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THE GEOMETRY OF UNDAMPED HARMONIC OSCILLATORS

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Abstract

In this article, we present a geometrical approach to understanding undamped harmonic oscillators. We investigate the phase-plane behavior of both unforced and forced oscillations. The methods explained in this article are not commonly found in differential equation texts and provide an illustrative way for understanding harmonic oscillators. A conservation of energy approach and a substitution method that allows solution patterns to be examined without explicitly having to find the time series is presented.

Keywords: Harmonic oscillator; conservation of energy; phase plane; envelopes; beats; resonance

2010 Mathematics Subject Classification: Primary 34A26
Secondary 34C25

1. Introduction

The geometrical theory of differential equations was originally initiated by the pioneering work of Poincaré in his celebrated *Les Méthodes Nouvelles de la Mécanique Céleste* (*The New Methods of Celestial Mechanics*) [6]. Motivated by the problem of the stability of the solar system he began his fundamental research on the qualitative behavior of differential equations. In this work, he introduced the concept of a trajectory, i.e. a curve in the phase plane (x, \dot{x}) parametrized by the time variable t . This idea enabled him to set up an elegant geometrical framework in which solutions were viewed as curves rather than analytical expressions [1]. In the same spirit, we wish to develop a similar approach to linear vibration problems that will appeal to readers who prefer to think in geometrical terms. We believe that this approach will provide an insight into the behavior of these systems for different ranges of parameters and additionally provide some understanding of how these systems are interconnected. Our contribution to the subject consists of identifying the solutions of harmonic oscillators using the geometry of phase-plane orbits and in detailing these ideas with pictorial interpretations.

Although these ideas are presented for the commonly studied linear spring-mass system, the generalization of the methods may be employed for more complex systems. We use this approach to investigate the solutions of the linear spring-mass system in terms of the energy of the system highlighting the solution patterns for a multitude of cases.

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2. The basics

The well-known equation of motion of a freely oscillating mass m suspended by a spring having stiffness k is given by

$$m\ddot{x} + kx = 0, \quad (1)$$

where $x = x(t)$ is the position of the mass at time t . Traditionally, equations with constant coefficients are solved using the substitution $x(t) = e^{\lambda t}$, which leads to solving an algebraic equation in λ called the characteristic equation. In order to address more complex situations, we approach this problem in a different way.

Typically, for equations which do not explicitly depend on time, the total energy is conserved. Equation (1) does not explicitly depend on time, therefore using the chain rule we have

$$m\dot{x} \frac{d\dot{x}}{dx} + kx = 0. \quad (2)$$

Equation (2) can be integrated as follows:

$$\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = E,$$

where the two terms on the left represent the kinetic and the potential energies respectively, and the constant of integration E represents the total energy. More generally, if

$$m\ddot{x} = f(x),$$

then the force is conservative and there exists a potential function $V(x)$ such that $dV(x)/dx = -f(x)$. This approach is well known and has been explored in [3], [5], and [8], for example.

Although an explicit time-series solution can be found, it is convenient to investigate the solutions (orbits) in the phase plane (x vs. \dot{x}). Notice that in the phase plane, the orbits are defined as ellipses of the form

$$\frac{m}{2E}\dot{x}^2 + \frac{k}{2E}x^2 = 1.$$

In many applications, such as in machines with rotating components, the position of the mass often depends on an external force. Consequently, in the presence of sinusoidal forcing (1) becomes

$$m\ddot{x} + kx = F_0 \cos \omega t, \quad (3)$$

where F_0 is the amplitude of the external force and ω is the frequency of the external force. Solutions of (3) have been extensively studied and discussed in texts such as [2] and [4].

3. Nondimensionalization

We can simplify our analysis by rewriting (3) in a nondimensional form and introducing the following nondimensional variables:

$$\begin{aligned} \bar{x} &= \frac{x}{L}, \\ \bar{t} &= \omega_0 t, \\ \bar{F}_0 &= \frac{F_0}{kL}, \\ \bar{\omega} &= \frac{\omega}{\omega_0}, \end{aligned} \quad (4)$$

where L is a characteristic length scale and $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system. Using the chain rule, the time derivatives become

$$\frac{dx}{dt} = \frac{dx}{d\bar{x}} \frac{d\bar{x}}{d\bar{t}} \frac{d\bar{t}}{dt} = L\omega_0 \frac{d\bar{x}}{d\bar{t}} \quad \text{and} \quad \frac{d^2x}{dt^2} = L\omega_0^2 \frac{d^2\bar{x}}{d\bar{t}^2}.$$

Substituting the new variables given in (4) along with the time derivative into (3) we arrive at the following nondimensional equation:

$$\ddot{\bar{x}} + \bar{x} = \bar{F}_0 \cos \bar{\omega} \bar{t},$$

now where a dot represents differentiation with respect to \bar{t} . In order to simplify notation, we will drop the *superimposed bar* making it clear that all the variables are nondimensional. Therefore, without loss of generality, we can now investigate the behavior of the system by analyzing the following nondimensional equation:

$$\ddot{x} + x = F_0 \cos \omega t. \quad (5)$$

Notice that in nondimensional form, the unforced equation (1) becomes

$$\ddot{x} + x = 0,$$

which integrates to

$$\dot{x}^2 + x^2 = R^2. \quad (6)$$

Equation (6) describes circular orbits in phase space with a constant radius R .

4. Forced oscillator

Recall that solutions to a forced oscillator can exhibit either beating patterns (when $\omega \approx 1$) or resonance (when $\omega = 1$). Beating patterns are described as having rapid oscillations within a slowly varying periodic amplitude. An example of a beating solution is illustrated in Figure 1(a). Resonance occurs when the amplitude of the solution increases without bound. A solution exhibiting resonance is illustrated in Figure 1(b). Notice that the solutions illustrated in Figure 1 are time-series solutions. We now explore the qualitative features of the phase-plane solutions to three different periodically forced linear equations of the form (3) using a geometrical approach.

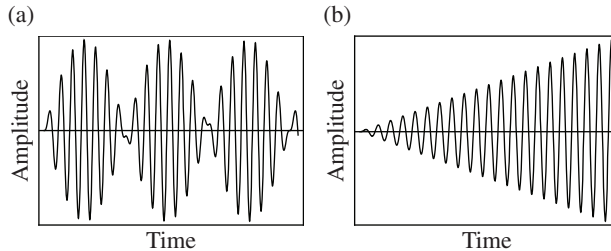


FIGURE 1: Example of a beating solution (a) and a solution exhibiting resonance (b).

4.1. The case $\omega \neq 1$

Typically, the solutions of (5), where $\omega \neq 1$, are written as

$$x(t) = c_1 \sin t + c_2 \cos t + \frac{F_0}{1 - \omega^2} \cos \omega t, \quad (7)$$

where c_1 and c_2 are constants determined by initial conditions and $c_1 \sin t + c_2 \cos t$ represents the homogeneous solution while

$$\frac{F_0}{1 - \omega^2} \cos \omega t$$

represents the particular solution. Notice that (7) can be rewritten with the homogeneous solution in amplitude phase form as

$$x(t) = R \cos(t + \phi) + \frac{F_0}{1 - \omega^2} \cos \omega t, \quad (8)$$

where R is the amplitude of the homogeneous oscillations and ϕ is a phase shift. Depending on initial conditions, the solution can be explicitly plotted as a time series in state space.

Another way to view the solution is to consider the variable t as a parameter in the corresponding phase plane. Doing this allows us to observe some interesting geometric properties inherent in the solutions. For example, upon differentiation, (8) becomes

$$\dot{x}(t) = -R \sin(t + \phi) - \frac{F_0 \omega}{1 - \omega^2} \sin \omega t. \quad (9)$$

Let $a = F_0/(1 - \omega^2)$, $b = -a\omega = -F_0\omega/(1 - \omega^2)$, and $y = \dot{x}$, then we can rewrite (8)–(9) as the following parametric equations:

$$x = a \cos \omega t + R \cos(t + \phi), \quad (10)$$

$$y = b \sin \omega t - R \sin(t + \phi). \quad (11)$$

Equations (10) and (11) combine to give the following algebraic equation:

$$(x - a \cos \omega t)^2 + (y - b \sin \omega t)^2 = R^2. \quad (12)$$

This is the equation of a circle in the xy -plane with center $(X, Y) = (a \cos \omega t, b \sin \omega t)$.

The solution to (5) in the phase plane is a trajectory that lies on the periphery of the circle (12), while the circle itself is orbiting the origin. This orbit can be described parametrically as follows:

$$X(t) = a \cos \omega t, \quad Y(t) = b \sin \omega t. \quad (13)$$

By squaring and adding the equations in (13), we arrive at the following relationship:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1. \quad (14)$$

This ellipse describes the path of the center of the circles defined in (12). Notice that the trajectory lies between an inner and an outer envelope. This geometrical interpretation is illustrated in Figure 2.

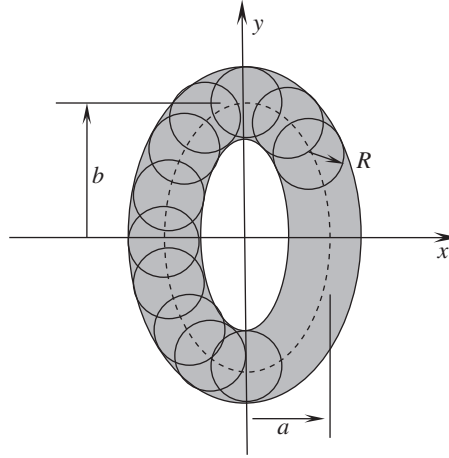


FIGURE 2: Trajectory of phase flow in which the phase circles are confined to be between two envelopes.

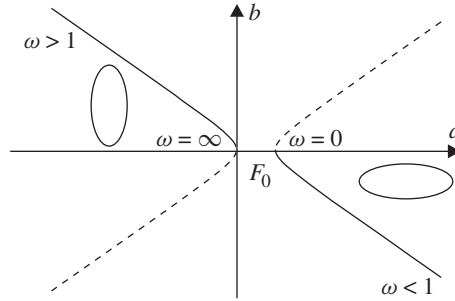


FIGURE 3: This illustrates the hyperbolic relationship between the parameters a and b . The solid curves correspond to positive values of ω while the dashed curves correspond to negative values of ω . When $0 < \omega < 1$, the ellipse described in (14) has a horizontal orientation, and when $\omega > 1$ it has a vertical orientation.

In order to determine these envelopes, the parameter $\theta = \omega t$ should be eliminated from the following two equations:

$$F(x, y, \theta) = (x - a \cos \theta)^2 + (y - b \sin \theta)^2 - R^2 = 0,$$

$$\frac{\partial F}{\partial \theta} = 2a \sin \theta (x - a \cos \theta) - 2b \cos \theta (y - b \sin \theta) = 0.$$

Unfortunately, doing this is not practical and it is easier to consider both these equations as a parametric representation of the envelopes, where each point of the envelope is determined by a specific value of θ . A more detailed explanation of this can be found in [7].

The shape of the ellipse given in (14) is solely determined from the parameters ω and F_0 . Recall that $a = F_0/(1 - \omega^2)$ and $b = -a\omega$ from which we have the relation $b/a = -\omega$. Thus,

$$a = \frac{F_0}{1 - (b/a)^2},$$

TABLE 1: Summary of the parameter regions where different solution patterns occur. For each of the parameter regions (a)–(g), the associated conditions and effect on the envelope are given.

Parameter region	Conditions	Effect on inner envelope
(a)	If $F_0 > R$, $\omega < 1$, with $ a > b > R$	inner envelope exists
(b)	If $F_0 > R$, $\omega < 1$, with $ a > R > b $	no inner envelope
(c)	If $F_0 < R$, $\omega < 1$, with $R > a > b $	special inner envelope exists
(d)	If $F_0 < R$, $\omega < 1$, with $ a > b > R$	inner envelope exists
(e)	If $1 < \omega$, with $ b > a > R$	inner envelope exists
(f)	If $1 < \omega$, with $R > b > a $	special inner envelope exists
(g)	If $1 < \omega$, with $ b > R > a $	no inner envelope

which can be rewritten as

$$\left(a - \frac{F_0}{2}\right)^2 - b^2 = \frac{F_0^2}{4}, \quad (15)$$

an equation of a hyperbola in the ab -plane. The hyperbola described by (15) is illustrated in Figure 3. There, the solid lines reflect the corresponding (a, b) pairs when $\omega \geq 0$ while the dashed lines correspond to (a, b) pairs when $\omega < 0$. For (a, b) pairs on the left-hand side of the hyperbola, the ellipse in (14) will have a verticle orientation, while for (a, b) pairs on the right-hand side of the hyperbola, the ellipse will have a horizontal orientation. Without loss of generality we consider positive ω values.

Note that the inner envelope as described above will only occur when both $|a| > R$ and $|b| > R$. If $|b| < R$, then a horizontal collapse of the inner envelop occurs. If $|a| < R$, then a vertical collapse occurs. The relationship between (a, b) -pairs and R has been summarized

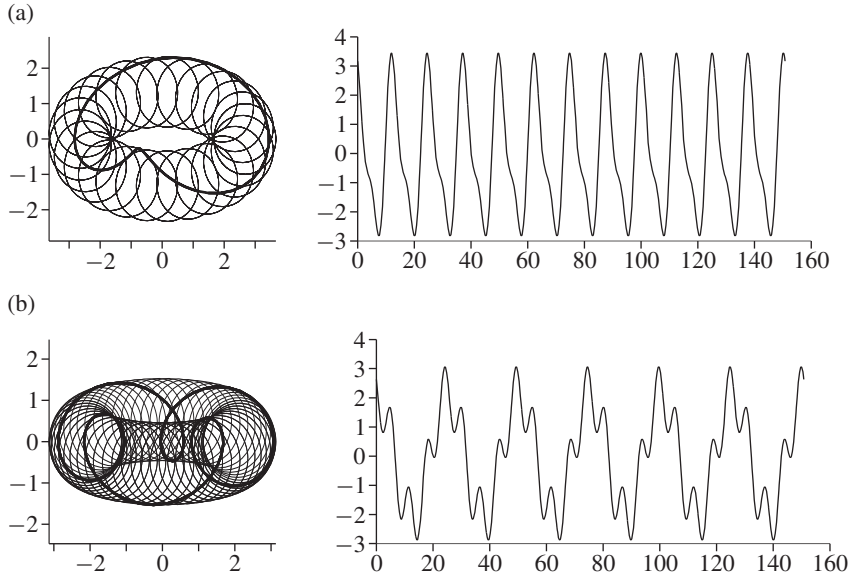


FIGURE 4: The left-hand side shows different phase-plane orbits for the parameter cases (a) and (b) in Table 1. For (a) we have $F_0 = 2$, $R = 1$, and $\omega = 0.5$ implies that $a = 2.6667$ and $b = -1.3333$; for (b) we have $F_0 = 2$, $R = 1$, and $\omega = 0.25$ implies that $a = 2.1333$ and $b = -0.5333$.

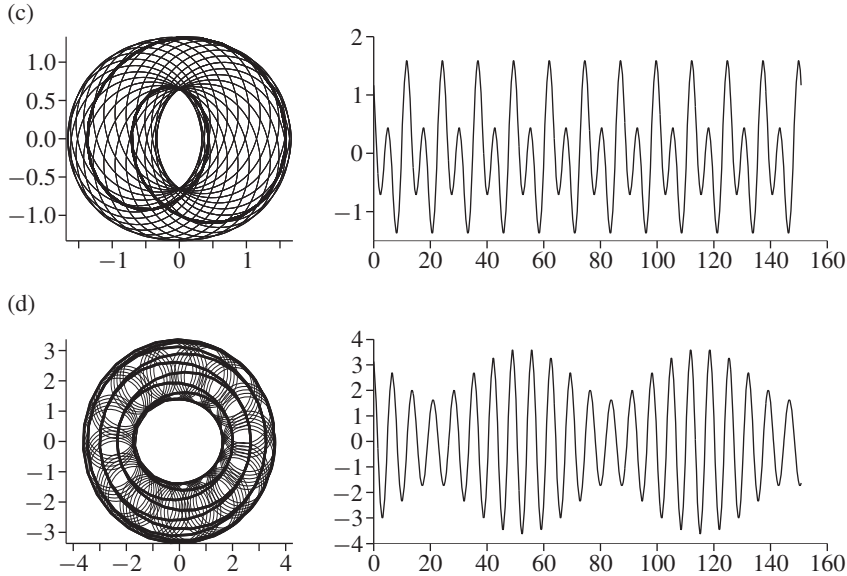


FIGURE 5: The left-hand side shows different phase-plane orbits for the parameter cases (c) and (d) in Table 1. For (c) we have $F_0 = 0.5$, $R = 1$, and $\omega = 0.5$ implies that $a = 0.6667$ and $b = -0.3333$; for (d) we have $F_0 = 0.5$, $R = 1$, and $\omega = 0.9$ implies that $a = 2.6316$ and $b = -2.3684$.

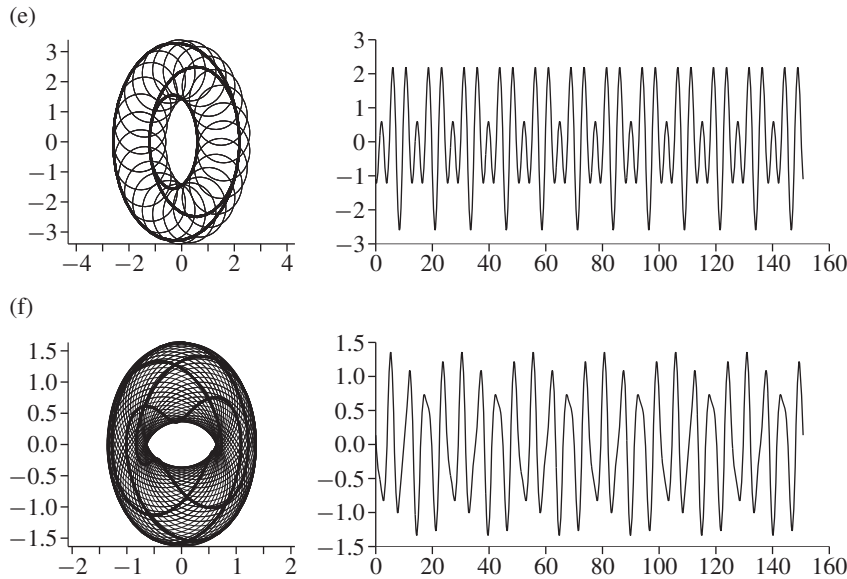


FIGURE 6: The left-hand side shows different phase-plane orbits for the parameter cases (e) and (f) in Table 1. For (e) we have $F_0 = 0.75$, $R = 1$, and $\omega = 1.2$ implies that $a = -1.3229$ and $b = 2.0455$; for (f) we have $F_0 = 0.75$, $R = 1$, and $\omega = 1.75$ implies that $a = -0.3636$ and $b = 0.6364$.

in Table 1, where the effect on the inner envelope has been noted. Since a and b both depend on F_0 and ω , the results will depend on these parameters. This dependence is illustrated in

Figures 4, 5, and 6 where both the phase-plane diagrams and time-series solutions have been plotted. The two cases illustrated in each of these figures show the solutions corresponding to the parameter regions, (a)–(b), (c)–(d), and (e)–(f) described in Table 1, respectively. An illustration of case (g) has been omitted since it is similar to case (b).

Special inner regions occur when $R > \max(a, b)$. Examples of these special inner regions are illustrated in Figure 5(c) and Figure 6(f). The investigation of the inner envelopes can be related to the energy of the system. When the solutions (in phase plane) are bounded away from the origin, then the total energy of the system stays away from zero.

A key element in our investigation of these solutions in the phase plane is the derivative of the time-series solution. For a more general class of equations, a more appropriate method would be to achieve the results without first finding the solution. This is in general very difficult to do. We now present such an approach for these simple linear systems using an appropriate substitution.

Consider (5), and let

$$z = x - \frac{F_0}{1 - \omega^2} \cos \omega t. \quad (16)$$

By substituting (16) into (5), we arrive at the relationship

$$\ddot{z} + z = 0. \quad (17)$$

Recall that a solution to (17) is given by

$$\dot{z}^2 + z^2 = R^2, \quad (18)$$

where R is a constant. Replacing the substitution (16) into (18) gives us the relationship

$$\left(x - \frac{F_0}{1 - \omega^2} \cos \omega t \right)^2 + \left(\dot{x} + \frac{F_0 \omega}{1 - \omega^2} \sin \omega t \right)^2 = R^2,$$

which is (12). Thus we have attained the same solutions in the phase plane without explicitly computing the solution. The key to this method lies in finding an appropriate substitution.

4.2. The case $\omega = 1$

If we consider (5) when $\omega = 1$, we can use a similar geometrical approach to describe the solutions in the phase plane. The solution to (5) when $\omega = 1$ is given by

$$x(t) = c_1 \sin t + c_2 \cos t + \frac{F_0}{2} t \sin t.$$

Rewriting the homogeneous solution in amplitude phase form, we obtain

$$x(t) = R \cos(t + \phi) + \frac{F_0}{2} t \sin t.$$

Letting $y = \dot{x}$, the equations for x and y can be written as

$$x = \frac{F_0}{2} t \sin t + R \cos(t + \phi) \quad (19)$$

and

$$y = \frac{F_0}{2} t \cos t + \frac{F_0}{2} \sin t - R \sin(t + \phi). \quad (20)$$

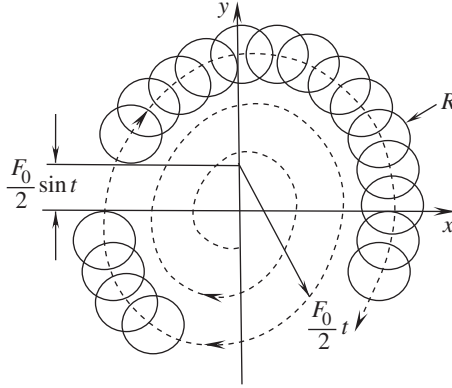


FIGURE 7: An unstable limit cycle in which the phase circles spiral outwards.

Combining (19) and (20) we attain the following relationship:

$$\left(x - \frac{F_0}{2}t \sin t\right)^2 + \left(y - \frac{F_0}{2}t \cos t - \frac{F_0}{2} \sin t\right)^2 = R^2 \cos^2(t + \phi) + R^2 \sin^2(t + \phi). \quad (21)$$

Similarly, (21) takes the form

$$\left(x - \frac{F_0}{2}t \sin t\right)^2 + \left(y - \frac{F_0}{2}t \cos t - \frac{F_0}{2} \sin t\right)^2 = R^2. \quad (22)$$

In this case, the centers of the circles follow a spiraling path that is linearly expanding in time. Figure 7 illustrates the path of the center shown as the spiraling trajectory and the individual circles that the orbits of the solution must lie on.

Figure 8 shows two different numerical simulations which illustrate the geometry of resonance as it pertains to phase-plane analysis. Notice that, in each case, the solution settles into its intrinsic oscillation with an increasing amplitude.

Here again we show how to generate the result illustrated in (22) using an appropriate substitution. Let

$$z = x - \frac{F_0}{2}t \sin t. \quad (23)$$

By substituting (23) into (5), we arrive at the relationship

$$\ddot{z} + z = 0,$$

whose solution leads us to

$$\dot{z}^2 + z^2 = R^2, \quad (24)$$

for constant R . Substituting (23) into (24) gives us the relationship

$$\left(x - \frac{F_0}{2}t \sin t\right)^2 + \left(\dot{x} - \frac{F_0}{2}t \cos t - \frac{F_0}{2} \sin t\right)^2 = R^2,$$

which is (22).

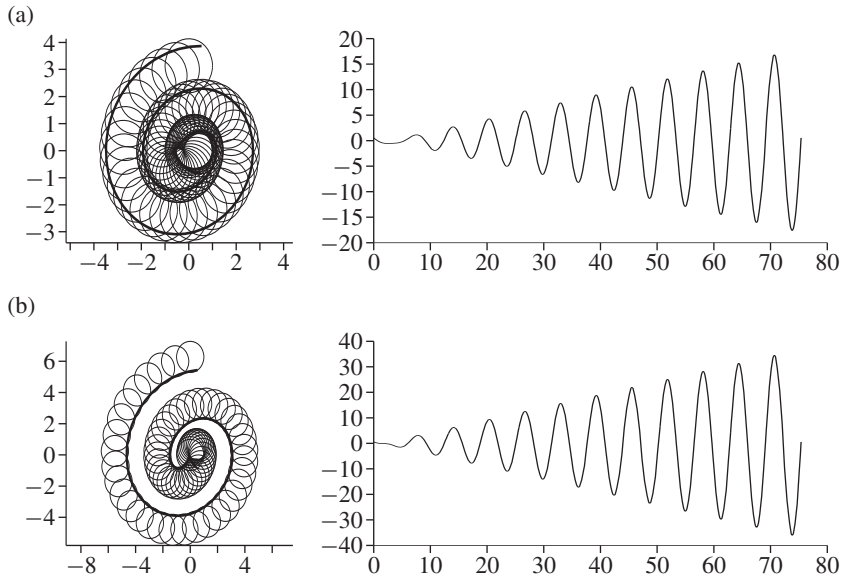


FIGURE 8: The left-hand side shows different phase-plane orbits for two different parameter cases along with the corresponding temporal solution on the right-hand side. These simulations illustrate the phase-plane pattern for a resonating time-series solution. For (a) we have $F_0 = 0.5$ and $R = 1$; for (b) we have $F_0 = 1$ and $R = 1$.

5. Conclusion

Although solutions of harmonic oscillators have been extensively studied, the geometric approach presented here provides a way for understanding the relationship between time-series and phase-plane solutions. By using an appropriate substitution, the geometric patterns can be derived from the differential equation only. This idea can be extended to problems where a particular solution is known, allowing one to gain an understanding without having to find an explicit time-series solution.

Some details have been left out for readability. For example, the parameter relationships between R , F_0 , and ω , in the forced case where $\omega \neq 1$, is very intricate. Instead of including each case carefully, our aim was to provide a summary of the geometrical effect each case has on the solutions. An interesting project would be to determine the various ways that these parameters interconnect and the role that the initial condition plays on this relationship. Similarly, in the resonance case, when investigating the phase-plane solutions, the overlap of the circles illustrated on the left-hand side of Figure 8 can also be a difficult treatment. Nonetheless, the basic geometrical forms of these solutions in the phase plane have been achieved, an approach that is omitted in the differential equations texts we have consulted. The interested reader might want to use a similar approach to investigate the free oscillator in the presence of damping. In this case, the solutions in the phase plane will spiral towards an elliptical limit cycle.

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