defined in terms of the structure functions $[e_i, e_j] = -c^k_{ij} e_k$ as

$$\gamma^i_{jk} = \frac{1}{2}(c^i_{[jk]} + c^j_{[ki]} - c^k_{[ij]}), \qquad c^i_{[jk]} = c^i_{jk} - c^i_{kj}. \tag{4.5}$$

It follows that $\gamma^i_{jk} = -\gamma^j_{ik}$. Exterior derivative of (4.2) gives $0 = (d\theta^i_j + \theta^i_k \wedge \theta^k_j) \wedge \omega^j$. This implies that

$$\Omega^i_j = d\theta^i_j + \theta^i_k \wedge \theta^k_j, \quad \Omega^i_j = -\Omega^j_i. \tag{4.6}$$

$\Omega = (\Omega^i_j)$ is called curvature 2-form or the Riemannian curvature tensor associated to the connection 1-form θ and it is given by

$$\Omega^i_j = \sum_{k<l} R^i_{jkl}\omega^k \wedge \omega^l. \tag{4.7}$$

The scalar curvature is defined by trace of Ricci tensor:

$$R = \sum_j R_{jj}. \tag{4.8}$$

Here R_{jl} is the Ricci tensor defined by $R_{jl} = \sum_k R^k_{jkl}$.