# Spinor L-Functions, Theta Correspondence, and Bessel Coefficients

Ramin Takloo-Bighash September 22, 2018

The purpose of this note is to establish the entireness of spinor L-function of certain automorphic cuspidal representations of the group similitude symplectic group of order four over the rational numbers. Our study of spinor L-function is based on an integral representation which works only for generic representations. For this reason, while methods of this papers do not directly apply to the most interesting case of interest, ie. Siegel modular forms of genus two, they show what "should" be true for holomorphic forms; after all, generic forms are expected to be in a certain sense typical. These integrals which were introduced by M. Novodvorsky in the Corvallis conference [10] serve as one of the few available integral representations for the Spinor L-function of GSp(4). Some of the details missing in Novodvorsky's original paper have been reproduced in Daniel Bump's survey article [1]. Further details have been supplied by [19]. David Soudry has generalized the integrals considered in Novodvorsky's paper to orthogonal groups of arbitrary odd degree.

In light of the results of [19], it is sufficient to study the integral of Novodvorsky at the archimedean place. Archimedean computations are often forbidding, and unless one expects major simplifications due to the nature of the parameters, the resulting integrals are often quite hard to manage. In our case of interest, the work of Moriyama [12] benefits from exactly such simplifications when he treats the case of cuspidal representations with archimedean components in the generic (limit of) discrete series. In this work, we concentrate on those archimedean representations for which direct computations have yielded very little. For this reason, our methods are a bit indirect, in fact somewhat more indirect that what at first seems necessary. Our

method is based on the theta correspondence. First we observe that Novodvorsky's integral is in fact a split Bessel functional. Then we pull back the Bessel functional via the theta correspondence for the dual reductive pair (GO(2,2),GSp(4)), and prove that the resulting functional on GO(2,2) is Eulerian. On the other hand, one can prove that the integral of Novodvorsky itself is Eulerian, with an Euler product involving the Whittaker functions. Next obvious step is to pull back the Whittaker function via the theta correspondence. Now we have obtained two different Euler product expansions which represent the same object, but do not look the same. Then one uses the standard technique of twisting with highly ramified characters to isolate the archimedean place to obtain an identity expressing the local Novodvorsky integral at the archimedean place in terms of an expression which does not go through the local Whittaker functions. The advantage of using this expression is that, first it avoids Whittaker function, so it is effectively more elementary, and second one can devise a two complex variable zeta function to study its analytic properties. This identity, at first, is established only for those representations which appear as archimedean components of global theta lifts from GO(2, 2). Then one uses various density arguments to extend the identity to other cases. The next natural step is to examine the identity to see what representations of the archimedean group have been covered. I suspect that at least all unramified tempered representations are included.

As mentioned above, the main contribution of this work, if any, is the archimedean analysis. Some of the results of this paper, especially in the case of discrete series representations, were announced in [21]. As stated above, the appearance of [12] has made our results for discrete series representations obsolete; Moriyama has obtained better and more explicit results for generic (limits of) discrete series, and some other representations, using more direct methods. Also we have recently learned that Asgari and Shahidi are preparing a manuscript which contains, among other things, the lifting of generic automorphic forms from spinor groups to general linear groups; if established, our results would be trivialized, as GSp(4) is nothing but GSin<sub>5</sub>. It appears that Brooks Roberts has used methods very similar to ours in [14] to study various non-archimedean questions. It turns out that both of us were influenced by Masaaki Furusawa, and communication with him and Shalika was our common source of inspiration. I learned about Bessel functionals and theta correspondence from J. A. Shalika while a graduate student at Johns Hopkins. Here we thank Shalika for continued support and encouragement over the past few years. Most of preliminary computations that led

to the writing of this paper were also performed at Hopkins under his supervision. The author has benefited from conversations with Freydoon Shahidi, Mahdi Asgari, Jeffrey Adams, Akshay Venkatesh, Peter Sarnak, and Brooks Roberts for answering many questions regarding his papers. Also the author wishes to thank the Clay Mathematical Institute for partial support of the project.

### Contents

1	The work of Jacquet and Langlands	5
<b>2</b>	Preliminaries on $GSp(4)$	11
	2.1 Bessel functionals	11
	2.2 Theta	13
	2.3 The Spinor L-function for $GSp(4)$	18
3	Bessel Functionals and Integral Representations	24
	3.1 The zeta integral of two complex variables; Euler product	30
4	The pull-back of the Whittaker function	42
	4.1 The Whittaker function	42
	4.2 Local Whittaker functions	49
5	Archimedean Zeta function	51
	5.1 Analytic continuation	54

#### Notation

In this paper, the group  $\operatorname{GSp}(4)$  over an arbitrary field K is the group of all matrices  $g \in \operatorname{GL}_4(K)$  that satisfy the following equation for some scalar  $\nu(g) \in K$ :

$${}^{t}gJg = \nu(g)J,$$

where 
$$J = \begin{pmatrix} & & 1 \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$$
. It is a standard fact that  $G = \mathrm{GSp}(4)$  is a

reductive group. The map  $(F^{\times})^3 \longrightarrow G$ , given by

$$(a, b, \lambda) \mapsto \operatorname{diag}(a, b, \lambda a^{-1}, \lambda b^{-1})$$

gives a parameterization of a maximal torus T in G. Let  $\chi_1$ ,  $\chi_2$  and  $\chi_3$  be quasi-characters of  $F^{\times}$ . We define the character  $\chi_1 \otimes \chi_2 \otimes \chi_3$  of T by

$$(\chi_1 \otimes \chi_2 \otimes \chi_3)(\operatorname{diag}(a, b, \lambda a^{-1}, \lambda b^{-1})) = \chi_1(a)\chi_2(b)\chi_3(\lambda).$$

The Weyl group is a dihedral group of order eight. We have three standard parabolic subgroups: The Borel subgroup B, The Siegel subgroup P, and the Klingen subgroup Q with the following Levi decompositions:

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1}\lambda & \\ & & b^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & s & r \\ & 1 & r & t \\ & & 1 \\ & & & 1 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} g & & \\ & \alpha^T g^{-1} \end{pmatrix} \begin{pmatrix} 1 & s & r \\ & 1 & r & t \\ & & 1 \\ & & & 1 \end{pmatrix}; g \in GL(2) \right\},$$

and finally Q is the maximal parabolic subgroup with non-abelian unipotent radical associated to the long simple root. Over a local field, we will use the notation  $\chi_1 \times \chi_2 \rtimes \chi_3$  for the parabolically induced representation from the minimal parabolic subgroup, by the character  $\chi_1 \otimes \chi_2 \otimes \chi_3$ . If  $\pi$  is a smooth representation of GL(2), and  $\chi$  a quasi-character of  $F^{\times}$ , then  $\pi \rtimes \chi$  (respectively  $\chi \rtimes \pi$ ) is the parabolically induced representation from the Levi subgroup of the Siegel (resp. Klingen) parabolic subgroup. We define a character of the unipotent radical N(B) of the Borel subgroup by the following:

$$\theta(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(x+t).$$

We call an irreducible representation  $(\Pi, V_{\Pi})$  of GSp(4) over a local field generic, if there is a functional  $\lambda_{\Pi}$  on  $V_{\Pi}$  such that

$$\lambda_{\Pi}(\Pi(n)v) = \theta(n)v,$$

for all  $v \in V_{\Pi}$  and  $n \in N(B)$ . If such a functional exists, it is unique up to a constant [18]. Freydoon Shahidi has given canonical constructions of these functionals in [16] for representations induced from generic representations. We define Whittaker functions on  $G \times V_{\Pi}$  by

$$W(\Pi, v, g) = \lambda_{\Pi}(\Pi(g)v).$$

When there is no danger of confusion, after fixing v and suppressing  $\Pi$ , we write W(g) instead of  $W(\Pi, v, g)$ . For any representation  $\pi$ , we will denote by  $\omega_{\pi}$  the central character of  $\pi$ .

## 1 The work of Jacquet and Langlands

In this section, we concentrate on the simpler group GL(2). This section serves as motivation for later sections which contain the main results of the paper. Our exposition is heavily based on [4], to the point of copying, especially pages 5-19. For the sake of familiarity and simplicity, we work over  $\mathbb{Q}$ .

Let  $G = \mathrm{GL}(2)$ . Suppose  $\chi$  is a unitary character of  $\mathbb{A}^{\times}$ . By a  $\chi$ -cusp form  $\varphi$  on GL(2), we mean an  $L^2(Z_{\mathbb{A}}G(\mathbb{Q})\backslash G(\mathbb{A}))$  function satisfying

$$\varphi(\begin{pmatrix} a & \\ & a \end{pmatrix}g) = \chi(a)\varphi(g),$$

and

$$\int_{\mathbb{Q}\backslash\mathbb{A}}\varphi(\begin{pmatrix}1&x\\&1\end{pmatrix}g)\,dx=0,$$

for almost all  $g \in G(\mathbb{A})$ . It is clear that if  $\varphi$  is a  $\chi$ -cusp form, and  $g \in G(\mathbb{A})$ , then the function  $g.\varphi$  on  $G(\mathbb{A})$  defined by

$$g.\varphi(h) = \varphi(hg),$$

is again a  $\chi$ -cusp form. This defines a representation of  $G(\mathbb{A})$  on the vector space of  $\chi$ -cusp forms  $L_0^2(\chi)$ . It is a fundamental fact that  $L_0^2(\chi)$  is a discrete

direct sum of irreducible subspaces each of which appears with multiplicity one. An irreducible representation  $\pi$  of  $GL_2(\mathbb{A})$  which is realized as an irreducible subspace  $H_{\pi}$  of  $L_0^2(\chi)$  is called a *cuspidal automorphic representation*.

Suppose  $\pi$  is an irreducible cuspidal automorphic representation of  $GL_2(\mathbb{A})$ , and  $\varphi \in H_{\pi}$ . We introduce a global zeta integral

$$\mathcal{Z}(\varphi, s) = \int_{\mathbb{Q}^{\times} \backslash \mathbb{A}^{\times}} \varphi(\begin{pmatrix} a \\ 1 \end{pmatrix}) |a|_{\mathbb{A}}^{s - \frac{1}{2}} d^{\times} a. \tag{1}$$

If  $\varphi$  is "nice enough",  $\mathcal{Z}(\varphi, s)$  defines an entire function in  $\mathbb{C}$ . Also, the zeta function  $\mathcal{Z}(\varphi, s)$  satisfies a functional equation:

$$\mathcal{Z}(\varphi, s) = \tilde{\mathcal{Z}}(\varphi^w, 1 - s),$$

where  $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , and  $\varphi^w(g) = \varphi(gw)$ . Also

$$\tilde{\mathcal{Z}}(\varphi, s) = \int_{\mathbb{Q}^{\times} \backslash \mathbb{A}^{\times}} \varphi(\begin{pmatrix} a \\ 1 \end{pmatrix}) |a|^{s - \frac{1}{2}} \chi^{-1}(a) d^{\times} a, \tag{2}$$

with  $\chi$  the central character of  $\pi$ .

The problem is to relate the function  $\mathcal{Z}(\varphi, s)$  to an automorphic L-function  $L(s, \pi, r)$  for some representation r of  ${}^LG = \mathrm{GL}_2(\mathbb{C})$ . For this purpose, we start by the Fourier expansion of  $\varphi$ 

$$\varphi(g) = \sum_{\xi \in \mathbb{O}^{\times}} W_{\varphi}^{\psi}(\begin{pmatrix} \xi \\ 1 \end{pmatrix} g). \tag{3}$$

Here

$$W_{\varphi}^{\psi}(g) = \int_{\mathbb{Q}\setminus\mathbb{A}} \varphi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \overline{\psi(x)} \, dx, \tag{4}$$

where  $\psi$  is a non-trivial character of  $\mathbb{Q}\setminus\mathbb{A}$ . It follows from (1) and (3) that

$$\mathcal{Z}(\varphi,s) = \int_{\mathbb{A}^{\times}} W_{\varphi}^{\psi}(\begin{pmatrix} a \\ 1 \end{pmatrix}) |a|^{s-\frac{1}{2}} d^{\times}a,$$

for  $\Re s$  large enough.

Now we recall some of the properties of the Whittaker functions  $W_{\varphi}^{\psi}$ . From now on we suppress  $\psi$ . We assume that  $\varphi$  is right K-finite. In this situation,  $W_{\varphi}$  is rapidly decreasing at infinity and satisfies

$$W_{\varphi}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g = \psi(x)W_{\varphi}(g), \tag{5}$$

for all  $x \in \mathbb{A}$ . The space of all such  $W_{\varphi}$  provides the  $\psi$ -Whittaker model of  $\pi$ . It is known that such a model is unique ([7], or [18]), and it is equal to the restricted tensor product of local Whittaker models  $\mathcal{W}(\pi_p, \psi_p)$ , where  $\pi = \otimes_p \pi_p$  and  $\psi = \prod_p \psi_p$ . In particular, we can assume that

$$W_{\varphi}(g) = \prod_{p} W_{p}(g_{p}), \tag{6}$$

where each  $W_p \in \mathcal{W}(\pi_p, \psi_p)$  and for almost all finite p,  $W_p$  is unramified, i.e.  $W_p(k) = 1$  for  $k \in K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ .

Finally we obtain for  $\Re s$  large

$$\mathcal{Z}(\varphi, s) = \prod_{p} \mathcal{Z}(W_{p}, s), \tag{7}$$

where

$$\mathcal{Z}(W_p, s) = \int_{\mathbb{Q}_p^{\times}} W_p(\begin{pmatrix} a \\ 1 \end{pmatrix}) |a|^{s - \frac{1}{2}} d^{\times} a. \tag{8}$$

First, we collect some of the properties of the local zeta functions  $\mathcal{Z}(W,s)$ . The fundamental fact for  $p < \infty$  is the following: There are a finite number of finite functions  $c_1, \ldots, c_N$  on  $\mathbb{Q}_p^{\times}$ , depending only on  $\pi_p$ , such that for every  $W \in \mathcal{W}(\pi_p, \psi_p)$ , there are Schwartz-Bruhat functions  $\Phi_1, \ldots, \Phi_N$  on  $\mathbb{Q}_p$  satisfying

$$W(\begin{pmatrix} a \\ 1 \end{pmatrix}) = \sum_{i=1}^{N} c_i(a)\Phi_i(a). \tag{9}$$

Here, a finite function is a function whose space of right translates by  $\mathbb{Q}_p^{\times}$  is finite dimensional; finite functions on  $\mathbb{Q}_p^{\times}$  are thus characters, integer powers of the valuation function, or products and linear combinations thereof. Taking the asymptotic expansion just mentioned for granted, we obtain from Tate's thesis that the integral defining  $\mathcal{Z}(W,s)$  converges for  $\Re s$  large (independent of s), and in the domain of convergence is equal to a rational function of  $p^{-s}$ . In particular, the integral has a meromorphic continuation to all of  $\mathbb{C}$ . Furthermore, the family of rational functions  $\{\mathcal{Z}(W,s) \mid W \in \mathcal{W}(\pi_p,\psi_p)\}$  admits a common denominator, i.e. a polynomial P such that  $P(p^{-s})\mathcal{Z}(W,s) \in \mathbb{C}[p^{-s},p^s]$ , for all W. Also, there exists a  $W^*$  in  $\mathcal{W}(\pi_p,\psi_p)$  with the property that  $\mathcal{Z}(W^*,s)=1$ . The analogous result in the archimedean situation is that there is a  $W^*$  such that  $\mathcal{Z}(W^*,s)$  has no poles or zeroes.

As in Tate's thesis, we also need to establish the functional equation and perform the local unramified computations. The functional equation asserts that there exists a meromorphic function  $\gamma(\pi_p, \psi_p, s)$  (rational function in  $p^{-s}$  when  $p < \infty$ ) such that

$$\tilde{\mathcal{Z}}(W^w, 1 - s) = \gamma(\pi_p, \psi_p, s) \mathcal{Z}(W, s). \tag{10}$$

Here  $W^w(g) = W(gw)$  and

$$\tilde{\mathcal{Z}}(W,s) = \int_{\mathbb{Q}_p^{\times}} W(\begin{pmatrix} a \\ 1 \end{pmatrix}) |a|^{s-\frac{1}{2}} \chi_p^{-1}(a) d^{\times} a,$$

with  $\chi_p$  the central character of  $\pi_p$ . The (non-trivial) proof of this equation follows from the fact that the integrals  $\mathcal{Z}$  and  $\tilde{\mathcal{Z}}$  define functionals (depending on s) satisfying a certain invariance property. Next one proves that the space of such functionals is at most one dimensional, implying that  $\mathcal{Z}$  and  $\tilde{\mathcal{Z}}$  must be proportional. The factor  $\gamma$  is simply the factor of proportionality.

We recall  $\pi_p$  is called unramified when

$$\pi_p = \operatorname{Ind}(\mu_1 \otimes \mu_2),$$

with  $\mu_1$  and  $\mu_2$  unramified characters of  $\mathbb{Q}_p^{\times}$ . We also suppose that  $W^0$  is the unique  $K_p$ -invariant function in  $\mathcal{W}(\pi_p, \psi_p)$ . We also suppose that  $\psi_p$  is unramified. Then a direct calculation, using the results of [2] for example, shows that

$$\mathcal{Z}(W^0, s) = \frac{1}{(1 - \mu_1(\varpi_p)p^{-s})(1 - \mu_2(\varpi_p)p^{-s})},$$

where  $\varpi_p$  is the local uniformizer at p. Next the conjugacy class in  ${}^LG = \operatorname{GL}_2(\mathbb{C})$  canonically associated with  $\pi_p$  is  $t_p = \begin{pmatrix} \mu_1(\varpi_p) \\ \mu_2(\varpi_p) \end{pmatrix}$ . In particular, we have

$$\mathcal{Z}(W^0, s) = L_p(s, \pi, r), \tag{11}$$

with r the standard two dimensional representation of  $GL_2(\mathbb{C})$ .

After this preparation, we can prove the conjecture of Langlands for  $L(s, \pi, r)$  with r as above. For simplicity, we write  $L(s, \pi)$  instead of  $L(s, \pi, r)$ . We start by extending the definition of  $L_v(s, \pi)$  to the ramified and archimedean places. We observe that in equation (11), the right hand side is indeed the

greatest common denominator of the family of rational functions  $\{\mathcal{Z}(W,s)\}$ . Since we have already noted that such a g.c.d. exists, even when the given representation is not unramified, we set

$$L_v(s,\pi) = \text{g.c.d. } \{\mathcal{Z}(W,s)\},\tag{12}$$

when  $v < \infty$ . Also, when  $v = \infty$ , we can choose an appropriate product of Tate's archimedean L-functions, denoted by  $L_{\infty}(s, \pi, r)$ , such that the ratio

$$\frac{\mathcal{Z}(W,s)}{L_{\infty}(s,\pi)} \tag{13}$$

is an entire function for all  $W \in \mathcal{W}(\pi_v, \psi_v)$ , and it is a nowhere vanishing function for some choice of W.

With this extension, we now proceed to outline the proof. Let S be a set of places, including the place at infinity, such that for  $v \notin S$ , all the data is unramified. We set

$$L_S(s,\pi) = \prod_{v \notin S} L_v(s,\pi). \tag{14}$$

For each "ramified" non-archimedean place p, we choose  $W_p$  such that  $\mathcal{Z}(W_p, s) = 1$ . Also for the archimedean v, we choose  $W_v$  such that  $\mathcal{Z}(W_v, s)$  is a non-vanishing entire function  $e^{g(s)}$ . If we set  $W = \prod_v W_v$ , with  $W_p = W_p^0$  for  $p \in S$ , we have

$$\mathcal{Z}(\varphi, s) = e^{g(s)} L_S(s, \pi), \tag{15}$$

implying the holomorphicity of  $L_S$ . This immediately implies the continuation of L to a meromorphic function with only a finite number of poles. It is this point with which the present note is concerned.

We finally turn to the functional equation of the completed L-function.

Choosing  $W_p$  so that  $\mathcal{Z}(W_p, s) = L_p(s, \pi)$ , we have

$$\begin{split} L(s,\pi) &= \mathcal{Z}(\varphi,s) \\ &= \tilde{\mathcal{Z}}(\varphi^w,1-s) \\ &= (\prod_{p \in S} \tilde{\mathcal{Z}}(W_p^w,1-s))L_S(1-s,\tilde{\pi}) \\ &= (\prod_{p \in S} \frac{\tilde{\mathcal{Z}}(W_p^w,1-s)}{L_p(1-s,\tilde{\pi})})L(1-s,\tilde{\pi}) \\ &= (\prod_{p \in S} \frac{\gamma(\pi_p,\psi_p,s)\mathcal{Z}(W_p,s)}{L_p(1-s,\tilde{\pi})})L(1-s,\tilde{\pi}) \\ &= (\prod_{p \in S} \frac{\gamma(\pi_p,\psi_p,s)L_p(s,\pi)}{L_p(1-s,\tilde{\pi})})L(1-s,\tilde{\pi}) \\ &= (\prod_{p \in S} \frac{\gamma(\pi_p,\psi_p,s)L_p(s,\pi)}{L_p(1-s,\tilde{\pi})})L(1-s,\tilde{\pi}), \end{split}$$

where

$$\epsilon(s, \pi_p, \psi_p) = \frac{\gamma(\pi_p, \psi_p, s) L_p(s, \pi)}{L_p(1 - s, \tilde{\pi})}.$$

Hence, if we set

$$\epsilon(s,\pi) = \prod_{p \in S} \epsilon(s,\pi_p,\psi_p),$$

we have the functional equation

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\tilde{\pi}),\tag{16}$$

as anticipated by Langlands. One last note is that the function  $\epsilon(s, \pi)$  is a monomial function of s. In particular, it has no poles or zeroes.

**Remark 1.1** In equation (9), if  $c_i = \mu_i v^{r_i}$ , with  $\mu_i$  a quasi-character, we have

$$L_p(s,\pi) = \prod_{i=1}^{N} L(s,\mu_i)^{r_i}.$$

The L-functions appearing on the right hand side are Tate's local L-factors for the quasi-characters  $\mu_i$ . This implies that in order to give an explicit calculations of the local L-factors, we need to determine the finite functions  $c_i$ . In [7], this is established by a case by case analysis of representation types for  $\pi_p$ , i.e. principal series vs. special representations vs. supercuspidals.

## 2 Preliminaries on GSp(4)

#### 2.1 Bessel functionals

We recall the notion of Bessel model introduced by Novodvorsky and Piatetski-Shapiro [11]. We follow the exposition of [3]. Let  $S \in M_2(\mathbb{Q})$  be such that  $S = {}^tS$ . We define the discriminant d = d(S) of S by  $d(S) = -4 \det S$ . Let us define a subgroup  $T = T_S$  of GL(2) by

$$T = \{ g \in \operatorname{GL}(2) \mid {}^{t}gSg = \det g.S \}.$$

Then we consider T as a subgroup of GSp(4) via

$$t \mapsto \begin{pmatrix} t & \\ & \det t \cdot {}^t t^{-1} \end{pmatrix},$$

 $t \in T$ .

Let us denote by U the subgroup of GSp(4) defined by

$$U = \{ u(X) = \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \mid X = {}^tX \}.$$

Finally, we define a subgroup R of GSp(4) by R = TU.

Let  $\psi$  be a non-trivial character of  $\mathbb{Q}\backslash\mathbb{A}$ . Then we define a character  $\psi_S$  on  $U(\mathbb{A})$  by  $\psi_S(u(X)) = \psi(\operatorname{tr}(SX))$  for  $X = {}^tX \in \mathsf{M}_2(\mathbb{A})$ . Usually when there is no danger of confusion, we abbreviate  $\psi_S$  to  $\psi$ . Let  $\Lambda$  be a character of  $T(\mathbb{Q})\backslash T(\mathbb{A})$ . Denote by  $\Lambda \otimes \psi_S$  the character of  $R(\mathbb{A})$  defined by  $(\Lambda \otimes \psi)(tu) = \Lambda(t)\psi_S(u)$  for  $t \in T(\mathbb{A})$  and  $u \in U(\mathbb{A})$ .

Let  $\pi$  be an automorphic cuspidal representation of  $\mathrm{GSp}_4(\mathbb{A})$  and  $V_{\pi}$  its space of automorphic functions. We assume that

$$\Lambda|_{\mathbb{A}^{\times}} = \omega_{\pi}.\tag{17}$$

Then for  $\varphi \in V_{\pi}$ , we define a function  $B_{\varphi}$  on  $GSp_4(\mathbb{A})$  by

$$B_{\varphi}(g) = \int_{Z_{\mathbb{A}}R_{\mathbb{Q}} \backslash R_{\mathbb{A}}} (\Lambda \otimes \psi_S)(r)^{-1} . \varphi(rh) \, dh.$$
 (18)

We say that  $\pi$  has a global Bessel model of type  $(S, \Lambda, \psi)$  for  $\pi$  if for some  $\varphi \in V_{\pi}$ , the function  $B_{\varphi}$  is non-zero. In this case, the  $\mathbb{C}$ -vector space of

functions on  $GSp_4(\mathbb{A})$  spanned by  $\{B_{\varphi} \mid \varphi \in V_{\pi}\}$  is called the space of the global Bessel model of  $\pi$ .

Similarly, one can consider local Bessel models. Fix a local field  $\mathbb{Q}_v$ . Define the algebraic groups  $T_S$ , U, and R as above. Also, consider the characters  $\Lambda$ ,  $\psi$ ,  $\psi_S$ , and  $\Lambda \otimes \psi_S$  of the corresponding local groups. Let  $(\pi, V_\pi)$  be an irreducible admissible representation of the group  $\mathrm{GSp}(4)$  over  $\mathbb{Q}_v$ . Then we say that the representation  $\pi$  has a local Bessel model of type  $(S, \Lambda, \psi)$  if there is a functional  $\lambda_B$  on  $(V_\pi^\infty)'$ , a continuous linear functional on  $V_\pi^\infty$  in such a way that

$$\lambda_B(\pi(r)v) = (\Lambda \otimes \psi_S)(r)\lambda_B(v),$$

for all  $r \in R(\mathbb{Q}_v)$ ,  $v \in V_{\pi}$ . Also, we require that  $\lambda_B$  would have some continuity properties similar to the ones satisfied by local Whittaker functionals.

In this work, we will be interested in two different types of Bessel models corresponding to two choices of the symmetric matrix S. The two choices of S are:

1. 
$$S = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
,

2. 
$$S = \begin{pmatrix} 1 \\ d \end{pmatrix}$$
, with d a positive square-free rational number.

Below, we will determine the subgroups  $T_S$ , and R, and explicitly write down the corresponding global Bessel functionals. We fix an irreducible automorphic cuspidal representation  $\pi$  of  $\mathrm{GSp}_4(\mathbb{A})$  and a unitary character  $\psi$  of  $\mathbb{A}$ throughout.

(1)  $S = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . This is the case of interest for us in this work. In this case, the subgroup  $T_S$  is equal to the subgroup consisting of diagonal matrices. A straightforward analysis then shows that for every character  $\Lambda$  of  $T_S(\mathbb{Q})\backslash T_S(\mathbb{A})$  subject to (17), there is a Hecke character of  $\mathbb{A}^\times$  such that the global Bessel functional (18) is given by

$$B_{\chi}^{\mathrm{split}}(g;\varphi) = \int_{F^{\times} \backslash \mathbb{A}^{\times}} \varphi^{U}(\begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix}) \chi(y) \, d^{\times}y.$$

Here when  $\phi$  is a cusp form on GSp(4), we have set

$$\phi^{U}(g) = \int_{(F \setminus \mathbb{A})^{3}} \phi(\begin{pmatrix} 1 & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w) \, du \, dv \, dw.$$

(2)  $S = \begin{pmatrix} 1 \\ d \end{pmatrix}$ . In this case, the subgroup  $T_S$  is equal to a non-split torus. Then there is a Hecke character of the torus  $T_S$ , say  $\chi$ , in such a way that

$$B_{\chi}(g;\varphi) = \int_{T_S(F)\mathbb{A}^{\times} \setminus T_S(\mathbb{A})} \varphi^U(\begin{pmatrix} \alpha \\ \det \alpha . {}^t \alpha^{-1} \end{pmatrix}) \chi(\alpha) \, d\alpha,$$

with  $\phi^U$  defined as before. The case of immediate interest is the case where d=1, in which case,

$$T_S = \{ g \in \operatorname{GL}_2 \mid {}^t g.g = \det g \}$$
$$= \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \in \operatorname{GL}_1 \}.$$

The problems of existence of Bessel functionals for this choice of the matrix S seem to be more delicate. We have considered these problems in [20].

#### 2.2 Theta

In this section we collect various results on theta correspondence that we will use in the sequel. In fact, this paper is a review of [13]. We have adapted the results of that paper to the case of our interest, split orthogonal spaces of signature (2,2). Other references of interest are [5,6].

Let V be the vector space  $M_2$ , of the two by two matrices, equipped with the quadratic form det. Let (,) be the associated non-degenerate inner product, and H = GO(V, (,)) be the group of orthogonal similitudes of V, (,). The group  $GL(2) \times GL(2)$  has a natural involution t defined by  $t(g_1, g_2) = (tb_2^{-1}, tb_1^{-1})$ , where the superscript t stands for the transposition. Let  $\tilde{H} = (GL(2) \times GL(2)) \times (t > be$  the semi-direct product of  $GL(2) \times GL(2)$  with the group of order two generated by t. There is a sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1, \tag{19}$$

where the homomorphism  $\rho: \tilde{H} \to H$  is defined by  $\rho(g_1, g_2)(v) = g_1 v g_2^{-1}$ , and  $\rho(t)v = {}^t v$ , for all  $g_1, g_2 \in GL(2)$  and  $v \in V$ . Also,  $\mathbb{G}_m \to \tilde{H}$  is the natural map  $z \mapsto (z, z) \times 1$ . It follows that the image of the subgroup  $GL(2) \times GL(2) \subset \tilde{H}$  under  $\rho$  is the connected component of the identity of H.

Let F be a local field of characteristic zero, with  $F = \mathbb{R}$  if F is archimedean. Fix a non-trivial unitary character  $\psi$  of F. The Weil representation  $\omega$  of  $\operatorname{Sp}(4,F) \times \operatorname{O}(V,F)$  defined with respect to  $\psi$  is the unitary representation on  $L^2(V^2)$  given by

$$\omega(1,h)\varphi(x) = \varphi(h^{-1}x),$$

$$\omega\left(\begin{pmatrix} a & \\ & t_{a^{-1}} \end{pmatrix}\right)\varphi(x) = |\det a|^{2}\varphi(xa),$$

$$\omega\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right)\varphi(x) = \psi\left(\frac{1}{2}\mathrm{tr}(bx,x)\right)\varphi(x),$$

$$\omega\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\varphi(x) = \gamma\hat{\varphi}(x).$$

Here,  $\hat{\varphi}$  is the Fourier transform defined by

$$\varphi(x) = \int_{V^2} \varphi(x') \psi(\operatorname{tr}(x, x')) \, dx'$$

with dx' self-dual, and  $\gamma$  is a certain fourth root of unity on  $\psi$ . If  $h \in O(V, F)$ ,  $a \in GL(2, F)$ ,  $b \in M_n(F)$  with  ${}^tb = b$  and  $x = (x_1, x_2), x' = (x'_1, x'_2) \in V^2$ , we write  $h^{-1}x = (h^{-1}x_1, h^{-1}x_2), xa = (x_1, x_2)(a_{ij}), (x, x') = ((x_i, x'_j)), bx = b^t(x_1, x_2)$ .

If F is non-archimedean,  $\omega$  preserves the space  $\mathcal{S}(V^2)$ ; by  $\omega$  we mean  $\omega$  acting on the latter space. When  $F = \mathbb{R}$ , we will work with Harish-Chandra modules of real reductive groups. Fix  $K_1 = \mathrm{Sp}(4,\mathbb{R}) \cap \mathrm{O}(4,\mathbb{R})$  as a maximal compact subgroup of  $\mathrm{Sp}(4,\mathbb{R})$ . We denote the Lie algebra of  $\mathrm{Sp}(4,\mathbb{R})$  by  $\mathfrak{g}_1 = \mathfrak{sp}(4,\mathbb{R})$ . Let  $V^+$  and  $V^-$  be positive and negative definite subspaces of X, respectively, such that  $V = V^+ \perp V^-$ . Then a maximal compact subgroup of  $\mathrm{O}(V,\mathbb{R})$  is  $\mathrm{O}(V^+,\mathbb{R}) \times \mathrm{V}(V^-,\mathbb{R}) \simeq \mathrm{O}(2,\mathbb{R}) \times \mathrm{O}(2,\mathbb{R})$ . The Lie algebra of  $\mathrm{O}(V,\mathbb{R})$  is  $\mathfrak{h}_1 = \mathfrak{o}(V,\mathbb{R})$ . Let  $\mathcal{S}(V^2) = \mathcal{S}_{\psi}(V^2)$  be the subspace of  $L^2(V^2)$  consisting of the functions

$$p(x) \exp \left[ -\frac{1}{2} |c| \left( \operatorname{tr}(x^+, x^+) - \operatorname{tr}(x^-, x^-) \right) \right].$$

Here p is a polynomial, and  $(x^+, x^+)$  and  $(x^-, x^-)$  are  $2 \times 2$  matrices with (i, j)-th entries  $(x_i^+, x_j^+)$  and  $(x_i^+, x_j^+)$  respectively, where  $x_i = x_i^+ + x_i^-$  corresponding to the decomposition of V;  $c \in \mathbb{R}^\times$  is such that  $\psi(t) = \exp(ict)$ . Then  $\mathcal{S}(V^2)$  is a  $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1, J_1)$  module under  $\omega$ ; this is the Harish-Chandra module we will work with throughout. Often, for the sake of uniformity in presentation, one uses the notation and terminology of genuine representations for archimedean places as well. The reader has to keep on mind, however, that this is just a matter of convenience.

Let  $\mathcal{R}(O(V, F))$  be the set of elements of Irr (O(V, F)) which are non-zero quotients of  $\omega$ , and define  $\mathcal{R}(\operatorname{Sp}(4, F))$  similarly. Again, the reader will have to keep in mind that at the archimedean place, we are working with underlying Harish-Chandra modules. Suppose F is real or non-archimedean of odd residual characteristic. Then the set

$$\{(\pi, \sigma) \in \mathcal{R}(\mathrm{Sp}(4, F)) \times \mathcal{R}(\mathrm{O}(\mathrm{V}, F)) \mid \mathrm{Hom}_{\mathrm{Sp}(4, F) \times \mathrm{O}(\mathrm{V}, F)}(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection, denoted by  $\theta$  in either direction, between the corresponding sets. When F is non-archimedean of even residual characteristic, one can establish the same for tempered representations. We refer the reader to [13], section 1, for more information.

We now recall the extended Weil representation for similitude groups. Define

$$R_V(F) = \{(g, h) \in \mathrm{GSp}(4, F) \times \mathrm{GO}(V, F) \mid \nu(g) = \nu(h)\}.$$

The Weil representation of  $\operatorname{Sp}(4, F) \times \operatorname{O}(V, F)$  on  $L^2(V^2)$  extends to a unitary representation of  $R_V(F)$  via

$$\omega(g,h)\varphi = |\nu(h)|^{-2}\omega(g\begin{pmatrix}1&\\&\nu(g)\end{pmatrix}^{-1},1)(\varphi\circ h^{-1}).$$

We would still like to consider the action of  $R_V(F)$  on  $\mathcal{S}(V^2)$ , but one has to take some care when considering the archimedean place, as in this case  $\mathcal{S}(V^2)$  is preserved only at the level of Harish-Chandra modules; we refer the reader to [13] for details. We denote the resulting genuine representation of  $R_V$ , in the non-archimedean case, or the  $(\mathfrak{r}_{\infty}, L_{\infty})$  Harish-Chandra module, in the archimedean case, again by  $\omega$ .

In analogy with the isometry case, one can ask when  $\operatorname{Hom}_{R_V}(\omega, \pi \otimes \sigma) \neq 0$  for  $\pi \in \operatorname{Irr}(\operatorname{GSp}(4, F))$  and  $\sigma \in \operatorname{Irr}(\operatorname{GO}(V, F))$ . Here  $\mathcal{R}$  for each group is the

collection of representations of the similitude group which when restricted to the corresponding isometry group have a non-zero component in  $\mathcal{R}$ . Then by theorem 1.8 of [13], parts 1, 3, 5,  $\operatorname{Hom}_{R_V}(\omega, \pi \otimes \sigma) \neq 0$  defines a bijection between  $\mathcal{R}(\operatorname{GSp}(4,F))$  and  $\mathcal{R}(\operatorname{GO}(V,F))$ . Again, over a non-archimedean field of even residual characteristic one has to restrict to an appropriate class of representations. Again, one denotes the resulting bijection by  $\theta$ . Proposition 1.11 of [13] states that  $\theta$  maps unramified representations to unramified representations.

Let  $(\pi_1, \pi_2)$  be a pair of representations of  $GL_2$  over the local field F with  $\omega_{\pi_1}.\omega_{\pi_2} = 1$ . Roberts [13] has associated to  $(\pi_1, \pi_2)$  an L-packet in GSp(4). Essentially, the idea is to consider the representation  $\pi = \pi_1 \otimes \pi_2$  of GSO(V, F) and then consider all possible extensions of  $\pi$  to GO(V, F); then consider the theta lifts of all such extended representations to GSp(4, F). We describe the L-parameter giving this packet in the archimedean situation. If

$$g_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, i = 1, 2, \text{ we set}$$

$$S(g_1, g_2) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix}.$$

For i = 1, 2, let  $\rho_i : W_{\mathbb{R}} \to \mathrm{GL}_2(\mathbb{C})$  be the L parameter of  $\pi$ . Then define an L-parameter  $\varphi(\rho_1, \rho_2) : W_{\mathbb{R}} \to \mathrm{GSp}(4, \mathbb{C})$  by

$$\varphi(\rho_1, \rho_2)(z) = S(\rho_1(z), \rho_2(z)^{-1}),$$

 $z \in W_{\mathbb{R}}$ . We take for granted the fact that the L packet defined by Roberts in the archimedean situation is the L packet associated to  $\varphi(\rho_1, \rho_2)$  by Langlands. We refer the reader to section 4 of [13], in particular pages 283-285 for basic properties of the L packets.

We now turn our attention to global theta correspondence for the similitude groups [13], section 5. In order to define global theta correspondence we need a global Weil representation. Fix a non-trivial unitary character of  $\mathbb{A}$  trivial on  $\mathbb{Q}$ . For a place v of  $\mathbb{Q}$ , let  $\omega_v$  be the representation defined above. Let  $x_1, \ldots, x_4$  be a vector space basis of  $M_2(\mathbb{Q})$  over  $\mathbb{Q}$ . Let  $(g, h) \in R_V(\mathbb{A})$ . Then for almost all places v,  $\omega_v(g_v, h_v)$  fixes the characteristic function of  $\mathcal{O}_v x_1 + \cdots + \mathcal{O}_v x_4$ . Let  $\mathcal{S}(V(\mathbb{A})^2)$  be the restricted algebraic direct product  $\otimes_v \mathcal{S}(V(\mathbb{Q}_v)^2)$  which is naturally an  $R_V(\mathbb{A}_f) \times (\mathfrak{r}_{\infty}, L_{\infty})$ -module. For

 $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$  and  $(g,h) \in R_V(\mathbb{A})$ , define

$$\theta(g,h;\varphi) = \sum_{x \in V(\mathbb{Q})^2} \omega(g,h)\varphi(x).$$

This series converges absolutely and is left  $R(\mathbb{Q})$  invariant. Fix a right invariant quotient measure on  $O(V,\mathbb{Q})\backslash O(V,\mathbb{A})$ . Let f be a cusp form on  $GO(V,\mathbb{A})$ . For  $g \in GSp(4,\mathbb{A})$  define

$$\theta(f,\varphi)(g) = \int_{\mathcal{O}(V,\mathbb{Q})\backslash\mathcal{O}(V,\mathbb{A})} \theta(g,h_1h;\varphi)f(h_1h) dh_1,$$

where  $h \in GO(V, \mathbb{A})$  is any element such that  $(g, h) \in R_V(\mathbb{A})$ . This integral converges absolutely, does depend on the choice of h, and the function  $\theta(f, \varphi)$  on  $GSp(4, \mathbb{A})$  is left  $GSp(4, \mathbb{Q})$  invariant. The function  $\theta(f, \varphi)$  is an automorphic function on  $GSp(4, \mathbb{A})$  of central character equal to the central character of f. If V is a  $GO(V, \mathbb{A}) \times (h_{\infty}, J_{\infty})$  subspace of the space of cusp forms on  $GO(V, \mathbb{A})$  of central character  $\chi$ , then we denote by  $\Theta(V)$  the  $GSp(4, \mathbb{A}_f) \times (g_{\infty}, K_{\infty})$  subspace of the space of automorphic forms on  $GSp(4, \mathbb{A})$  of central character  $\chi$  generated by all the  $\theta(f, \varphi)$  for  $f \in V$  and  $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$ .

For computational purposes, we need to make the above considerations explicit. Here the notation may be slightly different from above. Suppose  $\pi_1$  and  $\pi_2$  are two irreducible cuspidal automorphic representations of  $GL_2(\mathbb{A})$  satisfying

$$\omega_{\pi_1}.\omega_{\pi_2}=1.$$

Then for  $\varphi_1$  and  $\varphi_2$  cusp forms in the spaces of  $\pi_1$  and  $\pi_2$ , respectively, one can think of

$$\varphi(h_1, h_2) = \varphi_1(h_1)\varphi_2(h_2),$$

as a cusp form on the algebraic group  $\rho(H)$ . We extend the definition of  $\varphi$  to H by defining it to be right invariant under the compact totally disconnected group  $< t > (\mathbb{A}) = \prod_{v} < t >$ .

Define the subgroup  $H_1$  consisting of elements  $(h_1, h_2)$  satisfying

$$\det(h_1) = \det(h_2).$$

Then if  $\pi_1$  and  $\pi_2$  are two automorphic cuspidal representations of the group GL(2) with

$$\omega_{\pi_1}.\omega_{\pi_2}=1,$$

and

$$\pi_1 \neq \tilde{\pi}_2$$
,

then one can naturally think of the pair  $(\pi_1, \pi_2)$  as an automorphic cuspidal representation of the group H. If  $\varphi_1$  and  $\varphi_2$  are cusp forms on  $GL_2(\mathbb{A})$ , belonging to the spaces of the representations  $\pi_1$  and  $\pi_2$ , respectively, we define a cuspidal function  $\theta(\varphi_1, \varphi_2; f)$  on  $GSp(4, \mathbb{A})$  by

$$\theta(\varphi_1, \varphi_2; f)(g) = \int_{H_1(F) \setminus H_1(\mathbb{A})} \theta(g; h_1 h^1, h_2 h^2; f) \varphi_1(h_1 h^1) \varphi_2(h_2 h^2) d(h_1, h_2),$$

where the pair  $(h^1, h^2)$  is chosen such that

$$\det h^{1}(\det h^{2})^{-1} = \nu(g).$$

Here f is a Bruhat-Schwartz function on  $M_2(\mathbb{A}) \times M_2(\mathbb{A})$ , and

$$\theta(g; h_1 h^1, h_2 h^2; f) = \sum_{M_1, M_2 \in \mathsf{M}_2(F)} \omega(g; h_1 h^1, h_2 h^2) f(M_1, M_2),$$

where  $\omega$  is the Weil representation of [5]. We note this is different from the definition given earlier. Let  $\Theta(\pi_1, \pi_2)$  be the vector space generated by the functions  $\theta(\varphi_1, \varphi_2; f)$  for all choices of  $\varphi_1, \varphi_2$ , and f as above. Then  $\Theta(\pi_1, \pi_2)$  is an irreducible generic automorphic cuspidal representation of GSp(4). In fact, this is the generic element of the global L packet defined by Roberts [13]. If  $\Theta(\pi_1, \pi_2) = \bigotimes_v \Theta_v(\pi_1, \pi_2)$ , then  $\Theta_v(\pi_1, \pi_2)$  depends only on the v components of  $\pi_1, \pi_2$ , and is the generic element of corresponding local L packet.

### 2.3 The Spinor L-function for GSp(4)

In this section, we review the integral representation given by Novodvorsky [10] for G = GSp(4). The details of the material in the following paragraphs appear in [1], [19].

Let  $\varphi$  be a cusp form on  $\mathrm{GSp}(4,\mathbb{A})$ , belonging to the space of an irreducible cuspidal automorphic representation  $\pi$ . Consider the integral

$$Z_{N}(s,\phi,\mu) = \int_{\mathbb{A}^{\times}/\mathbb{Q}^{\times}} \int_{(\mathbb{A}/\mathbb{Q})^{3}} \phi \begin{pmatrix} 1 & x_{2} & x_{4} \\ 1 & \\ & 1 \\ z & -x_{2} & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y \\ & & 1 \\ & & 1 \end{pmatrix} \\ \times \psi(-x_{2})\mu(y)|y|^{s-\frac{1}{2}} dz dx_{2} dx_{4} d^{\times}y.$$

Since  $\phi$  is left invariant under the matrix

$$\begin{pmatrix} & & & 1 \\ & & 1 \\ & -1 & \\ -1 & & \end{pmatrix},$$

this integral has a functional equation  $s \to 1-s$ . A usual unfolding process as sketched in [1] then shows that

$$\mathbb{Z}_{N}(s,\phi,\mu) = \int_{\mathbb{A}^{\times}} \int_{\mathbb{A}} W_{\phi} \begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \mu(y)|y|^{s-\frac{3}{2}} dx d^{\times}y.$$
 (20)

Here the Whittaker function  $W_{\varphi}$  is given by

$$W_{\phi}(g) = \int_{(\mathbb{A}/\mathbb{Q})^4} \phi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 \\ & & & 1 \end{pmatrix} g\right) \times \psi^{-1}(x_1 + x_2) dx_1 dx_2 dx_3 dx_4$$

Equation (20) implies that, in order for  $Z_N(\varphi, s)$  to be non-zero, we need to assume that  $W_{\varphi}$  is not identically equal to zero. A representation satisfying this condition is called "generic." Every irreducible cuspidal representation of  $\mathrm{GL}(2)$  is generic. On other groups, however, there may exist non-generic cuspidal representations. In fact, those representations of  $\mathrm{GSp}(4)$  which correspond to holomorphic cuspidal Siegel modular forms are not generic.

From this point on, we assume that all the representations of GSp(4), local or global, which appear in the text are generic.

If  $\varphi$  is chosen correctly, the Whittaker function may be assumed to decompose locally as  $W(g) = \prod_v W_v(g_v)$ , a product of local Whittaker functions. Hence, for  $\Re s$  large, we obtain

$$\mathcal{Z}(\varphi, s) = \prod_{v} \mathcal{Z}(W_v, s), \tag{21}$$

where

$$Z_N(W_v, s) = \int_{F_v^{\times}} \int_{F_v} W_v \begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & x & & 1 \end{pmatrix} |y|^{s - \frac{3}{2}} dx d^{\times} y.$$
 (22)

As usual, we have a functional equation: There exists a meromorphic function  $\gamma(\pi_v, \psi_v, s)$  (rational function in  $\mathbb{N}v^{-s}$  when  $v < \infty$ ) such that

$$Z_N(W_v, s) = \gamma(\pi_v, \psi_v, s)\tilde{\mathcal{Z}}(W_v^w, 1 - s), \tag{23}$$

with w as above,

$$\tilde{\mathcal{Z}}(W_v, s) = \int_{F_v^{\times}} \int_{F_v} W_v \begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & x & & 1 \end{pmatrix} \chi_v^{-1}(y) |y|^{s - \frac{3}{2}} dx d^{\times} y,$$

and  $\chi_v$  the central character of  $\pi_v$ .

We also consider the unramified calculations. Suppose v is any nonarchimedean place of F such that  $W_v$  is right invariant by  $GSp(4, \mathcal{O}_v)$  and such that the largest fractional ideal on which  $\psi_v$  is trivial is  $\mathcal{O}$ . Then the Casselman-Shalika formula [2] allows us to calculate the last integral (cf. [1]). The result is the following:

$$\mathcal{Z}(W_v, s) = L(s, \pi_v, \text{Spin}). \tag{24}$$

Let us explain the notation. The connected L-group  ${}^LG^0$  is  $\mathrm{GSp}(\mathbb{C})$ . Let  ${}^LT$  be the maximal torus of elements of the form

$$t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix},$$

where  $\alpha_1\alpha_4 = \alpha_2\alpha_3$ . The fundamental dominant weights of the torus are  $\lambda_1$  and  $\lambda_2$  where

$$\lambda_1 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1,$$

and

$$\lambda_2 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 \alpha_3^{-1}.$$

The dimensions of the representation spaces associated with these dominant weights are four and five, respectively. In our notation, Spin is the representation of  $GSp(4, \mathbb{C})$  associated with the dominant weight  $\lambda_1$ , i.e. the standard representation of  $GSp(4, \mathbb{C})$  on  $\mathbb{C}^4$ . The L-function  $L(s, \pi, Spin)$  is called the Spinor, or simply the Spin, L-function of GSp(4).

Next step is to use the integral introduced above to extend the definition of the Spinor L-function to ramified non-archimedean and archimedean places.

We now sketch the computation of the local non-archimedean Euler factors of the Spin L-function given by the integral representation introduced above. In order for this to make sense, we need the following lemma:

**Lemma 2.1 (Theorem 2.1 of [19])** Suppose  $\Pi$  is a generic representation of GSp(4) over a non-archimedean local field K, q order of the residue field. For each  $W \in \mathcal{W}(\Pi, \psi)$ , the function  $\mathcal{Z}(W, s)$  is a rational function of  $q^{-s}$ , and the ideal  $\{\mathcal{Z}(W, s)\}$  is principal.

Sketch of proof. For  $W \in \mathcal{W}(\Pi, \psi)$ , we set

$$Z(W,s) = \int_{K} W \begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & & & 1 \end{pmatrix} |y|^{s-\frac{3}{2}} d^{\times} y.$$
 (25)

The first step of the proof is to show that the vector space  $\{Z(W,s)\}$  is the same as  $\{Z(W,s)\}$  (cf. Proposition 3.2 of [19]). Next, we use the asymptotic expansions of the Whittaker functions along the torus to prove the existence of the g.c.d. for the ideal  $\{Z(W,s)\}$ . Indeed, Proposition 3.5 of [19] (originally a theorem in [2]) states that there is a finite set of finite functions  $S_{\Pi}$ , depending only on  $\Pi$ , with the following property: for any  $W \in \mathcal{W}(\Pi, \psi)$ , and  $c \in S_{\Pi}$ , there is a Schwartz-Bruhat function  $\Phi_{c,W}$  on K such that

$$W\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \sum_{c \in S_{\Pi}} \Phi_{c,W}(y)c(y)|y|^{\frac{3}{2}}.$$

The lemma is now immediate.  $\square$ 

We have the following theorem:

**Theorem 2.2** Suppose  $\Pi$  is a generic representation of the group GSp(4) over a non-archimedean local field K. Then

1. If  $\Pi$  is supercuspidal, or is a sub-quotient of a representation induced from a supercuspidal representation of the Klingen parabolic subgroup, then  $L(s, \pi, \text{Spin}) = 1$ .

2. If  $\pi$  is a supercuspidal representation of GL(2) and  $\chi$  a quasi-character of  $K^{\times}$ , and  $\Pi = \pi \rtimes \chi$  is irreducible, we have

$$L(s, \Pi, \text{Spin}) = L(s, \chi).L(s, \chi.\omega_{\pi}).$$

3. If  $\chi_1$ ,  $\chi_2$ , and  $\chi_3$  are quasi-characters of  $K^{\times}$ , and  $\Pi = \chi_1 \times \chi_2 \times \chi_3$  is irreducible, we have

$$L(s, \Pi, \text{Spin}) = L(s, \chi_3).L(s, \chi_1\chi_3).L(s, \chi_2\chi_3).L(s, \chi_1\chi_2\chi_3).$$

4. When  $\Pi$  is not irreducible, one can prove similar statements for the generic subquotients of  $\Pi = \pi \rtimes \chi$  (resp.  $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$ ) according to the classification theorems of Sally-Tadic [15] and Shahidi [17] (cf. theorems 4.1 and 5.1 of [19]).

Remark 2.3 Sally and Tadic [15] and Shahidi [17] have completed the classification of representations supported in the Borel and Siegel parabolic subgroups. In particular, they have determined for which representations the parabolic induction is reducible. From their result, one can immediately establish a classification for all the generic representations supported in the Borel or Siegel parabolic subgroups.

Sketch of proof. By the proof of the lemma, we need to determine the asymptotic expansion of the Whittaker functions in each case. The argument consists of several steps:

Step 1 Bound the size of  $S_{\Pi}$ . Fix  $c \in S_{\Pi}$ , and define a functional  $\Lambda_c$  on  $\mathcal{W}(\Pi, \psi)$  by

$$\Lambda_c(W) = \Phi_{c,W}(0). \tag{26}$$

If  $c, c' \in S_{\Pi}$ , and  $c \neq c'$ , the two functionals  $\Lambda_c$  and  $\Lambda_{c'}$  are linearly independent. Furthermore, the functionals  $\Lambda_c$  belong to the dual of a certain twisted Jacquet module  $\Pi_{N,\bar{\theta}}$  (notation from [19], page 1095). Hence  $\#S_{\Pi} = \dim \Pi_{N,\bar{\theta}}$ . Then one uses an argument similar to those of [18], distribution theory on p-adic manifolds, to bound the dimension of the Jacquet module. The result (proposition 3.9 of [19]) is that if  $\Pi$  is supercuspidal or supported in the Klingen parabolic subgroup (resp. Siegel parabolic, resp. Borel parabolic), then  $\#S_{\Pi} = 0$  (resp.  $\leq 2$ , resp.  $\leq 4$ ). Note that this already implies the first part of the theorem.

From this point on, we concentrate on the Siegel parabolic subgroup, the Borel subgroup case being similar. We fix some notation. Suppose  $\Pi = \pi \rtimes \chi$ , with  $\pi$  supercuspidal of GL(2). Let  $\lambda_{\Pi}$  (resp.  $\lambda_{\pi}$ ) be the Whittaker functional of  $\Pi$  (resp.  $\pi$ ) from [16]. It follows from the proof of the lemma 2.1 that, for  $f \in \Pi$ , there is a positive number  $\delta(f)$ , such that

$$\lambda_{\Pi}(\Pi\begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & & & 1 \end{pmatrix})f) = \sum_{c \in S_{\Pi}} \Lambda_{c}(f)c(y)|y|^{\frac{3}{2}},$$

for  $|y| < \delta(f)$ . Here,  $\Lambda_c$  is the obvious functional on the space of  $\Pi$ .

**Step 2** Uniformity. For  $f \in Ind(\pi \times \chi | P \cap K, K)$ , and  $\tau \in \mathbb{C}$ , define  $f_{\tau}$  on G by

$$f_{\tau}(pk) = \delta_P(p)^{\tau + \frac{1}{2}} \pi \otimes \chi(p) f(k).$$

It is clear that  $f_{\tau}$  is a well-defined function on G, and that it belongs to the space of a certain induced representation  $\Pi_{\tau}$ . The Uniformity Theorem (Proposition 3.9 of [19]) asserts that one can take  $\delta(f_{\tau}) = \delta(f)$ .

**Step 3** Regular representations. This is the case where  $\omega_{\pi} \neq 1$ . In this situation, we have

$$\lambda_{\Pi}(\Pi\begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & & & 1 \end{pmatrix})f) = \tag{27}$$

$$\lambda_{\pi}(A(w,\Pi)(f)(e))\chi(y)|y|^{\frac{3}{2}} + C(w\Pi,w^{-1})^{-1}\lambda_{\pi}(f(e))\chi(y)\omega_{\pi}(y)|y|^{\frac{3}{2}},$$

for 
$$|y| < \delta(f)$$
. Here  $w = \begin{pmatrix} & & 1 \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$ ,  $A(w, \Pi)$  is the intertwining

integral of [16], and  $C(w\Pi, w^{-1})$  is the local coefficient of [16]. The proof of this identity follows from the the above lemma 2.1, and the Multiplicity One Theorem [18]. The idea is to find one term of the asymptotic expansion using the open cell; then apply the long intertwining operator to find the other term.

Note that the identity of *Step* 3 also applies to reducible cases. For example, if  $f \in \Pi$  is in the kernel of the intertwining operator  $A(w, \Pi)$ , the first term of the right hand side vanishes.

Step 4 Irregular Representations. The idea is the following: we twist everything in Step 3 by the complex number  $\tau$ , so that the resulting representation  $\Pi_{\tau}$  is regular. By Step 2, the identity still holds uniformly for all  $\tau$ . By a theorem of Shahidi [16] (essentially due to Casselman and Shalika [2]), we know that the left hand side of the identity is an entire function of  $\tau$ . This implies that the poles of the right hand side, coming from the intertwining operator and the local coefficient, must cancel out. Next, we let  $\tau \to 0$ . An easy argument (l'Hopital's rule!) shows the appearance of  $\chi(y)|y|^{\frac{3}{2}}$  and  $\chi(y)|y|^{\frac{3}{2}}\log_q|y|$  in the asymptotic expansion.

This finishes the sketch of proof of the theorem.  $\square$ 

Corollary 2.4 Let  $\pi$  be an irreducible generic representation of GSp(4) over a non-archimedean local field K. Let  $\mu$  be a quasi-character of  $K^{\times}$ . If  $\mu$  is highly ramified, we have

$$L(s, \pi \otimes \mu) = 1.$$

## 3 Bessel Functionals and Integral Representations

In the global situation, there is a simple relationship between the integral representation of the previous section and split Bessel functionals. The following simple observation which for the ease of reference we separate as a lemma forms the fundamental idea of this paper:

#### Lemma 3.1 We have

$$B_{\mu|.|^{s-\frac{1}{2}}}^{\text{split}}(I_4;\phi) = \int_{\mathbb{A}^{\times}} \int_{\mathbb{A}} W_{\phi} \begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & x & & 1 \end{pmatrix} w^{-1} \mu(y)|y|^{s-\frac{3}{2}} dx d^{\times}y,$$

with

$$w = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}.$$

This motivates the following definition.

**Definition 3.2** For  $\varphi_1$ ,  $\varphi_2$ , and f as above and  $\mu$  a Hecke character, we define

$$\begin{split} \mathcal{Z}(\varphi_1, \varphi_2, f; \mu) &= B_{\mu|.\,|^{-\frac{1}{2}}}^{\text{split}}(I_4; \theta(\varphi_1, \varphi_2; f)) \\ &= \int_{F^{\times} \backslash \mathbb{A}^{\times}} \theta(\varphi_1, \varphi_2; f)^U \begin{pmatrix} y & & \\ & 1 & \\ & & 1 \end{pmatrix}) \mu(y) |y|^{-\frac{1}{2}} d^{\times}y. \end{split}$$

Here if  $\phi$  is a cusp form on GSp(4), we have set

$$\phi^{U}(g) = \int_{(F \setminus \mathbb{A})^{3}} \phi(\begin{pmatrix} 1 & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w) \, du \, dv \, dw.$$

We prove that the above integral is an infinite product of local integrals. We do so by finding an expression relating our function  $\mathcal{Z}(\varphi_1, \varphi_2, f; s)$  to the Jacquet-Langlands zeta functions of  $\varphi_1$ , and  $\varphi_2$ .

Before stating our proposition, we recall a notation from [7]. If  $\phi$  is a cusp form on  $GL_2(\mathbb{A}_F)$ , in the space of a representation  $\pi$ ,  $\mu$  a Hecke character, and  $h \in GL_2(\mathbb{A}_F)$ , we set

$$Z(\phi, h, \mu) = \int_{F^{\times} \backslash \mathbb{A}^{\times}} \phi(\begin{pmatrix} a \\ 1 \end{pmatrix} h) \mu(a) |a|^{-\frac{1}{2}} d^{\times} a,$$

and

$$\tilde{Z}(\phi, h, \mu) = \int_{F^{\times} \backslash \mathbb{A}^{\times}} \phi(\begin{pmatrix} a \\ 1 \end{pmatrix} h) \omega_{\pi}(a)^{-1} \mu(a) |a|^{-\frac{1}{2}} d^{\times} a$$

Then, we have the following proposition:

**Proposition 3.3** For  $\varphi_1$ ,  $\varphi_2$ , and f as above, we have

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} Z(\varphi_1, h_1, \mu) Z(\varphi_2, h_2, \mu^{-1}|.|)$$
$$L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 dh_2$$

*Proof.* First, we obtain an expression for  $\theta(\varphi_1, \varphi_2; f)^U$ . We start by the following:

$$\theta(\varphi_{1}, \varphi_{2}; f)^{U}(g)$$

$$= \int_{(F \setminus \mathbb{A})^{3}} \theta(\varphi_{1}, \varphi_{2}; f) \begin{pmatrix} 1 & u & w \\ 1 & w & v \\ & 1 \\ & & 1 \end{pmatrix} g) \psi^{-1}(w) du dv dw$$

$$= \int_{(F \setminus \mathbb{A})^{3}} \int_{H_{1}(F) \setminus H_{1}(\mathbb{A})} \theta(\begin{pmatrix} 1 & u & w \\ 1 & w & v \\ & 1 \\ & & 1 \end{pmatrix} g; h_{1}h^{1}, h_{2}h^{2}; f)$$

$$\varphi_{1}(h_{1}h^{1}) \varphi_{2}(h_{2}h^{2}) d(h_{1}, h_{2}) \psi^{-1}(w) du dv dw,$$

where  $h^1$  and  $h^2$  are chosen in such a way that

$$\det h^1 \cdot (\det h^2)^{-1} = \nu(g).$$

Next, it follows from the definition of  $\theta$  that

$$\theta(\varphi_1, \varphi_2; f)^U(g) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1 h^1) \varphi_2(h_2 h^2) G_f(h_1 h^1, h_2 h^2; g) dh_1 dh_2,$$
(28)

where

$$G_f(h_1h^1, h_2h^2; g) = \sum_{M_1, M_2} \int_{(F \setminus \mathbb{A})^3} \omega\begin{pmatrix} 1 & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g, h_1h^1, h_2h^2) f(M_1, M_2)$$

$$\psi^{-1}(w) \, du \, dv \, dw.$$

Next, for fixed  $M_1$  and  $M_2$  we have

$$\int_{(F \setminus \mathbb{A})^3} \omega\begin{pmatrix} 1 & u & w \\ 1 & w & v \\ & 1 \\ & & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(M_1, M_2) \psi^{-1}(w) du dv dw 
= \omega(g, h_1 h^1, h_2 h^2) f(M_1, M_2) 
\int_{(F \setminus \mathbb{A})^3} \psi(tr\begin{pmatrix} u & w \\ w & v \end{pmatrix} \begin{pmatrix} \det M_1 & B(M_1, M_2) - \frac{1}{2} \\ B(M_2, M_1) - \frac{1}{2} & \det M_2 \end{pmatrix}) du dv dw.$$

Next, we have the following straightforward lemma:

**Lemma 3.4** For any  $2 \times 2$  matrix  $A \in M_2(\mathbb{A})$ , we have

$$\int_{(F\setminus\mathbb{A})^3} \psi(tr\begin{pmatrix} u & w \\ w & v \end{pmatrix} A) du dv dw = 0,$$

unless  $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , in which case the value of the integral is equal to 1.

The lemma implies that

$$G_f(h_1h^1, h_2h^2; g) = \sum_{(M_1, M_2) \in \mathcal{S}} \omega(g, h_1h^1, h_2h^2) f(M_1, M_2),$$

where

$$S = \{(X, Y) \in M_2(F) \times M_2(F) \mid \det X = 0, \det Y = 0, \det(X + Y) = 1\}.$$

**Lemma 3.5** The set S consists of a single orbit under the action of  $H_1(F)$ . The point  $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ) belongs to S. The stabilizer of P in  $H_1(F)$  is the subgroup D(F).

Consequently,

$$G_f(h_1 h^1, h_2 h^2; g) = \sum_{\gamma \in D(F) \backslash H_1(F)} \omega(1, \gamma) \omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}).$$

Inserting the right hand side of this expression for  $G_f$  in equation (28) gives

$$\theta(\varphi_{1}, \varphi_{2}; f)^{U}(g) = \int_{D(F)\backslash H_{1}(\mathbb{A})} \varphi_{1}(h_{1}h^{1})\varphi_{2}(h_{2}h^{2})\omega(g, h_{1}h^{1}, h_{2}h^{2})f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_{1} dh_{2},$$
(29)

We now turn our attention to  $\mathcal{Z}(\varphi_1, \varphi_2, f; s)$ . For this purpose, we need to first simplify  $\omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$ , when  $g = \begin{pmatrix} y & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ , and  $h^2 = \text{identity}$ , say. We have

$$\omega\begin{pmatrix} y \\ 1 \\ y \end{pmatrix}, h_1 \begin{pmatrix} y \\ 1 \end{pmatrix}, h_2) f\begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$$

$$= \omega\begin{pmatrix} y \\ 1 \\ 1 \\ y \end{pmatrix} \begin{pmatrix} 1 \\ y^{-1} \\ y^{-1} \end{pmatrix} L(h_1 \begin{pmatrix} y \\ 1 \end{pmatrix}, h_2) f\begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$$

$$= |y|^2 L(h_1 \begin{pmatrix} y \\ 1 \end{pmatrix}, h_2) f\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$$

$$= f\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_1^{-1} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} h_2, \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} h_2)$$

$$= f\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_1^{-1} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h_2, \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_1^{-1} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} h_2).$$

Hence, for the choices of g,  $h^1$ , and  $h^2$  as above, we have

$$\begin{split} \omega(g,h_1h^1,h_2h^2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \\ L(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}h_1\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix},h_2)f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}). \end{split}$$

This equation combined with equation (29) gives

$$\theta(\varphi_1, \varphi_2; f)^U \begin{pmatrix} y & & \\ & 1 & \\ & & 1 \end{pmatrix} = \int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1 \begin{pmatrix} y & \\ & 1 \end{pmatrix}) \varphi_2(h_2)$$

$$L(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_1 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 dh_2.$$

Next, we make a change of variables

$$(h_1, h_2) \mapsto \begin{pmatrix} y \\ 1 \end{pmatrix} h_1 \begin{pmatrix} y^{-1} \\ 1 \end{pmatrix}, h_2$$

to obtain

$$\theta(\varphi_1, \varphi_2; f)^U \begin{pmatrix} y & & \\ & 1 & \\ & & 1 \end{pmatrix} = \int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1 \begin{pmatrix} y & \\ & 1 \end{pmatrix} h_1 \varphi_2(h_2) L(h_1, h_2) f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 dh_2.$$

Next,

$$\begin{split} \mathcal{Z}(\varphi_1,\varphi_2,f;\mu) \\ &= \int_{F^\times \backslash \mathbb{A}^\times} \theta(\varphi_1,\varphi_2;f)^U \begin{pmatrix} y & & \\ & 1 & \\ & & 1 \\ & & y \end{pmatrix}) \mu(y)|y|^{-\frac{1}{2}} \, d^\times y \\ &= \int_{F^\times \backslash \mathbb{A}^\times} \int_{D(F)\backslash H_1(\mathbb{A})} \varphi_1 \begin{pmatrix} y & \\ & 1 \end{pmatrix} h_1) \varphi_2(h_2) \\ &L(h_1,h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \mu(y)|y|^{-\frac{1}{2}} \, dh_1 \, dh_2 \, d^\times y. \end{split}$$

At this stage, we use the obvious isomorphism

$$F^{\times} \backslash \mathbb{A}^{\times} \longrightarrow D(F) \backslash D(\mathbb{A}),$$

given by

$$a \mapsto \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ 1 \end{pmatrix}$$

to obtain

$$\begin{split} &\mathcal{Z}(\varphi_{1},\varphi_{2},f;\mu) \\ &= \int_{F^{\times}\backslash\mathbb{A}^{\times}} \int_{D(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F^{\times}\backslash\mathbb{A}^{\times}} \varphi_{1}\begin{pmatrix} y \\ 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} h_{1}) \varphi_{2}\begin{pmatrix} a \\ 1 \end{pmatrix} h_{2}) \\ &L\begin{pmatrix} a \\ 1 \end{pmatrix} h_{1}, \begin{pmatrix} y \\ 1 \end{pmatrix} h_{2}) f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \mu(y) |y|^{-\frac{1}{2}} d^{\times} a \, dh_{1} \, dh_{2} \, d^{\times} y \\ &= \int_{F^{\times}\backslash\mathbb{A}^{\times}} \int_{D(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F^{\times}\backslash\mathbb{A}^{\times}} \varphi_{1}\begin{pmatrix} ya \\ 1 \end{pmatrix} h_{1}) \varphi_{2}\begin{pmatrix} a \\ 1 \end{pmatrix} h_{2}) \\ &L(h_{1}, h_{2}) f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \mu(y) |y|^{-\frac{1}{2}} d^{\times} a \, dh_{1} \, dh_{2} \, d^{\times} y \\ &= \int_{F^{\times}\backslash\mathbb{A}^{\times}} \int_{D(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F^{\times}\backslash\mathbb{A}^{\times}} \varphi_{1}\begin{pmatrix} y \\ 1 \end{pmatrix} h_{1}) \varphi_{2}\begin{pmatrix} a \\ 1 \end{pmatrix} h_{2}) \\ &L(h_{1}, h_{2}) f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \mu(y) |y|^{-\frac{1}{2}} \mu^{-1}(a) |a|^{\frac{1}{2}} d^{\times} a \, dh_{1} \, dh_{2} \, d^{\times} y, \end{split}$$

after a change of variable  $y \mapsto ya^{-1}$ . The proposition now follows from a simple re-arrangement of the last expression.  $\square$ 

## 3.1 The zeta integral of two complex variables; Euler product

In order to study the zeta integral  $\mathcal{Z}(\varphi_1, \varphi_2, f; \mu)$ , we would have liked to introduce a function of two complex variables  $s_1$ ,  $s_2$  as follows: For  $\varphi_1$ ,  $\varphi_2$ , and f as above, and  $\mu$  Hecke character, we set

$$Z(\varphi_1, \varphi_2, f; \mu, |.|^{s_1}, |.|^{s_2}) = \int_{D(\mathbb{A})\backslash H_1(\mathbb{A})} Z(\varphi_1, h_1, \mu |.|^{s_1}) Z(\varphi_2, h_2, \mu^{-1} |.|^{s_2})$$
$$L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_1 dh_2,$$

with  $s_1, s_2 \in \mathbb{C}$ . Unfortunately, however, this integral is not well-defined for  $s_2 \neq 1 - s_1$ . In order to circumvent this problem we proceed as follows.

If  $\phi$  is a cusp form on  $GL_2(\mathbb{A}_F)$ , we define its Whittaker function by

$$W_{\phi}(g) = \int_{F \setminus \mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \psi(x)^{-1} dx,$$

for  $g \in GL_2(\mathbb{A}_F)$ . Then, we have the Fourier expansion

$$\phi(g) = \sum_{\alpha \in F^{\times}} W_{\phi}(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g),$$

with the right hand side a uniformly convergent series on compact sets in  $GL_2(A)$ . It is then a classical observation of [7] that for  $\Re s$  large, we have

$$Z(\phi, h, \mu|.|^s) = \int_{\mathbb{A}} W_{\phi}(\begin{pmatrix} a \\ 1 \end{pmatrix} h) \mu(a) |a|^{s-\frac{1}{2}} d^{\times} a.$$

We denote the right hand side of this equation by  $Z(W_{\phi}, h, s)$ .

We have a formal identity as follows:

$$Z(\varphi_{1}, \varphi_{2}, f; \mu, |.|^{s_{1}}, |.|^{s_{2}}) = \int_{D(\mathbb{A})\backslash H_{1}(\mathbb{A})} Z(W_{\varphi_{1}}, h_{1}, \mu |.|^{s_{1}}) Z(W_{\varphi_{2}}, h_{2}, \mu^{-1} |.|^{s_{2}})$$
$$L(h_{1}, h_{2}) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_{1} dh_{2}.$$

Next, we consider the Euler product. We choose  $\varphi_i$ , for i = 1, 2, so that

$$W_{\varphi_i} = \otimes_{v \in \mathcal{M}_F} W_v^i.$$

Also, we choose f to be a pure tensor of the form

$$\otimes_{v \in \mathcal{M}_F} f_v$$

with  $f_v$  unramified for almost all v.

With this choice of the data, we have yet another formal identity

$$\mathcal{Z}(\varphi_1, \varphi_2, f; \mu, |.|^{s_1}, |.|^{s_2}) = \prod_{v \in \mathcal{M}_F} \mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |.|_v^{s_1}, |.|_v^{s_2}).$$
(30)

Here, we have set

$$\mathcal{Z}_{v}(W_{v}^{1}, W_{v}^{2}, f_{v}; \mu_{v}, |.|^{s_{1}}, |.|^{s_{2}}) = \int_{D(F_{v})\backslash H_{1}(F_{v})} Z(W_{v}^{1}, h_{1}, \mu_{v}|.|^{s_{1}}_{v}) Z(W_{v}^{2}, h_{2}, \mu_{v}^{-1}|.|^{s_{2}}_{v})$$

$$L(h_{1}, h_{2}) f_{v}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) dh_{1} dh_{2}.$$

Also, for  $W_v$  a Whittaker function on a local group  $GL_2(F_v)$ , and  $h \in GL_2(F_v)$ , we have used the notation  $Z(W_v, h, \mu_v)$  to denote

$$\int_{F_v^\times} W_v(\begin{pmatrix} a \\ 1 \end{pmatrix} h) \mu_v(a) |a|^{-\frac{1}{2}} d^{\times} a.$$

The idea is to make sense out of the expression for

$$\mathcal{Z}_v(W_v^1, W_v^2, f_v; \mu_v, |.|^{s_1}, |.|^{s_2})$$

for  $\Re s_1, \Re s_2$  large. For this we use the following lemma:

**Lemma 3.6** Let  $v \in \mathcal{M}_F$ , and  $\Psi$  a continuous function of compact support on  $D(F_v)\backslash H_1(F_v)$ . Choose an arbitrary lift  $\Phi'$  of  $\Phi$  to  $GL_2(F_v) \times GL_2(F_v)$ . The functional  $\mu(\Phi)$  defined by

$$\int_{K_v} \int_{F_v^2} \int_{F_v^{\times}} \Phi'(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} k_2) |\epsilon|^{-1} d^{\times} \epsilon du dv dk_1 dk_2,$$

for an appropriate choice of a local maximal compact (and open for v non-archimedean), defines an invariant measure on  $D(F_v)\backslash H_1(F_v)$ . Furthermore, this measure has the following property: Fix a Haar measure  $\mu_D$  on  $D(F_v)$ , and for any continuous function of compact support  $\Psi$  on  $H_1(F_v)$ , set

$$\Psi_D(x) = \int_{D(F_n)} \Psi(yx) \, d\mu_1(y),$$

for  $x \in D(F_v) \backslash H_1(F_v)$ . Then the functional  $\mu_2$  defined by

$$\mu_2(\Psi) = \mu(\Psi_D),$$

with  $\Psi$  as above defines a Haar measure on  $H_1(F_v)$ .

**Definition 3.7** We set

$$\begin{split} & \mathcal{Z}_{v}(W_{v}^{1}, W_{v}^{2}, f_{v}; \mu_{v}, |.|^{s_{1}}, |.|^{s_{2}}) \\ & = \int_{u,v \in F_{v}} \int_{\epsilon \in F_{v}^{\times}} \int_{K_{v}^{2}} f(k_{1}^{-1} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix} k_{2}, k_{1}^{-1} \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix} k_{2}) \\ & \omega_{\pi_{2}}(\epsilon) |\epsilon|^{2s_{2}-2} (\int_{F_{v}^{\times}} W_{1}(\begin{pmatrix} \alpha \\ 1 \end{pmatrix} k_{1}) \mathbf{e}(u\alpha) \mu(\alpha) |\alpha|^{s_{1}-\frac{1}{2}} d^{\times}\alpha) \\ & (\int_{F_{v}^{\times}} W_{2}(\begin{pmatrix} \beta \\ 1 \end{pmatrix} k_{2}) \mathbf{e}(v\beta) \mu^{-1}(\beta) |\beta|^{s_{2}-\frac{1}{2}} d^{\times}\beta) du dv d^{\times}\epsilon dk_{1} dk_{2}. \end{split}$$

We immediately observe that if the integral is convergent, it is well-defined.

**Proposition 3.8** Suppose  $W_1, W_2$  are two Whittaker functions of  $GL_2(F_v)$  belonging to the spaces of representations  $\pi_1, \pi_2$ , respectively, with  $\omega_{\pi_1}.\omega_{\pi_2} = 1$ . Then the integral  $\mathcal{Z}(W_1, W_2, f; \mu_v, |.|_v^{s_1}, |.|_v^{s_2})$  converges absolutely for  $\Re s_1, \Re s_2 \gg 0$ .

*Proof.* We give a complete proof only for the case where v is a real place, the proof of the non-archimedean statement being identical. Also it is clear that we may assume that the quasi-character  $\mu_v$  is trivial. By definition, we need to show that the integral

$$\int_{u,v\in\mathbb{R}} \int_{\epsilon\in\mathbb{R}_{+}^{\times}} \int_{K_{v}^{2}} f(k_{1}^{-1} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix}) k_{2}, k_{1}^{-1} \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix}) k_{2} dk_{2} dk_{2$$

converges absolutely. By lemma 8.3.3 of [8], there are gauge functions  $\xi_1, \xi_2$  such that

$$|W_1| \le \xi_1$$
, and  $|W_2| \le \xi_2$ .

This implies that

$$\int_{\mathbb{R}^{\times}} |W_1(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} k_1) \mathbf{e}(u\alpha) |\alpha|^{s_1 - \frac{1}{2}} |d^{\times} \alpha \leq \int_{\mathbb{R}^{\times}} \xi_1(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}) |\alpha|^{\sigma_1 - \frac{1}{2}} d^{\times} \alpha,$$

and

$$\int_{\mathbb{R}^{\times}} |W_2(\begin{pmatrix} \beta & \\ & 1 \end{pmatrix} k_2) \mathbf{e}(v\beta) |\beta|^{s_2 - \frac{1}{2}} |d^{\times}\beta| \leq \int_{\mathbb{R}^{\times}} \xi_2(\begin{pmatrix} \beta & \\ & 1 \end{pmatrix}) |\beta|^{\sigma_2 - \frac{1}{2}} d^{\times}\beta.$$

The latter integrals converge absolutely for  $\sigma_1, \sigma_2$  large. In order to conclude the proof, we need to study the convergence of

$$\int_{u,v\in\mathbb{R}} \int_{\epsilon\in\mathbb{R}_{+}^{\times}} \int_{K_{v}^{2}} f\left(k_{1}^{-1} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix} k_{2}, k_{1}^{-1} \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix} k_{2}\right)$$
$$\omega_{\pi_{2}}(\epsilon)|\epsilon|^{2s_{2}-2} du dv d^{\times}\epsilon dk_{1} dk_{2}.$$

We claim that this integral converges absolutely for all values of  $s_2$ . In fact, if  $f \in \mathcal{S}(M_2(\mathbb{R}) \times M_2(\mathbb{R}))$ , the function g defined by

$$g(X,Y) = \int_{K_n^2} f(k_1^{-1}Xk_2, k_1^{-1}Yk_2) dk_1 dk_2$$

is in  $\mathcal{S}(M_2(\mathbb{R}) \times M_2(\mathbb{R}))$ . Thus, we must show that

$$\int_{u,v\in\mathbb{R}} \int_{\epsilon\in\mathbb{R}_{+}^{\times}} f\begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix}) \omega_{\pi_{2}}(\epsilon) |\epsilon|^{2s_{2}-2} du dv d^{\times}\epsilon$$

converges absolutely for all  $s_2$ . The first observation, due to Weil, is that the absolute value of a Schwartz-Bruhat function is bounded by a Schwartz-Bruhat function. Consequently, we can assume that f is a positive Schwartz-Bruhat function. But now it is clear that the function  $\Xi$  defined by

$$\Xi(\epsilon) = \int_{u,v \in \mathbb{R}} f(\begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix}) du dv$$

is in the space  $\mathcal{S}(\mathbb{R}^{\times})$ . Since our original integral is bounded by

$$\int_{\mathbb{R}} \Xi(\epsilon) \omega_{\pi_2}(\epsilon) |\epsilon|^{2\sigma_2 - 2} d^{\times} \epsilon,$$

the proposition is immediate.  $\Box$ 

Then we have the following proposition:

**Proposition 3.9** Let v be a non-archimedean place. Let  $W_1$  and  $W_2$  be given. Then there is a choice of f such that

$$\mathcal{Z}(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2}) = Z(W_1, \mu|.|_v^{s_1}) Z(W_2, \mu^{-1}|.|_v^{s_2}).$$

*Proof.* Let M be a very large positive integer. Let  $f=g\otimes h$  be a Schwartz function such that

Support 
$$g \subset \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathfrak{p}^M & \mathfrak{p}^M \\ \mathfrak{p}^M & \mathfrak{p}^M \end{pmatrix}$$
,

and

Support 
$$h \subset \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \mathfrak{p}^M & \mathfrak{p}^M \\ \mathfrak{p}^M & \mathfrak{p}^M \end{pmatrix}$$
.

Then upon setting,

$$h_1 = \begin{pmatrix} 1 & -u \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1},$$

$$h_2 = \begin{pmatrix} \epsilon \\ \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$f\left(\begin{pmatrix} \alpha\epsilon(a+vc) & \alpha\epsilon(b+vd) \\ \gamma\epsilon(a+vc) & \gamma\epsilon(b+vd) \end{pmatrix}, \begin{pmatrix} c\epsilon^{-1}(\alpha u+\beta) & d\epsilon^{-1}(\alpha u+\beta) \\ c\epsilon^{-1}(\gamma u+\delta) & d\epsilon^{-1}(\gamma u+\delta) \end{pmatrix}\right) \neq 0.$$

With the choice of f, it is not hard to draw the following conclusions:

- 1.  $\gamma, c \in \mathfrak{p}^M$ ,
- 2. u, v are integral,
- 3.  $\epsilon$  is a unit,
- 4. b + vd,  $\alpha u + \beta \in \mathfrak{p}^M$ .
- 5.  $\alpha \epsilon a, d\epsilon^{-1}\delta \in 1 + \mathfrak{p}^M$ .

Next,

$$Z(W_1, h_1, \mu_1|.|_v^{s_1}) = \int_{\mathbb{Q}_v^\times} W_1\left(\begin{pmatrix} x & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & -u \\ & 1\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}\right) \mu(x)|x|^{s_1 - \frac{1}{2}} d^\times x;$$

but

$$\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} \\ & \alpha(\alpha\delta - \beta\gamma)^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & -(\beta + u\alpha)\alpha(\alpha\delta - \beta\gamma)^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\alpha^{-1}\gamma & 1 \end{pmatrix},$$

implying that for M large, we have

$$Z(W_1, h_1, \mu | . |_v^{s_1}) = \int_{\mathbb{Q}_v^{\times}} W_1 \left( \begin{pmatrix} x \alpha^{-1} \\ \alpha (\alpha \delta - \beta \gamma)^{-1} \end{pmatrix} \right) \mu_1(x) |x|^{s_1 - \frac{1}{2}} d^{\times} x$$
$$= (\omega_{\pi_1} \mu) (\alpha^2 (\alpha \delta - \beta \gamma)^{-1}) Z(W_1, \mu | . |_v^{s_1}).$$

Similarly, for M large,

$$Z(W_1, h_2, \mu^{-1}|.|_v^{s_2}) = \mu^{-1}(\epsilon^{-1}d(ad - bc)^{-1})(\omega_{\pi_2}\mu^{-1})(\epsilon^{-1}d)Z(W_2, \mu^{-1}|.|_v^{s_2}).$$

The proposition is now immediate.  $\square$ 

Corollary 3.10 There is a choice of  $W_1, W_2, f$  such that

$$\mathcal{Z}(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2}) \equiv 1.$$

When  $W_1, W_2$  are spherical, the situation is particularly nice:

**Proposition 3.11** Suppose v is a non-archimedean place, and  $\pi_1, \pi_2$  are spherical representations of  $GL_2(F_v)$  with  $\omega_{\pi_1}.\omega_{\pi_2} = 1$ . Also, suppose that  $W_i \in \mathcal{W}(\pi_i, \psi)$ , i = 1, 2, is the normalized  $K_v$ -fixed vector. Furthermore, let f be the characteristic function of  $M_2(\mathcal{O}_v) \times M_2(\mathcal{O}_v)$ . Then for unramified quasi-character  $\mu$  we have

$$\mathcal{Z}(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2}) = L_v(s_1, \pi_1, \mu) L(s_2, \pi_2, \mu^{-1}).$$

*Proof.* In order to see this, we need to verify that if

$$L(h_1, h_2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \neq 0,$$

for  $(h_1, h_2) \in H_1(F_v)$ , we must have  $(h_1, h_2) \in D(F_v)(\mathrm{GL}_2(\mathcal{O}_v) \times \mathrm{GL}_2(\mathcal{O}_v))$ . For this, we start by the observation that one can take as a set  $\mathcal{R}$  of representatives for

$$D(F_v)\backslash H_1(F_v)/(\mathrm{GL}_2(\mathcal{O}_v)\times\mathrm{GL}_2(\mathcal{O}_v)),$$

the set of pairs of the form

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \begin{pmatrix} \epsilon \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} \end{pmatrix}.$$

Hence, we need to verify our claim only for elements  $(h_1, h_2)$  of the above form. We have

$$L(h_1, h_2) f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = f\begin{pmatrix} \epsilon & \epsilon v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -u\epsilon^{-1} \\ 0 & \epsilon^{-1} \end{pmatrix}).$$

Since f is the characteristic function of  $\mathsf{M}_2(\mathcal{O}_v) \times \mathsf{M}_v(\mathcal{O}_v)$ , for this last expression to be non-zero, we must have  $\epsilon^{\pm 1} \in \mathcal{O}_v$ ,  $\epsilon v \in \mathcal{O}_v$ , and  $\epsilon^{-1}u \in \mathcal{O}_v$ . This in turn implies that  $\epsilon \in \mathcal{O}_v^{\times}$ , and  $u, v \in \mathcal{O}_v$ . Now an application of lemma 3.6 gives the result.  $\square$ 

We can now proceed to collect information about the analytic properties of our two variable zeta function. we prove the following proposition:

**Proposition 3.12** For  $W_1$ ,  $W_2$  Whittaker functions, and f as above, the function  $\mathcal{Z}(W_1, W_2, f; \mu, |.|^{s_1}, |.|^{s_2})$  has an analytic continuation to a meromorphic function on  $\mathbb{C}^2$ . Furthermore, the ratio

$$\Psi(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2}) = \frac{\mathcal{Z}(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2})}{L(s_1, \pi_1, \mu)L(s_2, \pi_2, \mu^{-1})}$$

extends to an entire function on the entire  $\mathbb{C}^2$ . There is a choice of  $W_1$ ,  $W_2$ , and f such that the above ratio is a nowhere vanishing entire function.

*Proof.* We prove only the analyticity statement; the non-vanishing follows from proposition 3.9 and the corresponding GL(2) statement. We write out the details for the archimedean place. For simplicity, we will assume that  $\pi_1$  and  $\pi_2$  are irreducible principal series representations. Also we will assume that the quasi-character  $\mu$  is trivial. By lemma 3.6, we need to consider the integral

$$\int_{u,v\in\mathbb{R}} \int_{\epsilon\in\mathbb{R}_{+}^{\times}} \int_{K_{v}^{2}} f\left(k_{1}^{-1} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1}v \\ 0 & 0 \end{pmatrix} k_{2}, k_{1}^{-1} \begin{pmatrix} 0 & -u\epsilon \\ 0 & \epsilon \end{pmatrix} k_{2}\right) 
\omega_{\pi_{2}}(\epsilon)|\epsilon|^{2s_{2}-2} \left(\int_{\mathbb{R}^{\times}} W_{1}(\begin{pmatrix} \alpha \\ 1 \end{pmatrix} k_{1})\mathbf{e}(u\alpha)|\alpha|^{s_{1}-\frac{1}{2}} d^{\times}\alpha\right) 
\left(\int_{\mathbb{R}^{\times}} W_{2}(\begin{pmatrix} \beta \\ 1 \end{pmatrix} k_{2})\mathbf{e}(v\beta)|\beta|^{s_{2}-\frac{1}{2}} d^{\times}\beta\right) du dv d^{\times}\epsilon dk_{1} dk_{2}.$$
(31)

For this purpose, we use the description of the Whittaker model of a principal series representation from [7], page 101-102. Suppose  $\pi_1 = \pi(\mu_1, \mu_2)$ , and  $\pi_2 = \pi(\mu_3, \mu_4)$ . Then there is a Schwartz function  $P_i(x, y)$ , i = 1, 2, such that  $W_1 = W_{P_i}$  by the following recipe. Let

$$f_1(g) = (\mu_1 \nu^{\frac{1}{2}})(\det g) \int_{\mathbb{R}^{\times}} P_1[(0,1)\gamma g](\mu_1 \mu_2^{-1} \nu)(\gamma) d^{\times} \gamma,$$

and

$$f_2(g) = (\mu_3 \nu^{\frac{1}{2}})(\det g) \int_{\mathbb{R}^{\times}} P_2[(0,1)\delta g](\mu_3 \mu_4^{-1} \nu)(\delta) d^{\times} \delta,$$

when the integrals converge. Next, we set for i = 1, 2

$$W_{P_i}(g) = \int_{\mathbb{R}} f_{P_i}(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 \end{pmatrix} g) \mathbf{e}(x) dx.$$

In particular,

$$W_{P_1}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} k_1 = \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} (\mu_1 v^{\frac{1}{2}})(\alpha) (\mu_1 \mu_2^{-1} \nu)(\gamma) P_1((-\alpha \gamma, -x \gamma) k_1) \mathbf{e}(x) \, dx \, d^{\times} \gamma,$$

and

$$W_{P_2}\begin{pmatrix} \beta \\ 1 \end{pmatrix} k_2 = \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} (\mu_3 v^{\frac{1}{2}})(\beta) (\mu_3 \mu_4^{-1} \nu)(\delta) P_2((-\beta \delta, -y \delta) k_2) \mathbf{e}(y) \, dy \, d^{\times} \delta.$$

These integrals may not converge, but they have analytic continuations to entire functions of the characters  $\mu_i$ , i = 1, ..., 4.

We need a lemma/notation:

**Lemma 3.13** Suppose  $P_1$ ,  $P_2$ , and f are Schwartz-Bruhat functions as above. Then the function  $\Gamma$  whose value at

$$(X, Y, m, n, p, q) \in \mathsf{M}_2(\mathbb{R}) \times \mathsf{M}_2(\mathbb{R}) \times \mathbb{R}^4$$

is given by

$$\Gamma(X,Y,m,n,p,q) = \int_{K^2} f(k_1^{-1}Xk_2, k_1^{-1}Yk_2) P_1((m,n)k_1) P_2((p,q)k_2) dk_1 dk_2$$

is a Schwartz-Bruhat function.

The integral (31) is now equal to

$$\int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} \int_{u \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{v$$

$$= \int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} \int_{u \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{$$

We will abbreviate the inner  $\Gamma$ -expression appearing above to

$$\Gamma(\epsilon^{-1}, \epsilon^{-1}v, -u\epsilon, \epsilon, -\alpha\gamma, -x\gamma, -\beta\delta, -y\delta).$$

Next we consider the integral

$$\int_{u \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \Gamma(\epsilon^{-1}, \epsilon^{-1}v, -u\epsilon, \epsilon, -\alpha\gamma, -x\gamma, -\beta\delta, -y\delta) 
\mathbf{e}(x)\mathbf{e}(y)\mathbf{e}(u\alpha)\mathbf{e}(v\beta) \, dy \, dx \, dv \, du 
= |\gamma|^{-1} |\delta|^{-1} \int_{u \in \mathbb{R}} \int_{v \in \mathbb{R}} \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \Gamma(\epsilon^{-1}, v, u, \epsilon, -\alpha\gamma, x, -\beta\delta, y) 
\mathbf{e}(-\frac{x}{\gamma})\mathbf{e}(-\frac{y}{\delta})\mathbf{e}(-u\frac{\alpha}{\epsilon})\mathbf{e}(v\beta\epsilon) \, dy \, dx \, dv \, du 
= |\gamma|^{-1} |\delta|^{-1} \widetilde{\Gamma}(\epsilon^{-1}, -\beta\epsilon, \alpha\epsilon^{-1}, \epsilon, -\alpha\gamma, \gamma^{-1}, -\beta\delta, \delta^{-1}),$$

where  $\widetilde{\Gamma}$  is the appropriate Fourier transform of  $\Gamma$ .

Going back to (32), we obtain

$$\int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} |\gamma|^{-1} |\delta|^{-1} \widetilde{\Gamma}(\epsilon^{-1}, -\beta \epsilon, \alpha \epsilon^{-1}, \epsilon, -\alpha \gamma, \gamma^{-1}, -\beta \delta, \delta^{-1}) 
\omega_{\pi_{2}}(\epsilon) |\epsilon|^{2s_{2}-2} |\alpha|^{s_{1}} |\beta|^{s_{2}} \mu_{1}(\alpha) (\mu_{1} \mu_{2}^{-1} \nu)(\gamma) \mu_{3}(\beta) (\mu_{3} \mu_{4}^{-1} \nu)(\delta) 
d^{\times} \epsilon d^{\times} \delta d^{\times} \gamma d^{\times} \beta d^{\times} \alpha.$$

$$= \int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} \widetilde{\Gamma}(\epsilon^{-1}, -\beta \epsilon, \alpha \epsilon^{-1}, \epsilon, -\alpha \gamma^{-1}, \gamma, -\beta \delta^{-1}, \delta) 
\omega_{\pi_{2}}(\epsilon) |\epsilon|^{2s_{2}-2} |\alpha|^{s_{1}} |\beta|^{s_{2}} \mu_{1}(\alpha) (\mu_{1} \mu_{2}^{-1})(\gamma^{-1}) \mu_{3}(\beta) (\mu_{3} \mu_{4}^{-1})(\delta^{-1}) 
d^{\times} \epsilon d^{\times} \delta d^{\times} \gamma d^{\times} \beta d^{\times} \alpha.$$

$$= \int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} \widetilde{\Gamma}(\epsilon^{-1}, -\beta \delta \epsilon, \alpha \gamma \epsilon^{-1}, \epsilon, -\alpha, \gamma, -\beta, \delta) 
\omega_{\pi_{2}}(\epsilon) |\epsilon|^{2s_{2}-2} |\alpha|^{s_{1}} |\gamma|^{s_{1}} |\beta|^{s_{2}} |\delta|^{s_{2}} \mu_{1}(\alpha) \mu_{2}(\gamma) \mu_{3}(\beta) \mu_{4}(\delta) 
d^{\times} \epsilon d^{\times} \delta d^{\times} \gamma d^{\times} \beta d^{\times} \alpha$$

$$= \int_{\alpha \in \mathbb{R}^{\times}} \int_{\beta \in \mathbb{R}^{\times}} \int_{\gamma \in \mathbb{R}^{\times}} \int_{\delta \in \mathbb{R}^{\times}} \int_{\epsilon \in \mathbb{R}^{\times}_{+}} \widetilde{\Gamma}(\epsilon^{-1}, -\beta \delta \epsilon, \alpha \gamma \epsilon^{-1}, \epsilon, -\alpha, \gamma, -\beta, \delta) 
(\mu_{1} \nu^{s_{1}}) (\alpha) (\mu_{2} \nu^{s_{1}}) (\gamma) (\mu_{3} \nu^{s_{2}}) (\beta) (\mu_{4} \nu^{s_{2}}) (\delta) (\omega_{\pi_{2}} \nu^{2s_{2}-2}) (\epsilon) 
d^{\times} \epsilon d^{\times} \delta d^{\times} \gamma d^{\times} \beta d^{\times} \alpha$$
(33)

after obvious changes of variables, and simple re-arrangement of terms. Our result now follows from the following standard lemma:

**Lemma 3.14** Let  $\Phi$  be a Schwartz-Bruhat function on  $\mathbb{R}^n$ . Suppose  $\gamma_1, \ldots, \gamma_n$  are quasi-characters. Define the function  $Z(s_1, \ldots, s_n) = Z(\Phi; \gamma_1, \ldots, \gamma_n; s_1, \ldots, s_n)$  of the complex variables  $s_1, \ldots, s_n$  by

$$Z(s_1, \ldots, s_n) = \int_{(\mathbb{R}^{\times})^n} \Phi(\alpha_1, \ldots, \alpha_n) \prod_i \gamma_i(\alpha_i) |\alpha_i|^{s_i} d^{\times} \alpha_i,$$

whenever the integral converges. Then the integral converges for  $\Re s_i$  large enough, for i = 1, ..., n. The ratio

$$\frac{Z(\Phi; \gamma_1, \dots, \gamma_n; s_1, \dots, s_n)}{\prod_{i=1}^n L(s_i, \gamma_i)}$$

extends to an entire function. If  $\Phi \in \mathcal{S}(\mathbb{R}^{\times} \times \mathbb{R}^{n-1})$ , then the ratio

$$\frac{Z(\Phi; \gamma_1, \dots, \gamma_n; s_1, \dots, s_n)}{\prod_{i=2}^n L(s_i, \gamma_i)}$$

has an analytic continuation to an entire function.

Corollary 3.15 Let v be a non-archimedean place. Then in the above situation for  $\mu$  highly ramified  $\mathcal{Z}(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2})$  extends to an entire function of  $s_1, s_2$ .

Corollary 3.16 Let  $W_1, W_2$  be flat sections of Whittaker spaces as in the last section. Then the function  $\Psi(W_1, W_2, f; \mu, |.|_v^{s_1}, |.|_v^{s_2})$  is holomorphic in the parameters of  $W_1, W_2$ .

Summarizing,

**Proposition 3.17** Let the data be as above. Let S a finite collection of places containing the archimedean place such that for  $v \notin S$ , the local data at v is unramified. Then we have

$$\mathcal{Z}(\varphi_1, \varphi_2, \mu | . |^s) = L(s, \pi_1, \mu) L(1 - s, \pi_2, \mu^{-1})$$

$$\left\{ \prod_v \Psi(W_1, W_2, f; \mu_v, | . |^s_v, | . |^{1-s}_v) \right\}$$

where by lemmas 3.11 and 3.12 the expression in curly braces is a finite product and is entire.

## 4 The pull-back of the Whittaker function

In this section, we aim to relate the local Euler factor of the integral of Novodvorsky at the archimedean place to the corresponding Euler factor of the integral considered in Section 2.2. For this purpose, we start by studying the Whittaker function associated to  $\theta(\varphi_1, \varphi_2; f)$ , and from that we derive formulae for the corresponding local Whittaker functions.

### 4.1 The Whittaker function

In this section we compute the Whittaker function of a cuspidal function  $\theta(\varphi_1, \varphi_2; f)$ . Fix a non-trivial character  $\psi$  of  $F \setminus A$ . Define a character, again denoted by  $\psi$ , of the unipotent radical of the Borel subgroup of GSp(4) by the following

$$\psi\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(v+t).$$

Then we set

$$W(g) = \int_{N(F)\backslash N(\mathbb{A})} \theta(\varphi_1, \varphi_2; f)(ng)\psi^{-1}(n) dn.$$

The  $h^1$  and  $h^2$  above can be taken to be  $\begin{pmatrix} v(g) \\ 1 \end{pmatrix}$  and the identity matrix, respectively. Then we have

**Theorem 4.1** If  $\tilde{\pi}_1 \neq \bar{\pi}_2$ , we have

$$W(g) = \int_{\hat{N}(\mathbb{A})\backslash H_1(\mathbb{A})} W_1(\epsilon h_1 h^1) W_2(h_2 h^2)$$
$$\omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I_{2\times 2}) dh_1 dh_2,$$

where

$$\hat{N} = \{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} ) \mid x \in \mathbb{G}_a \}.$$

*Proof.* We start by

$$W(g) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \varphi_1(h_1 h^1) \varphi_2(h_2 h^2)$$

$$(\sum_{M_1, M_2} \int_{N(F)\backslash N(\mathbb{A})} \omega(ng; h_1 h^1, h_2 h^2) f(M_1, M_2) \psi^{-1}(n) dn)$$

$$d(h_1, h_2).$$

Therefore, we have to study the expression

$$I(M_1, M_2) = \int_{N(F) \setminus N(\mathbb{A})} \omega(ng; h_1 h^1, h_2 h^2) f(M_1, M_2) \psi^{-1}(n) dn.$$

For this we have

But the inner most integral

$$\int_{(F \setminus \mathbb{A})^3} \psi(tr(\begin{pmatrix} s & r \\ r & t \end{pmatrix} \begin{pmatrix} \det M_1 & B(M_1, M_2) \\ B(M_2, M_1) & \det M_2 - 1 \end{pmatrix}) dr ds dt = 0$$

unless det  $M_1 = 0$ , det  $M_2 = 1$ , and  $B(M_1, M_2) = 0$ , in which case it is equal to 1.

**Lemma 4.2** Under the action of  $H_1(F)$ , the set S consisting of the pairs of matrices  $(M_1, M_2)$  satisfying the conditions just mentioned is the union of the following two orbits:

- 1. The orbit of (O, I). The stabilizer of this element is the diagonal embedding of PGL(2) into  $H_1$ .
- 2. The orbit of  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ). The stabilizer of this element is the subgroup  $\tilde{N}$  of  $H_1$  consisting of pairs of matrices of the form

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, w \begin{pmatrix} 1 & x \\ 1 & \end{pmatrix} w^{-1}$$
.

*Proof.* Since det  $M_1 = 0$ , there are two cases to be considered:

- 1.  $M_1 = 0$ ,
- 2.  $M_1 \neq 0$ .

It's obvious that the first case corresponds to the first orbit in the statement of the lemma. Also the statement regarding the stabilizer is immediate. Next we consider the case when  $M_1 \neq 0$ . It is clear that under the action of  $H_1$ ,  $M_1$  is equivalent to the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Next suppose  $M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Since  $B(M_1, M_2) = 0$  and  $\det M_1 = 0$ , we obtain that  $\det(M_1 + M_2) = 1$ . This then implies that d = 0. But then since  $\det M_2 = 1$ , we obtain  $c = -b^{-1}$ . Hence  $M_2 = \begin{pmatrix} a & b \\ -b^{-1} \end{pmatrix}$ . Next consider the element

$$h = (\begin{pmatrix} 1 & b^{-1} \end{pmatrix} \begin{pmatrix} b^{-1} & a \\ b \end{pmatrix}, \begin{pmatrix} b^{-1} & b \end{pmatrix}) \in H_1(F).$$

Then it is easy to check that

$$h.(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}a&b\\-b^{-1}\end{pmatrix})=(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}&1\\-1&\end{pmatrix}).$$

The statement regarding the stabilizer is straightforward.  $\Box$  Next we study the contribution of each orbit to the Whittaker integral. Cor-

responding to the two orbits obtained above, we have the following two integrals:

$$I_{1}(g) = \int_{G(F)\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega\begin{pmatrix} 1 & v \\ & 1 \\ & & 1 \\ & -v & 1 \end{pmatrix} g, h_{1}h^{1}, h_{2}h^{2})$$
$$f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ & 1 \end{pmatrix})\varphi_{1}(h_{1}h^{1})\varphi_{2}(h_{2}h^{2})\psi^{-1}(v) dv d(h_{1}, h_{2}),$$

and

$$I_{2}(g) = \int_{\tilde{N}(F)\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega(\begin{pmatrix} 1 & v & \\ & 1 & \\ & & 1 \\ & -v & 1 \end{pmatrix} g, h_{1}h^{1}, h_{2}h^{2})$$
$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})\varphi_{1}(h_{1}h^{1})\varphi_{2}(h_{2}h^{2})\psi^{-1}(v) dv d(h_{1}, h_{2}).$$

Then it is clear that

$$W(g) = I_1(g) + I_2(g).$$

#### Lemma 4.3 We have

$$I_1(q) = 0,$$

except when  $\tilde{\pi}_1 = \bar{\pi}_2$ .

*Proof.* By [5], we have

$$I_{1}(g) = \int_{G(F)\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega\begin{pmatrix} 1 & v & \\ & 1 & \\ & & 1 \\ & -v & 1 \end{pmatrix} g \begin{pmatrix} I & \\ & \nu(g)^{-1}I \end{pmatrix})$$

$$L(h_{1}h^{1}, h_{2}h^{2})f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})\varphi_{1}(h_{1}h^{1})\varphi_{2}(h_{2}h^{2})$$

$$\psi^{-1}(v) dv d(h_{1}, h_{2})$$

$$= \int_{G(\mathbb{A})\backslash H_1(\mathbb{A})} \int_{PGL_2(F)\backslash PGL_2(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega(\begin{pmatrix} 1 & v \\ 1 & \\ & 1 \\ & -v & 1 \end{pmatrix}) g \begin{pmatrix} I & \\ & \nu(g)^{-1}I \end{pmatrix})$$

$$L(\gamma h_1 h^1, \gamma h_2 h^2) f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}) \varphi_1(\gamma h_1 h^1) \varphi_2(\gamma h_2 h^2)$$

$$\psi^{-1}(v) dv d\gamma d(h_1, h_2)$$

$$= \int_{G(\mathbb{A})\backslash H_1(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega(\begin{pmatrix} 1 & v \\ & 1 \\ & & 1 \\ & -v & 1 \end{pmatrix}) g \begin{pmatrix} I \\ & \nu(g)^{-1}I \end{pmatrix})$$

$$L(h_1h^1, h_2h^2) f(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ & 1 \end{pmatrix}) \psi^{-1}(v)$$

$$\left(\int_{PGL_2(F)\backslash PGL_2(\mathbb{A})} \varphi_1(\gamma h_1h^1) \varphi_2(\gamma h_2h^2) d\gamma \right) dv d(h_1, h_2).$$

The inner most integral

$$\int_{\mathrm{PGL}_{2}(F)\backslash\mathrm{PGL}_{2}(\mathbb{A})} \varphi_{1}(\gamma h_{1}h^{1})\varphi_{2}(\gamma h_{2}h^{2}) d\gamma =$$

$$<\pi_{1}(h_{1}h^{1})\varphi_{1}, \overline{\pi_{2}(h_{2}h^{2})\varphi_{2}}>_{L^{2}(\mathrm{PGL}_{2}(F)\backslash\mathrm{PGL}_{2}(\mathbb{A}))}.$$

The statement of the lemma is now obvious.  $\square$ 

Next we study the contribution of the second orbit.

#### Lemma 4.4 We have

$$I_{2}(g) = \int_{\hat{N}(\mathbb{A})\backslash H_{1}(\mathbb{A})} W_{\varphi_{1}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} h_{1} \begin{pmatrix} \nu(g) \\ 1 \end{pmatrix}) W_{\varphi_{2}}(h_{2})$$
$$\omega(g, h_{1} \begin{pmatrix} \nu(g) \\ 1 \end{pmatrix}, h_{2}) f\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I) d(h_{1}, h_{2}).$$

In this lemma,  $\hat{N}$  is the diagonal embedding of the unipotent upper triangular matrices in GL(2) in  $H_1$ . Also if  $\varphi$  is a cuspidal automorphic function on  $GL_2(\mathbb{A})$ , we have set

$$W_{\varphi}(g) = \int_{F \setminus \mathbb{A}} \varphi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \psi^{-1}(x) \, dx.$$

*Proof.* The proof consists of simple manipulations of the original expression for  $I_2(g)$ . We have

$$I_{2}(g) = \int_{\tilde{N}(F)\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega(\begin{pmatrix} 1 & v & \\ & 1 & \\ & & 1 \\ & -v & 1 \end{pmatrix} g, h_{1}h^{1}, h_{2}h^{2})$$
$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) \varphi_{1}(h_{1}h^{1}) \varphi_{2}(h_{2}h^{2}) \psi^{-1}(v) dv d(h_{1}, h_{2}).$$

We recall that  $\tilde{N}(F) = \{ \begin{pmatrix} 1 & x \\ 1 \end{pmatrix}, w \begin{pmatrix} 1 & x \\ 1 \end{pmatrix} w^{-1} \},$  and also that  $h^1 = \begin{pmatrix} \nu(g) \\ 1 \end{pmatrix}$  and  $h^2 = I$ . Using the formulae in [5], we have

$$\omega\begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & -v & 1 \end{pmatrix} g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) = \omega(g, h_1 h^1, h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ 1 & \end{pmatrix}) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}).$$

Hence

$$I_{2}(g) = \int_{\tilde{N}(F)\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \omega(g, h_{1}h^{1}, h_{2}h^{2}) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix}) \begin{pmatrix} 1 \\ -1 \end{pmatrix})$$

$$\varphi_{1}(h_{1}h^{1})\varphi_{2}(h_{2}h^{2})\psi^{-1}(v) dv d(h_{1}, h_{2})$$

$$= \int_{\tilde{N}(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \int_{F\backslash \mathbb{A}} \omega(g, \begin{pmatrix} 1 & u \\ 1 \end{pmatrix}) h_{1}h^{1}, w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} w^{-1}h_{2}h^{2})$$

$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}) \varphi_{1}(\begin{pmatrix} 1 & u \\ 1 \end{pmatrix}) h_{1}h^{1})\varphi_{2}(w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix}) w^{-1}h_{2}h^{2})$$

$$\psi^{-1}(v) du dv d(h_{1}, h_{2})$$

Next by definition and Lemma 5.1.2 of [5]

$$\omega(g, \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w^{-1} h_2 h^2)$$

$$= \omega \left(g \begin{pmatrix} I \\ \nu(g)^{-1}I \end{pmatrix}\right) L \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} w^{-1} h_2 h^2)$$
$$= L \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} w^{-1} h_2 h^2) \omega \begin{pmatrix} I \\ \nu(g)^{-1}I \end{pmatrix} g).$$

This identity implies that

$$\omega(g, \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} h_1 h^1, w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} w^{-1} h_2 h^2) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}))$$

$$= L(h_1 h^1, h_2 h^2) \omega(\begin{pmatrix} I \\ \nu(g)^{-1}I \end{pmatrix} g) f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 & 0 \end{pmatrix}, I)$$

$$= L(h_1 h^1, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} h_2 h^2) \omega(\begin{pmatrix} I \\ \nu(g)^{-1}I \end{pmatrix} g) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$

$$= \omega(g, h_1 h^1, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} h_2 h^2) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I).$$

Going back to  $I_2(g)$ , we obtain

$$I_{2}(g) = \int_{\tilde{N}(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \int_{F\backslash \mathbb{A}} \omega(g, h_{1}h^{1}, \begin{pmatrix} 1 & -v \\ 1 \end{pmatrix} w h_{2}h^{2}) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$
$$\varphi_{1}(\begin{pmatrix} 1 & u \\ 1 \end{pmatrix} h_{1}h^{1}) \varphi_{2}(w \begin{pmatrix} 1 & u \\ 1 \end{pmatrix} w^{-1}h_{2}h^{2}) \psi^{-1}(v) du dv d(h_{1}, h_{2})$$

Next we make a change of variables  $(h_1, h_2) \mapsto (h_1, w^{-1} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} h_2)$ , to obtain

$$I_{2}(g) = \int_{\hat{N}(\mathbb{A})\backslash H_{1}(\mathbb{A})} \int_{F\backslash \mathbb{A}} \int_{F\backslash \mathbb{A}} \omega(g, h_{1}h^{1}, h_{2}h^{2}) f\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I)$$
$$\varphi_{1}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} h_{1}h^{1}) \varphi_{2}\begin{pmatrix} 1 & u+v \\ & 1 \end{pmatrix} h_{2}h^{2}) \psi^{-1}(v) du dv d(h_{1}, h_{2}).$$

Now a change of variables  $v \mapsto v - u$  and re-arranging the order of integrals gives the result.  $\square$ 

Combining everything finishes the proof of the theorem.  $\Box$ 

#### 4.2 Local Whittaker functions

In this paragraph, we study the integrals of the previous section in some detail.

Suppose  $\pi_1$  and  $\pi_2$  are two irreducible admissible representations of the group GL(2) over a local field, such that  $\tilde{\pi}_1 \neq \pi_2, \bar{\pi}_2$ , and  $\omega_{\pi_1}.\omega_{\pi_2} = 1$ . For  $W_i \in \mathcal{W}(\pi_i, \psi)$ , for i = 1, 2, set

$$W_{v}(W_{1}, W_{2}; f)(g) = \int_{\hat{N}(F_{v})\backslash H_{1}(F_{v})} W_{1}(\epsilon h_{1} \begin{pmatrix} \nu(g) \\ 1 \end{pmatrix}) W_{2}(h_{2})$$

$$\omega(g, h_{1} \begin{pmatrix} \nu(g) \\ 1 \end{pmatrix}, h_{2}) f(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, I_{2\times 2}) dh_{1} dh_{2}.$$

**Proposition 4.5** For all  $W_i \in \mathcal{W}(\pi_i, \psi)$ , i = 1, 2, K-finite f in the space of Schwartz-Bruhat functions, and  $g \in \mathrm{GSp}_4(F_v)$ , the integral defining  $\mathbb{W}(W_1, W_2; f)(g)$  is absolutely convergent.

*Proof.* As usual we prove the proposition for the archimedean place. It is clear that we only need to prove the absolute convergence for  $g = I_{4\times 4}$ . In order to do this, we start by identifying a measurable set of representatives for  $\hat{N}(\mathbb{R})\backslash H_1(\mathbb{R})$ , and identifying the corresponding measure. On  $H_1(\mathbb{R})$ , we have the following natural set of representatives

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2 \end{pmatrix},$$

with  $x, y \in \mathbb{R}$ ,  $\epsilon \in \mathbb{R}^{\times}$ , and  $k_1, k_2 \in SO(2)$ . Also the corresponding measure is

$$|\eta|^{-2} dx dy d^{\times} \eta dk_1 dk_2.$$

This statement implies that the set of elements of the form

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1, \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} k_2 \end{pmatrix},$$

constitutes a measurable set of representatives for  $\hat{N}(\mathbb{R})\backslash H_1(\mathbb{R})$ . Also with this normalization the measure is

$$|\eta|^{-2} dx d^{\times} \eta dk_1 dk_2.$$

Hence we are reduced to proving the convergence of the following integral:

$$\int_{K} \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} \left| W_{1}(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_{1}) W_{2}(\begin{pmatrix} \eta \\ & \eta^{-1} \end{pmatrix} k_{2}) \right| .$$

$$\left| f(k_{1}^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ & \eta^{-1} \end{pmatrix} k_{2}, k_{1}^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} \eta \\ & \eta^{-1} \end{pmatrix} k_{2}) \right|$$

$$d^{\times} \eta \, dx \, dk_{1} \, dk_{2}.$$

Next we observe that in order to prove the absolute convergence of this integral, we just need to prove the absolute convergence of the integral over  $\eta$  and x. Also since

$$W_1(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1) = \psi(-x)W_1(\epsilon k_1),$$

we obtain

$$\left| W_1(\epsilon \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k_1) \right| = \left| W_1(\epsilon k_1) \right|.$$

Hence we are reduced to proving the convergence of the following integral:

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} \left| W_2(\begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}) \right|.$$

$$\left| f(\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}, \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}) \right| d^{\times} \eta dx.$$

But this integral is equal to

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} \left| \omega_{\pi_2}(\eta^{-1}) W_2(\begin{pmatrix} \eta^2 \\ 1 \end{pmatrix}) f(\begin{pmatrix} 0 & -\eta^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \eta & -x\eta^{-1} \\ 0 & \eta^{-1} \end{pmatrix}) \right| \, d^{\times} \eta \, dx$$

Now we write

$$f(\begin{pmatrix} 0 & -\eta^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \eta & -x\eta^{-1} \\ 0 & \eta^{-1} \end{pmatrix}) = q(\eta, \eta^{-1}, x\eta^{-1}),$$

where q is some Schwartz-Bruhat function in three variables. We then need to prove the convergence of the integral

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}^{\times}} \left| \omega_{\pi_2}(\eta^{-1}) W_2(\begin{pmatrix} \eta^2 \\ 1 \end{pmatrix}) q(\eta, \eta^{-1}, x \eta^{-1}) \right| d^{\times} \eta dx,$$

which after a change of variables  $x \mapsto x\eta$  and integration over x is equivalent to the convergence of an integral of the form

$$\int_{\mathbb{R}_{+}^{\times}} \left| W(\begin{pmatrix} \eta & \\ & 1 \end{pmatrix} \xi(\eta) \right| \eta^{\sigma} d^{\times} \eta$$

for  $\xi \in \mathcal{S}(\mathbb{R}^{\times})$ . Such an integral always converges by the moderate growth of the Whittaker function.  $\square$ 

Going back to the global situation, we choose  $\varphi_i$ , for i = 1, 2, so that

$$W_{\varphi_i} = \otimes_{v \in \mathcal{M}_F} W_v^i.$$

We also choose f to be a pure tensor of the form  $\otimes_v f_v$ . Then theorem 4.1 can be written in the form

$$W(g) = \prod_{v} W_{v}(W_{v}^{1}, W_{v}^{2}; f_{v})(g_{v}).$$

under appropriate conditions. This implies that for each local place v, if  $W_v$  is a  $K_v$ -finite vector in the local Whittaker model, there is a choice of the data such that  $W_v = W_v(W_v^1, W_v^2; f_v)$ . It is clear from the construction that, in the archimedean situation, the space of all such W's forms a  $(\mathfrak{g}, K)$ -module.

## 5 Archimedean Zeta function

In this section, we use the results of the previous paragraphs to obtain information about the archimedean zeta function. We have by lemma 3.1

$$B(\phi, \chi_s) = \int_{\mathbb{A}^{\times}} \int_{\mathbb{A}} W_{\phi} \begin{pmatrix} y & & \\ & y & \\ & & 1 \\ & x & & 1 \end{pmatrix} w^{-1} \mu(y)|y|^{s-\frac{3}{2}} dx d^{\times}y, \qquad (34)$$

with

$$w = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}$$

and

$$\chi_s(y) = \mu(y)|y|^{s-\frac{1}{2}}.$$

If we set  $\phi = \theta(\varphi_1, \varphi_2; f)$ , the left hand side of the above identity will be equal to what we have called  $\mathcal{Z}(\varphi_1, \varphi_2, f; \mu|.|^s)$ . We saw in 3.17 that

$$\mathcal{Z}(\varphi_1, \varphi_2, \mu | . |^s) = L(s, \pi_1, \mu) L(1 - s, \pi_2, \mu^{-1})$$

$$\left\{ \prod_v \Psi(W_1^v, W_2^v, f; \mu_v | . |^s_v, \mu_v^{-1} | . |^{1-s}_v) \right\}.$$

If we choose our vectors appropriately, that is factorizable, the right hand side of (34) is now equal to

$$\prod_{v} Z_{v,N}(s, \pi_{v}(w^{-1}) \mathbb{W}_{v}(W_{v}^{1}, W_{v}^{2}; f_{v}), \mu_{\infty})$$

$$= Z_{\infty,N}(s, \pi_{\infty}(w^{-1}) \mathbb{W}_{\infty}(W_{\infty}^{1}, W_{\infty}^{2}; f_{\infty}), \mu_{\infty})$$

$$\times L_{S}(s, \Pi, \mu) \prod_{v \in S \setminus \{\infty\}} Z_{v,N}(s, \pi_{v}(w^{-1}) \mathbb{W}_{v}(W_{v}^{1}, W_{v}^{2}; f_{v}), \mu_{v})$$

By the main result of [19], for each local place  $v \in S \setminus \{\infty\}$ , we can choose  $W_v^{sp} \in \mathcal{W}(\Pi_v)$  in such a way that

$$Z_{v,N}(s,\Pi_v^{-1}(w^{-1})W_v^{\text{sp}},\mu_v) = L_v(s,\Pi_v,\mu_v).$$

By the remark at the end of 4.2, we can choose the local data such that

$$\mathbb{W}_v(W_v^1, W_v^2; f_v) = W_v^{\mathrm{sp}}.$$

With this choice of the local data, we have

$$Z_{\infty,N}(s,\pi_{\infty}(w^{-1})\mathbb{W}_{\infty}(W_{\infty}^{1},W_{\infty}^{2};f_{\infty}),\mu_{\infty})$$

$$=\Phi_{S}^{\text{finite}}(\pi_{1},\pi_{2},\mu,s;W_{1},W_{2},f)L_{\infty}(s,\pi_{1},\mu)L_{\infty}(1-s,\pi_{2},\mu^{-1})$$

$$\times\Psi(W_{1}^{\infty},W_{2}^{\infty},f_{\infty};\mu_{\infty}|.|_{\infty}^{s},\mu_{\infty}^{-1}|.|_{\infty}^{1-s}),$$
(35)

with

$$\begin{split} \Phi_S^{\text{finite}}(\pi_1, \pi_2, \mu, s; W_1, W_2, f) \\ &= \frac{L^{\infty}(s, \pi_1, \mu) L^{\infty}(1 - s, \pi_2, \mu^{-1})}{L^{\infty}(s, \Pi, \mu)} \prod_{v \in S \backslash \{\infty\}} \Psi(W_1^v, W_2^v, f; \mu_v | \, . \, |_v^s, \mu_v^{-1} | \, . \, |_v^{1-s}) \\ &= \prod_{v \in S \backslash \{\infty\}} \Psi(W_1^v, W_2^v, f; \mu_v | \, . \, |_v^s, \mu_v^{-1} | \, . \, |_v^{1-s}), \end{split}$$

if  $\mu$  is chosen in such a way that for  $v \in S \setminus \{\infty\}$ , the local quasi-character  $\mu_v$  is highly ramified. Combining everything proves the first statement of the following theorem:

**Theorem 5.1** In the above situation, for each K-finite  $W \in \mathcal{W}(\Pi_{\infty})$ , the ratio

$$\frac{Z(s, W, \mu_{\infty})}{L_{\infty}(s, \pi_{1}^{\infty}, \mu_{\infty})L_{\infty}(s, \tilde{\pi}_{2}^{\infty}, \mu_{\infty})}$$

extends to an entire function of s. Furthermore, for each s, there is a choice of W such that the above expression does not vanish at s.

*Proof.* We only need to prove the second statement. In order to do this, we prove the existence of an entire function  $\Phi(s)$  such that

$$Z_{\infty,N}(s,\pi_{\infty}(w^{-1})\mathbb{W}_{\infty}(W_{\infty}^{1},W_{\infty}^{2};f_{\infty}),\mu_{\infty})$$

$$=\frac{1}{\Phi(s)}L_{\infty}(s,\pi_{1},\mu)L_{\infty}(1-s,\pi_{2},\mu^{-1})$$

$$\times\Psi(W_{1}^{\infty},W_{2}^{\infty},f_{\infty};\mu_{\infty}|.|_{\infty}^{s},\mu_{\infty}^{-1}|.|_{\infty}^{1-s}).$$
(36)

By proposition 3.12 there is a choice of the data with the required property. Again we assume that  $\mu$  is highly ramified for non-archimedean  $v \in S$ , and unramified outside S. In order to show the existence of  $\Phi(s)$  it is not hard to see that if we can show the existence of local non-archimedean data with the property that

$$L_v(s, \pi_1, \mu) L_v(1 - s, \pi_2, \mu^{-1}) \Psi(W_1^v, W_2^v, f; \mu_v | . |_v^s, \mu_v^{-1}| . |_v^{1-s})$$

is a constant, then we can take

$$\Phi(s) = C \prod_{v \in S \setminus \{\infty\}} Z_{v,N}(s, \pi_v(w^{-1}) \mathbb{W}_v(W_v^1, W_v^2; f_v), \mu_v),$$

with C the obvious non-zero constant. The existence of such data is the statement of Corollary 3.10.

We claim that the function  $\Phi(s)$  is nowhere vanishing. To see this, we set

$$F_1(W_{\infty}^1, W_{\infty}^2, s) = \frac{Z_{\infty, N}(s, \pi_{\infty}(w^{-1}) \mathbb{W}_{\infty}(W_{\infty}^1, W_{\infty}^2; f_{\infty}), \mu_{\infty})}{L_{\infty}(s, \pi_1, \mu) L_{\infty}(1 - s, \pi_2, \mu^{-1})}$$

$$F_2(W^1_{\infty}, W^2_{\infty}, s) = \Psi(W^{\infty}_1, W^{\infty}_2, f_{\infty}; \mu_{\infty} | . |_{\infty}^s, \mu_{\infty}^{-1} | . |_{\infty}^{1-s}).$$

So far we know that given any  $W_{\infty}^1, W_{\infty}^2$ , the complex functions  $F_1(s), F_2(s)$  are both entire. Next, let  $s_0$  be given and suppose  $\Phi(s_0) = 0$ ; but,

$$F_2(s) = \Phi(s)F_1(s), \tag{37}$$

which would then imply that for all choices of data we must have  $F_2(s_0) = 0$  which, by proposition 3.12, is not true. This finishes the proof of the theorem.

Remark 5.2 We observe that the function  $\Phi(s)$  defined in the proof of the theorem does not depend on  $W_{\infty}^1, W_{\infty}^2$ , and its dependence on  $\pi_1^{\infty}, \pi_2^{\infty}$  is merely through the non-archimedean components of the automorphic representations  $\pi_1, \pi_2$ . As  $\Phi(s)$  is the product of polynomials of  $q_v^{-s}$ , for  $v \in S$ , and as it nowhere vanishing, it is a function of the form

$$AB^{-s}$$

with B rational. Also prime numbers appearing in the decomposition of B are all from the set S. We will see later that  $\Phi(s)$  is in fact a constant.

## 5.1 Analytic continuation

Let  $\tau$  be a complex number with  $\Re \tau > 0$ . Then one can consider the archimedean principal series representation  $\pi(\tau) = \operatorname{Ind}(|.|^{\tau} \otimes |.|^{-\tau})$ . Let  $\rho_{\tau}: W_{\mathbb{R}} \to \operatorname{GL}_{2}(\mathbb{C})$  be the L parameter associated with the representation  $\pi(\tau)$ . We observe that if  $\pi(\tau)$  is irreducible, the corresponding L packet has a single element. Then as described in [2] one can consider a continuous map

$$P(\tau): \mathcal{S}(\mathrm{GL}_2(\mathbb{R})) \longrightarrow \pi(\tau).$$

Also for  $v \in \pi(\tau)$ , we set

$$W(v,g) = \int_{N(\mathbb{R})} v(ng)\psi^{-1}(n) dn$$

when the integral converges. Fix a Schwartz function f, and set

$$W_{\tau}(f;g) := W(P(\tau)(f),g).$$

A theorem of Shahidi asserts that  $W_{\tau}$  extends to an entire function of  $\tau$ . Usually, suppressing f, we simply write  $W_{\tau}$ . Fix two sections of  $W_{\tau}$ , say  $W_{\tau_1}$  and  $W_{\tau_2}$ . Next, consider the function

$$W_f(\tau_1, \tau_2) := W(W_{\tau_1}, W_{\tau_2}; f)$$

as before. We write  $F_i(\tau_1, \tau_2, s)$ , i = 1, 2, instead of the functions of the previous paragraph.

Let  $\mathbb{C}_{\text{aut}}$  be the collection of those complex numbers  $\tau$  with the property that  $\pi(\tau)$  occurs as the archimedean component of some automorphic cuspidal representation of the group GL(2). It is well-known that  $\mathbb{C}_{\text{temp}} := \mathbb{C}_{\text{aut}} \cap i\mathbb{R}$  is dense in  $i\mathbb{R}$ .

The function  $\mathbb{W}_f(\tau_1, \tau_2)$  is entire on  $\mathbb{C}^2$ , and for fixed  $(\tau_1, \tau_2) \in \mathbb{C}^2$  defines a Whittaker function on  $\mathrm{GSp}(4, \mathbb{R})$ . Also by construction if  $\tau_1, \tau_2 \in \mathbb{C}_{\mathrm{temp}}$ , the function  $\mathbb{W}_f(\tau_1, \tau_2)$  will make up the K-finite Whittaker model of the unique element of the local L packet  $\varphi(\rho_{\tau_1}, \rho_{\tau_2})$ . In fact, if we stay away from the points of reducibility, the unique element of the L packet given by  $\varphi(\rho_{\tau_1}, \rho_{\tau_2})$  is generic.

We have established the identity

$$F_1(\tau_1, \tau_2, s) = \Phi(s) F_2(\tau_1, \tau_2, s)$$

whenever  $(\tau_1, \tau_2) \in \mathbb{C}_{\text{aut}} \times \mathbb{C}_{\text{aut}}$ , and  $\Re s > b(\tau_1, \tau_2)$ . Presumably, the function  $\Phi(s)$  depends on s, and, though we have suppressed the dependence, on  $\tau_1, \tau_2$ . We now show that for  $\tau_1, \tau_2 \in \mathbb{C}_{\text{temp}}$ ,  $\Phi(s)$  is an absolute constant independent of all variables. For this we follow the argument of lemma 5 of [22], which is in the spirit of Burger-Li-Sarnak. The proof of Lemma 5 of [22] implies that given  $\tau \in i\mathbb{R}$  one can find an automorphic cuspidal representation of GL(2) with archimedean component arbitrarily close to  $\pi(\tau)$  and ramified only at one prescribed place. This, applied to a pair of tempered representations of GL(2) considered as a representation of GO(2, 2), implies that given a tempered representation of GO(2, 2)( $\mathbb{R}$ ) one can find two automorphic cuspidal representations with disjoint sets S. This observation combined with remark 5.2 proves that  $\Phi(s)$  must be a constant. Next, we have

$$F_1(\tau_1, \tau_2, s) = \Phi F_2(\tau_1, \tau_2, s)$$

whenever  $\tau_1, \tau_2 \in \mathbb{C}_{\text{temp}}$  and  $\Re s > b(s_1, s_2)$ . The density of  $\mathbb{C}_{\text{temp}}$  in  $i\mathbb{R}$  then implies that the identity must hold for all  $\tau_1, \tau_2$ , whenever  $\Re s > b(\tau_1, \tau_2)$ . But

we have seen that  $F_2$  is entire as a function of three complex variables; consequently, as  $F_1$  and  $F_2$  agree on an open set,  $F_2$  is the analytic continuation of  $F_1$ . Consequently, whatever we proved about  $F_2$  carries over to  $F_1$ .

# References

- [1] D. Bump, *The Rankin-Selberg Method: A survey*, Number Theory, trace formulas, and discrete groups (K. E. Bombieri and D. Goldfeld, eds.), Academic Press, San Diego, 1989.
- [2] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function.* Compositio Math. 41 (1980), no. 2, 207–231.
- [3] M. Furusawa, On L-functions for  $GSp(4) \times GL(2)$  and their special values. J. Reine Angew. Math. 438 (1993), 187–218.
- [4] S. Gelbart and F. Shahidi, Analytic Properties of Automorphic L-Functions, Prospectives in Mathematics, Academic Press, 1988.
- [5] M. Harris and S.S. Kudla, Arithmetic automorphic forms for non-holomorphic discrete series of GSp(2), Duke Math. J. 66 (1992), 59-121.
- [6] M. Harris, D. Soudry, and R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields I: lifting to* GSp<sub>4</sub>(Q), Invent. Math. 112 (1993), 377-411.
- [7] H. Jacquet and R. Langlands, Automorphic Forms on GL(2), Lecture Notes in Mathematics, Vol. 114, Springer, New York, 1970.
- [8] H. Jacquet, I. Piatetski-Shapiro, and J. A. Shalika, Automorphic Forms on GL(3) I, Ann. of Math., 109 (1979), 169-212.
- [9] R. P. Langlands, Classification of representations of real algebraic groups, mimeographed notes, IAS.
- [10] M. Novodvorsky, Automorphic L-Functions for the Symplectic Group GSp(4), AMS Proc. Symp. Pure Math., vol 33(2), 1979, 87-95.
- [11] M. Novodvorsky and I. Piatetski-Shapiro, Generalized Bessel Models for the Symplectic Group of Rank 2, Math. Sb., 90(2), 1973, 246-256.

- [12] T. Moriyama, Entireness of the spinor L-Functions for Certain Generic Cusp Forms on GSp(2), preprint.
- [13] B. Roberts, Global L-packets for GSp(2) and theta lifts, Documenta Mathematica 6 (2001), 247-314.
- [14] B. Roberts, Epsilon factors for some representations of GSp(2) and Bessel coefficients, in preparation.
- [15] P. Sally, Jr. and M. Tadic, Induced representations and classifications for  $\mathrm{GSp}(2,F)$  and  $\mathrm{Sp}(2,F)$ , Mem. Soc. Math. France (N.S.) No. 52 (1993), 75–133.
- [16] F. Shahidi, On certain L-functions, Amer. J. Math. 103 (1981), no. 2, 297–355.
- [17] F. Shahidi, A Proof of Langlands' Conjecture on Plancheral Measures; Complementary Series for p-adic Groups, Ann. of Math., 132 (1990), 273-330.
- [18] J. A. Shalika, Multiplicity One for GL(n), Ann. of Math. (2) 100 (1974), 171–193.
- [19] R. Takloo-Bighash, *L-functions for the p-adic group* GSp(4), American Journal of Mathematics, 122 (2000), 1085-1120.
- [20] R. Takloo-Bighash, The existence of Bessel functionals, preprint 2004.
- [21] R. Takloo-Bighash, , in Shahyad, A volume dedicated to S. Shahshahani's 60th birthday (ed. Lajevardi, P. Safari, and Y. Tabesh).
- [22] A. Venkatesh, The Burger-Sarnak method and operations on the unitary dual of GL(n), preprint.

Address: Department of Mathematics, Princeton University, Princeton, NJ 08544.

Email: rtakloo@math.princeton.edu