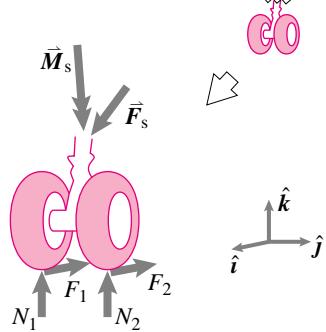


NO PROBLEMS

Introduction to  
**STATICS**  
and  
**DYNAMICS**

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**Andy Ruina and Rudra Pratap**

Pre-print for Oxford University Press, January 2002

# Summary of Mechanics

- 0) **The laws of mechanics apply to any collection of material or ‘body.’** This body could be the overall system of study or any part of it. In the equations below, the forces and moments are those that show on a free body diagram. Interacting bodies cause equal and opposite forces and moments on each other.

## I) Linear Momentum Balance (LMB)/Force Balance

Equation of Motion

$$\sum \vec{F}_i = \dot{\vec{L}}$$

The total force on a body is equal (I)  
to its rate of change of linear  
momentum.

Impulse-momentum  
(integrating in time)

$$\int_{t_1}^{t_2} \sum \vec{F}_i \cdot dt = \Delta \vec{L}$$

Net impulse is equal to the change in (Ia)  
momentum.

Conservation of momentum  
(if  $\sum \vec{F}_i = \vec{0}$ )

$$\dot{\vec{L}} = \vec{0} \Rightarrow \Delta \vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{0}$$

When there is no net force the linear (Ib)  
momentum does not change.

Statics  
(if  $\dot{\vec{L}}$  is negligible)

$$\sum \vec{F}_i = \vec{0}$$

If the inertial terms are zero the (Ic)  
net force on system is zero.

## II) Angular Momentum Balance (AMB)/Moment Balance

Equation of motion

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

The sum of moments is equal to the (II)  
rate of change of angular momentum.

Impulse-momentum (angular)  
(integrating in time)

$$\int_{t_1}^{t_2} \sum \vec{M}_C dt = \Delta \vec{H}_C$$

The net angular impulse is equal to (IIa)  
the change in angular momentum.

Conservation of angular momentum  
(if  $\sum \vec{M}_C = \vec{0}$ )

$$\dot{\vec{H}}_C = \vec{0} \Rightarrow \Delta \vec{H}_C = \vec{H}_{C2} - \vec{H}_{C1} = \vec{0}$$

If there is no net moment about point (IIb)  
C then the angular momentum about  
point C does not change.

Statics  
(if  $\dot{\vec{H}}_C$  is negligible)

$$\sum \vec{M}_C = \vec{0}$$

If the inertial terms are zero then the (IIc)  
total moment on the system is zero.

## III) Power Balance (1st law of thermodynamics)

Equation of motion

$$\dot{Q} + P = \underbrace{\dot{E}_K + \dot{E}_P + \dot{E}_{int}}_{\dot{E}}$$

Heat flow plus mechanical power (III)  
into a system is equal to its change  
in energy (kinetic + potential +  
internal).

for finite time

$$\int_{t_1}^{t_2} \dot{Q} dt + \int_{t_1}^{t_2} P dt = \Delta E$$

The net energy flow going in is equal (IIIa)  
to the net change in energy.

Conservation of Energy  
(if  $\dot{Q} = P = 0$ )

$$\dot{E} = 0 \Rightarrow \Delta E = E_2 - E_1 = 0$$

If no energy flows into a system, (IIIb)  
then its energy does not change.

Statics  
(if  $\dot{E}_K$  is negligible)

$$\dot{Q} + P = \dot{E}_P + \dot{E}_{int}$$

If there is no change of kinetic energy (IIIc)  
then the change of potential and  
internal energy is due to mechanical  
work and heat flow.

Pure Mechanics  
(if heat flow and dissipation  
are negligible)

$$P = \dot{E}_K + \dot{E}_P$$

In a system well modeled as purely (IIIId)  
mechanical the change of kinetic  
and potential energy is due to mechanical  
work.

# Some Definitions

(Please also look at the tables inside the back cover.)

$\vec{r}$ or $\vec{x}$	Position	.e.g., $\vec{r}_i \equiv \vec{r}_{i/O}$ is the position of a point i relative to the origin, O)
$\vec{v} \equiv \frac{d\vec{r}}{dt}$	Velocity	.e.g., $\vec{v}_i \equiv \vec{v}_{i/O}$ is the velocity of a point i relative to O, measured in a non-rotating reference frame)
$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$	Acceleration	.e.g., $\vec{a}_i \equiv \vec{a}_{i/O}$ is the acceleration of a point i relative to O, measured in a Newtonian frame)
$\vec{\omega}$	Angular velocity	A measure of rotational velocity of a rigid body.
$\vec{\alpha} \equiv \dot{\vec{\omega}}$	Angular acceleration	A measure of rotational acceleration of a rigid body.
$\vec{L} \equiv \begin{cases} \sum m_i \vec{v}_i & \text{discrete} \\ \int \vec{v} dm & \text{continuous} \end{cases}$ = $m_{\text{tot}} \vec{v}_{\text{cm}}$	Linear momentum	A measure of a system's net translational rate (weighted by mass).
$\dot{\vec{L}} \equiv \begin{cases} \sum m_i \vec{a}_i & \text{discrete} \\ \int \vec{a} dm & \text{continuous} \end{cases}$ = $m_{\text{tot}} \vec{a}_{\text{cm}}$	Rate of change of linear momentum	The aspect of motion that balances the net force on a system.
$\vec{H}_C \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{v}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{v} dm & \text{continuous} \end{cases}$	Angular momentum about point C	A measure of the rotational rate of a system about a point C (weighted by mass and distance from C).
$\dot{\vec{H}}_C \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{a}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{a} dm & \text{continuous} \end{cases}$	Rate of change of angular momentum about point C	The aspect of motion that balances the net torque on a system about a point C.
$E_K \equiv \begin{cases} \frac{1}{2} \sum m_i v_i^2 & \text{discrete} \\ \frac{1}{2} \int v^2 dm & \text{continuous} \end{cases}$	Kinetic energy	A scalar measure of net system motion.
$E_{\text{int}} =$ (heat-like terms)	Internal energy	The non-kinetic non-potential part of a system's total energy.
$P \equiv \sum \vec{F}_i \cdot \vec{v}_i + \sum \vec{M}_i \cdot \vec{\omega}_i$	Power of forces and torques	The mechanical energy flow into a system. Also, $P \equiv \dot{W}$ , rate of work.
$[\mathbf{I}^{\text{cm}}] \equiv \begin{bmatrix} I_{xx}^{\text{cm}} & I_{xy}^{\text{cm}} & I_{xz}^{\text{cm}} \\ I_{xy}^{\text{cm}} & I_{yy}^{\text{cm}} & I_{yz}^{\text{cm}} \\ I_{xz}^{\text{cm}} & I_{yz}^{\text{cm}} & I_{zz}^{\text{cm}} \end{bmatrix}$	Moment of inertia matrix about cm	A measure of how mass is distributed in a rigid body.

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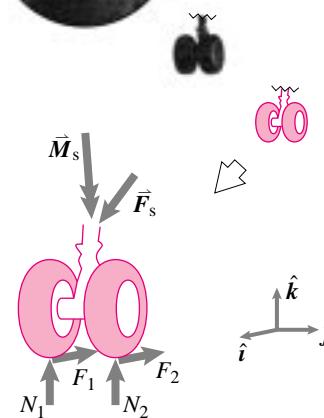
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Introduction to

# STATICS

and

# DYNAMICS



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# Preface

This is a statics and dynamics text for second or third year engineering students with an emphasis on vectors, free body diagrams, the basic momentum balance principles, and the utility of computation. Students often start a course like this thinking of mechanics reasoning as being vague and complicated. Our aim is to replace this loose thinking with concrete and simple mechanics problem-solving skills that live harmoniously with a useful mechanical intuition.

Knowledge of freshman calculus is assumed. Although most students have seen vector dot and cross products, vector topics are introduced from scratch in the context of mechanics. The use of matrices (to tidy-up systems of linear equations) and of differential equations (for describing motion in dynamics) are presented to the extent needed. The set-up of equations for computer solutions is presented in a pseudo-language easily translated by a student into one or another computation package that the student knows.

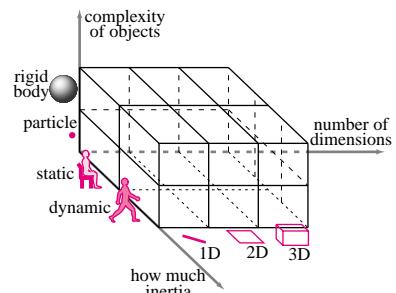
## Organization

We have aimed here to better unify the subject, in part, by an improved organization. Mechanics can be subdivided in various ways: statics vs dynamics, particles vs rigid bodies, and 1 vs 2 vs 3 spatial dimensions. Thus a 12 chapter mechanics table of contents could look like this

I. Statics	II. Dynamics
A. particles	C. particles
1) 1D	7) 1D
2) 2D	8) 2D
3) 3D	9) 3D
B. rigid bodies	D. rigid bodies
4) 1D	10) 1D
5) 2D	11) 2D
6) 3D	12) 3D

However, these topics are far from equal in their difficulty or in the number of subtopics they contain. Further, there are various concepts and skills that are common to many of the 12 sub-topics. Dividing mechanics into these bits distracts from the unity of the subject. Although some vestiges of the scheme above remain, our book has evolved to a different organization through trial and error, thought and rethought, review and revision, and nine semesters of student testing.

The first four chapters cover the basics of statics. Dynamics of particles and rigid bodies, based on progressively more difficult motions, is presented in chapters five to eleven. Relatively harder topics, that might be skipped in quicker courses, are identifiable by chapter, section or subsection titles containing words like “three dimensional” or “advanced”. In more detail:



**Chapter 1** defines mechanics as a subject which makes predictions about forces and motions using models of mechanical behavior, geometry, and the basic balance laws. The **laws of mechanics** are informally summarized.

**Chapter 2** introduces **vector skills** in the context of mechanics. Notational clarity is emphasized because correct calculation is impossible without distinguishing vectors from scalars. Vector addition is motivated by the need to add forces and relative positions, dot products are motivated as the tool which reduces vector equations to scalar equations, and cross products are motivated as the formula which correctly calculates the heuristically motivated concept of moment and moment about an axis.

**Chapter 3** is about **free body diagrams**. It is a separate chapter because, in our experience, good use of free body diagrams is almost synonymous with correct mechanics problem solution. To emphasize this to students we recommend that, to get any credit for a problem that uses balance laws in the rest of the course, a good free body diagram must be drawn.

**Chapter 4** makes up a short course in **statics** including an introduction to trusses, mechanisms, beams and hydrostatics. The emphasis is on two-dimensional problems until the last, more advanced section. Solution methods that depend on kinematics (*i.e.*, work methods) are deferred until the dynamics chapters. But for the stretch of linear springs, deformations are not covered.

**Chapter 5** is about **unconstrained motion of one or more particles**. It shows how far you can go using  $\vec{F} = m\vec{a}$  and Cartesian coordinates in 1, 2 and 3 dimensions in the absence of kinematic constraints. The first five sections are a thorough introduction to motion of one particle in one dimension, so called scalar physics, namely the equation  $F(x, v, t) = ma$  and special cases thereof. The chapter includes some review of freshman calculus as well as an introduction to energy methods. A few special cases are emphasized, namely, constant acceleration, force dependent on position (thus motivating energy methods), and the harmonic oscillator. After one section on coupled motions in 1 dimension, sections seven to ten discuss motion in two and three dimensions. The easy set up for computation of trajectories, with various force laws, and even with multiple particles, is emphasized. The chapter ends with a mostly theoretical section on the center-of-mass simplifications for systems of particles.

**Chapter 6** is the first chapter that concerns **kinematic constraint** in its simplest context, systems that are constrained to move without rotation in a **straight line**. In one dimension pulley problems provide the main example. Two and three dimensional problems are covered, such as finding structural support forces in accelerating vehicles and the slowing or incipient capsizing of a braking car. Angular momentum balance is introduced as a needed tool but without the usual complexities of curvilinear motion.

**Chapter 7** treats **pure rotation** about a fixed axis in two dimensions. Polar coordinates and base vectors are first used here in their simplest possible context. The primary applications are pendulums, gear trains, and rotationally accelerating motors or brakes.

**Chapter 8** treats **general planar motion** of a (planar) rigid body including rolling, sliding and free flight. Multi-body systems are also considered so long as they do not involve constraint (*i.e.*, collisions and spring connections but not hinges or prismatic joints).

**Chapter 9** is entirely about kinematics of particle motion. The over-riding theme is the use of base vectors which change with time. First, the discussion of polar coordinates started in chapter 7 is completed. Then path coordinates are introduced. The kinematics of relative motion, a topic that many students find difficult, is treated carefully but not elaborately in two stages. First using rotating base

vectors connected to a moving rigid body and then using the more abstract notation associated with the famous “five term acceleration formula.”

**Chapter 10** is about the mechanics of particles and rigid bodies utilizing the relative motion kinematics ideas from chapter 9. This is the capstone chapter for a two-dimensional dynamics course. After this chapter a good student should be able to navigate through and use most of the skills in the concept map on page 582.

**Chapter 11** is an introduction to 3D rigid body motion. It extends chapter 7 to **fixed axis rotation in three dimensions**. The key new kinematic tool here is the non-trivial use of the cross product for calculating velocities and accelerations. Fixed axis rotation is the simplest motion with which one can introduce the full moment of inertia matrix, where the diagonal terms are analogous to the scalar 2D moment of inertia and the off-diagonal terms have a “centripetal” interpretation. The main new application is dynamic balance. In our experience going past this is too much for most engineering students in the first mechanics course after freshman physics, so the book ends here.

**Appendix A** on **units** and dimensions is for reference. Because students are immune to preaching about units out of context, such as in an early or late chapter like this one, the main messages are presented by example throughout the book (the book may be unique amongst mechanics texts in this regard):

- All engineering calculations using dimensional quantities must be dimensionally ‘balanced’.
- Units are ‘carried’ from one line of calculation to the next by the same rules as go numbers and variables.

**Appendix B** on **contact laws** (friction and collisions) is for reference for students who puzzle over these issues.

A leisurely one semester statics course, or a more fast-paced half semester prelude to strength of materials should use chapters 1-4. A typical one semester dynamics course should cover most of chapters 5-11 preceded by topics from chapters 1-4, as needed. A one semester statics and dynamics course should cover about two thirds of chapters 1-6 and 8. A full year statics and dynamics course should cover most of the book.

### *Organization and formatting*

Each subject is covered in various ways.

- Every section starts with **descriptive text** and short *examples* motivating and describing the theory;
- More detailed explanations of the **theory** are in boxes interspersed in the text. For example, one box explains the common derivation of angular momentum balance from linear momentum balance, one explains the genius of the wheel, and another connects  $\vec{\omega}$  based kinematics to  $\hat{e}_r$  and  $\hat{e}_\theta$  based kinematics;
- **Sample problems** (marked with a gray border) at the end of most sections show how to do homework-like calculations. These set an example to the student in their consistent use of free body diagrams, systematic application of basic principles, vector notation, units, and checks against intuition and special cases;
- **Homework problems** at the end of each chapter give students a chance to practice mechanics calculations. The first problems for each section build a student’s confidence with the basic ideas. The problems are ranked in approximate order of difficulty, with theoretical questions last. Problems marked with an \* have an answer at the back of the book;

- **Reference tables** on the inside covers and end pages concisely summarize much of the content in the book. These tables can save students the time of hunting for formulas and definitions. They also serve to visibly demonstrate the basically simple structure of the whole subject of mechanics.

### *Notation*

Clear vector notation helps students do problems. Students sometimes mistakenly transcribe a conventionally printed bold vector  $\mathbf{F}$  the same way they transcribe a plain-text scalar  $F$ . To help minimize this error we use a redundant vector notation in this book (bold and harpooned  $\vec{\mathbf{F}}$ ).

As for all authors and teachers concerned with motion in two and three dimensions we have struggled with the tradeoffs between a precise notation and a simple notation. Beautifully clear notations are intimidating. Perfectly simple notations are ambiguous. Our attempt to find clarity without clutter is summarized in the box on page 9.

## Relation to other mechanics books

This book is in some ways original in organization and approach. It also contains some important but not sufficiently well known concepts, for example that angular momentum balance applies relative to any point, not just an arcane list of points. But there is little mechanics here that cannot be found in other books, including freshman physics texts, other engineering texts, and hundreds of classics.

Mastery of freshman physics (*e.g.*, from Halliday & Resnick, Tipler, or Serway) would encompass some part of this book's contents. However freshman physics generally leaves students with a vague notion of what mechanics is, and how it can be used. For example many students leave freshman physics with the sense that a free body diagram (or ‘force diagram’) is an vague conceptual picture with arrows for various forces and motions drawn on it this way and that. Even the book pictures sometimes do not make clear what force is acting on what body. Also, because freshman physics tends to avoid use of college math, many students end up with no sense of how to use vectors or calculus to solve mechanics problems. This book aims to lead students who may start with these fuzzy freshman physics notions into a world of intuitive yet precise mechanics.

There are many statics and dynamics textbooks which cover about the same material as this one. These textbooks have modern applications, ample samples, lots of pictures, and lots of homework problems. Many are good (or even excellent) in their own ways. Most of today's engineering professors learned from one of these books. We wrote this book with the intent of doing still better in a few ways:

- better showing the unity of the subject,
- more clear notation in figures and equations,
- better integration of the applicability of computers,
- more clear use of units throughout,
- introduction of various insights into how things work,
- a more informal and less intimidating writing style.

We intend that through this book students will come to see mechanics as a coherent network of basic ideas rather than a collection of ad-hoc recipes and tricks that one need memorize or hope to discover by divine inspiration.

There are hundreds of older books with titles like *statics*, *engineering mechanics*, *dynamics*, *machines*, *mechanisms*, *kinematics*, or *elementary physics* that cover aspects of the material here<sup>①</sup>. Although many mechanics books written from 1689-

① Here are three nice older books on mechanics:

J.P. Den Hartog's *Mechanics* originally published in 1948 but still available as an inexpensive reprint (well written and insightful);

J.L. Synge and B.A. Griffith, *Principles of Mechanics* through page 408. Originally published in 1942, reprinted in 1959 (good pedagogy but dry); and

E.J. Routh's, *Dynamics of a System of rigid bodies*, Vol I (the “elementary” part through chapter 7. Originally published in 1905, but reprinted in 1960 (a dense gold mine). Routh also has 5 other idea packed statics and dynamics books.

1960, are amazingly thoughtful and complete, none are good modern textbooks. They lack an appropriate pace, style of speech, and organization. They are too reliant on geometry skills and not enough on vectors and numerical computation skills. They lack sufficient modern applications, sample calculations, illustrations, and homework problems for a modern text book.

## Thank you

We have attempted to write a book which will help make the teaching and learning of mechanics more fun and more effective. We have tried to present the truth as we know it and as we think it is most effectively communicated. But we have undoubtedly left various technical and strategic errors. We thank you in advance for letting us know your thoughts so that we can improve future editions.

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## To the student

Mother nature is so strict that, to the extent we know her rules, we can make reliable predictions about the behavior of her children, the world of physical objects. In particular, for essentially all practical purposes all objects that engineers study strictly follow the laws of Newtonian mechanics. So, if you learn the laws of mechanics, as this book should help you do, you will be able to make quantitative calculations that predict how things stand, move, and fall. You will also gain intuition about how the physical world works.

## How to use this book

Most of you will naturally get help with homework by looking at similar examples and samples in the text or lecture notes, by looking up formulas in the front and back covers, or by asking questions of friends, teaching assistants and professors. What good are books, notes, classmates or teachers if they don't help you do homework problems? All the examples and sample problems in this book, for example, are just for this purpose. But too-much use of these resources while solving problems can lead to self deception. To see if you have learned to do a problem, do it again, justifying each step, *without looking up even one small thing*. If you can't do this, you have a new opportunity to learn at two levels. First, you can learn the missing skill or idea. More deeply, by getting stuck after you have been able to get through a problem with guidance, you can learn things about your learning process. Often the real source of difficulty isn't a key formula or fact, but something more subtle. We have tried to bring out some of these more subtle ideas in the text discussions which we hope you read, sooner or later.

Some of you are science and math school-smart, mechanically inclined, or are especially motivated to learn mechanics. Others of you are reluctantly taking this class to fulfil a requirement. We have written this book with both of you in mind. The sections start with generally accessible introductory material and include simple examples. The early sample problems in each section are also easy. But we also have discussions of the theory and other more advanced asides to challenge more motivated students.

## Calculation strategies and skills

In this book we try here to show you a systematic approach to solving problems. But it is not possible to reduce the world of mechanics problem solutions to one clear set of steps to follow. There is an art to solving problems, whether homework problems or engineering design problems. Art and human insight, as opposed to precise algorithm or recipe, is what makes engineering require humans and not just computers. Through discussion and examples, we will try to teach you some of this systematic art. Here are a few general guidelines that apply to many problems.

*Understand the question*

You may be tempted to start writing equations and quoting principles when you first see a problem. But it is generally worth a few minutes (and sometimes a few hours) to try to get an intuitive sense of a problem before jumping to equations. Before you draw any sketches or write equations, think: does the problem make sense? What information has been given? What are you trying to find? Is what you are trying to find determined by what is given? What physical laws make the problem solvable? What extra information do you think you need? What information have you been given that you don't need? Your general sense of the problem will steer you through the technical details.

Some students find they can read every line of sample problems yet cannot do test problems, or, later on, cannot do applied design work effectively. This failing may come from following details without spending time, thinking, gaining an overall sense of the problems.

*Think through your solution strategy*

For the problem solutions we present in this book or in class, there was a time when we had to think about the order of our work. You also have to think about the order of your work. You will find some tips in the text and samples. But it is your job to own the material, to learn how to think about it your own way, to become an expert in your own style, and to do the work in the way that makes things most clear to you and your readers.

**What's in your toolbox?**

In the toolbox of someone who can solve lots of mechanics problems are two well worn tools:

- A vector calculator that always keeps vectors and scalars distinct, and
- A reliable and clear free body diagram drawing tool.

Because many of the terms in mechanics equations are vectors, the ability to do vector calculations is essential. Because the concept of an isolated system is at the core of mechanics, every mechanics practitioner needs the ability to draw a good free body diagram. Would that we could write

“Click on WWW.MECH.TOOL today and order your own professional vector calculator and expert free body diagram drawing tool!”,

but we can't. After we informally introduce mechanics in the first chapter, the second and third chapters help you build your own set of these two most-important tools.

**Guarantee:** if you learn to do clear correct vector algebra and to draw good free body diagrams you will do well at mechanics.

**Think hard**

We do mechanics because we like mechanics. We get pleasure from thinking about how things work, and satisfaction from doing calculations that make realistic predictions. Our hope is that you also will enjoy idly thinking about mechanics and that you will be proud of your new modeling and calculation skills. You will get there if you think hard. And you will get there more easily if you learn to enjoy thinking hard. Often the best places to study are away from books, notes, pencil or paper when you are walking, washing or resting.

## A note on computation

Mechanics is a physical subject. The concepts in mechanics do not depend on computers. But mechanics is also a quantitative and applied subject described with numbers. Computers are very good with numbers. Thus the modern practice of engineering mechanics depends on computers. The most-needed computer skills for mechanics are:

- solution of simultaneous algebraic equations,
- plotting, and
- numerical solution of ODEs (Ordinary Differential Equations).

More basically, an engineer also needs the ability to routinely evaluate standard functions ( $x^3$ ,  $\cos^{-1} \theta$ , etc.), to enter and manipulate lists and arrays of numbers, and to write short programs.

### *Classical languages, applied packages, and simulators*

Programming in standard languages such as Fortran, Basic, C, Pascal, or Java probably take too much time to use in solving simple mechanics problems. Thus an engineer needs to learn to use one or another widely available computational package (e.g., MATLAB, OCTAVE, MAPLE, MATHEMATICA, MATHCAD, TKSOLVER, LABVIEW, etc). We assume that students have learned, or are learning such a package. We also encourage the use of packaged mechanics simulators (e.g., WORKING MODEL, ADAMS, DADS, etc) for building intuition, but none of the homework here depends on access to such a packaged simulator.

### *How we explain computation in this book.*

Solving a mechanics problem involves these major steps

- (a) Reducing a physical problem to a well posed mathematical problem;
- (b) Solving the math problem using some combination of pencil and paper and numerical computation; and
- (c) Giving physical interpretation of the mathematical solution.

This book is primarily about setup (a) and interpretation (c), which are the same, no matter what method is used to solve the equations. If a problem requires computation, the exact computer commands vary from package to package. So we express our computer calculations in this book using an informal pseudo computer language. For reference, typical commands are summarized in box on page xii.

### *Required computer skills.*

Here, in a little more detail, are the primary computer skills you need.

- Many mechanics problems are statics or ‘instantaneous mechanics’ problems. These problems involve trying to find some forces or accelerations at a given configuration of a system. These problems can generally be reduced to the **solution of linear algebraic equations** of this general type: solve

$$\begin{array}{rcl} 3 & x & + & 4 & y & = & 8 \\ -7 & x & + & \sqrt{2} & y & = & 3.5 \end{array}$$

for  $x$  and  $y$ . Some computer packages will let you enter equations almost as written above. In our pseudo language we would write:

```

set = { 3*x +      4*y = 8
       -7*x + sqrt(2)*y = 3.5  }
solve set for x and y

```

Other packages may require you to write the equations in matrix form something like this (see, or wait for, page ?? for an explanation of the matrix form of algebraic equations):

```

A = [ 3      4
      -7    sqrt(2) ]
b = [ 8 3.5 ]'
solve A*z=b for z

```

where  $A$  is a  $2 \times 2$  matrix,  $b$  is a column of 2 numbers, and the two elements of  $z$  are  $x$  and  $y$ . For systems of two equations, like above, a computer is hardly needed. But for systems of three equations pencil and paper work is sometimes error prone. Most often pencil and paper solution of four or more equations is too tedious and error prone.

- In order to see how a result depends on a parameter, or to see how a quantity varies with position or time, it is useful to see a **plot**. Any plot based on more than a few data points or a complex formula is far more easily drawn using a computer than by hand. Most often you can organize your data into a set of  $(x, y)$  pairs stored in an  $X$  list and a corresponding  $Y$  list. A simple computer command will then plot  $x$  vs  $y$ . The pseudo-code below, for example, plots a circle using 100 points

```

npoints = [1 2 3 ... 100]
theta   = npoints * 2 * pi / 100
X       = cos(theta)
Y       = sin(theta)
plot Y vs X

```

where  $npoints$  is the list of numbers from 1 to 100,  $\theta$  is a list of 100 numbers evenly spaced between 0 and  $2\pi$  and  $X$  and  $Y$  are lists of 100 corresponding  $x, y$  coordinate points on a circle.

- The result of using the laws of dynamics is often a set of differential equations which need to be solved. A simple example would be:

Find  $x$  at  $t = 5$  given that  $\frac{dx}{dt} = x$  and that at  $t = 0, x = 1$ .

The solution to this problem can be found easily enough by hand to be  $e^5$ . But often the differential equations are just too hard for pencil and paper solution. Fortunately the **numerical solution of ordinary differential equations** is already programmed into scientific and engineering computer packages. The simple problem above is solved with computer code analogous to this:

```

ODES = { xdot = x }
ICS  = { xzero = 1 }
solve ODES with ICS until t=5

```

Examples of many calculations of these types will be shown, starting on page ??.

## 0.1 Summary of informal computer commands

Computer commands are given informally and descriptively in this book. The commands below are not as precise as any real computer package. You should be able to use your package's documentation to translate the informal commands below. Many of the commands below depend on mathematical ideas which are introduced in the text. At first reading a student is not expected to absorb this table.

<code>x=7</code>	Set the variable $x$ to 7.	<code>cos(w)</code>	Make a new list, each element of which is the cosine of the corresponding element of $[w]$ .
<code>omega=13</code>	Set $\omega$ to 13.	<code>.....</code>	<code>.....</code>
<code>u=[1 0 -1 0] v=[2 3 4 pi]</code>	Define $u$ and $v$ to be the lists shown.	<code>mag(u)</code>	The square root of the sum of the squares of the elements in $[u]$ , in this case 1.41421...
<code>t= [.1 .2 .3 ... 5]</code>	Set $t$ to the list of 50 numbers implied by the expression.	<code>.....</code>	<code>.....</code>
<code>y=v(3)</code>	sets $y$ to the third value of $v$ (in this case 4).	<code>u dot v</code>	The vector dot product of component lists $[u]$ and $[v]$ , (we could also write $\text{sum}(A^*B)$ ).
<code>A=[1 2 3 6.9 5 0 1 12 ]</code>	Set $A$ to the array shown.	<code>.....</code>	<code>.....</code>
<code>z= A(2,3)</code>	Set $z$ to the element of $A$ in the second row and third column.	<code>C cross D</code>	The vector cross product of $\vec{C}$ and $\vec{D}$ , assuming the three element component lists for $[C]$ and $[D]$ have been defined.
<code>w=[ 3 4 2 5 ]</code>	Define $w$ to be a column vector.	<code>.....</code>	<code>.....</code>
<code>w = [ 3 4 2 5 ]'</code>	Same as above. ' means transpose.	<code>A matmult w</code>	Use the rules of matrix multiplication to multiply $[A]$ and $[w]$ .
<code>u+v</code>	Vector addition. In this case the result is $[3 \ 3 \ 3 \ \pi]$ .	<code>eqset = {3x + 2y = 6 6x + 7y = 8}</code>	Define 'eqset' to stand for the set of 2 equations in braces.
<code>u*v</code>	Element by element multiplication, in this case $[2 \ 0 \ -4 \ 0]$ .	<code>solve eqset for x and y</code>	Solve the equations in 'eqset' for $x$ and $y$ .
<code>sum(w)</code>	Add the elements of $w$ , in this case 14.	<code>solve Ax=b for x</code>	Solve the matrix equation $[A][x] = [b]$ for the list of numbers $x$ . This assumes $A$ and $b$ have already been defined.
<code>.....</code>	<code>.....</code>	<code>for i = 1 to N such and such end</code>	Execute the commands 'such and such' $N$ times, the first time with $i = 1$ , the second with $i = 2$ , etc
<code>.....</code>	<code>.....</code>	<code>plot y vs x</code>	Assuming $x$ and $y$ are two lists of numbers of the same length, plot the $y$ values vs the $x$ values.
<code>.....</code>	<code>.....</code>	<code>solve ODEs with ICs until t=5</code>	Assuming a set of ODEs and ICs have been defined, use numerical integration to solve them and evaluate the result at $t = 5$ .
<code>.....</code>	<code>.....</code>	<code>.....</code>	<code>.....</code>

With an informality consistent with what is written above, other commands are introduced here and there as needed.



---

# 1

# What is mechanics?

---

*Mechanics* is the study of force, deformation, and motion, and the relations between them. We care about forces because we want to know how hard to push something to move it or whether it will break when we push on it for other reasons. We care about deformation and motion because we want things to move or not move in certain ways. Towards these ends we are confronted with this general mechanics problem:

Given some (possibly idealized) information about the properties, forces, deformations, and motions of a mechanical system, make useful predictions about other aspects of its properties, forces, deformations, and motions.

By *system*, we mean a tangible thing such as a wheel, a gear, a car, a human finger, a butterfly, a skateboard and rider, a quartz timing crystal, a building in an earthquake, a piano string, and a space shuttle. Will a wheel slip? a gear tooth break? a car tip over? What muscles are used when you hit a key on your computer? How do people balance on skateboards? Which buildings are more likely to fall in what kinds of earthquakes? Why are low pitch piano strings made with helical windings instead of straight wires? How fast is the space shuttle moving when in low earth orbit?

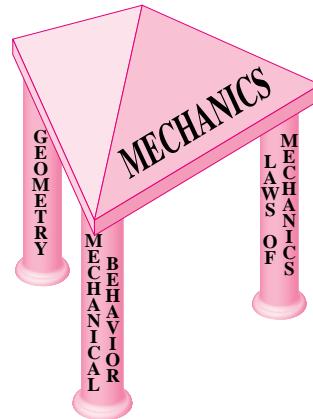
In mechanics we try to solve special cases of the general mechanics problem above by idealizing the system, using classical Euclidean geometry to describe deformation and motion, and assuming that the relation between force and motion is described with Newtonian mechanics, or “Newton’s Laws”. Newtonian mechanics has held up, with minor refinement, for over three hundred years. Those who want to know how machines, structures, plants, animals and planets hold together and move about

① The laws of classical mechanics, however expressed, are named for Isaac Newton because his theory of the world, the *Principia* published in 1689, contains much of the still-used theory. Newton used his theory to explain the motions of planets, the trajectory of a cannon ball, why there are tides, and many other things.

need to know Newtonian mechanics. In another two or three hundred years people who want to design robots, buildings, airplanes, boats, prosthetic devices, and large or microscopic machines will probably still use the equations and principles we now call Newtonian mechanics<sup>①</sup>

Any mechanics problem can be divided into 3 parts which we think of as the 3 pillars that hold up the subject:

1. the mechanical behavior of objects and materials (*constitutive laws*);
2. the geometry of motion and distortion (*kinematics*); and
3. the laws of mechanics ( $\vec{F} = m\vec{a}$ , etc.).



Let's discuss each of these ideas a little more, although somewhat informally, so you can get an overview of the subject before digging into the details.

### *Mechanical behavior*

The first pillar of mechanics is mechanical behavior. The *Mechanical behavior* of something is the description of how loads cause deformation (or vice versa). When something carries a force it stretches, shortens, shears, bends, or breaks. Your finger tip squishes when you poke something. Too large a force on a gear in an engine causes it to break. The force of air on an insect wing makes it bend. Various geologic forces bend, compress and break rock.

This relation between force and deformation can be viewed in a few ways. First, it gives us a definition of force. In fact, force can be defined by the amount of spring stretch it causes. Thus most modern force measurement devices measure force indirectly by measuring the deformation it causes in a calibrated spring. This is one justification for calling ‘mechanical behavior’ the first pillar. It gives us a notion of force even before we introduce the laws of mechanics.

Second, a piece of steel distorts under a given load differently than a same-sized piece of chewing gum. This observation that different objects deform differently with the same loads implies that the properties of the object affect the solution of mechanics problems. The relations of an object’s deformations to the forces that are applied are called the *mechanical properties* of the object. Mechanical properties are sometimes called *constitutive laws* because the mechanical properties describe how an object is constituted (at least from a mechanics point of view). The classic example of a constitutive law is that of a linear spring which you remember from your

elementary physics classes: ' $F = kx$ '. When solving mechanics problems one has to make assumptions and idealizations about the constitutive laws applicable to the parts of a system. How stretchy (elastic) or gooey (viscous) or otherwise deformable is an object? The set of assumptions about the mechanical behavior of the system is sometimes called the *constitutive model*.

Distortion in the presence of forces is easy to see on squeezed fingertips, or when thin pieces of wood bend. But with pieces of rock or metal the deformation is essentially invisible and sometimes hard to imagine. With the exceptions of things like rubber, flesh, or compliant springs, solid objects that are not in the process of breaking typically change their dimensions much less than 1% when loaded. Most structural materials deform less than one part per thousand with working loads. But even these small deformations can be important because they are enough to break bones and collapse bridges.

When deformations are not of consequence engineers often idealize them away. Mechanics, where deformation is neglected, is called *rigid body* mechanics because a rigid (infinitely stiff) solid would not deform. Rigidity is an extreme constitutive assumption. The assumption of rigidity greatly simplifies many calculations while generating adequate predictions for many practical problems. The assumption of rigidity also simplifies the introduction of more general mechanics concepts. Thus for understanding the steering dynamics of a car we might model it as a rigid body, whereas for crash analysis where rigidity is clearly a poor approximation, we might model a car as highly deformable.

Most constitutive models describe the material inside an object. But to solve a mechanics problem involving friction or collisions one also has to have a constitutive model for the contact interactions. The standard friction model (or idealization) ' $F \leq \mu N$ ' is an example of a contact constitutive model.

In all of mechanics, one needs constitutive models of a system and its components before one can make useful predictions.

### *The geometry of deformation and motion*

The second pillar of mechanics concerns the geometry of deformation and motion. Classical Greek (Euclidean) geometry concepts are used. Deformation is defined by changes of lengths and angles between sets of points. Motion is defined by the changes of the position of points in time. Concepts of length, angle, similar triangles, the curves that particles follow and so on can be studied and understood without Newton's laws and thus make up an independent pillar of the subject.

We mentioned that understanding small deformations is often important to predict when things break. But large motions are also of interest. In fact many machines and machine parts are designed to move something. Bicycles, planes, elevators, and hearses are designed to move people; a clockwork, to move clock hands; insect wings, to move insect bodies; and forks, to move potatoes. A connecting rod is designed to move a crankshaft; a crankshaft, to move a transmission; and a transmission, to move a wheel. And wheels are designed to move bicycles, cars, and skateboards.

The description of the motion of these things, of how the positions of the pieces change with time, of how the connections between pieces restrict the motion, of the curves traversed by the parts of a machine, and of the relations of these curves to each other is called *kinematics*. Kinematics is the study of the geometry of motion (or geometry in motion).

For the most part we think of deformations as involving small changes of distance between points on one body, and of net motion as involving large changes of distance between points on different bodies. Sometimes one is most interested in deformation (you would like the stretch between the two ends of a bridge brace to be small) and sometimes in the net motion (you would like all points on a plane to travel

about the same large distance from Chicago to New York). Really, deformation and motion are not distinct topics, both involve keeping track of the positions of points. The distinction we have made is for simplicity. Trying to simultaneously describe deformations and large motions is just too complicated for beginners. So the ideas are kept (somewhat artificially) separate in elementary mechanics courses such as this one. As separate topics, both the geometry needed to understand small deformations and the geometry needed to understand large motions of rigid bodies are basic parts of mechanics.

## Relation of force to motion, the laws of mechanics

The third pillar of mechanics is loosely called *Newton's laws*. One of Newton's brilliant insights was that the same intuitive 'force' that causes deformation also causes motion, or more precisely, acceleration of mass. Force is related to deformation by material properties (elasticity, viscosity, etc.) and to motion by the laws of mechanics summarized in the front cover. In words and informally, these are:<sup>①</sup>

<sup>①</sup> Isaac Newton's original three laws are: 1) an object in motion tends to stay in motion, 2)  $\vec{F} = m\vec{a}$  for a particle, and 3) the principle of action and reaction. These could be used as a starting point for study of mechanics. The more modern approach we take here leads to the same end.

- 0) The laws of mechanics apply to any system (rigid or not):
  - a) Force and moment are *the* measure of mechanical interaction; and
  - b) Action = minus reaction applies to all interactions, ('every action has an equal and opposite reaction');
- I) The net force on a system causes a net linear acceleration (*linear momentum balance*),
- II) The net turning effect of forces on system causes it to rotationally accelerate (*angular momentum balance*), and
- III) The change of energy of a system is due to the energy flow into the system (*energy balance*).

The principles of action and reaction, linear momentum balance, angular momentum balance, and energy balance, are actually redundant various ways. Linear momentum balance can be derived from angular momentum balance and, sometimes (see section ??), vice-versa. Energy balance equations can often be derived from the momentum balance equations. The principle of action and reaction can also be derived from the momentum balance equations. In the practice of solving mechanics problems, however, the ideas are generally considered independently without much concern for which idea could be derived from the others for the problem under consideration. That is, the four assumptions in I-III above are not a mathematically minimal set, but they are all accepted truths in Newtonian mechanics.

A lot follows from the laws of Newtonian mechanics, including the contents of this book. When these ideas are supplemented with models of particular systems (e.g., of machines, buildings or human bodies) and with Euclidean geometry, they lead to predictions about the motions of these systems and about the forces which act upon them. There is an endless stream of results about the mechanics of one or another special system. Some of these results are classified into entire fields of research such as 'fluid mechanics,' 'vibrations,' 'seismology,' 'granular flow,' 'biomechanics,' or 'celestial mechanics.'

The four basic ideas also lead to other more mathematically advanced formulations of mechanics with names like 'Lagrange's equations,' 'Hamilton's equations,' 'virtual work', and 'variational principles.' Should you take an interest in theoretical mechanics, you may learn these approaches in more advanced courses and books, most likely in graduate school.

## Statics, dynamics, and strength of materials

Elementary mechanics is traditionally partitioned into three courses named ‘statics’, ‘dynamics’, and ‘strength of materials’. These subjects vary in how much they emphasize material properties, geometry, and Newton’s laws.

*Statics* is mechanics with the idealization that the acceleration of mass is negligible in Newton’s laws. The first four chapters of this book provide a thorough introduction to statics. Strictly speaking things need not be standing still to be well idealized with statics. But, as the name implies, statics is generally about things that don’t move much. The first pillar of mechanics, constitutive laws, is generally introduced without fanfare into statics problems by the (implicit) assumption of rigidity. Other constitutive assumptions used include inextensible ropes, linear springs, and frictional contact. The material properties used as examples in elementary statics are very simple. Also, because things don’t move or deform much in statics, the geometry of deformation and motion are all but ignored. Despite the commonly applied vast simplifications, statics is useful, for example, for the analysis of structures, slow machines or the light parts of fast machines, and the stability of boats.

*Dynamics* concerns motion associated with the non-negligible acceleration of mass. Chapters 5-11 of this book introduce dynamics. As with statics, the first pillar of mechanics, constitutive laws, is given a relatively minor role in the elementary dynamics presented here. For the most part, the same library of elementary properties are used with little fanfare (rigidity, inextensibility, linear elasticity, and friction). Dynamics thus concerns the two pillars that are labelled by the confusingly similar words kinematics and kinetics. *Kinematics* concerns geometry with no mention of force and *kinetics* concerns the relation of force to motion. Once one has mastered statics, the hard part of dynamics is the kinematics. Dynamics is useful for the analysis of, for example, fast machines, vibrations, and ballistics.

*Strength of materials* expands statics to include material properties and also pays more attention to distributed forces (traction and stress). This book only occasionally touches lightly on strength of materials topics like stress (loosely, force per unit area), strain (a way to measure deformation), and linear elasticity (a commonly used constitutive model of solids). Strength of materials gives equal emphasis to all three pillars of mechanics. Strength of materials is useful for predicting the amount of deformation in a structure or machine and whether or not it is likely to break with a given load.

## How accurate is Newtonian mechanics?

In popular science culture we are repeatedly reminded that Newtonian ideas have been overthrown by relativity and quantum mechanics. So why should you read this book and learn ideas which are known to be wrong?

First off, this criticism is maybe inappropriate because general relativity and quantum mechanics are inconsistent with each other, not yet united by a universally accepted deeper theory of everything. But how big are the errors we make when we do classical mechanics, neglecting various modern physics theories?

- The errors from neglecting the effects of special relativity are on the order of  $v^2/c^2$  where  $v$  is a typical speed in your problem and  $c$  is the speed of light. The biggest errors are associated with the fastest objects. For, say, calculating space shuttle trajectories this leads to an error of about

$$\frac{v^2}{c^2} \approx \left( \frac{5 \text{ mi/s}}{3 \times 10^8 \text{ m/s}} \right)^2 \approx .000000001 \approx \text{one millionth of one percent}$$

- In classical mechanics we assume we can know exactly where something is and how fast it is going. But according to quantum mechanics this is impossible. The product of the uncertainty  $\delta x$  in position of an object and the the uncertainty  $\delta p$  of its momentum must be greater than Planck's constant  $\hbar$ . Planck's constant is small;  $\hbar \approx 1 \times 10^{-34}$  joule· s. The fractional error so required is biggest for small objects moving slowly. So if one measures the location of a computer chip with mass  $m = 10^{-4}$  kg to within  $\delta x = 10^{-6}$  m  $\approx$  a twenty fifth of a thousands of an inch, the uncertainty in its velocity  $\delta v = \delta p/m$  is only

$$\delta x \delta p = \hbar \Rightarrow \delta v = m \hbar / \delta x \approx 10^{-24} \text{ m/s} \approx 10^{-12} \text{ thousandths of an inch per year.}$$

- In classical mechanics we usually neglect fluctuations associated with the thermal vibrations of atoms. But any object in thermal equilibrium with its surroundings constantly undergoes changes in size, pressure, and energy, as it interacts with the environment. For example, the internal energy per particle of a sample at temperature  $T$  fluctuates with amplitude

$$\frac{\Delta E}{N} = \frac{1}{\sqrt{N}} \sqrt{k_B T^2 c_V},$$

where  $k_B$  is Boltzmann's constant,  $T$  is the absolute temperature,  $N$  is the number of particles in the sample, and  $c_V$  is the specific heat. Water has a specific heat of 1 cal/K, or around 4 Joule/K. At room temperature of 300 Kelvin, for  $10^{23}$  molecules of water, these values lead to an uncertainty of only  $7.2 \times 10^{-21}$  Joule in the the internal energy of the water. Thermal fluctuations are big enough to visibly move pieces of dust in an optical microscope, and to generate variations in electric currents that are easily measured, but for most engineering mechanics purposes they are negligible.

- general relativity errors having to do with the non-flatness of space are so small that the genius Einstein had trouble finding a place where the deviations from Newtonian mechanics could possibly be observed. Finally he predicted a small, barely measurable effect on the predicted motion of the planet Mercury.

On the other hand, the errors within mechanics, due to imperfect modeling or inaccurate measurement, are, except in extreme situations, far greater than the errors due to the imperfection of mechanics theory. For example, mechanical force measurements are typically off by a percent or so, distance measurements by a part in a thousand, and material properties are rarely known to one part in a hundred and often not one part in 10.

If your engineering mechanics calculations make inaccurate predictions it will surely be because of errors in modeling or measurement, not inaccuracies in the laws of mechanics. Newtonian mechanics, if not perfect, is still rather accurate while relatively much simpler to use than the theories which have 'overthrown' it. To seriously consider mechanics errors as due to neglect of relativity, quantum mechanics, or statistical mechanics, is to pretend to an accuracy that can only be obtained in the rarest of circumstances. You have trusted your life many times to engineers who treated classical mechanics as 'truth' and in the future if you do such, your engineering mechanics work will justly be based on classical mechanics concepts.

# 2 Vectors for mechanics

This book is about the laws of mechanics which were informally introduced in Chapter 1. The most fundamental quantities in mechanics, used to define all the others, are the two scalars, mass  $m$  and time  $t$ , and the two vectors, relative position  $\vec{r}_{i/O}$ , and force  $\vec{F}$ . Scalars are typed with an ordinary font ( $t$  and  $m$ ) and vectors are typed in bold with a harpoon on top ( $\vec{r}_{i/O}$ ,  $\vec{F}$ ). All of the other quantities we use in mechanics are defined in terms of these four. A list of all the scalars and vectors used in mechanics are given in boxes 2 and 2.2 on pages 8 and page 9. Scalar arithmetic has already been your lifelong friend. For mechanics you also need facility with vector arithmetic. Lets start at the beginning.

## What is a vector?

A vector is a (possibly dimensional) quantity that is fully described by its magnitude and direction

whereas scalars are just (possibly dimensional) single numbers<sup>①</sup> As a first vector example, consider a line segment with head and tail ends and a length (magnitude) of 2 cm and pointed Northeast. Lets call this vector  $\vec{A}$  (see fig. 2.1).

$$\vec{A} \stackrel{\text{def}}{=} \text{2 cm long line segment pointed Northeast}$$

Every vector in mechanics is well visualized as an arrow. The direction of the arrow is the direction of the vector. The length of the arrow is proportional to the magnitude of the vector. The magnitude of  $\vec{A}$  is a positive scalar indicated by  $|\vec{A}|$ . A vector does not lose its identity if it is picked up and moved around in space (so long as it is not rotated or stretched). Thus both vectors drawn in fig. 2.1 are  $\vec{A}$ .

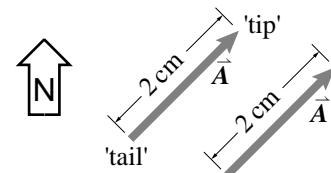


Figure 2.1: Vector  $\vec{A}$  is 2 cm long and points Northeast. Two copies of  $\vec{A}$  are shown.

(Filename: tfigure.northeast)

① By ‘dimensional’ we mean ‘with units’ like meters, Newtons, or kg. We don’t mean having an abstract vector-space dimension, as in one, two or three dimensional.

## Vector arithmetic makes sense

We have oversimplified. We said that a vector is something with magnitude and direction. In fact, by common modern convention, that's not enough. A one way street sign, for example, is not considered a vector even though it has a magnitude (its mass is, say, half a kilogram) and a direction (the direction of most of the traffic). A thing is only called a vector if elementary vector arithmetic, vector addition in particular, has a sensible meaning<sup>②</sup>.

The following sentence summarizes centuries of thought and also motivates this chapter:

The vectors in mechanics have magnitude and direction *and* elementary vector arithmetic with them has a sensible physical meaning.

This chapter is about vector arithmetic. In the rest of this chapter you will learn how to add and subtract vectors, how to stretch them, how to find their components, and how to multiply them with each other two different ways. Each of these operations has use in mechanics and, in particular, the concept of vector addition always has a physical interpretation.

## 2.1 Vector notation and vector addition

Facility with vectors has several aspects.

1. You must recognize which quantities are vectors (such as force) and which are scalars (such as length).
2. You have to use a notation that distinguishes between vectors and scalars using, for example,  $\vec{a}$ , or  $\underline{a}$  for acceleration and  $a$  or  $|\vec{a}|$  for the magnitude of acceleration.
3. You need skills in vector arithmetic, maybe a little more than you have learned in your previous math and physics courses.

In this first section (2.1) we start with notation and go on to the basics of vector arithmetic.

### 2.1 The scalars in mechanics

The scalar quantities used in this book, and their dimensions in brackets [ ], are listed below ( $M$  for mass,  $L$  for length,  $T$  for time,  $F$  for force, and  $E$  for energy).

- mass  $m$ ,  $[M]$ ;
- length or distance  $\ell$ ,  $w$ ,  $x$ ,  $r$ ,  $\rho$ ,  $d$ , or  $s$ ,  $[L]$ ;
- time  $t$ ,  $[T]$ ;
- pressure  $p$ ,  $[F/L^2] = [M/(L \cdot T^2)]$ ;
- angles  $\theta$  ‘theta’,  $\phi$  ‘phi’,  $\gamma$  ‘gamma’, and  $\psi$  ‘psi’, [dimensionless];
- energy  $E$ , kinetic energy  $E_K$ , potential energy  $E_P$ ,  $[E] = [F \cdot L] = [M \cdot L^2/T^2]$ ;
- work  $W$ ,  $[E] = [F \cdot L] = [M \cdot L^2/T^2]$ ;
- tension  $T$ ,  $[M \cdot L/T^2] = [F]$ ;
- power  $P$ ,  $[E/T] = [M \cdot L^2/T^3]$ ;

- the magnitudes of all the vector quantities are also scalars, for example

- speed  $|\vec{v}|$ ,  $[L/T]$ ;
- magnitude of acceleration  $|\vec{a}|$ ,  $[L/T^2]$ ;
- magnitude of angular momentum  $|\vec{H}|$ ,  $[M \cdot L^2/T]$ ;

- the components of vectors, for example

- $r_x$  (where  $\vec{r} = r_x \hat{i} + r_y \hat{j}$ ), or
- $L_{x'}$  (where  $\vec{L} = L_{x'} \hat{i}' + L_{y'} \hat{j}'$ );

- coefficient of friction  $\mu$  ‘mu’, or friction angle  $\phi$  ‘phi’;
- coefficient of restitution  $e$ ;
- mass per unit length, area, or volume  $\rho$ ;
- oscillation frequency  $\beta$  or  $\lambda$ .

### How to write vectors

A scalar is written as a single English or Greek letter. This book uses slanted type for scalars (*e.g.*,  $m$  for mass) but ordinary printing is fine for hand work (*e.g.*,  $m$  for mass). A vector is also represented by a single letter of the alphabet, either English or Greek, but ornamented to indicate that it is a vector and not a scalar. The common ornamentalations are described below.

*Use one of these vector notations in all of your work.*

Various ways of representing vectors in printing and writing are described below.<sup>①</sup>.

- Putting a harpoon (or arrow) over the letter  $F$  is the suggestive notation used in in this book for vectors.
- F** In most texts a bold  $F$  represents the vector  $\vec{F}$ . But bold face is inconvenient for hand written work. The lack of bold face pens and pencils tempts students to transcribe a bold  $F$  as  $F$ . But  $F$  with no adornment represents a scalar and not a vector. Learning how to work with vectors and scalars is hard enough without the added confusion of not being able to tell at a glance which terms in your equations are vectors and which are scalars.
- F Underlining or undersquiggle ( $\underline{F}$ ) is an easy and unambiguous notation for hand writing vectors. A recent poll found that 14 out of 17 mechanics professors use this notation. These professors would copy a  $\vec{F}$  from this book by writing  $\underline{F}$ . Also, in typesetting, an author indicates that a letter should be printed in bold by underlining.
- It is a stroke simpler to put a bar rather than a harpoon over a symbol. But the saved effort causes ambiguity since an over-bar is often used to indicate average.

<sup>①</sup>**Caution:** Be careful to distinguish vectors from scalars all the time. Clear notation helps clear thinking and will help you solve problems. If you notice that you are not using clear vector notation, *stop*, determine which quantities are vectors and which scalars, and fix your notation.

## 2.2 The Vectors in Mechanics

The vector quantities used in mechanics and the notations used in this book are shown below. The dimensions of each are shown in brackets [ ]. Some of these quantities are also shown in figure ??.

- position  $\vec{r}$  or  $\bar{x}$ , [L];
- velocity  $\vec{v}$  or  $\dot{\bar{x}}$  or  $\vec{r}$ , [L/t];
- acceleration  $\vec{a}$  or  $\vec{v}$  or  $\ddot{\bar{x}}$ , [L/t<sup>2</sup>];
- angular velocity  $\vec{\omega}$  ‘omega’ (or, if aligned with the  $\hat{k}$  axis,  $\vec{\theta}\hat{k}$ ), [1/t];
- rate of change of angular velocity  $\vec{\alpha}$  ‘alpha’ or  $\vec{\dot{\omega}}$  (or, if aligned with the  $\hat{k}$  axis,  $\vec{\theta}\hat{k}$ ), [1/t<sup>2</sup>];
- force  $\vec{F}$  or  $\bar{N}$ , [m · L/t<sup>2</sup>] = [F];
- moment or torque  $\vec{M}$ , [m · L<sup>2</sup>/t<sup>2</sup>] = [F · L];
- linear momentum  $\vec{L}$ , [m · L/t] and its rate of change  $\dot{\vec{L}}$ , [m · L/t<sup>2</sup>];
- angular momentum  $\vec{H}$ , [m · L<sup>2</sup>/t]; and its rate of change  $\dot{\vec{H}}$ , [m · L<sup>2</sup>/t<sup>2</sup>].
- unit vectors to help write other vectors [dimensionless]:
  - $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  for cartesian coordinates,

- $\hat{i}'$ ,  $\hat{j}'$ , and  $\hat{k}'$  for crooked cartesian coordinates,
- $\hat{e}_r$  and  $\hat{e}_\theta$  for polar coordinates,
- $\hat{e}_\tau$  and  $\hat{e}_n$  for path coordinates, and
- $\hat{\lambda}$  ‘lambda’ and  $\hat{n}$  as miscellaneous unit vectors.

Subscripts and superscripts are often added to indicate the point, points, body, or bodies the vectors are describing. Upper case letters (O, A, B, C,...) are used to denote points. Upper case calligraphic (or script if you are writing by hand) letters ( $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ... $\mathcal{F}$ ...) are for labeling rigid bodies or reference frames.  $\mathcal{F}$  is the fixed, Newtonian, or ‘absolute’ reference frame (think of  $\mathcal{F}$  as the ground if you are a first time reader).

For example,  $\vec{r}_{AB}$  or  $\vec{r}_{B/A}$  is the position of the point  $B$  relative to the point  $A$ .  $\vec{\omega}_{\mathcal{B}}$  is the absolute angular velocity of the body called  $\mathcal{B}$  ( $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$  is short hand for  $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$ ). And  $\vec{H}_{\mathcal{A}/C}$  is the angular momentum of body  $\mathcal{A}$  relative to point  $C$ .

The notation is further complicated when we want to take derivatives with respect to moving frames, a topic which comes up later in the book. For completeness:  $\mathcal{B}\vec{\omega}_{\mathcal{D}/\mathcal{E}}$  is the time derivative with respect to reference frame  $\mathcal{B}$  of the angular velocity of body  $\mathcal{D}$  with respect to body (or frame)  $\mathcal{E}$ . If this paragraph doesn’t read like gibberish to you, you probably already know dynamics!

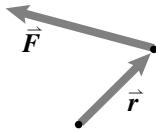


Figure 2.2: Position and force vectors are drawn with different scales.

(Filename:tfigure.posandforce)

There could be confusion, say, between the velocity  $\bar{v}$  and the average speed  $\bar{v}$ .

$\hat{i}$  Over-hat. Putting a hat on top is like an over-arrow or over-bar. In this book we reserve the hat for unit vectors. For example, we use  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ , or  $\hat{e}_1$ ,  $\hat{e}_2$ , and  $\hat{e}_3$  for unit vectors parallel to the  $x$ ,  $y$ , and  $z$  axis, respectively. The same poll of 17 mechanics professors found that 11 of them used no special notation for unit vectors and just wrote them like, e.g.,  $\underline{i}$ .

### Drawing vectors

In fig. 2.1, the magnitude of  $\vec{A}$  was used as the drawing length. But drawing a vector using its magnitude as length would be awkward if, say, we were interested in vector  $\vec{B}$  that points Northwest and has a magnitude of 2 m. To well contain  $\vec{B}$  in a drawing would require a piece of paper about 2 meters square (each edge the length of a basketball player). This situation moves from difficult to ridiculous if the magnitude of the vector of interest is 2 km and it would take half an hour to stroll from tail to tip dragging a purple crayon. Thus in pictures we merely make scale drawings of vectors with, say, one centimeter of graph paper representing 1 kilometer of vector magnitude.

The need for scale drawings to represent vectors is apparent for a vector whose magnitude is not length. Force is a vector since it has magnitude and direction. Say  $\vec{F}_{\text{gr}}$  is the 700 N force that the ground pushes up on your feet as you stand still. We can't draw a line segment with length 700 N for  $\vec{F}_{\text{gr}}$  because a Newton is a unit of force not length. A scale drawing is needed.

One often needs to draw vectors with different units on the same picture, as for showing the position  $\vec{r}$  at which a force  $\vec{F}$  is applied (see fig. 2.2). In this case different scale factors are used for the drawing of the vectors that have different units.

Drawing and measuring are tedious and also not very accurate. And drawing in 3 dimensions is particularly hard (given the short supply of 3D graph paper nowadays). So the magnitudes and directions of vectors are usually defined with numbers and units rather than scale drawings. Nonetheless, the drawing rules, and the geometric descriptions in general, still define vector concepts.

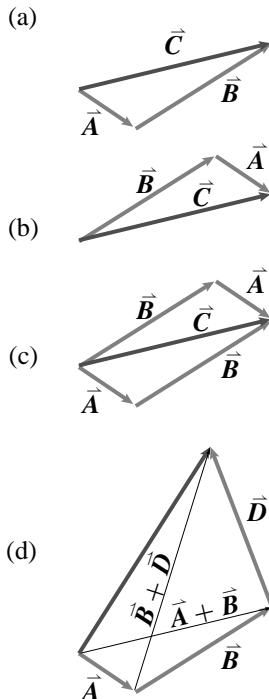


Figure 2.3: (a) tip to tail addition of  $\vec{A} + \vec{B}$ , (b) tip to tail addition of  $\vec{B} + \vec{A}$ , (c) the parallelogram interpretation of vector addition, and (d) The associative law of vector addition.

(Filename:tfigure.tiptotail)

### Adding vectors

The sum of two vectors  $\vec{A}$  and  $\vec{B}$  is defined by the *tip to tail* rule of vector addition shown in fig. 2.3a for the sum  $\vec{C} = \vec{A} + \vec{B}$ . Vector  $\vec{A}$  is drawn. Then vector  $\vec{B}$  is drawn with its tail at the tip (or head) of  $\vec{A}$ . The sum  $\vec{C}$  is the vector from the tail of  $\vec{A}$  to the tip of  $\vec{B}$ .

The same sum is achieved if  $\vec{B}$  is drawn first, as shown in fig. 2.3b. Putting both of ways of adding  $\vec{A}$  and  $\vec{B}$  on the same picture draws a parallelogram as shown in fig. 2.3c. Hence the tip to tail rule of vector addition is also called the *parallelogram* rule. The parallelogram construction shows the *commutative* property of vector addition, namely that  $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ . Note that you can view figs. 2.3a-c as 3D pictures. In 3D, the parallelogram will generally be on some tilted plane.

Three vectors are added by the same tip to tail rule. The construction shown in fig. 2.3d shows that  $(\vec{A} + \vec{B}) + \vec{D} = \vec{A} + (\vec{B} + \vec{D})$  so that the expression  $\vec{A} + \vec{B} + \vec{D}$  is unambiguous. This is the *associative* property of vector addition. This picture is also sensible in 3D where the 6 vectors drawn make up the edges of a tetrahedron which are generally not coplanar.

With these two laws we see that the sum  $\vec{A} + \vec{B} + \vec{D} + \dots$  can be permuted to  $\vec{D} + \vec{A} + \vec{B} + \dots$  or any which way without changing the result. So vector addition shares the associativity and commutivity of scalar addition that you are used to e.g., that  $3 + (7 + \pi) = (\pi + 3) + 7$ .

We can reconsider the statement ‘force is a vector’ and see that it hides one of the basic assumptions in mechanics, namely:

If forces  $\vec{F}_1$  and  $\vec{F}_2$  are applied to a point on a structure they can be replaced, for all mechanics considerations, with a single force  $\vec{F} = \vec{F}_1 + \vec{F}_2$  applied to that point

as illustrated in fig. 2.4. The force  $\vec{F}$  is said to be *equivalent* to the *concurrent* (acting at one point) force system consisting of  $\vec{F}_1$  and  $\vec{F}_2$ .

Note that two vectors with different dimensions cannot be added. Figure 2.2 on page 10 can no more sensibly be taken to represent meaningful vector addition than can the scalar sum of a length and a weight, “2 ft + 3 N”, be taken as meaningful.

## Subtraction and the zero vector $\vec{0}$

Subtraction is most simply defined by inverse addition. Find  $\vec{C} - \vec{A}$  means find the vector which when added to  $\vec{A}$  gives  $\vec{C}$ . We can draw  $\vec{C}$ , draw  $\vec{A}$  and then find the vector which, when added tip to tail to  $\vec{A}$  give  $\vec{C}$ . Fig. 2.3a shows that  $\vec{B}$  answers the question. Another interpretation comes from defining the negative of a vector  $-\vec{A}$  as  $\vec{A}$  with the head and tail switched. Again you can see from fig. 2.3b, by imagining that the head and tail on  $\vec{A}$  were switched that  $\vec{C} + (-\vec{A}) = \vec{B}$ . The negative of a vector evidently has the expected property that  $\vec{A} + (-\vec{A}) = \vec{0}$ , where  $\vec{0}$  is the vector with no magnitude so that  $\vec{C} + \vec{0} = \vec{C}$  for all vectors  $\vec{C}$ .

## Relative position vectors

The concept of relative position permeates most mechanics equations. The position of point B relative to point A is represented by the vector  $\vec{r}_{B/A}$  (pronounced ‘r of B relative to A’) drawn from A and to B (as shown in fig. 2.5). An alternate notation for this vector is  $\vec{r}_{AB}$  (pronounced ‘r A B’ or ‘r A to B’). You can think of the position of B relative to A as being the position of B relative to you if you were standing on A. Similarly  $\vec{r}_{C/B} = \vec{r}_{BC}$  is the position of C relative to B.

Figure 2.5a shows that relative positions add by the tip to tail rule. That is,

$$\vec{r}_{C/A} = \vec{r}_{B/A} + \vec{r}_{C/B} \quad \text{or} \quad \vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$$

so vector addition has a sensible meaning for relative position vectors.

Often when doing problems we pick a distinguished point in space, say a prominent point or corner of a machine or structure, and use it as the origin of a coordinate system O. The position of point A relative to O is  $\vec{r}_{A/O}$  or  $\vec{r}_{OA}$  but we often adopt the shorthand notation  $\vec{r}_A$  (pronounced ‘r A’) leaving the reference point O as implied. Figure 2.5b shows that

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

which rolls off the tongue easily and makes the concept of relative position easier to remember.<sup>①</sup>

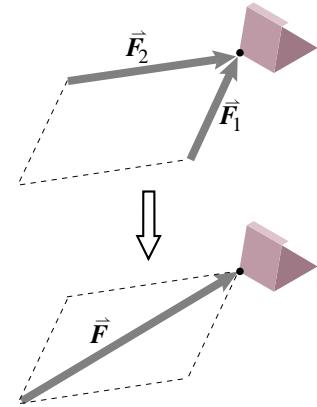


Figure 2.4: Two forces acting at a point may be replaced by their sum for all mechanics purposes.

(Filename:tfigure.forcesadd)

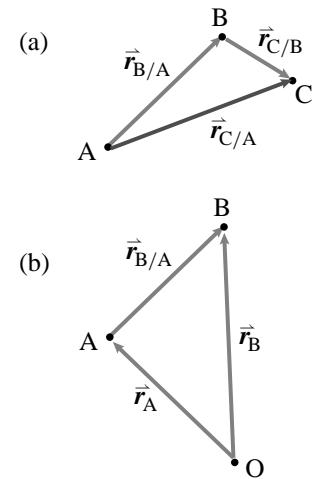


Figure 2.5: a) Relative position of points A, B, and C; b) Relative position of points O, A, and B.

(Filename:tfigure.relpos)

<sup>①</sup> For the first 7 chapters of this book you can just translate ‘relative to’ to mean ‘minus’ as in english. ‘How much money does Rudra have *relative to* Andy?’ means what is Rudra’s wealth *minus* Andy’s wealth? What is the position of B *relative to* A? It is the position of B *minus* the position of A.

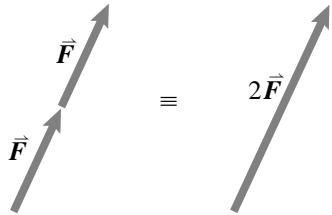


Figure 2.6: Multiplying a vector by a scalar stretches it.

(Filename:tfigure.stretch)

## Multiplying by a scalar stretches a vector

Naturally enough  $2\vec{F}$  means  $\vec{F} + \vec{F}$  (see fig. 2.6) and  $127\vec{A}$  means  $\vec{A}$  added to itself 127 times. Similarly  $\vec{A}/7$  or  $\frac{1}{7}\vec{A}$  means a vector in the direction of  $\vec{A}$  that when added to itself 7 times gives  $\vec{A}$ . By combining these two ideas we can define any rational multiple of  $\vec{A}$ . For example  $\frac{29}{13}\vec{A}$  means add 29 copies of the vector that when added 13 times to itself gives  $\vec{A}$ . We skip the mathematical fine point of extending the definition to  $c\vec{A}$  for  $c$  that are irrational.

We can define  $-17\vec{A}$  as  $17(-\vec{A})$ , combining our abilities to negate a vector and multiply it by a positive scalar. In general, for any positive scalar  $c$  we define  $c\vec{A}$  as the vector that is in the same direction as  $\vec{A}$  but whose magnitude is multiplied by  $c$ . Five times a 5 N force pointed Northeast is a 25 N force pointed Northeast. If  $c$  is negative the direction is changed and the magnitude multiplied by  $|c|$ . Minus 5 times a 5 N force pointed Northeast is a 25 N force pointed SouthWest.

If you imagine stretching a vector addition diagram (e.g., fig. 2.3a on page 10) equally in all directions the distributive rule for scalar multiplication is apparent:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

## Unit vectors have magnitude 1

*Unit vectors* are vectors with a magnitude of one. Unit vectors are useful for indicating direction. Key examples are the unit vectors pointed in the positive  $x$ ,  $y$  and  $z$  directions  $\hat{i}$  (called ‘i hat’ or just ‘i’),  $\hat{j}$ , and  $\hat{k}$ . We distinguish unit vectors by hating them but any undistinguished vector notation will do (e.g., using  $\underline{i}$ ).

An easy way to find a unit vector in the direction of a vector  $\vec{A}$  is to divide  $\vec{A}$  by its magnitude. Thus

$$\hat{\lambda}_A \equiv \frac{\vec{A}}{|\vec{A}|}$$

is a unit vector in the  $\vec{A}$  direction. You can check that this defines a unit vector by looking up at the rules for multiplication by a scalar. Multiplying  $\vec{A}$  by the scalar  $1/|\vec{A}|$  gives a new vector with magnitude  $|\vec{A}|/|\vec{A}| = 1$ .

A common situation is to know that a force  $\vec{F}$  is a yet unknown scalar  $F$  multiplied by a unit vector pointing between known points A and B. (fig. 2.7). We can then write  $\vec{F}$  as

$$\vec{F} = F\hat{\lambda}_{AB} = F \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = F \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

where we have used  $\hat{\lambda}_{AB}$  as the unit vector pointing from A to B.

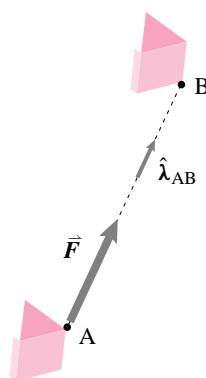


Figure 2.7: (Filename:tfigure.FAtoB)

## Vectors in pictures and sketches.

Some options for drawing vectors are shown in sample ?? on page ?? . The two notations below are the most common.

**Symbolic: labeling an arrow with a vector symbol.** Indicate a vector, say a force

$\vec{F}$ , by drawing an arrow and then labeling it with one of the symbolic notations above as in figure 2.8a. *In this notation, the arrow is only schematic*, the magnitude and direction are determined by the algebraic symbol  $\vec{F}$ . It is sometimes helpful to draw the arrow in the direction of the vector and approximately to scale, but this is not necessary.

**Graphical: a scalar multiplies an arrow.** Indicate a vector's direction by drawing an arrow with direction indicated by marked angles or slopes. The scalar multiple with a nearby scalar symbol, say  $F$ , as shown in figure 2.8b. This means  $F$  times a unit vector in the direction of the arrow. (Because  $F$  might be negative, sign confusion is common amongst beginners. Please see sample 2.1.)

**Combined: graphical representation used to define a symbolic vector.** The full symbolic notation can be used in a picture with the graphical information as a way of defining the symbol. For example if the arrow in fig. 2.8b were labeled with an  $\vec{F}$  instead of just  $F$  we would be showing that  $\vec{F}$  is a scalar multiplied by a unit vector in the direction shown.

## The components of a vector

A given vector, say  $\vec{F}$ , can be described as the sum of vectors each of which is parallel to a coordinate axis. Thus  $\vec{F} = \vec{F}_x + \vec{F}_y$  in 2D and  $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$  in 3D. Each of these vectors can in turn be written as the product of a scalar and a unit vector along the positive axes, e.g.,  $\vec{F}_x = F_x \hat{i}$  (see fig. 2.9). So

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x \hat{i} + F_y \hat{j} \quad (2D)$$

or

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}. \quad (3D)$$

The scalars  $F_x$ ,  $F_y$ , and  $F_z$  are called the components of the vector with respect to the axes  $xyz$ . The components may also be thought of as the orthogonal projections (the shadows) of the vector onto the coordinate axes.

Because the list of components is such a handy way to describe a vector we have a special notation for it. The bracketed expression  $[\vec{F}]_{xyz}$  stands for the list of components of  $\vec{F}$  presented as a horizontal or vertical array (depending on context), as shown below.

$$[\vec{F}]_{xyz} = [F_x, F_y, F_z] \quad \text{or} \quad [\vec{F}]_{xyz} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

If we had an  $xy$  coordinate system with  $x$  pointing East and  $y$  pointing North we could write the components of a 5N force pointed Northeast as  $[\vec{F}]_{xy} = [(5/\sqrt{2}) \text{ N}, (5/\sqrt{2}) \text{ N}]$ .

Note that the components of a vector in some crooked coordinate system  $x'y'z'$  are different than the coordinates for the same vector in the coordinate system  $xyz$  because the projections are different. Even though  $\vec{F} = \vec{F}$  it is *not true* that  $[\vec{F}]_{xyz} = [\vec{F}]_{x'y'z'}$  (see fig. 2.19 on page 25). In mechanics we often make use of multiple coordinate systems. So to define a vector by its components the coordinate system used must be specified.

Rather than using up letters to repeat the same concept we sometimes label the coordinate axes  $x_1$ ,  $x_2$  and  $x_3$  and the unit vectors along them  $\hat{e}_1$ ,  $\hat{e}_2$ , and  $\hat{e}_3$  (thus freeing our minds of the silently pronounced letters  $y,z,j$ , and  $k$ ).

## Manipulating vectors by manipulating components

Because a vector can be represented by its components (once given a coordinate system) we should be able to relate our geometric understanding of vectors to their components. In practice, when push comes to shove, most calculations with vectors are done with components.

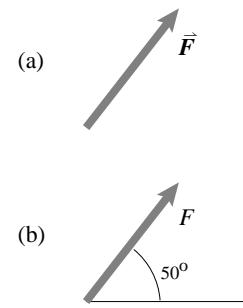


Figure 2.8: Two different ways of drawing a vector (a) shows a labeled arrow. The magnitude and direction of the vector is given by the symbol  $\vec{F}$ , the drawn arrow has no quantitative information. (b) shows an arrow with clearly indicated orientation next to the scalar  $F$ . This means a unit vector in the direction of the arrow multiplied by the scalar  $F$ .

(Filename:tfigure1.d)

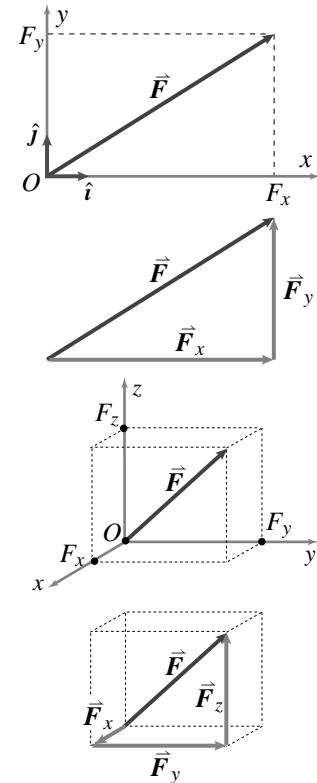


Figure 2.9: A vector can be broken into a sum of vectors, each parallel to the axis of a coordinate system. Each of these is a component multiplied by a unit vector along the coordinate axis, e.g.,  $\vec{F}_x = F_x \hat{i}$ .

(Filename:tfigure.vectproj)

### *Adding and subtracting with components*

Because a vector can be broken into a sum of orthogonal vectors, because addition is associative, and because each orthogonal vector can be written as a component times a unit vector we get the addition rule:

$$[\vec{A} + \vec{B}]_{xyz} = [(A_x + B_x), (A_y + B_y), (A_z + B_z)]$$

which can be described by the tricky words ‘the components of the sum of two vectors are given by the sums of the corresponding components.’ Similarly,

$$[\vec{A} - \vec{B}]_{xyz} = [(A_x - B_x), (A_y - B_y), (A_z - B_z)]$$

### *Multiplying a vector by a scalar using components*

The vector  $\vec{A}$  can be decomposed into the sum of three orthogonal vectors. If  $\vec{A}$  is multiplied by 7 than so must be each of the component vectors. Thus

$$[c\vec{A}]_{xyz} = [cA_x, cA_y, cA_z].$$

The components of a scaled vector are the corresponding scaled components.

## **Magnitude of a vector using components**

The Pythagorean theorem for right triangles ( $A^2 + B^2 = C^2$ ) tells us that

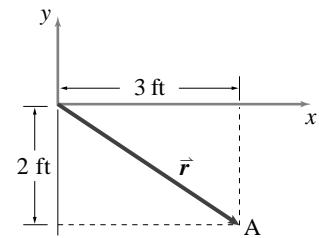
$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}, \quad (2D)$$

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}. \quad (3D)$$

To get the result in 3D the 2D Pythagorean theorem needs to be applied twice successively, first to get the magnitude of the sum  $\vec{F}_x + \vec{F}_y$  and once more to add in  $\vec{F}_z$ .

**SAMPLE 2.1** Drawing a vector from its components: Draw the vector  $\vec{r} = 3 \text{ ft} \hat{i} - 2 \text{ ft} \hat{j}$  using its components.

**Solution** To draw  $\vec{r}$  using its components, we first draw the axes and measure 3 units (any units that we choose on the ruler) along the  $x$ -axis and 2 units along the negative  $y$ -axis. We mark this point as  $A$  (say) on the paper and draw a line from the origin to the point  $A$ . We write the dimensions ‘3 ft’ and ‘2 ft’ on the figure. Finally, we put an arrowhead on this line pointing towards  $A$ .



□ Figure 2.10: A vector  $\vec{r} = 3 \text{ ft} \hat{i} - 2 \text{ ft} \hat{j}$  is drawn by locating its end point which is 3 units away along the  $x$ -axis and 2 units away along the negative  $y$ -axis.

(Filename:sfig1.2.4a)

**SAMPLE 2.2** Drawing a vector from its length and direction: A vector  $\vec{r}$  is 3.6 ft long and is directed  $33.7^\circ$  from the  $x$ -axis towards the negative  $y$ -axis. Draw  $\vec{r}$ .

**Solution** We first draw the  $x$  and  $y$  axes and then draw  $\vec{r}$  as a line from the origin at an angle  $-33.7^\circ$  from the  $x$ -axis (minus sign means measuring clockwise), measure 3.6 units (magnitude of  $\vec{r}$ ) along this line and finally put an arrowhead pointing away from the origin.

□

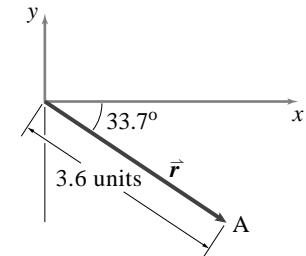


Figure 2.11: A vector  $\vec{r}$  with a given length (3.6 ft) and direction (slope angle  $\theta = -33.7^\circ$ ) is drawn by measuring its length along a line drawn at angle  $\theta$  from the positive  $x$ -axis.

(Filename:sfig1.2.4b)

**Comments:** Note that this is the same vector as in Sample 2.1. In fact, you can easily verify that

$$r_x = r \cos \theta = 3.6 \text{ ft} \cdot \cos(-33.7^\circ) = 3 \text{ ft}$$

and

$$r_y = r \sin \theta = 3.6 \text{ ft} \cdot \sin(-33.7^\circ) = -2 \text{ ft}.$$

Thus

$$\vec{r} = r_x \hat{i} + r_y \hat{j} = (3 \text{ ft}) \hat{i} - (2 \text{ ft}) \hat{j}$$

as given in Sample 2.1.

**SAMPLE 2.3** Various ways of representing a vector: A vector  $\vec{F} = 3\hat{i} + 3\hat{j}$  is represented in various ways, some incorrect, in the following figures. The base vectors used are shown first. Comment on each representation, whether it is correct or incorrect, and why.

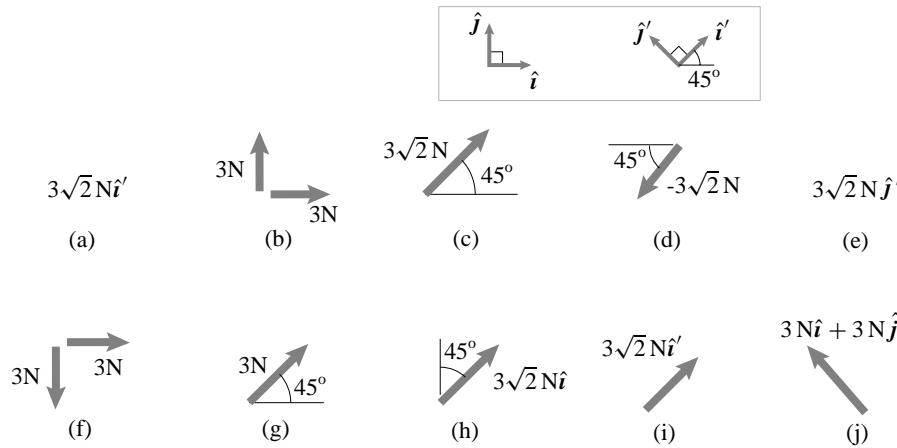


Figure 2.12: (Filename:sfig2.vectors.rep)

**Solution** The given vector is a force with components of 3 N each in the positive  $\hat{i}$  and  $\hat{j}$  directions using the unit vectors  $\hat{i}$  and  $\hat{j}$  shown in the box above. The unit vectors  $\hat{i}'$ , and  $\hat{j}'$  are also shown.

**a) Correct:**  $3\sqrt{2} N \hat{i}'$ . From the picture defining  $\hat{i}'$ , you can see that  $\hat{i}'$  is a unit vector with equal components in the  $\hat{i}$  and  $\hat{j}$  directions; *i.e.*, it is parallel to  $\vec{F}$ . So  $\vec{F}$  is given by its magnitude  $\sqrt{(3 N)^2 + (3 N)^2}$  times a unit vector in its direction, in this case  $\hat{i}'$ . It is the same vector.

**b) Correct:** Here two vectors are shown: one with magnitude 3 N in the direction of the horizontal arrow  $\hat{i}$ , and one with magnitude 3 N in the direction of the vertical arrow  $\hat{j}$ . When two forces act on an object at a point, their effect is additive. So the net vector is the sum of the vectors shown. That is,  $3\hat{i} + 3\hat{j}$ . It is the same vector.

**c) Correct:** Here we have a scalar  $3\sqrt{2} N$  next to an arrow. The vector described is the scalar multiplied by a unit vector in the direction of the arrow. Since the arrow's direction is marked as the same direction as  $\hat{i}'$ , which we already know is parallel to  $\vec{F}$ , this vector represents the same vector  $\vec{F}$ . It is the same vector.

**d) Correct:** The scalar  $-3\sqrt{2} N$  is multiplied by a unit vector in the direction indicated,  $-\hat{i}'$ . So we get  $(-3\sqrt{2} N)(-\hat{i}')$  which is  $3\sqrt{2} N \hat{i}'$  as before. It is the same vector.

**e) Incorrect:**  $3\sqrt{2} N \hat{j}'$ . The magnitude is right, but the direction is off by 90 degrees. It is a different vector.

**f) Incorrect:**  $3N\hat{i} - 3N\hat{j}$ . The  $\hat{i}$  component of the vector is correct but the  $\hat{j}$  component is in the opposite direction. The vector is in the wrong direction by 90 degrees. It is a different vector.

**g) Incorrect:** Right direction but the magnitude is off by a factor of  $\sqrt{2}$ .

**h) Incorrect:** The magnitude is right. The direction indicated is right. But, the algebraic symbol  $3\sqrt{2} N\hat{i}$  takes precedence and it is in the wrong direction ( $\hat{i}$  instead of  $\hat{i}'$ ). It is a different vector.

**i) Correct:** A labeled arrow. The arrow is only schematic. The algebraic symbols  $3\sqrt{2} N\hat{i}'$  define the vector. We draw the arrow to remind us that there is a vector to represent. The tip or tail of the arrow would be drawn at the point of the force application. In this case, the arrow is drawn in the direction of  $\vec{F}$  but it need not.

**j) Correct:** Like (i) above, the directional and magnitude information is in the algebraic symbols  $3 N\hat{i} + 3 N\hat{j}$ . The arrow is there to indicate a vector. In this case, it points in the wrong direction so is not ideally communicative. But (j) still correctly represents the given vector. It is the same vector.

□

**SAMPLE 2.4 Adding vectors:** Three forces,  $\vec{F}_1 = 2 \text{ N}\hat{i} + 3 \text{ N}\hat{j}$ ,  $\vec{F}_2 = -10 \text{ N}\hat{j}$ , and  $\vec{F}_3 = 3 \text{ N}\hat{i} + 1 \text{ N}\hat{j} - 5 \text{ N}\hat{k}$ , act on a particle. Find the net force on the particle.

**Solution** The net force on the particle is the vector sum of all the forces, *i.e.*,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= (2 \text{ N}\hat{i} + 3 \text{ N}\hat{j}) + (-10 \text{ N}\hat{j}) + (3 \text{ N}\hat{i} + 1 \text{ N}\hat{j} - 5 \text{ N}\hat{k}) \\ &= 2 \text{ N}\hat{i} + 3 \text{ N}\hat{j} + 0\hat{k} \\ &\quad + 0\hat{i} - 10 \text{ N}\hat{j} + 0\hat{k} \\ &\quad + 3 \text{ N}\hat{i} + 1 \text{ N}\hat{j} - 5 \hat{k} \\ &= (2 \text{ N} + 3 \text{ N})\hat{i} + (3 \text{ N} - 10 \text{ N} + 1 \text{ N})\hat{j} + (-5 \text{ N})\hat{k} \\ &= 5 \text{ N}\hat{i} - 6 \text{ N}\hat{j} - 5 \text{ N}\hat{k}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 5 \text{ N}\hat{i} - 6 \text{ N}\hat{j} - 5 \text{ N}\hat{k}}$$

**Comments:** In general, we do not need to write the summation so elaborately. Once you feel comfortable with the idea of summing only similar components in a vector sum, you can do the calculation in two lines.

**SAMPLE 2.5 Subtracting vectors:** Two forces  $\vec{F}_1$  and  $\vec{F}_2$  act on a body. The net force on the body is  $\vec{F}_{\text{net}} = 2 \text{ N}\hat{i}$ . If  $\vec{F}_1 = 10 \text{ N}\hat{i} - 10 \text{ N}\hat{j}$ , find the other force  $\vec{F}_2$ .

**Solution**

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 \\ \Rightarrow \vec{F}_2 &= \vec{F}_{\text{net}} - \vec{F}_1 \\ &= 2 \text{ N}\hat{i} - (10 \text{ N}\hat{i} - 10 \text{ N}\hat{j}) \\ &= (2 \text{ N} - 10 \text{ N})\hat{i} - (-10 \text{ N})\hat{j} \\ &= -8 \text{ N}\hat{i} + 10 \text{ N}\hat{j}.\end{aligned}$$

$$\boxed{\vec{F}_2 = -8 \text{ N}\hat{i} + 10 \text{ N}\hat{j}}$$

**SAMPLE 2.6 Scalar times a vector:** Two forces acting on a particle are  $\vec{F}_1 = 100 \text{ N}\hat{i} - 20 \text{ N}\hat{j}$  and  $\vec{F}_2 = 40 \text{ N}\hat{j}$ . If  $\vec{F}_1$  is doubled, does the net force double?

**Solution**

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = (100 \text{ N}\hat{i} - 20 \text{ N}\hat{j}) + (40 \text{ N}\hat{j}) \\ &= 100 \text{ N}\hat{i} + 20 \text{ N}\hat{j}\end{aligned}$$

After  $\vec{F}_1$  is doubled, the new net force  $\vec{F}_{(\text{net})_2}$  is

$$\begin{aligned}\vec{F}_{(\text{net})_2} &= 2\vec{F}_1 + \vec{F}_2 = 2(100 \text{ N}\hat{i} - 20 \text{ N}\hat{j}) + (40 \text{ N}\hat{j}) \\ &= 200 \text{ N}\hat{i} - 40 \text{ N}\hat{j} + 40 \text{ N}\hat{j} \\ &= 200 \text{ N}\hat{i} \neq 2 \underbrace{(100 \text{ N}\hat{i} + 20 \text{ N}\hat{j})}_{\vec{F}_{\text{net}}}\end{aligned}$$

No, the net force does not double.

**SAMPLE 2.7 Magnitude and direction of a vector:** The velocity of a car is given by  $\vec{v} = (30\hat{i} + 40\hat{j})$  mph.

- Find the speed (magnitude of  $\vec{v}$ ) of the car.
- Find a unit vector in the direction of  $\vec{v}$ .
- Write the velocity vector as a product of its magnitude and the unit vector.

### Solution

- (a) **Magnitude of  $\vec{v}$ :** The magnitude of a vector is the length of the vector. It is a scalar quantity, usually represented by the same letter as the vector but without the vector notation (i.e. no bold face, no underbar). It is also represented by the modulus of the vector (the vector written between two vertical lines). The length of a vector is the square root of the sum of squares of its components. Therefore, for

$$\begin{aligned}\vec{v} &= 30 \text{ mph} \hat{i} + 40 \text{ mph} \hat{j}, \\ v = |\vec{v}| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(30 \text{ mph})^2 + (40 \text{ mph})^2} \\ &= 50 \text{ mph}\end{aligned}$$

which is the speed of the car.

$$\boxed{\text{speed} = 50 \text{ mph}}$$

- (b) **Direction of  $\vec{v}$  as a unit vector along  $\vec{v}$ :** The direction of a vector can be specified by specifying a unit vector along the given vector. In many applications you will encounter in dynamics, this concept is useful. The unit vector along a given vector is found by dividing the given vector with its magnitude. Let  $\hat{\lambda}_v$  be the unit vector along  $\vec{v}$ . Then,

$$\begin{aligned}\hat{\lambda}_v &= \frac{\vec{v}}{|\vec{v}|} = \frac{30 \text{ mph} \hat{i} + 40 \text{ mph} \hat{j}}{50 \text{ mph}} \\ &= 0.6\hat{i} + 0.8\hat{j}. \quad (\text{unit vectors have no units!})\end{aligned}$$

$$\boxed{\hat{\lambda}_v = 0.6\hat{i} + 0.8\hat{j}}$$

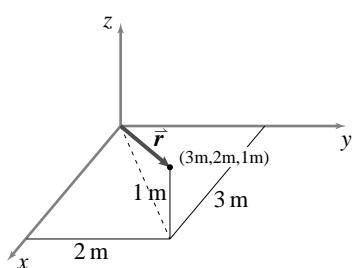
- (c)  **$\vec{v}$  as a product of its magnitude and the unit vector  $\hat{\lambda}_v$ :** A vector can be written in terms of its components, as given in this problem, or as a product of its magnitude and direction (given by a unit vector). Thus we may write,

$$\vec{v} = |\vec{v}|\hat{\lambda}_v = 50 \text{ mph}(0.6\hat{i} + 0.8\hat{j})$$

which, of course, is the same vector as given in the problem.

$$\boxed{\vec{v} = 50 \text{ mph}(0.6\hat{i} + 0.8\hat{j})}$$

**SAMPLE 2.8 Position vector from the origin:** In the  $xyz$  coordinate system, a particle is located at the coordinate  $(3\text{m}, 2\text{m}, 1\text{m})$ . Find the position vector of the particle.



**Solution** The position vector of the particle at P is a vector drawn from the origin of the coordinate system to the position P of the particle. See Fig. 2.13. We can write this vector as

$$\vec{r}_P = (3\text{m})\hat{i} + (2\text{m})\hat{j} + (1\text{m})\hat{k}$$

or       $\vec{r}_P = (3\hat{i} + 2\hat{j} + \hat{k})\text{m}.$

$$\boxed{\vec{r}_P = 3\hat{i} + 2\hat{j} + \hat{k}}$$

Figure 2.13: The position vector of the particle is a vector drawn from the origin of the coordinate system to the position of the particle.

(Filename:sfig2.vec1.6)

**SAMPLE 2.9 Relative position vector:** Let A  $(2\text{m}, 1\text{m}, 0)$  and B  $(0, 3\text{m}, 2\text{m})$  be two points in the  $xyz$  coordinate system. Find the position vector of point B with respect to point A, i.e., find  $\vec{r}_{AB}$  (or  $\vec{r}_{B/A}$ ).

**Solution** From the geometry of the position vectors shown in Fig. 2.14 and the rules of vector sums, we can write,

$$\begin{aligned}\vec{r}_B &= \vec{r}_A + \vec{r}_{AB} \\ \Rightarrow \vec{r}_{AB} &= \vec{r}_B - \vec{r}_A \\ &= (0\hat{i} + 3\hat{m}\hat{j} + 2\hat{m}\hat{k}) - (2\hat{m}\hat{i} + 1\hat{m}\hat{j} + 0\hat{k}) \\ &= -2\hat{m}\hat{i} + 2\hat{m}\hat{j} + 2\hat{m}\hat{k}.\end{aligned}$$

$$\boxed{\vec{r}_{AB} \equiv \vec{r}_{B/A} = -2\hat{m}\hat{i} + 2\hat{m}\hat{j} + 2\hat{m}\hat{k}}$$

Figure 2.14: The position vector of B with respect to A is found from  $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$ .

(Filename:sfig2.vec1.7)

**SAMPLE 2.10** *Finding a force vector given its magnitude and line of action:* A string is pulled with a force  $F = 100 \text{ N}$  as shown in the Fig. 2.15. Write  $F$  as a vector.

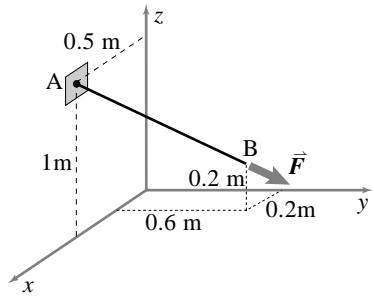


Figure 2.15: (Filename:sfig1.2.2)

**Solution** A vector can be written, as we just showed in the previous sample problem, as the product of its magnitude and a unit vector along the given vector. Here, the magnitude of the force is given and we know it acts along  $\overrightarrow{AB}$ . Therefore, we may write

$$\vec{F} = F \hat{\lambda}_{AB}$$

where  $\hat{\lambda}_{AB}$  is a unit vector along  $\overrightarrow{AB}$ . So now we need to find  $\hat{\lambda}_{AB}$ . We can easily find  $\hat{\lambda}_{AB}$  if we know vector  $\overrightarrow{AB}$ . Let us denote vector  $\overrightarrow{AB}$  by  $\vec{r}_{AB}$  (sometimes we will also write it as  $\vec{r}_{B/A}$  to represent the *position of B with respect to A* as a vector). Then,

$$\hat{\lambda}_{AB} = \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|}.$$

To find  $\vec{r}_{AB}$ , we note that (see Fig. 2.16)

$$\vec{r}_A + \vec{r}_{AB} = \vec{r}_B$$

where  $\vec{r}_A$  and  $\vec{r}_B$  are the position vectors of point A and point B respectively. Hence,

$$\begin{aligned} \vec{r}_{B/A} &= \vec{r}_{AB} = \vec{r}_B - \vec{r}_A \\ &= (0.2\hat{m} + 0.6\hat{m}\hat{j} + 0.2\hat{m}\hat{k}) - (0.5\hat{m}\hat{i} + 1.0\hat{m}\hat{j}) \\ &= -0.3\hat{m}\hat{i} + 0.6\hat{m}\hat{j} - 0.8\hat{m}\hat{k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\lambda}_{AB} &= \frac{-0.3\hat{m}\hat{i} + 0.6\hat{m}\hat{j} - 0.8\hat{m}\hat{k}}{\sqrt{(-0.3)^2 + (0.6)^2 + (-0.8)^2}} \text{ m} \\ &= -0.29\hat{i} + 0.57\hat{j} - 0.77\hat{k}, \end{aligned}$$

and, finally,

$$\begin{aligned} \vec{F} &= (\overbrace{100 \text{ N}}^F) \hat{\lambda}_{AB} \\ &= -29 \text{ N}\hat{i} + 57 \text{ N}\hat{j} - 77 \text{ N}\hat{k}. \end{aligned}$$

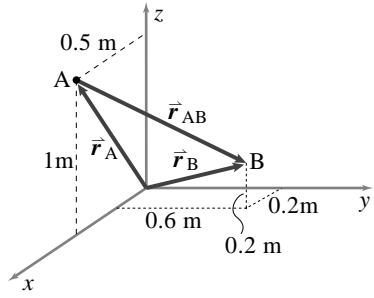


Figure 2.16:  $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$ .

(Filename:sfig1.2.2b)

$$\boxed{\vec{F} = -29 \text{ N}\hat{i} + 57 \text{ N}\hat{j} - 77 \text{ N}\hat{k}}$$

**SAMPLE 2.11 Adding vectors on computers:** The following six forces act at different points of a structure.  $\vec{F}_1 = -3 \text{ N} \hat{j}$ ,  $\vec{F}_2 = 20 \text{ N} \hat{i} - 10 \text{ N} \hat{j}$ ,  $\vec{F}_3 = 1 \text{ N} \hat{i} + 20 \text{ N} \hat{j} - 5 \text{ N} \hat{k}$ ,  $\vec{F}_4 = 10 \text{ N} \hat{i}$ ,  $\vec{F}_5 = 5 \text{ N} (\hat{i} + \hat{j} + \hat{k})$ ,  $\vec{F}_6 = -10 \text{ N} \hat{i} - 10 \text{ N} \hat{j} + 2 \text{ N} \hat{k}$ .

- Write all the force vectors in column form.
- Find the net force by hand calculation.
- Write a computer program to sum  $n$  vectors, each of length 3. Use your program to compute the net force.

### Solution

- (a) The 3-D vector  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$  is represented as a column (or a row) as follows:

$$[\vec{F}] = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz}$$

Following this convention, we write the given forces as

$$[\vec{F}_1] = \begin{pmatrix} 0 \\ -3 \text{ N} \\ 0 \end{pmatrix}_{xyz}, [\vec{F}_2] = \begin{pmatrix} 20 \text{ N} \\ -10 \text{ N} \\ 0 \end{pmatrix}_{xyz}, \dots, [\vec{F}_6] = \begin{pmatrix} -10 \text{ N} \\ -10 \text{ N} \\ 2 \text{ N} \end{pmatrix}_{xyz}$$

- (b) The net force,  $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$  or

$$\begin{aligned} [\vec{F}_{\text{net}}] &= \begin{pmatrix} 0 & 20 & 1 & 10 & 5 & -10 \\ -3 & + & -10 & + & 20 & + \\ 0 & 0 & -5 & 0 & 5 & 2 \end{pmatrix}_{xyz} \text{ N} \\ &= \begin{pmatrix} 26 \\ 2 \\ 2 \end{pmatrix}_{xyz} \text{ N} \end{aligned}$$

- (c) The steps to do this addition on computers are as follows.

- Enter the vectors as rows or columns:

```
F1 = [ 0  -3  0 ]
F2 = [ 20 -10  0 ]
F3 = [ 1  20 -5 ]
F4 = [ 10  0  0 ]
F5 = [ 5   5  5 ]
F6 = [ -10 -10  2 ]
```

- Sum the vectors, using a summing operation that automatically does element by element addition of vectors:

```
Fnet = F1 + F2 + F3 + F4 + F5 + F6
```

- The computer generated answer is:

```
Fnet = [ 26  2  2 ].
```

$$\boxed{\vec{F}_{\text{net}} = 26 \text{ N} \hat{i} + 2 \text{ N} \hat{j} + 2 \text{ N} \hat{k}}$$

## 2.2 The dot product of two vectors

The dot product is used to project a vector in a given direction, to reduce a vector to components, to reduce vector equations to scalar equations, to define work and power, and to help solve geometry problems.

The *dot product* of two vectors  $\vec{A}$  and  $\vec{B}$  is written  $\vec{A} \cdot \vec{B}$  (pronounced ‘A dot B’). The dot product of  $\vec{A}$  and  $\vec{B}$  is the product of the magnitudes of the two vectors times a number that expresses the degree to which  $\vec{A}$  and  $\vec{B}$  are parallel:  $\cos \theta_{AB}$ , where  $\theta_{AB}$  is the angle between  $\vec{A}$  and  $\vec{B}$ . That is,

$$\vec{A} \cdot \vec{B} \stackrel{\text{def}}{=} |\vec{A}| |\vec{B}| \cos \theta_{AB}$$

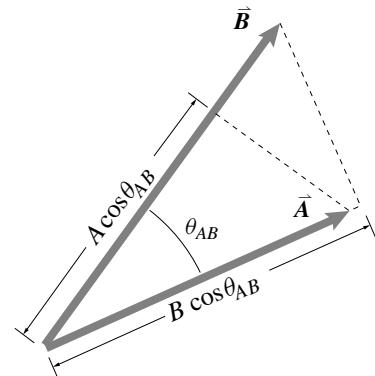


Figure 2.17: The dot product of  $\vec{A}$  and  $\vec{B}$  is a scalar and so is not easily drawn. It is given by  $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$  which is  $A$  times the projection of  $\vec{B}$  in the  $\vec{A}$  direction and also  $B$  times the projection of  $\vec{A}$  in the  $\vec{B}$  direction.

(Filename:tfigure1.11)

which is sometimes written more concisely as  $\vec{A} \cdot \vec{B} = AB \cos \theta$ . One special case is when  $\cos \theta_{AB} = 1$ ,  $\vec{A}$  and  $\vec{B}$  are parallel, and  $\vec{A} \cdot \vec{B} = AB$ . Another is when  $\cos \theta_{AB} = 0$ ,  $\vec{A}$  and  $\vec{B}$  are perpendicular, and  $\vec{A} \cdot \vec{B} = 0$ .<sup>①</sup>

The dot product of two vectors is a scalar. So the dot product is sometimes called the *scalar product*. Using the geometric definition of dot product, and the rules for vector addition we have already discussed, you can convince yourself of (or believe) the following properties of dot products.

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$  commutative law,  
 $AB \cos \theta = BA \cos \theta$
- $(a\vec{A}) \cdot \vec{B} = \vec{A} \cdot (a\vec{B}) = a(\vec{A} \cdot \vec{B})$  a distributive law,  
 $(aA)B \cos \theta = A(aB) \cos \theta$
- $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$  another distributive law,  
the projection of  $\vec{B} + \vec{C}$  onto  $\vec{A}$  is the sum of the two separate projections
- $\vec{A} \cdot \vec{B} = 0$  if  $\vec{A} \perp \vec{B}$  perpendicular vectors have zero for a dot product,  $AB \cos \pi/2 = 0$
- $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$  if  $\vec{A} \parallel \vec{B}$  parallel vectors have the product of their magnitudes for a dot product,  
 $AB \cos 0 = AB$ . In particular,  $\vec{A} \cdot \vec{A} = A^2$  or  $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$
- $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ ,  
 $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$  The standard base vectors used with cartesian coordinates are unit vectors and they are perpendicular to each other. In math language they are ‘orthonormal.’
- $\hat{i}' \cdot \hat{i}' = \hat{j}' \cdot \hat{j}' = \hat{k}' \cdot \hat{k}' = 1$ ,  
 $\hat{i}' \cdot \hat{j}' = \hat{j}' \cdot \hat{k}' = \hat{k}' \cdot \hat{i}' = 0$  The standard crooked base vectors are orthonormal.

The identities above lead to the following equivalent ways of expressing the dot product of  $\vec{A}$  and  $\vec{B}$  (see box 2.2 on page 24 to see how the component formula follows from the geometric definition above):

<sup>①</sup> If you don’t know, almost without a thought, that  $\cos 0 = 1$ ,  $\cos \pi/2 = 0$ ,  $\sin 0 = 0$ , and  $\sin \pi/2 = 1$  now is as good a time as any to draw as many triangles and unit circles as it takes to cement these special cases into your head.

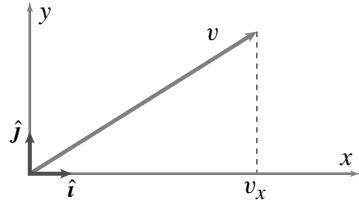


Figure 2.18: The dot product with unit vectors gives projection. For example,  $v_x = \vec{v} \cdot \hat{i}$ .

(Filename:figure1.3.dotprod)

$$\begin{aligned}
 \vec{A} \cdot \vec{B} &= |\vec{A}| |\vec{B}| \cos \theta_{AB} \\
 &= A_x B_x + A_y B_y + A_z B_z \quad (\text{component formula for dot product}) \\
 &= A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'} \\
 &= |\vec{A}| \cdot [\text{projection of } \vec{B} \text{ in the } \vec{A} \text{ direction}] \\
 &= |\vec{B}| \cdot [\text{projection of } \vec{A} \text{ in the } \vec{B} \text{ direction}]
 \end{aligned}$$

## Using the dot product to find components

To find the  $x$  component of a vector or vector expression one can use the dot product of the vector (or expression) with a unit vector in the  $x$  direction as in figure 2.18. In particular,

$$v_x = \vec{v} \cdot \hat{i}.$$

This idea can be used for finding components in any direction. If one knows the orientation of the crooked unit vectors  $\hat{i}', \hat{j}', \hat{k}'$  relative to the standard bases  $\hat{i}, \hat{j}, \hat{k}$  then all the angles between the base vectors are known. So one can evaluate the dot products between the standard base vectors and the crooked base vectors. In 2-D

## 2.3 THEORY

### Using the geometric definition of the dot product to find the dot product in terms of components

Vectors are essentially a geometric concept and we have consequently defined the dot product geometrically as  $\vec{A} \cdot \vec{B} = AB \cos \theta$ . Almost 400 years ago René Descartes discovered that you could do geometry by doing algebra on the coordinates of points.

So we should be able to figure out the dot product of two vectors by knowing their components. The central key to finding this component formula is the distributive law  $(\vec{A} \cdot (\vec{B} + \vec{C})) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ . If we write  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$  and  $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$  then we just repeatedly use the distributive law as follows.

$$\begin{aligned}
 \vec{A} \cdot \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\
 &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot B_x \hat{i} + \\
 &\quad (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot B_y \hat{j} + \\
 &\quad (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot B_z \hat{k} \\
 &= A_x B_x \hat{i} \cdot \hat{i} + A_y B_x \hat{j} \cdot \hat{i} + A_z B_x \hat{k} \cdot \hat{i} + \\
 &\quad A_x B_y \hat{i} \cdot \hat{j} + A_y B_y \hat{j} \cdot \hat{j} + A_z B_y \hat{k} \cdot \hat{j} + \\
 &\quad A_x B_z \hat{i} \cdot \hat{k} + A_y B_z \hat{j} \cdot \hat{k} + A_z B_z \hat{k} \cdot \hat{k} \\
 &= A_x B_x (1) + A_y B_x (0) + A_z B_x (0) +
 \end{aligned}$$

$$\begin{aligned}
 &A_x B_y (0) + A_y B_y (1) + A_z B_y (0) + \\
 &A_x B_z (0) + A_y B_z (0) + A_z B_z (1)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \quad \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \quad (3D). \\
 \Rightarrow \quad \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y \quad (2D).
 \end{aligned}$$

The demonstration above could have been carried out using a different orthogonal coordinate system  $x'y'z'$  that was crooked with respect to the  $xyz$  system. By identical reasoning we would find that  $\vec{A} \cdot \vec{B} = A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'}$ . Even though all of the numbers in the list  $[A_x, A_y, A_z]$  might be different from the numbers in the list  $[A_{x'}, A_{y'}, A_{z'}]$  and similarly all the list  $[\vec{B}]_{xyz}$  might be different than the list  $[\vec{B}]_{x'y'z'}$ , so (somewhat remarkably),

$$A_x B_x + A_y B_y + A_z B_z = A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'}.$$

If we call our coordinate  $x_1, x_2$ , and  $x_3$ ; and our unit base vectors  $\hat{e}_1, \hat{e}_2$ , and  $\hat{e}_3$  we would have  $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$  and  $\vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3$  and the dot product has the tidy form:  $\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_{i=1}^3 A_i B_i$ .

assume that the dot products between the standard base vectors and the vector  $\hat{j}'$  (i.e.,  $\hat{i} \cdot \hat{j}'$ ,  $\hat{j} \cdot \hat{j}'$ ) are known. One can then use the dot product to find the  $x'y'$  components ( $A_{x'}$ ,  $A_{y'}$ ) from the  $xy$  coordinates ( $A_x$ ,  $A_y$ ). For example, as shown in 2-D in figure 2.19, we can start with the obvious equation

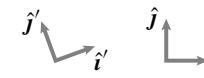
$$\vec{A} = \vec{A}$$

and dot both sides with  $\hat{j}'$  to get:

$$\begin{aligned}\underbrace{\vec{A} \cdot \hat{j}'}_0 &= \underbrace{\vec{A} \cdot \hat{j}'}_0 \\ \underbrace{(A_x \hat{i}' + A_y \hat{j}') \cdot \hat{j}'}_A &= \underbrace{(A_x \hat{i} + A_y \hat{j}) \cdot \hat{j}'}_A \\ A_x' \underbrace{\hat{i}' \cdot \hat{j}'}_0 + A_{y'} \underbrace{\hat{j}' \cdot \hat{j}'}_1 &= A_x \hat{i} \cdot \hat{j}' + A_y \hat{j} \cdot \hat{j}' \\ A_{y'} &= A_x \underbrace{(\hat{i} \cdot \hat{j}')}_{-\sin \theta} + A_y \underbrace{(\hat{j} \cdot \hat{j}')}_{\cos \theta}\end{aligned}$$

Similarly, one could find the component  $A_{x'}$  using a dot product with  $\hat{i}'$ .

This technique of finding components is useful when one problem uses more than one base vector system.



$$\begin{aligned}\vec{A} &= A_x \hat{i} + A_y \hat{j} \\ \vec{A} &= A_x \hat{i}' + A_{y'} \hat{j}'\end{aligned}$$

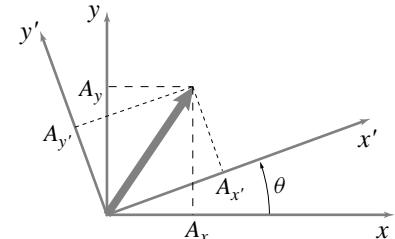


Figure 2.19: The dot product helps find components in terms of crooked unit vectors. For example,  $A_{y'} = \vec{A} \cdot \hat{j}' = A_x(\hat{i} \cdot \hat{j}') + A_y(\hat{j} \cdot \hat{j}') = A_x(-\sin \theta) + A_y(\cos \theta)$ .

(Filename: tfigure1.3.dotprod.a)

## Using dot products with other than $\hat{i}$ , $\hat{j}$ , or $\hat{k}$

It is often useful to use dot products to get scalar equations using vectors other than  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ .

**Example: Getting scalar equations without dotting with  $\hat{i}$ ,  $\hat{j}$ , or  $\hat{k}$**

Given the vector equation.

$$-mg\hat{j} + N\hat{n} = ma\hat{\lambda}$$

where it is known that the unit vector  $\hat{n}$  is perpendicular to the unit vector  $\hat{\lambda}$ , we can get a scalar equation by dotting both sides with  $\hat{\lambda}$  which we write as follows

$$\begin{aligned}\left\{ (-mg\hat{j} + N\hat{n}) = (ma\hat{\lambda}) \right\} \cdot \hat{\lambda} \\ (-mg\hat{j} + N\hat{n}) \cdot \hat{\lambda} = (ma\hat{\lambda}) \cdot \hat{\lambda} \\ -mg\hat{j} \cdot \hat{\lambda} + N \underbrace{\hat{n} \cdot \hat{\lambda}}_0 = ma \underbrace{\hat{\lambda} \cdot \hat{\lambda}}_1 \\ -mg\hat{j} \cdot \hat{\lambda} = ma.\end{aligned}$$

Then we find  $\hat{j} \cdot \hat{\lambda}$  as the cosine of the angle between  $\hat{j}$  and  $\hat{\lambda}$ . We have thus turned our vector equation into a scalar equation and eliminated the unknown  $N$  at the same time.  $\square$

## Using dot products to solve geometry problems

We have seen how a vector can be broken down into a sum of components each parallel to one of the orthogonal base vectors. Another useful decomposition is this: Given any vector  $\vec{A}$  and a unit vector  $\hat{\lambda}$  the vector  $\vec{A}$  can be written as the sum of two parts,

$$\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$$

where  $\vec{A}^{\parallel}$  is parallel to  $\hat{\lambda}$  and  $\vec{A}^{\perp}$  is perpendicular to  $\hat{\lambda}$  (see fig. 2.20). The part parallel to  $\hat{\lambda}$  is a vector pointed in the  $\hat{\lambda}$  direction that has the magnitude of the projection of  $\vec{A}$  in that direction,

$$\vec{A}^{\parallel} = (\vec{A} \cdot \hat{\lambda})\hat{\lambda}.$$

The perpendicular part of  $\vec{A}$  is just what you get when you subtract out the parallel part, namely,

$$\vec{A}^{\perp} = \vec{A} - \vec{A}^{\parallel} = \vec{A} - (\vec{A} \cdot \hat{\lambda})\hat{\lambda}$$

The claimed properties of the decomposition can now be checked, namely that  $\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$  (just add the 2 equations above and see), that  $\vec{A}^{\parallel}$  is in the direction of  $\hat{\lambda}$  (its a scalar multiple), and that  $\vec{A}^{\perp}$  is perpendicular to  $\hat{\lambda}$  (evaluate  $\vec{A}^{\perp} \cdot \hat{\lambda}$  and find 0).

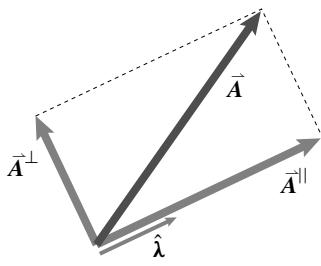
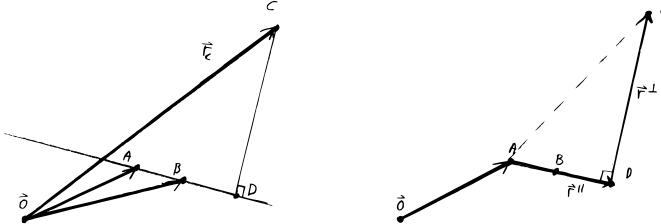


Figure 2.20: For any  $\vec{A}$  and  $\hat{\lambda}$ ,  $\vec{A}$  can be decomposed into a part parallel to  $\hat{\lambda}$  and a part perpendicular to  $\hat{\lambda}$ .

(Filename:figure.Graham1)



**Example.** Given the positions of three points  $\vec{r}_A$ ,  $\vec{r}_B$ , and  $\vec{r}_C$  what is the position of the point D on the line AB that is closest to C? The answer is,

$$\vec{r}_D = \vec{r}_A + \vec{r}_{C/A}^{\parallel}$$

where  $\vec{r}_{C/A}^{\parallel}$  is the part of  $\vec{r}_{C/A}$  that is parallel to the line segment AB. Thus,

$$\vec{r}_D = \vec{r}_A + (\vec{r}_C - \vec{r}_A) \cdot \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}.$$

□

Likewise we could find the parts of a vector  $\vec{A}$  in and perpendicular to a given plane. If the plane is defined by two vectors that are not necessarily orthogonal we could follow these steps. First find two vectors in the plane that are orthogonal, using the method above. Next subtract from  $\vec{A}$  the part of it that is parallel to each of the two orthogonal vectors in the plane. In math lingo the execution of this process goes by the intimidating name ‘Graham Schmidt orthogonalization.’

## A Given vector can be written as various sums and products

A vector  $\vec{A}$  has many representations. The equivalence of different representations of a vector is partially analogous to the case of a dimensional scalar which has the same value no matter what units are used (*e.g.*, the mass  $m = 4.41 \text{ lbm}$  is equal to  $m = 2 \text{ kg}$ ). Here are some common representations of vectors.

**Scalar times a unit vector in the vector’s direction.**  $\vec{F} = F\hat{\lambda}$  means the scalar  $F$  multiplied by the unit vector  $\hat{\lambda}$ .

**Sum of orthogonal component vectors.**  $\vec{F} = \vec{F}_x + \vec{F}_y$  is a sum of two vectors parallel to the  $x$  and  $y$  axis, respectively. In three dimensions,  $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$ .

**Components times unit base vectors.**  $\vec{F} = F_x \hat{i} + F_y \hat{j}$  or  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$  in three dimensions. One way to think of this sum is to realize that  $\vec{F}_x = F_x \hat{i}$ ,  $\vec{F}_y = F_y \hat{j}$  and  $\vec{F}_z = F_z \hat{k}$ .

**Components times rotated unit base vectors.**  $\vec{F} = F'_x \hat{i}' + F'_y \hat{j}'$  or  $\vec{F} = F'_x \hat{i}' + F'_y \hat{j}' + F'_z \hat{k}'$  in three dimensions. Here the base vectors marked with primes,  $\hat{i}', \hat{j}'$  and  $\hat{k}'$ , are unit vectors parallel to some mutually orthogonal  $x'$ ,  $y'$ , and  $z'$  axes. These  $x'$ ,  $y'$ , and  $z'$  axes may be crooked in relation to the  $x$ ,  $y$ , and  $z$  axis. That is, the  $x'$  axis need not be parallel to the  $x$  axis, the  $y'$  not parallel to the  $y$  axis, and the  $z'$  axis not parallel to the  $z$  axis.

**Components times other unit base vectors.** If you use polar or cylindrical coordinates the unit base vectors are  $\hat{e}_\theta$  and  $\hat{e}_R$ , so in 2-D,  $\vec{F} = F_R \hat{e}_R + F_\theta \hat{e}_\theta$  and in 3-D,  $\vec{F} = F_R \hat{e}_R + F_\theta \hat{e}_\theta + F_z \hat{k}$ . If you use ‘path’ coordinates, you will use the path-defined unit vectors  $\hat{e}_t$ ,  $\hat{e}_n$ , and  $\hat{e}_b$  so in 2-D  $\vec{F} = F_t \hat{e}_t + F_n \hat{e}_n$ . In 3-D  $\vec{F} = F_t \hat{e}_t + F_n \hat{e}_n + F_b \hat{e}_b$ .

**A list of components.**  $[\vec{F}]_{xy} = [F_x, F_y]$  or  $[\vec{F}]_{xyz} = [F_x, F_y, F_z]$  in three dimensions. This form coincides best with the way computers handle vectors. The row vector  $[F_x, F_y]$  coincides with  $F_x \hat{i} + F_y \hat{j}$  and the row vector  $[F_x, F_y, F_z]$  coincides with  $F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ .

In summary:

$$\begin{aligned}
 \vec{A} &= \vec{A} \\
 &= |\vec{A}| \hat{\lambda}_A = A \hat{\lambda}_A, && \text{where } \hat{\lambda}_A \parallel \vec{A}, A = |\vec{A}| \text{ and } |\hat{\lambda}_A| = 1 \\
 &= \vec{A}_x + \vec{A}_y + \vec{A}_z && \text{where } \vec{A}_x, \vec{A}_y, \vec{A}_z \text{ are parallel to the } x, y, z \text{ axis} \\
 &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, && \text{where } \hat{i}, \hat{j}, \hat{k} \text{ are parallel to the } x, y, z \text{ axis} \\
 &= A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}', && \text{where } \hat{i}', \hat{j}', \hat{k}' \text{ are } \parallel \text{ to skewed } x', y', z' \text{ axes} \\
 &= A_R \hat{e}_R + A_\theta \hat{e}_\theta + A_z \hat{k}, && \text{using polar coordinate basis vectors.} \\
 [\vec{A}]_{xyz} &= [A_x, A_y, A_z] && [\vec{A}]_{xyz} \text{ stands for the component list in } xyz \\
 [\vec{A}]_{x'y'z'} &= [A_x, A_y, A_z] && [\vec{A}]_{x'y'z'} \text{ stands for the component list in } x'y'z'
 \end{aligned}$$

## Vector algebra

Vectors are algebraic quantities and manipulated algebraically in equations. The rules for vector algebra are similar to the rules for ordinary (scalar) algebra. For example, if vector  $\vec{A}$  is the same as the vector  $\vec{B}$ ,  $\vec{A} = \vec{B}$ . For any scalar  $a$  and any vector  $\vec{C}$ , we then

$$\begin{aligned}
 \vec{A} + \vec{C} &= \vec{B} + \vec{C}, \\
 a \vec{A} &= a \vec{B}, \text{ and} \\
 \vec{A} \cdot \vec{C} &= \vec{B} \cdot \vec{C},
 \end{aligned}$$

because performing the same operation on equal quantities maintains the equality. The vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  might themselves be expressions involving other vectors.

The equations above show the allowable manipulations of vector equations: adding a common term to both sides, multiplying both sides by a common scalar, taking the dot product of both sides with a common vector.

All the distributive, associative, and commutative laws of ordinary addition and multiplication hold.<sup>①</sup>

<sup>①</sup>**Caution:** But you cannot divide a vector by a vector or a scalar by a vector:  $7/\vec{i}$  and  $\vec{A}/\vec{C}$  are *nonsense* expressions. And it does not make sense to add a vector and a scalar,  $7 + \vec{A}$  is a *nonsense* expression.

## Vector calculations on the computer

Most computer programs deal conveniently with lists of numbers, but not with vector notation and units. Thus our computer calculations will be in terms of vector components with the units left off. For example, when we write on the computer

$$\mathbf{F} = [ \begin{array}{ccc} 3 & 5 & -7 \end{array} ]$$

we take that to be the plain computer typing for  $\vec{F}_{xyz} = [3 \text{ N}, 5 \text{ N}, -7 \text{ N}]$ . This assumes that we are clear about what units and what coordinate system we are using. In particular, at this point in the course, you should only use one coordinate system in one problem in computer calculations.

Most computer languages will allow vector addition by a sequence of lines something like this:

$$\begin{aligned} \mathbf{A} &= [ \begin{array}{ccc} 1 & 2 & 5 \end{array} ] \\ \mathbf{B} &= [ \begin{array}{ccc} -2 & 4 & 19 \end{array} ] \\ \mathbf{C} &= \mathbf{A} + \mathbf{B} \end{aligned}$$

scaling (stretching) like this:

$$\begin{aligned} \mathbf{A} &= [ \begin{array}{ccc} 1 & 2 & 5 \end{array} ] \\ \mathbf{C} &= 3 * \mathbf{A} \end{aligned}$$

and dot products like this:

$$\begin{aligned} \mathbf{A} &= [ \begin{array}{ccc} 1 & 2 & 5 \end{array} ] \\ \mathbf{B} &= [ \begin{array}{ccc} -2 & 4 & 19 \end{array} ] \\ \mathbf{D} &= \mathbf{A}(1)*\mathbf{B}(1) + \mathbf{A}(2)*\mathbf{B}(2) + \mathbf{A}(3)*\mathbf{B}(3). \end{aligned}$$

In our pseudo code we write  $\mathbf{D} = \mathbf{A} \cdot \mathbf{B}$ . Many computer languages have a shorter way to write the dot product like  $\text{dot}(\mathbf{A}, \mathbf{B})$ . In a language built for linear algebra  $\mathbf{D} = \mathbf{A} * \mathbf{B}'$ <sup>①</sup> will work because the rules of matrix multiplication are then the same as the component formula for the dot product.

<sup>①</sup>  $\mathbf{B}'$  is a common notation for the transpose of  $\mathbf{B}$ , which means, in this case, to turn the row of numbers  $\mathbf{B}$  into a column of numbers.

**SAMPLE 2.12** *Calculating dot products:* Find the dot product of the two vectors  $\vec{a} = 2\hat{i} + 3\hat{j} - 2\hat{k}$  and  $\vec{r} = 5\text{m}\hat{i} - 2\text{m}\hat{j}$ .

**Solution** The dot product of the two vectors is

$$\begin{aligned}\vec{a} \cdot \vec{r} &= (2\hat{i} + 3\hat{j} - 2\hat{k}) \cdot (5\text{m}\hat{i} - 2\text{m}\hat{j}) \\ &= (2 \cdot 5\text{m}) \underbrace{\hat{i} \cdot \hat{i}}_1 - (2 \cdot 2\text{m}) \underbrace{\hat{i} \cdot \hat{j}}_0 \\ &\quad + (3 \cdot 5\text{m}) \underbrace{\hat{j} \cdot \hat{i}}_0 - (3 \cdot 2\text{m}) \underbrace{\hat{j} \cdot \hat{j}}_1 \\ &\quad - (2 \cdot 5\text{m}) \underbrace{\hat{k} \cdot \hat{i}}_0 + (2 \cdot 2\text{m}) \underbrace{\hat{k} \cdot \hat{j}}_0 \\ &= 10\text{m} - 6\text{m} \\ &= 4\text{m}.\end{aligned}$$

$$\boxed{\vec{a} \cdot \vec{r} = 4\text{m}}$$

**Comments:** Note that with just a little bit of foresight, we could totally ignore the  $\hat{k}$  component of  $\vec{a}$  since  $\vec{r}$  has no  $\hat{k}$  component, i.e.,  $\hat{k} \cdot \vec{r} = 0$ . Also, if we keep in mind that  $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0$ , we could compute the above dot product in one line:

$$\vec{a} \cdot \vec{r} = (2\hat{i} + 3\hat{j}) \cdot (5\text{m}\hat{i} - 2\text{m}\hat{j}) = (2 \cdot 5\text{m}) \underbrace{\hat{i} \cdot \hat{i}}_1 - (3 \cdot 2\text{m}) \underbrace{\hat{j} \cdot \hat{j}}_1 = 4\text{m}.$$

**SAMPLE 2.13** What is the  $y$ -component of  $\vec{F} = 5\text{N}\hat{i} + 3\text{N}\hat{j} + 2\text{N}\hat{k}$ ?

**Solution** Although it is perhaps obvious that the  $y$ -component of  $\vec{F}$  is 3 N, the scalar multiplying the unit vector  $\hat{j}$ , we calculate it below in a formal way using the dot product between two vectors. We will use this method later to find components of vectors in arbitrary directions.

$$\begin{aligned}F_y &= \vec{F} \cdot (\text{a unit vector along } y\text{-axis}) \\ &= (5\text{N}\hat{i} + 3\text{N}\hat{j} + 2\text{N}\hat{k}) \cdot \hat{j} \\ &= 5\text{N} \underbrace{\hat{i} \cdot \hat{j}}_0 + 3\text{N} \underbrace{\hat{j} \cdot \hat{j}}_1 + 2\text{N} \underbrace{\hat{k} \cdot \hat{j}}_0 \\ &= 3\text{N}.\end{aligned}$$

$$\boxed{F_y = \vec{F} \cdot \hat{j} = 3\text{N}.}$$

**SAMPLE 2.14** *Finding angle between two vectors using dot product:* Find the angle between the vectors  $\vec{r}_1 = 2\hat{i} + 3\hat{j}$  and  $\vec{r}_2 = 2\hat{i} - \hat{j}$ .

**Solution** From the definition of dot product between two vectors

$$\begin{aligned}\vec{r}_1 \cdot \vec{r}_2 &= |\vec{r}_1||\vec{r}_2| \cos \theta \\ \text{or } \cos \theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} \\ &= \frac{(2\hat{i} + 3\hat{j}) \cdot (2\hat{i} - \hat{j})}{(\sqrt{2^2 + 3^2})(\sqrt{2^2 + 1^2})} \\ &= \frac{4 - 3}{\sqrt{13}\sqrt{5}} = 0.124 \\ \text{Therefore, } \theta &= \cos^{-1}(0.124) = 82.87^\circ.\end{aligned}$$

$$\boxed{\theta = 83^\circ}$$

**SAMPLE 2.15** *Finding direction cosines from unit vectors:* Find the angles (from direction cosines) between  $\vec{F} = 4N\hat{i} + 6N\hat{j} + 7N\hat{k}$  and each of the three axes.

**Solution**

$$\begin{aligned}\vec{F} &= F\hat{\lambda} \\ \hat{\lambda} &= \frac{\vec{F}}{F} \\ &= \frac{4N\hat{i} + 6N\hat{j} + 7N\hat{k}}{\sqrt{4^2 + 6^2 + 7^2} N} \\ &= 0.4\hat{i} + 0.6\hat{j} + 0.7\hat{k}.\end{aligned}$$

Let the angles between  $\hat{\lambda}$  and the  $x$ ,  $y$ , and  $z$  axes be  $\theta$ ,  $\phi$  and  $\psi$  respectively. Then

$$\begin{aligned}\cos \theta &= \frac{\hat{i} \cdot \hat{\lambda}}{|\hat{i}||\hat{\lambda}|} = \frac{0.4}{|1||1|} = 0.4. \\ \Rightarrow \theta &= \cos^{-1}(0.4) = 66.4^\circ.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos \phi &= 0.6 \quad \text{or} \quad \phi = 53.1^\circ \\ \cos \psi &= 0.7 \quad \text{or} \quad \psi = 45.6^\circ.\end{aligned}$$

$$\boxed{\theta = 66.4^\circ, \phi = 53.1^\circ, \psi = 45.6^\circ}$$

**Comments:** The components of a unit vector give the direction cosines with the respective axes. That is, if the angle between the unit vector and the  $x$ ,  $y$ , and  $z$  axes are  $\theta$ ,  $\phi$  and  $\psi$ , respectively (as above), then

$$\hat{\lambda} = \underbrace{\cos \theta}_{\lambda_x} \hat{i} + \underbrace{\cos \phi}_{\lambda_y} \hat{j} + \underbrace{\cos \psi}_{\lambda_z} \hat{k}.$$

**SAMPLE 2.16** *Projection of a vector in the direction of another vector:* Find the component of  $\vec{F} = 5\hat{i} + 3\hat{j} + 2\hat{k}$  along the vector  $\vec{r} = 3\hat{i} - 4\hat{j}$ .

**Solution** The dot product of a vector  $\vec{a}$  with a unit vector  $\hat{\lambda}$  gives the projection of the vector  $\vec{a}$  in the direction of the unit vector  $\hat{\lambda}$ . Therefore, to find the component of  $\vec{F}$  along  $\vec{r}$ , we first find a unit vector  $\hat{\lambda}_r$  along  $\vec{r}$  and dot it with  $\vec{F}$ .

$$\begin{aligned}\hat{\lambda}_r &= \frac{\vec{r}}{|\vec{r}|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2} \text{ m}} = 0.6\hat{i} - 0.8\hat{j} \\ F_r &= \vec{F} \cdot \hat{\lambda}_r \\ &= (5\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (0.6\hat{i} - 0.8\hat{j}) \\ &= 3.0 \text{ N} + 2.4 \text{ N} = 5.4 \text{ N}.\end{aligned}$$

$$F_r = 5.4 \text{ N}$$

**SAMPLE 2.17** Assume that after writing the equation  $\sum \vec{F} = m\vec{a}$  in a particular problem, a student finds  $\sum \vec{F} = (20 \text{ N} - P_1)\hat{i} + 7\hat{j} - P_2\hat{k}$  and  $\vec{a} = 2.4 \text{ m/s}^2\hat{i} + a_3\hat{j}$ . Separate the scalar equations in the  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  directions.

**Solution**

$$\sum \vec{F} = m\vec{a}$$

Taking the dot product of both sides of this equation with  $\hat{i}$ , we write

$$\begin{aligned}\hat{i} \cdot \sum \vec{F} &= \hat{i} \cdot m\vec{a} \\ \hat{i} \cdot [(20 \text{ N} - P_1)\hat{i} + 7\hat{j} - P_2\hat{k}] &= m(2.4 \text{ m/s}^2\hat{i} + a_3\hat{j}) \\ \Rightarrow \underbrace{(20 \text{ N} - P_1)}_{F_x} \underbrace{\hat{i} \cdot \hat{i}}_1 + 7 \underbrace{\hat{j} \cdot \hat{i}}_0 - P_2 \underbrace{\hat{k} \cdot \hat{i}}_0 &= m \underbrace{(2.4 \text{ m/s}^2 \hat{i} \cdot \hat{i})}_1 + a_3 \underbrace{\hat{j} \cdot \hat{i}}_0 \\ \Rightarrow \sum F_x &= ma_x \\ \Rightarrow 20 \text{ N} - P_1 &= m(2.4 \text{ m/s}^2)\end{aligned}$$

Similarly,

$$\hat{j} \cdot [\sum \vec{F} = m\vec{a}] \Rightarrow \sum F_y = ma_y \quad (2.1)$$

$$\hat{k} \cdot [\sum \vec{F} = m\vec{a}] \Rightarrow \sum F_z = ma_z. \quad (2.2)$$

Substituting the given components of  $\vec{F}$  and  $\vec{a}$  in the remaining Eqns. (2.1) and (2.2) we get

$$\begin{aligned}7 \text{ N} &= ma_y \\ -P_2 &= 0.\end{aligned}$$

**Comments:** As long as both sides of a vector equation are in the same basis, separating the scalar equations is trivial—simply equate the respective components from both sides. The technique of taking the dot product of both sides with a vector is quite general and powerful. It gives a scalar equation valid in any direction that one desires. You will appreciate this technique more if the vector equation uses more than one basis.

## 2.3 Cross product, moment, and moment about an axis

When you try to move something you can push it and you can turn it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*. In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product, a second way of multiplying vectors together. The cross product is used to define (and calculate) moment and to calculate various quantities in dynamics. The cross product also sometimes helps solve three-dimensional geometry problems.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient principle of the lever is the basic idea incorporated by moments. The principle of the lever can be viewed as the root of all mechanics.

Ultimately you can take on faith the vector definition of moment (given opposite the inside cover) and its role in eqs. II. But we can more or less deduce the definition by generalizing from common experience.

### Teeter totter mechanics

The two people weighing down on the teeter totter in Fig. 2.21 tend to rotate it about its hinge, the right one clockwise and the left one counterclockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.

As is verified a million times a year by young future engineering students, to balance a teeter-totter the smaller person needs to be further from the hinge. If two people are on one side then the teeter totter is balanced by two similar people an equal distance from the hinge on the other side. Two people can balance one similar person by scooting twice as close to the hinge. These proportionalities generalize to this: the tendency of a force to cause rotation is proportional to the size of the force and to its distance from the hinge (for forces perpendicular to the teeter totter).

If someone standing nearby adds a force that is directed towards the hinge it causes no tendency to rotate. Because any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because we assume that the affect of the sum of these forces is the sum of the affects of each separately, and because the force towards the hinge has no tendency to rotate, we have deduced:

The moment of a force about a hinge is the product of its distance from the hinge and the component of the force perpendicular to the line from the hinge to the force.

Here then is the formula for 2D *moment about C* or *moment with respect to C*.<sup>①</sup>

$$M_{/C} = |\vec{r}| (|\vec{F}| \sin \theta) = (|\vec{r}| \sin \theta) |\vec{F}|. \quad (2.3)$$

Here,  $\theta$  is the angle between  $\vec{r}$  (the position of the point of force application relative to the hinge) and  $\vec{F}$  (see fig. 2.22). This formula for moment has all the teeter totter deduced properties. Moment is proportional to  $r$ , and to the part of  $\vec{F}$  that is perpendicular to  $\vec{r}$ . The re-grouping as  $(|\vec{r}| \sin \theta)$  shows that a force  $\vec{F}$  has the same effect if it is applied at a new location that is displaced in the direction of  $\vec{F}$ . That is,

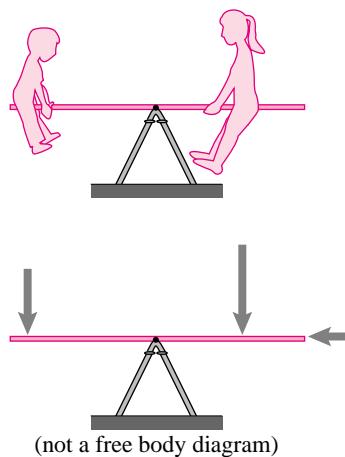


Figure 2.21: On a balanced teeter totter the bigger person gets the short end of the stick. A sideways force directed towards the hinge has no effect on the balance.

(Filename:tfigure.teeter)

<sup>①</sup> The ‘/’ in the subscript of  $\vec{M}$  reads as ‘relative to’ or ‘about’. For simplicity we often leave the / out and just write  $\vec{M}_C$ .

the force  $\vec{F}$  can slide along its length without changing its  $M_{/C}$  and is *equivalent* in its effect on the teeter totter. The quantity  $|\vec{r}| \sin \theta$  is sometimes called the *lever arm* of the force.

By common convention we define as positive a moment that causes a counter-clockwise rotation. A moment that causes a clockwise rotation is negative. If we define  $\theta$  appropriately then eqn. (2.3) obeys this sign convention. We define  $\theta$  as the angle from the positive vector  $\vec{r}$  to the positive vector  $\vec{F}$  measured counterclockwise. Point the thumb of your right hand towards yourself. Point the fingers of your right hand along  $\vec{r}$  and curl them towards the direction of  $\vec{F}$  and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has  $\theta = 90^\circ$  so  $\sin \theta = 1$  and the formula 2.3 gives a positive counterclockwise  $M$ . The force of the person on the right has  $\theta = 270^\circ$  ( $3/4$  of a revolution) so  $\sin \theta = -1$  and the formula 2.3 gives a negative  $M$ .

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the  $xy$  plane, the 2D moment is a vector in the  $\hat{k}$  direction (straight out of the plane). So eqn. 2.3 becomes

$$\vec{M}_{/C} = |\vec{r}| |\vec{F}| \sin \theta \hat{k}. \quad (2.4)$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

## The 2D cross product

The expression we have found for the right side of eqn. 2.4 is the 2D cross product of vectors  $\vec{r}$  and  $\vec{F}$ . We can now apply the concept to any pair of vectors whether or not they represent force and position. The 2D cross product is defined as :

$$\underbrace{\vec{A} \times \vec{B}}_{\text{'A cross B'}} \stackrel{\text{def}}{=} |\vec{A}| |\vec{B}| \sin \theta \hat{k}. \quad (2.5)$$

where  $\theta$  is the amount that  $\vec{A}$  would need to be rotated counterclockwise to point in the same direction as  $\vec{B}$ . An equivalent alternative approach is to define the cross product as

$$\vec{A} \times \vec{B} \stackrel{\text{def}}{=} |\vec{A}| |\vec{B}| \sin \theta \hat{n}. \quad (2.6)$$

with  $\theta$  defined to be less than  $180^\circ$  and  $\hat{n}$  defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of  $\vec{A}$  towards the direction of  $\vec{B}$ . For the  $\vec{r}$  and  $\vec{F}$  on the right of the teeter totter this definition forces us to point our thumb into the plane (in the negative  $\hat{k}$  direction). With this definition  $\sin \theta$  is always positive and the negative moments come from  $\hat{n}$  being in the  $-\hat{k}$  direction.

With a few sketches you could convince yourself that the definition of cross product in eqn.2.5 obeys these standard algebra rules (for any 2D vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  and any scalar  $d$ ):

$$\begin{aligned} d(\vec{A} \times \vec{B}) &= (d\vec{A}) \times \vec{B} = \vec{A} \times (d\vec{B}) \\ \vec{A} \times (\vec{B} + \vec{C}) &= \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \end{aligned}$$

A difference between the algebra rules for scalar multiplication and vector cross product multiplication is that for scalar multiplication  $AB = BA$  whereas for the cross product  $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$  (because the definition of  $\theta$  in eqn. 2.5 and  $\hat{n}$  in 2.6 depends on order). In particular  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ .

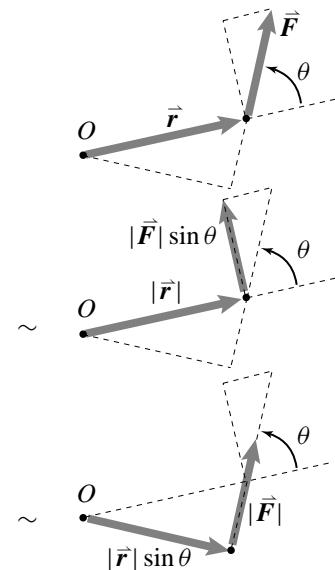


Figure 2.22: The moment of a force is either the product of its radius with its perpendicular component or of its lever arm and the full force. The  $\sim$  indicates that the lower two forces and positions have the same moment.

(Filename:tfigure.slidevector)

Because the magnitude of the cross product of  $\vec{A}$  and  $\vec{B}$  is the magnitude of  $\vec{A}$  times the magnitude of the projection of  $\vec{B}$  in the direction perpendicular to  $\vec{A}$  (as shown in the top two illustrations of fig. 2.22) you can think of the cross product as a measure of how much two vectors are perpendicular to each other. In particular

$$\begin{aligned} \text{if } \vec{A} \perp \vec{B} &\Rightarrow |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}|, \text{ and} \\ \text{if } \vec{A} \parallel \vec{B} &\Rightarrow |\vec{A} \times \vec{B}| = \vec{0}. \end{aligned}$$

For example,  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{j} \times \hat{i} = -\hat{k}$ ,  $\hat{i} \times \hat{i} = \vec{0}$ , and  $\hat{j} \times \hat{j} = \vec{0}$ .

### Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing  $\vec{A} = A_x \hat{i} + A_y \hat{j}$ , and  $\vec{B} = B_x \hat{i} + B_y \hat{j}$  and using the distributive rules:

$$\vec{A} \times \vec{B} = (A_x B_y - B_x A_y) \hat{k}. \quad (2.7)$$

Some people remember this formula by putting the components of  $\vec{A}$  and  $\vec{B}$  into a matrix and calculating the determinant  $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$ . If you number the components of  $\vec{A}$  and  $\vec{B}$  (e.g.,  $[\vec{A}]_{x_1 x_2} = [A_1, A_2]$ ), the cross product is  $\vec{A} \times \vec{B} = (A_1 B_2 - B_1 A_2) \hat{e}_3$ . This you might remember as “first times second minus second times first.”

**Example:** Given that  $\vec{A} = 1\hat{i} + 2\hat{j}$  and  $\vec{B} = 10\hat{i} + 20\hat{j}$  then  $\vec{A} \times \vec{B} = (1 \cdot 20 - 2 \cdot 10)\hat{k} = 0\hat{k} = \vec{0}$ .  $\square$

For vectors with just a few components it is often most convenient to use the distributive rule directly.

**Example:** Given that  $\vec{A} = 7\hat{i}$  and  $\vec{B} = 37.6\hat{i} + 10\hat{j}$  then  $\vec{A} \times \vec{B} = (7\hat{i}) \times (37.6\hat{i} + 10\hat{j}) = (7\hat{i}) \times (37.6\hat{i}) + (7\hat{i}) \times (10\hat{j}) = \vec{0} + 70\hat{k} = 70\hat{k}$ .  $\square$

*There are many ways of calculating a 2D cross product*

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use the geometric definition directly, the first times the perpendicular part of the second (distance times perpendicular component of force), the second times the perpendicular part of the first (lever arm times the force), components, or break each of the vectors into a sum of vectors and use the distributive rule.

### 2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment  $\vec{M}_{/C}$  of eqn. 2.4. Given  $\vec{F} = F_x \hat{i} + F_y \hat{j}$  acting at P, where  $\vec{r}_{P/C} = r_x \hat{i} + r_y \hat{j}$ , the moment of the force about C is

$$\vec{M}_{/C} = (r_x F_y - r_y F_x) \hat{k}$$

or the moment of  $\vec{F}$  about the axis at C is

$$M_C = r_x F_y - r_y F_x \quad (2.8)$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 2.23.

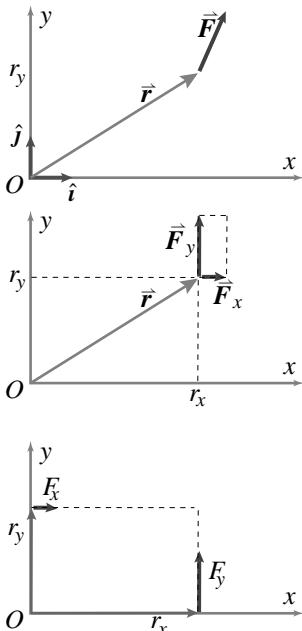


Figure 2.23: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the x and y axis, and adding the moments of the two components.

(Filename:tfigure1.2Dcrosscomps)

### 3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics or dynamics problem by considering moment about a judiciously chosen axis.

Lets start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world (see Fig. 2.24). We now re-interpret the 2D moment  $M$  as the moment of the 2D forces about the  $\hat{\lambda}$  axis of rotation at the hinge. It is plain that a force  $\vec{F}^{\parallel}$  pushing a teeter totter parallel to the axle causes no tendency to rotate. And we already agreed that a radial force  $\vec{F}^r$  causes no rotation. So we see that the moment a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 2.25. Here an imagined axis of rotation is defined as the line through C that is in the  $\hat{\lambda}$  direction. A force  $\vec{F}$  is applied at P. We can break  $\vec{F}$  into a sum of three vectors

$$\vec{F} = \vec{F}^{\parallel} + \vec{F}^r + \vec{F}^{\perp}$$

where  $\vec{F}^{\parallel}$  is parallel to the axis,  $\vec{F}^r$  is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and  $\vec{F}^{\perp}$  is out of the plane defined by C, P and  $\hat{\lambda}$ . By analogy with the teeter totter we see that  $\vec{F}^r$  and  $\vec{F}^{\parallel}$  cause no tendency to rotate about the axis. So only the  $\vec{F}^{\perp}$  contributes.

**Example:** Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand. Press down and note that the door is not opened or closed. Push towards the hinge and note that the door is not opened or closed. Push and pull away and towards your body and note how easily you cause the door to rotate. Thus the only force component that tends to rotate the door is perpendicular to the plane of the door (which is the plane of the hinge and line from the hinge to your hand). Now move your hand to the middle of the door, half the distance from the hinge. Note that it takes more force to rotate the door with the same authority (push with your pinky if you have trouble feeling the difference).

Thus the only potent force for rotation is perpendicular to the plane of the hinge and point of force application, and its potency is increased with distance from the hinge.  $\square$

We can also decompose  $\vec{r} = \vec{r}_{P/C}$  into two parts, one parallel to the hinge and one radial, as

$$\vec{r} = \vec{r}^{\parallel} + \vec{r}^r.$$

Clearly  $\vec{r}^{\parallel}$  has no affect on how much rotation  $\vec{F}$  causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force  $\vec{F}$  about the axis  $\hat{\lambda}$  through C is given by

$$M_{\lambda C} = r^r F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is  $|\vec{r}^r|$  and  $\vec{F}^{\perp}$  is the part of the force that causes right-handed rotation about the axis. A

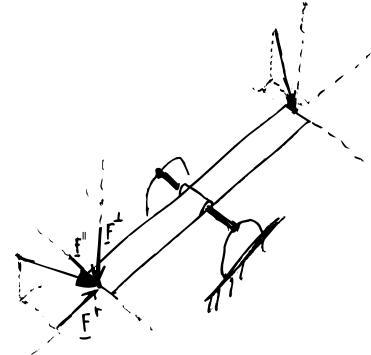


Figure 2.24: Teeter totter with applied forces broken into components parallel to the axis  $\vec{F}^{\parallel}$ , radial  $\vec{F}^r$ , and perpendicular to the plane containing the axis and the point of force application  $\vec{F}^{\perp}$ .

(Filename:tfigure.3Dteeter)

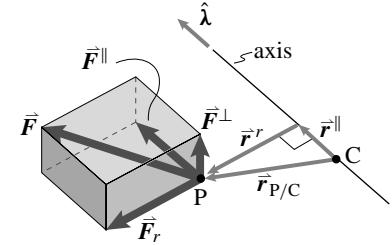


Figure 2.25: Moment about an axis

(Filename:tfigure2.mom.axis)

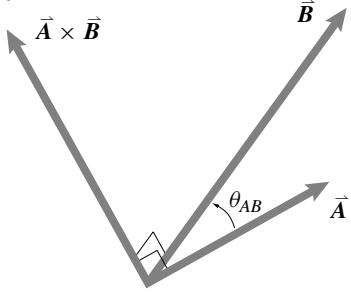


Figure 2.26: The cross product of  $\vec{A}$  and  $\vec{B}$  is perpendicular to  $\vec{A}$  and  $\vec{B}$  in the direction given by the right hand rule. The magnitude of  $\vec{A} \times \vec{B}$  is  $AB \sin \theta_{AB}$ .

(Filename:tfigure1.12)

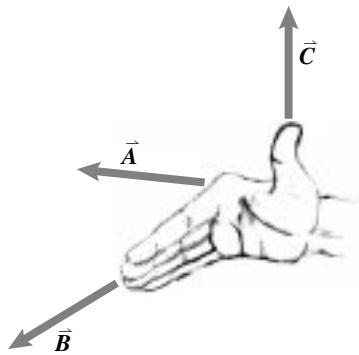


Figure 2.27: The right hand rule for determining the direction of the cross product of two vectors.  $\vec{C} = \vec{A} \times \vec{B}$ .

(Filename:tfigure.rhr)

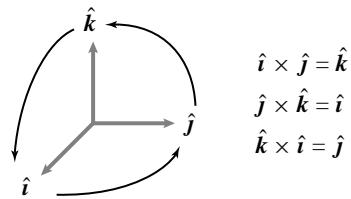


Figure 2.28: Mnemonic device to remember the cross product of the standard base unit vectors.

(Filename:tfigure1.e)

moment about an axis is defined as positive if curling the fingers of your right hand give the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 2.25). The force of the left person on the teeter totter causes a positive moment about the  $\hat{k}$  axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

*“Moment is distance ( $|\vec{r}^r|$ ) times force ( $|\vec{F}^\perp|$ )”*

perfectly defines moment about an axis.

Three dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector defined by the 3D cross product which we introduce now.

### The 3D cross product (or vector product)

The cross product of two vectors  $\vec{A}$  and  $\vec{B}$  is written  $\vec{A} \times \vec{B}$  and pronounced ‘A cross B.’ In contrast to the dot product, which gives a scalar and measures how much two vectors are parallel, the cross product is a vector and measures how much they are perpendicular. The cross product is also called *the vector product*.

The cross product is defined by:

$$\vec{A} \times \vec{B} \stackrel{\text{def}}{=} |\vec{A}| |\vec{B}| \sin \theta_{AB} \hat{n}$$

where  $|\hat{n}| = 1$ ,

$$\hat{n} \perp \vec{A},$$

$$\hat{n} \perp \vec{B},$$

$$0 \leq \theta_{AB} \leq \pi,$$

$\hat{n}$  is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of  $\vec{A}$  and then wrapped towards the direction of  $\vec{B}$ .

If  $\vec{A}$  and  $\vec{B}$  are perpendicular then  $\theta_{AB} = \pi/2$ ,  $\sin \theta_{AB} = 1$ , and the magnitude of the cross product is  $AB$ . If  $\vec{A}$  and  $\vec{B}$  are parallel then  $\theta_{AB} = 0$ ,  $\sin \theta_{AB} = 0$  and the cross product is  $\vec{0}$  (the zero vector). This is why we say the cross product is a measure of the degree of orthogonality of two vectors.

Using the definition above you should be able to verify to your own satisfaction that  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ . Applying the definition to the standard base unit vectors you can see that  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{j} \times \hat{k} = \hat{i}$ , and  $\hat{k} \times \hat{i} = \hat{j}$  (figure 2.28).

The geometric definition above and the geometric (tip to tale) definition of vector addition imply that the cross product follows the distributive rule (see box 2.4 on page 41).

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Applying the distributive rule to the cross products of  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$  and  $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$  leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned} \vec{A} \times \vec{B} &= [A_y B_z - A_z B_y] \hat{i} \\ &\quad + [A_z B_x - A_x B_z] \hat{j} \\ &\quad + [A_x B_y - A_y B_x] \hat{k} \end{aligned}$$

There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a ‘determinant’ of the  $3 \times 3$  matrix with one row given by  $\hat{i}, \hat{j}, \hat{k}$  and the other two rows the components of  $\vec{A}$  and  $\vec{B}$ .

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B}) = a(\vec{A} \times \vec{B})$  (a distributive law)
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$  (the cross product is not commutative!)
- $\vec{A} \times \vec{B} = \vec{0}$  if  $\vec{A} \parallel \vec{B}$  (parallel vectors have zero cross product)
- $|\vec{A} \times \vec{B}| = AB$  if  $\vec{A} \perp \vec{B}$
- $\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$  (assuming the  $x, y, z$  coordinate system is right handed — if you use your right hand and point your fingers along the positive  $x$  axis, then curl them towards the positive  $y$  axis, your thumb will point in the same direction as the positive  $z$  axis. )
- $\hat{i}' \times \hat{j}' = \hat{k}', \quad \hat{j}' \times \hat{k}' = \hat{i}', \quad \hat{k}' \times \hat{i}' = \hat{j}'$   
(assuming the  $x'y'z'$  coordinate system is also right handed.)
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}, \quad \hat{i}' \times \hat{i}' = \hat{j}' \times \hat{j}' = \hat{k}' \times \hat{k}' = \vec{0}$

## The moment vector

We now define the moment of a force  $\vec{F}$  applied at P, relative to point C as

$$\vec{M}_{/C} = \vec{r}_{P/C} \times \vec{F}$$

which we read in short as ‘M is r cross F.’ The moment vector is admittedly a difficult idea to intuit. A look at its components is helpful.

$$\vec{M}_{/C} = (r_y F_z - r_z F_y) \hat{i} + (r_z F_x - r_x F_z) \hat{j} + (r_x F_y - r_y F_x) \hat{k}$$

You can recognize the  $z$  component of the moment vector is the moment of the force about the  $\hat{k}$  axis through C (eqn. 2.8). Similarly the  $x$  and  $y$  components of  $\vec{M}_C$  are the moments about the  $\hat{i}$  and  $\hat{j}$  axis through C. So at least the components of  $\vec{M}_C$  have intuitive meaning. They are the moments around the  $x, y$ , and  $z$  axes respectively.

Starting with this moment-about-the-coordinate-axes interpretation of the moment vector, each of the three components can be deduced graphically by the moves shown in fig. 2.30. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture you can see, say, that the moment about the  $+y$  axis gets a positive contribution from  $F_x$  with lever arm  $r_z$  and a negative contribution from  $F_z$  with lever arm  $r_x$ . Thus the  $y$  component of  $\vec{M}$  is  $r_z F_x - r_x F_z$ .

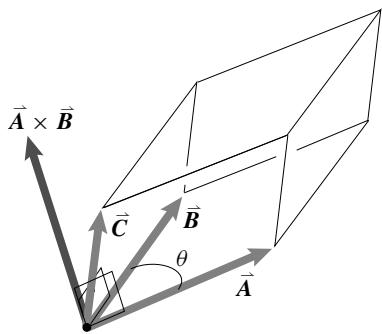


Figure 2.29: One interpretation of the mixed triple product of  $\vec{A} \times \vec{B} \cdot \vec{C}$  is as the volume (a scalar) of a parallelepiped with  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  as the three edges emanating from one corner. This interpretation only works if  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  are taken in the appropriate order, otherwise  $\vec{A} \times \vec{B} \cdot \vec{C}$  is minus the volume which is calculated.

(Filename:tfigure1.13)

- ① In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

## The mixed triple product

The ‘mixed triple product’ of  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The mixed triple product is useful for calculating the moment of a force about an axis and for related dynamics quantities. The *mixed triple product* of  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  is defined by and written as

$$\vec{A} \cdot (\vec{B} \times \vec{C})$$

and pronounced ‘A dot B cross C.’ The parentheses () are sometimes omitted (*i.e.*,  $\vec{A} \cdot \vec{B} \times \vec{C}$ ) because the wrong grouping  $(\vec{A} \cdot \vec{B}) \times \vec{C}$  is nonsense (you can’t take the cross product of a scalar with a vector). It is apparent that one way of calculating the mixed triple product is to calculate the cross product of  $\vec{B}$  and  $\vec{C}$  and then to take the dot product of that result with  $\vec{A}$ . Some people use the notation  $(\vec{A}, \vec{B}, \vec{C})$  for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of  $\vec{A}$  and  $\vec{B}$  and then the dot product of the result with  $\vec{C}$ . That is  $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$ . This identity can be verified using the geometric description below, or by looking at the (complicated) expression for the mixed triple product of three general vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \times \vec{A} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C} = \dots$$

The minus signs in the above expressions follow from the cross product identity that  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ .

The mixed triple product has various geometric interpretations, one of them is that  $\vec{A} \cdot \vec{B} \times \vec{C}$  is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  as shown in figure 2.29.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x)$$

The mixed triple product of three vectors is zero if<sup>①</sup>

- any two of them are parallel, or
- if all three of the vectors have one common plane.

A different triple product, sometimes called the vector triple product is defined by  $\vec{A} \times (\vec{B} \times \vec{C})$  which is discussed later in the text when it is needed (see box 11.1 on page 643).

## More on moment about an axis

We defined moment about an axis geometrically using fig. 2.25 on page 35 as  $M_{\lambda} = r^r F^{\perp}$ . We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda C} = \hat{\lambda} \cdot \vec{M}_{/C} \quad (\text{An inspired guess...}) \quad (2.9)$$

We break both  $\vec{r}$  and  $\vec{F}$  into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of the vectors are parallel. Thus,

$$\begin{aligned}\hat{\lambda} \cdot \vec{M}_{P/C} &= \hat{\lambda} \cdot \vec{r}_{P/C} \times \vec{F} \\ &= \hat{\lambda} \cdot (\vec{r}^r + \vec{r}^\parallel) \times (\vec{F}^\perp + \vec{F}^\parallel + \vec{F}^r) \\ &= \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^\perp + \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^\parallel + \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^r \dots \\ &\quad + \hat{\lambda} \cdot \vec{r}^\parallel \times \vec{F}^\perp + \hat{\lambda} \cdot \vec{r}^\parallel \times \vec{F}^\parallel + \hat{\lambda} \cdot \vec{r}^\parallel \times \vec{F}^r \\ &= r^r F^\perp + 0 + 0 + 0 + 0 + 0 \\ &= r^r F^\perp. \quad (\dots \text{ and a good guess too.})\end{aligned}$$

We can calculate the cross and dot product any convenient way, say using vector components.

#### Example: Moment about an axis

Given a force,  $\vec{F}_1 = (5\hat{i} - 3\hat{j} + 4\hat{k})$  N acting at a point  $P$  whose position is given by  $\vec{r}_{P/O} = (3\hat{i} + 2\hat{j} - 2\hat{k})$  m, what is the moment about an axis through the origin  $O$  with direction  $\hat{\lambda} = \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$ ?

$$\begin{aligned}M_{\hat{\lambda}} &= (\vec{r}_{P/O} \times \vec{F}_1) \cdot \hat{\lambda} \\ &= [(3\hat{i} + 2\hat{j} - 2\hat{k}) \text{ m} \times (5\hat{i} - 3\hat{j} + 4\hat{k}) \text{ N}] \cdot \left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) \\ &= -\frac{17}{\sqrt{2}} \text{ mN.}\end{aligned}$$

□

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 2.3 will have different components in the different systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector  $\vec{M}_C$  is the direction of the axis through  $C$  about which  $\vec{F}$  has the greatest moment. The magnitude of  $\vec{M}_C$  is the moment of  $\vec{F}$  about that axis.

#### Special optional ways to draw moment vectors

Neither of the special rotation notations below is needed because moment (and later, angular velocity, and angular momentum) is a vector like any other. None-the-less, sometimes it is nice to use a notation that suggests the rotational nature of these quantities.

**Arched arrow for 2-D moment and angular velocity.** In 2D problems in the  $xy$  plane, the relevant moment, angular velocity, and angular momentum point straight out of the plane in the  $z$  ( $\hat{k}$ ) direction. A way of drawing this is to use an arced arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in Fig. 2.31a indicate the same moment vector.

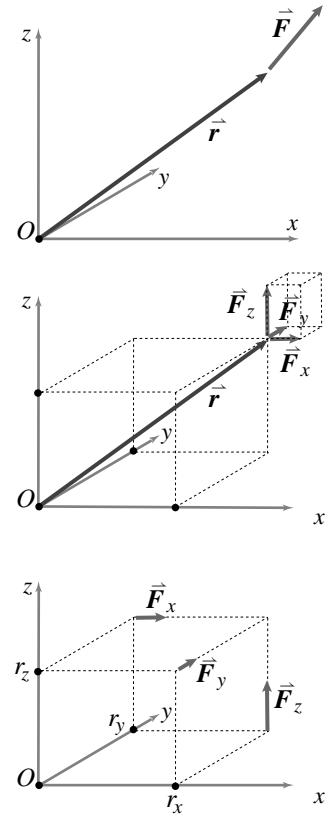


Figure 2.30: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis.

(Filename:tfigure1.3Dcrosscomps)

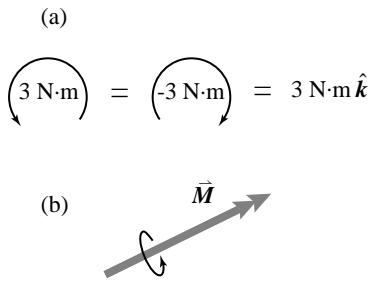


Figure 2.31: Optional drawing method for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantities in three dimensions.

(Filename: tfigure1rot.d)

**Double headed arrow for 3-D rotations and moments.** Some people like to distinguish vectors for rotational motion and torque from other vectors. Two ways of making this distinction are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in Fig. 2.31b.

## Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [ 1 2 5 ]
B = [ -2 4 19 ]
C = [ ( A(2)*B(3) - A(3)*B(2) ) ...
      ( A(3)*B(1) - A(1)*B(3) ) ...
      ( A(1)*B(2) - A(2)*B(1) ) ]
```

giving the result  $C = [18 \quad -29 \quad 8]$ . Many computer languages have a shorter way to write the cross product like `cross(A, B)`. The mixed triple product might be calculated by assembling a  $3 \times 3$  matrix of rows and then taking a determinant like this:

```
A     = [ 1 2 5 ]
B     = [ -2 4 19 ]
C     = [ 32 4 5 ]
matrix = [A ; B ; C]
mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile language might well allow the command `dot( A, cross(B, C) )` to calculate the mixed triple product.

## 2.4 THEORY

The 3D cross product is distributive over sums; calculation with components

Finding the component formula for the cross product from the geometric definition depends on the geometric definition obeying the distributive rule.

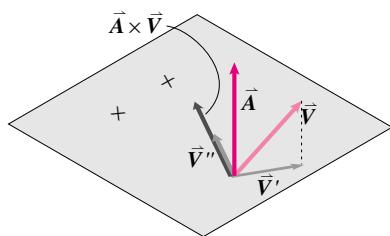
### The distributive rule

We would like to demonstrate that the geometrically defined cross product obeys the rule

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Here is a 3D construction demonstrating this fact. It is a bit trickier than the demonstration of most of the algebra manipulation rules for vectors.

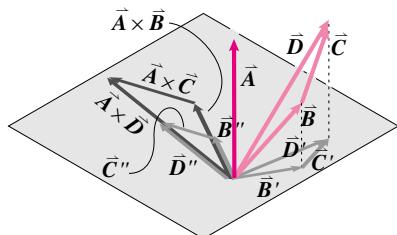
First we present another geometric definition of the cross product of  $\vec{A}$  and any vector  $\vec{V}$ . Consider a plane P that is perpendicular to  $\vec{A}$ .



Now look at  $\vec{V}'$ , the projection of  $\vec{V}$  on to P. The right-hand-rule normal of  $\vec{A}$  and  $\vec{V}$  is the same as the normal of  $\vec{A}$  and  $\vec{V}'$ . Also,  $|\vec{V}'| = |\vec{V}| \sin \theta_{AV}$ . So  $\vec{A} \times \vec{V} = \vec{A} \times \vec{V}'$ . Now consider  $\vec{V}''$  which is the rotation of  $\vec{V}'$  by  $90^\circ$  around  $\vec{A}$ . Note that  $\vec{V}''$  is still in the plane P. Finally stretch  $\vec{V}''$  by  $|\vec{A}|$ . The result is a vector in the P plane that is  $\vec{A} \times \vec{V}$  since it has the correct magnitude and direction.

Thus  $\vec{A} \times \vec{V}$  is given by projecting  $\vec{V}$  onto P, rotating that projection by  $90^\circ$  about  $\vec{A}$ , and stretching that by  $|\vec{A}|$ .

Now consider  $\vec{B}$ ,  $\vec{C}$ , and  $\vec{D} \equiv \vec{B} + \vec{C}$ .



All three cross products

$$\vec{A} \times \vec{B}, \quad \vec{A} \times \vec{C}, \quad \text{and} \quad \vec{A} \times \vec{D},$$

can be calculated by this projection, rotation, and stretch. But each of these three operations is distributive since

- the projection of a sum is the sum of the projections ( $\vec{D}' = \vec{B}' + \vec{C}'$ );
- the sum of two  $90^\circ$  rotated vectors is the rotation of the sum ( $\vec{D}'' = \vec{B}'' + \vec{C}''$ ); and

- stretched  $\vec{D}''$  is (stretched  $\vec{B}''$ ) + (stretched  $\vec{C}''$ ).

Thus the act of taking the cross product of  $\vec{A}$  with  $\vec{B}$  and adding that to the cross product of  $\vec{A}$  with  $\vec{C}$  gives the same result as taking the cross product of  $\vec{A}$  with  $\vec{D}$  ( $\equiv \vec{B} + \vec{C}$ ), demonstrating the distributive law.

### Calculation of the cross product with components

Application of the distributive rule to vectors expressed in terms of the standard unit base vectors yields the oft-used component expression for the cross product as follows

$$\begin{aligned} \vec{A} \times \vec{B} &= [A_x \hat{i} + A_y \hat{j} + A_z \hat{k}] \times [B_x \hat{i} + B_y \hat{j} + B_z \hat{k}] \\ &= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k} \\ &\quad + A_y B_x \hat{j} \times \hat{i} + A_y B_y \hat{j} \times \hat{j} + A_y B_z \hat{j} \times \hat{k} \\ &\quad + A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j} + A_z B_z \hat{k} \times \hat{k} \\ &= A_x B_x \vec{0} + A_x B_y \hat{k} - A_x B_z \hat{j} \\ &\quad - A_y B_x \hat{k} + A_y B_y \vec{0} + A_y B_z \hat{i} \\ &\quad + A_z B_x \hat{j} - A_z B_y \hat{i} + A_z B_z \vec{0} \\ &= [A_y B_z - A_z B_y] \hat{i} \\ &\quad + [A_z B_x - A_x B_z] \hat{j} \\ &\quad + [A_x B_y - A_y B_x] \hat{k} \end{aligned}$$

**SAMPLE 2.18** *Cross product in 2-D:* Two vectors  $\vec{a}$  and  $\vec{b}$  of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is  $\theta = 60^\circ$ . Find the cross product of the two vectors.

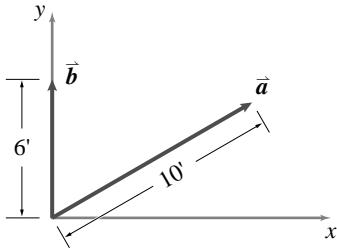


Figure 2.32: (Filename:sfig2.vec2.cross1)

**Solution** Both vectors  $\vec{a}$  and  $\vec{b}$  are in the  $xy$  plane. Therefore, their cross product is,

$$\begin{aligned}\vec{a} \times \vec{b} &= |\vec{a}| |\vec{b}| \sin \theta \hat{n} \\ &= (10 \text{ ft}) \cdot (6 \text{ ft}) \cdot \sin 60^\circ \hat{k} \\ &= 60 \text{ ft}^2 \cdot \frac{\sqrt{3}}{2} \hat{k} \\ &= 30\sqrt{3} \text{ ft}^2 \hat{k}.\end{aligned}$$

$$\boxed{\vec{a} \times \vec{b} = 30\sqrt{3} \text{ ft}^2 \hat{k}}$$

**SAMPLE 2.19** *Computing 2-D cross product in different ways:* The two vectors shown in the figure are  $\vec{a} = 2\hat{i} - \hat{j}$  and  $\vec{b} = 4\hat{i} + 2\hat{j}$ . The angle between the two vectors is  $\theta = \sin^{-1}(4/5)$  (this information can be found out from the given vectors). Find the cross product of the two vectors

- (a) using the angle  $\theta$ , and
- (b) using the components of the vectors.

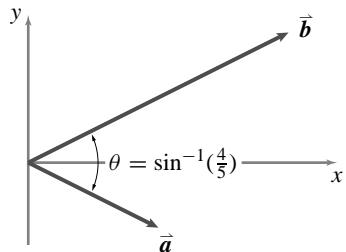


Figure 2.33: (Filename:sfig2.vec2.cross2)

**Solution**

- (a) Cross product using the angle  $\theta$ :

$$\begin{aligned}\vec{a} \times \vec{b} &= |\vec{a}| |\vec{b}| \sin \theta \hat{n} \\ &= |2\hat{i} - \hat{j}| |4\hat{i} + 2\hat{j}| \cdot \sin(\sin^{-1} \frac{4}{5}) \hat{k} \\ &= (\sqrt{2^2 + 1^2})(\sqrt{4^2 + 2^2}) \cdot \frac{4}{5} \hat{k} \\ &= \sqrt{5} \cdot \sqrt{20} \cdot \frac{4}{5} \hat{k} = 10 \cdot \frac{4}{5} \hat{k} \\ &= 8\hat{k}.\end{aligned}$$

- (b) Cross product using components:

$$\begin{aligned}\vec{a} \times \vec{b} &= (2\hat{i} - \hat{j}) \times (4\hat{i} + 2\hat{j}) \\ &= 2\hat{i} \times (4\hat{i} + 2\hat{j}) - \hat{j} \times (4\hat{i} + 2\hat{j}) \\ &= 8\underbrace{\hat{i} \times \hat{i}}_{\mathbf{0}} + 4\underbrace{\hat{i} \times \hat{j}}_{\hat{k}} - 4\underbrace{\hat{j} \times \hat{i}}_{-\hat{k}} - 2\underbrace{\hat{j} \times \hat{j}}_{\mathbf{0}} \\ &= 4\hat{k} + 4\hat{k} \\ &= 8\hat{k}.\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{a} \times \vec{b} = 8\hat{k}}$$

**SAMPLE 2.20** *Finding the minimum distance from a point to a line:* A straight line passes through two points, A (-1,4) and B (2,2), in the  $xy$  plane. Find the shortest distance from the origin to the line.

**Solution** Let  $\hat{\lambda}_{AB}$  be a unit vector along line AB. Then,

$$\hat{\lambda}_{AB} \times \vec{r}_B = \underbrace{|\hat{\lambda}_{AB}|}_{1} |\vec{r}_B| \sin \theta \hat{n} = |\vec{r}_B| \sin \theta \hat{k}.$$

Now  $|\vec{r}_B| \sin \theta$  is the component of  $\vec{r}_B$  that is perpendicular to  $\hat{\lambda}_{AB}$  or line AB, i.e., it is the perpendicular, and hence the shortest, distance from the origin (the root of vector  $\vec{r}_B$ ) to the line AB. Thus, the shortest distance  $d$  from the origin to the line AB is computed from,

$$\begin{aligned} d &= |\hat{\lambda}_{AB} \times \vec{r}_B| \\ &= \left| \left( \frac{3\hat{i} + \hat{j}}{\sqrt{3^2 + 1^2}} \right) \times (2\hat{i} + 2\hat{j}) \right| = \left| \frac{6}{\sqrt{10}}\hat{k} - \frac{2}{\sqrt{10}}\hat{k} \right| = \left| \frac{4}{\sqrt{10}}\hat{k} \right| \\ &= \frac{4}{\sqrt{10}}. \end{aligned}$$

$$d = 4/\sqrt{10}$$

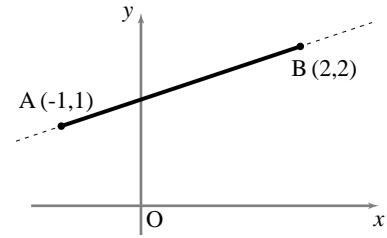


Figure 2.34: (Filename:sfig2.vec2.perp2D)

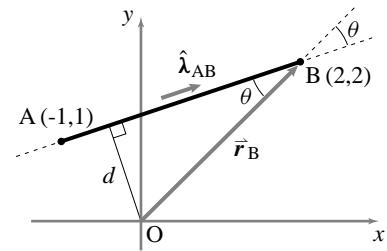


Figure 2.35: (Filename:sfig2.vec2.perp2Da)

**Comments:** In this calculation,  $\vec{r}_B$  is an arbitrary vector from the origin to some point on line AB. You can take any convenient vector. Since the shortest distance is unique, any such vector will give you the same answer. In fact, you can check your answer by selecting another vector and repeating the calculations, e.g., vector  $\vec{r}_A$ .

**SAMPLE 2.21** *Moment of a force:* Find the moment of force  $\vec{F} = 1\text{N}\hat{i} + 20\text{N}\hat{j}$  shown in the figure about point O.

**Solution** The force acts through point A on the body. Therefore, we can compute its moment about O as follows.

$$\begin{aligned} \vec{M}_O &= \vec{r}_{OA} \times \vec{F} \\ &= \underbrace{(-2\text{m} \cdot \cos 60^\circ \hat{i} - 2\text{m} \cdot \sin 60^\circ \hat{j})}_{\vec{r}_{OA}} \times \underbrace{(1\text{N}\hat{i} + 20\text{N}\hat{j})}_{\vec{F}} \\ &= (-1\text{m}\hat{i} - \sqrt{3}\text{m}\hat{j}) \times (1\text{N}\hat{i} + 20\text{N}\hat{j}) \\ &= -20\text{N}\cdot\text{m}\hat{k} + 1.732\text{N}\cdot\text{m}\hat{k} \\ &= -18.268\text{N}\cdot\text{m}\hat{k}. \end{aligned}$$

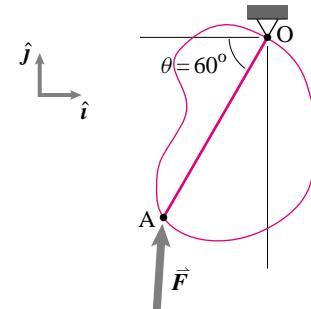


Figure 2.36: (Filename:sfig2.vec2.mom1)

$$\vec{M}_O = -18.268\text{N}\cdot\text{m}\hat{k}$$

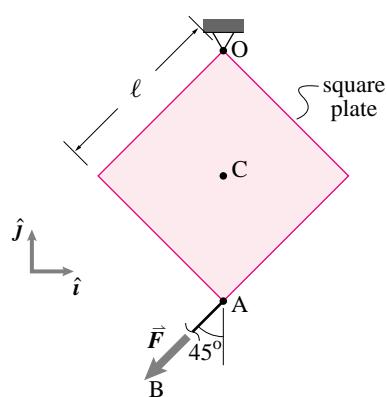


Figure 2.37: (filename:sfig2.vec2.mom2)

**SAMPLE 2.22** A 2 m square plate hangs from one of its corners as shown in the figure. At the diagonally opposite end, a force of 50 N is applied by pulling on the string AB. Find the moment of the applied force about the center C of the plate.

**Solution** The moment of  $\vec{F}$  about point C is

$$\vec{M}_C = \vec{r}_{A/C} \times \vec{F}.$$

So, to compute  $\vec{M}_C$ , we need to find the vectors  $\vec{r}_{A/C}$  and  $\vec{F}$ .

$$\begin{aligned}\vec{r}_{A/C} &= -CA\hat{j} = -\frac{\ell}{\sqrt{2}}\hat{j} \quad (\text{since } OA = 2 CA = \sqrt{2}\ell) \\ \vec{F} &= F(-\cos \theta \hat{i} - \sin \theta \hat{j}) = -F(\cos \theta \hat{i} + \sin \theta \hat{j})\end{aligned}$$

Hence,

$$\begin{aligned}\vec{M}_C &= -\frac{\ell}{\sqrt{2}}\hat{j} \times [-F(\cos \theta \hat{i} + \sin \theta \hat{j})] \\ &= \frac{\ell}{\sqrt{2}}F \underbrace{(\cos \theta \hat{j} \times \hat{i} + \sin \theta \hat{j} \times \hat{j})}_{-\hat{k}} \\ &= -\frac{\ell}{\sqrt{2}}F \cos \theta \hat{k} \\ &= -\frac{2 \text{ m}}{\sqrt{2}} \cdot 50 \text{ N} \cdot \cos 45^\circ \hat{k} = -50 \text{ N} \cdot \text{m} \hat{k}.\end{aligned}$$

$$\boxed{\vec{M}_C = -50 \text{ N} \cdot \text{m} \hat{k}}$$

**SAMPLE 2.23** Computing cross product in 3-D: Compute  $\vec{a} \times \vec{b}$ , where  $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$  and  $\vec{b} = 3\hat{i} + -4\hat{j} + \hat{k}$ .

**Solution** The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$\begin{aligned}\vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \hat{i} + \hat{j} - 2\hat{k}, \\ \vec{b} &= b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = 3\hat{i} + -4\hat{j} + \hat{k}.\end{aligned}$$

- **Calculation using the determinant formula:** In this method, we first write a  $3 \times 3$  matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{i}(1 - 8) + \hat{j}(-6 - 1) + \hat{k}(-4 - 3) \\ &= -7(\hat{i} + \hat{j} + \hat{k}).\end{aligned}$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

$$\begin{aligned}\vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1\hat{i} \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + \\ &\quad a_2\hat{j} \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + \\ &\quad a_3\hat{k} \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1b_1(\underbrace{\hat{i} \times \hat{i}}_0) + a_1b_2(\underbrace{\hat{i} \times \hat{j}}_{\hat{k}}) + a_1b_3(\underbrace{\hat{i} \times \hat{k}}_{-\hat{j}}) + \\ &\quad a_2b_1(\underbrace{\hat{j} \times \hat{i}}_{-\hat{k}}) + a_2b_2(\underbrace{\hat{j} \times \hat{j}}_0) + a_2b_3(\underbrace{\hat{j} \times \hat{k}}_{\hat{i}}) + \\ &\quad a_3b_1(\underbrace{\hat{k} \times \hat{i}}_{\hat{j}}) + a_3b_2(\underbrace{\hat{k} \times \hat{j}}_{-\hat{i}}) + a_3b_3(\underbrace{\hat{k} \times \hat{k}}_0) \\ &= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{i}(1 - 8) + \hat{j}(-6 - 1) + \hat{k}(-4 - 3) \\ &= -7(\hat{i} + \hat{j} + \hat{k})\end{aligned}$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 2.38 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g.,  $\hat{i} \times \hat{j} = \hat{k}$  but  $\hat{j} \times \hat{i} = -\hat{k}$ .

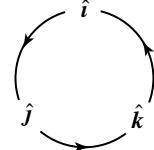


Figure 2.38: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g.  $\hat{i} \times \hat{j} = \hat{k}$  but  $\hat{j} \times \hat{i} = -\hat{k}$ .

(Filename:ijkcircle)

$$\boxed{\vec{a} \times \vec{b} = -7(\hat{i} + \hat{j} + \hat{k})}$$

**SAMPLE 2.24** *Finding a vector normal to two given vectors:* Find a unit vector perpendicular to the vectors  $\vec{r}_A = \hat{i} - 2\hat{j} + \hat{k}$  and  $\vec{r}_B = 3\hat{j} + 2\hat{k}$ .

**Solution** The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let  $\vec{N} = N\hat{\lambda}$  be the perpendicular vector.

$$\begin{aligned}\vec{N} &= \vec{r}_A \times \vec{r}_B \\ &= (\hat{i} - 2\hat{j} + \hat{k}) \times (3\hat{j} + 2\hat{k}) \\ &= -7\hat{i} - 2\hat{j} + 3\hat{k}.\end{aligned}$$

This calculation can be done in any of the two ways shown in the previous sample problem.

Therefore,

$$\begin{aligned}\hat{\lambda} &= \frac{\vec{N}}{N} \\ &= \frac{-7\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{7^2 + 2^2 + 3^2}} \\ &= -0.89\hat{i} - 0.25\hat{j} + 0.38\hat{k}\end{aligned}$$

$$\boxed{\hat{\lambda} = -0.89\hat{i} - 0.25\hat{j} + 0.38\hat{k}}$$

Check:

- $|\hat{\lambda}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \stackrel{?}{=} 1$ . (it is a unit vector)
- $\hat{\lambda} \cdot \vec{r}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \stackrel{?}{=} 0$ . ( $\hat{\lambda} \perp \vec{r}_A$ ).
- $\hat{\lambda} \cdot \vec{r}_B = 3(-0.25) + 2(0.38) \stackrel{?}{=} 0$ . ( $\hat{\lambda} \perp \vec{r}_B$ ).

**Comments:** If  $\hat{\lambda}$  is perpendicular to  $\vec{r}_A$  and  $\vec{r}_B$ , then so is  $-\hat{\lambda}$ . The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if  $\hat{\lambda}$  is perpendicular to a vector  $\vec{r}$  then any scalar multiple of  $\hat{\lambda}$ , i.e.,  $\alpha\hat{\lambda}$ , is also perpendicular to  $\vec{r}$ . This follows from the fact that

$$\alpha\hat{\lambda} \cdot \vec{r} = \alpha(\hat{\lambda} \cdot \vec{r}) = \alpha(0) = 0.$$

The case of  $-\hat{\lambda}$  is just a particular instance of this rule with  $\alpha = -1$ .

**SAMPLE 2.25** *Finding a vector normal to a plane:* Find a unit vector normal to the plane ABC shown in the figure.

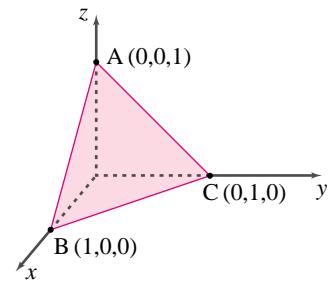


Figure 2.39: (Filename:sfig2.vec2.normal)

**Solution** A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say  $\vec{r}_{AB}$  and  $\vec{r}_{AC}$ , the normal to the plane would be perpendicular to both  $\vec{r}_{AB}$  and  $\vec{r}_{AC}$ . Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of  $\vec{r}_{AB}$  and  $\vec{r}_{AC}$ . Thus,

$$\begin{aligned}\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\ &= (\hat{i} - \hat{k}) \times (\hat{j} - \hat{k}) \\ &= \underbrace{\hat{i} \times \hat{j}}_{\hat{k}} - \underbrace{\hat{i} \times \hat{k}}_{-\hat{j}} - \underbrace{\hat{k} \times \hat{j}}_{-\hat{i}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\ &= \hat{i} + \hat{j} + \hat{k} \\ \Rightarrow \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\ &= \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}).\end{aligned}$$

$$\boxed{\hat{n} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})}$$

Check: Now let us check if  $\hat{n}$  is normal to any vector in the plane ABC. It is fairly easy to show that  $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$ . It is, however, not a surprise; it better be since we found  $\hat{n}$  from the cross product of  $\vec{r}_{AB}$  and  $\vec{r}_{AC}$ . Let us check if  $\hat{n}$  is normal to  $\vec{r}_{BC}$ :

$$\begin{aligned}\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \cdot (-\hat{i} + \hat{j}) \\ &= \frac{1}{\sqrt{3}}(-\hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j}) \\ &= \frac{1}{\sqrt{3}}(-1 + 1) \stackrel{!}{=} 0.\end{aligned}$$

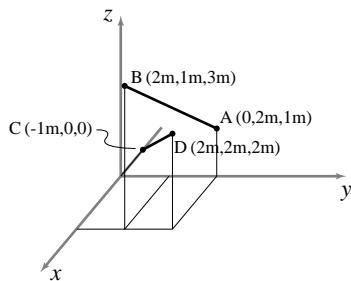


Figure 2.40: (Filename:sfig2\_vec2\_perp3D)

**SAMPLE 2.26** *The shortest distance between two lines:* Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) as shown in the figure. Find the shortest distance between the two lines.

**Solution** The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

- First find a vector that is perpendicular to both the lines. This is easy. Take two vectors  $\vec{r}_1$  and  $\vec{r}_2$ , one along each of the two given lines. Take the cross product of the two unit vectors and make the resulting vector a unit vector  $\hat{n}$ .
- Find a vector parallel to  $\hat{n}$  that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along  $\hat{n}$ .
- Find the length (magnitude) of the vector just found (in the direction of  $\hat{n}$ ). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

- Step-1: Find a unit vector  $\hat{n}$  that is perpendicular to both the lines.

$$\begin{aligned}\vec{r}_{AB} &= 2\hat{i} - 1\hat{j} + 2\hat{k} \\ \vec{r}_{CD} &= 3\hat{i} + 2\hat{j} + 2\hat{k} \\ \Rightarrow \vec{r}_{AB} \times \vec{r}_{CD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} \text{ m}^2 \\ &= \hat{i}(-2 - 4) \text{ m}^2 + \hat{j}(6 - 4) \text{ m}^2 + \hat{k}(4 + 3) \text{ m}^2 \\ &= (-6\hat{i} + 2\hat{j} + 7\hat{k}) \text{ m}^2\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{n} &= \frac{\vec{r}_{AB} \times \vec{r}_{CD}}{|\vec{r}_{AB} \times \vec{r}_{CD}|} \\ &= \frac{1}{\sqrt{89}}(-6\hat{i} + 2\hat{j} + 7\hat{k}).\end{aligned}$$

- Step-2: Find any vector from one line to the other line and find its component along  $\hat{n}$ .

$$\begin{aligned}\vec{r}_{AC} &= -1\hat{i} - 2\hat{j} - 1\hat{k} \\ \vec{r}_{AC} \cdot \hat{n} &= -(\hat{i} + 2\hat{j} + \hat{k}) \text{ m} \cdot \frac{1}{\sqrt{89}}(-6\hat{i} + 2\hat{j} + 7\hat{k}) \\ &= \frac{1}{\sqrt{89}}(6 - 4 - 7) \text{ m} = -\frac{5}{\sqrt{89}} \text{ m}.\end{aligned}$$

- Step-3: Find the required distance  $d$  by taking the magnitude of the component along  $\hat{n}$ .

$$d = |\vec{r}_{AC} \cdot \hat{n}| = \left| -\frac{5}{\sqrt{89}} \text{ m} \right| = 0.53 \text{ m}$$

$d = 0.53 \text{ m}$

**SAMPLE 2.27** *The mixed triple product:* Calculate the mixed triple product  $\hat{\lambda} \cdot (\vec{a} \times \vec{b})$  for  $\hat{\lambda} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ ,  $\vec{a} = 3\hat{i}$ , and  $\vec{b} = \hat{i} + \hat{j} + 3\hat{k}$ .

**Solution** We compute the given mixed triple product in two ways here:

- Method-1: Straight calculation using cross product and dot product.

$$\begin{aligned} \text{Let } \vec{c} &= \vec{a} \times \vec{b} \\ &= (3\hat{i}) \times (\hat{i} + \hat{j} + 3\hat{k}) \\ &= 3\underbrace{\hat{i} \times \hat{i}}_{\vec{0}} + 3\underbrace{\hat{i} \times \hat{j}}_{\hat{k}} + 9\underbrace{\hat{i} \times \hat{k}}_{-\hat{j}} = -9\hat{j} + 3\hat{k} \end{aligned}$$

$$\begin{aligned} \text{So, } \hat{\lambda} \cdot (\vec{a} \times \vec{b}) &= \hat{\lambda} \cdot \vec{c} \\ &= \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) \cdot (-9\hat{j} + 3\hat{k}) = -\frac{9}{\sqrt{2}}. \end{aligned}$$

- Method-2: Using the determinant formula for mixed product.

$$\begin{aligned} \hat{\lambda} \cdot (\vec{a} \times \vec{b}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\ &= \frac{1}{\sqrt{2}}(0 - 0) + \frac{1}{\sqrt{2}}(0 - 9) + 0 = -\frac{9}{\sqrt{2}}. \end{aligned}$$

$$\boxed{\hat{\lambda} \cdot (\vec{a} \times \vec{b}) = -\frac{9}{\sqrt{2}}}$$

**SAMPLE 2.28** *Moment about an axis:* A vertical force of unknown magnitude  $F$  acts at point B of a triangular plate ABC shown in the figure. Find the moment of the force about edge CA of the plate.

**Solution** The moment of a force  $\vec{F}$  about an axis x-x is given by

$$M_{xx} = \hat{\lambda}_{xx} \cdot (\vec{r} \times \vec{F})$$

where  $\hat{\lambda}_{xx}$  is a unit vector along the axis x-x,  $\vec{r}$  is a position vector from any point on the axis to the applied force. In this problem, the given axis is CA. Therefore, we can take  $\vec{r}$  to be  $\vec{r}_{AB}$  or  $\vec{r}_{CB}$ . Here,

$$\hat{\lambda}_{CA} = \frac{\vec{r}_{CA}}{|\vec{r}_{CA}|} = \frac{3(-\hat{i} + \hat{j})}{\sqrt{9+9}} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

Now, moment about point A is

$$\begin{aligned} \vec{M}_A &= \vec{r}_{AB} \times \vec{F} \\ &= (-2\hat{i} - 3\hat{j}) \times F\hat{k} = 2F\hat{j} - 3F\hat{i}. \end{aligned}$$

Therefore, the moment about CA is

$$\begin{aligned} M_{CA} &= \hat{\lambda}_{CA} \cdot (\vec{r}_{AB} \times \vec{F}) = \hat{\lambda}_{CA} \cdot \vec{M}_A \\ &= \left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) \cdot (-3F\hat{i} + 2F\hat{j}) \\ &= \left(\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)F = \frac{5}{\sqrt{2}}F. \end{aligned}$$

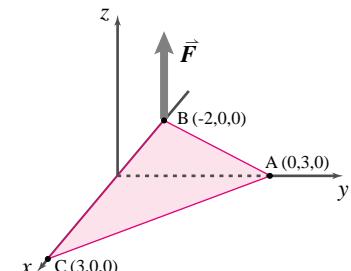


Figure 2.41: (Filename:sfig2.vec2.momxx)

$$\boxed{M_{CA} = \frac{5}{\sqrt{2}}F}$$

## 2.4 Solving vector equations

If as an engineer you knew all quantities of interest you would not need to calculate. But as a rule in life you know less than you would like to know. And you naturally try to figure out more. In engineering mechanics analysis you find more quantities of interest from others that you already know (or assume) using the laws of mechanics (including geometry and kinematics). Because many of the laws of these laws are vector equations, engineering analysis often requires the solving of vector equations.

The methods involved are much the same whether the problems are in geometry, kinematics, statics, dynamics or a combination of these. In this section we will show a few methods for solving some of the more common vector equations. In a sense there are no new concepts in this section; if you are already adept at vector manipulations you will find yourself reading quickly.

### Vector algebra

We will be concerned with manipulating equations that involve vectors (like  $\vec{A}$ ,  $\vec{B}$ ,  $\vec{C}$ , and  $\vec{0}$ ) and scalars (like  $a$ ,  $b$ ,  $c$ , and 0). Without knowing anything about mechanics or the geometric meaning of vectors, one can learn to do correct vector algebra by just following the manipulation rules below, these are elaborations of elementary scalar algebra to accommodate vectors and the three new kinds of multiplication (scalar times vector, dot product, and cross product). Here is a summary.

Addition and all three kinds of multiplication (scalar multiplication, dot product, cross product) all follow the usual commutative, associative, and distributive laws of scalar addition and multiplication with the following exceptions:

- $a + \vec{A}$  is nonsense,
- $a/\vec{A}$  is nonsense,
- $\vec{A}/\vec{B}$  is nonsense,
- $a \cdot \vec{A}$  is nonsense (unless you mean by it  $a\vec{A}$ ),
- $a \times \vec{A}$  is nonsense,
- $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ ,

and the following extra simplification rules:

- $a\vec{A}$  is a vector,
- $\vec{A} \cdot \vec{B}$  is a scalar,
- $\vec{A} \times \vec{B}$  is a vector,
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$  (so  $\vec{A} \times \vec{A} = \vec{0}$ )
- $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$ .

Following these rules automatically enforces correct manipulations. Armed with insight you can direct these manipulations towards a desired end.

**Example.** Say you know  $\vec{A}$ ,  $\vec{B}$ ,  $\vec{C}$  and  $\vec{D}$  and you know that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

but you don't know  $a$ ,  $b$ , and  $c$ . How could you find  $a$ ? First dot both sides with  $\vec{B} \times \vec{C}$  and then blindly follow the rules:

$$\begin{aligned} a\vec{A} + b\vec{B} + c\vec{C} &= \vec{D} \cdot (\vec{B} \times \vec{C})0 \\ a\vec{A} \cdot (\vec{B} \times \vec{C}) + b \underbrace{\vec{B} \cdot (\vec{B} \times \vec{C})}_{0} + c \underbrace{\vec{C} \cdot (\vec{B} \times \vec{C})}_{0} &= \vec{D} \cdot (\vec{B} \times \vec{C}) \\ a &= \frac{\vec{D} \cdot (\vec{B} \times \vec{C})}{\vec{A} \cdot (\vec{B} \times \vec{C})}. \end{aligned}$$

The two zeros followed from the general rules that  $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$  and  $\vec{A} \times \vec{A} = \vec{0}$ . The last line of the calculation assumes that  $\vec{A} \cdot (\vec{B} \times \vec{C}) \neq 0$ . (The linear algebra savvy reader will recognize this thoughtless manipulation as a derivation of Cramer's rule for  $3 \times 3$  matrices.) Note the derivation above breaks down if the vectors  $\vec{A}$ ,  $b\vec{B}$ , and  $c\vec{C}$  are co-planar (see box 2.5).  $\square$

The point of the example above was to show the vector algebra rules at work. However, to get to the end took the first 'move' of dotting the equation with the appropriate vector. That move could be motivated this way. We are trying to find  $a$  and not  $b$  or  $c$ . We can get rid of the terms in the equation that contain  $b$  and  $c$  if we can dot  $\vec{B}$  and  $\vec{C}$  with a vector perpendicular to both of them.  $\vec{B} \times \vec{C}$  is perpendicular to both  $\vec{B}$  and  $\vec{C}$  so can be used to kill them off with a dot product. The 0s in the example calculation were thus expected for geometric reasons.

## Count equations and unknowns.

One cannot (usually) find more unknowns than one has scalar equations. Before you do lots of algebra, you should check that you have as many equations as unknowns. If not, you probably can't find all the unknowns. How do you count vector equations and vector unknowns? A two-dimensional vector is fully described by two numbers. For example, a 2D vector is described by its  $x$  and  $y$  components or its magnitude and the angle it makes with the positive  $x$  axis. A three-dimensional vector is described by three numbers. So a vector equation counts as 2 or 3 equations in 2 or 3 dimensional problems, respectively. And an unknown vector counts as 2 or 3 unknowns in 2 or 3 dimensions, respectively. If the direction of a vector is known but its magnitude is not, then the magnitude is the only unknown. Magnitude is a scalar, so it counts as one unknown.

### *Example: Counting equations*

Say you are doing a 2-D problem where you already know the vector  $\hat{\lambda} = \sqrt{2}\hat{i} + \sqrt{2}\hat{j}$  and you are given the vector equation

$$C\hat{\lambda} = \vec{a}.$$

You then have two equations (a vector equation in 2-D) and three unknowns (the scalar  $C$  and the vector  $\vec{a}$ ). There are more unknowns than equations so this vector equation is not sufficient for finding  $C$  and  $\vec{a}$ .  $\square$

Most often when you have as many equations as unknowns the equations have a unique solution. When you have more equations than unknowns there is most often no solution to the equations. When you have more unknowns than equations most often you have a whole family of solutions.

However these are only guidelines, no matter how many equations and unknowns you have, you could have no solutions, many solutions or a unique solution. The geometric significance of some cases that satisfy and that don't satisfy these guidelines is given in box 2.5 on page 63.

## Vector triangles

In 2D one often wants to know all three vectors in a vector triangle, the diagram for expressions like

$$\vec{A} + \vec{B} = \vec{C} \quad \text{or} \quad \vec{A} - \vec{C} = \vec{B} \quad \text{or} \quad \vec{A} + \vec{B} + \vec{C} = \vec{0} \quad \text{etc.}$$

Usually at least one vector is given and some information is given about the others. The situation is much like the geometry problem of drawing a triangle given various bits of information about the lengths of its sides and its interior angles. If enough information is given to prove triangle congruence, then enough information is given to determine all angles and sides. A difference between vector triangles and proofs of triangle congruence is that triangle congruence does not depend on the overall orientation, whereas vector triangles need to have the correct orientation. Nonetheless, the tools used to solve triangles are useful for solving vector equations.

## Vector addition

We start with a problem that is in some sense solved at the start. Say  $\vec{A}$  and  $\vec{B}$  are known and you want to find  $\vec{C}$  given that

$$\vec{C} = \vec{A} + \vec{B}.$$

The obvious and correct answer is that you find  $\vec{C}$  by vector addition. You could do this addition graphically by drawing a scale picture, or by adding corresponding vector components. Suppose now that  $\vec{A}$  and  $\vec{B}$  are given to you in terms of magnitude and direction and that you are interested in the direction of  $\vec{C}$ .

**Example: adding vectors defined by magnitude and direction**

Say direction is indicated by angle measured counterclockwise from the positive  $x$  axis and that  $A = 5\sqrt{2}$ ,  $\theta_A = \pi/4$ ,  $B = 4$ , and  $\theta_B = 2\pi/3$ . So

$$\begin{aligned}\vec{A} &= A(\cos \theta_A \hat{i} + \sin \theta_A \hat{j}) \\ &= 5\sqrt{2}(\cos(\pi/4)\hat{i} + \sin(\pi/4)\hat{j}) = 5\hat{i} + 5\hat{j} \\ \vec{B} &= B(\cos \theta_B \hat{i} + \sin \theta_B \hat{j}) \\ &= 4(\cos(2\pi/3)\hat{i} + \sin(2\pi/3)\hat{j}) = -2\hat{i} + 2\sqrt{3}\hat{j} \\ \vec{C} &= \vec{A} + \vec{B} = (5\hat{i} + 5\hat{j}) + (-2\hat{i} + 2\sqrt{3}\hat{j}) \\ &= 3\hat{i} + (5 + 2\sqrt{3})\hat{j} \\ \Rightarrow \theta_C &= \tan^{-1}(C_y/C_x) = \tan^{-1}((5 + 2\sqrt{3})/3) \approx 1.23 \approx 70.5^\circ \\ \text{and } C &= \sqrt{3^2 + (5 + 2\sqrt{3})^2} \approx 8.98\end{aligned}$$

□

To find  $\theta_C$  we used the arctan (or  $\tan^{-1}$ ) function which can be off by  $\pi$ <sup>①</sup>. To find the angle of  $\vec{C}$  we had to convert  $\vec{A}$  and  $\vec{B}$  to coordinate form, add components, and then convert back to find the angle of  $\vec{C}$ . That is, even though the desired answer is given by a sum, carrying out the sum takes a bit of effort. An alternative approach avoids some work.

**Example: Same as above, different method**

Start with picture of the situation, Fig. 2.42. By adding angles,

$$\theta_2 = \pi/4 + \pi/3 = 7\pi/12.$$

From the law of cosines (see box 2.6 on page 64),

$$\begin{aligned}C^2 &= A^2 + B^2 - 2AB \cos \theta_2 \\ \Rightarrow C &= \sqrt{(5\sqrt{2})^2 + 4^2 - 2(5\sqrt{2}) \cdot 4 \cdot \cos(7\pi/12)} \\ &\approx 8.98 \quad (\text{as before})\end{aligned}$$

And from the law of sines (see box 2.6),

$$\begin{aligned}\frac{\sin \theta_1}{B} &= \frac{\sin \theta_2}{C} \\ \Rightarrow \theta_1 &= \sin^{-1}\left(\frac{B \sin \theta_2}{C}\right) \approx \sin^{-1}\left(\frac{4 \sin(7\pi/12)}{8.98}\right) \\ &\approx .445 \\ \Rightarrow \theta_C &= \theta_A + \theta_2 \approx \pi/4 + .445 \approx 1.23 \quad (\text{as before}).\end{aligned}$$

□

<sup>①</sup> The problem is that, measuring angles between 0 and  $2\pi$  (or equivalently between  $-\pi$  and  $\pi$ ) there are always two different angles that have the same tangent. The inverse tangent function picks one. Some computers or calculators always pick an angle between 0 and  $\pi$  and some always pick a value between  $-\pi/2$  and  $\pi/2$ . Both of these could be the wrong answer. So you need to check and possibly add  $\pi$  to your answer, or, alternatively use one of these two commands: 1) the two-argument inverse tangent ( $\arctan(x, y)$ ) or 2) rectangular-to-polar coordinate conversion, using the angle as the desired arctangent.

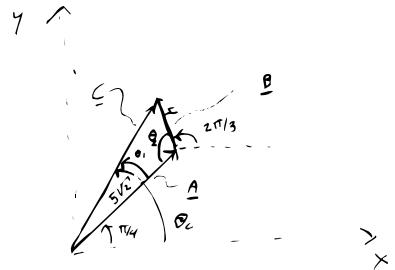


Figure 2.42: Using trig to solve vector triangles  
(Filename:tfigure.cosinetriangle)

This second approach is somewhat more direct in some situations.

The determination of a third vector by vector addition is analogous to the determination of a triangle in geometry by “side-angle-side”.

## Vector subtraction

Say you want to find  $\vec{C}$  given  $\vec{A}$  and  $\vec{B}$  and that  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  add to zero. So, subtracting  $\vec{C}$  from both sides and multiplying through by -1 we get

$$\begin{aligned}\vec{A} + \vec{B} + \vec{C} &= \vec{0} \\ \Rightarrow \quad \vec{C} &= -\vec{A} - \vec{B}.\end{aligned}$$

The problem has now been reduced to one of addition which can be done by drawing, components, or trig as shown above.

## Find the magnitude of two vectors given their directions and their sum (2D)

Often one knows that 2 vectors  $\vec{A}$  and  $\vec{B}$  add to a given third vector  $\vec{C}$ . The directions of  $\vec{A}$  and  $\vec{B}$  are known but not their magnitudes. That is, given  $\hat{\lambda}_A$ ,  $\hat{\lambda}_B$  and  $\vec{C}$  and that

$$\begin{aligned}\vec{A} + \vec{B} &= \vec{C} \\ A\hat{\lambda}_A + B\hat{\lambda}_B &= \vec{C}\end{aligned}\tag{2.11}$$

you would like to find  $\vec{A}$  and  $\vec{B}$  (which you will know if you find  $A$  and  $B$ ).

### Example: A walk

You walked SW (half way between South and West) for a while and NNW (half way between North and NorthWest,  $22.5^\circ$  West of North) for a while and ended up going a net distance of 200 m East.  $\vec{A}$  and  $\vec{B}$  are your displacements on the first and second parts of your walk.

So, taking  $xy$  axes aligned with East and North, the directions are

$$\hat{\lambda}_A = \frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j} \text{ and } \hat{\lambda}_B = -\sin(\frac{\pi}{8})\hat{i} + \cos(\frac{\pi}{8})\hat{j}$$

and the given sum is  $\vec{C} = 200 \text{ m}\hat{i}$ . Still unknown are the distances  $A$  and  $B$ .  $\square$

Here are four ways to solve eqn. (2.11) which will be illustrated with “a walk”.

### Method I: Use dot products with $\hat{i}$ and $\hat{j}$

If we take the dot product of both sides of eqn. (2.11) with  $\hat{i}$  and then again with  $\hat{j}$  we get:

$$\begin{aligned}\hat{i} \cdot \{\text{eqn. (2.11)}\} &\Rightarrow A\lambda_{Ax} + B\lambda_{Bx} = C_x, \text{ and} \\ \hat{j} \cdot \{\text{eqn. (2.11)}\} &\Rightarrow A\lambda_{Ay} + B\lambda_{By} = C_y\end{aligned}\tag{2.12}$$

where the components of the vectors  $\hat{\lambda}_A$ ,  $\hat{\lambda}_B$ , and  $\vec{C}$  are known, or easily determined, because the vectors are known (however they are represented). Eqns. 2.12 are two scalar equations in the unknowns  $A$  and  $B$ . You can solve these any way that pleases you. One method would be to write the equations in matrix form

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} \\ \lambda_{Ay} & \lambda_{By} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix}\tag{2.13}$$

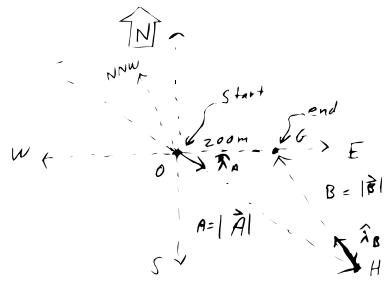


Figure 2.43: An indirect walk from O to D via C.  
(Filename:tfigure.2.walk)

**Example: Solving “A walk”: method I, simultaneous equations**

For the walk example above we would have

$$\begin{bmatrix} \sqrt{2}/2 & -\sin(\frac{\pi}{8}) \\ -\sqrt{2}/2 & \cos(\frac{\pi}{8}) \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 200 \text{ m} \\ 0 \end{bmatrix}$$

which solves (on a computer or calculator) to  $A \approx 483 \text{ m}$  and  $B \approx 370$   
(with the total walked distance being about 852 m).  $\square$

Taking dot products of a vector equation with  $\hat{i}$  and  $\hat{j}$  is equivalent to extracting the  $x$  and  $y$  components of the equation. But we use the dot product notation to highlight that you could dot both sides of the vector equation with any vector that pleases you and you would get a legitimate scalar equation. Use any other vector that pleases you (not parallel with the first) and you will get a second independent equation. And the two resulting equations will have the same solution for  $A$  and  $B$  as the  $x$  and  $y$  (or “ $\hat{i}$ ” and “ $\hat{j}$ ”) equations above.

*Method II: pick a vector for dot product that kills terms you don’t know.*

Pretend for a paragraph that you only want to find  $A$  in eqn. (2.11), for example that you only wanted to know the distance walked on the first leg of the indirect walk in the example above. It would be nice to reduce eqn. (2.11) to a single scalar equation in the single unknown  $A$ . We’d like to get rid of the term with  $B$ , a quantity that we do not know. Suppose we knew a vector  $\hat{n}_B$  that was perpendicular to  $\hat{\lambda}_B$ . If we dotted both sides of eqn. (2.11) we’d get:

$$\begin{aligned} \hat{n} \cdot \{\text{eqn. (2.11)}\} &\Rightarrow \hat{n}_B \cdot (A\hat{\lambda}_A + B\hat{\lambda}_B) = \hat{n}_B \cdot \vec{C} \\ &\Rightarrow \hat{n}_B \cdot (A\hat{\lambda}_A) + \hat{n}_B \cdot (B\hat{\lambda}_B) = \hat{n}_B \cdot \vec{C} \\ \hat{n}_B \perp \hat{\lambda}_B \text{ so } \hat{n}_B \cdot \hat{\lambda}_B = 0 &\Rightarrow (\hat{n}_B \cdot \hat{\lambda}_A) A = \hat{n}_B \cdot \vec{C} \\ &\Rightarrow A = \frac{\hat{n}_B \cdot \vec{C}}{\hat{n}_B \cdot \hat{\lambda}_A}. \end{aligned}$$

To make use of this method we have to cook up a vector  $\hat{n}_B$  that is perpendicular to  $\hat{\lambda}_B$ <sup>①</sup>. The vector  $\hat{n}_B = \hat{k} \times \hat{\lambda}_B$  serves the purpose. So we get

$$A = \frac{(\hat{k} \times \hat{\lambda}_B) \cdot \vec{C}}{(\hat{k} \times \hat{\lambda}_B) \cdot \hat{\lambda}_A} = \frac{\lambda_{By} C_x - \lambda_{Bx} C_y}{\lambda_{By} \lambda_{Ax} - \lambda_{Bx} \lambda_{Ay}}$$

which, if you learned such, you may recognize as the Cramer’s rule solution of eqn. (2.19). Summarizing<sup>②</sup>.

To reduce eqn. (2.11) to one scalar equation in the one unknown  $A$ , kill the  $\hat{\lambda}_B$  or  $\vec{B}$  term by dotting both sides of with  $\hat{k} \times \hat{\lambda}_B$  or  $\hat{k} \times \vec{B}$

① The vector  $\hat{k}$  (the unit vector out of the page) is perpendicular to  $\hat{\lambda}_B$  but is unfortunately not suitable because it is also perpendicular to  $\hat{\lambda}_A$  and  $\vec{C}$  so only yields the equation  $0 + 0 = 0$  or the nonsense that  $A = 0/0$ .

② Its the modern way, kill the things you don’t know about (and thus don’t like) using the most powerful weapons at your disposal.

Altogether you can think of this method as something like the “component” method. But we are taking components of the vectors in the direction perpendicular to  $\vec{B}$ . Alternatively you can think of this method as taking the projection of the vector equation onto a line perpendicular to  $\vec{B}$ .

Similarly dotting both sides of eqn. (2.11) with  $\hat{k} \times \hat{\lambda}_A$  gives

$$B = \frac{(\hat{k} \times \hat{\lambda}_A) \cdot \vec{C}}{(\hat{k} \times \hat{\lambda}_A) \cdot \hat{\lambda}_B}.$$

**Example: Solving “A walk”: method II, judicious dot products**

You should be able to derive the formulas above as needed. Dotting, for example, both sides of eqn. (2.11) with  $\hat{k} \times \hat{\lambda}_B$  and plugging in the known components yields

$$\begin{aligned} A = \frac{(\hat{k} \times \hat{\lambda}_B) \cdot \vec{C}}{(\hat{k} \times \hat{\lambda}_B) \cdot \hat{\lambda}_A} &= \frac{\lambda_{By} C_x - \lambda_{Bx} C_y}{\lambda_{By} \lambda_{Ax} - \lambda_{Bx} \lambda_{Ay}} \\ &= \frac{\cos(\pi/8) \cdot 200 \text{ m} - (-\sin(\pi/8)) \cdot 0}{\cos(\pi/8) \cdot (\sqrt{2}/2) - (-\sin(\pi/8)) \cdot (-\sqrt{2}/2)} \\ &\approx 483 \text{ m} \quad (\text{as before}) \end{aligned}$$

□

**Method III, graphical solution**

On the vector triangle defined by  $\vec{A} + \vec{B} = \vec{C}$  we call O the tail end of  $\vec{A}$ . The location of the tip of  $\vec{C}$  at G can be drawn to scale. Then the point H can be located as at the intersection of two lines: one emanating from O and in the direction of  $\hat{\lambda}_A$  and one emanating from G and in the direction of  $\hat{\lambda}_B$ . Once the point H is located, the lengths A and B can be measured.

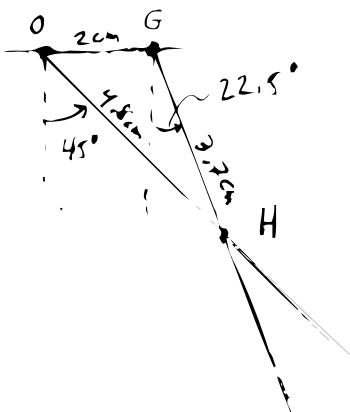


Figure 2.44: Method III: Find point H as the intersection of two known lines.

(Filename:tfigure.2.walkIII)

**Example: Solving “A walk”: method III, graphing**

Taking 100 mas drawn to scale as, say 1 cm, point G is drawn 2 cm to the right of O. The location of the point H is found as the intersection of two lines: one emanating from O and pointing  $45^\circ$  counterclockwise from the  $-\hat{j}$  axis, and the other emanating from G and pointing  $22.5^\circ$  counterclockwise from the  $-\hat{j}$  axis. The distance from O to H can be measured as about 4.8 cm yielding  $A \approx 480 \text{ m}$ .

This construction can be done with pencil and paper or with a computer drawing program. □

**Method IV, trigonometry**

The final method, the classical method used predominantly before vector notation was well accepted, is to treat the vector triangle as a triangle with some known sides and some known angles, and to use the *law of sines* (discussed in box 2.6).

Because  $\vec{C}$  and the directions of  $\vec{A}$  and  $\vec{B}$  are assumed known, the angles  $a$  (opposite side A) and  $b$  (opposite side B) are known. Because the sum of interior angles in a triangle is  $\pi$  we know the angle  $c = \pi - a - b$ . The law of sines tells us that

$$\frac{\sin a}{A} = \frac{\sin c}{C} \quad \text{and} \quad \frac{\sin b}{B} = \frac{\sin c}{C}$$

which we can rewrite as

$$A = \frac{C \sin a}{\sin c} \quad \text{and} \quad B = \frac{C \sin b}{\sin c}.$$

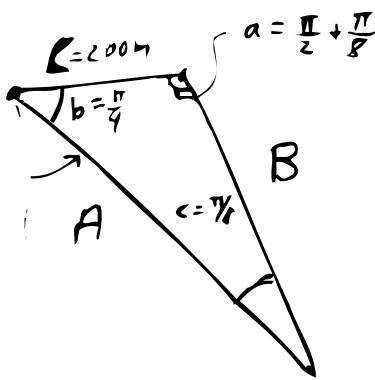


Figure 2.45: Solving “A walk” using the law of sines.

(Filename:tfigure.2.walkIV)

**Example: Solving “A walk”: method IV, the law of sines**

Referring to Fig. 2.45 we get

$$A = \frac{C \sin a}{\sin c} = \frac{200 \text{ m} \cdot \sin(5\pi/8)}{\sin(\pi/8)} \approx 483 \text{ m}$$

$$\text{and } B = \frac{C \sin b}{\sin c} = \frac{200 \text{ m} \cdot \sin(\pi/4)}{\sin(\pi/8)} \approx 370 \text{ m}$$

as we have found three times already.  $\square$

The determination of two vectors by knowing their directions and their sum is analogous to determination of a triangle by “angle-side-angle”.

**The magnitudes and sum of two vectors are known (2D)**

Two vectors  $\vec{A}$  and  $\vec{B}$  in the plane have known magnitudes  $A$  and  $B$  but unknown directions  $\hat{\lambda}_A$  and  $\hat{\lambda}_B$ . Their sum  $\vec{C}$  is known. So, measuring angles counterclockwise relative to the positive  $x$  axis, we have:

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{C} \\ A\hat{\lambda}_A + B\hat{\lambda}_B &= \vec{C} \\ A(\cos \theta_A \hat{i} + \sin \theta_A \hat{j}) + B(\cos \theta_B \hat{i} + \sin \theta_B \hat{j}) &= \vec{C} \end{aligned} \quad (2.14)$$

where eqn. (2.14) is one 2D vector equation in 2 unknowns:  $\theta_A$  and  $\theta_B$ .

*Method 1: using an appropriate dot product*

This problem is really best solved with trig (see below) and getting it right with component method is a matter of hindsight. Eqn. 2.14 can be rewritten as

$$C(\cos \theta_C \hat{i} + \sin \theta_C \hat{j}) - A(\cos \theta_A \hat{i} + \sin \theta_A \hat{j}) = B(\cos \theta_B \hat{i} + \sin \theta_B \hat{j})$$

Taking the dot product of each side with itself gives

$$C^2 + A^2 - 2AC \underbrace{(\cos \theta_C \cos \theta_A + \sin \theta_C \sin \theta_A)}_{\cos(\theta_C - \theta_A)} = B^2$$

so

$$\theta_A = \theta_C - \arccos \left( \frac{C^2 + A^2 - B^2}{2AC} \right).$$

Now  $\vec{A}$  is fully determined and  $\vec{B}$  can be found by vector subtraction. Note that the arccos function is always double valued (the negative of any arccos is also a legitimate arccos), so that the solution of this problem is not unique. Also, if the argument of the arccos function is greater than 1 in magnitude, there is no solution; this happens if any two of  $A$ ,  $B$ , and  $C$  is greater than the third (that is, if the so-called “triangle inequality” is violated) and there is no way of making a triangle with the given lengths.)

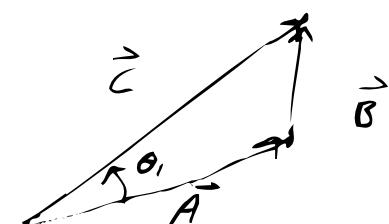


Figure 2.46: For use in using the law of cosines to solve a vector triangle.  
(Filename:tfigure.cosinestriangle)

*Method II: The law of cosines*

Referring to Fig. 2.46, we can apply the law of cosines directly to get

$$B^2 = A^2 + C^2 - 2AB \cos \theta_B \quad (2.15)$$

$$\text{which we can solve to get} \quad \theta_1 = \arccos \left( \frac{C^2 + A^2 - B^2}{2AC} \right). \quad (2.16)$$

Thus the orientation of  $\vec{A}$  is determined in relation to  $\vec{C}$ . This method is a bit quicker than the component method above because it skips the steps where, in effect, the component method derives the law of cosines.

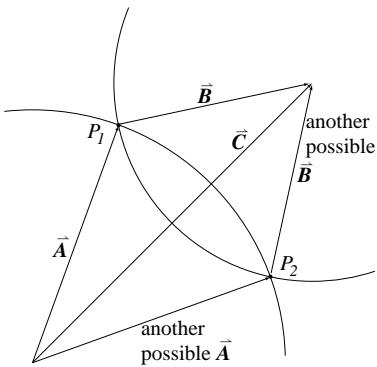
*Method III: graphical construction*

Figure 2.47: Solving a vector triangle where vectors  $\vec{A}$  and  $\vec{B}$  have known magnitude but unknown direction.

(Filename:tfigure.2circs)

### Find the magnitude of three vectors given their directions and their sum (3D)

This problem is close in approach to its junior 2D cousin on page 54 and to the example on page 50. It is the most common of the 3D vector equation problems. Assume that you know the directions of three vectors  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  (given, say, as the unit vectors  $\hat{\lambda}_A$ ,  $\hat{\lambda}_B$ , and  $\hat{\lambda}_C$ ), as well as their sum  $\vec{D}$ . So we have

$$\begin{aligned} \vec{A} &+ \vec{B} + \vec{C} = \vec{D} \\ A\hat{\lambda}_A &+ B\hat{\lambda}_B + C\hat{\lambda}_C = \vec{D} \end{aligned} \quad (2.17)$$

and we want to find  $A$ ,  $B$ , and  $C$  from which we can find  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  (e.g.,  $\vec{A} = A\hat{\lambda}_A$ ). We can think of the last of eqn. (2.17) as one 3D vector equation in three unknowns.

In three dimensions the graphical approach is essentially impossible. And the trigonometric approach is awkward to say the least, and probably only generally practical for people with British accents who are long dead. The general ideas in the first two methods still stand, however. Thus the use of vector concepts is basically unavoidable in 3D problems.

**Method I: doting with  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ .**

We can dot the left and right sides of eqn. (2.17) with  $\hat{i}$  or  $\hat{j}$  or  $\hat{k}$ . This is equivalent to taking the  $x$ ,  $y$  and  $z$  components of the equation. We get then

$$\begin{aligned}\hat{i} \cdot \{\text{eqn. (2.17)}\} &\Rightarrow A\lambda_{Ax} + B\lambda_{Bx} + C\lambda_{Cx} = D_x, \\ \hat{j} \cdot \{\text{eqn. (2.17)}\} &\Rightarrow A\lambda_{Ay} + B\lambda_{By} + C\lambda_{Cy} = D_y, \text{ and} \\ \hat{k} \cdot \{\text{eqn. (2.17)}\} &\Rightarrow A\lambda_{Az} + B\lambda_{Bz} + C\lambda_{Cz} = D_z\end{aligned}\quad (2.18)$$

which can be written in matrix form as

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} & \lambda_{Cx} \\ \lambda_{Ay} & \lambda_{By} & \lambda_{Cy} \\ \lambda_{Az} & \lambda_{Bz} & \lambda_{Cz} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}. \quad (2.19)$$

Unless the matrix is sparse (has a lot of zeros as entries) it is probably best to solve such a set of equations for  $A$ ,  $B$  and  $C$  on a computer or calculator.

**Method II: pick a vector for dot product that kills terms you don't know.**

The philosophy here is the same as for method II in 2D (page 55). Pretend for a paragraph that you only want to find  $A$  in eqn. (2.17). We can kill the terms involving the unknowns  $B$  and  $C$  by dotting both sides of the equation with a vector perpendicular to  $\hat{\lambda}_B$  and  $\hat{\lambda}_C$ . Such a vector is  $\hat{\lambda}_B \times \hat{\lambda}_C$ . Thus

$$\begin{aligned}(\hat{\lambda}_B \times \hat{\lambda}_C) \cdot \{\text{eqn. (2.11)}\} &= (\hat{\lambda}_B \times \hat{\lambda}_C) \cdot (A\hat{\lambda}_A + B\hat{\lambda}_B + C\hat{\lambda}_C) = (\hat{\lambda}_B \times \hat{\lambda}_C) \cdot \vec{D} \\ \Rightarrow (\hat{\lambda}_B \times \hat{\lambda}_C) \cdot (A\hat{\lambda}_A) + \vec{0} + \vec{0} &= (\hat{\lambda}_B \times \hat{\lambda}_C) \cdot \vec{D} \\ \Rightarrow A &= \frac{\vec{D} \cdot (\hat{\lambda}_B \times \hat{\lambda}_C)}{\hat{\lambda}_A \cdot (\hat{\lambda}_B \times \hat{\lambda}_C)}.\end{aligned}$$

If you use a matrix determinant to evaluate the mixed triple product you can recognize this formula (like the formula solving the example on 50) as Cramer's rule. By a judicious dot product we have reduced the vector equation to a scalar equation in one unknown. Similarly we could get one equation in one unknown for  $B$  or for  $C$  by doting eqn. (2.17) with  $\hat{\lambda}_A \times \hat{\lambda}_C$  and  $\hat{\lambda}_A \times \hat{\lambda}_B$ , respectively<sup>①</sup>.

## Parametric equations for lines and planes

### A line in 2D

In geometry a line on a plane is often describe as the set of  $x$  and  $y$  points that satisfy an equation like

$$Ax + By = D \quad \text{or} \quad y = mx + b$$

for given  $A$ ,  $B$ , and  $D$  or  $m$  and  $b$ . However a line is a “one dimensional” object and it is nice to describe it that way. The parametric form that is often useful is:

$$\vec{r} = \vec{r}_0 + s\vec{v} \quad (2.20)$$

where  $\vec{r}$  are the set of points on the line, one point for each value of the scalar parameter  $s$ .  $\vec{r}_0$  is any one given point on the line and  $\vec{v}$  is a vector parallel to the line. In the special case that  $\vec{v}$  is a unit vector,  $s$  is the distance from the point at  $\vec{r}_0$  to the point at  $\vec{r}$ . If the vector  $\vec{v}$  was the velocity of a point moving on the line then  $s$  would be the time since it was at the point  $\vec{r}_0$ .

<sup>①</sup> Note that the key to the method was doting with a vector in an appropriate direction, the magnitude of the vector did not matter. So if, for example, you knew any vector  $\vec{v}_B$  in the direction of  $\hat{\lambda}_B$  and any vector  $\vec{v}_C$  in the direction of  $\hat{\lambda}_C$  you could dot both sides of eqn. (2.17) with  $\vec{v}_B \times \vec{v}_C$  to get one scalar equation for  $A$ . This can simplify calculations by avoiding the square roots (which cancel in the end) that you calculate to find unit vectors.

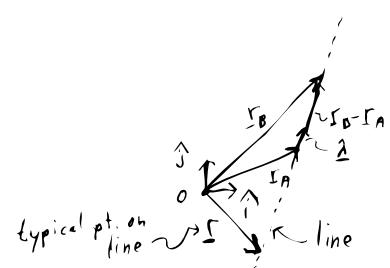


Figure 2.48: Parametric description of a line using vectors.  
(Filename:figure.parametricline)

**Example: Parametric equation of a line**

A parametric equation for the line going through the points with position vectors  $\vec{r}_A$  and  $\vec{r}_B$  is

$$\vec{r} = \vec{r}_A + s \left( \underbrace{\vec{r}_B - \vec{r}_A}_{\vec{v}} \right) \quad \text{or better} \quad \vec{r} = \vec{r}_A + s \hat{\lambda}_A$$

where  $\hat{\lambda}_A = (\vec{r}_B - \vec{r}_A) / |\vec{r}_B - \vec{r}_A|$

□

*A line in 3D*

In three dimensions a line is often described geometrically as the intersection of two planes. But a line in three dimensions is still a one dimensional object so the parametric form eqn. (2.20), applicable in three dimensions as well as two, is nice.

*A plane*

A plane in three dimensions can be described as the set of points  $x, y$ , and  $z$  that satisfy an equation like:

$$Ax + By + Cz = D$$

for a given A,B,C, and D. The parametric description of a plane uses two parameters  $s_1$  and  $s_2$  and is

$$\vec{r} = \vec{r}_O + s_1 \vec{v}_1 + s_2 \vec{v}_2 \quad (2.21)$$

where  $\vec{r}$  is a typical point on the plane,  $\vec{v}_1$  and  $\vec{v}_2$  are any two non-parallel vectors that lie in the plane and  $s_1$  and  $s_2$  are any two real numbers. Each pair  $(s_1, s_2)$  corresponds to one point in the plane and vice versa. The numbers  $s_1$  and  $s_2$  can be thought of as in-plane distance coordinates if the vectors  $\vec{v}_1$  and  $\vec{v}_2$  are mutually orthogonal unit vectors.

**Example: A plane**

A parametric equation for the plane going through the three points  $\vec{r}_A$ ,  $\vec{r}_B$ , and  $\vec{r}_C$  is

$$\vec{r} = \underbrace{\vec{r}_0}_{\vec{r}_0} + s_1 \underbrace{(\vec{r}_B - \vec{r}_A)}_{\vec{v}_1} + s_2 \underbrace{(\vec{r}_C - \vec{r}_A)}_{\vec{v}_2}$$

You can check that when  $s_1 = s_2 = 0$  the point on the plane  $\vec{r}_A$  is given. And when one of the  $s$  values is one and the other zero the points  $\vec{r}_B$  and  $\vec{r}_C$  are given. □

**Vectors, matrices, and linear algebraic equations**

Once has drawn a free body diagram and written the force and moment balance equations one is left with vector equations to solve for various unknowns. The vector equations of mechanics can be reduced to scalar equations by using dot products. The simplest dot product to use is with the unit vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ . This use of dot products is equivalent to taking the  $x$ ,  $y$ , and  $z$  components of the vector equation. The two vector equations

$$\begin{aligned} a\hat{i} + b\hat{j} &= (c - 5)\hat{i} + (d + 7)\hat{j} \\ (a - c)\hat{i} + (a + b)\hat{j} &= (c + b)\hat{i} + (2a + c)\hat{j} \end{aligned}$$

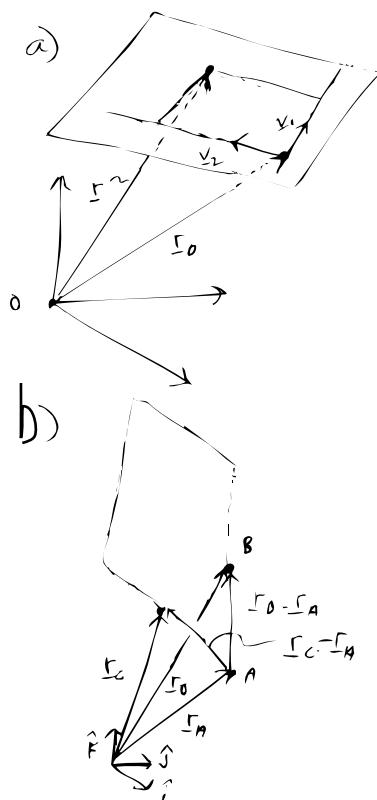


Figure 2.49: a) Parametric equation of a plane. b) the plane through the points A,B, and C

(filename:tfigure.parametricplane)

with four scalar unknowns  $a$ ,  $b$ ,  $c$ , and  $d$ , can be rewritten as four scalar equations, two from each two-dimensional vector equation. Taking the dot product of the first equation with  $\hat{i}$  gives  $a = c - 5$ . Similarly dotting with  $\hat{j}$  gives  $b = d + 7$ . Repeating the procedure with the second equation gives 4 scalar equations:

$$\begin{aligned} a &= c - 5 \\ b &= d + 7 \\ a - c &= c + b \\ a + b &= 2a + c. \end{aligned}$$

These equations can be re-arranged putting unknowns on the left side and knowns on the right side:

$$\begin{array}{rccccccccc} 1a & + & 0b & + & -1c & + & 0d & = & -5 \\ 0a & + & 1b & + & 0c & + & -1d & = & 7 \\ 1a & + & -1b & + & -2c & + & 0d & = & 0 \\ -1a & + & 1b & + & -1c & + & 0d & = & 0 \end{array}$$

These equations can in turn be written in standard matrix form. The standard matrix form is a short hand notation for writing (linear) equations, such as the equations above:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[A]} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[\mathbf{x}]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[\mathbf{y}]}$$

$$\Rightarrow [A] \cdot [\mathbf{x}] = [\mathbf{y}].$$

The matrix equation  $[A] \cdot [\mathbf{x}] = [\mathbf{y}]$  is in a form that is easy to input to any of several programs that solve linear equations. The computer (or a do-able but probably untrustworthy hand calculation) should return the following solution for  $[\mathbf{x}]$  ( $a$ ,  $b$ ,  $c$ , and  $d$ ).

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix}.$$

That is,  $a = -5$ ,  $b = -5$ ,  $c = 0$ , and  $d = -12$ . If you doubt the solution, check it. To check the answer, plug it back into the original equation and note the equality (or lack thereof!). In this case, we have done our calculations correctly and

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[A]} \cdot \underbrace{\begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix}}_{[\mathbf{x}]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[\mathbf{y}]}.$$

Going back to the original vector equations we can also check that

$$\begin{aligned} -5\hat{i} + -5\hat{j} &= (0 - 5)\hat{i} + (-12 + 7)\hat{j} \\ (-5 - 0)\hat{i} + (-5 + -5)\hat{j} &= (0 + -5)\hat{i} + (2 \cdot -5 + 0)\hat{j}. \end{aligned}$$

## ‘Physical’ vectors and row or column vectors

The word ‘vector’ has two related but subtly different meanings. One is a physical vector like  $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ , a quantity with magnitude and direction. The other meaning is a list of numbers like the row vector

$$[\mathbf{x}] = [x_1, x_2, x_3]$$

or the column vector

$$[\mathbf{y}] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Once you have picked a basis, like  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , and  $\hat{\mathbf{k}}$ , you can represent a physical vector  $\vec{\mathbf{F}}$  as a row vector  $[F_x, F_y, F_z]$  or a column vector  $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$ . But the components of a given vector depend on the base coordinate system (or base vectors) that are used. For clarity it is best to distinguish a physical vector from a list of components using a notation like the following:

$$[\vec{\mathbf{F}}]_{XYZ} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The square brackets around  $\vec{\mathbf{F}}$  indicate that we are looking at its components. The subscript  $XYZ$  identifies what coordinate system or base vectors are being used. The right side is a list of three numbers (in this case arranged as a column, the default arrangement in linear algebra).

## 2.5 THEORY

### *Existence, uniqueness, and geometry*

As mentioned in the text, sometimes vector equations can have no solutions, sometimes a unique solution and sometimes multiple solutions. In some of the common types of vector equations these cases often have simple geometric interpretation.

#### Example 1

Consider a very simple equation

$$a\vec{v}_1 = \vec{w}$$

where  $\vec{v}_1$  and  $\vec{w}$  are given and you are to find  $a$ . The left hand side is a parametric expression for the points on a line through the origin in the direction  $\vec{v}_1$ . So the equation only has a solution if  $\vec{w}$  is on this line. In other words  $\vec{w}$  must be parallel to  $\vec{v}_1$ . This vector equation is equivalent to 2 scalar equations (3 in 3D) with one scalar unknown and we expect generally to find no solution. That is, two random vectors  $\vec{v}_1$  and  $\vec{w}$  are unlikely to be parallel either in 2D or 3D.

#### Example 2

Now consider this vector equation in two unknown scalars  $a$  and  $b$  with all vectors in the plane

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

If  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel it is apparent that  $a\vec{v}_1 + b\vec{v}_2$  could be any vector on the plane. So there would be a unique solution for every possible  $\vec{w}$ . But if  $\vec{v}_1$  and  $\vec{v}_2$  are parallel then the expression  $a\vec{v}_1 + b\vec{v}_2$  just describes a line. If  $\vec{w}$  is on this line there are many solutions for  $a$  and  $b$  because the two vectors  $a\vec{v}_1$  and  $\vec{v}_2$  can be added various ways that partially cancel.

in 2D a test to see if two vectors are parallel is to take their cross product. So, if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \hat{k} = v_{1x}v_{2y} - v_{1y}v_{2x} = \det \begin{bmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{bmatrix} = 0$$

then  $\vec{v}_1$  and  $\vec{v}_2$  are parallel and there are either many solutions or no solutions depending on whether or not  $\vec{w}$  is also parallel to  $\vec{v}_1$  and  $\vec{v}_2$ .

#### Example 3

Consider the same example as above

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

but where  $\vec{v}_1$ ,  $\vec{v}_2$  and  $\vec{w}$  are vectors in 3D. Now the question is whether the vector  $\vec{w}$  is in the plane described parametrically by  $a\vec{v}_1 + b\vec{v}_2$ . If we count equations (3) and unknowns (2) we see that solution should be unlikely. Or, given 3 random vectors in 3D  $\vec{v}_1$ ,  $\vec{v}_2$  and  $\vec{w}$ , it is unlikely that  $\vec{w}$  would be in the plane determined by  $\vec{v}_1$  and  $\vec{v}_2$ .

#### Example 4

Finally consider this common equation in 3D.

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{w}. \quad (2.22)$$

where  $\vec{v}_1$ ,  $\vec{v}_2$ ,  $\vec{v}_3$ , and  $\vec{w}$  are given vectors and  $a$ ,  $b$  and  $c$  are unknowns. If  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  are not co-planar, then it should be clear that any point in space  $\vec{w}$  can be reached by some value of  $a$ ,  $b$  and  $c$ . On the other hand, if  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  are co-planar than there is only a solution if  $\vec{w}$  is on the plane and then there are many solutions because there are many ways for  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  to cancel each other out.

We can test for coplanarity with geometric reasoning and cross products. The vector  $\vec{v}_1 \times \vec{v}_2$  is orthogonal to the plane of  $\vec{v}_1$  and  $\vec{v}_2$ . So, if  $\vec{v}_3$  is in the same plane it will be orthogonal to  $\vec{v}_1 \times \vec{v}_2$ . Thus if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = 0$$

the three vectors are co-planar. But this test can also be written as

$$\det \begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} = 0$$

which is what we would expect from considering the matrix form of eqn. (2.22)

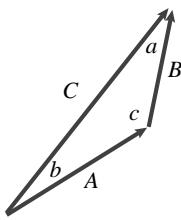
$$\begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

and checking to see if the  $3 \times 3$  matrix is “singular” (a linear algebra word meaning that the determinant is zero).

## 2.6 THEORY

### Vector triangles and the laws of sines and cosines

The tip to tail rule of vector addition defines a triangle. Given some information about the vectors in this triangle how does one figure out the rest? One traditional approach is to use the laws of sines and cosines.



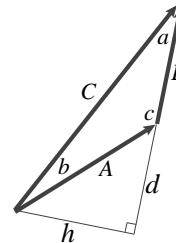
Consider the vector sum  $\vec{A} + \vec{B} = \vec{C}$  represented by the triangle shown with traditionally labeled sides  $A$ ,  $B$ , and  $C$  and internal angles  $a$ ,  $b$ , and  $c$ .

The sides and angles are related by

$$\begin{aligned}\frac{\sin a}{A} &= \frac{\sin b}{B} = \frac{\sin c}{C} && \text{the law of sines, and} \\ C^2 &= A^2 + B^2 - 2AB \cos c && \text{the law of cosines.}\end{aligned}$$

### Proof of the law of cosines

The proof of the law of cosines is similar in spirit. So as to end up with the usual lettering lets look at altitude  $h$  of the triangle.



This is the base of two different right triangles. So by the pythagorean theorem we have on the one hand that

$$h^2 = A^2 - d^2$$

and on the other that

$$h^2 = C^2 - (B + d)^2.$$

Equating these expressions and expanding the square we get

$$\begin{aligned}A^2 - d^2 &= C^2 - (B^2 + d^2 - 2dB) \\ \Rightarrow A^2 + B^2 + 2dB &= C^2\end{aligned}\quad (2.24)$$

But  $d = -A \cos c$  so

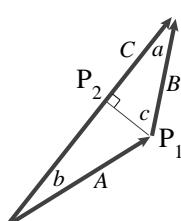
$$C^2 = A^2 + B^2 - 2AB \cos c.$$

Sometimes the angle we call here  $c$  is called  $\theta$ .

These laws, applied to various sides and angles of a triangle are useful when you want to figure out the shape and size of a triangle when, of the six triangle quantities (three sides and three angles), only are given. At least one of these three has to be a length.

As noted, it is possible to give problems of this type that have no solutions. And it is possible to give problems that have either 1 or 2 solutions. No example was given in the text of the "side-side-angle" case because it has infinitely many solutions.

In this era where vector algebra is popular and so is the representation of vectors in terms of their components, the laws of sines and cosines are not used that often. But as shown in the section, there are cases where the laws of sines and cosines are the easiest approach.



On the one hand length  $P_1P_2$  is given by

$$P_1P_2 = B \sin a$$

and on the other hand by

$$P_1P_2 = A \sin b$$

so

$$B \sin a = A \sin b \Rightarrow \frac{\sin a}{A} = \frac{\sin b}{B}.$$

We could do likewise with all three altitudes thus proving the triple equality.

**SAMPLE 2.29 Plain vanilla vector equation in 2-D:** Three forces act on a particle as shown in the figure. The equilibrium condition of the particle requires that  $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$ . It is given that  $\vec{W} = -20\text{ N}\hat{j}$ . Find the magnitudes of forces  $\vec{F}_1$  and  $\vec{F}_2$ .

**Solution** We are given a vector equation,  $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$ , in which one vector  $\vec{W}$  is completely known and the directions of the other two vectors are given. We need to find their magnitudes. Let us write the vectors as

$$\vec{F}_1 = F_1 \hat{\lambda}_1, \quad \vec{F}_2 = F_2 \hat{\lambda}_2, \quad \vec{W} = -W \hat{j},$$

where  $\hat{\lambda}_1$  and  $\hat{\lambda}_2$  are unit vectors along  $\vec{F}_1$  and  $\vec{F}_2$ , respectively (their directions are specified by the given angles in the figure), and  $W = 20\text{ N}$  as given. We can write the unit vectors in component form as

$$\hat{\lambda}_1 = \lambda_{1x} \hat{i} + \lambda_{1y} \hat{j} \quad \text{and} \quad \hat{\lambda}_2 = \lambda_{2x} \hat{i} + \lambda_{2y} \hat{j}.$$

Now we can write the given vector equation as

$$F_1(\lambda_{1x} \hat{i} + \lambda_{1y} \hat{j}) + F_2(\lambda_{2x} \hat{i} + \lambda_{2y} \hat{j}) = W \hat{j}. \quad (2.25)$$

Dotting both sides of eqn. (2.25) with  $\hat{i}$  and  $\hat{j}$  respectively, we get

$$\lambda_{1x} F_1 + \lambda_{2x} F_2 = 0 \quad (2.26)$$

$$\lambda_{1y} F_1 + \lambda_{2y} F_2 = W. \quad (2.27)$$

Here, we have two equations in two unknowns ( $F_1$  and  $F_2$ ). We can solve these equations for the unknowns. Let us solve these two linear equations by first putting them into a matrix form and then solving the matrix equation. The matrix equation is

$$\begin{bmatrix} \lambda_{1x} & \lambda_{2x} \\ \lambda_{1y} & \lambda_{2y} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Using Cramer's rule for matrix inversion, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\lambda_{1x}\lambda_{2y} - \lambda_{2x}\lambda_{1y}} \begin{bmatrix} \lambda_{2y} & -\lambda_{2x} \\ -\lambda_{1y} & \lambda_{1x} \end{bmatrix} \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Substituting the numerical values of  $\lambda_{1x} = -\cos 30^\circ = -\sqrt{3}/2$ ,  $\lambda_{1y} = -\sin 30^\circ = 1/2$  and similarly,  $\lambda_{2x} = 1/\sqrt{2}$ ,  $\lambda_{2y} = 1/\sqrt{2}$ , and  $W = 20\text{ N}$ , we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 14.64 \\ 17.93 \end{pmatrix} \text{ N}.$$

$$F_1 = 14.64 \text{ N}, \quad F_2 = 17.93 \text{ N}$$

**Check:** We can easily check if the values we have got are correct. For example, substituting the numerical values in eqn. (2.26), we get

$$14.64 \text{ N} \cdot \left(-\frac{\sqrt{3}}{2}\right) + 17.93 \text{ N} \cdot \frac{1}{\sqrt{2}} \stackrel{?}{=} 0.$$

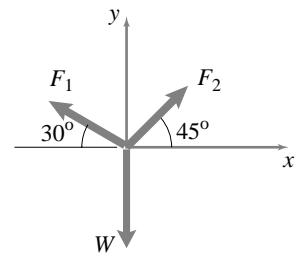


Figure 2.50: (Filename:fig2.4.veceqn.1)

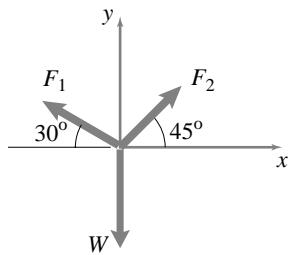


Figure 2.51: (filename:sfig2.4.vecceqn.2)

**SAMPLE 2.30** *Solving for a single unknown from a 2-D vector equation:* Consider the same problem as in Sample 2.29. That is, you are given that  $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$  where  $\vec{W} = -20 \text{ N} \hat{j}$  and  $\vec{F}_1$  and  $\vec{F}_2$  act along the directions shown in the figure. Find the magnitude of  $\vec{F}_2$ .

**Solution** Once again, we write the given vector equation as

$$F_1 \hat{\lambda}_1 + F_2 \hat{\lambda}_2 = W \hat{j},$$

where  $W = 20 \text{ N}$ ,  $\hat{\lambda}_1 = \lambda_{1x} \hat{i} + \lambda_{1y} \hat{j} = -\sqrt{3}/2 \hat{i} + 1/2 \hat{j}$ , and  $\hat{\lambda}_2 = \lambda_{2x} \hat{i} + \lambda_{2y} \hat{j} = 1/\sqrt{2}(\hat{i} + \hat{j})$ . We are interested in finding  $F_2$  only. So, let us take a dot product of this equation with a vector that gets rid of the  $F_1$  term. Any such vector would have to be perpendicular to  $\hat{\lambda}_1$ . One such vector is  $\hat{k} \times \hat{\lambda}_1$ . Let us call this vector  $\hat{n}_1$ , that is,  $\hat{n}_1 = \hat{k} \times (\lambda_{1x} \hat{i} + \lambda_{1y} \hat{j}) = \lambda_{1x} \hat{j} - \lambda_{1y} \hat{i}$ . Now, dotting the given vector equation with  $\hat{n}_1$ , we get

$$\begin{aligned} F_1 \underbrace{(\hat{n}_1 \cdot \hat{\lambda}_1)}_0 + F_2 (\hat{n}_1 \cdot \hat{\lambda}_2) &= W (\hat{n}_1 \cdot \hat{j}) \\ \Rightarrow F_2 &= W \frac{\hat{n}_1 \cdot \hat{j}}{\hat{n}_1 \cdot \hat{\lambda}_2} \\ &= W \frac{(\lambda_{1x} \hat{j} - \lambda_{1y} \hat{i}) \cdot \hat{j}}{(\lambda_{1x} \hat{j} - \lambda_{1y} \hat{i}) \cdot (\lambda_{2x} \hat{i} + \lambda_{2y} \hat{j})} \\ &= W \frac{\lambda_{1x}}{\lambda_{1x} \lambda_{2y} - \lambda_{1y} \lambda_{2x}} \\ &= 20 \text{ N} \frac{-\sqrt{3}/2}{-\sqrt{3}/2 \cdot 1/\sqrt{2} - 1/2 \cdot 1/\sqrt{2}} \\ &= 20 \text{ N} \frac{\sqrt{6}}{\sqrt{3} + 1} = 17.93 \text{ N} \end{aligned}$$

which, of course, is the same value we got in Sample 2.29. Note that here we obtained one scalar equation in one unknown by dotting the 2-D vector equation with an appropriate vector to get rid of the other unknown  $F_1$ .

$F_2 = 17.93 \text{ N}$

**SAMPLE 2.31** *Solving a 3-D vector equation on a computer:* Four forces,  $\vec{F}_1$ ,  $\vec{F}_2$ ,  $\vec{F}_3$  and  $\vec{N}$  are in equilibrium, that is,  $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{N} = \vec{0}$  where  $\vec{N} = -100 \text{ kN} \hat{k}$  is known and the directions of the other three forces are known.  $\vec{F}_1$  is directed from (0,0,0) to (1,-1,1),  $\vec{F}_2$  from (0,0,0) to (-1,-1,1), and  $\vec{F}_3$  from (0,0,0) to (0,1,1). Find the magnitudes of  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$ .

**Solution** Let  $\vec{F}_1 = F_1 \hat{\lambda}_1$ ,  $\vec{F}_2 = F_2 \hat{\lambda}_2$ , and  $\vec{F}_3 = F_3 \hat{\lambda}_3$ , where  $\hat{\lambda}_1$ ,  $\hat{\lambda}_2$  and  $\hat{\lambda}_3$  are unit vectors in the directions of  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$ , respectively. Then the given vector equation can be written as

$$F_1 \hat{\lambda}_1 + F_2 \hat{\lambda}_2 + F_3 \hat{\lambda}_3 = -\vec{N} = -N \hat{k}$$

where  $N = -100 \text{ kN}$ . Dotting this equation with  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  respectively, and realizing that  $\hat{i} \cdot \hat{\lambda}_1 = \lambda_{1x}$ ,  $\hat{j} \cdot \hat{\lambda}_1 = \lambda_{1y}$ , etc., we get the following three scalar equations.

$$\begin{aligned}\lambda_{1x} F_1 + \lambda_{2x} F_2 + \lambda_{3x} F_3 &= 0 \\ \lambda_{1y} F_1 + \lambda_{2y} F_2 + \lambda_{3y} F_3 &= 0 \\ \lambda_{1z} F_1 + \lambda_{2z} F_2 + \lambda_{3z} F_3 &= -N.\end{aligned}$$

Thus we get a system of three linear equations in three unknowns. To solve for the unknowns, we set up these equations as a matrix equation and then use a computer to solve it. In matrix form these equations are

$$\begin{bmatrix} \lambda_{1x} & \lambda_{2x} & \lambda_{3x} \\ \lambda_{1y} & \lambda_{2y} & \lambda_{3y} \\ \lambda_{1z} & \lambda_{2z} & \lambda_{3z} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -N \end{pmatrix}.$$

To solve this equation on a computer, we need to input the matrix of unit vector components and the known vector on the right hand side. From the given coordinates for the directions of forces, we have  $\hat{\lambda}_1 = (\hat{i} - \hat{j} + \hat{k})/\sqrt{3}$ ,  $\hat{\lambda}_2 = (-\hat{i} - \hat{j} + \hat{k})/\sqrt{3}$ , and  $\hat{\lambda}_3 = (\hat{j} + \hat{k})/\sqrt{2}$ . <sup>①</sup>We are also given that  $N = -100 \text{ kN}$ . Now, we use the following pseudo-code to find the solution on a computer.

```
Let s2 = sqrt(2), s3 = sqrt(3)
A = [ 1/s3 -1/s3 0
      -1/s3 -1/s3 1/s2
      1/s3 1/s3 1/s2 ]
b = [ 0 0 100 ]
solve A*F = b for F
```

Using this pseudo-code we find the solution to be

```
F = [ 43.3013
      43.3013
      70.7107 ]
```

That is,  $F_1 = F_2 = 43.3 \text{ kN}$  and  $F_3 = 70.7 \text{ kN}$ .

$$F_1 = 43.3 \text{ kN}, F_2 = 43.3 \text{ kN}, F_3 = 70.7 \text{ kN}$$

<sup>①</sup> These unit vectors are computed by taking a vector from one end point to the other end point (as given) and then dividing by its magnitude. For example, we find  $\hat{\lambda}_1$  by first finding  $\vec{r}_1 = (1)\hat{i} + (-1)\hat{j} + (1)\hat{k}$ , a vector from (0,0,0) to (1,-1,1), and then  $\hat{\lambda}_1 = \frac{\vec{r}_1}{|\vec{r}_1|}$ .

**SAMPLE 2.32 Vector operations on a computer:** Consider the problem of Sample 2.31 again. That is, you are given the vector equation  $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{N} = \vec{0}$  where  $\vec{N} = -100 \text{ kN} \hat{k}$  and the directions of  $\vec{F}_1$ ,  $\vec{F}_2$  and  $\vec{F}_3$  are given by the unit vectors  $\hat{\lambda}_1 = (\hat{i} - \hat{j} + \hat{k})/\sqrt{3}$ ,  $\hat{\lambda}_2 = (-\hat{i} - \hat{j} + \hat{k})/\sqrt{3}$ , and  $\hat{\lambda}_3 = (\hat{j} + \hat{k})/\sqrt{2}$ , respectively. Find  $F_1$ .

**Solution** We can, of course, solve the problem as we did in Sample 2.31 and we get the answer as a part of the unknown forces we solved for. However, we would like to show here that we can extract one scalar equation in just one unknown ( $F_1$ ) from the given 3-D vector equation and solve for the unknown without solving a matrix equation. Although we can carry out all required calculations by hand, we will show how we can use a computer to do these operations.

We can write the given vector equation as

$$F_1 \hat{\lambda}_1 + F_2 \hat{\lambda}_2 + F_3 \hat{\lambda}_3 = -\vec{N}. \quad (2.28)$$

We want to find  $F_1$ . Therefore, we should dot this equation with a vector that gets rid of both  $F_2$  and  $F_3$ , i.e., with a vector which is perpendicular to both  $\vec{F}_2$  and  $\vec{F}_3$ . One such vector is  $\vec{F}_2 \times \vec{F}_3$  or  $\hat{\lambda}_2 \times \hat{\lambda}_3$ . Let  $\hat{n} = \hat{\lambda}_2 \times \hat{\lambda}_3$ . Now, dotting both sides of eqn. (2.28) with  $\hat{n}$ , we get

$$F_1(\hat{\lambda}_1 \cdot \hat{n}) + F_2(\hat{\lambda}_2 \cdot \hat{n}) + F_3(\hat{\lambda}_3 \cdot \hat{n}) = -\vec{N} \cdot \hat{n}$$

Since  $\hat{\lambda}_2 \cdot \hat{n} = 0$  and  $\hat{\lambda}_3 \cdot \hat{n} = 0$  ( $\hat{n}$  is normal to both  $\hat{\lambda}_2$  and  $\hat{\lambda}_3$ ), we get

$$\begin{aligned} F_1(\hat{\lambda}_1 \cdot \hat{n}) &= -\vec{N} \cdot \hat{n} \\ \Rightarrow F_1 &= \frac{-\vec{N} \cdot \hat{n}}{\hat{\lambda}_1 \cdot \hat{n}}. \end{aligned}$$

Thus we have found the solution. To compute the expression on the right hand side of the above equation we use the following pseudo-code which assumes that you have written (or have access to) two functions, `dot` and `cross`, that compute the dot and cross product of two given vectors.

```
lambda_1 = 1/sqrt(3)*[1 -1 1]';
lambda_2 = 1/sqrt(3)*[-1 -1 1]';
lambda_3 = 1/sqrt(2)*[0 1 1]';
N = [0 0 -100]';
n = cross(lambda_2, lambda_3);
F1 = - dot(N, n)/dot(lambda_1, n)
```

By following these steps on a computer, we get the output  $F_1 = 43.3013$ , that is,  $F_1 = 43.3 \text{ kN}$ , which, of course, is the same answer we obtained in Sample 2.31.

$F_1 = 43.3 \text{ kN}$

## 2.5 Equivalent force systems

Most often one does not want to know the complete details of all the forces acting on a system. When you think of the force of the ground on your bare foot you do not think of the thousands of little forces at each micro-asperity or the billions and billions of molecular interactions between the wood (say) and your skin. Instead you think of some kind of equivalent force. In what way equivalent? Well, because all that the equations of mechanics know about forces is their net force and net moment, you have a criterion. You replace the actual force system with a simpler force system, possibly just a single well-placed force, that has the same total force and same total moment with respect to a reference point C.

The replacement of one system with an equivalent system is often used to help simplify or solve mechanics problems. Further, the concept of equivalent force systems allows us to define a *couple*, a concept we will use throughout the book. Here is the definition of the word *equivalent*<sup>①</sup> when applied to force systems in mechanics.

Two force systems are said to be *equivalent* if they have the same sum (the same *resultant*) and the same net moment about any one point C.

<sup>①</sup> Other phrases used to describe the same concept in other books include: *statically equivalent*, *mechanically equivalent*, and *equipollent*.

We have already discussed two important cases of equivalent force systems. On page 11 we stated the mechanics assumption that a set of forces applied at one point is equivalent to a single resultant force, their sum, applied at that point. Thus when doing a mechanics analysis you can replace a collection of forces at a point with their sum. If you think of your whole foot as a ‘point’ this justifies the replacement of the billions of little atomic ground contact forces with a single force.

On page 33 we discovered that a force applied at a different point is equivalent to the same force applied at a point displaced in the direction of the force. You can thus harmlessly slide the point of force application along the line of the force.

More generally, we can compare two sets of forces. The first set consists of  $\vec{F}_1^{(1)}, \vec{F}_2^{(1)}, \vec{F}_3^{(1)}, \text{etc.}$  applied at positions  $\vec{r}_{1/C}^{(1)}, \vec{r}_{2/C}^{(1)}, \vec{r}_{3/C}^{(1)}, \text{etc.}$  In short hand, these forces are  $\vec{F}_i^{(1)}$  applied at positions  $\vec{r}_{i/C}^{(1)}$ , where each value of  $i$  describes a different force ( $i = 7$  refers to the seventh force in the set). The second set of forces consists of  $\vec{F}_j^{(2)}$  applied at positions  $\vec{r}_{j/C}^{(2)}$  where each value of  $j$  describes a different force in the second set.

Now we compare the net (resultant) force and net moment of the two sets. If

$$\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)} \quad \text{and} \quad \vec{M}_C^{(1)} = \vec{M}_C^{(2)} \quad (2.29)$$

then the two sets are *equivalent*. Here we have defined the net forces and net moments by

$$\begin{aligned} \vec{F}_{\text{tot}}^{(1)} &= \sum_{\text{all forces } i} \vec{F}_i^{(1)}, & \vec{M}_C^{(1)} &= \sum_{\text{all forces } i} \vec{r}_{i/C}^{(1)} \times \vec{F}_i^{(1)}, \\ \vec{F}_{\text{tot}}^{(2)} &= \sum_{\text{all forces } j} \vec{F}_j^{(2)}, & \text{and} \quad \vec{M}_C^{(2)} &= \sum_{\text{all forces } j} \vec{r}_{j/C}^{(2)} \times \vec{F}_j^{(2)}. \end{aligned}$$

If you find the  $\sum$  (sum) symbol intimidating see box 2.5 on page 70.

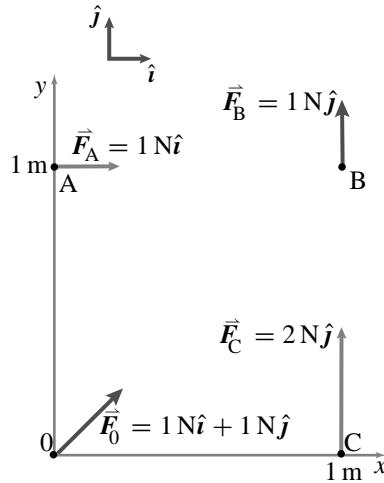


Figure 2.52: The force system  $\vec{F}_A, \vec{F}_C$  is equivalent to the force system  $\vec{F}_0, \vec{F}_B$ .

(Filename:tfigure.equivforcepair)

*Example:*

Consider force system (1) with forces  $\vec{F}_A$  and  $\vec{F}_C$  and force system (2) with forces  $\vec{F}_0$  and  $\vec{F}_B$  as shown in fig. 2.52. Are the systems equivalent? First check the sum of forces.

$$\begin{aligned}\vec{F}_{\text{tot}}^{(1)} &\stackrel{?}{=} \vec{F}_{\text{tot}}^{(2)} \\ \sum \vec{F}_i^{(1)} &\stackrel{?}{=} \sum \vec{F}_j^{(2)} \\ \vec{F}_A + \vec{F}_C &\stackrel{?}{=} \vec{F}_0 + \vec{F}_B \\ 1 \text{ N} \hat{i} + 2 \text{ N} \hat{j} &\stackrel{\checkmark}{=} (1 \text{ N} \hat{i} + 1 \text{ N} \hat{j}) + 1 \text{ N} \hat{j}\end{aligned}$$

Then check the sum of moments about C.

$$\begin{aligned}\vec{M}_C^{(1)} &\stackrel{?}{=} \vec{M}_C^{(2)} \\ \sum \vec{r}_{i/C}^{(1)} \times \vec{F}_i^{(1)} &\stackrel{?}{=} \sum \vec{r}_{j/C}^{(2)} \times \vec{F}_j^{(2)} \\ \vec{r}_{A/C} \times \vec{F}_A + \vec{r}_{C/C} \times \vec{F}_C &\stackrel{?}{=} \vec{r}_{0/C} \times \vec{F}_0 + \vec{r}_{B/C} \times \vec{F}_B \\ (-1 \text{ m} \hat{i} + 1 \text{ m} \hat{j}) \times 1 \text{ N} \hat{i} + \vec{0} \times 2 \text{ N} \hat{j} &\stackrel{?}{=} (-1 \text{ m} \hat{i}) \times (1 \text{ N} \hat{i} + 1 \text{ N} \hat{j}) + 1 \text{ m} \hat{j} \times 1 \text{ N} \hat{j} \\ -1 \text{ m} \hat{k} &\stackrel{\checkmark}{=} -1 \text{ m} \hat{k}\end{aligned}$$

So the two force systems are indeed equivalent.  $\square$

What is so special about the point C in the example above? Nothing.

## 2.7 $\sum$ means add

In mechanics we often need to add up lots of things: all the forces on a body, all the moments they cause, all the mass of a system, *etc.* One notation for adding up all 14 forces on some body is

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 + \vec{F}_7 \\ &\quad + \vec{F}_8 + \vec{F}_9 + \vec{F}_{10} + \vec{F}_{11} + \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}.\end{aligned}$$

which is a bit long, so we might abbreviate it as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_{14}.$$

But this is definition by pattern recognition. A more explicit statement would be

$$\vec{F}_{\text{net}} = \text{The sum of all 14 forces } \vec{F}_i \text{ where } i = 1 \dots 14$$

which is too space consuming. This kind of summing is so important that mathematicians use up a whole letter of the greek alphabet as a short hand for ‘the sum of all’. They use the capital greek ‘S’ (for Sum) called sigma which looks like this:

$$\sum.$$

When you read  $\sum$  aloud you don’t say ‘S’ or ‘sigma’ but rather ‘the sum of.’ The  $\sum$  (sum) notation may remind you of infinite series, and convergence thereof. We will rarely be concerned with infinite sums in this book and never with convergence issues. So panic on those grounds is unjustified. We just want to easily write

about adding things. For example we use the  $\sum$  (sum) to write the sum of 14 forces  $\vec{F}_i$  explicitly and concisely as

$$\sum_{i=1}^{14} \vec{F}_i$$

and say ‘the sum of  $F$  sub  $i$  where  $i$  goes from one to fourteen’. Sometimes we don’t know, say, how many forces are being added. We just want to add all of them so we write (a little informally)

$$\sum \vec{F}_i \text{ meaning } \vec{F}_1 + \vec{F}_2 + \text{etc.,}$$

where the subscript  $i$  lets us know that the forces are numbered.

Rather than panic when you see something like  $\sum_{i=1}^{14}$ , just relax and think: oh, we want to add up a bunch of things all of which look like the next thing written. In general,

$$\sum (\text{thing})_i \text{ translates to } (\text{thing})_1 + (\text{thing})_2 + (\text{thing})_3 + \text{etc.}$$

no matter how intimidating the ‘thing’ is. In time you can skip writing out the translation and will enjoy the concise notation.

See box ?? for a similar discussion of integration ( $\int$ ) and addition.

If two force systems are equivalent with respect to some point C, they are equivalent with respect to *any* point.

For example, both of the force systems in the example above have the same moment of  $2 \text{ N m} \hat{k}$  about the point A. See box 2.5 for the proof of the general case.

### Example: Frictionless wheel bearing

If the contact of an axle with a bearing housing is perfectly frictionless then each of the contact forces has no moment about the center of the wheel. Thus the whole force system is equivalent to a single force at the center of the wheel.  $\square$

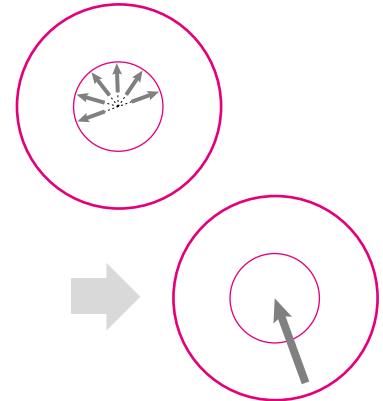


Figure 2.53: Frictionless wheel bearing. All the bearing forces are *equivalent* to a single force acting at the center of the wheel.

(Filename: tfigure2.wheelbearing)

## Couples

Consider a pair of equal and opposite forces that are not colinear. Such a pair is called a *couple*.<sup>①</sup> The net moment caused by a couple is the size of the force times the distance between the two lines of action and doesn't depend on the reference

<sup>①</sup>**Caution:** Just because a collection of forces adds to zero doesn't mean the net moment they cause adds to zero.

## 2.8 THEORY

*Two force systems that are equivalent for one reference point are equivalent for all reference points.*

Consider two sets of forces  $\vec{F}_i^{(1)}$  and  $\vec{F}_j^{(2)}$  with corresponding points of application  $P_i^{(1)}$  and  $P_j^{(2)}$  at positions relative to the origin of  $\vec{r}_i^{(1)}$  and  $\vec{r}_j^{(2)}$ . To simplify the discussion let's define the net forces of the two systems as

$$\vec{F}_{\text{tot}}^{(1)} \equiv \sum \vec{F}_i^{(1)} \quad \text{and} \quad \vec{F}_{\text{tot}}^{(2)} \equiv \sum \vec{F}_j^{(2)},$$

and the net moments about the origin as

$$\vec{M}_0^{(1)} \equiv \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} \quad \text{and} \quad \vec{M}_0^{(2)} \equiv \sum \vec{r}_j^{(2)} \times \vec{F}_j^{(2)}.$$

Using point 0 as a reference, the statement that the two systems are equivalent is then  $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$  and  $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$ . Now consider point C with position  $\vec{r}_C = \vec{r}_{C/0} = -\vec{r}_{0/C}$ . What is the net moment of force system (1) about point C?

$$\begin{aligned} \vec{M}_C^{(1)} &\equiv \sum \vec{r}_{i/C}^{(1)} \times \vec{F}_i^{(1)} \\ &= \sum (\vec{r}_i^{(1)} - \vec{r}_C) \times \vec{F}_i^{(1)} \\ &= \sum (\vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_C \times \vec{F}_i^{(1)}) \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \sum \vec{r}_C \times \vec{F}_i^{(1)} \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_C \times \left( \sum \vec{F}_i^{(1)} \right) \\ &= \vec{M}_0^{(1)} - \vec{r}_C \times \vec{F}_{\text{tot}}^{(1)}. \end{aligned}$$

$$= \vec{M}_0^{(1)} + \vec{r}_{0/C} \times \vec{F}_{\text{tot}}^{(1)}.$$

[ **Aside.** The calculation above uses the ‘move’ of factoring a constant out of a sum. This mathematical move will be used again and again in the development of the theory of mechanics. ]

Similarly, for force system (2)

$$\vec{M}_C^{(2)} = \vec{M}_0^{(2)} + \vec{r}_{0/C} \times \vec{F}_{\text{tot}}^{(2)}.$$

If the two force systems are equivalent for reference point 0 then  $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$  and  $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$  and the expressions above imply that  $\vec{M}_C^{(1)} = \vec{M}_C^{(2)}$ . Because we specified nothing special about the point C, the systems are equivalent for *any* reference point. Thus, to demonstrate equivalence we need to use a reference point, but once equivalence is demonstrated we need not name the point since the equivalence holds for all points.

By the same reasoning we find that once we know the net force and net moment of a force system ( $\vec{F}_{\text{tot}}$ ) relative to some point C (call it  $\vec{M}_C$ ), we know the net moment relative to point D as

$$\vec{M}_D = \vec{M}_C + \vec{r}_{C/D} \times \vec{F}_{\text{tot}}.$$

Note that if the net force is  $\vec{0}$  (and the force system is then called a couple) that  $\vec{M}_D = \vec{M}_C$  so the net moment is the same for all reference points.

② People who have been in difficult long term relationships don't need a mechanics text to know that a couple is a pair of equal and opposite forces that push each other round and round.



Figure 2.54: One couple. The forces add to zero. Then net moment they cause does not.

(Filename:tfigure.onecouple)

point. In fact, any force system that has  $\vec{F}_{\text{tot}} = \vec{0}$  causes the same moment about all different reference points (as shown at the end of box 2.5). So, in modern usage, any force system with any number of forces and with  $\vec{F}_{\text{tot}} = \vec{0}$  is called a couple. A couple is described by its net moment.②

A couple is any force system that has a total force of  $\vec{0}$ . It is described by the net moment  $\vec{M}$  that it causes.

We then think of  $\vec{M}$  as representing an equivalent force system that contributes  $\vec{0}$  to the net force and  $\vec{M}$  to the net moment with respect to every reference point.

The concept of a couple (also called an applied moment or an applied torque) is especially useful for representing the net effect of a complicated collection of forces that causes some turning. The complicated set of electromagnetic forces turning a motor shaft can be replaced by a couple.

## Every system of forces is equivalent to a force and a couple

Given any point C, we can calculate the net moment of a system of forces relative to C. We then can replace the sum of forces with a single force at C and the net moment with a couple at C and we have an equivalent force system.

A force system is equivalent to a force  $\vec{F} = \vec{F}_{\text{tot}}$  acting at C and a couple  $M$  equal to the net moment of the forces about C, i.e.,  $\vec{M} = \vec{M}_C$ .

If instead we want a force system at D we could recalculate the net moment about D or just use the translation formula (see box 2.5).

$$\begin{aligned}\vec{F}_{\text{tot}} &= \vec{F}_{\text{tot}}, \quad \text{and} \\ \vec{M}_D &= \vec{M}_C + \vec{r}_{C/D} \times \vec{F}_{\text{tot}}.\end{aligned}$$

stays the same and the moment at D is the moment at C plus the moment caused by  $\vec{F}_{\text{net}}$  acting at position C relative to D. The net effect of the forces of the ground on a tree, for example, is of a force and a couple acting on the base of the tree.

## The tidyest representation of a force system: a “wrench”

Any force system can be represented by an equivalent force and a couple at any point. But force systems can be reduced to simpler forms. That this is so is of more theoretical than practical import. We state the results here without proof (see problems 2.122 and 2.123 on page 731).

In 2D one of these two things is true:

- The system is equivalent to a couple, or

- There is a line of points for which the system can be described by an equivalent force with no couple.

In 3D one of these three things is true:

- The system is equivalent to a couple (applied anywhere), or
- The system is equivalent to a force (applied on a given line parallel to the force), or
- There is a line of points for which the system can be reduced to a force and a couple where the force, couple, and line are all parallel. The representation of the system of forces as a force and a parallel moment is called a *wrench*.

## Equivalent does not mean equivalent for all purposes

We have perhaps oversimplified.

Imagine you stayed up late studying and overslept. Your roommate was not so diligent; woke up on time and went to wake you by gently shaking you. Having read this chapter so far and no further, and being rather literal, your roommate gets down on the floor and presses on the linoleum underneath your bed applying a force that is *equivalent* to pressing on you. Obviously this is not equivalent in the ordinary sense of the word. It isn't even equivalent in all of its mechanics effects. One force moves you even if you don't wake up, and the other doesn't.

Any two force systems that are 'equivalent' but different *do* have different mechanical effects. So, in what sense are two force systems that have the same net force and the same net moment really equivalent? They are equivalent in their contributions to the equations of mechanics (equations 0-II on the inside cover) for any system to which they are both applied. But full mechanical analysis of a situation requires looking at the mechanics equations of many subsystems. In the mechanics equations for each subsystem, two 'equivalent' force systems are equivalent if they are both applied to that subsystem.

For the analysis of the subsystem that is you sleeping, the force of your roommate's hand on the floor isn't applied to you, so doesn't show up in the mechanics equations for you, and doesn't have the same effect as a force on you.

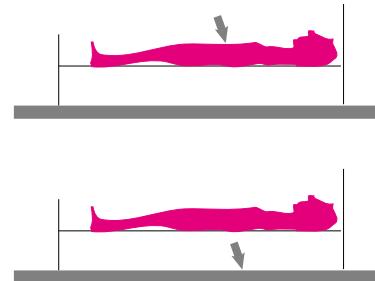


Figure 2.55: It feels different if someone presses on you or presses on the floor underneath you with an 'equivalent' force. The equivalence of 'equivalent' force systems depends on them both being applied to the same system.

(Filename: tfigure.inbed)

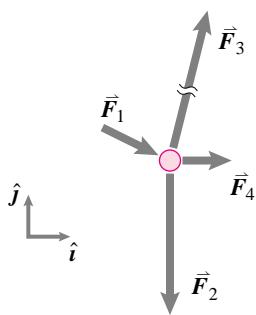


Figure 2.56: (Filename:sfig2.vec3.particle)

**SAMPLE 2.33 Equivalent force on a particle:** Four forces  $\vec{F}_1 = 2\hat{i} - 1\hat{j}$ ,  $\vec{F}_2 = -5\hat{j}$ ,  $\vec{F}_3 = 3\hat{i} + 12\hat{j}$ , and  $\vec{F}_4 = 1\hat{i}$  act on a particle. Find the equivalent force on the particle.

**Solution** The equivalent force on the particle is the net force, *i.e.*, the vector sum of all forces acting on the particle. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 \\ &= (2\hat{i} - 1\hat{j}) + (-5\hat{j}) + (3\hat{i} + 12\hat{j}) + (1\hat{i}) \\ &= 6\hat{i} + 6\hat{j}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 6\hat{i} + 6\hat{j}}$$

Note that there is no net couple since all the four forces act at the same point. This is always true for particles. Thus, the equivalent force-couple system for particles consists of only the net force.

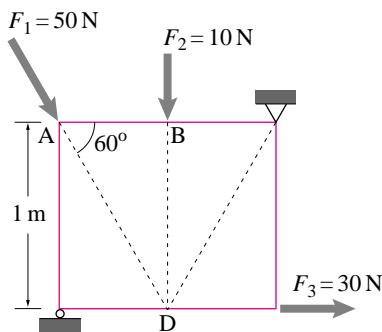


Figure 2.57: (Filename:sfig2.vec3.plate)

**SAMPLE 2.34 Equivalent force with no net moment:** In the figure shown,  $F_1 = 50\text{ N}$ ,  $F_2 = 10\text{ N}$ ,  $F_3 = 30\text{ N}$ , and  $\theta = 60^\circ$ . Find the equivalent force-couple system about point D of the structure.

**Solution** From the given geometry, we see that the three forces  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$  pass through point D. Thus they are *concurrent* forces. Since point D is on the line of action of these forces, we can simply slide the three forces to point D without altering their mechanical effect on the structure. Then the equivalent force-couple system at point D consists of only the net force,  $\vec{F}_{\text{net}}$ , with no couple (the three forces passing through point D produce no moment about D). This is true for all concurrent forces. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= F_1(\cos\theta\hat{i} - \sin\theta\hat{j}) - F_2\hat{j} + F_3\hat{i} \\ &= (F_1 \cos\theta + F_3)\hat{i} - (F_1 \sin\theta + F_2)\hat{j} \\ &= (50\text{ N} \cdot \frac{1}{2} + 30\text{ N})\hat{i} - (50\text{ N} \cdot \frac{\sqrt{3}}{2} + 10\text{ N})\hat{j} \\ &= 50\hat{i} - 53.3\hat{j},\end{aligned}$$

and       $\vec{M}_D = \vec{0}$ .

Graphically, the solution is shown in Fig. 2.58

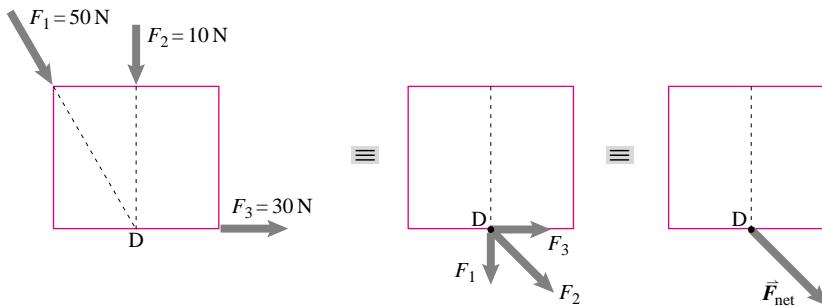


Figure 2.58: (Filename:sfig2.vec3.plate.a)

$$\boxed{\vec{F}_{\text{net}} = 50\hat{i} - 53.3\hat{j}, \vec{M}_D = \vec{0}}$$

**SAMPLE 2.35 An equivalent force-couple system:** Three forces  $F_1 = 100\text{ N}$ ,  $F_2 = 50\text{ N}$ , and  $F_3 = 30\text{ N}$  act on a structure as shown in the figure where  $\alpha = 30^\circ$ ,  $\theta = 60^\circ$ ,  $\ell = 1\text{ m}$  and  $h = 0.5\text{ m}$ . Find the equivalent force-couple system about point D.

**Solution** The net force is the sum of all applied forces, i.e.,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= F_1(-\sin \alpha \hat{i} - \cos \alpha \hat{j}) + F_2(\cos \theta \hat{i} - \sin \theta \hat{j}) + F_3 \hat{j} \\ &= (-F_1 \sin \alpha + F_2 \cos \theta) \hat{i} + (-F_1 \cos \alpha - F_2 \sin \theta + F_3) \hat{j} \\ &= (-100\text{ N} \cdot \frac{1}{2} + 50\text{ N} \cdot \frac{1}{2}) \hat{i} + (-100\text{ N} \cdot \frac{\sqrt{3}}{2} - 50\text{ N} \cdot \frac{\sqrt{3}}{2} + 30\text{ N}) \hat{j} \\ &= -25\text{ N} \hat{i} - 99.9\text{ N} \hat{j}.\end{aligned}$$

Forces  $\vec{F}_1$  and  $\vec{F}_3$  pass through point D. Therefore, they do not produce any moment about D. So, the net moment about D is the moment caused by force  $\vec{F}_2$ :

$$\begin{aligned}\vec{M}_D &= \vec{r}_{C/D} \times \vec{F}_2 \\ &= h \hat{j} \times F_2(\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= -F_2 h \cos \theta \hat{k} \\ &= -50\text{ N} \cdot 0.5\text{ m} \frac{1}{2} \hat{k} = -12.5\text{ N}\cdot\text{m} \hat{k}.\end{aligned}$$

The equivalent force-couple system is shown in Fig. 2.60

$$\boxed{\vec{F}_{\text{net}} = -25\text{ N} \hat{i} - 99.9\text{ N} \hat{j} \text{ and } \vec{M}_D = -12.5\text{ N}\cdot\text{m} \hat{k}}$$

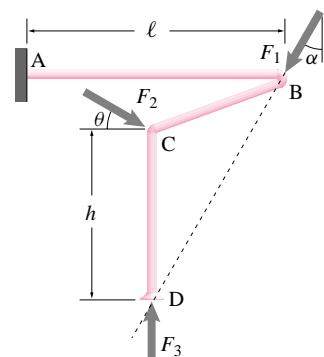


Figure 2.59: (Filename:sfig2.vec3.bar)

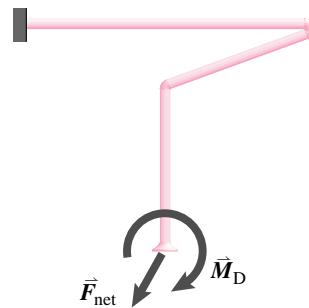


Figure 2.60: (Filename:sfig2.vec3.bar.a)

**SAMPLE 2.36 Translating a force-couple system:** The net force and couple acting about point B on the 'L' shaped bar shown in the figure are  $100\text{ N}$  and  $20\text{ N}\cdot\text{m}$ , respectively. Find the net force and moment about point G.

**Solution** The net force on a structure is the same about any point since it is just the vector sum of all the forces acting on the structure and is independent of their point of application. Therefore,

$$\vec{F}_{\text{net}} = \vec{F} = -100\text{ N} \hat{j}.$$

The net moment about a point, however, depends on the location of points of application of the forces with respect to that point. Thus,

$$\begin{aligned}\vec{M}_G &= \vec{M}_O + \vec{r}_{O/G} \times \vec{F} \\ &= M \hat{k} + (-\ell \hat{i} + h \hat{j}) \times (-F \hat{j}) \\ &= (M + F\ell) \hat{k} \\ &= (20\text{ N}\cdot\text{m} + 100\text{ N} \cdot 1\text{ m}) \hat{k} = 120\text{ N}\cdot\text{m} \hat{k}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -100\text{ N} \hat{j}, \text{ and } \vec{M}_G = 120\text{ N}\cdot\text{m} \hat{k}}$$

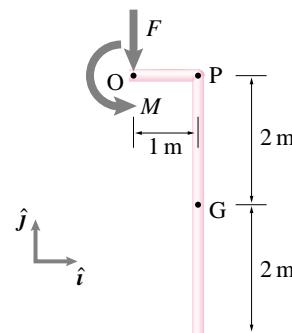


Figure 2.61: (Filename:sfig2.vec3.bentbar)

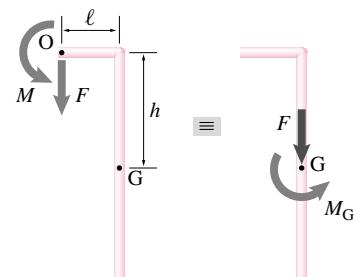


Figure 2.62: (Filename:sfig2.vec3.bentbar.a)

**SAMPLE 2.37 Checking equivalence of force-couple systems:** In the figure shown below, which of the force-couple systems shown in (b), (c), and (d) are equivalent to the force system shown in (a)?

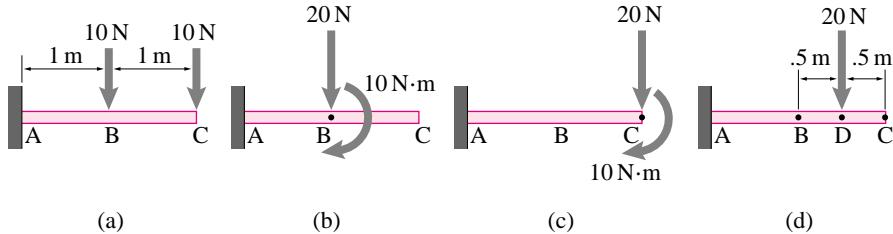


Figure 2.63: (Filename:sfig2.vec3.beam)

**Solution** The equivalence of force-couple systems require that (i) the net force be the same, and (ii) the net moment about any reference point be the same. For the given systems, let us choose point B as our reference point for comparing their equivalence. For the force system shown in Fig. 2.63(a), we have,

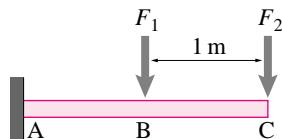


Figure 2.64: (Filename:sfig2.vec3.beam.a)

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = -10 \text{ N} \hat{j} - 10 \text{ N} \hat{j} = -20 \text{ N} \hat{j} \\ \vec{M}_{B_{\text{net}}} &= \vec{r}_{C/B} \times \vec{F}_2 = 1 \text{ m} \hat{i} \times (-10 \text{ N} \hat{j}) = -10 \text{ N} \cdot \text{m} \hat{k}.\end{aligned}$$

Now, we can compare the systems shown in (b), (c), and (d) against the computed equivalent force-couple system,  $\vec{F}_{\text{net}}$  and  $\vec{M}_B$ .

- Figure (b) shows exactly the system we calculated. Therefore, it represents an equivalent force-couple system.
- Figure (c): Let us calculate the net force and moment about point B for this system.

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_C \stackrel{?}{=} -20 \text{ N} \hat{j} \\ \vec{M}_B &= \vec{M}_C + \vec{r}_{C/B} \times \vec{F}_C \\ &= -10 \text{ N} \cdot \text{m} \hat{k} + 1 \text{ m} \hat{i} \times (-20 \text{ N} \hat{j}) = -30 \text{ N} \cdot \text{m} \hat{k} \neq \vec{M}_{B_{\text{net}}}.\end{aligned}$$

Thus, the given force-couple system in this case is not equivalent to the force system in (a).

- Figure (d): Again, we compute the net force and the net couple about point B:

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_D \stackrel{?}{=} -20 \text{ N} \hat{j} \\ \vec{M}_B &= \vec{r}_{D/B} \times \vec{F}_D \\ &= 0.5 \text{ m} \hat{i} \times (-20 \text{ N} \hat{j}) = -10 \text{ N} \cdot \text{m} \hat{k} \stackrel{?}{=} \vec{M}_{B_{\text{net}}}.\end{aligned}$$

Thus, the given force-couple system (with zero couple) at D is equivalent to the force system in (a).

(b) and (d) are equivalent to (a); (c) is not.

**SAMPLE 2.38** *Equivalent force with no couple:* For a body, an equivalent force-couple system at point A consists of a force  $\vec{F} = 20\text{N}\hat{i} + 16\text{N}\hat{j}$  and a couple  $\vec{M}_A = 10\text{N}\cdot\text{m}\hat{k}$ . Find a point on the body such that the equivalent force-couple system at that point consists of only a force (zero couple).

**Solution** The net force in the two equivalent force-couple systems has to be the same. Therefore, for the new system,  $\vec{F}_{\text{net}} = \vec{F} = 20\text{N}\hat{i} + 15\text{N}\hat{j}$ . Let B be the point at which the equivalent force-couple system consists of only the net force, with zero couple. We need to find the location of point B. Let A be the origin of a  $xy$  coordinate system in which the coordinates of B are  $(x, y)$ . Then, the moment about point B is,

$$\begin{aligned}\vec{M}_B &= \vec{M}_A + \vec{r}_{A/B} \times \vec{F} \\ &= M_A\hat{k} + (-x\hat{i} - y\hat{j}) \times (F_x\hat{i} + F_y\hat{j}) \\ &= M_A\hat{k} + (-F_yx + F_xy)\hat{k}.\end{aligned}$$

Since we require that  $\vec{M}_B$  be zero, we must have

$$\begin{aligned}F_yx - F_xy &= M_A \\ \Rightarrow y &= \frac{F_y}{F_x}x - \frac{M_A}{F_x} \\ &= \frac{15\text{ N}}{20\text{ N}}x - \frac{10\text{ N}\cdot\text{m}}{20\text{ N}} \\ &= 0.75x - 0.5\text{ m}.\end{aligned}$$

This is the equation of a line. Thus, we can select any point on this line and apply the force  $\vec{F} = 20\text{N}\hat{i} + 15\text{N}\hat{j}$  with zero couple as an equivalent force-couple system.

Any point on the line  $y = 0.75x - 0.5\text{ m}$ .

So, how or why does it work? The line we obtained is shown in gray in Fig. 2.66. Note that this line has the same slope as that of the given force vector (slope = 0.75 =  $F_y/F_x$ ) and the offset is such that shifting the force  $\vec{F}$  to this line counter balances the given couple at A. To see this clearly, let us select three points C, D, and E on the line as shown in Fig. 2.67. From the equation of the line, we find the coordinates of C(0,-.5m), D(.24m,.32m) and E(.67m,0). Now imagine moving the force  $\vec{F}$  to C, D, or E. In each case, it must produce the same moment  $\vec{M}_A$  about point A. Let us do a quick check.

- $\vec{F}$  at point C: The moment about point A is due to the horizontal component  $F_x = 20\text{ N}$ , since  $F_y$  passes through point A. The moment is  $F_x \cdot AC = 20\text{ N} \cdot 0.5\text{ m} = 10\text{ N}\cdot\text{m}$ , same as  $M_A$ . The direction is counterclockwise as required.
- $\vec{F}$  at point D: The moment about point A is  $|\vec{F}| \cdot AD = 25\text{ N} \cdot 0.4\text{ m} = 10\text{ N}\cdot\text{m}$ , same as  $M_A$ . The direction is counterclockwise as required.
- $\vec{F}$  at point E: The moment about point A is due to the vertical component  $F_y$ , since  $F_x$  passes through point A. The moment is  $F_y \cdot AE = 15\text{ N} \cdot 0.67\text{ m} = 10\text{ N}\cdot\text{m}$ , same as  $M_A$ . The direction here too is counterclockwise as required.

Once we check the calculation for one point on the line, we should not have to do any more checks since we know that sliding the force along its line of action (line CB) produces no couple and thus preserves the equivalence.

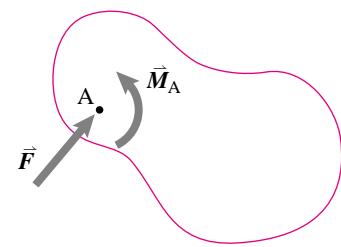


Figure 2.65: (Filename:sfig2.vec3.body)

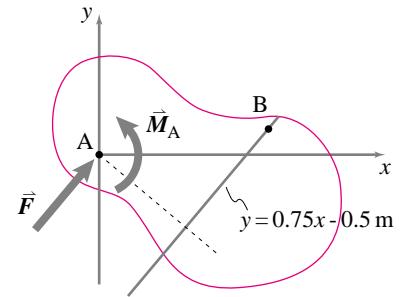


Figure 2.66: (Filename:sfig2.vec3.body.a)

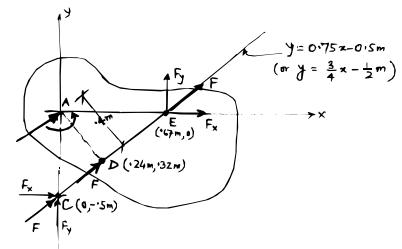


Figure 2.67: (Filename:sfig2.vec3.body.b)



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# 3 Free body diagrams

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## *The zeroth laws of mechanics*

One way to understand something is to isolate it, see how it behaves on its own, and see how it responds to various stimuli. Then, when the thing is not isolated, you still think of it as isolated, but think of the effects of all its surroundings as stimuli. We can also see its behavior as causing stimulus to other things around it, which themselves can be thought of as isolated and stimulating back, and so on.

This reductionist approach is used throughout the physical and social sciences. A tobacco plant is understood in terms of its response to light, heat flow, the chemical environment, insects, and viruses. The economy of Singapore is understood in terms of the flow of money and goods in and out of the country. And social behavior is regarded as being a result of individuals reacting to the sights, sounds, smells, and touch of other individuals and thus causing sights, sounds, smell and touch that the others react to in turn, etc.

The isolated system approach to understanding is made most clear in thermodynamics courses. A system, usually a fluid, is isolated with rigid walls that allow no heat, motion or material to pass. Then, bit by bit, as the subject is developed, the response of the system to certain interactions across the boundaries is allowed. Eventually, enough interactions are understood that the system can be viewed as isolated even when in a useful context. The gas expanding in a refrigerator follows the same rules of heat-flow and work as when it was expanded in its ‘isolated’ container.

The subject of mechanics is also firmly rooted in the idea of an isolated system. As in elementary thermodynamics we will be solely concerned with *closed* systems. A (closed) system, in mechanics, is a fixed collection of material. You can draw an imaginary boundary around a system, then in your mind paint all the atoms inside the

① The mechanics of *open systems*, where material crosses the system boundaries, is important in fluid mechanics and even in some elementary dynamics problems (like rockets), where material is allowed to cross the system boundaries. But the equations governing these *open* systems are deduced from careful application of the more fundamental governing mechanics equations of *closed* systems. So we have to master the mechanics of closed systems first.

① Why do we awkwardly number the first law as zero? Because it is really more of an underlying assumption, a background concept, than a law. As a law it is a little imprecise since force has not yet been defined. You could take the zeroth law as an implicit and partial definition of force. The phrase “zeroth law” means “important implicit assumption”. The second part of the zeroth law is usually called “Newton’s third law.”

② **Free-body Perkins.** At Cornell University, in the 1950’s, a professor Harold C. Perkins earned the nick name ‘Free-body Perkins’ by stopping random mechanics students in the hall and saying “You! Come in my office! Draw a free body diagram!” Students learned that they should draw free body diagrams, at least to please Free-body Perkins. But by learning to please Perkins they learned to get more right answers to mechanics problems, and they learned how to better explain their work.

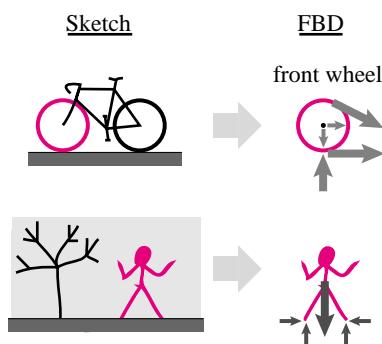


Figure 3.1: A sketch of a bicycle and a free body diagram of the braked front wheel. A sketch of a person and a free body diagram of a person.

(Filename: tfigure2.1)

boundary red, and then define the system as being the red atoms, no matter whether they cross the original boundary markers or not. Thus mechanics depends on bits of matter as being durable and non-ephemeral. A given bit of matter in a system exists forever, has the same mass forever, and is always in that system.①

Mechanics is based on the notion that any part of a system is itself a system and that all interactions between systems or subsystems have certain simple rules, most basically:

*The measure of mechanical interaction is force,*

and

*What one system does to another, the other does back to the first.*

Thus a person can be moved by forces, but not by the sight of a tree falling towards them or the attractive smell of a flower (these things may cause, by rules that fall outside of mechanics, forces that move a person). And when a person is moved by the force of the ground on her feet, the ground is pushed back just as hard. The two simple rules above, which we call the zeroth<sup>①</sup> laws of mechanics, imply that all the mechanical effects of interaction on a system can be represented by a sketch of the system with arrows showing the forces of interaction. If we want to know how the system in turn effects its surroundings we draw the opposite arrows on a sketch of the surroundings.

In mechanics a system is often called a *body* and when it is isolated it is *free* (as in *free* from its surroundings). In mechanics a sketch of an isolated system and the forces which act on it is called a *free body diagram*. A more descriptive phrase might have been “isolated system diagram”, but this latter phrase is not in common usage.

### 3.1 Free body diagrams

A *free body diagram* is a sketch of the system of interest and the forces that act on the system. A free body diagram precisely defines the system to which you are applying mechanics equations and the forces to be considered. Any reader of your calculations needs to see your free body diagrams. To put it directly, if you want to be right and be seen as right, then ②

*Draw a Free Body Diagram!*

The concept of the free body diagram is simple. In practice, however, drawing useful free body diagrams takes some thought, even for those practiced at the art. Here are some free body diagram properties and features:

- A free body diagram is a picture of the system for which you would like to apply linear or angular momentum balance (force and moment balance being special cases) or power balance. It shows the system isolated (‘free’) from its environment. That is, the free body diagram does *not* show things that are near or touching the system of interest. See figure 3.1.

- A free body diagram may show one or more particles, rigid bodies, deformable bodies, or parts thereof such as a machine, a component of a machine, or a part of a component of a machine. You can draw a free body diagram of any collection of material that you can identify. The word ‘body’ connotes a standard object in some people’s minds. In the context of free body diagrams, ‘body’ means system. The body in a free body diagram may be a subsystem of the overall system of interest.
  - The free body diagram of a system shows the forces and moments that the surroundings impose on the system. That is, since the only method of mechanical interaction that God has invented is force (and moment), the free body diagram shows what it would take to mechanically fool the system if it was literally cut free. That is, the motion of the system would be totally unchanged if it were cut free and the forces shown on the free body diagram were applied as a replacement for all external interactions.
  - The forces and moments are shown on the free body diagram at the points where they are applied. These places are where you made ‘cuts’ to *free* the *body*.
  - At places where the outside environment causes or restricts translation of the isolated system, a contact force is drawn on the free body diagram. Draw the contact force outside the sketch of the system for viewing clarity. A block supported by a hinge with friction in figure 3.2 illustrates how the reaction force on the block due to the hinge is best shown outside the block.
  - At connections to the outside world that cause or restrict rotation of the system a contact torque (or couple or moment) is drawn. Draw this moment outside the system for viewing clarity. Refer again to figure 3.2 to see how the moment on the block due to the friction of the hinge is best shown outside the block.
  - The free body diagram shows the system cut free from the source of any *body forces* applied to the system. Body forces are forces that act on the inside of a body from objects outside the body. It is best to draw the body forces on the interior of the body, at the center of mass if that correctly represents the net effect of the body forces. Figure 3.2 shows the cleanest way to represent the gravity force on the uniform block acting at the center of mass. ①.
  - The free body diagram shows all external forces acting on the system but *no* internal forces — forces between objects within the body are not shown.
  - The free body diagram shows nothing about the motion ②. It shows: no “centrifugal force”, no “acceleration force”, and no “inertial force”. For statics this is a non-issue because inertial terms are neglected for all purposes.
- Velocities, inertial forces, and acceleration forces do not show on a free body diagram.*

The prescription that you *not show inertial forces* is a practical lie. In the D'Alembert approach to dynamics you can show inertial forces on the free body diagram. The D'Alembert approach is discussed in box ?? on page ???. This legitimate and intuitive approach to dynamics is not followed in this book because of the frequent sign errors amongst beginners who use it.

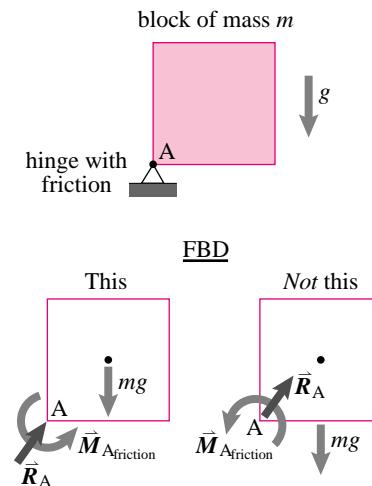


Figure 3.2: A uniform block of mass  $m$  supported by a hinge with friction in the presence of gravity. The free body diagram on the right is correct, just less clear than the one on the left.

(Filename:tfigure2.outside.loads)

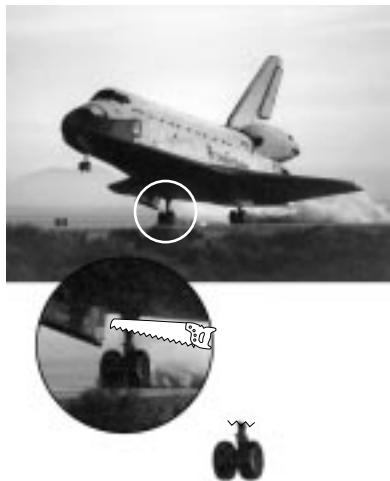
① **Body Forces.** In this book, the only body force we consider is gravity. For near-earth gravity, gravity forces show on the free body diagram as a single force at the center of gravity, or as a collection of forces at the center of gravity of each of the system parts. For parts of electric motors and generators, not covered here in detail, electrostatic or electro-dynamic body forces also need to be considered.

② **Caution:** A common *error* made by beginning dynamics students is to put velocity and/or acceleration arrows on the free body diagram.

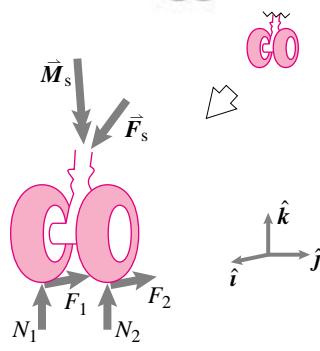
## How to draw a free body diagram

We suggest the following procedure for drawing a free body diagram, as shown schematically in fig. 3.3

- Define in your own mind what system or what collection of material, you would like to write momentum balance equations for. This subsystem may be part of your overall system of interest.



- (b) Draw a sketch of this system. Your sketch may include various cut marks to show how it is isolated from its environment. At each place the system has been cut free from its environment you imagine that you have cut the system free with a sharp scalpel or with a chain saw.
- (c) Look systematically at the picture at the places that the system interacts with material *not* shown in the picture, places where you made ‘cuts’.
- (d) Use forces and torques to fool the system into thinking it has not been cut. For example, if the system is being pushed in a given direction at a given contact point, then show a force in that direction at that point. If a system is being prevented from rotating by a (cut) rod, then show a torque at that cut.
- (e) To show that you have cut the system from the earth’s gravity force show the force of gravity on the system’s center of mass or on the centers of mass of its parts.



### How to draw forces on free body diagrams

How you draw a force on a free body diagram depends on

- How much you know about the force when you draw the free body diagram. Do you know its direction? its magnitude?; and
- Your choice of notation (which may vary from vector to vector within one free body diagram). See page 12 for a description of the ‘symbolic’ and ‘graphical’ vector notations.

Figure 3.3: The process of drawing a FBD is illustrated by the sequence shown.

(Filename:figure2.howtoFBD)

Some of the possibilities are shown in fig. 3.4 for three common notations for a 2D force in the cases when (a) any  $\vec{F}$  possible, (b) the direction of  $\vec{F}$  is fixed, and (c) everything about  $\vec{F}$  is fixed.

	(a) Nothing is known about $\vec{F}$	(b) Direction of $\vec{F}$ is known	(c) $\vec{F}$ is known
Symbolic	$\vec{F}$	$\vec{F} = F \frac{\hat{i} + \hat{j}}{\sqrt{2}}$	$10\hat{i} + 10\hat{j}$
Graphical	$F$ at angle $\theta$ from horizontal	Three cases: $F$ at $45^\circ$ from horizontal, $F$ at $45^\circ$ from vertical, and $F$ with horizontal component $\ell$ and vertical component $\ell$	$14.1\text{ N}$ at $45^\circ$ from horizontal
Components	$F_x$ and $F_y$	$F_1$ and $F_2$	Two $10\text{ N}$ forces at $90^\circ$ to each other

Figure 3.4: The various ways of notating a force on a free body diagram. (a) nothing is known or everything is variable (b) the direction is known. (c) Everything is known. In one free body diagram different notations can be used for different forces, as needed or convenient.

(Filename:figure.fbdvectnot)

## Equivalent force systems

The concept of ‘fooling’ a system with forces is somewhat subtle. If the free body diagram involves ‘cutting’ a rope what force should one show? A rope is made of many fibers so cutting the rope means cutting all of the rope fibers. Should one show hundreds of force vectors, one for each fiber that is cut? The answer is: yes and no. You would be correct to draw all of these hundreds of forces at the fiber cuts. But, since the equations that are used with any free body diagram involve only the total force and total moment, you are also allowed to replace these forces with an equivalent force system (see section 2.5).

*Any force system acting on a given free body diagram can be replaced by an equivalent force and couple.*

In the case of a rope, a single force directed nearly parallel to the rope and acting at about the center of the rope’s cross section is equivalent to the force system consisting of all the fiber forces. In the case of an ideal rope, the force is exactly parallel to the rope and acts exactly at its center.

## Action and reaction

For some systems you will want to draw free body diagrams of subsystems. For example, to study a machine, you may need to draw free body diagrams of its parts; for a building, you may draw free body diagrams of various structural components; and, for a biomechanics analysis, you may ‘cut up’ a human body. When separating a system into parts, you must take account of how the subsystems interact. Say these subsystems, e.g. two touching parts of a machine, are called  $\mathcal{A}$  and  $\mathcal{B}$ . We then have that

If       $\mathcal{A}$     feels force     $\vec{F}$     and couple     $\vec{M}$     from  $\mathcal{B}$ ,  
 then     $\mathcal{B}$     feels force     $-\vec{F}$     and couple     $-\vec{M}$     from  $\mathcal{A}$ .

To be precise we must make clear that  $\vec{F}$  and  $-\vec{F}$  have the same line of action.<sup>①</sup>

The principle of action and reaction doesn’t say anything about what force or moment acts on one object. It only says that the actor of a force and moment gets back the opposite force and moment.

It is easy to make mistakes when drawing free body diagrams involving action and reaction. Box 3.3 on page 96 shows some correct and incorrect partial FBD’s of interacting bodies  $\mathcal{A}$  and  $\mathcal{B}$ . Use notation consistent with box ?? on page ?? for the action and reaction vectors.

<sup>①</sup> The principle of action and reaction can be derived from the momentum balance laws by drawing free body diagrams of little slivers of material. Nonetheless, in practice you can think of the principle of action and reaction as a basic law of mechanics. Newton did. The principal of action and reaction is “Newton’s third law”.

## Interactions

The way objects interact mechanically is by the transmission of a force or a set of forces. If you want to show the effect of body  $\mathcal{B}$  on  $\mathcal{A}$ , in the most general case you can expect a force and a moment which are equivalent to the whole force system, however complex.

That is, the most general interaction of two bodies requires knowing

- six numbers in three dimensions (three force components and three moment components)
- and three numbers in two dimensions (two force components and one moment).

Many things often do not interact in this most general way so often fewer numbers are required. You will use what you know about the interaction of particular bodies to reduce the number of unknown quantities in your free body diagrams.

Some of the common ways in which mechanical things interact, or are assumed to interact, are described in the following sections. You can use these simplifications in your work.

## Constrained motion and free motion

One general principle of interaction forces and moments concerns constraints. Whenever a *motion* of  $\mathcal{A}$  is either caused *or* prevented by  $\mathcal{B}$  there is a corresponding *force* shown at the interaction point on the free body diagram of  $\mathcal{A}$ . Similarly if  $\mathcal{B}$  causes or prevents *rotation* there is a *moment* (or torque or couple) shown on the free body diagram of  $\mathcal{A}$  at the place of interaction.

The converse is also true. Many kinds of mechanical attachment gadgets are specifically designed to allow motion. If an attachment allows free motion in some direction the free body diagram shows no force in that direction. If the attachment allows free rotation about an axis then the free body diagram shows no moment (couple or torque) about that axis.

You can think of each attachment point as having a variety of jobs to do. For every possible direction of translation and rotation, the attachment has to either allow free motion or restrict the motion. In every way that motion is restricted (or caused) by the connection a force or moment is required. In every way that motion is free there is no force or couple. Motion of body  $\mathcal{A}$  is caused and restricted by forces and couples which act on  $\mathcal{A}$ . Motion is freely allowed by the absence of such forces and couples.

Here are some of the common connections and the free body diagrams with which they are associated.

## Cuts at rigid connections

Sometimes the body you draw in a free body diagram is firmly attached to another.

Figure 3.5 shows a cantilever structure on a building. The free body diagram of the cantilever has to show all possible force and load components. Since we have used vector notation for the force  $\vec{F}$  and the moment  $\vec{M}_C$  we can be ambiguous about whether we are doing a two or three dimensional analysis.

A common question by new mechanics students seeing a free body diagram like in figure 3.5 is: ‘gravity is pointing down, so why do we have to show a horizontal reaction force at C?’ Well, for a stationary building and cantilever a quick statics analysis reveals that  $\vec{F}_C$  must be vertical, so the question is reasonable. But one must remember: this book is about statics *and* dynamics and in dynamics *the forces on a body do not add to zero*. In fact, the building shown in figure 3.5 might be accelerating rapidly to the right due to the motions of a violent earthquake occurring at the instant pictured in the figure. Sometimes you know a force is going to turn out to be zero, as for the sideways force in this example if treated as a statics problem. In these cases it is a matter of taste whether or not you show the sideways force on the free body diagram (see box 3.1 on page 86).

The attachment of the cantilever to the building at C in figure 3.5 is surely intended to be rigid and prevent the cantilever from moving up or down (falling), from moving

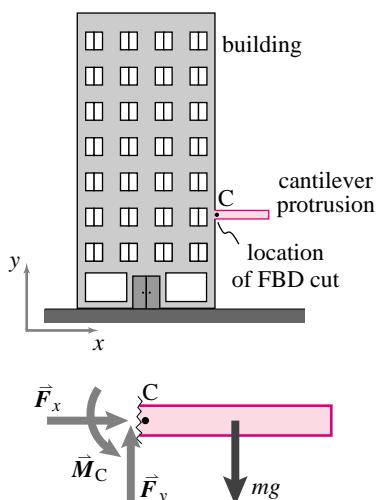


Figure 3.5: **A rigid connection: a cantilever structure on a building.** At the point C where the cantilever structure is connected to the building all motions are restricted so every possible force needs to be shown on the free body diagram cut at C.

(Filename:tfigure2.rigid)

sideways (and drifting into another building) or from rotating about point C. In most of the building's life, the horizontal reaction at C is small. But since the connection at C clearly prevents relative horizontal motion, a horizontal reaction force is drawn on the free body diagram. During an earthquake, this horizontal component will turn out to be not zero.

The situation with rigid connections is shown more abstractly in figure 3.6.

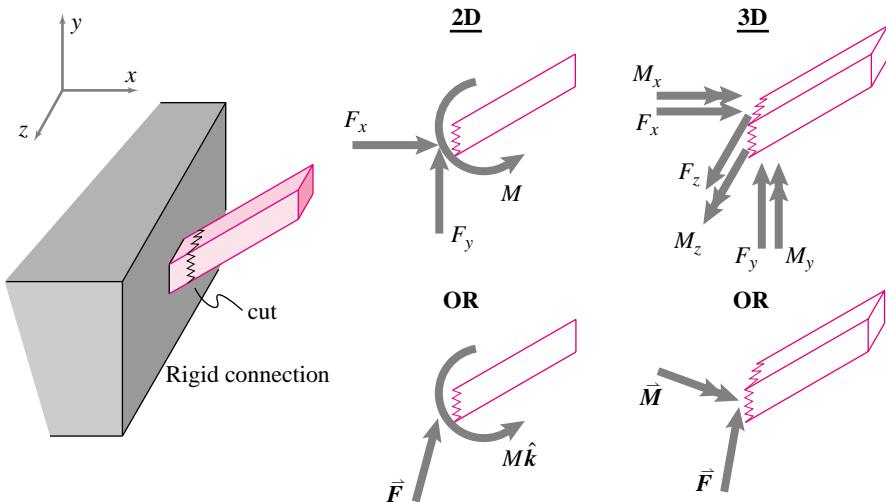


Figure 3.6: A rigid connection shown with partial free body diagrams in two and three dimensions. One has a choice between showing the separate force components (top) or using the vector notation for forces and moments (bottom). The double head on the moment vector is optional.

(Filename:tfigure2.rigidb)

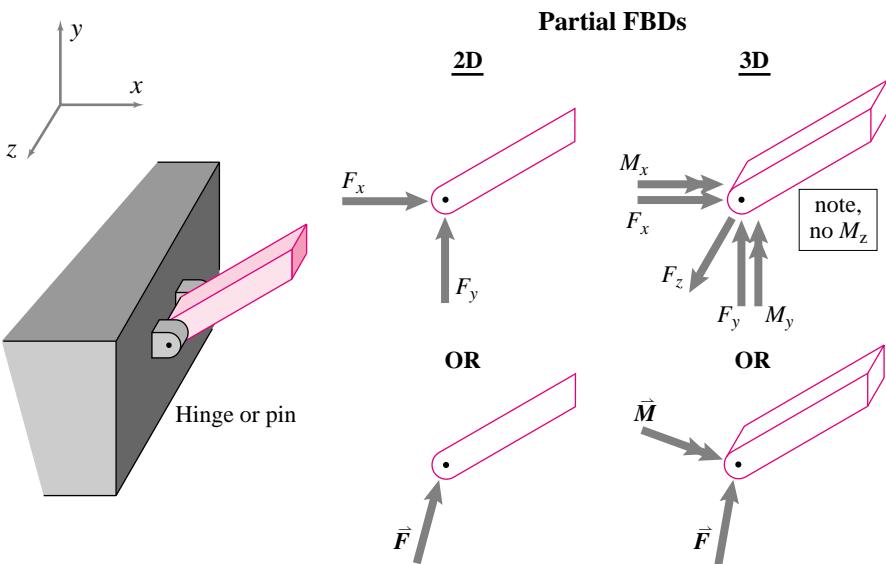


Figure 3.7: A hinge with partial free body diagrams in 2D and 3-D. A hinge joint is also called a pin joint because it is sometimes built by drilling a hole and inserting a pin.

(Filename:tfigure2.hinge)

## Cuts at hinges

A hinge, shown in figure 3.7, allows rotation and prevents translation. Thus, the free body diagram of an object cut at a hinge shows *no* torque about the hinge axis but does show the force or its components which prevent translation.

There is some ambiguity about how to model pin joints in three dimensions. The ambiguity is shown with reference to a hinged door (figure 3.8). Clearly, one hinge, if the sole attachment, prevents rotation of the door about the  $x$  and  $y$  axes shown. So, it is natural to show a couple (torque or moment) in the  $x$  direction,  $M_x$ , and in the  $y$  direction,  $M_y$ . But, the hinge does not provide very stiff resistance to rotations in these directions compared to the resistance of the other hinge. That is, even if both hinges are modeled as ball and socket joints (see the next sub-section), offering no resistance to rotation, the door still cannot rotate about the  $x$  and  $y$  axes.

If a connection between objects prevents relative translation or rotation that is already prevented by another stiffer connection, then the more compliant connection reaction is often neglected. Even without rotational constraints, the translational constraints at the hinges A and B restrict rotation of the door shown in figure 3.8. The hinges are probably well modeled — that is, they will lead to reasonably accurate

### 3.1 THEORY

#### *How much mechanics reasoning should you use when you draw a free body diagram?*

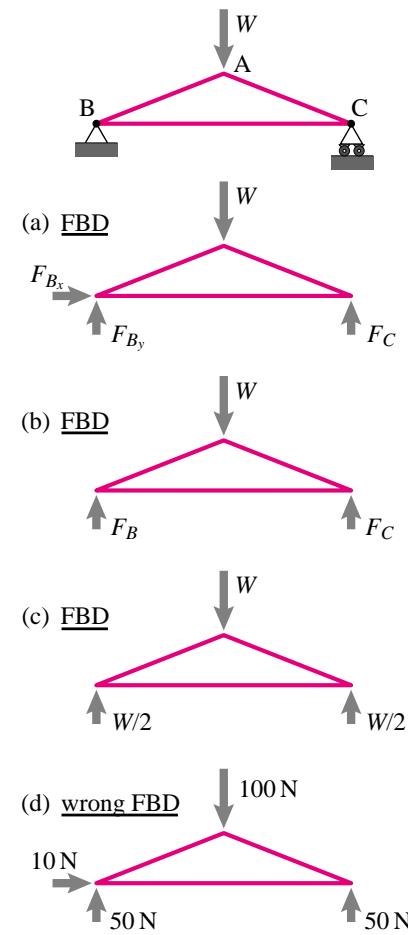
The simple rules for drawing free body diagrams prescribe an unknown force every place a motion is prevented and an unknown torque where rotation is prevented. Consider the simple symmetric truss with a load  $W$  in the middle. By this prescription the free body diagram to draw is shown as (a). There is an unknown force restricting both horizontal and vertical motion at the hinge at B.

However, a person who knows some statics will quickly deduce that the horizontal force at B is zero and thus draw the free body diagram in figure (b). Or if they really think ahead they will draw the free body diagram in (c). All three free body diagrams are correct. In particular diagram (a) is correct even though  $F_{Bx}$  turns out to be zero and (b) is correct even though  $F_B$  turns out to be equal to  $F_C$ .

Some people, thinking ahead, sometimes say that the free body diagram in (a) is wrong. But it should be pointed out that free body diagram (a) is correct because the force  $F_{Bx}$  is not specified and therefore could be zero. Free body diagram (d), on the other hand, explicitly and incorrectly assigns a non-zero value to  $F_{Bx}$ , so it is wrong.

A reasonable approach is to follow the naive rules, and then later use the force and momentum equations to find out more about the forces. That is use free body diagram (a) and discover (c) using the laws of mechanics. If you are confident about the anticipated results, it is sometimes a time saver to use diagrams analogous to (b) or (c) but beware of

- making assumptions that are not reasonable, and
- wasting time trying to think ahead when the force and momentum balance equations will tell all in the end anyway.



calculations of forces and motions — by ball and socket joints at A and B. In 2-D , a ball and socket joint is equivalent to a hinge or pin joint.

## Ball and socket joint

Sometimes one wishes to attach two objects in a way that allows no relative translation but for which all rotation is free. The device that is used for this purpose is called a ‘ball and socket’ joint. It is constructed by rigidly attaching a sphere (the ball) to one of the objects and rigidly attaching a partial spherical cavity (the socket) to the other object.

The human hip joint is a ball and socket joint. At the upper end of the femur bone is the femoral head, a sphere to within a few thousandths of an inch. The hip bone has a spherical cup that accurately fits the femoral head. Car suspensions are constructed from a three-dimensional truss-like mechanism. Some of the parts need free relative rotation in three dimensions and thus use a joint called a ‘ball joint’ or ‘rod end’ that is a ball and socket joint.

Since the ball and socket joint allows all rotations, no moment is shown at a cut ball and socket joint. Since a ball and socket joint prevents relative translation in all directions, the possibility of force in any direction is shown.

## String, rope, wires, and light chain

One way to keep a radio tower from falling over is with wire, as shown in figure 3.10. If the mass and weight of the wires seems small it is common to assume they can only transmit forces along their length. Moments are not shown because ropes, strings, and wires are generally assumed to be so compliant in bending that the bending moments are negligible. We define *tension* to be the force pulling away from a free body diagram cut. ①

①**Caution:** Sometimes string like things should not be treated as idealized strings. Short wires can be stiff so bending moments may not be negligible. The mass of chains can be significant so that the mass and weight may not be negligible, the direction of the tension force in a sagging chain is not in the direction connecting the two chain endpoints.

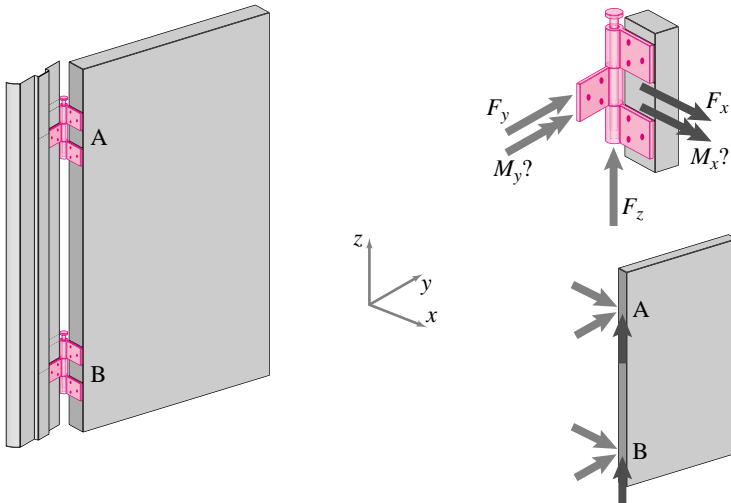


Figure 3.8: **A door held by hinges.** One must decide whether to model hinges as proper hinges or as ball and socket joints. The partial free body diagram of the door at the lower right neglects the couples at the hinges, effectively idealizing the hinges as ball and socket joints. This idealization is generally quite accurate since the rotations that each hinge might resist are already resisted by their being two connection points.

(Filename:tfigure2.door)

## Springs and dashpots

Springs are used in many machines to absorb and return small amounts of energy. Dashpots are used to absorb energy. They are shown schematically in fig. 3.12. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. Often they are attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Since they only have forces at their ends they are ‘two-force’ bodies and, by the reasoning of coming section 4.1, the forces at their ends are equal, opposite,

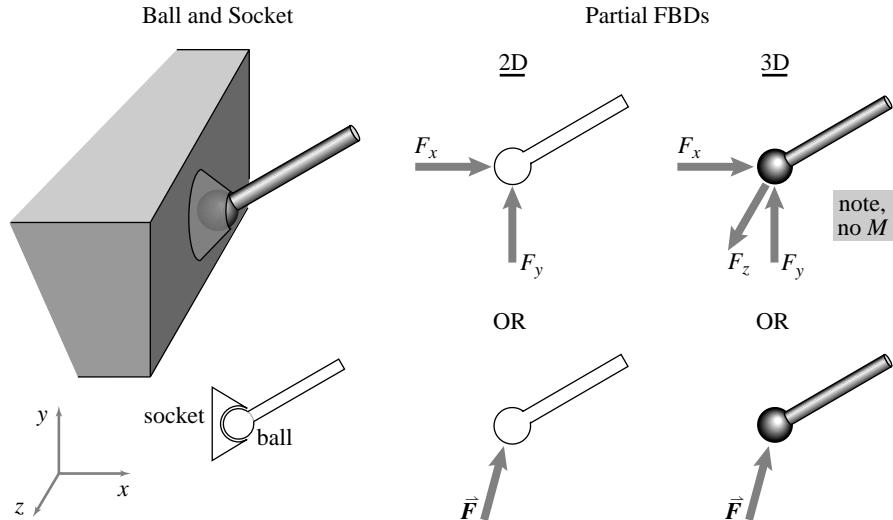


Figure 3.9: A ball and socket joint allows all relative rotations and no relative translations so reaction forces, but not moments, are shown on the partial free body diagrams. In two dimensions a ball and socket joint is just like a pin joint. The top partial free body diagrams show the reaction in component form. The bottom illustrations show the reaction in vector form.

(Filename:tfigure2.ballands)

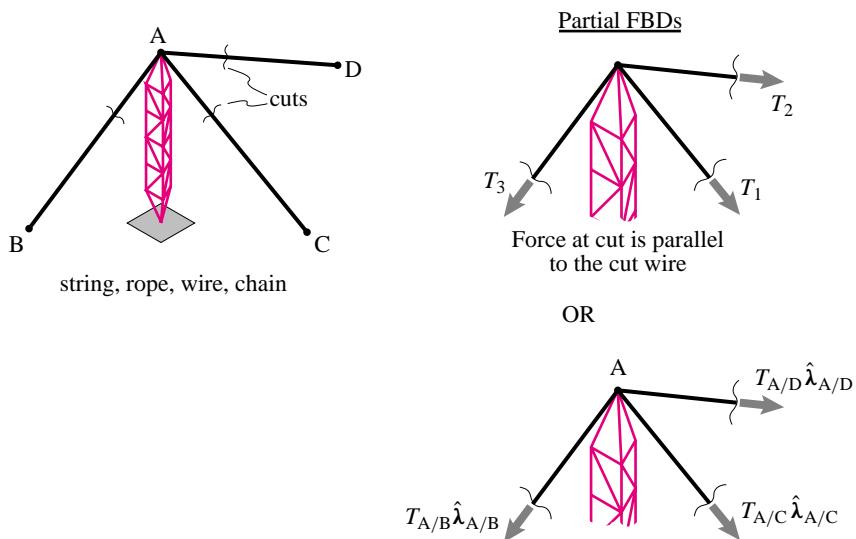


Figure 3.10: A radio tower kept from falling with three wires. A partial free body diagram of the tower is drawn two different ways. The upper figure shows three tensions that are parallel to the three wires. The lower partial free body diagram is more explicit, showing the forces to be in the directions of the  $\hat{\lambda}$ s, unit vectors parallel to the wires.

(Filename:tfigure2.string)

and along the line of connection.

### Springs

Springs often look like the standard spring drawing in figure 3.11.

If the tension in a spring is a function of its length alone, independent of its rate of lengthening, the spring is said to be ‘elastic.’ If the tension in the spring is proportional to its stretch the spring is said to be ‘linear.’ The assumption of linear elastic behavior is accurate for many physical springs. So, most often if one says one is using a spring, the linear and elastic properties are assumed.

The stretch of a spring is the amount by which the spring is longer than when it is relaxed. This relaxed length is also called the ‘unstretched’ length, the ‘rest’ length, or the ‘reference’ length. If we call the unstretched length, the length of the spring when its tension is zero,  $\ell_0$ , and the present length  $\ell$ , then the stretch of the spring is  $\Delta\ell = \ell - \ell_0$ . The tension in the spring is proportional to this stretch. Most often people use the letter  $k$  for the proportionality constant and say ‘the spring has constant  $k$ .’ So the basic equation defining a spring is

$$T = k\Delta\ell.$$

### Dashpots

Dashpots are used to absorb, or dampen, energy. The most familiar example is in the shock absorbers of a car. The symbol for a dashpot shown in figure 3.12 is meant to suggest the mechanism. A fluid in a cylinder leaks around a plunger as the dashpot gets longer and shorter. The dashpot resists motion in both directions.

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. The relation is assumed to hold for negative lengthening as well. So the compression (negative tension) is proportional to the rate at which the dashpot shortens (negative lengthens). The defining equation for a linear dashpot is:

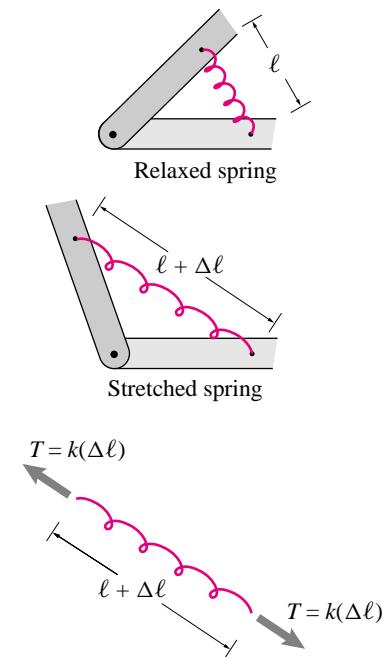
$$T = C\dot{\ell}$$

where  $C$  is the dashpot constant.

### Collisions

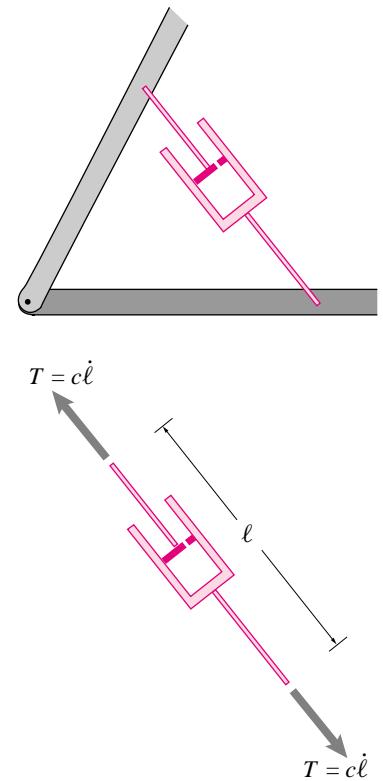
Two objects are said to *collide* when some interaction force or moment between them becomes very large, so large that other forces acting on the bodies become negligible. For example, in a car collision the force of interaction at the bumpers may be many times the weight of the car or the reaction forces acting on the wheels.

The analysis of collisions is a little different than the analysis of smooth motions, as will be discussed later in the text. But this analysis still depends on free body diagrams showing the non-negligible collision forces. See figure 3.13. Knowing which forces to include and which to ignore in a collision problem is an issue which can have great subtlety. Some rules of thumb:



**Figure 3.11: Spring connection.** The tension in a spring is usually assumed to be proportional to its change in length, with proportionality constant  $k$ :  $T = k\Delta\ell$ .

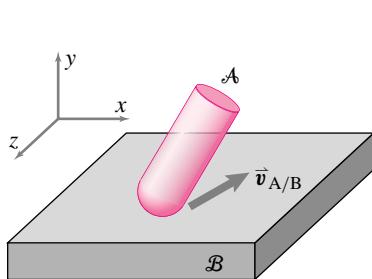
(Filename:tfigure2.spring)



**Figure 3.12: A dashpot.** A dashpot is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens.

(Filename:tfigure2.dashpot)

- ignore forces from gravity, springs, and at places where contact is broken in the collision, and
- include forces at places where new contact is made, or where contact is maintained.



$\mathcal{A}$  slides frictionally on  $\mathcal{B}$

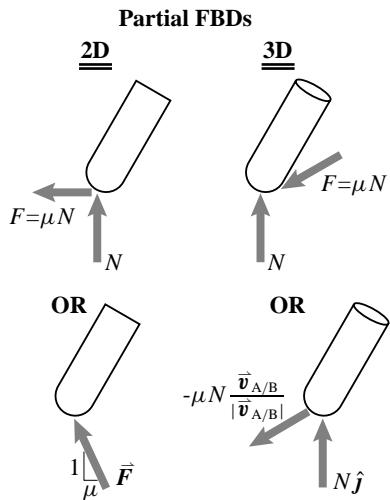


Figure 3.14: Object  $\mathcal{A}$  slides on the plane  $\mathcal{B}$ . The friction force on  $\mathcal{A}$  is in the direction that opposes the relative motion.

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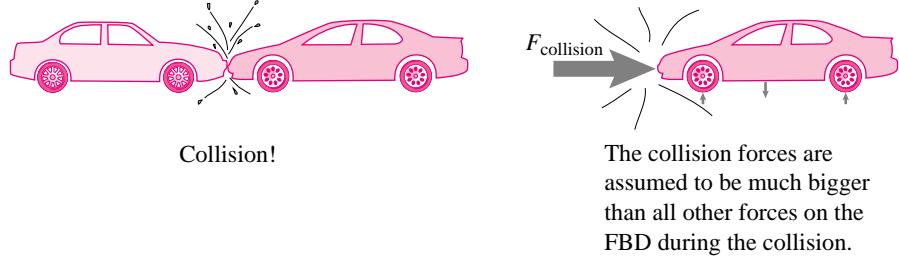


Figure 3.13: Here cars are shown colliding. A free body diagram of the right car shows the collision force and should not show other forces which are negligibly small. Here they are shown as negligibly small forces to give the idea that they may be much smaller than the collision force. The wheel reaction forces are neglected because of the spring compliance of the suspension and tires.

(Filename:tfigure2.collisions)

## Friction

When two independent solids are in contact relative slipping motion is resisted by *friction*. Friction can prevent slip and resists any slip which does occur.

The force on body  $\mathcal{A}$  from body  $\mathcal{B}$  is decomposed into a part which is tangent to the surface of contact  $\vec{F}$ , with  $|\vec{F}| = F$ , and a part which is normal to the surface  $N$ . The relation between these forces depends on the relative slip of the bodies  $\vec{v}_{A/B}$ . The magnitude of the frictional force is usually assumed to be proportional to the normal force with proportionality constant  $\mu$ . So the deceptively simple defining equation for the friction force  $F$  during slip is

$$F = \mu N$$

where  $N$  is the component of the interaction force in the inwards normal direction. The problem with this simple equation is that it assumes you have drawn the friction force in the direction opposing the slip of  $\mathcal{A}$  relative to  $\mathcal{B}$ . If the direction of the friction force has been drawn incorrectly then the formula gives the wrong answer.

If two bodies are in contact but are *not* sliding then the friction force can still keep the objects from sliding. The strength of the friction bond is often assumed to be proportional to the normal force with proportionality constant  $\mu$ . Thus if there is no slip we have that the force is something less than or equal to the strength,

$$|F| \leq \mu N.$$

Partial FBD's for the cases of slip and no slip are shown in figures 3.14 and 3.15, respectively. See the appendix for a further discussion of friction. To make things a little more precise, for those more formally inclined, we can write the friction equations as follows:

$$\begin{aligned} \vec{F}_{(\mathcal{B} \text{ acts on } \mathcal{A})} &= -\mu N \frac{\vec{v}_{A/B}}{|\vec{v}_{A/B}|}, \text{ if } \vec{v}_{A/B} \neq \vec{0}, \\ |\vec{F}_{(\mathcal{B} \text{ acts on } \mathcal{A})}| &\leq \mu N, \text{ if } \vec{v}_{A/B} = \vec{0}. \end{aligned}$$

The unit vector  $\frac{\vec{v}_{A/B}}{|\vec{v}_{A/B}|}$  is in the direction of relative slip. The principle of action and reaction, discussed previously, determines the force that  $A$  acts on  $B$ .

The simplest friction law, the one we use in this book, uses a single constant coefficient of friction  $\mu$ . Usually  $.05 \leq \mu \leq 1.2$ . We do not distinguish the static coefficient  $\mu_s$  from the dynamic coefficient  $\mu_d$  or  $\mu_k$ . That is  $\mu = \mu_s = \mu_k = \mu_d$  for our purposes. We promote the use of this simplest law for a few reasons.

- All friction laws used are quite approximate, no matter how complex. Unless the distinction between static and dynamic coefficients of friction is essential to the engineering calculation, using  $\mu_s \neq \mu_k$  doesn't add to the calculation's usefulness.
- The concept of a static coefficient of friction that is larger than a dynamic coefficient is, it turns out, not well defined if bodies have more than one point of contact, which they often do have.
- Students learning to do dynamics are often confused about how to handle problems with friction. Since the more complex friction laws are of questionable usefulness and correctness, it seems time is better spent understanding the simplest relations.

In summary, the simple model of friction we use is:

Friction resists relative slipping motion. *During slip the friction force opposes relative motion and has magnitude  $F = \mu N$ .* When there is no slip the magnitude of the friction force  $F$  cannot be determined from the friction law but it cannot exceed  $\mu N$ ,  $F \leq \mu N$ .

Sometimes people describe the friction coefficient with a *friction angle*  $\phi$  rather than the coefficient of friction (see fig. 3.16). The friction angle is the angle between the net interaction force (normal force plus friction force) and the normal to the sliding surface when slip is occurring. The relation between the friction coefficient  $\mu$  and the friction angle  $\phi$  is

$$\tan \phi = \mu.$$

The use of  $\phi$  or  $\mu$  to describe friction are equivalent. Which you use is a matter of taste and convenience.

### “Smooth” and “rough” surfaces

As a modeling simplification for situations where we would like to neglect friction forces we sometimes assume frictionless contact and thus set  $\mu = \phi = 0$ . In many books, but never in this one, the phrase “perfectly smooth” means frictionless. It is true that when separated by a little fluid (say water between your feet and the bathroom tile, or oil between pieces of a bearing) that smooth surfaces slide easily by each other. And even without a lubricant sometimes slipping can be reduced by roughening a surface. But making a surface progressively smoother does not diminish the friction to zero. In fact, extremely smooth surfaces sometimes have anomalously high friction. In general, there is no reliable relation correlation smoothness and low friction.

Similarly many books, but not this one, use the phrase “perfectly rough” to mean perfectly high friction ( $\mu \rightarrow \infty$  and  $\phi \rightarrow 90^\circ$ ) and hence that no slip is allowed. This is misleading twice over. First, as just stated, rougher surfaces do not reliably have more friction than smooth ones. Second, even when  $\mu \rightarrow \infty$  slip can proceed in some situations (see, for example, box 4.1 on page 120).

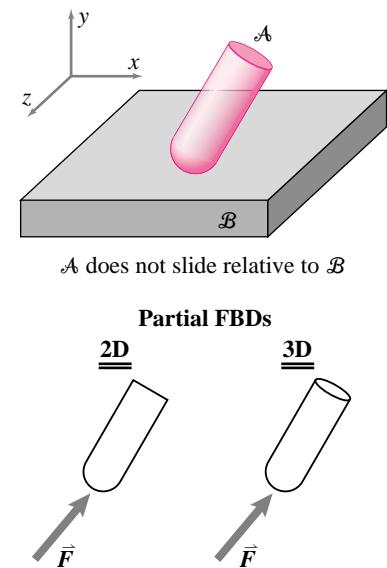


Figure 3.15: Object  $A$  does not slide relative to the plane  $B$ .

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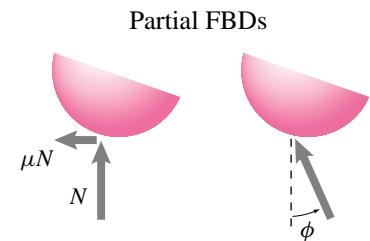


Figure 3.16: Two ways of characterizing friction: the friction coefficient  $\mu$  and friction angle  $\phi$ .

(Filename:tfigure2.friction.angle)

We use the phrase *frictionless* to mean that there is no tangential force component and not the misleading words “perfectly smooth”. We use the phrase *no slip* to mean that no tangential motion is allowed and not the misleading words “perfectly rough”.

## Rolling contact

An idealization for the non-skidding contact of balls, wheels, and the like is *pure rolling*.

*Objects  $\mathcal{A}$  and  $\mathcal{B}$  are in pure rolling contact when their (relatively convex) contacting points have equal velocity. They are not slipping, separating, or interpenetrating.*

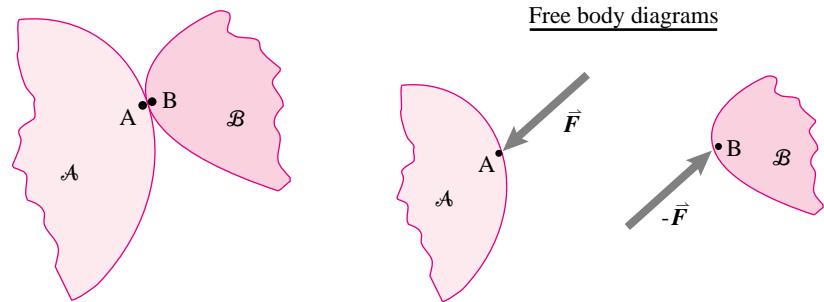


Figure 3.17: Rolling contact: Points of contact on adjoining bodies have the same velocity,  $\vec{v}_A = \vec{v}_B$ . But,  $\vec{\omega}_{\mathcal{A}}$  is not necessarily equal to  $\vec{\omega}_{\mathcal{B}}$ .

(Filename:tfigure2.rolling.contact)

Most often, we are interested in cases where the contacting bodies have some non-zero relative angular velocity — a ball sitting *still* on level ground may be technically in rolling contact, but not interestingly so.

The simplest common example is the rolling of a round wheel on a flat surface in two dimensions. See figure 3.18. In practice, there is often confusion about the

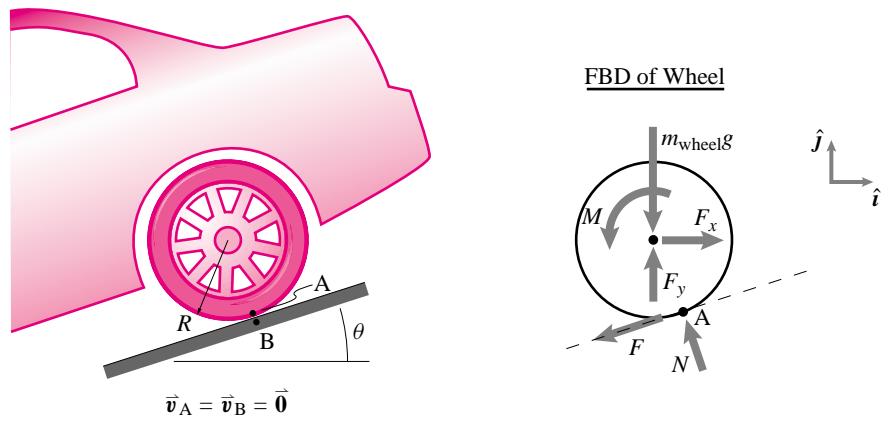


Figure 3.18: Pure rolling of a round wheel on a flat slope in two dimensions.

(Filename:tfigure2.pure.rolling.wheel)

direction and magnitude of the force  $F$  shown in the free body diagram in figure 3.18. Here is a recipe:

- 1.) Draw  $F$  as shown in any direction, tangent to the surface.

- 2.) Solve the statics or dynamics problem and find  $F$ . (It may turn out to be a negative force, which is fine.)
- 3.) Check that rolling is really possible; that is, that slip would not occur. If the force is greater than the frictional strength,  $|F| > \mu N$ , the assumption of rolling contact is not appropriate. In this case, you must assume that  $F = \mu N$  or  $F = -\mu N$  and that slip occurs; then, re-solve the problem.

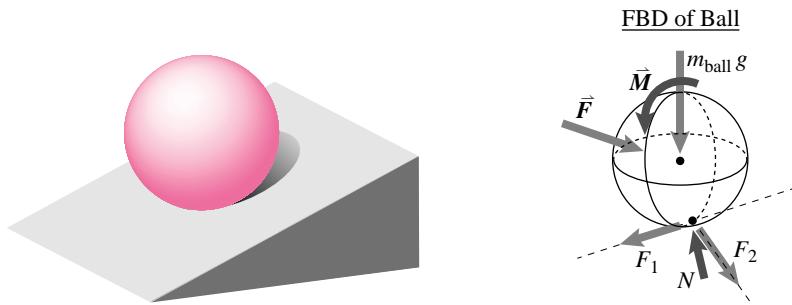


Figure 3.19: **Rolling ball in 3-D.** The force  $\vec{F}$  and moment  $\vec{M}$  are applied loads from, say, wind, gravity, and any attachments.  $N$  is the normal reaction and  $F_1$  and  $F_2$  are the in plane components of the frictional reaction. One must check the no-slip condition,  $\mu^2 N^2 \geq F_1^2 + F_2^2$ .

(Filename:tfigure2.3D.rolling)

In three-dimensional rolling contact, we have a free body diagram that again looks like a free body diagram for non-slipping frictional contact. Consider, for example, the ball shown in figure 3.19. For the friction force to be less than the friction coefficient times the normal force, we have

$$\sqrt{F_1^2 + F_2^2} \leq \mu N \quad \text{or} \quad F_1^2 + F_2^2 \leq \mu^2 N^2 \quad \text{no slip condition}$$

Rolling is just a special case of frictional contact. It is the case where bodies contact at a single point (or on a line, as with cylinders) and have relative rotation yet have no relative velocity at their contacting points. The tricky part about rolling is the kinematic analysis. This kinematics, we take up in section 8.3 on page 467 after you have learned more about angular velocity  $\vec{\omega}$ .

## Rolling resistance

Non-ideal rolling contact includes provision for *rolling resistance*. This resistance is simply represented by either moving the location of the point of contact force or by a contact couple. Rolling resistance leads to subtle questions which we would like to finesse here. A brief introduction is given in chapter 10.

## Ideal wheels

An ideal wheel is an approximation of a real wheel. It is a sensible approximation if the mass of the wheel is negligible, bearing friction is negligible, and rolling resistance is negligible. Free body diagrams of undriven ideal wheels in two and three dimensions are shown in figure 3.20. This idealization is rationalized in chapter 4 in box 4.1 on page 120. Note that if the wheel is not massless, the 2-D free body diagram looks more like the one in figure 3.20b with  $F_{\text{friction}} \leq \mu N$ .

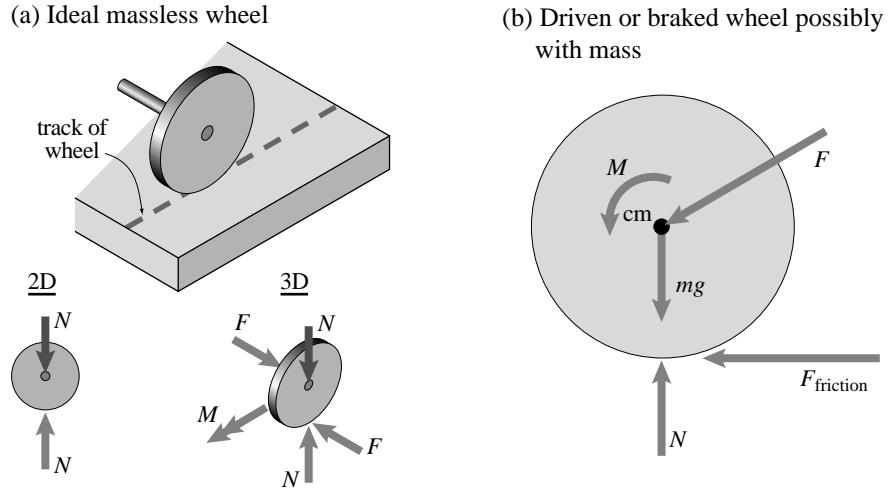


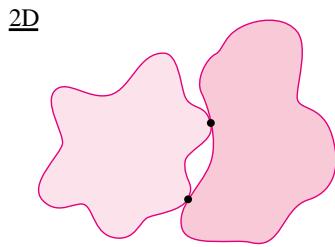
Figure 3.20: An ideal wheel is massless, rigid, undriven, round and rolls on flat rigid ground with no rolling resistance. Free body diagrams of ideal undriven wheels are shown in two and three dimensions. The force  $F$  shown in the three-dimensional picture is perpendicular to the path of the wheel. (b) 2D free body diagram of a wheel with mass, possibly driven or braked. If the wheel has mass but is not driven or braked the figure is unchanged but for the moment  $M$  being zero.

(Filename:figure2.idealwheel)

## 3.2 THEORY

### *Conformal contact of rigid bodies: a near impossibility*

If you take two arbitrary shaped rigid objects and make them touch without overlapping you will most often only be able to make contact at a few points (typically 1 to 3 points in 2D, and 1 to 6 points in 3D). Cut out two pieces of cardboard (leaving no straight edges) and slide them around on a table and you will see this.



But machines are not made of random parts. Many parts are made to conform, like an axle and a bearing, and many parts are machined with flat surfaces and thus seem to conform with each other whether or not by explicit intent. Many machined objects seem nominally (in name) conform. But do they really conform?

Let us consider the case of two rigid objects pressed together at their flat surfaces. We can think of a rigid object as a limiting case of stiffer and stiffer objects; and we can think of flat surfaces as the limiting case of less and less rough surfaces. Now imagine pressing two objects together that are not quite flat and are also not quite rigid.

On the one hand, no matter how stiff the objects so long as they have a little compliance, if you made them flatter and flatter,

eventually the little bit of deformation from your pressing would make them conform and they would make contact along the contact surfaces (where the details of the pressure distribution still would depend on the shape of the bodies away from the contact area).

On the other hand, no matter how flat the contact surfaces (so long as they were not perfectly flat), if you made them stiffer and stiffer, the deformation would be extinguished and eventually they would only make contact at a few points (as in the figure above).

To get the idea considering two springs in parallel that have almost equal length. Consider the limits as the lengths become matched and as the stiffnesses go to infinity (see problem. ?? on page ??).

That is, the meaning of the phrase ‘flat and rigid’ depends on whether you first think of the objects as flat and then rigid, or first rigid and then flat. In math language this dependence on the order of limits is called a *distinguished* limit. Here it means that the idea of rigid objects touching on flat surfaces is ill-defined.

This distinguished limit is not a mathematical fine point. It corresponds to the physical reality that things which look flat and hard touch each other with a pressure distribution that is highly dependent on fine details of construction and loading.

In many mechanics problems one can, by means of the equations of elementary mechanics taught here, find an equivalent force system to that of the micro-contact force distribution. Using more advanced mechanics reasoning (the theory of elasticity) and computers (finite element programs) one can estimate certain features of the details of the contact pressure distribution if one knows the surface shapes accurately. But in many mechanics calculations the details of the contact force distribution are left unknown.

## Extended contact

When things touch each other over an extended region, like the block on the plane of fig. 3.21a, it is not clear what forces to put where on the free body diagram. On the one hand one imagines reality to be somewhat reflected by millions of small forces as in fig. 3.21b which may or may not be divided into normal ( $n_i$ ) and frictional ( $f_i$ ) components. But one generally is not interested in such detail, and even if interested one cannot find it easily (see box 3.1 on page 94).

A simple approach is to replace the detailed force distribution with a single equivalent force, as shown in fig. 3.21c broken into components. The location of this force is not relevant for some problems.<sup>①</sup>

If one wants to make clear that the contact forces serve to keep the block from rotating, one may replace the contact force distribution with a pair of contacts at the corners as in fig. 3.21d.

## Summary of free body diagrams.

- Draw one or more clear free body diagrams!
- Forces and moments on the free body diagram show *all* mechanical interactions.
- Every point on the boundary of a body has a force in every direction that motion is either being caused *or* prevented. Similarly with torques.
- If you do not know the direction of a force, use vector notation to show that the direction is yet to be determined.
- Leave off the free body diagram forces that you think are negligible such as, possibly:
  - The force of air on small slowly moving bodies;
  - Forces that prevent motion that is already prevented by a much stiffer means (as for the torques at each of a pair of hinges);
  - Non-collisional forces, such as gravity, during a collision.

<sup>①</sup> In 3D, contact force distributions cannot always be replaced with an equivalent force at an appropriate location (see section 2.5). A couple may be required. Nonetheless, many people often make the approximation that a contact force distribution can be replaced by a force at an appropriate location. This approximation neglects any frictional resistance to twisting about the normal to the contact plane.

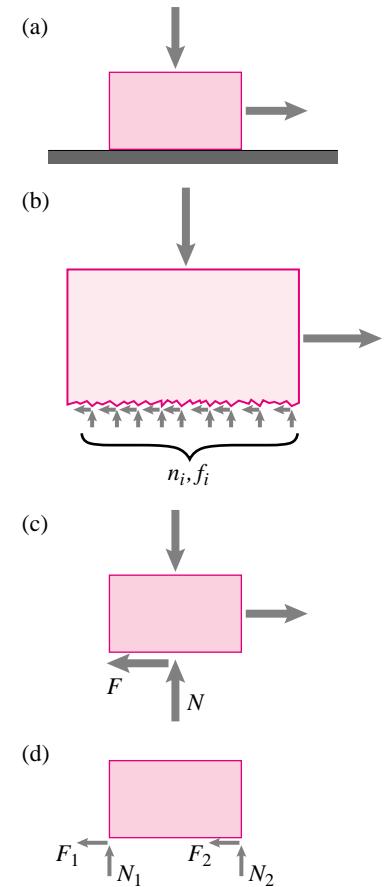
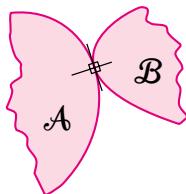


Figure 3.21: The contact forces of a block on a plane can be sensibly modeled in various ways.

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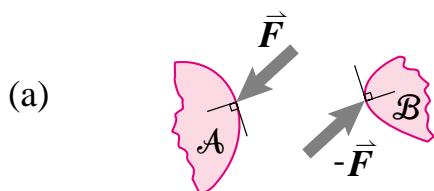
### 3.3 Action and reaction on partial FBD's of interacting bodies

Imagine bodies  $\mathcal{A}$  and  $\mathcal{B}$  are interacting and that you want to draw separate free body diagrams (FBD's) of each.

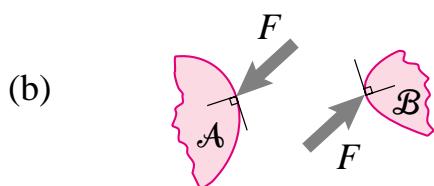


Part of the FBD of each shows the interaction force. The FBD of  $\mathcal{A}$  shows the force of  $\mathcal{B}$  on  $\mathcal{A}$  and the FBD of  $\mathcal{B}$  shows the force of  $\mathcal{A}$  on  $\mathcal{B}$ . To illustrate the concept, we show partial FBD's of both  $\mathcal{A}$  and  $\mathcal{B}$  using the principle of action and reaction. Items (a - d) are correct and items (e - g) are wrong.

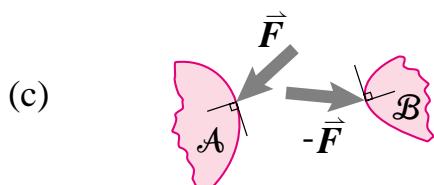
#### Correct partial FBD's



(a) Here are some good partial FBD's: the arrows are equal and opposite and the vector notations are opposite in sign.

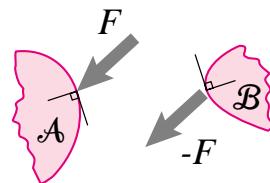


(b) These FBD's are also good since the opposite arrows multiplied by equal magnitude  $F$  produce net vectors that are equal and opposite.



(c) The FBD's may look wrong, and they are impractically misleading and not advised. But technically they are okay because we take the vector notation to have precedence over the drawing inaccuracy.

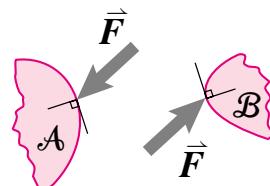
(d)



(d) The FBD's may look wrong but since no vector notation is used, the forces should be interpreted as in the direction of the drawn arrows and multiplied by the shown scalars. Since the same arrow is multiplied by  $F$  and  $-F$ , the net vectors are actually equal and opposite.

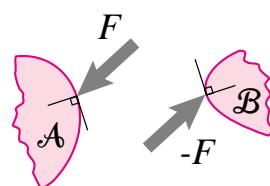
#### Wrong partial FBD's

(e)



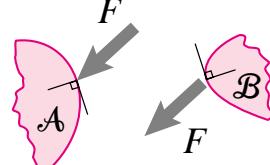
(e) The FBD's are wrong because the vector notation  $\vec{F}$  takes precedence over the drawn arrows. So the drawing shows the same force  $\vec{F}$  acting on both  $\mathcal{A}$  and  $\mathcal{B}$ , rather than the opposite force.

(f)



(f) Since the opposite arrow is multiplied by the negative scalars, the FBD's here show the same force acting on both  $\mathcal{A}$  and  $\mathcal{B}$ . Treating a double-negative as a negative is a common mistake.

(g)



(g) The FBD's are obviously wrong since they again show the same force acting on  $\mathcal{A}$  and  $\mathcal{B}$ . These FBD's would represent the principle of double action which applies to laundry detergents but not to mechanics.

**SAMPLE 3.1** A mass and a pulley. A block of mass  $m$  is held up by applying a force  $F$  through a massless pulley as shown in the figure. Assume the string to be massless. Draw free body diagrams of the mass and the pulley separately and as one system.

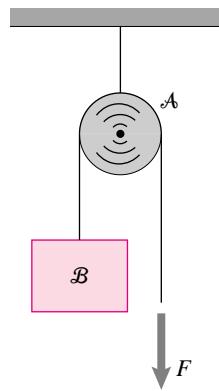


Figure 3.22: (Filename:sfig2.1.02)

**Solution** The free body diagrams of the block and the pulley are shown in Fig. 3.23. Since the string is massless and we assume an ideal massless pulley, the tension in the string is the same on both sides of the pulley. Therefore, the force applied by the string on the block is simply  $F$ . When the mass and the pulley are considered as one system, the force in the string on the left side of the pulley doesn't show because it is internal to the system.

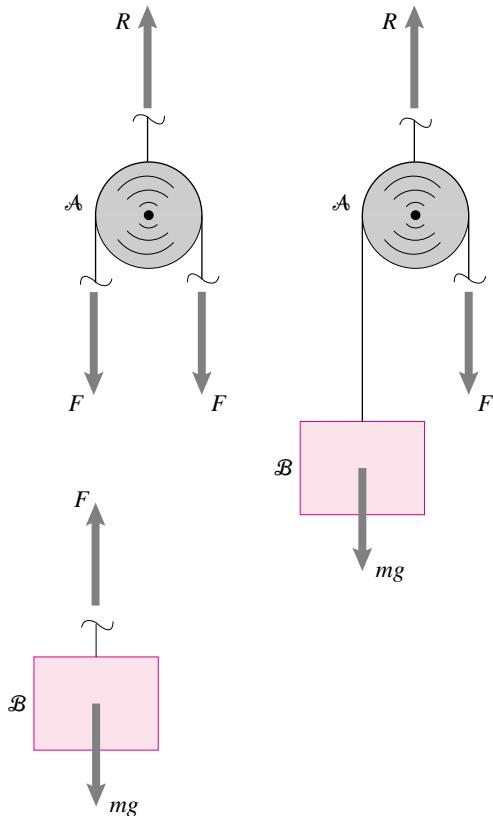


Figure 3.23: The free body diagrams of the mass, the pulley, and the mass-pulley system. Note that for the purpose of drawing the free body diagram we need not show that we know that  $R = 2F$ . Similarly, we could have chosen to show two different rope tensions on the sides of the pulley and reasoned that they are equal as is done in the text.

(Filename:sfig2.1.02a)

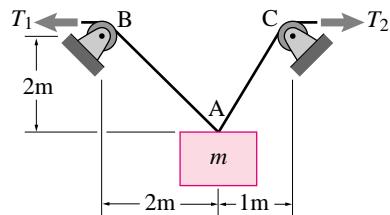
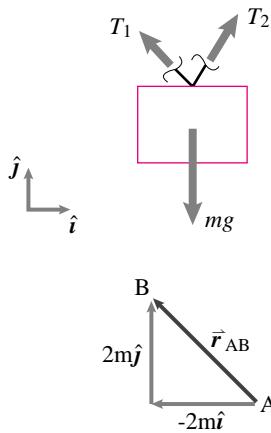


Figure 3.24: (Filename:sfig2.1.2a)

**SAMPLE 3.2 Forces in strings.** A block of mass  $m$  is held in position by strings  $AB$  and  $AC$  as shown in Fig. 3.24. Draw a free body diagram of the block and write the vector sum of all the forces shown on the diagram. Use a suitable coordinate system.

Figure 3.25: Free body diagram of the block and a diagram of the vector  $\vec{r}_{AB}$ .  
(Filename:sfig2.1.2b)

**Solution** To draw a free body diagram of the block, we first *free* the block. We cut strings  $AB$  and  $AC$  very close to point  $A$  and show the forces applied by the cut strings on the block. We also isolate the block from the earth and show the force due to gravity. (See Fig. 3.25.)

To write the vector sum of all the forces, we need to write the forces as vectors. To write these vectors, we first choose an  $xy$  coordinate system with basis vectors  $\hat{i}$  and  $\hat{j}$  as shown in Fig. 3.25. Then, we express each force as a product of its magnitude and a unit vector in the direction of the force. So,

$$\vec{T}_1 = T_1 \hat{\lambda}_{AB} = T_1 \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|},$$

where  $\vec{r}_{AB}$  is a vector from  $A$  to  $B$  and  $|\vec{r}_{AB}|$  is its magnitude. From the given geometry,

$$\begin{aligned} \vec{r}_{AB} &= -2m\hat{i} + 2m\hat{j} \\ \Rightarrow \hat{\lambda}_{AB} &= \frac{2m(-\hat{i} + \hat{j})}{\sqrt{2^2 + 2^2}} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}). \end{aligned}$$

Thus,

$$\vec{T}_1 = T_1 \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}).$$

Similarly,

$$\begin{aligned} \vec{T}_2 &= T_2 \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j}) \\ m\vec{g} &= -mg\hat{j}. \end{aligned}$$

Now, we write the sum of all the forces:

$$\begin{aligned} \sum \vec{F} &= \vec{T}_1 + \vec{T}_2 + m\vec{g} \\ &= \left( -\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}} \right) \hat{i} + \left( \frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg \right) \hat{j}. \end{aligned}$$

$$\boxed{\sum \vec{F} = \left( -\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}} \right) \hat{i} + \left( \frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg \right) \hat{j}}$$

**SAMPLE 3.3 Two bodies connected by a massless spring.** Two carts A and B are connected by a massless spring. The carts are pulled to the left with a force  $F$  and to the right with a force  $T$  as shown in Fig. 3.26. Assume the wheels of the carts to be massless and frictionless. Draw free body diagrams of

- cart A,
- cart B, and
- carts A and B together.

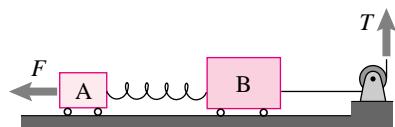
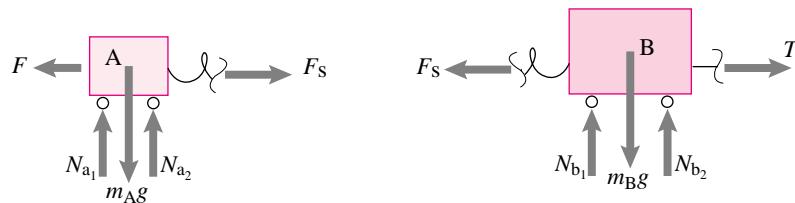


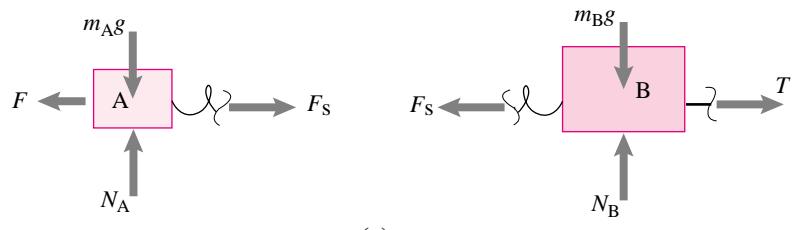
Figure 3.26: Two carts connected by a massless spring  
(Filename:sfig2.1.3a)

**Solution** The three free body diagrams are shown in Fig. 3.27 (a) and (b). In Fig. 3.27 (a) the force  $F_s$  is applied by the spring on the two carts. Why is this force the same on both carts? In Fig. 3.27(b) the spring is a part of the system. Therefore, the forces applied by the spring on the carts and the forces applied by the carts on the spring are internal to the system. Therefore these forces do not show on the free body diagram.

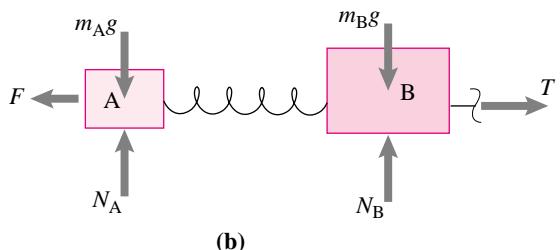
Note that the normal reaction of the ground can be shown either as separate forces on the two wheels of each cart or as a resultant reaction.



[or]



(a)



(b)

Figure 3.27: Free body diagrams of (a) cart A and cart B separately and (b) cart A and B together  
(Filename:sfig2.1.3b)

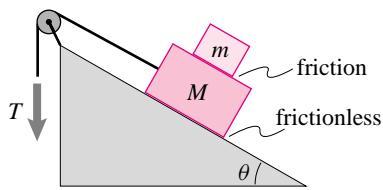


Figure 3.28: Two blocks held in place on a frictionless inclined surface  
 (Filename:sfig2.1.4a)

**SAMPLE 3.4** *Stacked blocks at rest on an inclined plane.* Blocks  $A$  and  $B$  with masses  $m$  and  $M$ , respectively, rest on a frictionless inclined surface with the help of force  $T$  as shown in Fig. 3.28. There is friction between the two blocks. Draw free body diagrams of each of the two blocks separately and a free body diagram of the two blocks as one system.

**Solution** The three free body diagrams are shown in Fig. 3.29 (a) and (b). Note the action and reaction pairs between the two blocks; the normal force  $N_A$  and the friction force  $F_f$  between the two bodies  $A$  and  $B$ . If we consider the two blocks together as a system, then the forces  $N_A$  and  $F_f$  do not show on the free body diagram of the system (See Fig. 3.29(b)), because now they are internal to the system.

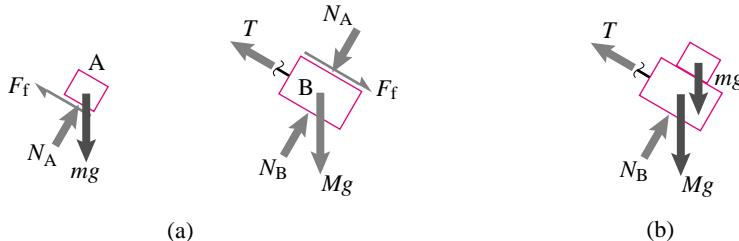


Figure 3.29: Free body diagrams of (a) block  $A$  and block  $B$  separately and (b) blocks  $A$  and  $B$  together.  
 (Filename:sfig2.1.4b)

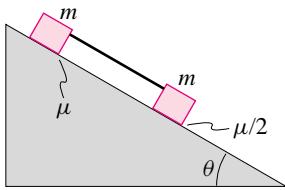


Figure 3.30: Two blocks slide down a frictional inclined plane. The blocks are connected by a light rigid rod.  
 (Filename:sfig2.1.15)

**SAMPLE 3.5** *Two blocks slide down a frictional inclined plane.* Two blocks of identical mass but different material properties are connected by a massless rigid rod. The system slides down an inclined plane which provides different friction to the two blocks. Draw free body diagrams of the two blocks separately and of the system (two blocks with the rod).

**Solution** The Free body diagrams are shown in Fig. 3.31. Note that the friction forces on the two blocks are different because the coefficients of friction are different for the two blocks. The normal reaction of the plane, however, is the same for each block (why?).

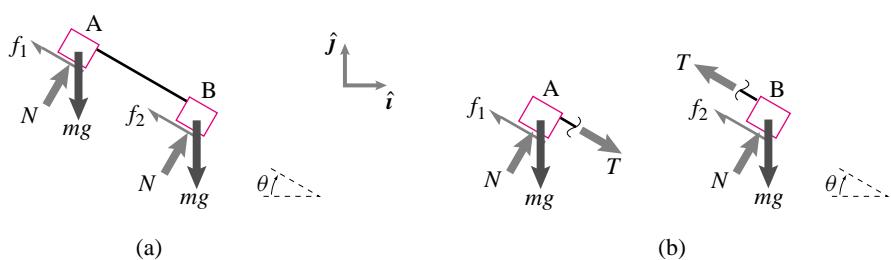


Figure 3.31: Free body diagrams of (a) the two blocks and the rod as a system and (b) the two blocks separately.  
 (Filename:sfig2.1.15a)

**SAMPLE 3.6** *Massless pulleys.* A force  $F$  is applied to the pulley arrangement connected to the cart of mass  $m$  shown in Fig. 3.32. All the pulleys are massless and frictionless. The wheels of the cart are also massless but there is friction between the wheels and the horizontal surface. Draw a free body diagram of the cart, its wheels, and the two pulleys attached to the cart, all as one system.

**Solution** The free body diagram of the cart system is shown in Fig. 3.33. The force in each part of the string is the same because it is the same string that passes over all the pulleys.

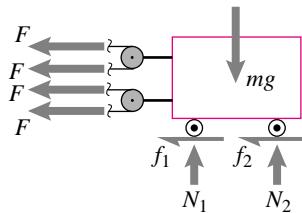


Figure 3.32: A cart with pulleys. (Filename:sfig2.1.7a)

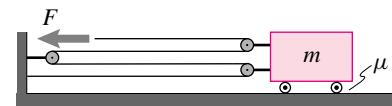


Figure 3.33: Free body diagram of the cart. (Filename:sfig2.1.7b)

**SAMPLE 3.7** *Two carts connected by pulleys.* The two masses shown in Fig. 3.34 have frictionless bases and round frictionless pulleys. The inextensible massless cord connecting them is always taut. Mass A is pulled to the left by force  $F$  and mass B is pulled to the right by force  $P$  as shown in the figure. Draw free body diagrams of each mass.

**Solution** Let the tension in the cord be  $T$ . Since the pulleys and the cord are massless, the tension is the same in each section of the cord. This equality is clearly shown in the Free body diagrams of the two masses below.

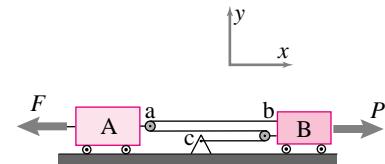


Figure 3.34: Two carts connected by massless pulleys. (Filename:sfig2.1.12)

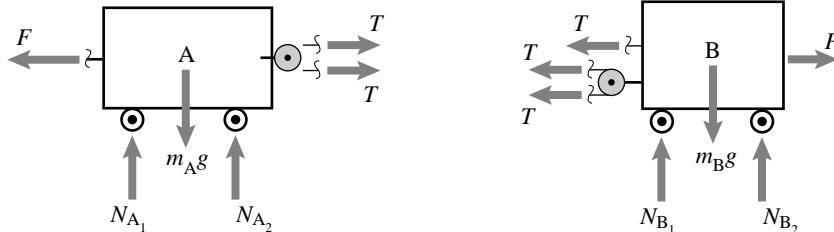


Figure 3.35: Free body diagrams of the two masses. (Filename:sfig2.1.12a)

**Comments:** We have shown unequal normal reactions on the wheels of mass B. In fact, the two reactions would be equal only if the forces applied by the cord on mass B satisfy a particular condition. Can you see what condition they must satisfy for, say,  $N_{A1} = N_{A2}$ . [Hint: think about the moment of forces about the center of mass A.]

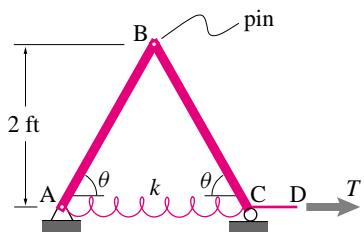


Figure 3.36: (Filename:sfig2.1.5)

**SAMPLE 3.8 Structures with pin connections.** A horizontal force  $T$  is applied on the structure shown in the figure. The structure has pin connections at A and B and a roller support at C. Bars AB and BC are rigid. Draw free body diagrams of each bar and the structure including the spring.

**Solution** The free body diagrams are shown in figure 3.37. Note that there are both vertical and horizontal forces at the pin connections because pins restrict translation in any direction. At the roller support at point C there is only vertical force from the support ( $T$  is an externally applied force).

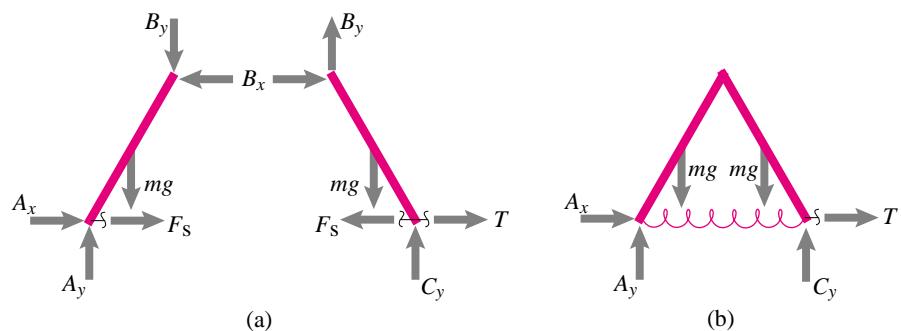


Figure 3.37: Free body diagrams of (a) the individual bars and (b) the structure as a whole.

(Filename:sfig2.1.5a)

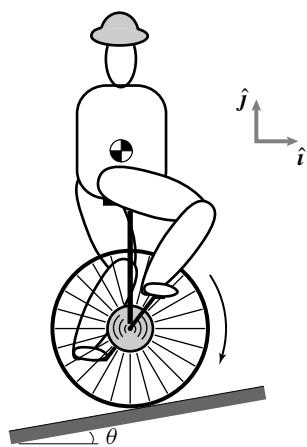


Figure 3.38: The unicyclist

(Filename:sfig2.1.8)

**SAMPLE 3.9 A unicyclist in action.** A unicyclist weighing 160 lbs exerts a force on the front pedal with a vertical component of 30 lbf at the instant shown in figure 3.38. The rear pedal barely touches the other foot. Assume the wheel and the frame are massless. Draw free body diagrams of the cyclist and the cycle. Make other reasonable assumptions if required.

**Solution** Let us assume, there is friction between the seat and the cyclist and between the pedal and the cyclist's foot. Let's also assume a 2-D analysis. The free body diagrams of the cyclist and the cycle are shown in Fig. 3.39. We assume no couple interaction at the seat.

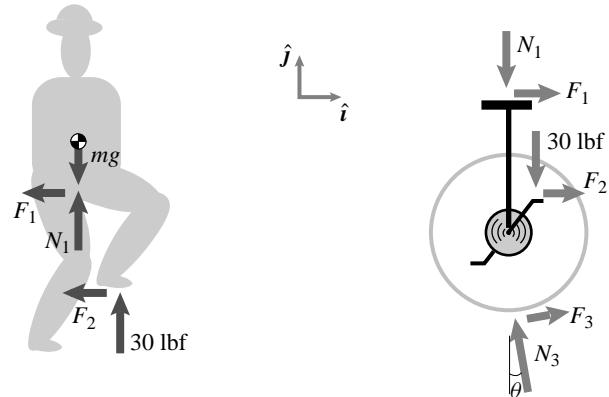


Figure 3.39: Free body diagram of the cyclist and the cycle. (Filename:sfig2.1.8a)



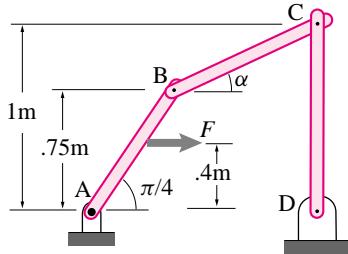


Figure 3.40: A four bar linkage.

(Filename:sfig2.2.1)

**SAMPLE 3.10** The four bar linkage shown in the figure is pushed to the right with a force  $F$  as shown in the figure. Pins A, C & D are frictionless but joint B is rusty and has friction. Neglect gravity; and assume that bar AB is massless. Draw free body diagrams of each of the bars separately and of the whole structure. Use consistent notation for the interaction forces and moments. Clearly mark the action-reaction pairs.

**Solution** A ‘good’ pin resists any translation of the pinned body, but allows free rotation of the body about an axis through the pin. The body reacts with an equal and opposite force on the pin. When two bodies are connected by a pin, the pin exerts separate forces on the two bodies. Ideally, in the free body diagram, we should show the pin, the first body, and the second body separately and draw the interaction forces. This process, however, results in too many free body diagrams. Therefore, usually, we let the pin be a part of one of the objects and draw the free body diagrams of the two objects.

Note that the pin at joint B is rusty, which means, it will resist a relative rotation of the two bars. Therefore, we show a moment, in addition to a force, at point B of each of the two rods AB and BC.

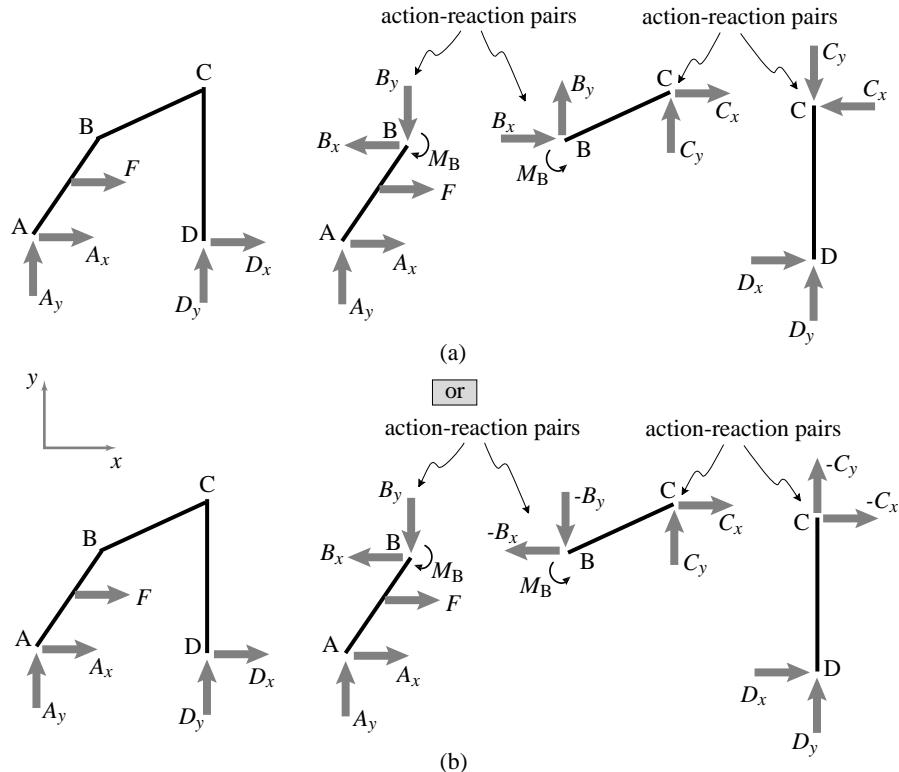


Figure 3.41: Style 1: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

(Filename:sfig2.2.1b)

Figure 3.41 shows the free body diagrams of the structure and the individual rods. In this figure, we show the forces in terms of their  $x$ - and  $y$ -components. The directions of the forces are shown by the arrows and the magnitude is labeled as  $A_x$ ,  $A_y$ , etc. Therefore, a force, shown as an arrow in the positive  $x$ -direction with ‘magnitude’  $A_x$ , is the same as that shown as an arrow in the negative  $x$ -direction with magnitude  $-A_x$ . Thus, the free body diagrams in Fig. 3.41(a) show exactly the

same forces as in Fig. 3.41(b).

In Fig. 3.42, we show the forces by an arrow in an arbitrary direction. The corresponding labels represent their magnitudes. The angles represent the unknown directions of the forces.

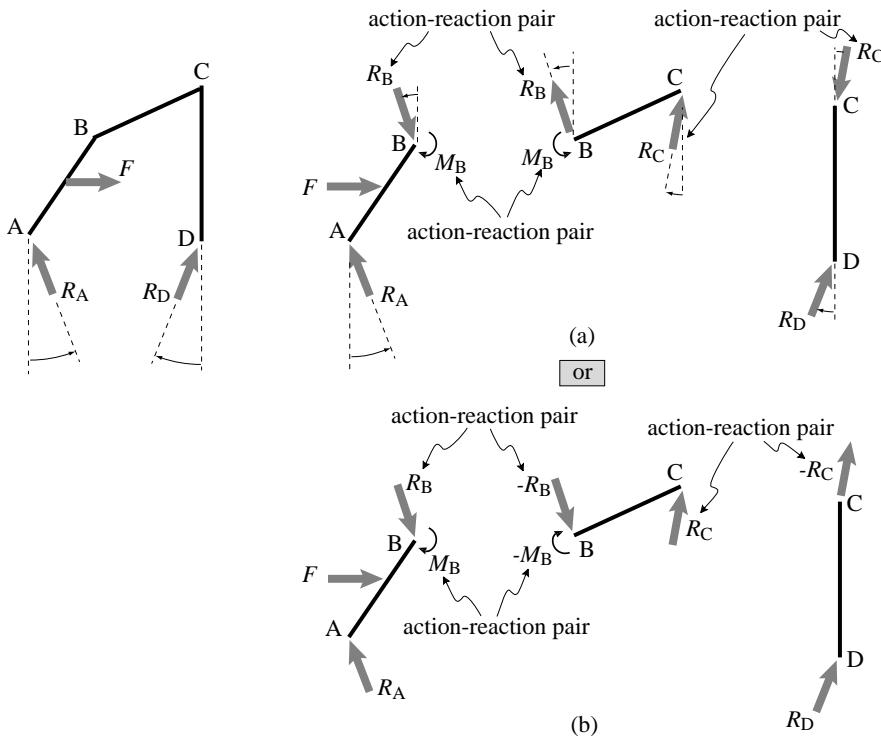


Figure 3.42: Style 2: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

(Filename:sfig2.2.1c)

In Fig. 3.43, we show yet another way of drawing and labeling the free body diagrams, where the forces are labeled as vectors.

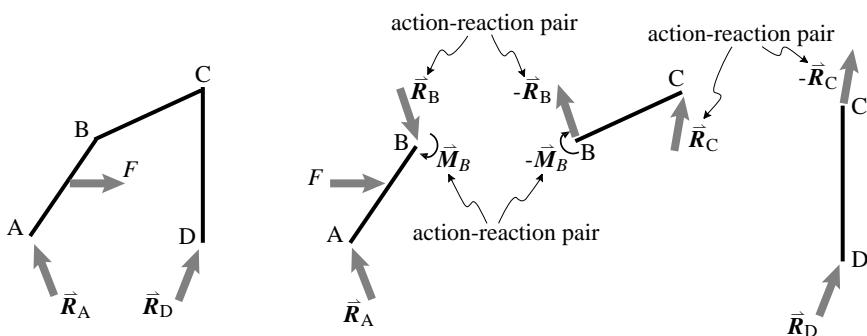


Figure 3.43: Style 3: Free body diagrams of the structure and the individual bars. The label of a force indicates both its magnitude and direction. The arrows are arbitrary and merely indicate that a force or a moment acts on those locations.

(Filename:sfig2.2.1d)

Note: There are *no* two-force bodies in this problem. Bar AB is massless but is not a two-force member because it has a couple at its end.



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# 4 Statics

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Statics is the mechanics of things that don't move. But everything does move, at least a little. So statics doesn't exactly apply to anything. The statics equations are, however, a very good approximation of the more general dynamics equations for many practical problems. The statics equations are also easier to manage than the dynamics equations. So with little loss of accuracy, sometimes very little loss, and a great saving of effort, sometimes a very great saving, many calculations can be performed using a statics model instead of a more general dynamics model. Thus it is not surprising that typical engineers perform many more statics calculations than dynamics calculations. Statics is the core of structural and strength analysis. And even for a moving system, say an accelerating car, statics calculations are appropriate for many of the parts. Simply put, and perhaps painful to remember when you complete this chapter and begin the chapters on dynamics, statics is more useful to most engineers than dynamics.

One possible motivation for studying statics is that the statics skills all carry over to dynamics which is a more general subject. But the opposite is maybe closer to truth. Statics is indeed a special case of dynamics. But for many engineers the benefit of going on from statics to dynamics is the sharpening of the more-useful statics skills that ensue.

## *How does general mechanics simplify to statics?*

The mechanics equations in the front cover are applicable to everything most engineers will ever encounter. The statics equations are a special case that apply only approximately to many things. In statics we set the right hand sides of equations I and II to zero. The neglected terms involve mass times acceleration and are called the *inertial terms*. For statics we set the inertial terms  $\dot{\mathbf{L}}$  and  $\dot{\mathbf{H}}_C$  to zero. Thus we replace

the linear and angular momentum balance equations with their simplified forms

$$\sum_{\text{All external forces}} \vec{F} = \vec{0} \quad \text{and} \quad \sum_{\text{All external torques}} \vec{M}_C = \vec{0} \quad (\text{Ic, IIc})$$

which are called the *force balance* and *moment balance* equations and together are called the *equilibrium* equations. The forces to be summed are those that show on a free body diagram of the system. The torques that are summed are those due to the same forces (by means of  $\vec{r}_{i/C} \times \vec{F}_i$ ) plus those due to any force systems that have been replaced with equivalent couples. If the forces on a system satisfy eqs. Ic and IIc the system is said to be in *static equilibrium* or just in equilibrium.

A system is in static equilibrium if the applied forces and moments add to zero.

Which can also be stated as

The forces on a system in static equilibrium, considered as a system, are equivalent to a zero force and a zero couple.

The approximating assumption that an object is in static equilibrium is that the forces mediated by an object are much larger than the forces needed to accelerate it. The statics equations are generally reasonably accurate for

- Things that a normal person would call “still” such as a building or bridge on a calm day, and a sleeping person; and
- Things that move slowly or with little acceleration, such as a tractor plowing a field or the arm of a person holding up a book while seated in a smooth-flying airplane; and for
- Parts that mediate the forces needed to accelerate more massive parts, such as gears in a transmission, the rear wheel of an accelerating bicycle, the strut in the landing gear of an airplane, and the individual structural members of a building swaying in an earthquake.

Quantitative estimation of the goodness of the statics approximation is not a statics problem, so we defer it until the chapters on dynamics.

## How is statics done?

The practice of statics involves:

- Drawing free body diagrams of the system of interest and of appropriate subsystems;
- writing equations Ic and IIc for each free body diagram; and
- using vector manipulation skills to solve for unknown features of the applied loads or geometry.

## The organization of this chapter

This whole chapter involves drawing free body diagrams and apply the force and moment balance equations. The chapter development is, roughly, the application of this procedure to more and more complex systems. We start with single bodies in the next key section. We then go on to the most useful examples of composite bodies, trusses. The relation between statics and the prediction of structural failure is next explained to be based on the concept of “internal” forces. Springs are ubiquitous in mechanics, so we devote a section to them. More difficult statics problems with composite bodies, mechanisms and frames, come next. Hydrostatics, useful for understanding the forces of water on a structure, is next. The final section serves as a cover for harder and three dimensional problems associated with all of the statics topics but has little new content.

Further statics skills will be developed later in the dynamics portion of the book. In particular, statics methods that depend on kinematics (work methods) are deferred.

## Two dimensional and three dimensional mechanics

The world we live in is three dimensional. So two dimensional models and equations are necessarily approximations. The theory of mechanics is a three dimensional theory that is simplified in two dimensional models. To appreciate the simplification one needs to understand 3D mechanics. But to understand 3D mechanics it is best to start practicing with 2D mechanics. Thus, until the last section of this chapter, we emphasize use of the two dimensional approximation and are intentionally casual about its precise meaning. We will think of cylinders and spheres as circles, of boxes as squares, and of cars as things with two wheels (one in front, one in back). In the last section on three dimensional statics we will look more closely at the meaning of the 2 dimensional approximation.

## 4.1 Static equilibrium of one body

A body is in static equilibrium if and only if the force balance and moment balance equations are hold:

$$\underbrace{\sum \vec{F} = \vec{0}}_{\substack{\text{All external forces} \\ \text{force balance}}} \quad \text{and} \quad \underbrace{\sum \vec{M}_C = \vec{0}}_{\substack{\text{All external torques} \\ \text{moment balance}}} \quad (\text{Ic,IIc})$$

for some point C. Is C a special point? No. Why? Because the statics equations say that the net force system is equivalent to a zero force and zero couple at C. We know from our study of equivalent force systems that this implies that the force system is equivalent to a zero force and zero couple at any and every point. So you can use any convenient point for the reference point in the moment balance equation.

**Example.** As you sit still reading, gravity is pulling you down and forces from the floor on your feet, the chair on your seat, and the table on your elbows hold you up. All of these forces add to zero. The net moment of these forces about the front-left corner of your desk adds to zero.  $\square$

In two dimensions the equilibrium equations make up 3 independent scalar equations (2 components of force, 1 of moment). In 3 dimensions the equilibrium equations make up 6 independent scalar equations (3 components of force and 3 components of moment).

We now proceed to consider a sequence of special loading situations. In principle you don't need to know any of them, force balance and moment balance spell out the whole statics story.

## Concurrent forces, equilibrium of a particle

The word particle usually means something small. In mechanics a *particle* is something whose spatial extent is ignored for one reason or another. If the ‘body’ in a free body diagram is a particle then all forces on it act at the same point, namely at the particle, and are said to be *concurrent* (see fig. 4.1). Force balance says that the forces add to zero. The moment balance equation adds no information because it is automatically satisfied (concurrent forces adding to zero have no moment about any point).

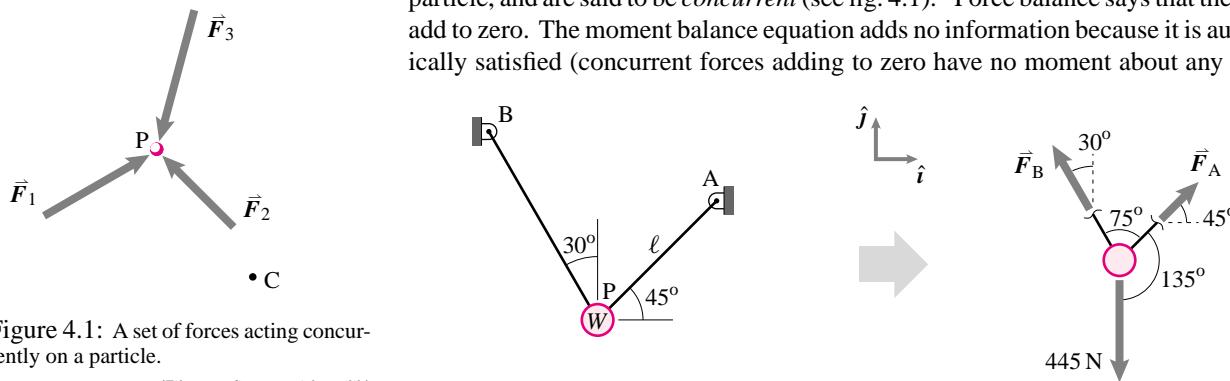


Figure 4.1: A set of forces acting concurrently on a particle.  
(Filename:tfigure.particleequilib)

**Example.** A 100 pound weight hangs from 2 lines. So

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad 445 \text{ N}(-\hat{j}) + F_A \frac{(\hat{i} + \hat{j})}{\sqrt{2}} + F_B \left( -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j} \right) = \vec{0}.$$

This can be solved any number of ways to get  $F_A = 230.3 \text{ N}$  and  $F_B = 325.8 \text{ N}$ .  $\square$

Although the moment balance equation has nothing to add in the case of concurrent forces, it can be used instead of force balance.

**Example.** Consider the same weight hanging from 2 strings. Moment balance about point A gives

$$\sum \vec{M}_A = \vec{0} \quad \Rightarrow \quad \vec{r}_{P/A} \times 445 \text{ N}(-\hat{j}) + \vec{r}_{P/A} \times F_B \left( -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j} \right) + \vec{0} = \vec{0}.$$

Evaluating the cross products one way or another one again gets  $F_B = 325.8 \text{ N}$ . Similarly moment balance about B could be used to find  $F_A = 230.3 \text{ N}$ .  $\square$

If we thought of moment balance first we could have solved this problem using moments and said the force balance had nothing to add. In either case, we only have two useful scalar equilibrium equations in 2D and 3 in 3D for concurrent force systems. The other equations are satisfied automatically because of the force concurrence.

## One-force body

Lets first dispose of the case of a “one-force” body. Consider a finite body with only one force acting on it. Assume it is in equilibrium. Force balance says that the sum of forces must be zero. So that force must be zero.

If only one force is acting on a body in equilibrium that force is zero.

That was too easy, but a count to 3 wouldn’t feel complete if it didn’t start at 1.

## Two-force body

When only two forces act on a system the situation is also simplified, though not so drastically as the case with one force. To determine the simplification, we apply the equilibrium equations of statics (Ic and IIc) to the body. Consider the free body diagram of a body  $\mathcal{B}$  in figure 4.2a. Forces  $\vec{F}_P$  and  $\vec{F}_Q$  are acting on  $\mathcal{B}$  at points  $P$  and  $Q$ . First, we have that the sum of all forces on the body are zero,

$$\begin{aligned} \sum_{\substack{\text{All external} \\ \text{forces}}} \vec{F} &= \vec{0} \\ \vec{F}_P + \vec{F}_Q &= \vec{0} \quad \Rightarrow \quad \vec{F}_P = -\vec{F}_Q. \end{aligned}$$

Thus, the two forces must be equal in magnitude and opposite in direction. So, thus far, we can conclude that the forces must be parallel as shown in figure 4.2b. But the forces still seem to have a net turning effect, thus still violating the concept of static equilibrium. The sum of all external torques on the body about any point are zero. So, summing moments about point  $P$ , we get,

$$\begin{aligned} \sum_{\substack{\text{All external} \\ \text{torques}}} \vec{M}_{/P} &= \vec{0} \\ \vec{r}_{Q/P} \times \vec{F}_Q &= \vec{0} \quad (\vec{F}_P \text{ produces no torque about } P.) \\ |\vec{r}_{Q/P}| (\hat{\lambda}_{Q/P} \times \vec{F}_Q) &= \vec{0} \quad (\hat{\lambda}_{Q/P} = \frac{\vec{r}_{Q/P}}{|\vec{r}_{Q/P}|} = -\frac{\vec{r}_{P/Q}}{|\vec{r}_{P/Q}|}) \end{aligned}$$

So  $\vec{F}_Q$  has to be parallel to the line connecting  $P$  and  $Q$ . Similarly, taking the sum of moments about point  $Q$ , we get

$$-\hat{\lambda}_{Q/P} \times \vec{F}_Q = \vec{0}$$

and  $\vec{F}_P$  also must be parallel to the line connecting  $P$  and  $Q$ . So, not only are  $\vec{F}_P$  and  $\vec{F}_Q$  equal and opposite, they are collinear as well since they are parallel to the axis passing through their points of action (see fig. 4.2c). Summarizing,

If a body in static equilibrium is acted on by two forces, then those forces are equal, opposite, and have a common line of action.

A body with only two forces acting on it is called a *two-force bodies* or *two-force member*. If you recognize a two-force body you can draw it in a free body diagram as in fig. 4.2c and the equations of force and moment balance applied to this body will provide no new information. This shortcut is sometimes useful for systems with several parts some of which are two-force members. Most often springs, dashpots, struts, and strings are idealized as two-force bodies as for bar BC in the example below.

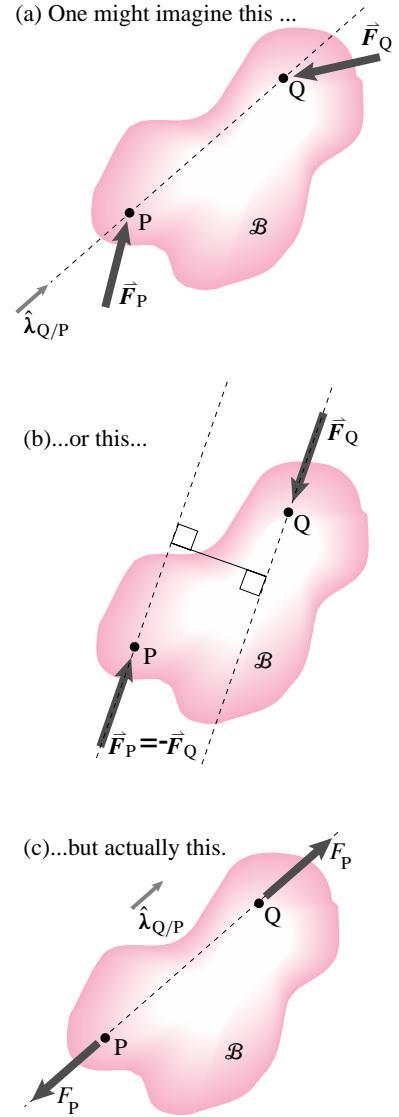
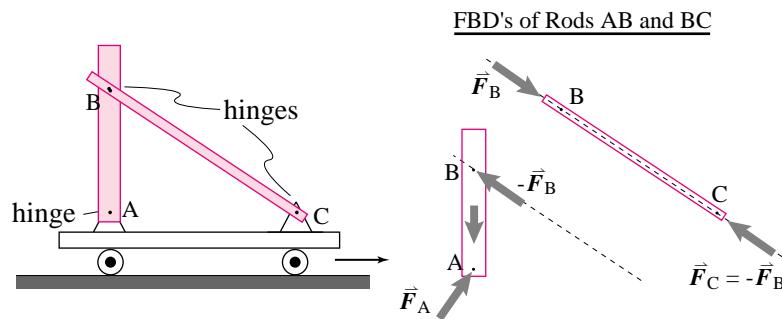


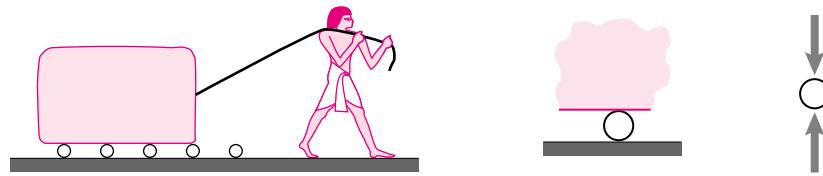
Figure 4.2: (a) Two forces acting on a body  $\mathcal{B}$ . (b) force balance implies that the forces are equal in magnitude and opposite in direction. (c) moment balance implies that the forces are collinear. Body  $\mathcal{B}$  is a two-force member; the two forces are equal, opposite, and collinear.

(Filename:tfigure2.two.force)

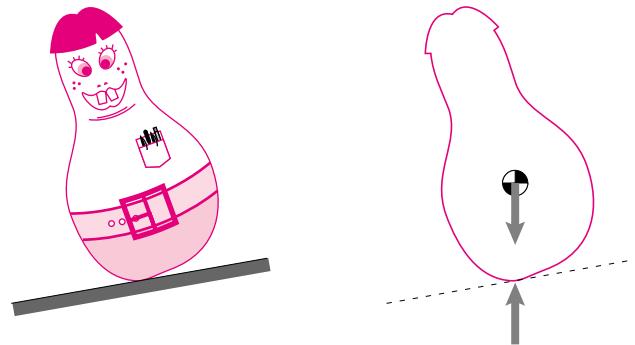
**Example: Tower and strut**

Consider an accelerating cart holding up massive tower  $AB$  which is pinned at  $A$  and braced by the light strut  $BC$ . The rod  $BC$  qualifies as a two-force member. The rod  $AB$  does not because it has three forces and is also not in static equilibrium (non-negligible accelerating mass). Thus, the free body diagram of rod  $BC$  shows the two equal and opposite colinear forces at each end parallel to the rod and the tower  $AB$  does not.

□

**Example: Logs as bearings**

Consider the ancient egyptian dragging a big stone. If the stone and ground are flat and rigid, and the log is round, rigid and much lighter than the stone we are led to the free body diagram of the log shown. *With these assumptions there can't be any resistance to rolling.* Note that this effectively frictionless rolling occurs no matter how big the friction coefficient between the contacting surfaces. That the egyptian got tired comes from logs, ground, stone, not being perfectly flat (or round) and rigid. (Also, it is tiring to keep replacing the logs in the front.) □

**Example: One point of support**

If an object with weight is supported at just one point, that point must be directly above or below the center of mass. Why? The gravity forces are

equivalent to a single force at the center of mass. The body is then a two force body. Since the direction of the gravity force is down, the support point and center of mass must be above one another. Similarly if a body is suspended from one point, the center of gravity must be directly above or below that point.  $\square$

## Three-force body

If a body in equilibrium has only three forces on it, again there is a general simplification that one can deduce from the general equations of statics

$$\sum_{\text{All external forces}} \vec{F} = \vec{0} \quad \text{and} \quad \sum_{\text{All external torques}} \vec{M}_C = \vec{0}.$$

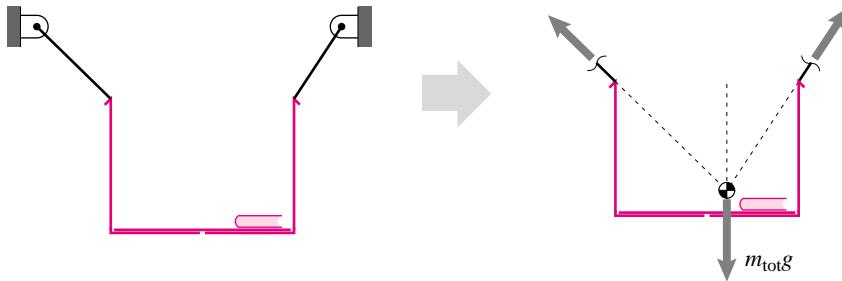
The simplification is not as great as for two-force bodies but is remarkably useful for more difficult statics problems. In box 4.1 on page 113 moment balance about various axes is used to prove that

*for a three-force body to be in equilibrium, the forces*

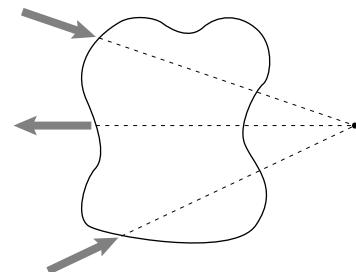
- (a) *must be coplanar, and*
- (b) *must either have lines of action which intersect at a single point, or the three forces are parallel.*

That is, one could imagine three random forces acting on a body. But, for equilibrium they must be coplanar and concurrent.

### Example: Hanging book box



2D



3D

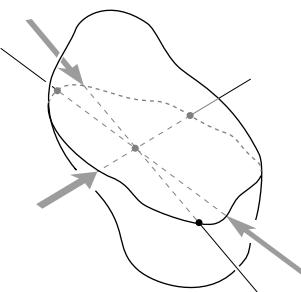


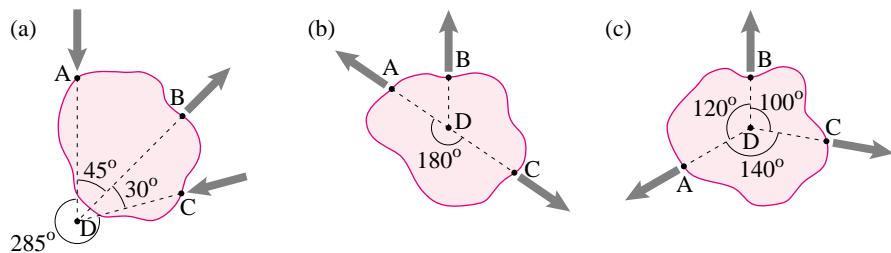
Figure 4.3: In a three-force body, the lines of action of the forces intersect at a single point and are coplanar. The point of intersection does not have to lie within the body.

(Filename:figure2.three.force)

## 4.1 THEORY Three-force bodies

Consider a body in static equilibrium with just three forces on it;  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$  acting at  $\vec{r}_1$ ,  $\vec{r}_2$ , and  $\vec{r}_3$ . Taking moment balance about the axis through  $\vec{r}_1$  and  $\vec{r}_2$ . So, the lines of action of all three forces are in the plane defined by the three points of action and the lines of action of  $\vec{F}_2$  and  $\vec{F}_3$  must intersect. Taking moment balance about this point of intersection implies that  $\vec{F}_1$  has line of action passing through the same point. (The exceptional case is when  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$  are parallel and have a common plane of action.)

A box with a book inside is hung by two strings so that it is in equilibrium on when level. The lines of action of the two strings must intersect directly above the center of mass of the box/book system.  $\square$



### Example: Which way do the forces go?

The maximum angle between pairs of forces can be (a) greater than, (b) equal to, or (c) less than  $180^\circ$ . In case (b) force balance in the direction perpendicular to line ADC shows that the odd force must be zero. In case (a) force balance perpendicular to the middle force implies that the outer two forces are both directed from D or both directed away from D. Force balance in the direction of the middle force shows that it has to have the opposite sense than the outer forces. If the others are pushing in then it is pulling away. If the outer forces are pulling away then it is pushing in. In case (c) application of force balance perpendicular to the force at C shows that the other two forces must both pull away towards D or both push in. Then force balance along C shows that all three forces must have the same sense. All three forces are pulling away from D or all three are pushing in.  $\square$

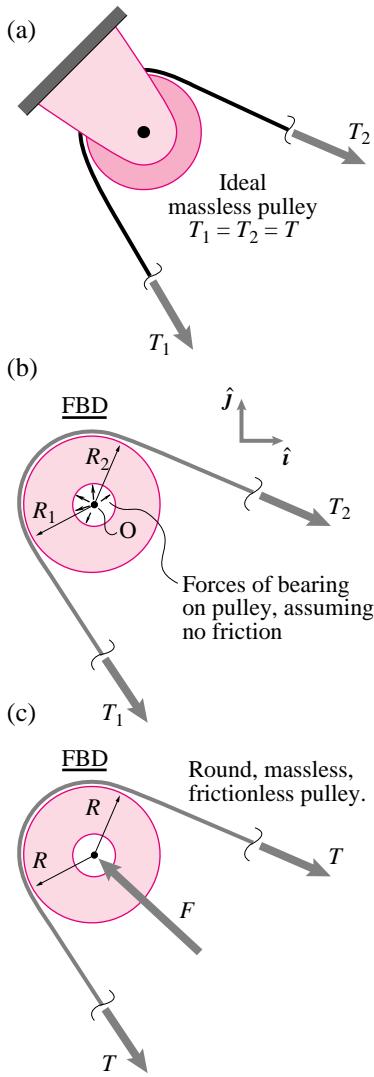


Figure 4.4: (a) An ideal massless pulley, (b) FBD of idealized massless pulley, detailing the frictionless bearing forces and showing forces at the cut strings, (c) final FBD after analysis.

(Filename:tfigure3.pulleytheory1)

### The idealized massless pulley

Both real machines and mechanical models are built of various building blocks. One of the standards is a pulley. We often draw pulleys schematically something like in figure 4.4a which shows that we believe that the tension in a string, line, cable, or rope that goes around an ideal pulley is the same on both sides,  $T_1 = T_2 = T$ . An ideal pulley is

- Round,
- Has frictionless bearings,
- Has negligible inertia, and
- Is wrapped with a line which only carries forces along its length.

We now show that these assumptions lead to the result that  $T_1 = T_2 = T$ . First, look at a free body diagram of the pulley with a little bit of string at both ends. Since we assume the bearing has no friction, the interaction between the pulley bearing shaft and the pulley has no component tangent to the bearing.

To find the relation between tensions, we apply angular momentum balance (equation II) about point O

$$\left\{ \sum \vec{M}_O = \hat{\vec{H}}_O \right\} \cdot \hat{k}. \quad (4.1)$$

Evaluating the left hand side of eqn. 4.1

$$\begin{aligned} \sum \vec{M}_O \cdot \hat{k} &= R_2 T_2 - R_1 T_1 + \underbrace{\text{bearing friction}}_0 \\ &= R(T_2 - T_1), \text{ since } R_1 = R_2 = R. \end{aligned}$$

Because there is no friction, the bearing forces acting perpendicular to the round bearing shaft have no moment about point O (see also the short example on page 71). Because the pulley is round,  $R_1 = R_2 = R$ .

When mass is negligible, dynamics reduces to statics because, for example, all the terms in the definition of angular momentum are multiplied by the mass of the system parts. So the right hand side of eqn. 4.1 reduces to  $\hat{\vec{H}}_O \cdot \hat{k} = 0$ .

Putting these assumptions and results together gives

$$\begin{aligned}\left\{ \sum \vec{M}_O = \hat{\vec{H}}_O \right\} \cdot \hat{k} \\ \Rightarrow R(T_2 - T_1) = 0 \\ \Rightarrow T_1 = T_2\end{aligned}$$

Thus, the tensions on the two lines of an ideal massless pulley are equal.

Lopsided pulleys are not often encountered, so it is usually satisfactory to assume round pulleys. But, in engineering practice, the assumption of frictionless bearings is often suspect. In dynamics, you may not want to neglect pulley mass.

### Lack of equilibrium as a sign of dynamics

Surprisingly, statics calculations often give useful information about dynamics. If, in a given problem, you find that forces cannot be balanced this is a sign that the related physical system will accelerate in the direction of imbalance. If you find that moments cannot be balanced, this is a sign of rotational acceleration in the physical system. The first example ('block on ramp') in the next subsection illustrates the point.

### Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For statics problems one would like to specify the object(s) the forces on the them, the nature of the interactions and then just write the force balance and moment balance equations and be sure that the solution follows by solving the equations.

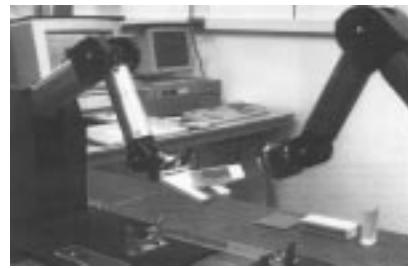
For better and worse, things are not always this simple. For better because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse because it means people have to think hard to do mechanics problems.

Many mechanics problems do have a solution, just one, that follows from the governing equations. But some reasonable looking problems have no solutions. And some problems have multiple solutions. When these mathematical anomalies arise, they usually have some physical importance. Even for problems with one solution, the route to finding that solution can involve more than simple equation manipulation.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

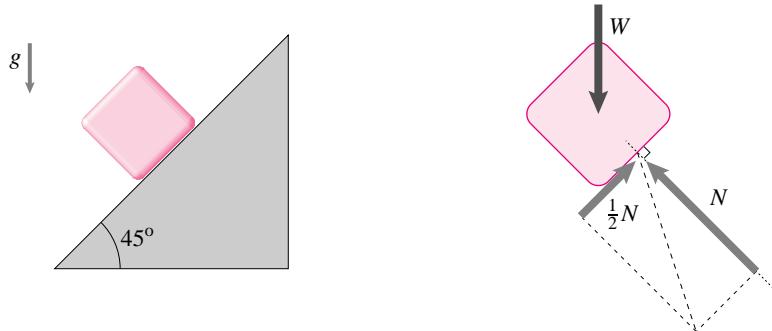
- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude  $\mu N$  *if* there is slip, *otherwise* the force of friction is something less than  $\mu N$  in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked.

*Example: Robot hand*

Roboticist Michael Erdmann has designed a palm manipulator that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one the other or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact.  $\square$

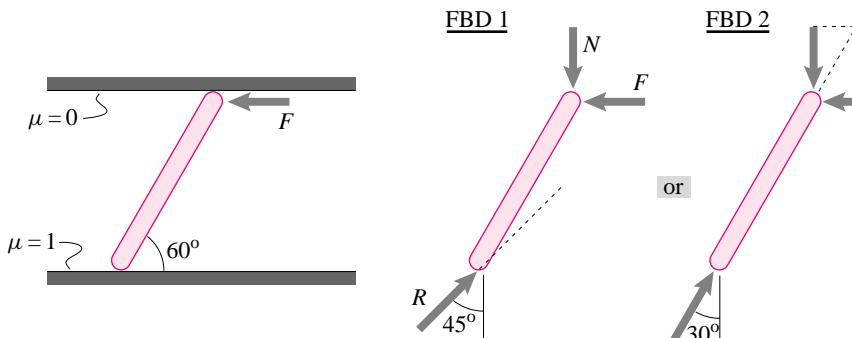
Sometimes there is no statics solution as the following simple example shows.

*Example: Block on ramp.*

A block with coefficient of friction  $\mu = .5$  is in static equilibrium sliding steadily down a  $45^\circ$  ramp. Not! The two forces in the free body diagram cannot add to zero (since they are not parallel). The assumptions are not consistent. They lead to a contradiction. Given the geometry and friction coefficient one could say that the assumption of equilibrium was inconsistent (and actually the block accelerates down the ramp). If equilibrium is demanded, say you saw the block just sitting there, then you can pin the contradiction on a mis-measured slope or a mis-estimated coefficient of friction.  $\square$

The following problem shows a case where a statics problem has multiple solutions.

**Example: Rod pushed in a channel.**



A light rod is just long enough to make a  $60^\circ$  angle with the walls of a channel. One channel wall is frictionless and the other has  $\mu = 1$ . What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free body diagram. The forces shown can be in equilibrium if all the forces are zero. Thus we have the solution that the rod slides in equilibrium with no force. If we assume that the block is not sliding the friction force on the lower wall can be at any angle between  $\pm 45^\circ$ . Thus we have equilibrium with the second FBD for arbitrary positive  $F$ . This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold arbitrary force  $F$  without slipping. That we have found freely slipping solutions with no force and jammed solutions with arbitrary force corresponds physically to one being able to easily slide a rod like this down a slot and then have it totally jam. Some rock-climbing equipment depends on such self-locking and easy release.  $\square$

One might not at first thing of string connections as being a form of contact, but the whether a string is taught or not is the same as whether contact is made with a frictionless spherical wall or not.

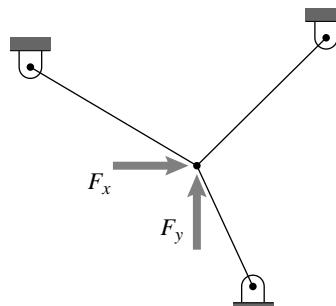
**Example: Particle held by two strings.**



Two inextensible strings are slightly slack when no load is applied to the knot in the middle. When a load is applied what is the tension in the strings? Force balance along the strings gives us one equation for the two unknown tensions. There are many solutions. There are even solutions where both tensions are positive. But geometry does not allow both of the strings to be at full length simultaneously. Thus we have to assume one of the strings has no tension when applying force balance. If we pick the wrong string we will get the contradiction that its tension is negative.  $\square$

The triviality of this example perhaps hides the problem, so here it is again with three strings.

*Example: Particle held by three strings.*



Three inextensible strings are just slightly slack when no load is applied to the knot in the middle. When a load is applied what is the tension in the strings? Planar force balance gives us two equations for the 3 unknown tensions. These equations have many solutions, even some with positive tension in all three strings. But geometry does not allow all three strings to be at their extended lengths simultaneously. So at least one string has to be slack and have no tension. If you guess the right one you will find positive tension in the other two strings. If you guess the wrong one you will get the contradiction that one of the strings has negative tension.  $\square$

If this example still seems too easy to demonstrate that sometimes you have to think about which of two or more conditionals needs to be enforced, try a case with four strings in three dimensions.

These examples, and one could construct many more, show that you have to look out for static equilibrium being not consistent with other information given. This contradiction could arise in an ill-posed problem, a problem that is really a dynamics problem, or as you eliminate possibilities that a given well-posed statics problem superficially allows.

## The general case

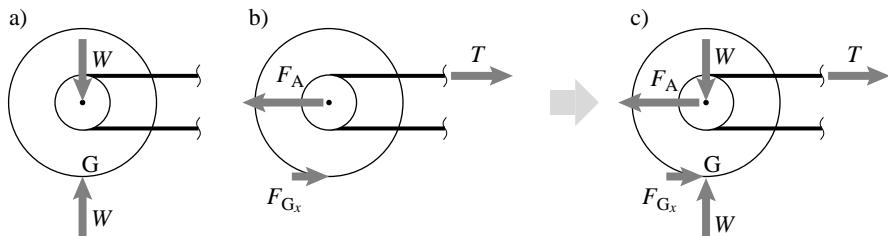
For one body, whether in 1D, 2D or 3D the equations of equilibrium are:

$$\underbrace{\sum_{\text{All external forces}} \vec{F} = \vec{0}}_{\text{force balance}} \quad \text{and} \quad \underbrace{\sum_{\text{All external torques}} \vec{M}_C = \vec{0}}_{\text{moment balance}} \quad (\text{Ic, IIc})$$

Solving a statics problem means using these equations, along with any available information about the forces involved, to find various unknowns. For some problems, the various tricks involving one-force, two-force, and three-force bodies can serve as a time saver for solving these equations and can help build your intuition. For some contact problems you may have to try various cases. But ultimately, always, statics means applying the force balance and moment balance equations.

## Linearity and superposition

For a given geometry the equilibrium equations are *linear*. That is: If for a given object you know a set of forces that is in equilibrium and you also know a second set of forces that is in equilibrium, then the sum of the two sets is also in equilibrium.

**Example: A bicycle wheel**

The free body diagram of an ideal massless bicycle wheel with a vertical load is shown in (a) above. The same wheel driven by a chain tension but with no weight is shown in equilibrium in (b) above. The sum of these two load sets (c) is therefore in equilibrium.  $\square$

The idea that you can add two solutions to a set of equations is called the *principle of superposition*, sometimes called the principle of superimposition<sup>①</sup>. The principle of superposition provides a useful shortcut for some mechanics problems.

<sup>①</sup> Here's a pun to help you remember the idea. When talkative Sam comes over you get bored. When hungry Sally comes over you reluctantly go get a snack for her. When Sam and Sally come over together you get bored *and* reluctantly go get a snack. Each one of them is imposing. By the principle of superimposition their effects add when they are together and they are super imposing.

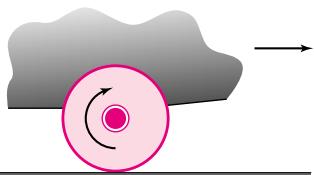
## 4.2 Wheels and two force bodies

One often hears whimsical reverence for the “invention of the wheel.” Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

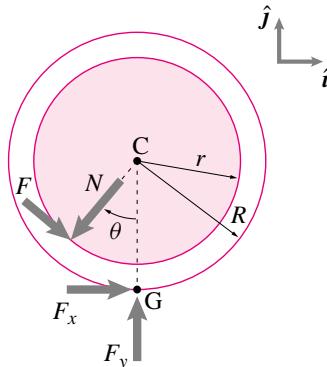
Without a wheel the force it takes to drag something is about  $\mu W$ . Since  $\mu$  ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 112). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again.

The simplest wheel design uses a dry “journal” bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



To figure out the forces involved we draw a free body diagram of the wheel. We neglect the wheels weight because it is generally much smaller than the forces it mediates. To make the situation clear the picture shows too-large a bearing hole  $r$ .



The force of the axle on the wheel has a normal component  $N$  and a frictional component  $F$ . The force of the ground on the wheel has a part holding the cart up  $F_y$  and a part along the ground  $F_x$  which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known and also the vertical part of the ground reaction force  $F_y$  we have as unknowns  $N$ ,  $F$ ,  $\theta$  and  $F_x$ . To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

$$F_x \hat{i} + F_y \hat{j} + N(-\sin \theta \hat{i} - \cos \theta \hat{j}) + F(\cos \theta \hat{i} - \sin \theta \hat{j}) = \vec{0},$$

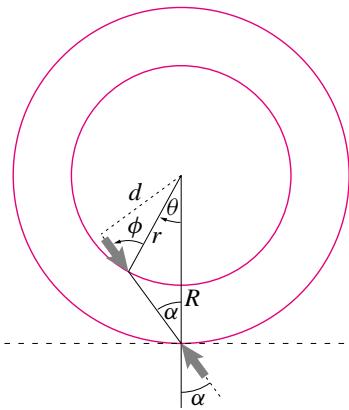
which could be reduced to 2 scalar equations by taking components or dot products; and moment balance which is easiest to see in terms of forces and perpendicular distances as

$$Fr + F_x R = 0.$$

Of key interest is finding the force resisting motion  $F_x$ . With a little mathematical manipulation we could solve the 4 scalar equations

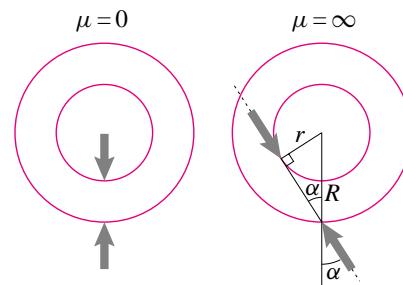
above for any of  $F_x$ ,  $N$ ,  $F$ , and  $\theta$  in terms of  $r$ ,  $R$ ,  $F_y$ , and  $\mu$ . We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free body diagram shows equal and opposite colinear forces at the two contact points.



The friction angle  $\phi$  describes the friction between the axle and wheel (with  $\tan \phi = \mu$ ). The angle  $\alpha$  describes the *effective friction* of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is  $\mu_{\text{eff}} = \tan \alpha$ . (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground  $\mu_{\text{eff}} = \mu$ .)

Lets consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient  $\mu \rightarrow \infty$  and  $\phi \rightarrow 90^\circ$ .



In the case that the wheel bearing has no friction we satisfactorily see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin \alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then  $\sin \alpha$  is less than .1 no matter how bad the bearing material. For such small values we can make the approximation  $\mu_{\text{eff}} = \tan \alpha \approx \sin \alpha$  so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

The genius of the wheel design is that it makes the effective friction less than  $r/R$  no matter how bad the bearing friction.

Going back to the two-force body free body diagram we can see that

$$\begin{aligned} d &= d \\ \Rightarrow r \sin \phi &= R \sin \alpha \\ \Rightarrow \sin \alpha &= \frac{r}{R} \sin \phi. \quad (*) \end{aligned}$$

From this formula we can extract the limiting cases discussed previously ( $\phi = 0$  and  $\phi = 90^\circ$ ). We can also plug in the small angle approximations ( $\sin \alpha \approx \tan \alpha$  and  $\sin \phi \approx \tan \phi$ ) if the friction coefficient is low to get

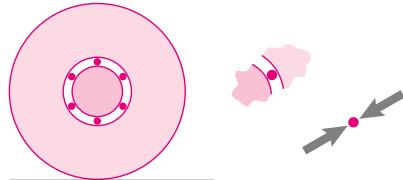
$$\mu_{\text{eff}} \approx \mu \frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity  $\sin = \sqrt{1 + \tan^{-2}}^{-1}$  to solve the exact equation (\*) for

$$\mu_{\text{eff}} = \mu \frac{r}{R} \left( \frac{1}{\sqrt{1 + \mu^2(1 - r^2/R^2)}} \right),$$

where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



Each ball is a two force body and thus only transmits radial loads. Its as if there were no friction on the bearing and we get a specific resistance of zero,  $\mu_{\text{eff}} = 0$ . Of course real ball bearings are not perfectly smooth or perfectly rigid, so its good to keep  $r/R$  small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.



**SAMPLE 4.1 Concurrent forces:** A block of mass  $m = 10\text{ kg}$  hangs from strings  $AB$  and  $AC$  in the vertical plane as shown in the figure. Find the tension in the strings.

**Solution** The free body diagram of the block is shown in figure 4.6. Since the block is at rest, the equation of force balance is

$$\begin{aligned} \sum \vec{F} &= \vec{0} \\ \text{or} \quad T_1 \hat{\lambda}_{AB} + T_2 \hat{\lambda}_{AC} - mg \hat{j} &= \vec{0}, \end{aligned} \quad (4.2)$$

where  $\hat{\lambda}_{AB}$  and  $\hat{\lambda}_{AC}$  are unit vectors in the  $AB$  and  $AC$  directions, respectively. From geometry,

$$\begin{aligned} \hat{\lambda}_{AB} &= \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{-2m\hat{i} + 2m\hat{j}}{2\sqrt{2}\text{ m}} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}) \\ \hat{\lambda}_{AC} &= \frac{\vec{r}_{AC}}{|\vec{r}_{AC}|} = \frac{1m\hat{i} + 2m\hat{j}}{\sqrt{5}\text{ m}} = \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j}) \end{aligned}$$

Dotting eqn. (4.2) with  $\hat{i}$  we get

$$\underbrace{T_1(\hat{\lambda}_{AB} \cdot \hat{i})}_{-1/\sqrt{2}} + \underbrace{T_2(\hat{\lambda}_{AC} \cdot \hat{i})}_{1/\sqrt{5}} = 0 \Rightarrow T_1 = \sqrt{\frac{2}{5}} T_2$$

Dotting eqn. (4.2) with  $\hat{j}$  and substituting  $T_1 = \sqrt{2/5} T_2$ , we get

$$\begin{aligned} \underbrace{\sqrt{\frac{2}{5}} T_2(\hat{\lambda}_{AB} \cdot \hat{j})}_{T_1} + \underbrace{T_2(\hat{\lambda}_{AC} \cdot \hat{j})}_{2/\sqrt{5}} - mg &= 0 \\ \Rightarrow \frac{3}{\sqrt{5}} T_2 - mg &= 0 \Rightarrow T_2 = \frac{\sqrt{5}}{3} mg = 73.12\text{ N} \end{aligned}$$

Substituting in  $T_1 = \sqrt{\frac{2}{5}} T_2$ , we have  $T_1 = \sqrt{\frac{2}{5}} \cdot (73.12\text{ N}) = 46.24\text{ N}$

$$T_1 = 46.24\text{ N}, T_2 = 73.12\text{ N}$$

**Note:** We could also write eqn. (4.2) in matrix form and solve the matrix equation to find  $T_1$  and  $T_2$ . Substituting  $\hat{\lambda}_{AB}$  and  $\hat{\lambda}_{AC}$  in terms of  $\hat{i}$  and  $\hat{j}$  in eqn. (4.2) and dotting the resulting equation with  $\hat{i}$  and  $\hat{j}$ , we can write eqn. (4.2) as

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ mg \end{pmatrix} \Rightarrow \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Using Cramer's rule for the inverse of a matrix, we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = -\frac{\sqrt{10}}{3} \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 0 \\ mg \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{3} mg \\ \frac{\sqrt{5}}{3} mg \end{pmatrix}$$

which is, of course, the same result as we got above.

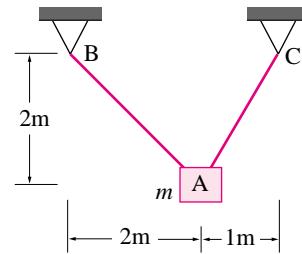


Figure 4.5: (Filename:sfig2.4new.1)

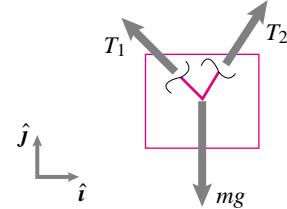


Figure 4.6: (Filename:sfig2.4new.1a)

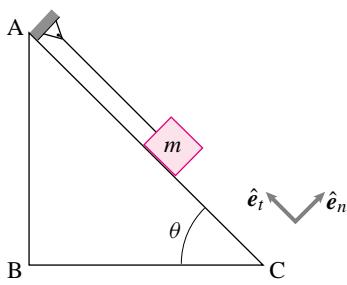


Figure 4.7: A mass-particle on an inclined plane.

(Filename:sfig2.1.11)

**SAMPLE 4.2** A small block of mass  $m$  rests on a frictionless inclined plane with the help of a string that connects the mass to a fixed support at A. Find the force in the string.

**Solution** The free body diagram of the mass is shown in Fig. 4.8. The string force  $F_s$  and the normal reaction of the plane  $N$  are unknown forces. To determine the

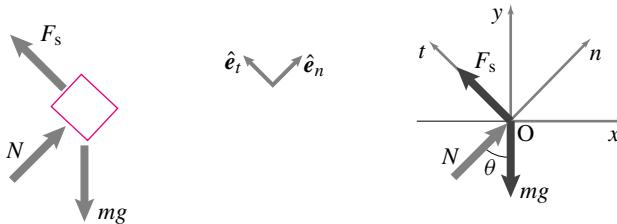


Figure 4.8: Free body diagram of the mass and the geometry of force vectors. (Filename:sfig2.1.11a)

unknown forces, we write the force balance equation,  $\sum \vec{F} = \vec{0}$ ,

$$\vec{F}_s + \vec{N} + m\vec{g} = \vec{0}$$

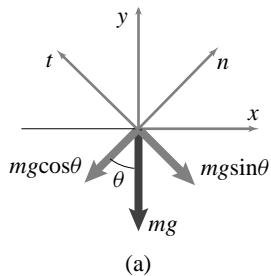
We can express the forces in terms of their components in various ways and then dot the vector equation with appropriate unit vectors to get two independent scalar equations. For example, let us draw two unit vectors  $\hat{e}_t$  and  $\hat{e}_n$  along and perpendicular to the plane. Now we write the force balance equation using mixed basis vectors  $\hat{e}_t$  and  $\hat{e}_n$ , and  $\hat{i}$  and  $\hat{j}$ :

$$F_s \hat{e}_t + N \hat{e}_n - mg \hat{j} = \vec{0} \quad (4.3)$$

We can now find  $F_s$  directly by taking the dot product of the above equation with  $\hat{e}_t$  since the other unknown  $N$  is in the  $\hat{e}_n$  direction and  $\hat{e}_n \cdot \hat{e}_t = 0$ :

$$[\text{eqn. (4.3)}] \cdot \hat{e}_t \Rightarrow F_s - mg (\hat{j} \cdot \hat{e}_t) = 0 \Rightarrow F_s = mg \sin \theta$$

$$F_s = mg \sin \theta$$



Note that we did not have to separate out two scalar equations and solve for  $F_s$  and  $N$  simultaneously. If we needed to find  $N$ , we could do that too from a single equation by taking the dot product of eqn. (4.3) with  $\hat{n}$ :

$$[\text{eqn. (4.3)}] \cdot \hat{e}_n \Rightarrow N - mg (\hat{j} \cdot \hat{e}_n) = 0 \Rightarrow N = mg \cos \theta$$

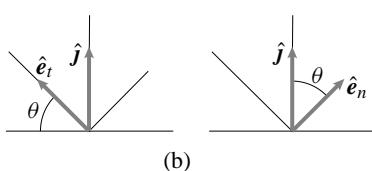


Figure 4.9: (a) Components of  $mg$  along  $t$  and  $n$  directions. (b) The mixed basis dot products:  $\hat{j} \cdot \hat{e}_t = \sin \theta$  and  $\hat{j} \cdot \hat{e}_n = \cos \theta$

(Filename:sfig2.1.11b)

**Writing direct scalar equations:** You are familiar with this method from your elementary physics courses. We resolve all forces into their components along the desired directions and then sum the forces. Here,  $F_s$  is along the plane and therefore, has no component perpendicular to the plane. Force  $N$  is perpendicular to the plane and therefore, has no component along the plane. We resolve the weight  $mg$  into two components: (1)  $mg \cos \theta$  perpendicular to the plane ( $n$  direction) and (2)  $mg \sin \theta$  along the plane ( $t$  direction). Now we can sum the forces:

$$\sum F_t = 0 \Rightarrow F_s - mg \sin \theta = 0; \quad \text{and} \quad \sum F_n = 0 \Rightarrow N - mg \cos \theta = 0$$

which, of course, is essentially the same as the equations obtained above.

**SAMPLE 4.3 A bar as a 2-force body:** A 4 ft long horizontal bar supports a load of 60 lbf at one of its ends. The other end is pinned to a wall. The bar is also supported by a string connected to the load-end of the bar and tied to the wall. Find the force in the bar and the tension in the string.

**Solution** Let us do this problem two ways — using equilibrium equations without much thought, and using those equations with some insight.

- (a) The free body diagram of the bar is shown in Fig. 4.11. The moment balance about point A,  $\sum M_A = 0$ , gives

$$\begin{aligned} \vec{r}_{C/A} \times T\hat{\lambda} + \vec{r}_{C/A} \times (-P\hat{j}) &= \vec{0} \\ \underbrace{\ell\hat{i} \times T(-\cos\theta\hat{i} + \sin\theta\hat{j})}_{\ell T \sin\theta\hat{k}} + \underbrace{\ell\hat{i} \times (-P\hat{j})}_{-\ell P\hat{k}} &= \vec{0} \\ (T\ell \sin\theta - P\ell)\hat{k} &= \vec{0} \quad (4.4) \\ [\text{eqn. (4.4)}] \cdot \hat{k} \Rightarrow T = \frac{P}{\sin\theta} = \frac{60 \text{ lbf}}{\frac{3}{5}} &= 100 \text{ lbf.} \end{aligned}$$

The force equilibrium,  $\sum \vec{F} = 0$ , gives

$$\begin{aligned} (A_x - T \cos\theta)\hat{i} + (A_y + T \sin\theta - P)\hat{j} &= \vec{0} \quad (4.5) \\ [\text{eqn. (4.5)}] \cdot \hat{i} \Rightarrow A_x = T \cos\theta = (100 \text{ lbf}) \cdot \frac{4}{5} &= 80 \text{ lbf} \\ [\text{eqn. (4.5)}] \cdot \hat{j} \Rightarrow A_y = P - T \sin\theta &= 0 \end{aligned}$$

where the last equation,  $A_y = P - T \sin\theta = 0$  follows from eqn. (4.4). Thus, the force in the rod is  $\vec{A} = 80 \text{ lbf}\hat{i}$ , i.e., a purely compressive force, and the tension in the string is 100 lbf.

$$\boxed{\vec{A} = 80 \text{ lbf}\hat{i}, \quad T = 100 \text{ lbf}}$$

- (b) From the free body diagram of the rod, we realize that the rod is a *two-force body*, since the forces act at only two points of the body, A and C. The reaction force at A is a single force  $\vec{A}$ , and the forces at end C, the tension  $\vec{T}$  and the load  $\vec{P}$ , sum up to a single net force, say  $\vec{F}$ . So, now using the fact that the rod is a two-force body, the equilibrium equation requires that  $\vec{F}$  and  $\vec{A}$  be equal, opposite, and colinear (along the longitudinal axis of the bar). Thus,

$$\vec{A} = -\vec{F} = -F\hat{i}.$$

Now,

$$\begin{aligned} \vec{F} &= \vec{P} + \vec{T} \\ -F\hat{i} &= -P\hat{j} + T \sin\theta\hat{j} - T \cos\theta\hat{i} \quad (4.6) \\ [\text{eqn. (4.6)}] \cdot \hat{j} \Rightarrow P &= T \sin\theta \\ \Rightarrow T &= \frac{P}{\sin\theta} = \frac{60 \text{ lbf}}{\frac{3}{5}} = 100 \text{ lbf} \\ [\text{eqn. (4.6)}] \cdot \hat{i} \Rightarrow F &= T \cos\theta = (100 \text{ lbf}) \cdot \frac{4}{5} = 80 \text{ lbf.} \end{aligned}$$

The answers, of course, are the same.

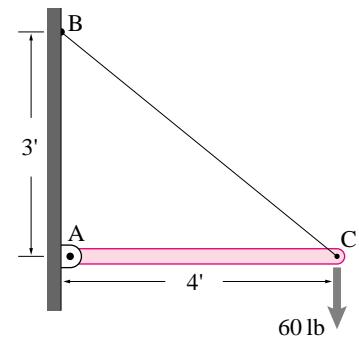


Figure 4.10: (Filename:sfig4.single.bar)

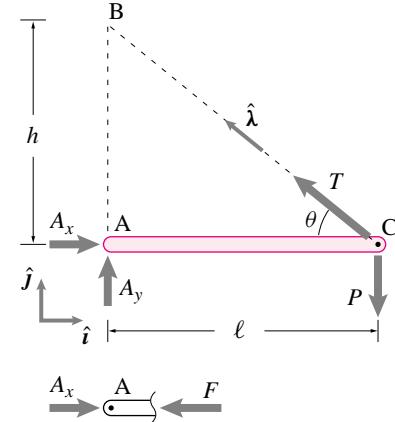


Figure 4.11: (Filename:sfig4.single.bar.a)

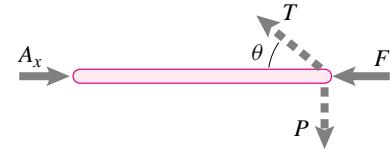


Figure 4.12: (Filename:sfig4.single.bar.b)

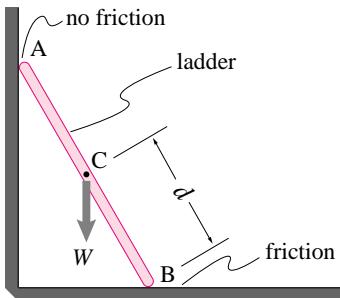


Figure 4.13: (Filename:sfig4.single.ladder)

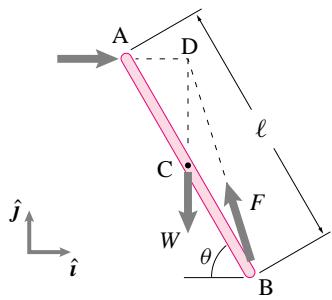


Figure 4.14: The free body diagram of the ladder indicates that it is a three force body. Since the direction of the forces acting at points B and C are known (the normal, horizontal reaction at B and the vertical gravity force at C), it is easy to find the direction of the net ground reaction at A — it must pass through point D. The ground reaction  $F$  at A can be decomposed into a normal reaction and a horizontal reaction (the force of friction,  $F_s$ ) at A.

(Filename:sfig4.single.ladder.a)

**SAMPLE 4.4 Will the ladder slip?** A ladder of length  $\ell = 4\text{ m}$  rests against a wall at  $\theta = 60^\circ$ . Assume that there is no friction between the ladder and the vertical wall but there is friction between the ground and the ladder with  $\mu = 0.5$ . A person weighing 700 N starts to climb up the ladder.

- Can the person make it to the top safely (without the ladder slipping)? If not, then find the distance  $d$  along the ladder that the person can climb safely. Ignore the weight of the ladder in comparison to the weight of the person.
- Does the “no slip” distance  $d$  depend on  $\theta$ ? If yes, then find the angle  $\theta$  which makes it safe for the person to reach the top.

### Solution

- The free body diagram of the ladder is shown in Fig. 4.14. There is only a normal reaction  $\vec{R} = R\hat{i}$  at A since there is no friction between the wall and the ladder. The force of friction at B is  $\vec{F}_s = -F_s\hat{i}$  where  $F_s \leq \mu N$ . To determine how far the person can climb the ladder without the ladder slipping, we take the critical case of impending slip. In this case,  $F_s = \mu N$ . Let the person be at point C, a distance  $d$  along the ladder from point B.

From moment balance about point B,  $\sum \vec{M}_B = \vec{0}$ , we find

$$\begin{aligned}\vec{r}_{A/B} \times \vec{R} + \vec{r}_{C/B} \times \vec{W} &= \vec{0} \\ -R\ell \sin \theta \hat{k} + Wd \cos \theta \hat{k} &= \vec{0} \\ \Rightarrow R &= W \frac{d \cos \theta}{\ell \sin \theta}\end{aligned}$$

From force equilibrium, we get

$$(R - \mu N)\hat{i} + (N - W)\hat{j} = \vec{0} \quad (4.7)$$

Dotting eqn. (4.7) with  $\hat{j}$  and  $\hat{i}$ , respectively, we get

$$\begin{aligned}N &= W \\ R &= \mu N = \mu W\end{aligned}$$

Substituting this value of  $R$  in eqn. (4.7) we get

$$\begin{aligned}\mu W &= W \frac{d \cos \theta}{\ell \sin \theta} \\ \Rightarrow d &= \mu \ell \tan \theta\end{aligned} \quad (4.8)$$

$$= 0.5 \cdot (4\text{ m}) \cdot \tan 60^\circ \quad (4.9)$$

$$= 3.46\text{ m}$$

Thus, the person cannot make it to the top safely.

$$d = 3.46\text{ m}$$

- The “no slip” distance  $d$  depends on the angle  $\theta$  via the relationship in eqn. (4.8). The person can climb the ladder safely up to the top (i.e.,  $d = \ell$ ), if

$$\tan \theta = \frac{1}{\mu} \Rightarrow \theta = \tan^{-1}(\mu^{-1}) = 63.43^\circ$$

Thus, any angle  $\theta \geq 64^\circ$  will allow the person to climb up to the top safely.

$$\theta \geq 64^\circ$$

**SAMPLE 4.5** How much friction does the ball need? A ball of mass  $m$  sits between an incline and a vertical wall as shown in the figure. There is no friction between the wall and the ball but there is friction between the incline and the ball. Take the coefficient of friction to be  $\mu$  and the angle of incline with the horizontal to be  $\theta$ . Find the force of friction on the ball from the incline.

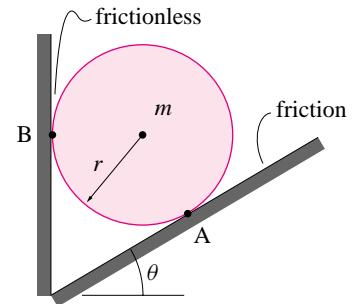


Figure 4.15: (Filename:sfig4.single.ball)

**Solution** The free body diagram of the ball is shown in Fig. 4.16. Note that the normal reaction of the vertical wall,  $N$ , the force of gravity,  $mg$ , and the normal reaction of the incline,  $R$ , all pass through the center C of the ball. Therefore, the moment balance about point C,  $\sum \vec{M}_C = \vec{0}$ , gives

$$\begin{aligned}\vec{r}_{A/C} \times F_s \hat{\lambda} &= \vec{0} \\ \Rightarrow F_s &= 0\end{aligned}$$

Thus the force of friction on the ball is zero! Note that  $F_s$  is independent of  $\theta$ , the angle of incline. Thus, irrespective of what the angle of incline is, in the static equilibrium condition, there is no force of friction on the ball.

$$F_s = 0$$

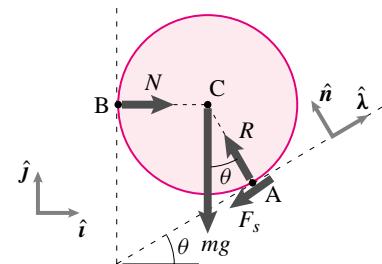


Figure 4.16: (Filename:sfig4.single.ball.a)

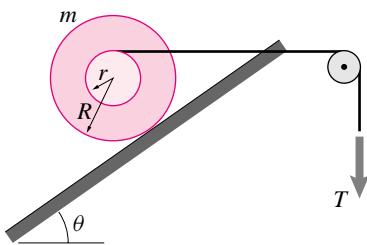


Figure 4.17: (Filename:fig4.single.spool)

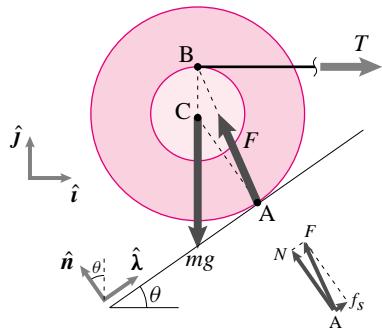


Figure 4.18: (Filename:fig4.single.spool.a)

**SAMPLE 4.6** *Can you balance this?* A spool of mass  $m = 2 \text{ kg}$  rests on an incline as shown in the figure. The inner radius of the spool is  $r = 200 \text{ mm}$  and the outer radius is  $R = 500 \text{ mm}$ . The coefficient of friction between the spool and the incline is  $\mu = 0.4$ , and the angle of incline  $\theta = 60^\circ$ .

- Which way does the force of friction act, up or down the incline?
- What is the required horizontal pull  $T$  to balance the spool on the incline?
- Is the spool about to slip?

### Solution

(a) The free body diagram of the spool is shown in Fig. 4.18. Note that the spool is a 3-force body. Therefore, in static equilibrium all the three forces — the force of gravity  $mg$ , the horizontal pull  $T$ , and the incline reaction  $F$  — must intersect at a point. Since  $T$  and  $mg$  intersect at the top of the inner drum (point B), the incline reaction force  $\vec{F}$  must be along the direction AB. Now the incline reaction  $\vec{F}$  is the vector sum of two forces — the normal (to the incline) reaction  $N$  and the friction force  $F_s$  (along the incline). The normal reaction force  $N$  passes through the center C of the spool. Therefore, the force of friction  $F_s$  must point up along the incline to make the resultant  $\vec{F}$  point along AB.

(b) From the moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , we get

$$\vec{r}_{C/A} \times (-mg\hat{j}) + \vec{r}_{B/A} \times (T\hat{i}) = \vec{0}$$

Substituting the cross products

$$\vec{r}_{C/A} \times (-mg\hat{j}) = mgR \sin \theta \hat{k} \quad \text{and} \quad \vec{r}_{B/A} \times (T\hat{i}) = -T(R \cos \theta + r)\hat{k}$$

and dotting the entire equation with  $\hat{k}$ , we get

$$\begin{aligned} mgR \sin \theta &= T(R \cos \theta + r) \\ \Rightarrow T &= mg \frac{\sin \theta}{\cos \theta + r/R} \\ &= 2 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{.2 \text{ m}}{.5 \text{ m}}} = 18.88 \text{ N} \end{aligned}$$

$$T = 18.88 \text{ N}$$

(c) To find if the spool is about to slip, we need to find the force of friction  $F_s$  and see if  $F_s = \mu N$ . The force balance on the spool,  $\sum \vec{F} = \vec{0}$  gives

$$T\hat{i} - mg\hat{j} + F_s\hat{\lambda} + N\hat{n} = \vec{0} \quad (4.10)$$

where  $\hat{\lambda}$  and  $\hat{n}$  are unit vectors along the incline and normal to the incline, respectively. Dotting eqn. (4.10) with  $\hat{\lambda}$  we get

$$\begin{aligned} F_s &= -T(\hat{i} \cdot \hat{\lambda}) + mg(\hat{j} \cdot \hat{\lambda}) = -T \cos \theta + mg \sin \theta \\ &= -18.88 \text{ N}(1/2) + 19.62 \text{ N}(\sqrt{3}/2) = 7.55 \text{ N} \end{aligned}$$

Similarly, we compute the normal force  $N$  by dotting eqn. (4.10) with  $\hat{n}$ :

$$\begin{aligned} N &= -T(\hat{i} \cdot \hat{n}) + mg(\hat{j} \cdot \hat{n}) = T \sin \theta + mg \cos \theta \\ &= 18.88 \text{ N}(\sqrt{3}/2) + 19.62 \text{ N}(1/2) = 26.16 \text{ N} \end{aligned}$$

Now we find that  $\mu N = 0.4(26.16 \text{ N}) = 10.46 \text{ N}$  which is greater than  $F_s = 7.55 \text{ N}$ . Thus  $F_s < \mu N$ , and therefore, the spool is not in the condition of impending slip.

## 4.2 Elementary truss analysis

Join two pencils (or pens, chopsticks, or popsicle sticks) tightly together with a rubber band as in fig. 4.19a. You can feel that the pencils rotate relative to each other relatively easily. But it is hard to slide one against the other. Add a third pencil to complete the triangle (fig. 4.19b). The relative rotation of the first two pencils is now almost totally prohibited. Now tightly strap four pencils (or whatever) into a square with rubber bands as in fig. 4.19c, making 4 rubber band joints at the corners. Put the square down on a table. The pencils don't stretch or bend visibly, nor do they slide much along each-other's lengths, but the connections allow the pencils to rotate relative to each other so the square easily distorts into a parallelogram. Because a triangle is fully determined by the lengths of its sides and a quadrilateral is not, the triangle is *much* harder to distort than the square. A triangle is sturdy even without restraint against rotation at the joints and a square is not.

Now add two more pencils to your triangle to make two triangles (fig. 4.19d). So long as you keep this structure flat on the table, it is also sturdy. You have just observed the essential inspiration of a truss: triangles make sturdy structures.

A different way to imagine discovering a truss is by means of swiss cheese. Imagine your first initial design for a bridge is to make it from one huge piece of solid steel. This would be heavy and expensive. So you could cut holes out of the chunk here and there, greatly diminishing the weight and amount of material used, but not much reducing the strength. Between these holes you would see other heavy regions of metal from which you might cut more holes leading to a more savings of weight at not much cost in strength. In fact, the reduced weight in the middle decreases the load on the outer parts of the structure possibly making the whole structure stronger. Eventually you would find yourself with a structure that looks much like a collection of bars attached from end to end in vaguely triangular patterns. As opposed to a solid block, a truss

- Uses less material;
- Puts less gravity load on other parts of the structure;
- Leaves space for other things of interest (*e.g.*, cars, cables, wires, people).

Real trusses are usually not made by removing material from a solid but by joining bars of steel, wood, or bamboo with welds, bolts, rivets, nails, screws, glue, or lashings. Now that you are aware you will probably notice trusses in bridges, radio towers, and large-scale construction equipment. Early airplanes were flying trusses.<sup>①</sup> Trusses have been used as scaffoldings for millennia. Birds have had bones whose internal structure is truss-like since they were dinosaurs. Trusses are worth study on their own, since they are a practical way to design sturdy light structures. But trusses also are useful

- As a first example of a complex mechanical system that a student can analyze;
- As an example showing the issues involved in structural analysis;
- As an intuition builder for understanding structures that are not really trusses (The engineering mind often sees an underlying conceptual truss where no physical truss is externally visible).

What is a truss?

A **truss** is a structure made from connecting long narrow elements at their ends.

The sturdiness of most trusses comes from the inextensibility of the bars, not the resistance to rotation at the joints. To make the analysis simpler the (generally small) resistance to rotation in the joints is totally neglected in truss analysis. Thus

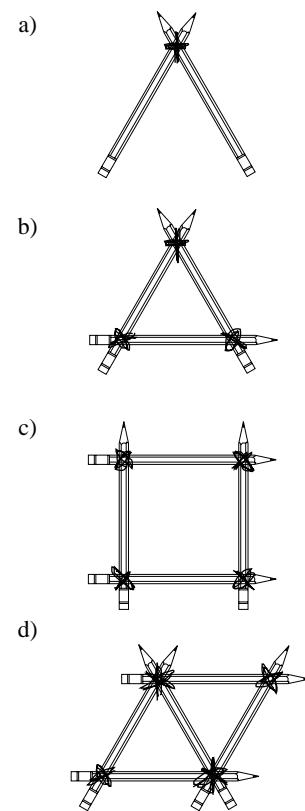


Figure 4.19: a) Two pencils strapped together with a rubber band are not sturdy. b) A triangle made of pencils feels sturdy. c) A square made of 4 pencils easily distorts into a parallelogram. d) A structure made of two triangles feels sturdy (if held on a table).

(Filename:tfigure.pencil)

<sup>①</sup> The Wright brothers first planes were near copies of the planes built a few years earlier by Octave Chanute, a retired bridge designer. With regard to structural design, these early biplanes were essentially flying bridges. Take away the outer skin from many small modern planes and you will also find trusses.

An *ideal truss* is an assembly of two force members.

Or, if you like, an ideal truss is a collection of bars connected at their ends with frictionless pins. Loads are only applied at the pins. In engineering analysis, the word ‘truss’ refers to an ideal truss even though the object of interest might have, say, welded joint connections. Had we assumed the presence of welding equipment in your room, the opening paragraph of this section would have described the welding of metal bars instead of the attachment of pencils with rubber bands. Even welded, you would have found that a triangle is more rigid than a square.

## Bars, joints, loads, and supports

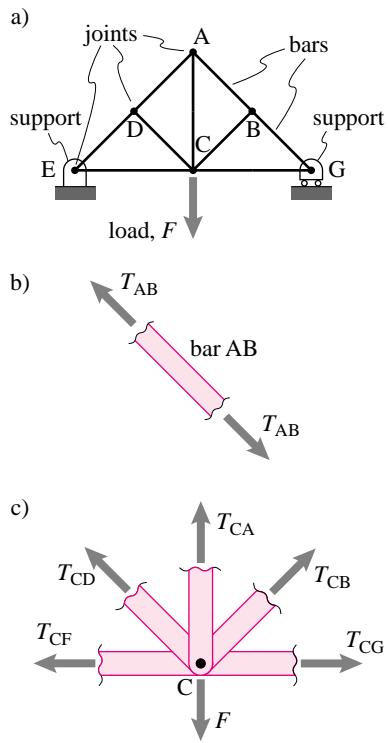


Figure 4.20: a) a truss, b) each bar is a two-force body, c) A joint is acted on by bar tensions and from applied loads.

(Filename:tfigure.trussdef)

## The method of free body diagrams

Trusses are always analyzed by the method of free body diagrams. Free body diagrams are drawn of the whole truss and of various parts of the truss, the equilibrium equations are applied to each free body diagram, and the resulting equations are solved for the unknown bar forces and reactions.

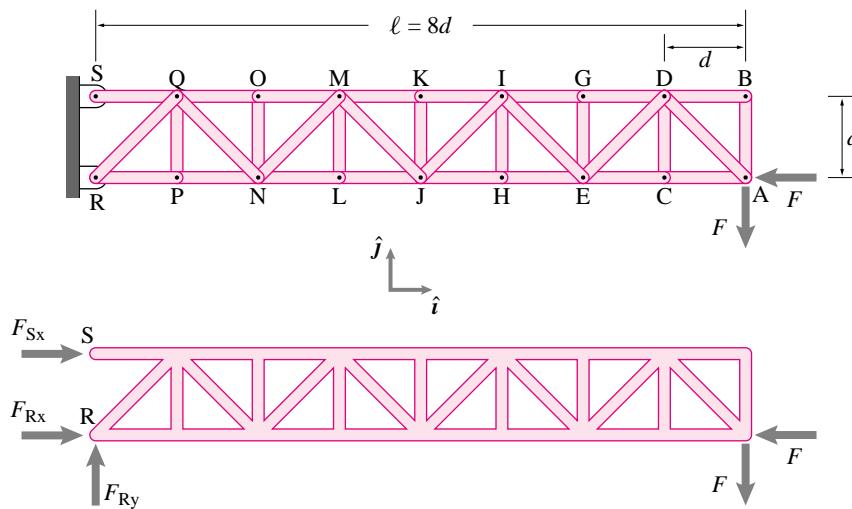
The method of free body diagrams is sometimes subdivided into two sub-methods.

- In the *method of joints* you draw free body diagrams of every joint and apply the force balance equations to each free body diagram. The method of joints is systematic and complete; if a truss can be solved, it can be solved with the method of joints.

- In the *method of sections* you draw a free body diagrams of one or more sections of the structure each of which includes 2 or more joints and apply force and moment balance to the section. The method of sections is powerful tool but is generally not applied systematically. Rather, the method of sections is a mostly used for determining 1-3 bar forces in trusses that have a simple aspect to them. The method of sections can add to your intuitive understanding of how a structure carries a load.

For either of these methods, it is often useful to first draw a free body diagram of the whole structure and use the equilibrium equations to determine what you can about the reaction forces.

Consider this planar approximation to the arm of a derrick used in construction where  $F$  and  $d$  are known (see fig.4.21). This truss has joints A-S (skipping 'F' to avoid confusion with the load). As is common in truss analysis, we totally neglect the force of gravity on the truss elements<sup>①</sup>. From the free body diagram of the whole



<sup>①</sup> To include the force of gravity on the truss elements replace the single gravity force at the center of each bar with a pair of equivalent forces at the ends. The gravity loads then all apply at the joints and the truss can still be analyzed as a collection of two-force members.

Figure 4.21: A truss. (Filename:tfigure.derrick)

structure we find that

$$\begin{aligned} \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow F_{Ry} = F \\ \left\{ \sum \vec{M}_S = \vec{0} \right\} \cdot \hat{k} &\Rightarrow F_{Rx} = 9F \\ \left\{ \sum \vec{M}_R = \vec{0} \right\} \cdot \hat{k} &\Rightarrow F_{Sx} = -8F. \end{aligned}$$

## The method of joints

The sure-fire approach to solve a truss is the brute force method of joints. For the truss above you draw 18 free body diagrams, one for each joint. For each joint free body diagram you write the force balance equations, each of which can be broken down into 2 scalar equations. You then solve these 36 equations for the 33 unknown bar tensions and the 3 reactions (which we found already, but need not have). In general solving 36 simultaneous equations is really only feasible with a computer, which is one way to go about things.

For simple triangulated structures, like the one in fig. 4.21, you can find a sequence of joints for which there are at most two unknown bar forces at each joint. So hand solution of the joint force balance equations is actually feasible. For this truss we could start at joint B (see fig. 4.22) where force balance tells us at a glance that

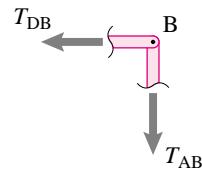


Figure 4.22: Free body diagram of joint B.  
(Filename:tfigure.derrickB)

$$T_{AB} = 0 \quad \text{and} \quad T_{DB} = 0.$$

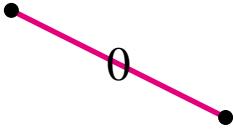


Figure 4.23: A zero force member is sometimes indicated by writing a zero on top of the bar.

(Filename:tfigure.zeroforce)

Just by looking at the joint and thinking about the free body diagram you could probably pick out these *zero force members*. Now you can draw a free body diagram of joint A where there are only two unknown tensions (since we just found  $T_{AB}$ ), namely  $T_{AD}$  and  $T_{AC}$ . Force balance will give two scalar equations which you can solve to find these. Now you can move on to joint C. Here, without drawing the free body diagram on paper, you might see that bar CD is also a zero force member (its the only thing pulling up on joint C and the net up force has to be zero). In any case force balance for joint C will tell you  $T_{CD}$  and  $T_{CE}$ . You can then work your way through the alphabet of joints and find all the bar tensions, using the bar tensions you have already found as you go on to new joints.

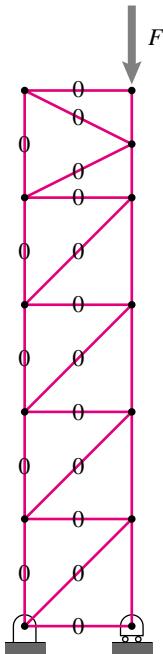


Figure 4.24: A tower with many zero force members. Although they carry no load they prevent structural collapse.

(Filename:tfigure.zeroforcetower)

## Zero force members

The unnecessary but useful trick of recognizing zero-force members, like we just did for bars AB,BD and CD in the truss of fig. 4.21, can be systematized. The basic idea is this: if there is any direction for which only one bar contributes a force, that bar tension must be zero. In particular:

- At any joint where there are no loads, where there are only two unknown non-parallel bar forces, and where all known bar-tensions are zero, the two new bar tensions are both zero (joint B in the example above).
- At any joint where all bars but one are in the same direction as the applied load (if any), the one bar is a zero-force member (see joints C, G, H, K, L, O, and P in the example above).

In the truss of fig. 4.21 bars AB, BD, CD, EG, IH, JK, ML, NO, and PQ are all zero force members. Sometimes it is useful to keep track of the zero force members by marking them with a zero (see fig. 4.23). Although zero-force members seem to do nothing, they are generally needed. For this or that reason there are small loads, imperfections, or load induced asymmetries in a structure that give the ‘zero-force’ bars a small job to do, a job not noticed by the equilibrium equations in elementary truss analysis, but one that can prevent total structural collapse. Imagine, for example, the tower of fig. 4.24 if all of the zero-force members were removed.

## The method of sections

Say you are interested in the truss of fig. 4.21, but only in the tension of bar KM. You already know how to find  $T_{KM}$  using the brute-force method of joints or by working through the joints one at time. The method of sections provides a shortcut.

You look for a way to isolate a section of the structure using a *section cut* that cuts the bar of interest and at most two other bars as in free body diagram 4.25. For the method of sections to bear easy fruit, the truss must be simple in that it has a place where it can be divided with only three bar cuts.

Because 2D statics of finite bodies gives three scalar equations we can find all three unknown tensions. In particular:

$$\{\sum \vec{M}_J = \vec{0}\} \cdot \hat{k} \quad \Rightarrow \quad T_{KM} = 4F.$$

Using this same section cut we can also find:

$$\begin{aligned} \{\sum \vec{M}_M = \vec{0}\} \cdot \hat{k} &\Rightarrow T_{JL} = -5F, \text{ and} \\ \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{j} &\Rightarrow T_{JM} = \sqrt{2}F. \end{aligned}$$

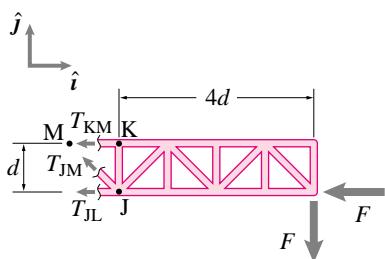


Figure 4.25: Free body diagram of a ‘section’ of the structure.

(Filename:tfigure.derricksection)

A traditional part of the shortcut in the method of sections is to avoid the solution of even two or three simultaneous equations by judicious choice of equilibrium equations following this general rule.

Use equilibrium equations that don't contain terms that you don't know and don't care about.

The two common implementations of this rule are:

- Use moment balance about points where the lines of action of two unknown forces meet. In the free body diagram of fig. 4.25 moment balance about point J eliminates  $T_{JM}$  and  $T_{JL}$  and gives one equation for  $T_{KM}$ .
- Use force balance perpendicular to the direction of a pair of parallel unknown forces. In the free body diagram of fig. 4.25 force balance in the  $\hat{j}$  direction eliminates  $T_{KM}$  and  $T_{JL}$  and gives one equation for  $T_{JM}$ .

In the method of joints, as you worked your way along the structure fig. 4.21 from right to left you would have found the tensions getting bigger and bigger on the top bars and the compressions (negative tensions) getting bigger and bigger on the bottom bars. With the method of sections you can see that this comes from the lever arm of the load  $F$  being bigger and bigger for longer and longer sections of truss. The moment caused by the vertical load  $F$  is carried by the tension in the top bars and compression in the bottom bars.



Figure 4.26: Many bridges are essentially trusses. Here's one that is partially obscured by the truss on a car bridge.

(Filename:tfigure.truss)

## Why aren't trusses everywhere?

Trusses can carry big loads with little use of material and can look nice (See fig. 4.27)., so why don't engineers use them for all structural designs? Here are some reasons to consider other designs:

- Trusses are relatively difficult to build and thus possibly expensive.
- They are sensitive to damage when loads are not applied at the anticipated joints. They are especially sensitive to loads on the middle of the bars.
- Trusses inevitably depend on the tension strength in some bars. Some common building materials (*e.g.*, concrete, stone, and clay) crack easily when pulled.
- Trusses usually have little or no redundancy, so failure in one part can lead to total structural failure.
- The triangulation that trusses require can use space that is needed for other purposes (*e.g.*, doorways or rooms)
- Trusses tend to be stiff, and sometimes more flexibility is desirable (*e.g.*, diving boards, car suspensions).
- In some places some people consider trusses unaesthetic.

None-the-less, for situations where you want a stiff, light structure that can carry known loads at pre-defined points, a truss is often a great design choice.



Figure 4.27: Sometimes trusses are used only because they look nice. The tensegrity structure 'Needle Tower' was designed by artist Kenneth Snelson and is on display in the Hirshhorn Museum in Washington, DC. Here you are looking straight up the middle. Photograph by Christopher Rywalt.

(Filename:tfigure.Needle)

## Summary

Using free body diagrams of the whole structure, sections of the structure, or the joints, you can find the tensions in the bars and the reaction forces for some elementary trusses. There are trusses that do not yield to this analysis, however, which are discussed in the next section.

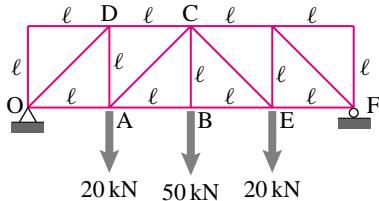


Figure 4.28: (Filename:sfig4.truss.simple)

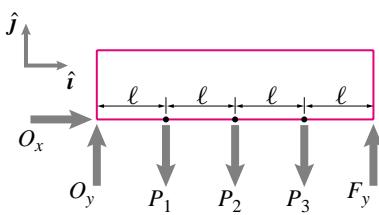


Figure 4.29: (Filename:sfig4.truss.simple.a)

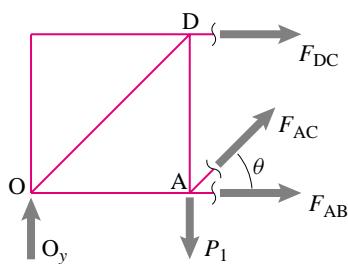
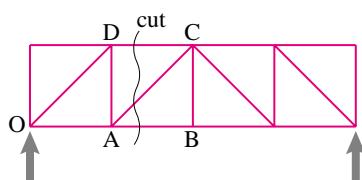


Figure 4.30: (Filename:sfig4.truss.simple.b)

**SAMPLE 4.7 A 2-D truss:** The box truss shown in the figure is loaded by three vertical forces acting at joints A, B, and E. All horizontal and vertical bars in the truss are of length 2 m. Find the forces in members AB, AC, and DC.

**Solution** First, we need to find the support reactions at points O and F. We do this by drawing the free body diagram of the whole truss and writing the equilibrium equations for it. Referring to Fig. 4.29, the force equilibrium,  $\sum \vec{F} = \vec{0}$  implies,

$$O_x \hat{i} + (O_y + F_y - P_1 - P_2 - P_3) \hat{j} = \vec{0} \quad (4.11)$$

Dotting eqn. (4.11) with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$\begin{aligned} O_x &= 0 \\ O_y + F_y &= P_1 + P_2 + P_3 \end{aligned} \quad (4.12)$$

The moment equilibrium about point O,  $\vec{M}_O = \vec{0}$ , gives

$$(-P_1\ell - P_2 2\ell - P_3 3\ell + F_y 4\ell) \hat{k} = \vec{0} \quad (4.13)$$

$$\text{or} \quad F_y = \frac{1}{4}(P_1 + 2P_2 + 3P_3) \quad (4.14)$$

Solving eqns. (4.12) and (4.14), we get

$$F_y = 45 \text{ kN}, \text{ and } O_y = 45 \text{ kN}.$$

In fact, from the symmetry of the structure and the loads, we could have guessed that the two vertical reactions must be equal, i.e.,  $O_y = F_y$ . Then, from eqn. (4.12) it follows that  $O_y = F_y = (P_1 + P_2 + P_3)/2 = 45 \text{ kN}$ .

Now, we proceed to find the forces in the members AB, AC, and DC. For this purpose, we make a cut in the truss such that it cuts members AD, AC, and DC, just to the right of joints A and D. Next, we draw the free body diagram of the left (or right) portion of the truss and use the equilibrium equations to find the required forces. Referring to Fig. 4.30, the force equilibrium requires that

$$(F_{AB} + F_{DC} \cos \theta) \hat{i} + (O_y - P_1 + F_{AC} \sin \theta) \hat{j} = \vec{0} \quad (4.15)$$

Dotting eqn. (4.15) with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$F_{AB} + F_{DC} \cos \theta = 0 \quad (4.16)$$

$$O_y - P_1 + F_{AC} \sin \theta = 0 \quad (4.17)$$

So far, we have two equations in three unknowns ( $F_{AB}$ ,  $F_{DC}$ ,  $F_{AC}$ ). We need one more independent equation to be able to solve for the unknown forces. We now write moment equilibrium equation about point A, i.e.,  $\sum \vec{M}_A = \vec{0}$ ,

$$\begin{aligned} (-O_y \ell - F_{DC} \ell) \hat{k} &= \vec{0} \\ \Rightarrow O_y + F_{DC} &= 0. \end{aligned} \quad (4.18)$$

We can now solve eqns. (4.16–4.18) any way we like, e.g., using elimination or a computer. The solution we get (see next page for details) is:

$$F_{AC} = -25\sqrt{2} \text{ kN}, \quad F_{DC} = -45 \text{ kN}, \quad \text{and} \quad F_{AB} = 70 \text{ kN}.$$

$F_{AC} = -25\sqrt{2} \text{ kN}, \quad F_{DC} = -45 \text{ kN}, \quad F_{AB} = 70 \text{ kN}.$
---

**Comments:**

- Note that the values of  $F_{AC}$  and  $F_{DC}$  are negative which means that bars AC and DC are in compression, not tension, as we initially assumed. Thus the solution takes care of our incorrect assumptions about the directionality of the forces.
- **Short-cuts:** In the solution above, we have not used any tricks or any special points for moment equilibrium. However, with just a little bit of mechanics intuition we can solve for the required forces in five short steps as shown below.
  - (i) No external force in  $\hat{i}$  direction implies  $O_x = 0$ .
  - (ii) Symmetry about the middle point B implies  $O_y = F_y$ . But,

$$O_y + F_y = \sum P_i = 90 \text{ kN} \Rightarrow O_y = F_y = 45 \text{ kN}.$$

(iii)  $(\sum \bar{M}_A = \bar{0}) \cdot \hat{k}$  gives

$$O_y \ell + F_{DC} \ell = 0 \Rightarrow F_{DC} = -O_y = -45 \text{ kN}.$$

(iv)  $(\sum \bar{M}_C = \bar{0}) \cdot \hat{k}$  gives

$$-O_y 2\ell + P_1 \ell + F_{AB} \ell = 0 \Rightarrow F_{AB} = 2O_y - P_1 = 70 \text{ kN}.$$

(v)  $(\sum \bar{F} = \bar{0}) \cdot \hat{j}$  gives

$$O_y - P_1 + F_{AC} \sin \theta = 0 \Rightarrow F_{AC} = (P_1 - O_y) / \sin \theta = -25\sqrt{2} \text{ kN}.$$

- **Solving equations:** On the previous page, we found  $F_{AB}$ ,  $F_{DC}$ , and  $F_{AC}$  by solving eqns. (4.15–4.17) simultaneously. Here, we show you two ways to solve those equations.

(a) *By elimination:* From eqn. (4.17), we have

$$F_{AC} = \frac{O_y - P_1}{\sin \theta} = \frac{20 \text{ kN} - 45 \text{ kN}}{1/\sqrt{2}} = -25\sqrt{2} \text{ kN}.$$

From eqn. (4.18), we get

$$F_{DC} = -O_y = -45 \text{ kN},$$

and finally, substituting the values found in eqn. (4.15), we get

$$F_{AB} = -F_{DC} - F_{AC} \cos \theta = 45 \text{ kN} + 25\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 70 \text{ kN}.$$

(b) *On a computer:* We can write the three equations in the matrix form:

$$\underbrace{\begin{bmatrix} 1 & 1 & \cos \theta \\ 0 & 0 & \sin \theta \\ 0 & 1 & 0 \end{bmatrix}}_A \underbrace{\begin{Bmatrix} F_{AB} \\ F_{DC} \\ F_{AC} \end{Bmatrix}}_x = \underbrace{\begin{Bmatrix} 0 \\ P_1 - O_y \\ -O_y \end{Bmatrix}}_b \quad \text{kN}$$

We can now solve this matrix equation on a computer by keying in matrix A (with  $\theta$  specified as  $\pi/4$ ) and vector b as input and solving for x.<sup>①</sup>

① Pseudocode:  
 $A = [1 1 \cos(\pi/4)$   
 $0 0 \sin(\pi/4)$   
 $0 1 0]$   
 $b = [0 -25 -45]$   
 $\text{solve } A*x = b \text{ for } x$

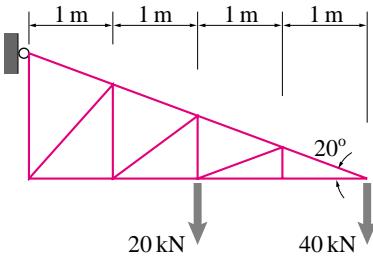


Figure 4.31: (filename:sfig4.truss.comp)

**SAMPLE 4.8** The truss shown in the figure has four horizontal bays, each of length 1 m. The top bars make 20° angle with the horizontal. The truss carries two loads of 40 kN and 20 kN as shown. Find the forces in each bar. In particular, find the bars that carry the maximum tensile and compressive forces.

**Solution** Since we need to find the forces in all the 15 bars, we need to find enough equations to solve for these 15 forces in addition to 3 unknown reactions  $A_x$ ,  $A_y$ , and  $I_x$ . Thus we have a total of 18 unknowns. Note that there are 9 joints and therefore, we can generate 18 scalar equations by writing force equilibrium equations (one vector equation per joint) for each joint.

$$\text{Number of unknowns } 15 + 3 = 18$$

$$\text{Number of joints } 9$$

$$\text{Number of equations } 9 \times 2 = 18$$

So, we go joint by joint, draw the free body diagram of each joint and write the equilibrium equations. After we get all the equations, we can solve them on a computer. All joint equations are just force equilibrium equations, *i.e.*,  $\sum \vec{F} = \vec{0}$ .

- Joint A:

$$(A_x + T_1 + T_{10} \cos \alpha_1) \hat{i} + (A_y + T_{11} + T_{10} \sin \alpha_1) \hat{j} = \vec{0} \quad (4.19)$$

- Joint B:

$$(-T_1 + T_2 + T_8 \cos \alpha_2) \hat{i} + (T_9 + T_8 \sin \alpha_2) \hat{j} = \vec{0} \quad (4.20)$$

- Joint C:

$$(-T_2 + T_3 + T_6 \cos \alpha_3) \hat{i} + (T_7 + T_6 \sin \alpha_3) \hat{j} = P \hat{j} \quad (4.21)$$

- Joint D:

$$(T_4 - T_3) \hat{i} + T_5 \hat{j} = \vec{0} \quad (4.22)$$

- Joint E:

$$(-T_4 - T_{15} \cos \theta) \hat{i} + T_{15} \sin \theta \hat{j} = 2P \hat{j} \quad (4.23)$$

- Joint F:

$$(-T_6 \cos \alpha_3 + (T_{15} - T_{14}) \cos \theta) \hat{i} + (-T_6 \sin \alpha_3 + (T_{14} - T_{15}) \sin \theta - T_5) \hat{j} = \vec{0} \quad (4.24)$$

- Joint G:

$$(-T_8 \cos \alpha_2 + (T_{14} - T_{13}) \cos \theta) \hat{i} + ((T_{13} - T_{14}) \sin \theta - T_8 \sin \alpha_2 - T_7) \hat{j} = \vec{0} \quad (4.25)$$

- Joint H:

$$(-T_{10} \cos \alpha_1 + (T_{13} - T_{12}) \cos \theta) \hat{i} + ((T_{12} - T_{13}) \sin \theta - T_{10} \sin \alpha_1 - T_9) \hat{j} = \vec{0} \quad (4.26)$$

- Joint I:

$$(-I_x + T_{12} \cos \theta) \hat{i} + (-T_{11} - T_{12} \sin \theta) \hat{j} = \vec{0} \quad (4.27)$$

Dotting each equation from (4.19) to (4.27) with  $\hat{i}$  and  $\hat{j}$ , we get the required 18 equations. We need to define all the angles that appear in these equations ( $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ , and  $\theta$ ) before we are ready to solve the equations on a computer.

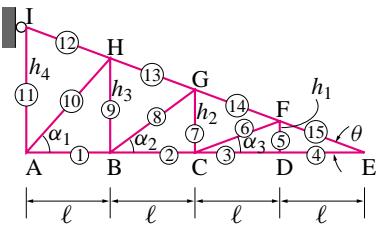


Figure 4.32: (filename:sfig4.truss.comp.a)

Let  $\ell$  be the length of each horizontal bar and let DF =  $h_1$ , CG =  $h_2$ , and BH =  $h_3$ . Then,  $h_1/\ell = h_2/2\ell = h_3/3\ell = \tan \theta$ . Therefore,

$$\begin{aligned}\tan \alpha_1 &= \frac{h_1}{\ell} = \tan \theta \Rightarrow \alpha_1 = \tan^{-1}(\tan \theta) = \theta \\ \tan \alpha_2 &= \frac{h_2}{\ell} = 2 \tan \theta \Rightarrow \alpha_2 = \tan^{-1}(2 \tan \theta) \\ \tan \alpha_3 &= \frac{h_3}{\ell} = 3 \tan \theta \Rightarrow \alpha_3 = \tan^{-1}(3 \tan \theta)\end{aligned}$$

Now, we are ready for a computer solution. You can enter the 18 equations in matrix form or as your favorite software package requires and get the solution by solving for the unknowns. Here are two examples of pseudocodes. Let us order the unknown forces in the form

$$x = [T_1 \ T_2 \ \dots \ T_{15} \ A_x \ A_y \ I_x]^T$$

so that  $x_1 - x_{15} = T_1 - T_{15}$ ,  $x_{16} = A_x$ ,  $x_{17} = A_y$ , and  $x_{18} = I_x$

(a) Entering full matrix equation:

```
theta = pi/9                      % specify theta in radians
alphal = theta                     % calculate alphal
alpha2 = atan(2*tan(theta))        % calculate alpha2 from arctan
alpha3 = atan(3*tan(theta))        % calculate alpha3 from arctan

C = cos(theta), S = sin(theta)      % compute all sines and cosines
C1 = cos(alphal), S1 = sin(alphal)
C2 = . . .

A = [1 0 0 0 0 0 0 0 C1 0 0 0 0 0 1 0 0           % enter matrix A row-wise
     0 0 0 0 0 0 0 0 S1 1 0 0 0 0 0 1 0
     .
     .
     0 0 0 0 0 0 0 0 0 0 -1 -S 0 0 0 0 0 0]
b = [0 0 0 0 20 0 0 0 40 0 0 0 0 0 0 0 0]'    % enter column vector b
solve A*x = b for x
```

(b) Entering each equation as part of matrix A and vector b:

```
A(1,[1 10 16]) = [1 C1 1]
A(2,[10 11 17]) = [S1 1 1]
.
.
A(18,[11 12]) = [-1 -S]
b(6,1) = 20
b(10,1) = 40
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is

$$\begin{aligned}T_1 &= -128.22 \text{ kN}, \quad T_2 = T_3 = T_4 = -109.9 \text{ kN}, \quad T_5 = T_6 = 0, \\ T_7 &= 20 \text{ kN}, \quad T_8 = -22.66 \text{ kN}, \quad T_9 = -T_{10} = 13.33 \text{ kN}, \quad T_{11} = -50 \text{ kN}, \\ T_{12} &= 146.19 \text{ kN}, \quad T_{13} = 136.44 \text{ kN}, \quad T_{14} = T_{15} = 116.95 \text{ kN}, \\ A_x &= 137.37 \text{ kN}, \quad A_y = 60 \text{ kN}, \quad I_x = -137.37 \text{ kN}.\end{aligned}$$

## 4.3 Advanced truss analysis: determinacy, rigidity, and redundancy

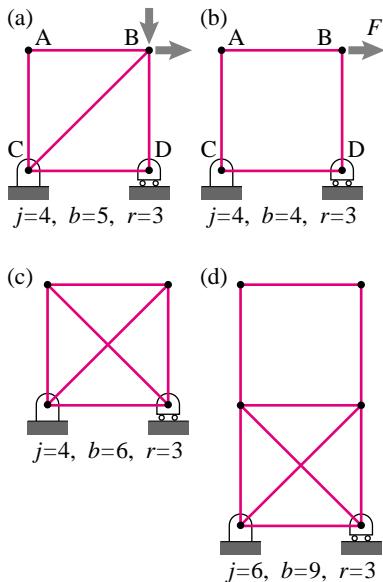


Figure 4.33: a) a statically determinate truss, b) a non-rigid truss, c) a redundant truss, and d) a non-rigid and redundant truss.

(Filename:tfigure.4cases)

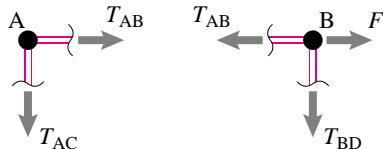


Figure 4.34: Free body diagrams of joints A and B of fig. 4.33b

(Filename:tfigure.squarejoints)

① As a curiosity notice that you could make the diagonals in fig. 4.33c both sticks and all of the outside square from cables and the truss would still carry all loads. This is the simplest ‘tensegrity’ structure. In a tensegrity structure no more than one bar in compression is connected to any one joint. (See fig. 4.27 for a more elegant example.). The label ‘Tensegrity structure’ was coined by the truss-pre-occupied designer Buckminster Fuller. Fuller is also responsible for re-inventing the “geodesic dome” a type of structure studied previously by Cauchy.

After you have mastered the elementary truss analysis of the previous section, namely the method of free body diagrams in its two incarnations (the method of joints and the method of sections) you might wonder if at least one of these methods always work. The answer is yes, if you just look at the homework problems for elementary truss analysis, but ‘no’ if you look at the variety of real (good and bad) structures in the world. In this section we discuss the classification of trusses into types. In the previous section all of the examples were from one of these types.

### Determinate, rigid, and redundant trusses

Your first concern when studying trusses is to develop the ability to solve a truss using free body diagrams and equilibrium equations. A truss that yields a solution, and only one solution, to such an analysis for all possible loadings is called *statically determinate* or just *determinate*. The braced box supported with one pin joint and one pin on rollers (see fig. 4.33a) is a classic statically determinate truss. A statically determinate truss is *rigid* and does not have *redundant* bars.

You should beware, however, that there are a few other possibilities.

Some trusses are *non-rigid*, like the one shown in fig. 4.33b, and can not carry arbitrary loads at the joints.

### Example: Joint equations and non-rigid structures

Free body diagrams of joints A and B of fig. 4.33b are shown in fig. 4.34.

$$\begin{aligned} \text{jointB : } & \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} \Rightarrow T_{AB} &= F \\ \text{jointA : } & \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} \Rightarrow T_{AB} &= 0 \end{aligned}$$

The contradiction that  $T_{AB}$  is both  $F$  and 0 implies that the equations of statics have no solution for a horizontal load at joint B.  $\square$

A non-rigid truss can carry some loads, and you can find the bar tensions using the joint equilibrium equations when these loads are applied. For example, the structure of fig. 4.33b can carry a vertical load at joint B. Engineers sometimes choose to design trusses that are not rigid, the simplest example being a single piece of cable hanging a weight. A more elaborate example is a suspension bridge which, when analyzed as a truss, is not rigid.

A *redundant* truss has more bars than needed for rigidity. As you can tell from inspection or analysis, the braced square of fig. 4.33a is rigid. None the less engineers will often choose to add extra redundant bracing as in fig. 4.33c for a variety of reasons.

- Redundancy is a safety feature. If one member breaks the whole structure holds up.
- Redundancy can increase a structure’s strength.
- Redundancy can allow tensile bracing. In the structure of Fig. 4.33a top load to the left puts bar BC in compression. Thus bar BC can’t be, say, a cable. But in structure fig. 4.33c both diagonals can be cables and neither need carry compression for any load①.

A property of redundant structures is that you can find more than one set of bar forces that satisfy the equilibrium equations. Even when the loads are all zero these structures can have non-zero *locked in* forces (sometimes called ('locked in stress', or 'self stress'). In the structure of fig. 4.33c, for example, if one of the diagonals got hot and stretched both it and the opposite diagonal would be put in compression while the outside was in tension. For structures whose parts are likely to expand or contract, or for which the foundation may shift, this locked in stress can be a contributor to structural failure. So redundancy is not all good.

Finally, a structure can be both non-rigid and redundant as shown in fig. 4.33d. This structure can't carry all loads, but the loads it can carry it can carry with various locked in bar forces.

More examples of statically determinate, non-rigid, and redundant truss are given on pages 143 and 144.

Note, one of the basic assumptions in elementary truss analysis which we have thus far used without comment is that motions and deformations of the structure are not taken into account when applying the equilibrium equations. If a bar is vertical in the drawing then it is taken as vertical for all joint equilibrium equations.

#### *Example: Hanging rope*

For elementary truss analysis, a hanging rope would be taken as hanging vertically even if side loads are applied to its end. This obviously ridiculous assumption manifests itself in truss analysis by the discovery that a hanging rope cannot carry any sideways loads (if it must stay vertical this is true).  $\square$

## Determining determinacy: counting equations and unknowns

How can you tell if a truss is statically determinate? The only sure test is to write all the joint force balance equations and see if they have a unique solution for all possible joint loads. Because this is an involved linear algebra calculation (which we skip in this book), it is nice to have shortcuts, even if not totally reliable. Here are three:

- See, using your intuition, if the structure can deform without any of the bars changing length. You can see that the structures of fig. 4.33b and d can distort. If a structure can distort it is not rigid and thus is not statically determinate.
- See, using your intuition, if there are any redundant bars. A redundant bar is one that prevents a structural deformation that already is prevented. It is easy to see that the second diagonal in structures of fig. 4.33c and d is clearly redundant so these structures are not statically determinate.
- Count the total number of joint equations, two for each joint. See if this is equal to the number of unknown bar forces and reactions. If not, the structure is not statically determinate.

The counting formula in the third criterion above is:

$$2j = b + r \quad (4.28)$$

where  $j$  is the number of joints, including joints at reaction points,  $b$  is the number of bars, and  $r$  is the number of reaction components that shows on a free body diagram of the whole structure (2 from pin joints, 1 from a pin on a roller).

① A non-rigid truss is sometimes called ‘over-determinate’ because there are more equations than unknowns. However, the term ‘over-determinate’ may incorrectly conjure up the image of there being too many bars (which we call *redundant*) rather than too many joints. So we avoid use of this phrase.

② In the language of mathematics we would say that satisfaction of the counting equation  $2j = b+r$  is a *necessary* condition for static determinacy but it is not *sufficient*.

If  $2j > b + r$  the structure is necessarily not rigid because then there are more equations than unknowns<sup>①</sup>. For such a structure there are some loads for which there is no set of bar forces and reactions that can satisfy the joint equilibrium equations. A structure that is non-redundant and non-rigid always has  $2j > b + r$  (see fig. 4.33b).

If  $2j < b + r$  the structure is redundant because there are not as many equations as unknowns; if the equations can be solved there is more than one combination of forces that solve them. A structure that is rigid and redundant always has  $2j < b + r$  (see fig. 4.33b).

But the possibility of structures that are both non-rigid and redundant makes the counting formulas an imperfect way to classify structures<sup>②</sup>. Non-rigid redundant structures can have  $2j < b + r$ ,  $2j = b + r$ , or  $2j > b + r$ . The redundant non-rigid structure in fig. 4.33d has  $2j = b + r$ .

The discussion above can be roughly summarized by this table (refer to fig. 4.33 for a simple example of each entry and to pages 143 and 144 for several more examples).

Truss Type	Rigid	Non-rigid
Non-redundant	a) $2j = b + r$ (Statically determinate)	b) $2j > b + r$
Redundant	c) $2j < b + r$	d) $2j = b + r$ , or $2j > b + r$

A basic summary is this:

If

- $2j = b + r$  and
- you cannot see any ways the structure can distort, and
- you cannot see any redundant bars

then the truss is likely statically determinate. But the only way you can know for sure is through either a detailed study of the joint equilibrium equations, or familiarity with similar structures.

On the other hand if

- $2j > b + r$ , or
- $2j < b + r$ , or
- you can see a way the structure can distort, or
- you can see one or more redundant bars,

then the truss is *not* statically determinate.

#### Example: The classic statically determinate structure

A *triangulated truss* can be drawn as follows:

- (a) draw one triangle,
- (b) then another by adding two bars to an edge,

- (c) then another by adding two bars to an existent edge
- (d) and so on, but never adding a triangle by adding just one bar, and
- (e) you hold this structure in place with a pin at one joint and one pin on roller at another joint

then the structure is statically determinate. Many elementary trusses are of exactly this type. (*Note:* if you violate the ‘but’ in rule (d) you can make a truss that looks ‘triangulated’ but is redundant and therefore not statically determinate.)  $\square$

## Floating trusses

Sometimes one wants to know if a structure is rigid and non-redundant when it is floating unconnected to the ground (but still in 2D, say). For example, a triangle is rigid when floating and a square is not. The truss of fig. 4.35a is rigid as connected but not when floating (fig. 4.35b). A way to find out if a floating structure is rigid is to connect one bar of the truss to the ground by connecting one end of the bar with a pin and the other with a pin on a roller, as in fig. 4.35c. All determinations of rigidity for the floating truss are the same as for a truss grounded this way. The counting formula eqn. 4.28, is reduced to

$$2j = b + 3$$

because this minimal way of holding the structure down uses  $r = 3$  reaction force components.

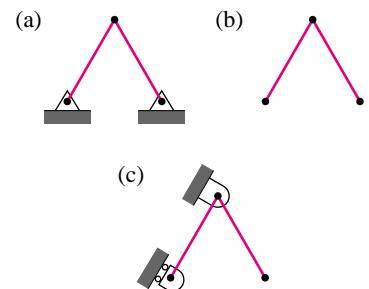


Figure 4.35: a) a determinate two bar truss connected to the ground, b) the same truss is not rigid when floating, which you can tell by seeing that c) it is not rigid when one bar is fixed to the ground.

(Filename:tfigure.rigidonground)

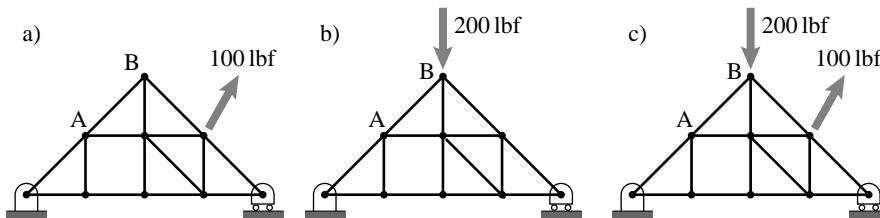
## The principle of superposition for trusses

Say you have solved a truss with a certain load and have also solved it with a different load. Then if both loads were applied the reactions would be the sums of the previously found reactions and the bar forces would be the sums of the previously found bar forces.

This useful fact follows from the linearity of the equilibrium equations<sup>①</sup>.

### Example: Superposition and a truss

<sup>①</sup> A careful derivation would also show that the linearity depends on the nature of the foundation. Linearity holds for pins and pins on rollers, but not for frictional contact.



If for the loading (a) you found  $T_{AB} = 50$  lbf and for loading (b) you found  $T_{AB} = -140$  lbf then for loading (c)  $T_{AB} = 50$  lbf  $- 140$  lbf  $= -90$  lbf  $\square$

### 4.3 Theory: Rigidity, redundancy, linear algebra and maps

This mathematical aside is only for people who have had a course in linear algebra. For definiteness this discussion is limited to 2D trusses, but the ideas also apply to 3D trusses.

For beginners trusses fall into two types, those that are uniquely solvable (statically determinate) and those that are not. Statically determinate trusses are rigid and non-redundant. However, a truss could be non-rigid and non-redundant, rigid and redundant, or non-rigid and redundant. These four possibilities are shown with a simple example each in figure 4.33 on page 138, as a simple table on page 140, and as a big table of examples on pages 143 and 144. The table below, which we now proceed to discuss in detail, is a more abstract mathematical representation of this same set of possibilities.

We can number the bars of the truss followed by the reaction components  $1, 2, \dots, n$ , where  $n = b + r$ . The bar tensions and support reaction forces can be put in a vertical list  $[F_1, F_2, \dots, F_n]'$ . The set of lists of all conceivable tensions and reaction forces we call the “vector space”  $V$  (it is also  $R^n$ ).

We can also make a list of all possible applied loads. In a 2D truss there can be a horizontal or vertical load at each joint. So, we can write a list of  $m = 2j$  numbers to represent the load. If there is only an applied load at a few joints most of the elements of this load vector will be zero. The set of all possible loads we call the vector space  $W$ .

If we use the method of joints we can write two scalar equilibrium equations for each joint. These are linear algebraic equations. Thus we can write them in matrix form as:

$$[A][v] = [w] \quad (4.29)$$

where  $[v]$  is the list of bar tensions and reaction forces, and  $[w]$  is the list of applied loads to the joints. The matrix  $[A]$  is determined by the geometry of the truss. The classification of trusses is really a statement about the solutions of eqn. 4.29. This classification follows, in turn, from the properties of the matrix  $[A]$ .

Another point of view is to think of eqn. 4.29 as a function that maps one vector space onto another. For any  $[v]$  eqn. 4.29 maps that  $[v]$  to some  $[w]$ . That is, if one were given all the bar tensions and reactions one could uniquely determine the applied loads from eqn. 4.29. This map, from  $V$  to  $W$  we call  $T$ .

We can now discuss each of the truss categorizations in turn, with reference to the table at the end of this box.

The first column of the table corresponds to rigid trusses. These trusses have at least one set of bar forces that can equilibrate any particular load. This means that for every  $[w]$  there is some  $[v]$  that maps to (whose image is)  $[w]$ . In these cases the map  $T$  is onto. And the columns space of  $[A]$  is  $W$ . Thus  $[A]$  needs to have at least as many columns as the dimension of  $W$  which is the number of rows of  $[A]$ .

On the other hand if the structure is not rigid there are some loads that cannot be equilibrated by any bar forces. This is the

second column of the table. There is at least some  $[w]$  with no pre-image  $[v]$ . Thus the map  $T$  is not onto and the column space of  $[A]$  is less than all of  $W$ .

The first row of the table describes trusses which are not redundant. Thus, any loads which can be equilibrated can be equilibrated with a unique set of bar tensions and reactions. Thus the columns of  $[A]$  are linearly independent and the map  $T$  is one to one. The matrix  $[A]$  must have at least as many rows as columns.

If a truss is redundant, as in the second row of the table, then there are various ways to equilibrate loads which can be carried. Points in  $W$  in the image of one, and the columns of  $A$  are linearly dependent.

•      •      •

We can now look at the four entries in the table. The top left case is the statically determinate case where the structure is rigid and non-redundant. The map  $T$  is one to one and onto,  $V = W$ , and the matrix  $[A]$  is square and non-singular.

The bottom left case corresponds to a truss that is rigid and redundant. The map  $T$  is onto but not one to one. The columns of  $[A]$  are linearly dependent and it has more columns than rows (it is wide).

The top right case is not rigid and not redundant. Some loads cannot be equilibrated and those that can be are equilibrated uniquely.  $T$  is one to one but not onto. The columns of  $[A]$  are linearly independent but they do not span  $W$ . The matrix  $[A]$  has more rows than columns and is thus tall.

The bottom right case is the most perverse. The structure is not rigid but is redundant. Not all loads can be equilibrated but those that can be equilibrated are equilibrated non-uniquely. The matrix  $[A]$  could have any shape but its columns are linearly dependent and do not span  $W$ . The map  $T$  is neither one to one nor onto.

•      •      •

	Rigid • $T$ is onto • $\text{col}(A) = W$	Not rigid • $T$ is not onto • $\text{col}(A) \neq W$
Not redundant • $T$ is one to one • columns of $A$ are linearly independent	<p><math>A</math> is square and invertible bar &amp; react. forces</p> <p><math>V</math>      <math>W</math></p> <p><math>T</math> is one to one and onto</p>	<p><math>A</math> is tall bar &amp; react. forces</p> <p><math>V</math>      <math>W</math></p> <p><math>T</math> is one to one but not onto</p>
Redundant • $T$ is not one to one • columns of $A$ are linearly dependent	<p><math>A</math> is wide bar &amp; react. forces</p> <p><math>V</math>      <math>W</math></p> <p><math>T</math> is onto but not one to one</p>	<p><math>A</math> can be wide, square, or tall bar &amp; react. forces</p> <p><math>V</math>      <math>W</math></p> <p><math>T</math> is neither one to one nor onto</p>

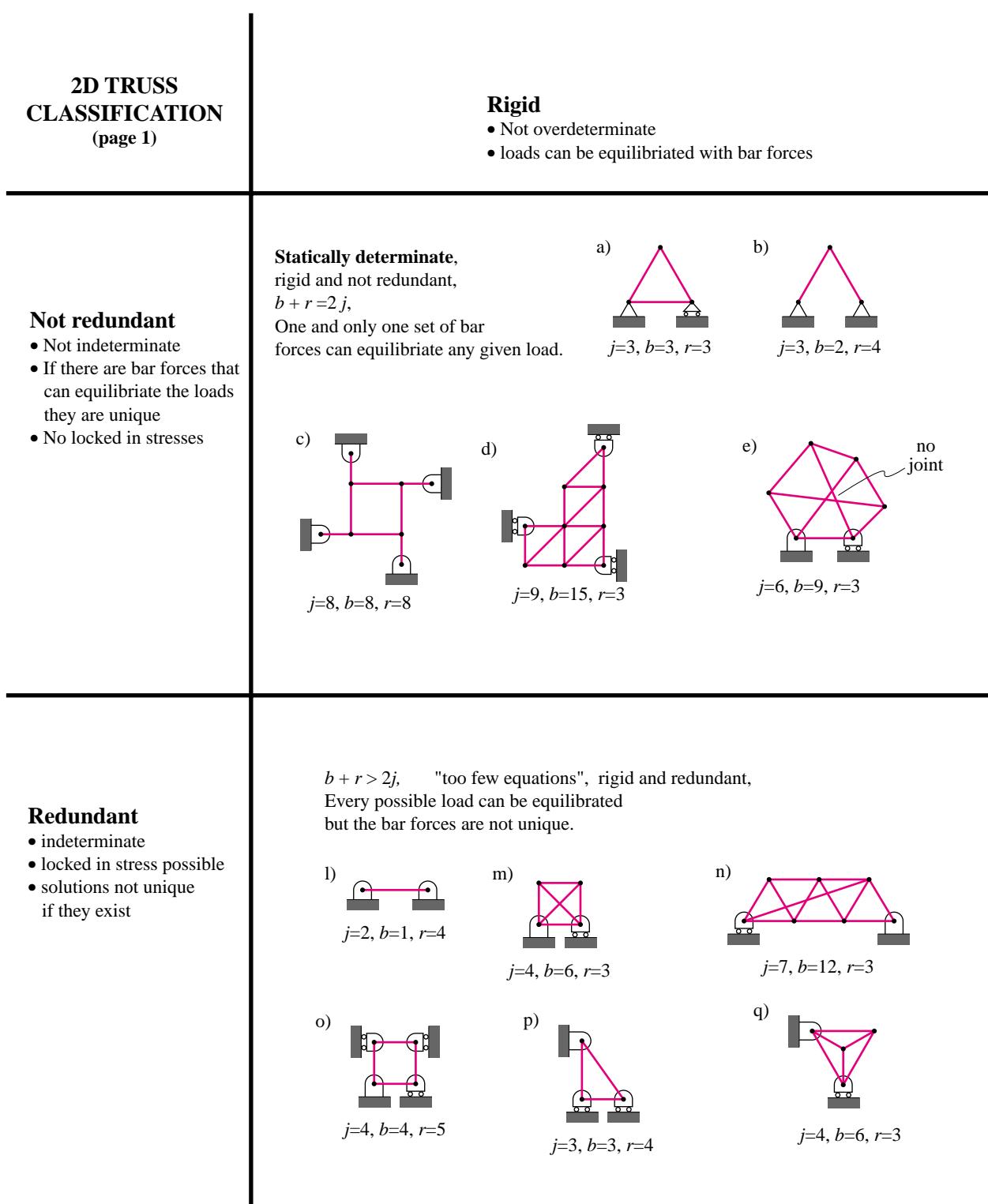


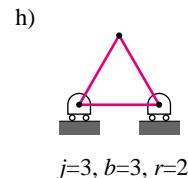
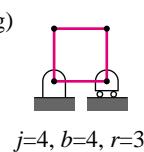
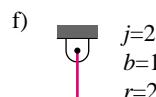
Figure 4.36: Examples of 2D trusses. These two pages concern the 2-fold system for identifying trusses. Trusses can be rigid or not rigid (the two columns) and they can be redundant or not redundant (the two rows). Elementary truss analysis is only concerned with rigid and not redundant trusses (*statically determinate* trusses). Note that the only difference between trusses (b) and (s) is a change of shape (likewise for the far more subtle examples (e) and (u)). Truss (e) is interesting as a rare example of a determinate truss with no triangles.

(Filename: tfigure.trussclass1)

**2D TRUSS  
CLASSIFICATION**  
(page 2)

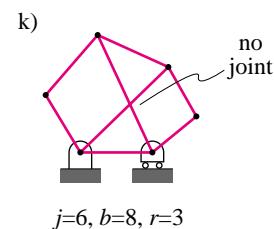
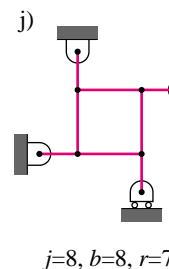
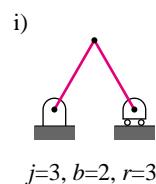
**Not rigid**  
• 'overdeterminate'

$b + r < 2j$ , not rigid and not redundant, "too many equations"  
Unique bar forces for some loads,  
no solution for other loads.



**Not redundant**

- Not indeterminate
- If there are bar forces that can equilibriate the loads they are unique
- No locked in stresses



Not rigid and redundant

**Redundant**

- indeterminate
- locked in stress possible
- solutions not unique if they exist

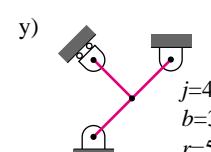
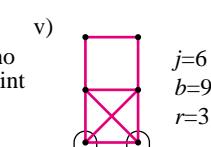
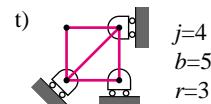
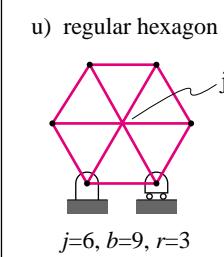
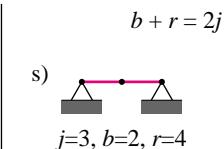
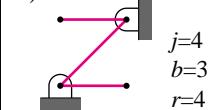
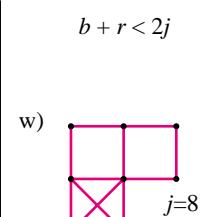
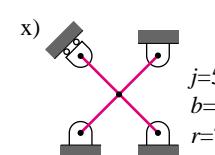
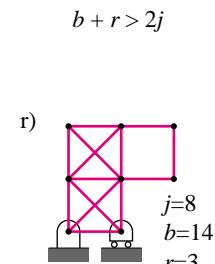


Figure 4.37: (Second page of a two page table.) (filename: tfigure.trussclass2)

**SAMPLE 4.9 An indeterminate truss:** For the truss shown in the figure, find all support reactions.

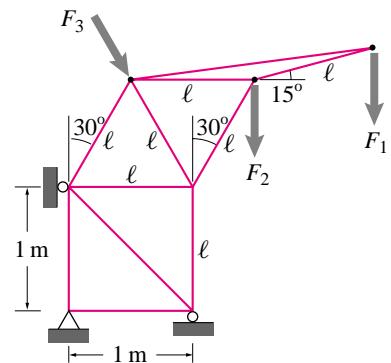


Figure 4.38: (Filename:sfig4.truss.over)

**Solution** The free body diagram of the truss is shown in Fig. 4.39. We need to find the support reactions  $A_x$ ,  $A_y$ ,  $B_y$ , and  $D_x$ .

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$(A_x + D_x + F_3 \cos \theta_1) \hat{i} + (A_y + B_y - F_3 \sin \theta_1 - F_2 - F_1) \hat{j} = \vec{0} \quad (4.30)$$

$$[\text{eqn. (4.30)}] \cdot \hat{i} \Rightarrow A_x + D_x = -F_3 \cos \theta_1 \quad (4.31)$$

$$[\text{eqn. (4.30)}] \cdot \hat{j} \Rightarrow A_y + B_y = F_1 + F_2 + F_3 \sin \theta_1 \quad (4.32)$$

Now we apply moment balance about point A,  $\sum \vec{M}_A = \vec{0}$ . Let A be the origin of our  $xy$ -coordinate system (so that we can write  $\vec{r}_{D/A} = \vec{r}_D$ , etc.).

$$\vec{r}_D \times \vec{D}_x + \vec{r}_F \times \vec{F}_3 + \vec{r}_G \times \vec{F}_1 + \vec{r}_E \times \vec{F}_2 + \vec{r}_B \times \vec{B}_y = 0$$

where,

$$\begin{aligned} \vec{r}_D \times \vec{D}_x &= \ell \hat{j} \times D_x \hat{i} = -D_x \ell \hat{k} \\ \vec{r}_F \times \vec{F}_3 &= (\vec{r}_D + \vec{r}_{F/D}) \times \vec{F}_3 = [\ell \hat{j} + \ell(\sin \theta_1 \hat{i} + \cos \theta_1 \hat{j})] \times F_3(\cos \theta_1 \hat{i} - \sin \theta_1 \hat{j}) \\ &= F_3 \ell \cos \theta_1 \hat{k} - F_3 \ell \hat{k} = -F_3 \ell(1 + \cos \theta_1) \hat{k} \\ \vec{r}_G \times \vec{F}_1 &= (r_{G_x} \hat{i} + r_{G_y} \hat{j}) \times (-F_1 \hat{j}) = -r_{G_x} F_1 \hat{k} \\ &= -F_1 \ell(1 + \sin \theta_1 + \cos \theta_2) \hat{k} \\ \vec{r}_E \times \vec{F}_2 &= -F_2(\ell + \ell \sin \theta_1 \hat{k} = -F_2 \ell(1 + \sin \theta_1) \hat{k} \\ \vec{r}_B \times \vec{B}_y &= \ell \hat{i} \times B_y \hat{j} = B_y \ell \hat{k} \end{aligned}$$

Adding them together and dotting with  $\hat{k}$  we get

$$-D_x \ell - F_3 \ell(1 + \cos \theta_1) - F_1 \ell(1 + \sin \theta_1 + \cos \theta_2) - F_2 \ell(1 + \sin \theta_1) + B_y \ell = 0$$

$$\Rightarrow B_y - D_x = F_1(1 + \sin \theta_1 + \cos \theta_2) + F_2(1 + \sin \theta_1) + F_3(1 + \cos \theta_1). \quad (4.33)$$

We have three equations (4.31–4.33) containing four unknowns  $A_x$ ,  $A_y$ ,  $B_y$ , and  $D_x$ . So, we cannot solve for the unknowns uniquely. This was expected as the truss is indeterminate. However, if we assume a value for one of the unknowns, we can solve for the rest in terms of the assumed one. For example, let  $D_x = \alpha$ . For simplicity let the right hand sides of eqns. (4.31, 4.32, and 4.33) be  $C_1$ ,  $C_2$ , and  $C_3$  (computed values), respectively. Then, we get  $A_x = C_1 - \alpha$ ,  $A_y = C_2 - C_3 - \alpha$ , and  $B_y = C_3 + \alpha$ . The equilibrium is satisfied for any value of  $\alpha$ . Thus there are infinite number of solutions! This is true for all indeterminate systems. However, when deformations of structures are taken into account (extra constraint equations), then solutions do turn out to be unique. You will learn about such things in courses dealing with strength of materials.

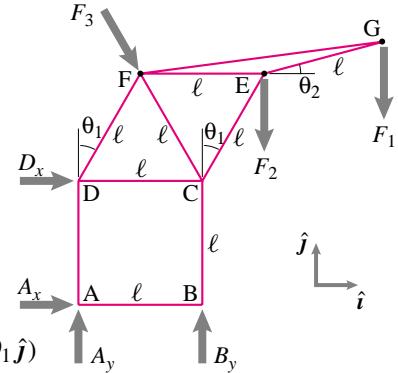


Figure 4.39: (Filename:sfig4.truss.over.a)

□

## 4.4 Internal forces

*“Take one.”*

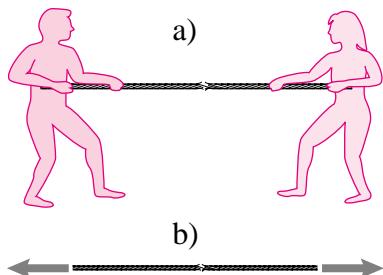


Figure 4.40: a) Two people pulling on a rope that is likely to break in the middle, b) A free body diagram of the rope.

(Filename:tfigure.ropeexternal)

Consider two people pulling on the frayed rope of fig. 4.40a. A free body diagram of the rope is shown in fig. 4.40b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free body diagram. For the rope these are the forces at the ends. The free body diagram does not include internal forces. Thus nothing about the ‘internal forces’ at the fraying part of the rope shows up in the mechanics equations describing the rope.

Mechanics has nothing to say about so called ‘internal forces’ and thus nothing to say about the rope breaking in the middle. ‘Internal forces’ are meaningless in mechanics. End of section.

*“Cut! There’s got to be more to it than that. Let’s try again.”*

## 4.4 Internal forces

“Take two.”

On page 1 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free body diagrams. And free body diagrams only show external forces. How can we use mechanics to describe the ‘forces’ inside a body? We use an idea whose simplicity hides its utility and depth:

You cut the body, and what was inside it is now the outside of a smaller body.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn’t cut using forces (remember, ‘forces are *the* measure of mechanical interaction’), one force, say, at each fiber that is cut. Then we get the free body diagram of fig. 4.41a. We can simplify this to the free body diagram of fig. 4.41b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope we see that

$$\begin{aligned} \text{Sum of vertical forces is zero} &\Rightarrow F_y = 0 \\ \text{Sum of horizontal forces is zero} &\Rightarrow F_x = -T \\ \text{Sum of moments about the cut is zero} &\Rightarrow M = 0. \end{aligned}$$

Thus we get the simpler free body diagram of fig. 4.41c as you probably already knew without using the equilibrium equations explicitly.

### Tension

We have just derived the concept of ‘tension in a rope’ also sometimes called the ‘axial force’. The tension is the pulling force on a free body diagram of the cut rope. If we had used the same cut for a free body diagram of the left half of the rope we would see the free body diagram of fig. 4.41d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension  $T$ . The force vector is the opposite of the force vector on the right half of the rope. So it doesn’t make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ( $-T\hat{i}$  on the left end of the right half and  $T\hat{i}$  on the right end of the left half). Instead we talk about the scalar tension  $T$  which expresses the force vector at the cut as

$$\vec{F} = T\hat{\lambda}$$

where  $\hat{\lambda}$  is a unit vector pointing out from the free body diagram cut. Because  $\hat{\lambda}$  switches direction depending on which half rope you are looking at, the same scalar  $T$  works for both pieces.

The tension in a rope, cable, or bar is the amount of force pulling out on a free body diagram of the cut rope, cable, or bar. Tension is a scalar.

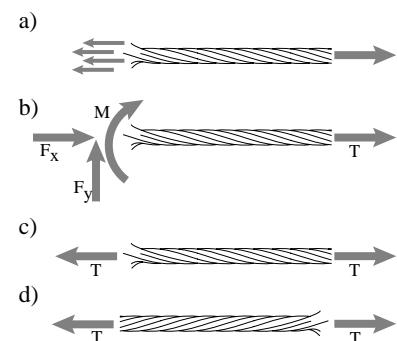


Figure 4.41: a) free body diagram of the right part of the rope, b) the same free body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free body diagram of the left portion of the rope.

(Filename: figure.ropeinternal)

Note our abuse of language: force is a vector, tension is an ‘internal force’ and tension is a scalar. What we call ‘internal forces’ are not really forces. We can’t talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for each string half. An ‘internal force’ isn’t a force unless it is made external by a free body diagram cut, in which case it is not internal. We use this confusing language because of its strong place in engineering practice and its constant reinforcement by our intuitions which sense ‘internal forces’. Whenever you see the phrase ‘internal force’ you should substitute in your mind ‘a scalar with dimensions of force from which you can find the force on a free body diagram cut’ ①.

① Calling tension a scalar is one of the practical lies we tell you for relative simplicity. The clearest representation of ‘internal forces’ is with tensors. But that idea is too advanced for this book.

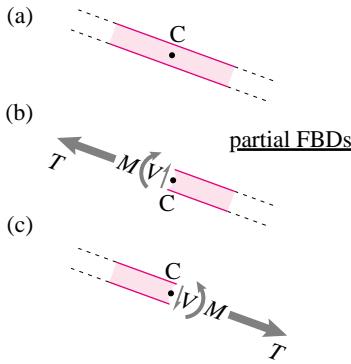
For a two force body the tension is a constant along the length (because we found  $T$  without ever using information about the location of the free body diagram cut). We used this idea without comment in trusses when we included a small stub of each bar in the free body diagrams of the joints and showed a tension force along the stub.

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A  $10kN$  cable can carry a tension of  $10,000\text{ N}$  all along its length. This means a free body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than  $10,000\text{ N}$ . If the rope is frayed it make break at, say, a tension of  $2,000\text{ N}$ , meaning a free body diagram with a cut at the fray can only show forces up to  $2,000\text{ N}$ .

As noted in the context of trusses, tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends).

## Shear force and bending moment

To characterize the strength of more than just 2-force bodies. we need to generalize the concept of tension. The main idea, which was emphasized in chapter 2, is this:



You can make a free body diagram cut anywhere on any body no matter how it is loaded.

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals (not just in trusses).
- Internal forces in long narrow things are easier to understand than in bulkier objects, and so are studied first.

For now we limit ourselves to 2D statics. At an arbitrary cut we break the force into two components (see fig. 4.42).

- The *tension*  $T$  is the scalar part of the force directed along the bar assumed positive when pulling away from the free body diagram cut.
- The *shear force*  $V$  is the force perpendicular to the bar (tangent to the free body diagram cut). Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

Since we are just doing 2D problems now, the moment is always in the out of plane (typically  $\hat{k}$ ) direction.

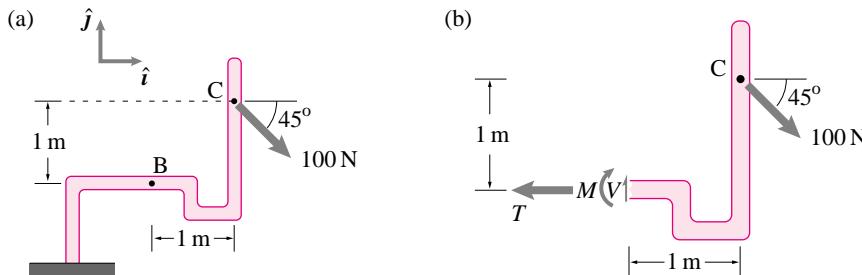
Figure 4.42: a) A piece of a structure, loads not shown; b) a partial free body diagram of the right part of the bar; c) a partial free body diagram of the left part of the bar.  
(Filename:tfigure.signs)



Figure 4.43: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.  
(Filename:tfigure.smilingbeam)

- The bending moment  $M$  is the scalar part of the bending moment. The sign convention is that for a smiling beam (Fig. 4.43): A clockwise ( $-\hat{k}$ ) couple is positive on a left cut and a counterclockwise ( $\hat{k}$ ) couple is positive on a right cut<sup>①</sup>.

The tension  $T$ , shear  $V$ , and bending moment  $M$  on fig. 4.42 follows these sign conventions.



<sup>①</sup> Note that neither  $V$  nor  $T$  changes if you rotate your paper until the picture is upside down. However, this definition for the sign convention for  $M$  has the disadvantage that the bending moment does change sign if you turn your paper upside down. Here is a more precise definition which gets rid of this flaw. Choose the  $x$  and  $\hat{i}$  direction to be along the bar. Bending moment is positive for a cut with normal in the  $-\hat{i}$  direction if clockwise. Bending moment is positive for a cut with a normal in the  $\hat{i}$  direction if counterclockwise. More concisely, if  $\hat{n}$  is the normal to the cut, bending moment is positive in the  $\hat{n} \times \hat{j}$  direction.

#### Example: Internal forces in a bent rod

The internal forces at B can be found by making a free body diagram of a portion of the structure with a cut at B.

$$\begin{aligned} \text{Sum of vertical forces is zero} &\Rightarrow V = (100/\sqrt{2}) \text{ N} \\ \text{Sum of horizontal forces is zero} &\Rightarrow T = (100/\sqrt{2}) \text{ N} \\ \text{Sum of moments about the cut at B is zero} &\Rightarrow M = -100\sqrt{2} \text{ N m.} \end{aligned}$$

□

## Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. If you want to know the internal forces at a variety of points you can draw a variety of free body diagrams with cuts at those points of interest. Another approach, which we present now, is to leave the position of the free body diagram cut a variable, and then calculate the internal forces in terms of that variable.

#### Example: Tension in a rod from its own weight.

The uniform  $1 \text{ cm}^2$  steel square rod with density  $\rho = 7.7 \text{ gm/cm}^3$  and length  $\ell = 100 \text{ m}$  has total weight  $W = mg = \rho\ell Ag$  (see fig. 4.44). What is the tension a distance  $x_D$  from the top? Using the free body diagram with cut at  $x_D$  we get:

$$\begin{aligned} \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} \\ \Rightarrow T &= \rho Ag(\ell - x_D) \\ &= (7.7 \text{ gm/cm}^3)(1 \text{ cm}^2)(9.8 \text{ N/kg})(100 \text{ m} - x_D) \\ &= 7.7 \cdot 9.8 \frac{\text{gm N m}}{\text{cm kg}} (100 - \frac{x_D}{\text{m}}) \underbrace{\left( \frac{1 \text{ kg}}{1000 \text{ gm}} \right)}_1 \underbrace{\left( \frac{100 \text{ cm}}{1 \text{ m}} \right)}_1 \\ &= 7.5 \left( 100 - \frac{x_D}{\text{m}} \right) \text{ N.} \end{aligned}$$

So, at the bottom end at  $x_D = 100 \text{ m}$  we get  $T = 0$  and at the top end where  $x_D = 0 \text{ m}$  we get  $T = 750 \text{ N}$  and in the middle at  $x_D = 50 \text{ m}$  we get  $T = 375 \text{ N}$ . □

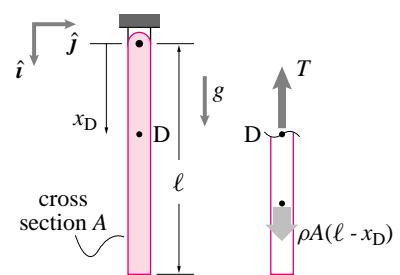


Figure 4.44: a) Rod hanging with gravity.  
b) free body diagram with cut at  $x_D$ .

(Filename:tfigure.tensioncut)

Because the free body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

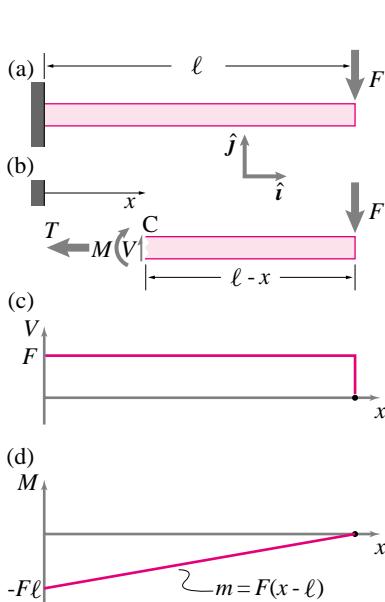


Figure 4.45: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

(Filename:tfigure.bendandsheardia)

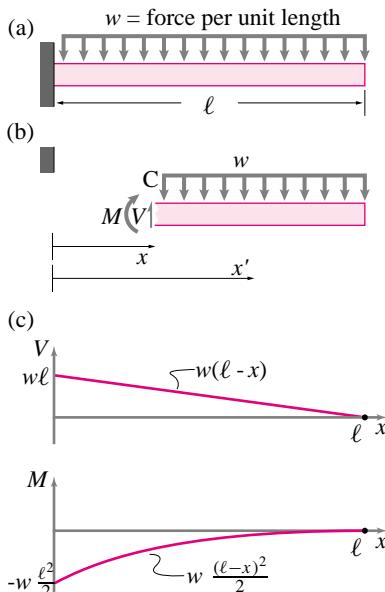


Figure 4.46: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

(Filename:tfigure.uniformcant)

### Example: Cantilever $M$ and $V$ diagram

A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load  $F$  at the end (fig. 4.45a). By drawing a free body diagram with a cut at the arbitrary point C (fig. 4.45b) we can find the internal forces as a function of the position of C.

$$\begin{aligned} \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow V = F \\ \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} &\Rightarrow T = 0 \\ \left\{ \sum \vec{M}_C = \vec{0} \right\} \cdot \hat{k} &\Rightarrow M(x) = F(x - \ell). \end{aligned}$$

That the tension is zero in these problems is so well known that the tension is often not drawn on the free body diagram and not calculated. We can now plot  $V(x)$  and  $M(x)$  as in figs. 4.45c and 4.45d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ( $M = -F\ell$ ) to 0 at the end. It is the big value of  $|M|$  at the fixed support that makes cantilever beams typically break there.  $\square$

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

### Example: Distributed load

A cantilever beam has a downwards uniformly distributed load of  $w$  per unit length (fig. 4.46a). Using the free body diagram shown (fig. 4.46b) we can find:

$$\begin{aligned} \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow \left\{ V(x)\hat{j} + \int d\vec{F} \right\} \cdot \hat{j} = 0 \\ &\Rightarrow V(x) = \int_x^\ell w dx' \\ &= w \cdot (\ell - x) \\ \left\{ \sum \vec{M}_C = \vec{0} \right\} \cdot \hat{k} &\Rightarrow \left\{ M(x)(-\hat{k}) + \int \vec{r}_{/C} \times d\vec{F} \right\} \cdot \hat{k} = 0 \\ &\Rightarrow M(x) = \int_x^\ell (x' - x) w dx' \\ &= w \cdot (x'^2/2 - x'x) \Big|_x^\ell \\ &= (\ell^2/2 - \ell x) - (x^2/2 - x^2) \\ &= -w \cdot (\ell - x)^2/2. \end{aligned}$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force  $w \cdot (\ell - x)$  applied a distance  $(\ell - x)/2$  to the right of the cut. Shear and bending moment diagrams are shown in figs. 4.46a and 4.46b.  $\square$

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

**Example: Superposition**

Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$\begin{aligned} V &= F + w(\ell - x) \\ M(x) &= F(x - \ell) + -w(\ell - x)^2/2. \end{aligned}$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments.  $\square$

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free body diagram of the cut bar, depending on the location of the cut.

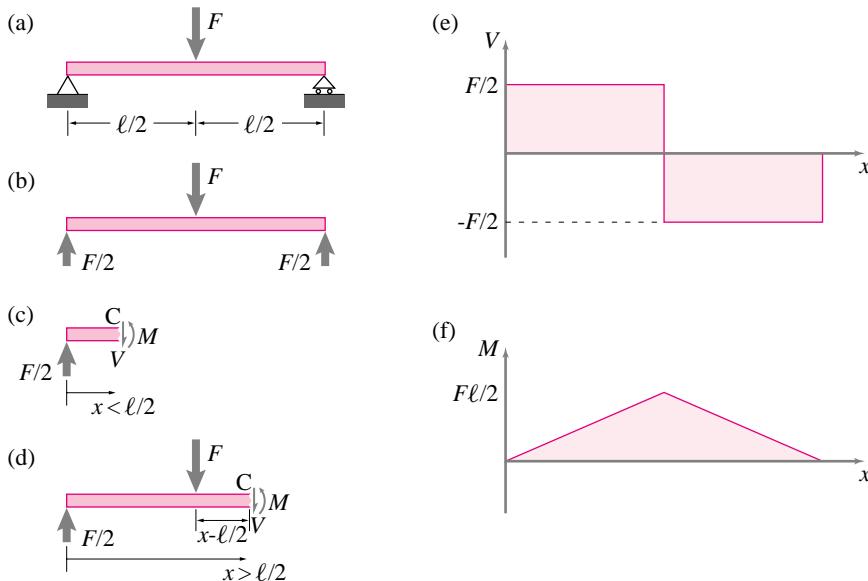
**Example: Simply supported beam with point load in the middle**

Figure 4.47: a) Simply supported beam, b) free body diagram of whole beam, c) free body diagram with cut to the left of the applied force, d) free body diagram with cut to the right of the applied force  
e) Shear force diagram, f) Bending moment diagram

(Filename:tfigure.simplesup)

A *simply supported* beam is mounted with pivots at both ends (fig. 4.47a). First we draw a free body diagram of the whole beam (fig. 4.47a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 4.47c and 4.47d). With the free body diagram 4.47c we can find  $V(x)$  and  $M(x)$  for  $x < \ell/2$  and with the free body diagram 4.47d we can find  $V(x)$  and  $M(x)$  for  $x > \ell/2$ .

$$\begin{aligned} \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{j} &\Rightarrow V = F/2 & \text{for } x < \ell/2 \\ &&= -F/2 & \text{for } x > \ell/2 \\ \{\sum \vec{M}_C = \vec{0}\} \cdot \hat{k} &\Rightarrow M(x) = Fx/2 & \text{for } x < \ell/2 \\ &&= F(\ell - x)/2 & \text{for } x > \ell/2 \end{aligned}$$

These relations can be plotted as in figs. 4.47e and 4.47f. Some observations: For this beam the biggest bending moment is in the middle, the

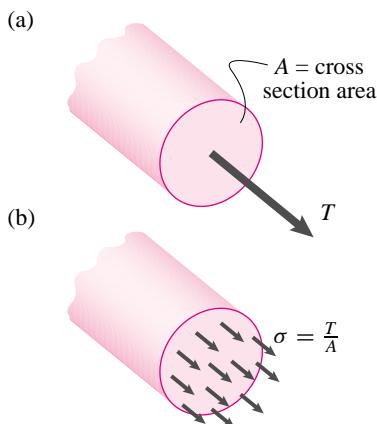
place where simply supported beams often break. Instead of the free body diagram shown in (c) and (d) we could have drawn a free body diagrams of the bar to the right of the cut and would have got the same  $V(x)$  and  $M(x)$ . We avoided drawing a free body diagram cut at the applied load where  $V(x)$  has a discontinuity.  $\square$

## How to find $T$ , $V$ , and $M$

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free body diagram of the whole bar.
- Using the free body diagram above find the reaction forces .
- Draw a free body diagram(s) of the cut bar of interest.
  - For each region between concentrated loads draw one free body diagram.
  - Show the piece from the cut to one or the other end (So that all but the internal forces are known).
  - Don't make cuts at intermediate points of connection or load application.
- Use the equilibrium equations to find  $T$ ,  $V$ , or  $M$  (Moment balance about a point at the cut is a good way to find  $M$ .)
- Use the results above to plot  $V(x)$  and  $M(x)$  ( $T(x)$  is rarely plotted).
  - Use the same  $x$  scale for this plot as for the free body diagram of the whole bar.
  - Put the plots directly under the free body diagram of the bar (so you can most easily relate features of the loads to features of the  $V$  and  $M$  diagrams).

## Stress is force per unit area



For a given load, if you replace one bar in tension with two bars side by side you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side by side bars together you would again have one bar but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load  $T$ , the force per unit area on an interior free body diagram cut is  $T/A$ . Force per unit area normal to an internal free body diagram cut is called *tension stress* and denoted  $\sigma$  (lower case ‘sigma’, the Greek letter s).

$$\sigma = T/A$$

Figure 4.48: a) Tension on a free body diagram cut is equivalent to b) uniform tension stress.

(Filename:tfigure.tension)

**Example: Stress in a hanging bar**

Look at the hanging bar in the example on page 149. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it.  $\square$

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress  $\frac{T}{A}$  is actually the tension stress all across the cross section.

We can similarly define the average *shear stress*  $\tau_{\text{ave}}$  ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress  $\tau_{\text{ave}}$  does give an indication of the actual shear stress in the bar (*e.g.*, for a rectangular elastic bar the peak shear stress is 50% larger than  $\tau_{\text{ave}}$ ).

The biggest stresses typically come from bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be big, gives you the important notion that bending moment is a common cause of structural failure.

## Internal force summary

'Internal forces' are the scalars which describe the force and moment on potential internal free body diagram cuts. They are found by applying the equilibrium equations to free body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.

"Cut. OK it's a take. Lets quit for the day."

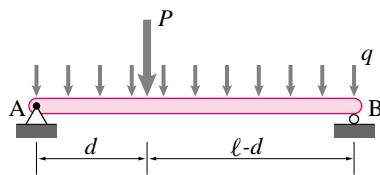


Figure 4.49: (Filename:sfig4.intern.ssup)

**SAMPLE 4.10 Support reactions on a simply supported beam:** A uniform beam of length 3 m is simply supported at A and B as shown in the figure. A uniformly distributed vertical load  $q = 100 \text{ N/m}$  acts over the entire length of the beam. In addition, a concentrated load  $P = 150 \text{ N}$  acts at a distance  $d = 1 \text{ m}$  from the left end. Find the support reactions.

**Solution** Since the beam is supported at A on a pin joint and at B on a roller, the unknown reactions are

$$\vec{A} = A_x \hat{i} + A_y \hat{j}, \quad \vec{B} = B_y \hat{j}$$

The uniformly distributed load  $q$  can be replaced by an equivalent concentrated load  $W = q\ell$  acting at the center of the beam span. The free body diagram of the beam, with the concentrated load replaced by the equivalent concentrated load is shown in Fig. 4.50. The moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , gives

$$\begin{aligned} (-Pd - W \frac{\ell}{2} + B_y \ell) \hat{k} &= \vec{0} \\ \Rightarrow B_y &= P \frac{d}{\ell} + \frac{1}{2} W \\ &= 150 \text{ N} \cdot \frac{1}{3} + \frac{1}{2} \cdot 300 \text{ N} = 200 \text{ N} \end{aligned}$$

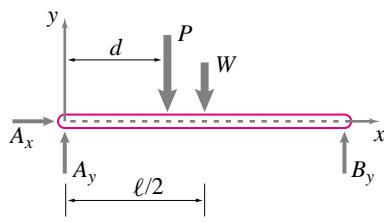


Figure 4.50: (Filename:sfig4.intern.ssup.a)

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$\begin{aligned} \vec{A} + B_y \hat{j} - P \hat{j} - W \hat{j} &= \vec{0} \\ \Rightarrow \vec{A} &= (-B_y + P + W) \hat{j} \\ &= (-200 \text{ N} + 150 \text{ N} + 300 \text{ N}) \hat{j} = 250 \text{ N} \hat{j} \end{aligned}$$

$$\boxed{\vec{A} = 250 \text{ N} \hat{j}, \quad \vec{B} = 200 \text{ N} \hat{j}}$$

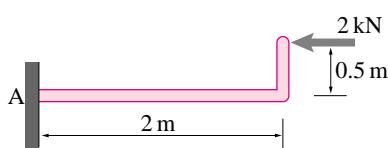


Figure 4.51: (Filename:sfig4.intern.cant)

**SAMPLE 4.11 Support reactions on a cantilever beam:** A 2 kN horizontal force acts at the tip of an 'L' shaped cantilever beam as shown in the figure. Find the support reactions at A.

**Solution** The free body diagram of the beam is shown in Fig. 4.52. The reaction force at A is  $\vec{A}$  and the reaction moment is  $\vec{M} = M \hat{k}$ . Writing moment balance equation about point A,  $\sum \vec{M}_A = \vec{0}$ , we get

$$\begin{aligned} \vec{M} + \vec{r}_{C/A} \times \vec{F} &= \vec{0} \\ \vec{M} + (\ell \hat{i} + h \hat{j}) \times (-F \hat{i}) &= \vec{0} \\ \Rightarrow \vec{M} &= -Fh \hat{k} \\ &= -2 \text{ kN} \cdot 0.5 \text{ m} \hat{k} \\ &= -1 \text{ kN} \cdot \text{m} \hat{k} \end{aligned}$$

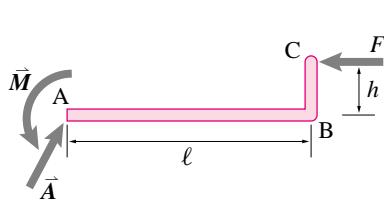


Figure 4.52: (Filename:sfig4.intern.cant.a)

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$\begin{aligned} \vec{A} + \vec{F} &= \vec{0} \\ \Rightarrow \vec{A} &= -\vec{F} = -(-2 \text{ kN} \hat{i}) = 2 \text{ kN} \hat{i} \end{aligned}$$

$$\boxed{\vec{A} = 2 \text{ kN} \hat{i}, \quad \vec{M} = -1 \text{ kN} \cdot \text{m} \hat{k}}$$

**SAMPLE 4.12 Net force of a uniformly distributed system:** A uniformly distributed vertical load of intensity 100 N/m acts on a beam of length  $\ell = 2 \text{ m}$  as shown in the figure.

- Find the net force acting on the beam.
- Find an equivalent force-couple system at the mid-point of the beam.
- Find an equivalent force-couple system at the right end of the beam.

### Solution

(a) **The net force:** Since the load is uniformly distributed along the length, we can find the total or the net load by calculating the load on an infinitesimal segment of length  $dx$  of the beam and then integrating over the entire length of the beam. Let the load intensity (load per unit length) be  $q$  ( $q = 100 \text{ N/m}$ , as given). Then the vertical load on segment  $dx$  is (see Fig. 4.54),

$$d\vec{F} = q dx (-\hat{j}).$$

Therefore, the net force is,

$$\vec{F}_{\text{net}} = \int_0^\ell q dx (-\hat{j}) = q \ell \hat{j} = -100 \text{ N/m} \cdot 2 \text{ m} \hat{j} = -200 \text{ N} \hat{j}.$$

$$\boxed{\vec{F}_{\text{net}} = -200 \text{ N} \hat{j}}$$

(b) **The equivalent system at the mid-point:** We have already calculated the net force that can replace the uniformly distributed load. Now we need to calculate the couple at the mid-point of the beam to get the equivalent force-couple system. Again, consider a small segment of the beam of length  $dx$  located at distance  $x$  from the mid-point C (see Fig. 4.55). The moment about point C due to the load on  $dx$  is  $(q dx)x(-\hat{k})$ . But, we can find a similar segment on the other side of C with exactly the same length  $dx$ , at exactly the same distance  $x$ , that produces a moment of  $(q dx)x(+\hat{k})$ . The two contributions cancel each other and we have a net zero moment about C. Now, you can imagine the whole beam made up of these pairs that contribute equal and opposite moment about C and thus the net moment about the mid-point is zero. You can also find the same result by straight integration:

$$\vec{M}_C = \int_{-\ell/2}^{+\ell/2} qx dx (-\hat{k}) = \frac{qx^2}{2} \Big|_{-\ell/2}^{+\ell/2} (-\hat{k}) = \vec{0}.$$

$$\boxed{\vec{F}_{\text{net}} = -200 \text{ N} \hat{j}, \text{ and } \vec{M}_C = \vec{0}}$$

(c) **The equivalent system at the end:** The net force remains the same as above. We compute the net moment about the end point B, referring to Fig. 4.56, as follows.

$$\begin{aligned} \vec{M}_B &= \int_0^\ell (-x \hat{i}) \times (-q dx \hat{j}) = -q \int_0^\ell x dx \hat{k} \\ &= -\frac{q \ell^2}{2} \hat{k} = -\frac{100 \text{ N/m} \cdot 4 \text{ m}^2}{2} \hat{k} = -200 \text{ N}\cdot\text{m} \hat{k}. \end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -200 \text{ N} \hat{j} \text{ and } \vec{M}_B = -200 \text{ N}\cdot\text{m} \hat{k}}$$

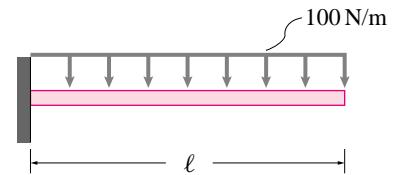


Figure 4.53: (Filename:sfig2.vec3.uniform)

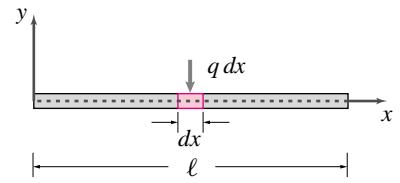


Figure 4.54: (Filename:sfig2.vec3.uniform.a)

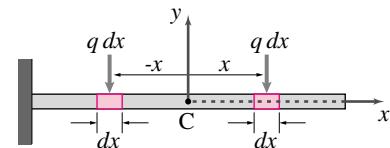


Figure 4.55: (Filename:sfig2.vec3.uniform.b)

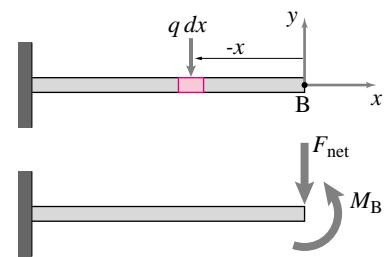


Figure 4.56: (Filename:sfig2.vec3.uniform.c)

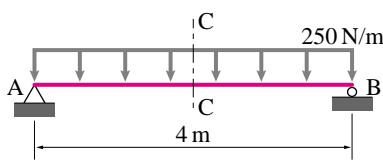


Figure 4.57: (Filename:sfig4.intern.ssvm)

**SAMPLE 4.13** For the uniformly loaded, simply supported beam shown in the figure, find the shear force and the bending moment at the mid-section c-c of the beam.

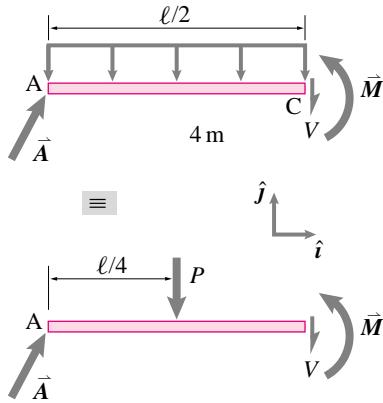


Figure 4.58: (Filename:sfig4.intern.ssvm.a)

**Solution** To determine the shear force  $V$  and the bending moment  $M$  at the mid-section c-c, we cut the beam at c-c and draw its free body diagram as shown in Fig. 4.58. For writing force and moment balance equations we use the second figure where we have replaced the distributed load with an equivalent single load  $F = (q\ell)/2$  acting vertically downward at distance  $\ell/4$  from end A.

The force balance,  $\sum \vec{F} = \vec{0}$ , implies that

$$A_x \hat{i} + A_y \hat{j} - V \hat{j} - F \hat{j} = \vec{0}$$

Dotting with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$\begin{aligned} A_x &= 0 \\ V &= A_y - F \end{aligned} \quad (4.34)$$

$$= A_y - \frac{q\ell}{2} \quad (4.35)$$

From the moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , we get

$$\begin{aligned} M \hat{k} - \left( \frac{q\ell}{2} \cdot \frac{\ell}{4} \right) \hat{k} - V \ell \hat{k} &= 0 \\ \Rightarrow M &= \frac{q\ell^2}{8} + V\ell \end{aligned} \quad (4.36)$$

Thus, to find  $V$  and  $M$  we need to know the support reaction  $\vec{A}$ . From the free body diagram of the beam in Fig. 4.59 and the moment equilibrium equation about point B,  $\sum \vec{M}_B = \vec{0}$ , we get

$$\begin{aligned} \vec{r}_{A/B} \times \vec{A} + \vec{r}_{C/B} \times \vec{Q} &= \vec{0} \\ (-A_y \ell + q\ell \frac{\ell}{2}) \hat{k} &= \vec{0} \\ \Rightarrow A_y &= \frac{q\ell}{2} = 500 \text{ N} \end{aligned}$$

Thus  $\vec{A} = 500 \text{ N} \hat{j}$ . Substituting  $\vec{A}$  in eqns. (4.35) and (4.36), we get

$$\begin{aligned} V &= 500 \text{ N} - 500 \text{ N} = 0 \\ M &= 250 \text{ N} \cdot \frac{(4 \text{ m})^2}{8} + 0 \\ &= 500 \text{ N}\cdot\text{m} \end{aligned}$$

$V = 0, \quad M = 500 \text{ N}\cdot\text{m}$

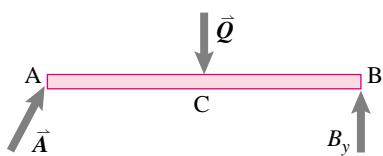


Figure 4.59: (Filename:sfig4.intern.ssvm.b)

**SAMPLE 4.14** The cantilever beam AD is loaded as shown in the figure where  $W = 200 \text{ lbf}$ . Find the shear force and bending moment on a section just left of point B and another section just right of point B.

**Solution** To find the desired internal forces, we need to make a cut at a section just to the left of B and one just to the right of B. We first take the one that is to the right of point B. The free body diagram of the right part of the cut beam is shown in Fig. 4.61. Note that if we selected the left part of the beam, we would need to determine support reactions at A. The uniformly distributed load  $2W$  of the block sitting on the beam can be replaced by an equivalent concentrated load  $2W$  acting at point E, at distance  $a/2$  from the end D of the beam.

Let us denote the the shear force by  $V^+$  and the bending moment by  $M^+$  at the section of our interest. Now, from the force equilibrium of the part-beam BD we get

$$\begin{aligned} V^+ \hat{j} - 2W \hat{j} &= \vec{0} \\ \Rightarrow V^+ &= 2W \\ &= 400 \text{ lbf} \end{aligned}$$

The moment equilibrium about point B,  $\sum \vec{M}_B = \vec{0}$ , gives

$$\begin{aligned} -M^+ \hat{k} - 2W \cdot \frac{3a}{2} \hat{k} &= \vec{0} \\ \Rightarrow M^+ &= -3Wa \\ &= -1200 \text{ lb}\cdot\text{ft} \end{aligned}$$

Now, we determine the internal forces at a section just to the left of point B. Let the shear and bending moment at this section be  $V^-$  and  $M^-$ , respectively, as shown in the free body diagram (Fig. 4.62). Note that load  $W$  acting at B is now included in the free body diagram since the beam is now cut just a teeny bit left of this load.

From the force equilibrium of the part-beam, we have

$$\begin{aligned} V^- \hat{j} - W \hat{j} - 2W \hat{j} &= \vec{0} \\ \Rightarrow V^- &= 3W \\ &= 600 \text{ lbf} \end{aligned}$$

and, from moment equilibrium about point B,  $\sum \vec{M}_B = \vec{0}$ , we get

$$\begin{aligned} -M^- \hat{k} - 2W \cdot \frac{3a}{2} \hat{k} &= \vec{0} \\ \Rightarrow M^- &= -3Wa \\ &= -1200 \text{ lb}\cdot\text{ft} \end{aligned}$$

$$M^+ = M^- = -1200 \text{ lb}\cdot\text{ft}, \quad V^+ = 400 \text{ lbf}, \quad V^- = 600 \text{ lbf}$$

Note that the bending moment remains the same on either side of point B but the shear force jumps by  $V^+ - V^- = 200 \text{ lbf} = W$  as we go from right to the left. This jump is expected because a concentrated load  $W$  acts at B, in between the two sections we consider. Concentrated external forces cause a jump in shear, and concentrated external moments cause a jump in the bending moment.

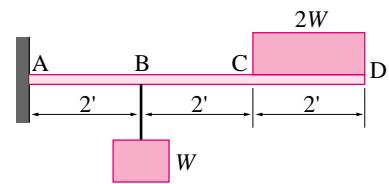


Figure 4.60: (Filename:sfig4.intern.cantvm)

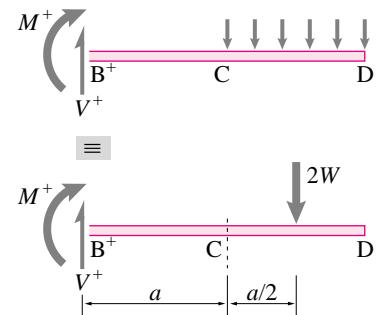


Figure 4.61: (Filename:sfig4.intern.cantvm.a)

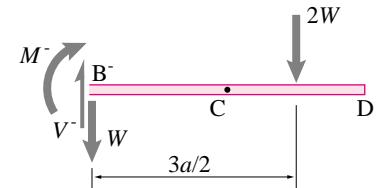


Figure 4.62: (Filename:sfig4.intern.cantvm.b)

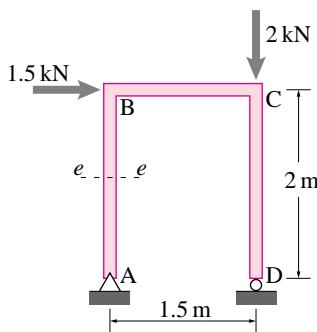


Figure 4.63: (Filename:sfig4.intern.frame)

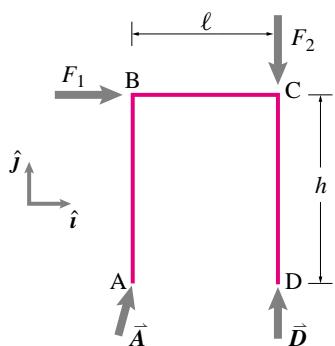


Figure 4.64: (Filename:sfig4.intern.frame.a)

**SAMPLE 4.15 A simple frame:** A 2 m high and 1.5 m wide rectangular frame ABCD is loaded with a 1.5 kN horizontal force at B and a 2 kN vertical force at C. Find the internal forces and moments at the mid-section e-e of the vertical leg AB.

**Solution** To find the internal forces and moments, we need to cut the frame at the specified section e-e and consider the free body diagram of either AE or EBCD. No matter which of the two we select, we will need the support reactions at A or D to determine the internal forces. Therefore, let us first find the support reactions at A and D by considering the free body diagram of the whole frame (Fig. 4.64). The moment balance about point A,  $\sum \vec{M}_A = \vec{0}$ , gives

$$\begin{aligned} \vec{r}_B \times \vec{F}_1 + \vec{r}_C \times \vec{F}_2 + \vec{r}_D \times \vec{D} &= \vec{0} \\ h\hat{j} \times F_1\hat{i} + (h\hat{j} + \ell\hat{i}) \times (-F_2\hat{j}) + \ell\hat{i} \times D\hat{j} &= \vec{0} \\ -F_1h\hat{k} - F_2\ell\hat{k} + D\ell\hat{k} &= \vec{0} \\ \Rightarrow D &= F_1 \frac{h}{\ell} + F_2 \\ &= 1.5 \text{ kN} \cdot \frac{2}{1.5} + 2 \text{ kN} \\ &= 4 \text{ kN} \end{aligned}$$

From force equilibrium,  $\sum \vec{F} = \vec{0}$ , we have

$$\begin{aligned} \vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{D} \\ &= -F_1\hat{i} + F_2\hat{j} - D\hat{j} \\ &= -1.5 \text{ kN}\hat{i} - 2 \text{ kN}\hat{j} \end{aligned}$$

Now we draw the free body diagram of AE to find the shear force  $V$ , axial (tensile) force  $T$ , and the bending moment  $M$  at section e-e.

From the force equilibrium of part AE, we get

$$\begin{aligned} \vec{A} - V\hat{i} + T\hat{j} &= \vec{0} \\ (A_x - V)\hat{i} + (A_y + T)\hat{j} &= \vec{0} \\ \Rightarrow V &= A_x = -1.5 \text{ kN} \\ T &= -A_y = 2 \text{ kN} \end{aligned}$$

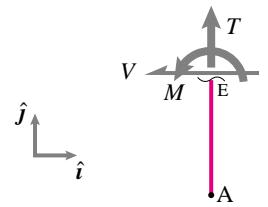


Figure 4.65: (Filename:sfig4.intern.frame.b)

From the moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , we have

$$\begin{aligned} M\hat{k} + \frac{h}{2}\hat{j} \times (-V\hat{i}) &= \vec{0} \\ M\hat{k} + V \frac{h}{2}\hat{k} &= \vec{0} \\ \Rightarrow M &= -V \frac{h}{2} \\ &= -(-1.5 \text{ kN}) \cdot \frac{2 \text{ m}}{2} \\ &= 1.5 \text{ kN}\cdot\text{m} \end{aligned}$$

$V = 1.5 \text{ kN}, \quad T = 2 \text{ kN}, \quad M = 1.5 \text{ kN}\cdot\text{m}$
---

**SAMPLE 4.16 Shear force and bending moment diagrams:** A simply supported beam of length  $\ell = 2$  m carries a concentrated vertical load  $F = 100$  N at a distance  $a$  from its left end. Find and plot the shear force and the bending moment along the length of the beam for  $a = \ell/4$ .

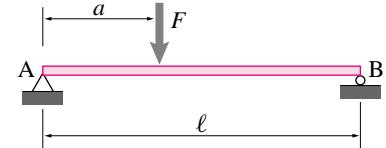


Figure 4.66: (Filename:sfig4.intern.ssvmx)

**Solution** We first find the support reactions by considering the free body diagram of the whole beam shown in Fig. 4.67. By now, we have developed enough intuition to know that the reaction at A will have no horizontal component since there is no external force in the horizontal direction. Therefore, we take the reactions at A and B to be only vertical. Now, from the moment equilibrium about point B,  $\sum \vec{M}_B = \vec{0}$ , we get

$$\begin{aligned} F(\ell - a)\hat{k} - A_y\ell\hat{k} &= \vec{0} \\ \Rightarrow A_y &= \frac{F(\ell - a)}{\ell} \\ &= F\left(1 - \frac{a}{\ell}\right) \end{aligned}$$

and from the force equilibrium in the vertical direction,  $(\sum \vec{F} = \vec{0}) \cdot \hat{j}$ , we get

$$B_y = F - A_y = F\frac{a}{\ell}$$

Now we make a cut at an arbitrary (variable) distance  $x$  from A where  $x < a$  (see Fig. 4.68). Carrying out the force balance and the moment balance about point A, we get, for  $0 \leq x < a$ ,

$$V = A_y = F\left(1 - \frac{a}{\ell}\right) \quad (4.37)$$

$$M = Vx = F\left(1 - \frac{a}{\ell}\right)x \quad (4.38)$$

Thus  $V$  is constant for all  $x < a$  but  $M$  varies linearly with  $x$ .

Now we make a cut at an arbitrary  $x$  to the right of load  $F$ , i.e.,  $a < x \leq \ell$ . Again, from the force balance in the vertical direction, we get

$$V = -F + F\left(1 - \frac{a}{\ell}\right) = -F\frac{a}{\ell} \quad (4.39)$$

and from the moment balance about point A,

$$\begin{aligned} M &= Fa + Vx \\ &= Fa - F\frac{a}{\ell}x \\ &= Fa\left(1 - \frac{x}{\ell}\right) \end{aligned} \quad (4.40)$$

Although eqn. (4.38) is strictly valid for  $x < a$  and eqn. (4.40) is strictly valid for  $x > a$ , substituting  $x = a$  in these two equations gives the same value for  $M (= Fa(1 - a/\ell))$  as it must because there is no reason to have a jump in the bending moment at any point along the length of the beam. The shear force  $V$ , however, does jump because of the concentrated load  $F$  at  $x = a$ .

Now, we plug in  $a = \ell/4 = 0.5$  m, and  $F = 100$  N, in eqns. (4.37)–(4.40) and plot  $V$  and  $M$  along the length of the beam by varying  $x$ . The plots of  $V(x)$  and  $M(x)$  are shown in Fig. 4.69.

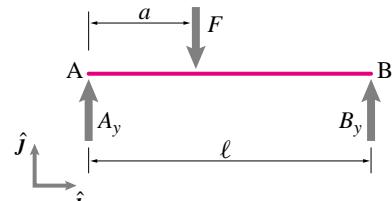


Figure 4.67: (Filename:sfig4.intern.ssvmx.a)

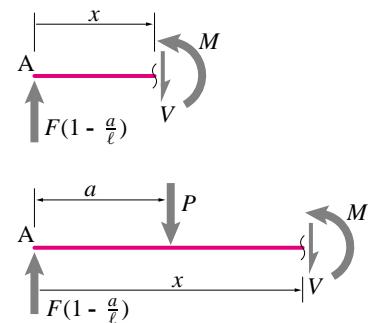


Figure 4.68: (Filename:sfig4.intern.ssvmx.b)

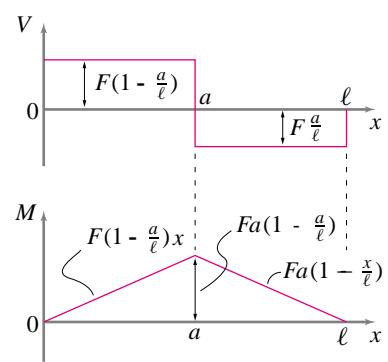


Figure 4.69: (Filename:sfig4.intern.ssvmx.c)

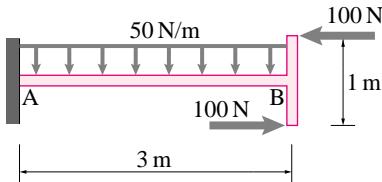


Figure 4.70: (Filename:sfig4.intern.cantvnx)

**SAMPLE 4.17 Shear force and bending moment diagrams by superposition:** For the cantilever beam and the loading shown in the figure, draw the shear force and the bending moment diagrams by

- considering all the loads together, and
- considering each load (of one type) at a time and using superposition.

### Solution

(a)  *$V(x)$  and  $M(x)$  with all forces considered together:* The horizontal forces acting at the end of the cantilever are equal and opposite and, therefore, produce a couple. So, we first replace these forces by an equivalent couple  $M_{\text{applied}} = 100 \text{ N} \cdot 1 \text{ m} = 100 \text{ N}\cdot\text{m}$ . Since we have a cantilever beam, we can consider the right hand side of the beam after making a cut anywhere for finding  $V$  and  $M$  without first finding the support reactions.

Let us cut the beam at an arbitrary distance  $x$  from the right hand side. The free body diagram of the right segment of the beam is shown in Fig. 4.71. From the force balance,  $\sum \vec{F} = \vec{0}$ , we find that

$$\begin{aligned} -V\hat{j} + qx\hat{j} &= \vec{0} \\ \Rightarrow V &= qx \\ &= (50 \text{ N/m})x \end{aligned} \quad (4.41)$$

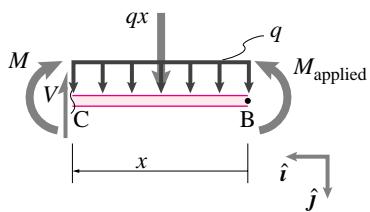


Figure 4.71: (Filename:sfig4.intern.cantvnx.a)

Thus the shear force varies linearly along the length of the beam with

$$\begin{aligned} V(x = 0) &= 0, \\ \text{and } V(x = 3 \text{ m}) &= 150 \text{ N} \end{aligned}$$

The moment balance about point C,  $\sum \vec{M}_C = \vec{0}$ , gives

$$-M\hat{k} - qx \cdot \frac{x}{2}\hat{k} + M_{\text{applied}}\hat{k} = \vec{0}$$

where the moment due to the distributed load is most easily computed by considering an equivalent concentrated load  $qx$  acting at  $x/2$  from the end B. Thus,

$$\Rightarrow M = M_{\text{applied}} - q\frac{x^2}{2} \quad (4.42)$$

$$= 100 \text{ N}\cdot\text{m} - 50 \text{ N/m} \cdot \frac{x^2}{2} \quad (4.43)$$

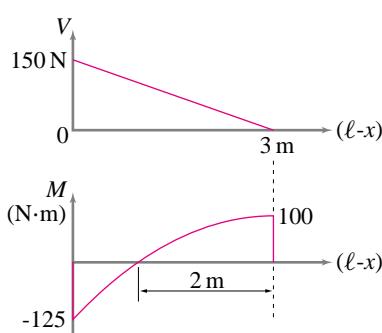


Figure 4.72: (Filename:sfig4.intern.cantvnx.b)

Thus, the bending moment varies quadratically with  $x$  along the length of the beam. In particular, the values at the ends are

$$\begin{aligned} M(x = 0) &= 100 \text{ N}\cdot\text{m} \\ \text{and } M(x = 3 \text{ m}) &= -125 \text{ N}\cdot\text{m} \end{aligned}$$

The shear force and the bending moment diagrams obtained from eqns. (4.41) and (4.42) are shown in Fig. 4.72. Note that  $M = 0$  at  $x = 2 \text{ m}$  as given by eqn. (4.42).

- (b)  $V(x)$  and  $M(x)$  by superposition: Now we consider the cantilever beam with only one type of load at a time. That is, we first consider the beam only with the uniformly distributed load and then only with the end couple. We draw the shear force and the bending moment diagrams for each case separately and then just *add them up*. That is superposition.

So, first let us consider the beam with the uniformly distributed load. The free body diagram of a segment CB, obtained by cutting the beam at a distance  $x$  from the end B, is shown in Fig. 4.73. Once again, from force balance, we get

$$V = qx \quad \text{for } 0 \leq x \leq \ell \quad (4.44)$$

and from the moment balance about point C,  $\sum \vec{M}_C = \vec{0}$ , we get

$$M = -qx \cdot \frac{x}{2} = -q \frac{x^2}{2} \quad \text{for } 0 \leq x \leq \ell \quad (4.45)$$

Figure 4.74 shows the plots of  $V$  and  $M$  obtained from eqns. (4.44) and (4.45), respectively, with the values computed from  $x = 0$  to  $x = 3$  m with  $q = 50 \text{ N/m}$  as given.

Now we take the beam with only the end couple and repeat our analysis. A cut section of the beam is shown in Fig. 4.75. In this case, it should be obvious that from force balance and moment balance about any point, we get

$$\begin{aligned} V &= 0 \\ \text{and } M &= M_{\text{applied}} \end{aligned}$$

Thus, both the shear force and the bending moment are constant along the length of the beam as shown in Fig. 4.75.

Now superimposing (adding) the shear force diagrams from Figs. 4.74 and 4.75, and similarly, the bending moment diagrams from Figs. 4.74 and 4.75, we get the same diagrams as in Fig. 4.76.

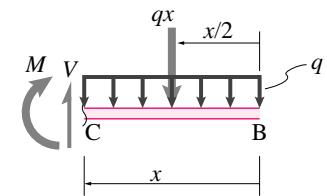


Figure 4.73: (Filename:sfig4.intern.cantvmx.c)

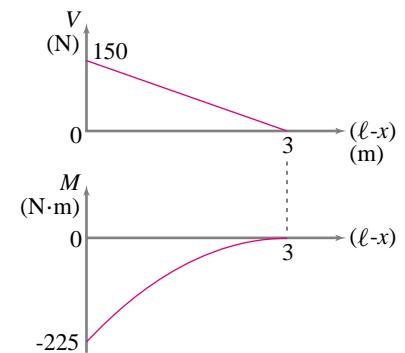


Figure 4.74: (Filename:sfig4.intern.cantvmx.d)

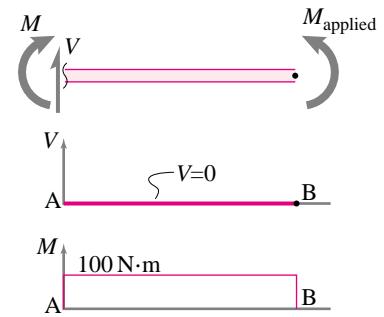


Figure 4.75: (Filename:sfig4.intern.cantvmx.e)

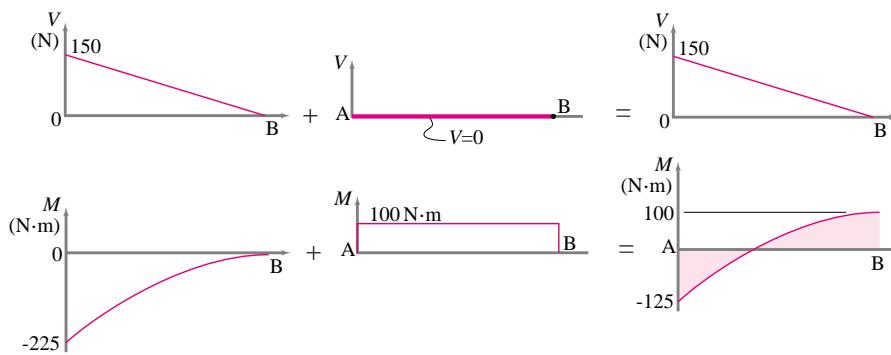


Figure 4.76: (Filename:sfig4.intern.cantvmx.f)

□

## 4.5 Springs

In the same way that machines and buildings are built from bricks, gears, beams, bolts and other standard pieces, elementary mechanics models of the world are made from a few elementary building blocks. Conspicuous so far, roughly categorized, are:

- Special objects:
  - Point masses.
  - Rigid bodies:
    - \* Two force bodies,
    - \* Three force bodies,
    - \* Pulleys, and
    - \* Wheels.
- Special connections:
  - Hinges,
  - Welds,
  - Sliding contact, and
  - Rolling contact.

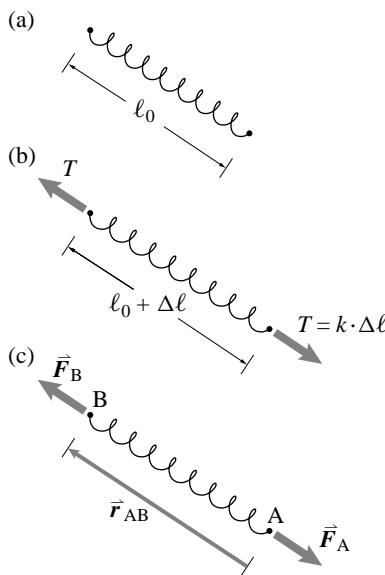


Figure 4.77: An ideal spring with rest length  $\ell_0$  and stretched length  $\ell_0 + \Delta\ell$ . The tension in the spring is  $T$  and the vector forces at the ends are  $\vec{F}_A$  and  $\vec{F}_B$ .

(Filename: figure2.spring2)

Each of these things has a dual life. On the one hand a mechanical hinge corresponds to a product you can buy in a hardware store called a hinge. On the other hand a hinge in mechanics represents a constraint that restricts certain motions and freely allows others. A hinge in a mechanics model may or may not correspond to hardware called a hinge. When considering a box balanced on an edge, we may *model* the contact as a hinge meaning we would use the same equations for the forces of contact as we would use for a hinge. We might buy a pulley, but we might model a rope sliding around a post as a rope on a pulley even though there was no literal pulley in sight. We might buy a brick because it is fairly rigid, and model it in mechanics as a rigid body. But a rigid body model might well be used for human body parts that we know deform noticeably. Thus the mechanics model for these things may correspond more or less with the properties of physical objects with the same names.

This section is devoted to a new building block that similarly has a dual personality: a spring.

Springs, in various forms but most characteristically as helices made of steel wire, can be purchased from hardware stores and mechanical parts suppliers. Springs are used to hold things in place (a clothes pin), to store energy (a clock or toy spring), to reduce contact forces (spring bumpers), and to isolate something from vibrations (a car suspension spring). You will find springs in most any complicated machine. Take apart a disposable camera, an expensive printer, a gas lawn mower, or a washing machine and you will find springs.

On the other hand, springs are used in mechanical ‘models’ of many things that are not explicitly springs. For much of this book we approximate solids as rigid. But sometimes the flexibility or *elasticity* of an object is an important part of its mechanics. The simplest way of accounting for this is to use a spring. So a tire may be modeled as a spring as might be the near-surface-material of a bouncing ball, a strut in a truss, the snap-back of the earth’s crust in an earthquake, your achilles tendon, or the give of soil under a concrete slab. Engineer Tom McMahon idealized the give of a running track as that of a spring when he designed the record breaking track used in the Harvard stadium.

In this section we consider an ideal spring (see also page 88 in section 3.1). You may view an ideal spring as an approximation to a hardware product or as an idealized building block for mechanical models.

An *ideal spring* is a massless two-force body characterized by its *rest length*  $\ell_0$  (also called the ‘relaxed length’, or ‘reference length’), its stiffness  $k$ , and defining equation (or constitutive law):

$$T = k \cdot (\ell - \ell_0) \quad \text{or} \quad T = k \cdot \Delta\ell$$

where  $\ell$  is the present length and  $\Delta\ell$  is the increase in length or *stretch* (see Fig. 4.77). This model of a spring goes by the name *Hooke’s law*.

This spring is *linear* because of the formula  $k\Delta\ell$  and not, say,  $k(\Delta\ell)^3$ . It is *elastic* because the tension only depends on length and not on, say, rate of extension. The spring formula is sometimes quoted as ‘ $F = kx$ ’<sup>①</sup>. A plot of tension versus length for an ideal spring is shown in Fig. 4.78a.

A comment on notation. Often in engineering we write  $\Delta(\text{something})$  to mean the change of ‘*something*.’ Most often one also has in mind a small change. In the context of springs, however,  $\Delta\ell$  is allowed to be a rather large change. We use the notation  $\delta\ell$  for small increments to avoid confusion. A useful way to think about springs is that increments of force are proportional to increments of length change, whether the force or length is already large or small:

$$\delta T = k\delta\ell \quad \text{or} \quad \frac{\delta T}{\delta\ell} = k \quad \text{or} \quad \frac{\delta\ell}{\delta T} = \frac{1}{k}.$$

The reciprocal of stiffness  $\frac{1}{k}$  is called the *compliance*. A compliant spring stretches a lot when the tension is changed. A compliant spring is not stiff. A stiff spring has small stretch when the tension is changed. A stiff spring is not compliant.

Because the spring force is along the spring, we can write a vector formula for the force on the B (say) end of the spring as ( see Fig. 4.77)

$$\vec{F}_B = k \cdot \underbrace{(|\vec{r}_{AB}| - \ell_0)}_{\Delta\ell} \underbrace{\left( \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} \right)}_{\hat{\lambda}_{AB}}. \quad (4.46)$$

where  $\hat{\lambda}_{AB}$  is a unit vector along the spring. This explicit formula is useful for, say, numerical calculations.

## Zero length springs

A special case of linear springs that has remarkable mechanical consequences is a *zero-length* spring that has rest length  $\ell_0 = 0$ . The defining equations in scalar and vector form are thus simplified to

$$T = k\ell \quad \text{and} \quad \vec{F}_B = k \cdot \vec{r}_{AB}.$$

The tension verses length curve for a zero-length spring is shown in Fig. 4.78b.

At first blush such a spring seems *non-physical*, meaning that it seems to represent something which is not a reasonable approximation to any real thing. If you take a coil spring all the metal gets in the way of the spring possibly relaxing to the point of the ends coinciding. In fact, however, there are many ways to build things which act something like zero length springs. For example, the tension verses length curve of a rubber band (or piece of surgical tubing) looks something like that shown in Fig. 4.78c. Over some portion of the curve the zero-length spring approximation may be reasonable. For other physical implementations of zero-length springs see box 4.5 on page 164.

<sup>①</sup> The form ‘ $F = kx$ ’ can lead to sign errors because the direction of the force is not evident. The safest way to avoid sign errors when dealing with springs is to

- Draw a free body diagram of the spring;
- Write the *increase* in length  $\Delta\ell$  in terms of geometry variables in your problem;
- Use  $T = k\Delta\ell$  to find the tension in the spring; and then
- Use the principle of action and reaction to find the forces on the objects to which the spring is connected.

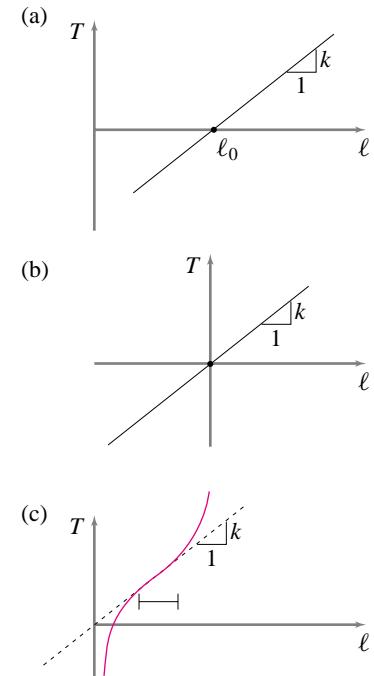


Figure 4.78: a) Tension verses length for an ideal spring, b) for a zero-length spring, and c) for a strip of rubber.

(Filename:tfigure.tensionvslength)

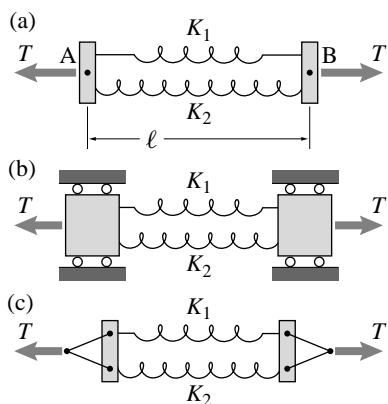


Figure 4.79: a) Schematic of parallel springs, b) genuinely parallel springs, c) a reasonable approximation of parallel springs.

## Assemblies of springs

Here and there throughout the rest of the book you will see how springs are put together with others of the basic building blocks in mechanics. Here we see how springs are put together with other springs.

In short, the result of attaching springs to each other in various ways is a new spring with a stiffness that depends on the stiffnesses of the components and on how the springs are connected.

## Springs in parallel

Two springs that share a load and stretch the same amount are said to be in *parallel*.

(Filename:tfigure.parallelsprings)

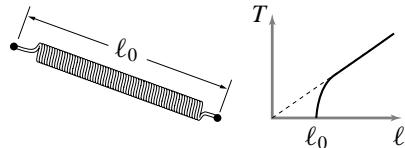
### 4.4 Examples of zero length springs

The mathematics in many mechanics problems is simplified by the zero-length spring approximation. When is it reasonable?

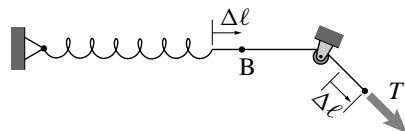
**Rubber bands.** As shown in Fig. 4.78c straps of rubber behave like zero-length springs over some of their length. If this is the working length of your mechanism then the zero-length spring approximation may be good.

**A stretchy conventional spring.** Some springs are so stretchy that they are used at lengths much larger than their rest lengths. Thus the approximation that  $k(\ell - \ell_0) = k\ell \left(1 - \frac{\ell_0}{\ell}\right) \approx k\ell$  may be reasonable.

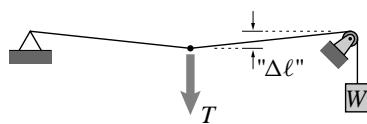
**A pre-stressed coil spring.** Some door springs and many springs used in desk lamps are made tightly wound so that each coil of wire is pressed against the next one. It takes some tension just to start to stretch such a spring. The tension versus length curve for such springs can look very much like a zero-length spring once stretch has started. In fact, in the original elegant 1930's patent, which commonly seen present-day parallelogram-mechanism lamps imitate, specifies that the spring should behave as a zero-length spring. Such a pre-stressed zero-length coil spring was a central part of the design of the long period seismometer featured on a 1959 Scientific American cover.



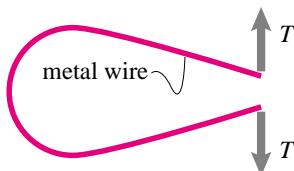
**A spring, string, and pulley.** If a spring is connected to a string that is wrapped around a pulley then the end of the string can feel like a zero force spring if the attachment point is at the pulley when the spring is relaxed.



**A string pulled from the side.** If a taught string is pulled from the side it acts like a zero-length spring in the plane orthogonal to the string.



**A 'U' clip.** If a springy piece of metal is bent so that its unloaded shape is a pinched 'U' then it acts very much like a zero length spring. This is perhaps the best example in that it needs no anchor (unlike a pulley) and can be relaxed to almost zero length (unlike a pre-stressed coil).



The standard schematic for this is shown in Fig. 4.79a where the springs are visibly parallel. This schematic is a non-physical cartoon since applied tension would cause the end-bars to rotate unless the attachment points A and B are located carefully. What is meant by the schematic in Fig. 4.79a is the somewhat clumsy constrained mechanism of Fig. 4.79b. In engineering practice one rarely builds such a structure. A simpler partial constraint against rotations is provided by the triangle of cables shown in Fig. 4.79c; rotations are quite limited if the triangles are much longer than wide. For the purposes of discussion here, we assume that any of Fig. 4.79abc represent a situation where the springs both stretch the same amount.

The stretches and tensions of the two springs are  $\Delta\ell_1$ ,  $\Delta\ell_2$ ,  $T_1$ , and  $T_2$ . For each spring we have the defining constitutive relation:

$$T_1 = k_1 \Delta\ell_1 \quad \text{and} \quad T_2 = k_2 \Delta\ell_2. \quad (4.47)$$

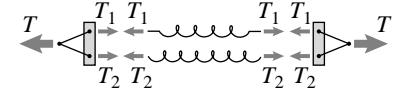


Figure 4.80: Free body diagrams of the components of a parallel spring arrangement.

(Filename: tfigure\_parallelfbds)

As usual, they key to understanding the situation is through appropriate free body diagrams (see Fig. 4.80). Force balance for one of the end supports shows that

$$T = T_1 + T_2 \quad (4.48)$$

showing that the load is shared by the two springs. Springs in parallel stretch the same amount thus we have the kinematic relation:

$$\Delta\ell_1 = \Delta\ell_2 = \Delta\ell. \quad (4.49)$$

Determining the relation between  $T$  and  $\Delta\ell$  is a matter of manipulating these equations:

$$\begin{aligned} T &= T_1 + T_2 \\ &= k_1 \Delta\ell_1 + k_2 \Delta\ell_2 \\ &= k_1 \Delta\ell + k_2 \Delta\ell \\ &= \underbrace{(k_1 + k_2)}_k \Delta\ell. \end{aligned}$$

Thus we get that the effective spring constant of the pair of springs in parallel is, intuitively:

$$k = k_1 + k_2. \quad (4.50)$$

The loads carried by the springs are

$$T_1 = \frac{k_1}{k_1 + k_2} T \quad \text{and} \quad T_2 = \frac{k_2}{k_1 + k_2} T$$

which add up to  $T$  as they must.

#### *Example: Two springs in parallel.*

Take  $k_1 = 99 \text{ N/cm}$  and  $k_2 = 1 \text{ N/cm}$ . The effective spring constant of the parallel combination is:

$$k = k_1 + k_2 = 99 \text{ N/cm} + 1 \text{ N/cm} = 100 \text{ N/cm}.$$

Note that  $T_1/T = .99$  so even though the two springs share the load, the stiffer one carries 99% of it. For practical purposes, or for the design of this system, it would be reasonable to remove the much less stiff spring.

□

The reasoning above with two springs in parallel is easy enough to reproduce with 3 or more springs. The result is:

$$k_{\text{tot}} = k_1 + k_2 + k_3 + \dots \quad \text{and} \quad T_1 = Tk_1/k_{\text{tot}}, \quad T_2 = Tk_2/k_{\text{tot}} \dots$$

That is,

- The net spring constant is the sum of the constants of the separate springs; and
- The load carried by springs is in proportion to their spring constants.

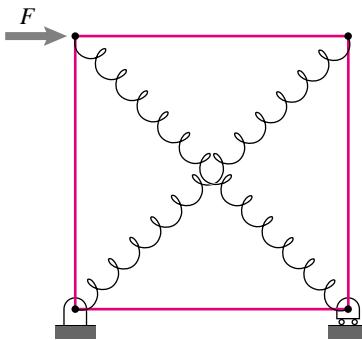


Figure 4.81: A mechanics joke to make a point. The bars in the open square above are rigid. The deformation into a diamond is resisted by the two springs shown. They share the load and they have stretches that are linked by the kinematics. Thus these two perpendicular springs are ‘in parallel.’ (By the way, you are not expected to be able to analyze the compliance of this structure at this point.)

(Filename:tfigure.paralleljoke)

### Some comments on parallel springs

Once you understand the basic ideas and calculations for two side-by-side springs connected to common ends, there are a few things to think about for context.

For the purposes of drawing pictures (*e.g.*, Fig. 4.79a) parallel springs are drawn side by side. But in the mechanics analysis we treated them as if they were on top of each other. A pair of parallel springs is like a two bar truss where the bars are on top of each other but connected at their ends. With 2 bars and 2 joints we have  $2j < b + 3$ , and a redundant truss. In fact this is the simplest redundant truss, as one spring (read bar) does exactly the same job as the other (carries the same loads, resists the same motions). With statics alone we can not find the tensions in the springs since the statics equation  $T_1 + T_2 = T$  has non-unique solutions.

In the context of trusses you may have had the following reasonable thought: The laws of statics allow multiple solutions to redundant problems. But a bar in a real physical structure has, at one instant of time, some unique bar tension. What determines this tension? Now we know the answer: the deformations and material properties. This is the first, and perhaps most conspicuous, occasion in this book that you see a problem where the three pillars of mechanics are assembled in such clear harmony, namely, material properties (eq. 4.47), the laws of mechanics (eq. 4.48), and the geometry of motion and deformation (eq. 4.49). In strength of materials calculations, where the distribution of stress is not determinable by statics alone, this threesome (geometry of deformation, material properties and statics) clearly come together in almost every calculation.

Finally, in the discussion above ‘in parallel’ corresponded to the springs being geometrically parallel. In common mechanics usage the words ‘in parallel’ are more general and mean that the net load is the sum of the loads carried by the two springs, and the stretches of the two springs are the same (or in a ratio restricted by kinematics). You will see cases where ‘in parallel’ springs are not the least bit parallel (*e.g.*, see Fig. 4.81).

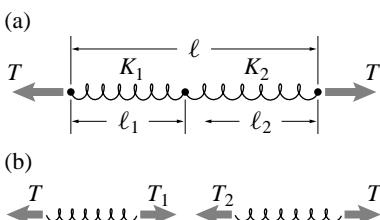


Figure 4.82: Schematic of springs in series.

(Filename:tfigure.seriessprings)

### Springs in series

Two springs that share a displacement and carry the same load are *in series*.

A schematic of two springs in series is shown in Fig. 4.82a where the springs are aligned serially, one after the other. To determine the net stiffness of this simple

spring network we again assemble the three pillars of mechanics, using the free body diagram of Fig. 4.82b.

$$\begin{array}{ll} \text{Constitutive law: } & T_1 = k_1(\ell_1 - \ell_{10}), \quad T_2 = k_2(\ell_2 - \ell_{20}), \\ \text{Kinematics: } & \ell_0 = \ell_{10} + \ell_{20}, \quad \ell = \ell_1 + \ell_2, \\ \text{Force Balance: } & T_1 = T, \quad \text{and} \quad T_2 = T. \end{array} \quad (4.51)$$

(where, e.g.,  $\ell_{10}$  reads ‘ell sub one zero’ and is the rest length of spring 1). We can manipulate these equations much as we did for the similar equations for springs in parallel. The manipulation differs in structure the same way the equations do. For springs in parallel the tensions add and the displacements are equal. For springs in series the displacements add and the tensions are equal.

$$\begin{aligned} \Delta\ell &= \ell - \ell_0 \\ &= (\ell_1 + \ell_2) - (\ell_{10} + \ell_{20}) \\ &= (\ell_1 - \ell_{10}) + (\ell_2 - \ell_{20}) \\ &= \Delta\ell_1 + \Delta\ell_2 \\ &= \frac{T_1}{k_1} + \frac{T_2}{k_2} \\ &= \underbrace{\left( \frac{1}{k_1} + \frac{1}{k_2} \right)}_{\frac{1}{k}} T. \end{aligned}$$

Thus we get that the net compliance is the sum of the compliances:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2},$$

which you might compare with springs in parallel (Eqn. 4.50). The sharing of the net stretch is in proportion to the compliances:

$$\Delta\ell_1 = \frac{1/k_1}{1/k_1 + 1/k_2} \Delta\ell \quad \text{and} \quad \Delta\ell_2 = \frac{1/k_2}{1/k_1 + 1/k_2} \Delta\ell$$

which add up to  $\Delta\ell$  as they must.

#### *Example: Two springs in series.*

Take  $1/k_1 = 99 \text{ cm/N}$  and  $1/k_2 = 1 \text{ cm/N}$ . The effective compliance of the parallel combination is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = 99 \text{ cm/N} + 1 \text{ cm/N} = 100 \text{ cm/N}.$$

Note that  $\Delta\ell_1/\Delta\ell = .99$  so even though the two springs share the displacement, the more compliant one has 99% of it. For design purposes, or for modeling this system, it would be fair to replace the much more stiff spring with a rigid link.  $\square$

Much of what you need to know about the words ‘in parallel’ and ‘in series’ follows easily from these phrases:

In parallel, forces and stiffnesses add.  
In series, displacements and compliances add.

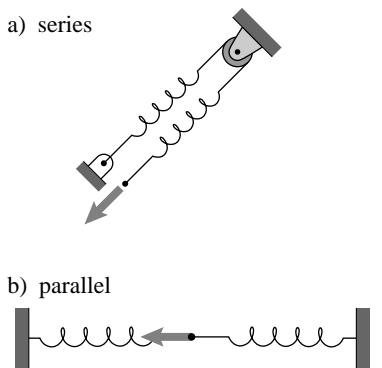


Figure 4.83: a) The two springs shown are in series because they carry the same load and their displacements add. b) These two springs are in parallel because they have a common displacement and their forces add.

(Filename:tfigure.parallelconfusion)

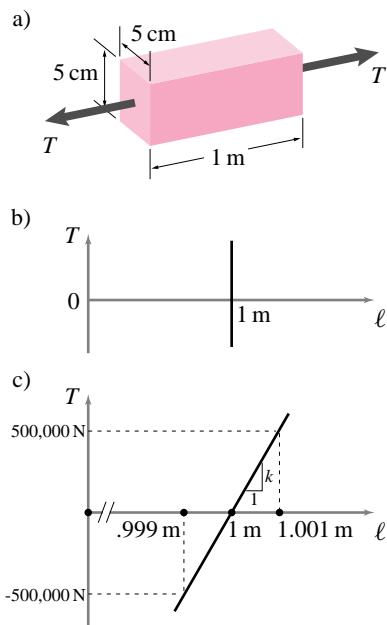


Figure 4.84: a) a steel rod in tension, b) tension versus length curve, c) zoom in on the tension versus length curve

(Filename:tfigure.steelrodspring)

① Because it is hard to picture steel deforming, your intuition may be helped by thinking of all solids as being rubber, or, if you want to look inside, like a chunk of Jello. (Jello is colored water held together by long gelatine molecules extracted from animal hooves. Those who are Kosher or vegetarian may substitute a sea-weed based Agar jell in their imagined deformation experiments.)

## Rigid bodies, springs and air

As the previous two examples illustrate, springs can sometimes be replaced with ‘air’ (nothing) or with rigid links without changing the system or model behavior much. One way to think about this is that in the limit as  $k \rightarrow \infty$  a spring becomes a rigid bar and in the limit  $k \rightarrow 0$  a spring becomes air.

These ideas are used by engineers, often intuitively or even subconsciously and with no substantiating calculations, when making a model of a mechanical system. If one of several pieces in series is much stiffer than the others it is often replaced with a rigid link. If one of several pieces in parallel is much more compliant than the others it is often replaced with air. For example:

- When a coil spring is connected to a linkage, the other pieces in the linkage, though undoubtedly somewhat compliant, are typically modelled as rigid. They are stiffer than the spring and in series with it.
- A single hinge resists rotation about axes perpendicular to the hinge axis. But a door connected at two points along its edge is stiffly prevented against such rotations. Thus the hinge stiffness is in parallel with the greater rotational stiffness of the two connection points and is thus often neglected (see the discussion and figures in section 3.1 starting on page 86).
- Welded joints in a determinate truss are modeled as frictionless pins. The rotational stiffness of the welds is ‘in parallel’ with the axial stiffness of the bars. To see this look at two bars welded together at an angle. Imagine trying to break this weld by pulling the two far bar ends apart. Now imagine trying to break the weld if the two far ends are connected to each other with a third bar. The third bar is ‘in parallel’ with the weld material. See the first few sentences of section 4.2 for a do-it-yourself demonstration of the idea.
- Human bones are often modeled as rigid because, in part, when they interact with the world they are in series with more compliant flesh.

Note, again, that the mechanics usage of the words ‘in parallel’ and ‘in series’ don’t always correspond to the geometric arrangement. For example the two springs in Fig. 4.83a are in series and the two springs in Fig. 4.83b are in parallel.

## Solid bars are linear springs

When a structure or machine is built with literal springs (*e.g.*, a wire helix) it is common to treat the other parts as rigid. But when a structure has no literal springs the small amount of deformation in rigid looking objects can be important, especially for determining how loads are shared in redundant structures.

Let’s consider a 1 m (about a yard) steel rod with a 5 cm square (about (2 in)<sup>2</sup>) cross section (Fig. 4.84a). If we plot the tension versus length we get a curve like Fig. 4.84b. The length just doesn’t visibly change (unless the tension got so large as to damage the rod, not shown.) But, when you pull on anything, it does deform at least a little. If we zoom in on the tension versus length plot we get Fig. 4.84c. To change the length by one part in a thousand we have to apply a tension of about 500,000 N (about 60 tons). Nonetheless the plot reveals that the solid steel rod behaves like a (very stiff) linear spring.

Surprisingly perhaps this little bit of compliance is important to structural engineers who often like to think of solid metal rods as linear springs. How does their stiffness depend on their shape and composition?①

Let’s take a reference bar with cross sectional area  $A_0$  and rest length  $\ell_0$  and pull it with tension  $T$  and measure the elongation  $\Delta\ell_0$  (Fig. 4.85ab). The stiffness of this reference rod is  $k_0 = T_0/\Delta\ell_0$ . Now put two such rods side by side and you have parallel springs. You might imagine this sequence: two bars are near each other,

then side by side, then touching each other, then glued together, then melted together into one rod with twice the cross section. The same tension in each causes the same elongation, or it takes twice the tension to cause the same elongation when you have twice the cross sectional area. Likewise with three side by side bars and so on, so for bars of equal length

$$k = \frac{A}{A_0} k_0.$$

On the other hand we could put the reference rods end to end in series. Then the same tension causes twice the elongation. We could be three or more rods together in series thus for bars with equal cross sections:

$$k = \frac{\ell_0}{\ell} k_0.$$

Putting these together we get:

$$k = \left( \frac{A}{A_0} \right) \left( \frac{\ell_0}{\ell} \right) k_0 = \left( \frac{k_0 \ell_0}{A_0} \right) \frac{A}{\ell}.$$

Now presumably if we took a rod with a given material, length, and cross section the stiffness would be  $k$ , no matter what the dimensions of the reference rod. So  $\left( \frac{k_0 \ell_0}{A_0} \right)$  has to be a material constant. It is called  $E$ , the *modulus of elasticity* or Young's modulus. For all steels  $E \approx 30 \cdot 10^6 \text{ lbf/in}^2 \approx 210 \cdot 10^9 \text{ N/m}^2$  (consistent with Fig. 4.84c). Aluminum has about a third this stiffness. So, a solid bar is a linear spring, obeying the spring equations:

$$k = \frac{EA}{\ell} \quad \text{or} \quad \Delta\ell = \frac{T\ell}{EA} \quad \text{or} \quad T = \frac{\Delta\ell EA}{\ell}$$

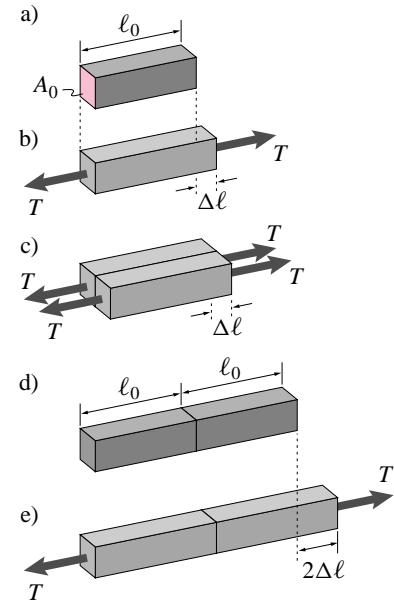


Figure 4.85: a) reference rod, b) reference rod in tension, c) two reference rods side by side, d) and e) two reference rods glued end to end.

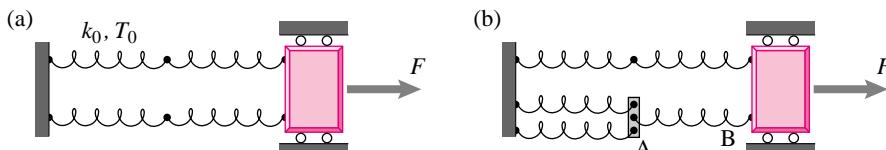
(Filename: figure.steelparallel)

## Strength and stiffness

Most often when you build a structure you want to make it stiff and strong. The ideas of stiffness and strength are so intimately related that it is sometimes hard to untangle them. For example, you might examine a product in discount store by putting your hand on it, applying small forces and observing the motion. Then you might say: "pretty shaky, I don't think it will hold up" meaning that the stiffness is low so you think the thing may break if the loads get high.

Although stiffness and strength are often correlated, they are distinct concepts. Something is stiff if the force to cause a given motion is high. Something is strong if the force to cause any part of it to break is high. In fact, it is possible for a structure to be made weaker by making it stiffer.

### Example: Stiffer but weaker.



Say all springs have stiffness  $k_0$  and break when the tension in them reaches  $T_0$ . Because of the mixture of parallel and series springs, the net stiffness of the structure in (a) is  $k_{\text{net}} = k_0$ . Its strength is  $2T_0$  because none of the springs reaches its breaking tension until  $F = T_0$ .

By doubling up one of the springs in (b) the structure is made 16% stiffer ( $k_{\text{net}} = 7k_0/6$ ). But spring AB now reaches its breaking point  $T_0$  when the applied load  $F = 21T_0/12$ , a 12.5% lower load than the  $2T_0$  the structure could carry before the stiffening.

The structure is made stiffer by reducing the deflection of point A. But this causes spring AB to stretch more and thus break at a smaller load. In some approximate sense, the load is thus concentrated in spring AB. This concentration of load into one part of structure is one reason that stiffness and strength need to be considered separately. Load concentration (or *stress concentration*) is a major cause of structural failure.  $\square$

## Why aren't springs in all mechanical models?

All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

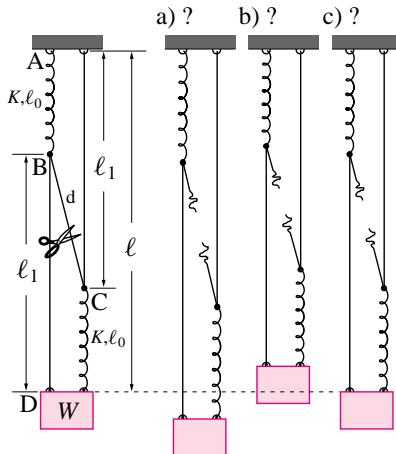
## Linear springs are just one way to model 'give'

If it is important to consider the deformability of an object, the linear spring model is just one simple model. It happens to be a good model for the small deformation of many solids. But the linear spring model is defined by the two words 'linear' and 'elastic'. For some purposes one might want to model the force due to deformation as being *non-linear*, like  $T = k_1(\Delta\ell) + k_2(\Delta\ell)^3$ . And one may want to take account of the dissipative or *in-elastic* nature of something. The most common example being a linear dashpot  $T = c\dot{\ell}$ . Various mixtures of non-linearity and inelasticity may be needed to model the large deformations of a yielding metal, for example.

## 4.5 A puzzle with two springs and three ropes.

This is a tricky puzzle.

Consider a weight hanging from 3 strings (BD, BC, and AC) and 2 springs (AB and CD) as in the left picture below. Point B is above point C and all ropes and springs are somewhat taught (none is slack).



When rope BC is cut does the weight go (a) down?, (b) up?, or (c) stay put?

If you have the energy and curiosity you should stop reading and try thinking, experimenting, or calculating when you see three dots.

In 15 minutes or so you can set up this experiment with 3 pieces of string, 2 rubber bands and a soda bottle. Hang the partially filled soda bottle from a door knob (or the top corner of a door, or a ruler cantilevered over the top of a refrigerator). Adjust the string lengths and amount of weight so that no strings or rubber bands are slack and make sure point B is above point C. The two points A can coincide as can the two points D. You might want to separate them a little with, say, a small wad of paper so you can see which string is which.

Looking at your experimental setup, but not pulling and poking at it, try to predict whether your bottle will go up down or not move when you cut the middle string.

The answer is, by experiment, that: When you cut the middle string the weight goes up a little. This violates many people's intuitions. In fact, this puzzle was published as one of a class of problems for which people have poor intuitions.

Now try to figure out why the experiment comes out the way it does? Also, try to figure out the error in your thinking if you got it wrong (like most people do).

All simple explanations are based on the assumption that the lengths of the strings AC and BD are constant at  $\ell_1$ .

To simplify the reasoning let's assume that springs AB and CD are identical and carry the same tension  $T_s$  and that the ropes AC and BD carry the same tension  $T_r$ . As usual, we need free body diagrams. (With the symmetry we have assumed diagrams (a) and (c) provide identical information.) The three free body diagrams can be considered before and after the middle string removal by having  $T_d > 0$  or  $T_d = 0$ , respectively. Vertical force balance gives (approximating  $T_d$  as vertical):

$$T_s + T_r = W \quad \text{and} \quad 2T_r + T_d = W \quad \Rightarrow \quad T_s = (W + T_d)/2.$$

Because we approximate AC as rigid with length  $\ell_1$ , the downwards position of the weight is the string length  $\ell_1$  plus the rest length of the spring  $\ell_0$  plus the stretch of the spring  $T_s/k$ :

$$\ell = \ell_1 + \ell_0 + T_s/k = \ell_1 + \ell_0 + (W + T_d)/(2k).$$

In the course of this experiment  $\ell_1$ ,  $\ell_0$ ,  $W$  and  $k$  are constants. So as the tension  $T_d$  goes from positive to zero (when the rope BC is cut)  $\ell$  decreases. So the weight goes up.

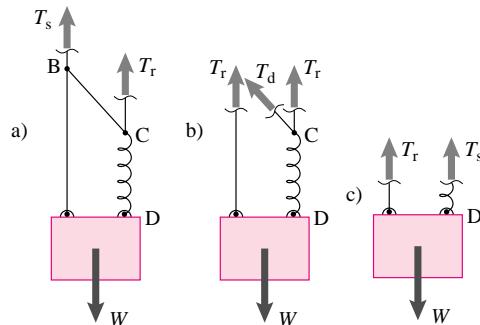
More intuitively, start with the configuration with the rope already cut and apply a small upwards force at C. It has no effect on the tension in spring CD thus the weight does not move. Now apply a small downwards force at B. This does stretch spring AB and thus lower point B, thus lowering the weight since  $\ell_1$  is constant. Applying both simultaneously is like attaching the middle rope. Thus attaching the middle rope lowers the weight and cutting the middle rope raises it again.

Here is another intuitive approach. Point C can't move. Point B moves up and down just as much as the weight does. Point B is a distance  $d$  above point C. Since the rope BC is taught, releasing it will allow B and C to separate, thus increasing  $d$  and raising the weight.

What about springs in parallel and series? Here is a quick but wrong explanation for the experimental result, though it happens to predict the right answer, or at least the right direction of motion.

"Before rope BC is cut the two springs are more or less in series because the load is carried from spring through BC to spring. Afterwards they are more or less in parallel because they have the same stretch and share the load. Two springs in parallel have 4 times the stiffness of the same two springs in series. So in the parallel arrangement the deflection is less. So the weight goes up when the springs switch from series to parallel."

What is the error in this thinking? The position of the weight comes from spring deflection added to the position when there is no weight. For the argument just presented to make sense, the rest position of the mass (with gravity switched off) would have to be the same for the supposed 'series' and 'parallel' cases, which it is not ( $\ell_1 + \ell_0 \neq \ell_0 + d + \ell_0$ ).



Another way to see the fallacy of this 'parallel versus series' argument is that the incremental stiffness of the system is, assuming inextensible ropes, infinite. That is, if you add or subtract a small load to the bottle it doesn't move until one or another rope goes slack. (The small deformation you do see has to do with the stretch of the ropes, something that none of the simple explanations take into account.) If the springs were in series or parallel we would expect an incremental stiffness that was related to spring stretch not rope stretch.

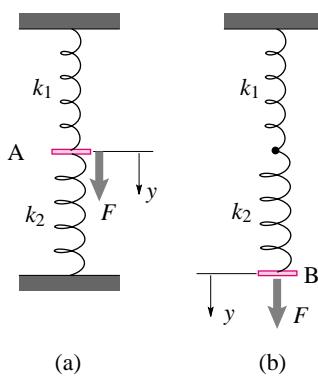
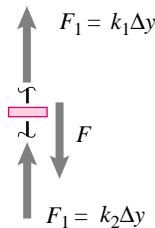
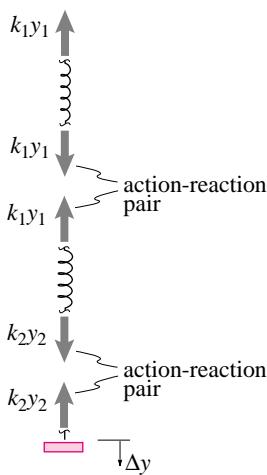


Figure 4.86: (Filename:sfig4.2springs)

Figure 4.87: Free body diagram of point A.  
(Filename:sfig4.2springs.a)Figure 4.88: Free body diagrams  
(Filename:sfig4.2springs.b)

**SAMPLE 4.18 Springs in series versus springs in parallel:** Two springs with spring constants  $k_1 = 100 \text{ N/m}$  and  $k_2 = 150 \text{ N/m}$  are attached together as shown in Fig. 4.86. In case (a), a vertical force  $F = 10 \text{ N}$  is applied at point A, and in case (b), the same force is applied at the end point B. Find the force in each spring for static equilibrium. Also, find the equivalent stiffness for (a) and (b).

**Solution** In static equilibrium, let  $\Delta y$  be the displacement of the point of application of the force in each case. We can figure out the forces in the springs by writing force balance equations in each case.

- **Case (a):** The free body diagram of point A is shown in Fig. 4.87. As point A is displaced downwards by  $\Delta y$ , spring 1 gets stretched by  $\Delta y$  whereas spring 2 gets compressed by  $\Delta y$ . Therefore, the forces applied by the two springs,  $k_1\Delta y$  and  $k_2\Delta y$ , are in the same direction. Then, the force balance in the vertical direction,  $\hat{j} \cdot (\sum \vec{F} = \vec{0})$ , gives:

$$\begin{aligned} F &= F_1 + F_2 = (k_1 + k_2)\Delta y \\ \Rightarrow \Delta y &= \frac{F}{k_1 + k_2} = \frac{10 \text{ N}}{(100 + 150) \text{ N/m}} = 0.04 \text{ m} \\ \Rightarrow F_1 &= k_1\Delta y = 100 \text{ N/m} \cdot 0.04 \text{ m} = 4 \text{ N} \\ \Rightarrow F_2 &= k_2\Delta y = 150 \text{ N/m} \cdot 0.04 \text{ m} = 6 \text{ N} \end{aligned}$$

The equivalent stiffness of the system is the stiffness of a single spring that will undergo the same displacement  $\Delta y$  under  $F$ . From the equilibrium equation above, it is easy to see that,

$$k_e = \frac{F}{\Delta y} = k_1 + k_2 = 250 \text{ N/m.}$$

$$F_1 = 4 \text{ N}, \quad F_2 = 6 \text{ N}, \quad k_e = 250 \text{ N/m}$$

- **Case (b):** The free body diagrams of the two springs is shown in Fig. 4.88 along with that of point B. In this case both springs stretch as point B is displaced downwards. Let the net stretch in spring 1 be  $y_1$  and in spring 2 be  $y_2$ .  $y_1$  and  $y_2$  are unknown, of course, but we know that

$$y_1 + y_2 = \Delta y \quad (4.52)$$

Now, using the free body diagram of point B and writing the force balance equation in the vertical direction, we get  $F = k_2 y_2$  and from the free body diagram of spring 2, we get  $k_2 y_2 = k_1 y_1$ . Thus the force in each spring is the same and equals the applied force, i.e.,

$$F_1 = k_1 y_1 = F = 10 \text{ N} \quad \text{and} \quad F_2 = k_2 y_2 = F = 10 \text{ N.}$$

The springs in this case are in series. Therefore, their equivalent stiffness,  $k_e$ , is

$$k_e = \left( \frac{1}{k_1} + \frac{1}{k_2} \right)^{-1} = \left( \frac{1}{100 \text{ N/m}} + \frac{1}{150 \text{ N/m}} \right)^{-1} = 60 \text{ N/m.}$$

Note that the displacements  $y_1$  and  $y_2$  are different in this case. They can be easily found from  $y_1 = F/k_1$  and  $y_2 = F/k_2$ .

$$F_1 = F_2 = 10 \text{ N}, \quad k_e = 60 \text{ N/m}$$

**Comments:** Although the springs attached to point A do not visually seem to be in parallel, from mechanics point of view they are parallel. Springs in *parallel* have the *same displacement* but *different forces*. Springs in *series* have *different displacements* but the *same force*.

**SAMPLE 4.19 Stiffness of three springs:** For the spring networks shown in Fig. 4.89(a) and (b), find the equivalent stiffness of the springs in each case, given that each spring has a stiffness of  $k = 20 \text{ N/m}$ .

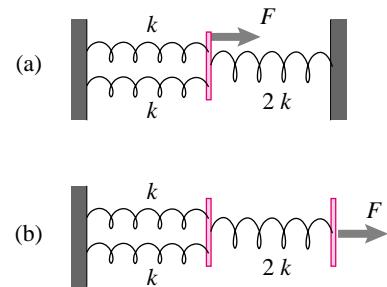


Figure 4.89: (Filename:sfig4.3springs)

### Solution

- (a) In Fig. 4.89(a), all springs are in parallel since all of them undergo the same displacement  $\Delta x$  in order to balance the applied force  $F$ . Each of the two springs on the left stretches by  $\Delta x$  and the spring on the right compresses by  $\Delta x$ . Therefore, the equivalent stiffness of the three springs is

$$k_p = k + k + 2k = 4k = 80 \text{ kN/m.}$$

Pictorially,

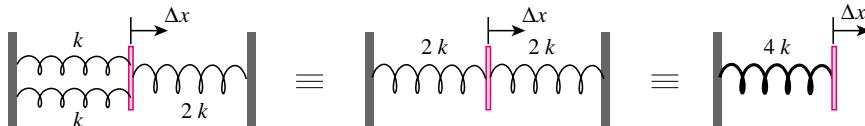


Figure 4.90: (Filename:sfig4.3springs.a)

$$k_{equiv} = 80 \text{ kN/m}$$

- (b) In Fig. 4.89(b), the first two springs (on the left) are in parallel but the third spring is in series with the first two. To see this, imagine that for equilibrium point A moves to the right by  $\Delta x_A$  and point B moves to the right by  $\Delta x_B$ . Then each of the first two springs has the same stretch  $\Delta x_A$  while the third spring has a net stretch  $= \Delta x_B - \Delta x_A$ . Therefore, to find the equivalent stiffness, we can first replace the two parallel springs by a single spring of equivalent stiffness  $k_p = k + k = 2k$ . Then the springs with stiffnesses  $k_p$  are  $2k$  are in series and therefore their equivalent stiffness  $k_s$  is found as follows.

$$\begin{aligned} \frac{1}{k_s} &= \frac{1}{k_p} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \\ \Rightarrow k_s &= k = 20 \text{ kN/m.} \end{aligned}$$

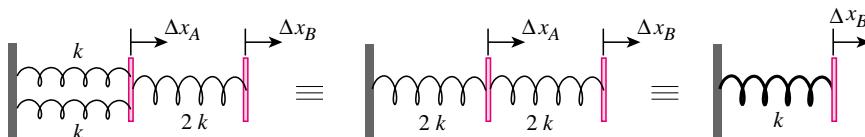


Figure 4.91: (Filename:sfig4.3springs.b)

$$k_{equiv} = 20 \text{ kN/m}$$

**SAMPLE 4.20 Stiffness vs strength:** Which of the two structures (network of springs) shown in the figure is stiffer and which one has more strength if each spring has stiffness  $k = 10 \text{ kN/m}$  and strength  $F_0 = 10 \text{ kN}$ .

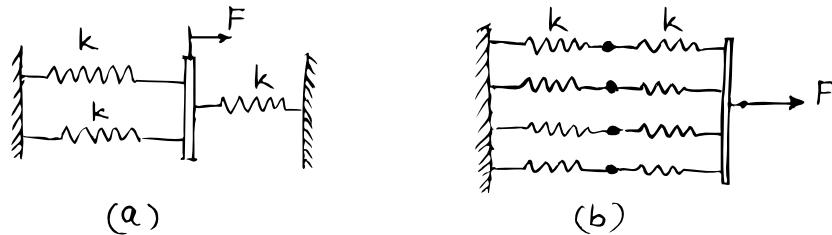


Figure 4.92: (Filename:sfig4.many springs)

**Solution** In structure (a), all the three springs are in parallel. Therefore, the equivalent stiffness of the three springs is

$$k_a = k + k + k = 3k = 30 \text{ kN/m.}$$

For figuring out the strength of the structure, we need to find the force in each spring. From the free body diagram in Fig. 4.93 we see that,

$$k\Delta x + k\Delta x + k\Delta x = F \Rightarrow \Delta x = \frac{F}{3k}$$

Therefore, the force in each spring is

$$F_s = k\Delta x = \frac{F}{3}$$

But the maximum force that a spring can take is  $(F_s)_{\max} = F_0 = 10 \text{ kN}$ . Therefore, the maximum force that the structure can take, i.e., the strength of the structure, is

$$F_{\max} = 3F_0 = 30 \text{ kN.}$$

Stiffness = 30 kN/m, Strength = 30 kN
---------------------------------------

Now we carry out a similar analysis for structure (b). There are four parallel chains in this structure, with each chain containing two springs in series. The stiffness of each chain,  $k_c$ , is found from

$$\frac{1}{k_c} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \Rightarrow k_c = \frac{k}{2} = 5 \text{ kN/m.}$$

So, the stiffness of the entire structure is

$$k_b = k_c + k_c + k_c + k_c = 4k_c = 20 \text{ kN/m.}$$

We find the force in each spring to be  $F/4$  from the free body diagram shown in Fig. 4.94. Therefore, the maximum force that the structure can take is

$$F_{\max} = 4F_0 = 40 \text{ kN.}$$

Stiffness = 20 kN/m, Strength = 40 kN
---------------------------------------

Thus, the structure in Fig. 4.92(a) is stiffer but the structure in (b) is stronger (more strength).

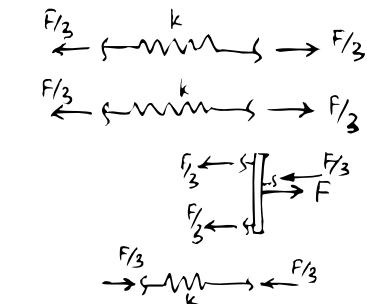


Figure 4.93: (Filename:sfig4.many springs.a)

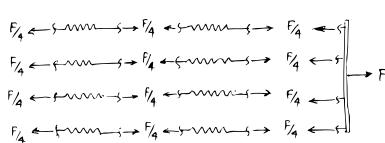


Figure 4.94: (Filename:sfig4.many springs.b)



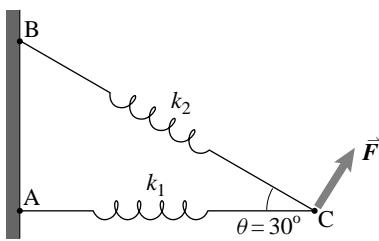


Figure 4.95: (filename:sfig4.spring.compl)

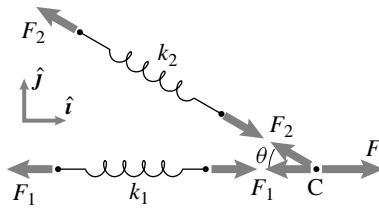


Figure 4.96: (filename:sfig4.spring.compl.a)

**SAMPLE 4.21 Compliance matrix of a structure:** For the two-spring structure shown in the figure, find the deflection of point C when

- (a)  $\vec{F} = 1 \text{ N} \hat{i}$ ,
- (b)  $\vec{F} = 1 \text{ N} \hat{j}$ ,
- (c)  $\vec{F} = 30 \text{ N} \hat{i} + 20 \text{ N} \hat{j}$ ,

The spring stiffnesses are  $k_1 = 10 \text{ kN/m}$  and  $k_2 = 20 \text{ kN/m}$ .

### Solution

(a) *Deflections with unit force in the x-direction:* Let  $\Delta \vec{r} = \Delta x \hat{i} + \Delta y \hat{j}$  be the displacement of point C of the structure due to the applied load. We can figure out the deflections in each spring as follows. Let  $\hat{\lambda}_{AC}$  and  $\hat{\lambda}_{BC}$  be the unit vectors along AC and BC, respectively. Then, the change in the length of spring AC due to the displacement of point C is

$$\begin{aligned}\Delta_{AC} &= \hat{\lambda}_{AC} \cdot \Delta \vec{r} \\ &= \hat{i} \cdot (\Delta x \hat{i} + \Delta y \hat{j}) = \Delta x\end{aligned}$$

Similarly, the change in the length of spring BC is

$$\begin{aligned}\Delta_{BC} &= \hat{\lambda}_{BC} \cdot \Delta \vec{r} \\ &= (\cos \theta \hat{i} - \sin \theta \hat{j}) \cdot (\Delta x \hat{i} + \Delta y \hat{j}) = \Delta x \cos \theta - \Delta y \sin \theta.\end{aligned}$$

Now we can find the force in each spring since we know the deflection in each spring.

$$\text{Force in spring AB} = F_1 = k_1 \Delta x \quad (4.53)$$

$$\text{Force in spring BC} = F_2 = k_2 (\Delta x \cos \theta - \Delta y \sin \theta). \quad (4.54)$$

The forces in the springs, however, depend on the applied force, since they must satisfy static equilibrium. Thus, we can determine the deflection by first finding  $F_1$  and  $F_2$  in terms of the applied load and substituting in the equations above to solve for the deflection components.

Let  $\vec{F} = f_x \hat{i} = 1 \text{ N} \hat{i}$ , (we have adopted a special symbol  $f_x$  for the unit load). Then, from the free body diagram of the springs and the end pin shown in Fig. 4.97 and the force equilibrium ( $\sum \vec{F} = \vec{0}$ ), we have,

$$f_x \hat{i} - F_1 \hat{i} + F_2 (-\cos \theta \hat{i} + \sin \theta \hat{j}) = \vec{0}$$

Dotting with  $\hat{j}$  and  $\hat{i}$  we get,

$$F_2 = 0$$

$$F_1 = f_x = 1 \text{ N}.$$

Substituting the values of  $F_1$  and  $F_2$  from above in eqns. (4.53 and 4.54), and solving for  $\Delta x$  and  $\Delta y$  we get,

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=f_x \hat{i}} = \begin{pmatrix} \frac{1}{k_1} \\ \frac{1}{k_1} \cot \theta \end{pmatrix} f_x. \quad (4.55)$$

Substituting the given values of  $\theta$  and  $k_1$  and  $f_x = 1 \text{ N}$ , we get

$$\Delta \vec{r} = \Delta x \hat{i} + \Delta y \hat{j} = (100 \hat{i} + 173 \hat{j}) \times 10^{-6} \text{ m}.$$

$$\Delta \vec{r} = (100 \hat{i} + 173 \hat{j}) \times 10^{-6} \text{ m}$$

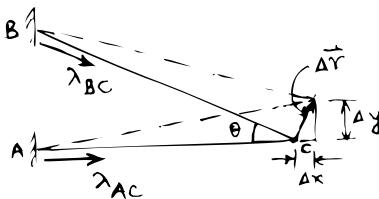


Figure 4.97: (filename:sfig4.spring.compl.b)

- (b) *Deflections with unit force in the y-direction:* We carry out a similar analysis for this case. We again assume the displacement of point C to be  $\Delta \vec{r} = \Delta x \hat{i} + \Delta y \hat{j}$ . Since the geometry of deformation and the associated results are the same, eqns. (4.53) and (4.54) remain valid. We only need to find the spring forces from the static equilibrium under the new load. From the free body diagram in Fig. 4.98 we have,

$$\begin{aligned} (-F_1 - F_2 \cos \theta) \hat{i} + (F_2 \sin \theta + F) \hat{j} &= \bar{0} \\ \Rightarrow F_2 &= -\frac{F}{\sin \theta} \\ \text{and } F_1 &= -F_2 \cos \theta = F \cot \theta. \end{aligned}$$

Substituting these values of  $F_1$  and  $F_2$  in terms of  $f_y$  in eqns. (4.53) and (4.54), we get

$$\begin{aligned} f_y \cot \theta &= k_1 \Delta x \Rightarrow \Delta x = \frac{f_y}{k_1} \cot \theta \\ -\frac{f_y}{\sin \theta} &= k_2 (\Delta x \cos \theta - \Delta y \sin \theta) \\ \Rightarrow \Delta y &= \frac{1}{\sin \theta} (\Delta x \cos \theta + \frac{f_y}{k_2 \sin \theta}) \\ &= f_y \left( \frac{1}{k_1} \cot^2 \theta + \frac{1}{k_2} \csc^2 \theta \right) \end{aligned}$$

Thus,

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=f_y \hat{j}} = \begin{pmatrix} \frac{1}{k_1} \cot \theta \\ \frac{1}{k_1} \cot^2 \theta + \frac{1}{k_2} \csc^2 \theta \end{pmatrix} f_y. \quad (4.56)$$

Substituting the values of  $\theta$ ,  $k_1$ , and  $k_2$ , and  $f_y = 1 \text{ N}$ , we get

$$\Delta \vec{r} = \Delta x \hat{i} + \Delta y \hat{j} = (173 \hat{i} + 500 \hat{j}) \times 10^{-6} \text{ m.}$$

$$\boxed{\Delta \vec{r} = (173 \hat{i} + 500 \hat{j}) \times 10^{-6} \text{ m}}$$

- (c) *Deflection under general load:* Since we have already got expressions for deflections in the  $x$  and  $y$ -directions under unit loads in the  $x$  and  $y$ -directions, we can now combine the results to find the deflection under any general load  $\vec{F} = F_x \hat{i} + F_y \hat{j}$  as follows.

$$\begin{aligned} \Delta \vec{r} &= \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = F_x \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=1 \hat{i}} + F_y \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}_{\vec{F}=1 \hat{j}} \\ &= \begin{bmatrix} k_1^{-1} & k_1^{-1} \cot \theta \\ k_1^{-1} \cot \theta & k_1^{-1} \cot^2 \theta + k_2^{-1} \csc^2 \theta \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}. \end{aligned}$$

Once again, substituting all given values and  $F_x = 30 \text{ N}$  and  $F_y = 20 \text{ N}$ , we get

$$\Delta \vec{r} = (6.4 \hat{i} + 15.2 \hat{j}) \times 10^{-3} \text{ m.}$$

$$\boxed{\Delta \vec{r} = (6.4 \hat{i} + 15.2 \hat{j}) \times 10^{-3} \text{ m}}$$

**Note:** The matrix obtained above for finding the deflection under general load is called the *compliance matrix* of the structure. Its inverse is known as the *stiffness matrix* of the structure and is used to find forces given deflections.

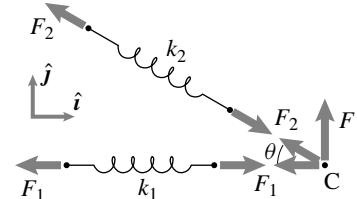


Figure 4.98: (Filename:sfig4.springs.compl.c)

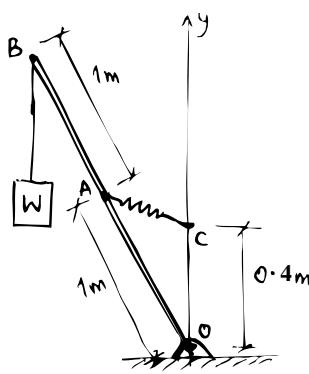


Figure 4.99: (Filename:sfig4.zerospring)

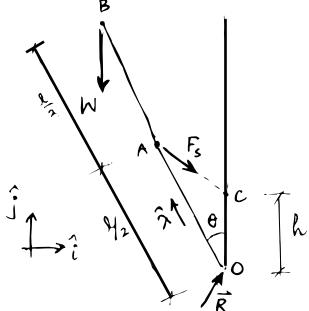
**SAMPLE 4.22** *Zero length springs are special!* A rigid and massless rod AB of length 2 m supports a weight  $W = 100\text{ kg}$  hung from point B. The rod is pinned at O and supported by a zero length (in relaxed state) spring attached at mid-point A and point C on the vertical wall. Find the equilibrium angle  $\theta$  and the force in the spring.

**Solution** The free body diagram of the rod is shown in Fig. 4.100 in an assumed equilibrium state. Let  $\hat{\lambda} = -\sin \theta \hat{i} + \cos \theta \hat{j}$  be a unit vector along OB. The spring force can be written as  $\vec{F}_s = k \vec{r}_{C/A}$ . We need to determine  $\theta$  and  $\delta$ .

Let us write moment equilibrium equation about point O, i.e.,  $\sum \vec{M}_O = \vec{0}$ ,

$$\vec{r}_{B/O} \times \vec{W} + \vec{r}_{A/O} \times \vec{F}_s = \vec{0}$$

Noting that



we get,

$$\begin{aligned}\vec{r}_{B/O} &= \ell \hat{\lambda}, \quad \vec{r}_{A/O} = \frac{\ell}{2} \hat{\lambda}, \\ \vec{F}_s &= k \vec{r}_{C/A} = k(\vec{r}_C - \vec{r}_A) \\ &= k(h \hat{j} - \frac{\ell}{2} \hat{\lambda}),\end{aligned}$$

$$\begin{aligned}\ell \hat{\lambda} \times (-W \hat{j}) + \frac{\ell}{2} \hat{\lambda} \times k(h \hat{j} - \frac{\ell}{2} \hat{\lambda}) &= \vec{0} \\ -W\ell(\hat{\lambda} \times \hat{j}) + kh\frac{\ell}{2}(\hat{\lambda} \times \hat{j}) &= \vec{0}\end{aligned}$$

Figure 4.100: (Filename:sfig4.zerospring.a)

Dotting this equation with  $(\hat{\lambda} \times \hat{j})$ , we get,

$$\begin{aligned}-W\ell + kh\frac{\ell}{2} &= 0 \\ \Rightarrow kh &= 2W.\end{aligned}$$

Thus the result is independent of  $\theta$ ! As long as the spring stiffness  $k$  and the height of point C,  $h$ , are such that their product equals  $2W$ , the system will be in equilibrium at any angle. This is why zero length springs are special.

Equilibrium is satisfied at any angle if  $kh = 2W$

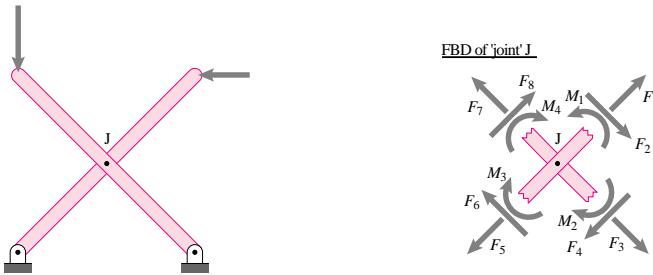
## 4.6 Structures and machines

The laws of mechanics apply to one body shown in one free body diagram. Yet engineers design things with many pieces each of which may be thought of as a body. One class of examples are trusses which you learned to analyze in section 4.2.

We would now like to analyze things built of pieces that are connected in a more complex way. These things include various structures which are designed to not move and various machines which are designed to move. Our general goal here is to find the interaction forces and the ‘internal’ forces in the components.

The secret to our success with trusses was that all of the pieces in a truss are two-force members. Thus free body diagrams of joints involved forces that were in known directions. Because now the pieces are not all two-force bodies, we will not know the directions of the interaction forces *a priori* and the method of joints will be nearly useless.

### Example: An X structure



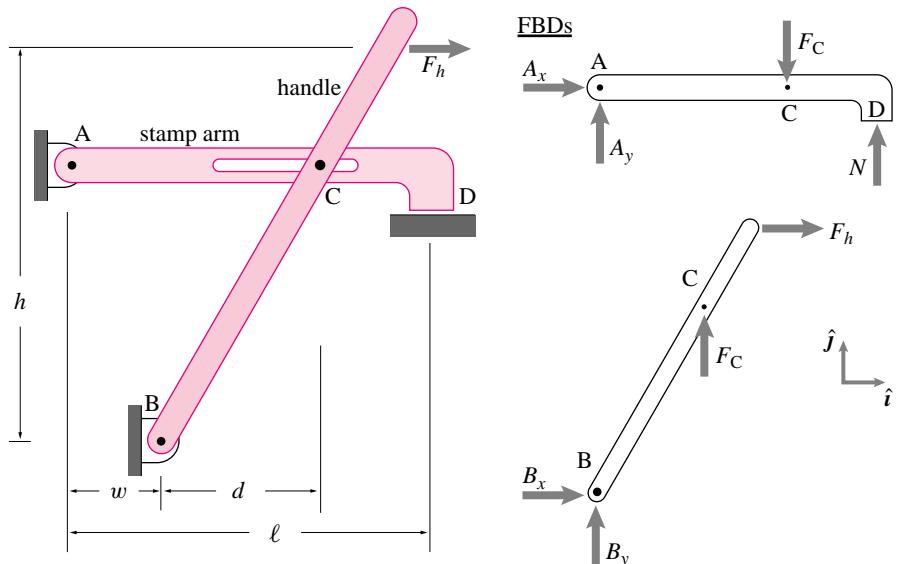
Two bars are joined in an ‘X’ by a pin at J. Neither of the bars is a two-force body so a free body diagram of the ‘joint’ at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components.  $\square$

Instead of drawing free body diagrams of the connections, our approach here is to draw free body diagrams of each of the structure or machine’s parts. Sometimes, as was the case with trusses, it is also useful to draw a free body diagram of a whole structure or of some multi-piece part of the structure<sup>①</sup>.

<sup>①</sup> You might wonder why we didn’t analyze trusses this way, by drawing free body diagrams of each of the bars. This seldom used approach to trusses, the ‘method of bars and pins’ is discussed in box 4.6 on 187.

### Example: Stamp machine

Pulling on the handle (below) causes the stamp arm to press down with a force  $N$  at D. We can find  $N$  in terms of  $F_h$  by drawing free body diagrams of the handle and stamp arm, writing three equilibrium equations for each piece and then solving these 6 equations for the 6 unknowns ( $A_x$ ,  $A_y$ ,  $F_C$ ,  $N$ ,  $B_x$ , and  $B_y$ ).



For this problem, the answer can be found more quickly with a judicious choice of equilibrium equations.

$$\begin{aligned} \text{For the handle, } \left\{ \sum \vec{M}_{/B} = \vec{0} \right\} \cdot \hat{k} &\Rightarrow -hF_h + dF_c = 0 \\ \text{For the stamp arm, } \left\{ \sum \vec{M}_{/A} = \vec{0} \right\} \cdot \hat{k} &\Rightarrow -(d+w)F_c + \ell N = 0 \end{aligned}$$

$$\text{eliminating } F_c \Rightarrow N = \frac{h(d+w)}{d\ell} F_h.$$

Note that the stamp force  $N$  can be made very large by making  $d$  small and thus the handle nearly vertical. Often in structural or machine design one or another force gets extremely large or small as the design is changed to put pieces in near alignment.  $\square$

## Static determinacy

A statically determinate structure has a solution for all possible applied loads, has only one solution, and this solution can be found by using equilibrium equations applied to each of the pieces. As for trusses, not all structures are statically determinate. The simple counting formula that is necessary for determinacy but does not guarantee determinacy is:

$$\text{number of equations} = \text{number of unknowns}$$

Where, in 2D, there are three equilibrium equations for each object. There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

**Example: 'X' structure counting**

In the 'X' structure above we can count as follows.

$$\begin{aligned} \text{number of equations} &\stackrel{?}{=} \text{number of unknowns} \\ (3 \text{ eqs per bar}) \cdot (2 \text{ bars}) &\stackrel{?}{=} (2 \text{ unknown force comps per pin}) \cdot (3 \text{ pins}) \\ 6 \text{ eqs} &\stackrel{\checkmark}{=} 6 \text{ unknown force components} \end{aligned}$$

So the 'X' structure passes the counting test for static determinacy.  $\square$

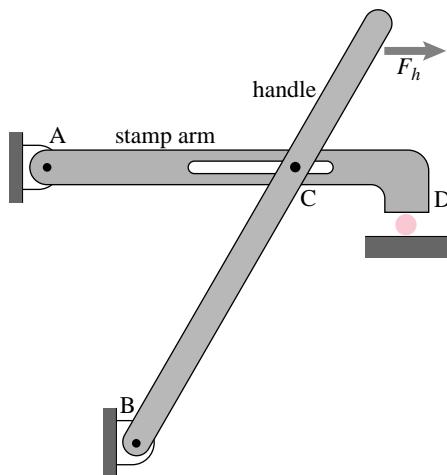
**Indeterminate structures are mechanisms**

An indeterminate structure cannot carry all loads and, if not also redundant, has more equilibrium equations than unknown reaction or interaction force components. Such a structure is also called a mechanism. The stamp machine above is a mechanism if there is assumed to be no contact at D. In particular the equilibrium equations cannot be satisfied unless  $F_h = 0$ . Mechanisms have variable configurations. That is, the constraints still allow relative motion.

An attempt to design a rigid structure that turns out to be a mechanism is a design failure. But for machine design, the mechanism aspect of a structure is essential. Even though mechanisms are called 'statically indeterminate' because they cannot carry all possible loads, the desired forces can often be determined using statics. For the stamp machine above the equilibrium equations are made solvable by treating one of the applied forces, say  $N$ , as an unknown, and the other,  $F$  in this case, as a known. This is a common situation in machine design where you want to determine the loads at one part of a mechanism in terms of loads at another part. For the purposes of analysis, a trick is to make a mechanism determinate by putting a pin on rollers connection to ground at the location of any forces with unknown magnitudes but known directions.

**Example: Stamp machine with roller**

Putting a roller at D, the location of the unknown stamp force, turns the stamp machine into a determinate structure.

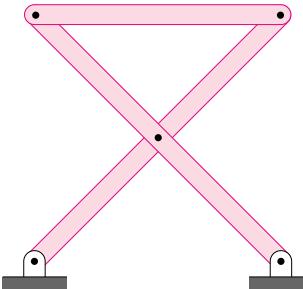


$\square$

## Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components.

We generally avoid trying to find those force components which cannot be found uniquely from the equilibrium equations. Finding them depends on modeling the deformation, a topic emphasized in advanced structural courses.



### Example: Overbraced 'X'

The structure above is evidently redundant because it has a bar added to a structure which was already statically determinate. By counting we get

$$\begin{aligned} \text{number of equations} &\stackrel{?}{=} \text{number of unknowns} \\ 3 \cdot (\underbrace{\text{number of bars}}_3) &\stackrel{?}{=} 2 \cdot (\underbrace{\text{number of joints}}_5) \\ 9 \text{ eqs} &< 10 \text{ unknown force components} \end{aligned}$$

thus demonstrating redundancy. □

## Some common mechanical designs

Rigid bodies can be connected in various arrangements for various purposes. Here we describe several basic machine fragments.

### A lever

Maybe the simplest machine, and one we have mentioned several times, is a lever (fig. 4.101). An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads. The hinge could be at point A, B or C and the free body diagram of fig. 4.101 is the same. The study of the lever precedes the change of mechanics from a taxonomic to a quantitative subject. So there is specialized antiquated vocabulary of levers, classifying them depending on where the pivot is located, and on which force you think of as input and which you think of as output. For historical curiosity: A ‘class one’ lever has the pivot in the middle; a ‘class two’ lever has the pivot at one end and the input force at the other; and a ‘class three’ lever has the pivot at one end and the input force in the middle.

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer pulling a nail, a boat oar, one half of a pair of tweezers, a break lever, a gear,

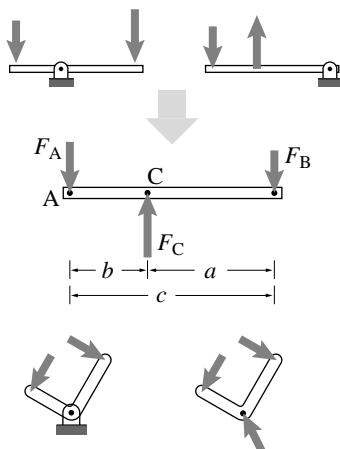


Figure 4.101: A lever can have the pivot in various places. The free body diagram looks the same in any case.

(Filename:tfigure.lever)

and, most generally, any three-force body. Using the equilibrium relations on the free body diagram in fig. 4.101 you can find that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you can find the relation between any pair of the forces. In practice it is undoubtedly easier to use moment balance about an appropriate point than to memorize this formula.

### Gears

A transmission is used to ‘transmit’ motion caused at one place to motion at another and, generally to also speed it up or slow it down. Simultaneously force or moment is transmitted from one point to another, generally being attenuated or amplified. One type of transmission is based on gears (Fig. 4.102a). If we think of the input and output as the moments on the two gears, we find from the free body diagram in Fig. 4.102b that

$$\text{For gear A, } \left\{ \sum \vec{M}_{i/A} = \vec{0} \right\} \cdot \hat{k} \Rightarrow -R_A F + M_A = 0$$

$$\text{For gear B, } \left\{ \sum \vec{M}_{i/B} = \vec{0} \right\} \cdot \hat{k} \Rightarrow -R_B F + M_B = 0$$

$$\text{eliminating } F \Rightarrow M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gears circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

Two gears pulled out of a bigger transmission are shown in Fig. 4.102c. Gear A has an inner part with radius  $R_{A_i}$  welded to an outer part with radius  $R_{A_o}$ . Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free body diagram in Fig. 4.102d gives that  $R_{A_i} F_A = R_{A_o} F$ . You can think of the one gear as a lever (see Fig. 4.103). Moment balance about B in the second free body diagram gives that  $R_{B_i} F = R_{B_o} F_B$ . Combining we get

$$F_B = \frac{R_{A_i}}{R_{A_o}} \frac{R_{B_i}}{R_{B_o}} F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}} \frac{R_{B_o}}{R_{B_i}} F_B$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of  $F_A$  as the input and amplifies the force if you think of  $F_B$  as the input. If the inner gears have one tenth the radius of the outer gears than the multiplication or attenuation is a factor of 100.

### An ideal wedge

Wedges are kind of machine. For an ideal wedge one neglects friction, effectively replacing sliding contact with rolling contact (see Fig. 4.104ab). Although this approximation may not be accurate, it is helpful for building intuition. For the free

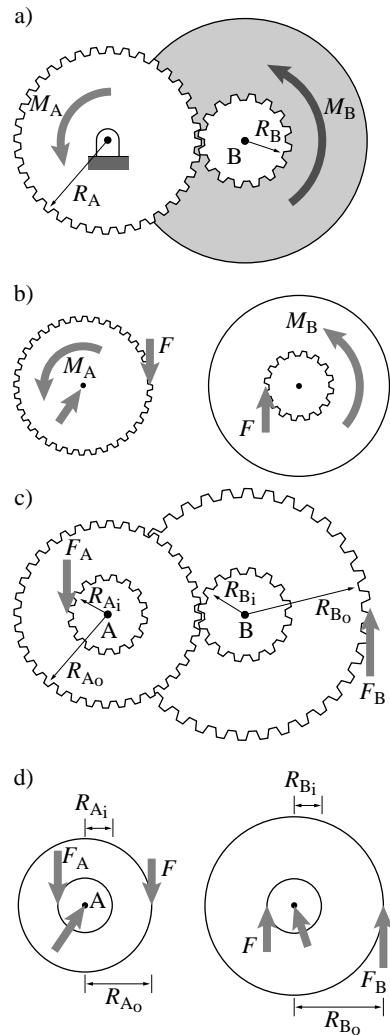


Figure 4.102: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free body diagrams.

(Filename:tfigure.gears)

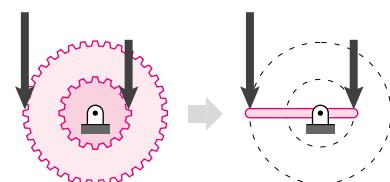


Figure 4.103: One gear may be thought of as a lever.

(Filename:tfigure.gearisalever)

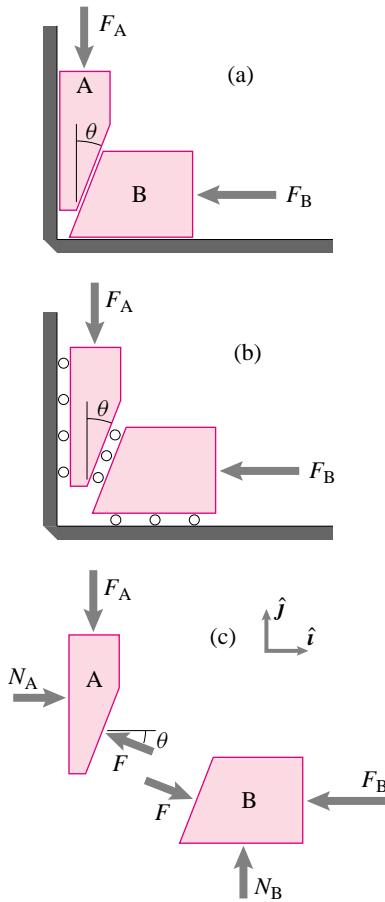


Figure 4.104: a) A wedge, b) treated as frictionless in the ideal case, c) free body diagrams

(Filename:tfigure.wedge)

body diagrams of Fig. 4.104c we have not fussed over the exact location of the contact forces since the key idea depends on force balance and not moment balance. Neglecting gravity,

$$\begin{aligned} \text{For block A, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow -F_A + F \sin \theta = 0 \\ \text{For block B, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} &\Rightarrow -F_B + F \cos \theta = 0 \end{aligned}$$

$$\text{eliminating } F \Rightarrow F_B = \frac{1}{\tan \theta} F_A.$$

To multiply the force  $F_A$  by 10 takes a wedge with a taper of  $\theta = \tan^{-1} 0.1 \approx 6^\circ$ . With this taper, an ideal wedge could also be viewed as a device to attenuate the force  $F_B$  by a factor of 10, although wedges are never used for force attenuation in practice, as we now explain.

#### A wedge with friction

In the real world frictionless things are hard to find. Nonetheless the concept of a frictionless bearing can be a reasonable idealization because of rollers, grease, and the big lever-arm that the wheel periphery has compared to the axle radius. In the case of wedges, neglecting friction is not generally an accurate model.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity we still take the wall and floor interactions to be frictionless.

Figure 4.105 shows free body diagrams of wedge blocks. We draw separate free body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the right. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan \phi}_{\mu} N$$

where Fig. 4.105 shows the resultant contact force (normal component plus frictional component) and its angle  $\phi$  to the surface normal.

Assuming block A is sliding down we get from free body diagram 4.105a that

$$\begin{aligned} \text{For block A, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow -F_A + F \sin(\theta + \phi) = 0 \\ \text{For block B, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} &\Rightarrow -F_B + F \cos(\theta + \phi) = 0 \end{aligned}$$

$$\text{eliminating } F \Rightarrow F_B = \frac{1}{\tan(\theta + \phi)} F_A. \quad (4.57)$$

If we take a taper of  $6^\circ$  and a friction coefficient of  $\mu = .3 \Rightarrow \phi \approx 17^\circ$  we get that  $F_B/F_A \approx 2.5$  instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe. Now lets consider the case when force  $F_B$  is pushing block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free body diagram 4.105b

$$\begin{aligned} \text{For block A, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{j} &\Rightarrow -F_A + F \sin(\theta - \phi) = 0 \\ \text{For block B, } \left\{ \sum \vec{F}_i = \vec{0} \right\} \cdot \hat{i} &\Rightarrow -F_B + F \cos(\theta - \phi) = 0 \end{aligned}$$

$$\text{eliminating } F \Rightarrow F_A = \tan(\theta - \phi) F_B. \quad (4.58)$$

Again using  $\theta = 6^\circ$  and  $\phi = 17^\circ$  we see that if  $F_B = 100 \text{ lbf}$  that  $F_A = \tan(-11^\circ) \cdot 100 \text{ lbf} \approx -20 \text{ lbf}$ . That is, the 100 pounds doesn't push block A up at all, but even with no gravity you need to pull up with a 20 pound force to get it to move. If we

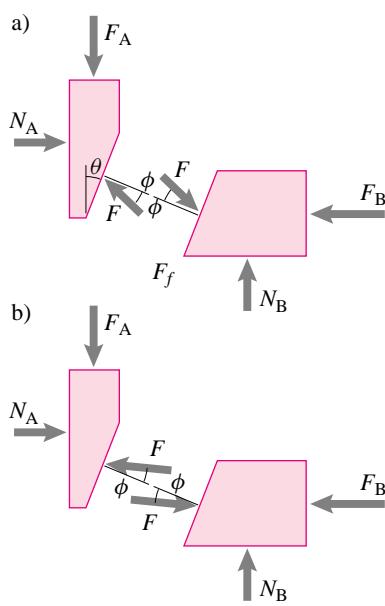


Figure 4.105: Free body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

(Filename:tfigure.wedgefriction)

insist that the downwards force  $F_A$  is positive or zero, that the pushing force  $F_B$  is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever  $\phi > \theta$ . (Actually we didn't need to do this second calculation at all. Eqn 4.57 shows the same paradox when  $\theta + \phi > 90^\circ$ . Trying to squeeze block B to the right for large  $\theta$  is exactly like trying to squeeze block A up for small  $\theta$ .)

This *selflocking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-backdrivable* whenever  $\phi > \theta$ . That is, pushing down on A can 'drive' block B to the right, but pushing to the left on block B cannot push block A 'back' up. Non-backdrivability is a feature or a defect depending on context.

The borderline case of backdrivability is when  $\theta = \phi$  and  $F_B = F_A / \tan 2\theta$ . Assuming  $\theta$  is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2} \frac{F_A}{\tan \theta} \approx \frac{1}{2} \cdot (\text{the value of } F_B \text{ had there been no friction}).$$

Thus the design guideline: non-back-drivable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is with fresh wet watermelon seed squeezed between two pinched fingers.

#### Pulley and chain drives

Chain and pulley drives are kind of like spread out gears (Fig. 4.106). The rotation of two shafts is coupled not by the contact of gear teeth but by a belt around a pulley or a chain around a sprocket. For simple analysis one draws free body diagrams for each sprocket or pulley with a little bit of chain as in Fig. 4.106b. Note that  $T_1 \neq T_2$ , unlike the case of an ideal undriven pulley. Applying moment balance we find,

$$\text{For gear A, } \left\{ \sum \vec{M}_{i/A} = \vec{0} \right\} \cdot \hat{k} \Rightarrow -R_A(T_2 - T_1) + M_A = 0$$

$$\text{For gear B, } \left\{ \sum \vec{M}_{i/B} = \vec{0} \right\} \cdot \hat{k} \Rightarrow R_B(T_1 - T_2) - M_B = 0$$

$$\text{eliminating } (T_2 - T_1) \Rightarrow M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

exactly as for a pair of gears. Note that we cannot find  $T_2$  or  $T_1$  but only their difference. Typically in design if, say,  $M_A$  is positive, one would try to keep  $T_1$  as small as possible without the belt slipping or the chain jumping teeth. If  $T_1$  grows then so must  $T_2$ , to preserve their difference. This increase in tension increases the loads on the bearings as well as the chain or belt itself.

#### 4-bar linkages

Four bar linkages often, confusingly, have 3 bars, the fourth piece is the something bigger. A planar mechanism with four pieces connected in a loop by hinges is a four bar linkage. Four bar linkages are remarkably common. After a single body connected at a hinge (like a gear or lever) a four bar linkage is one of the simplest mechanisms that can move in just one way (have just one degree of freedom).

A reasonable model of seated bicycle pedaling uses a 4-bar linkage (Fig. 4.107a). The whole bicycle frame is one bar, the human thigh is the second, the calf is the third, and the bicycle crank is the fourth. The four hinges are the hip joint, the knee joint, the pedal axle, and the bearing at the bicycle crank axle. A more sophisticated

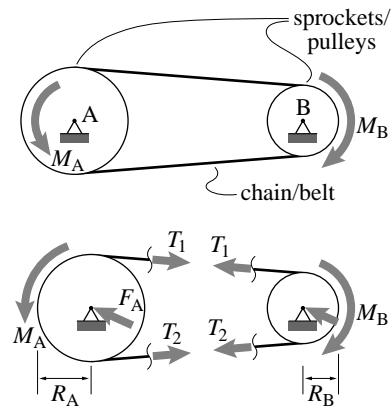


Figure 4.106: a) A chain or pulley drive involving two sprockets or pulleys and one chain or belt, b) free body diagrams of each of the sprockets/pulleys.  
(Filename:tfigure.chainpulley)

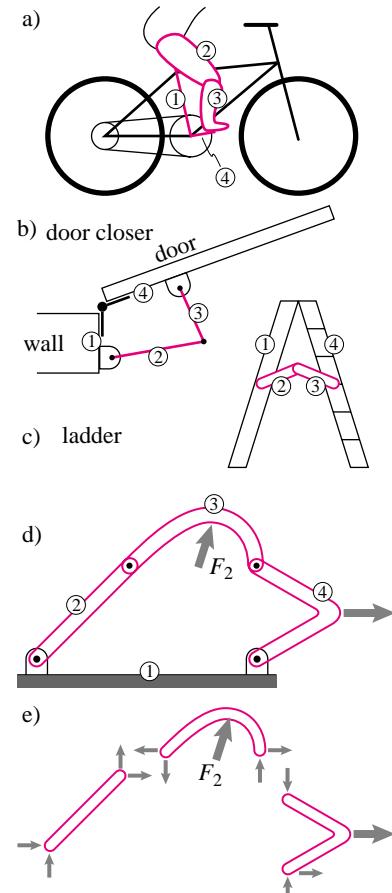


Figure 4.107: Four bar linkages. a) A bicycle, thigh, calf, and crank, b) a door closer, c) a folding ladder, d) a generic mechanism, e) free body diagrams of the parts of a generic mechanism.  
(Filename:tfigure.fourbar)

model of the system would include the ankle joint and the foot would make up a fifth bar.

A standard door closing mechanism is part of a 4-bar linkage (Fig. 4.107b). The door jamb and door are two bars and the mechanism pieces make up the other two.

A standard folding ladder design is, until locked open, a 4-bar linkage (Fig. 4.107c).

An abstracted 4-bar linkage with two loads is shown in Fig. 4.107d with free body diagrams in Fig. 4.107e. If one of the applied loads is given, then the other applied load along with interaction and reaction forces make up nine unknown components (after using the principle of action and reaction). With three equilibrium equations for each of the three bars, all these unknowns can be found.

### *Slider crank*

A mechanism closely related to a four bar linkage is a slider crank (Fig. 4.108a). An umbrella is one example (rotated 90° in Fig. 4.108b). If the sliding part is replaced by a bar, as in Fig. 4.108c, the point C moves in a circle instead of a straight line. If the height  $h$  is very large then the arc traversed by C is nearly a straight line so the motion of the four-bar linkage is almost the same as the slider crank. For this reason, slider cranks are sometimes regarded as a special case of a four-bar linkage in the limit as one of the bars gets infinitely long.

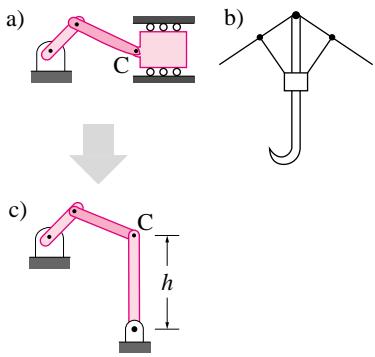


Figure 4.108: a) a slider crank, b) an umbrella has a slider-crank mechanism, c) the equivalent four-bar linkage, at least when  $h \rightarrow \infty$ .

(Filename:tfigure.slidercrank)

### *Summary of structures and machines*

The basic approach to the statics of structures and machines in 2D is straightforward and involves no tricks:

- Draw free body diagrams for each of the components.
- On the free body diagram use the principle of action and reaction to relate the forces on interacting components.
- Write three independent equilibrium equations for each piece (Say, force balance and moment balance, or moment balance about three non-colinear points).
- Solve these equations for the desired unknowns.

If you are lazy and resourceful, you can sometimes save work by

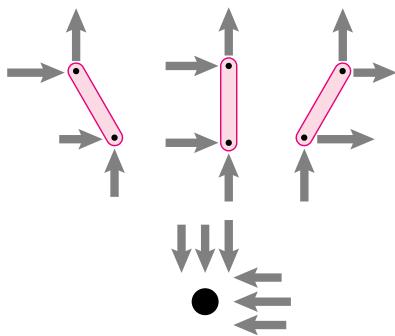
- drawing a free body diagram of the whole structure or some collection of pieces,  
or
- using appropriate equilibrium equations that avoid variables that you don't  
know and don't care about.

## 4.6 The ‘method of bars and pins’ for trusses

A statically determinate truss is a special case of the type of structure discussed in this section. So the methods of this section should work. They do and the resulting method, which is essentially never used in such detail, we will call ‘the method of bars and pins’.

In the method of bars and pins you treat a truss like any other structure. You draw a free body diagram of each bar and of each pin. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the equilibrium equations.

Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.



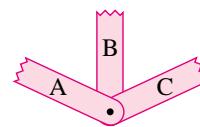
Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the ‘method of bars and pins’ reduces to the method of joints in the end.

Another approach is to ignore the pins and just think of a truss as bars that are connected with forces and no moments. Draw free body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

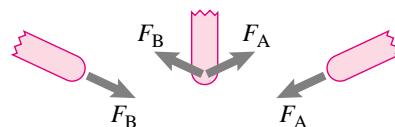
However this approach leads to a difficulty if more than two bars are connected at one hinge. The law of action and reaction is stated for pair-wise interactions not for triples or quadruples. Nonetheless, one can proceed by the following trick.

At each joint where say bars A, B, and C are connected, break the connection into pair-wise interactions. For example, imagine a frictionless hinge connecting A to B and one connecting B to C but ignore the connection of A with C. That the two connections are spatially coincident is confusing but not a problem. On the free body diagram of A a force will show from B. On the free body diagram of B forces will show from A and C. And on the free body diagram of C a force will show from B. (Beware not to assume that the force from B onto A or C is along B.) The truss is thus analyzable by writing the equilibrium equations for these bars in terms of the unknown interaction forces.

Partial Structure



Partial FBD's



The trick above can also be used for the analysis of structures and machines that have multiple pieces connected at one point. In the machines treated in this section we have avoided the difficulty above by only considering connections between pairs of bodies. This covers many mechanisms and structures but unfortunately does not cover many trusses. For trusses this trickiness can be avoided by use of the method of joints.

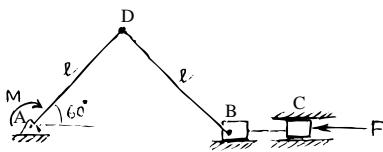


Figure 4.109: (Filename:sfig4.mech.slider)

**SAMPLE 4.23** A slider crank: A torque  $M = 20 \text{ N}\cdot\text{m}$  is applied at the bearing end A of the crank AD of length  $\ell = 0.2 \text{ m}$ . If the mechanism is in static equilibrium in the configuration shown, find the load F on the piston.

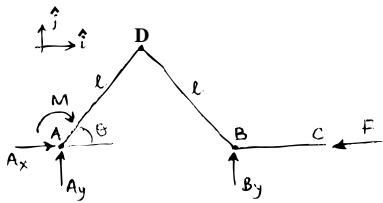


Figure 4.110: (Filename:sfig4.mech.slider.a)

**Solution** The free body diagram of the whole mechanism is shown in Fig. 4.110. From the moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , we get

$$\begin{aligned} \vec{M} + \vec{r}_{B/A} \times (\vec{B} + \vec{F}) &= \vec{0} \\ -M\hat{k} + 2\ell \cos \theta \hat{i} \times (B_y \hat{j} - F \hat{i}) &= \vec{0} \\ (-M + 2B_y \ell \cos \theta) \hat{k} &= \vec{0} \\ \Rightarrow B_y &= \frac{M}{2\ell \cos \theta} \end{aligned}$$

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$\begin{aligned} (A_x - F)\hat{i} + (A_y + B_y)\hat{j} &= 0 \\ A_x &= F \\ A_y &= -B_y \end{aligned}$$

Note that we still need to find  $F$  or  $A_x$ . So far, we have had only three equations in four unknowns ( $A_x$ ,  $A_y$ ,  $B_y$ ,  $F$ ). To solve for the unknowns, we need one more equation. We now consider the free body diagram of the mechanism without the crank, that is, the connecting rod DB and the piston BC together. See Fig. 4.111. Unfortunately, we introduce two more unknowns (the reactions) at D. However, we do not care about them. Therefore, we can write the moment equilibrium equation about point D,  $\sum \vec{M}_D = \vec{0}$  and get the required equation without involving  $D_x$  and  $D_y$ .

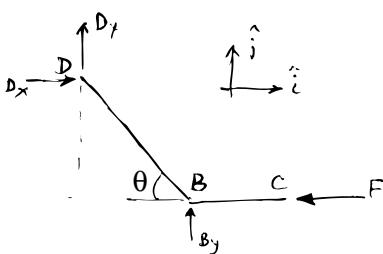


Figure 4.111: (Filename:sfig4.mech.slider.b)

$$\begin{aligned} \vec{r}_{B/D} \times (-F\hat{i} + B_y\hat{j}) &= \vec{0} \\ \ell(\cos \theta \hat{i} - \sin \theta \hat{j}) \times (-F\hat{i} + B_y\hat{j}) &= \vec{0} \\ B_y \ell \cos \theta \hat{k} - F \ell \sin \theta \hat{k} &= \vec{0} \end{aligned}$$

Dotting the last equation with  $\hat{k}$  we get

$$\begin{aligned} F &= B_y \frac{\cos \theta}{\sin \theta} \\ &= \frac{M}{2\ell \cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \\ &= \frac{M}{2\ell \sin \theta} \\ &= \frac{20 \text{ N}\cdot\text{m}}{2 \cdot 0.2 \text{ m} \cdot \sqrt{3}/2} \\ &= 57.74 \text{ N.} \end{aligned}$$

$$F = 57.74 \text{ N}$$

Note that the force equilibrium carried out above is not really useful since we are not interested in finding the reactions at A. We did it above to show that just one free body diagram of the whole mechanism was not sufficient to find  $F$ . On the other hand, writing moment equations about A for the whole mechanism and about D for the connecting rod plus the piston is enough to determine  $F$ .

**SAMPLE 4.24** There is more to it than meets the eye! A flyball governor is shown in the figure with all relevant masses and dimensions. The relaxed length of the spring is 0.15 m and its stiffness is 500 N/m.

- Find the static equilibrium position of the center collar.
- Find the force in the strut AB or CD.
- How does the spring force required to hold the collar depend on  $\theta$ ?

**Solution** Let  $\ell_0 (= 0.15 \text{ m})$  denote the relaxed length of the spring and let  $\ell$  be the stretched length in the static equilibrium configuration of the flyball, i.e., the collar is at a distance  $\ell$  from the fixed support EF. Then the net stretch in the spring is  $\delta \equiv \Delta\ell = \ell - \ell_0$ . We need to determine  $\ell$ , the spring force  $k\delta$ , and its dependence on the angle  $\theta$  of the ball-arm.

The free body diagram of the collar is shown in Fig. 4.113. Note that the struts AB and CD are two-force bodies (forces act only at the two end points on each strut). Therefore, the force at each end must act along the strut. From geometry ( $AB = BE = d$ ), then, the strut force  $F$  on the collar must act at angle  $\theta$  from the vertical. Now, the force balance in the vertical direction, i.e.,  $[\sum \vec{F} = \vec{0}] \cdot \hat{j}$ , gives

$$-2F \cos \theta + k\delta = mg \quad (4.59)$$

Thus to find  $\delta$  we need to find  $F$  and  $\theta$ . Now we draw the free body diagram of arm EBG as shown in Fig. 4.114. From the moment balance about point E, we get

$$\begin{aligned} \vec{r}_{G/E} \times (-2mg\hat{j}) + \vec{r}_{B/E} \times \vec{F} &= \vec{0} \\ 2d\hat{\lambda} \times (-2mg\hat{j}) + d\hat{\lambda} \times F(-\sin\theta\hat{i} + \cos\theta\hat{j}) &= \vec{0} \\ -4mgd(\hat{\lambda} \times \hat{j}) + Fd[-\sin\theta(\hat{\lambda} \times \hat{i}) + \cos\theta(\hat{\lambda} \times \hat{j})] &= \vec{0} \\ -\sin\theta\hat{k} & \\ \cos\theta\hat{k} & \\ -\sin\theta\hat{k} & \\ 4mgd \sin\theta\hat{k} + Fd(-\sin\theta \cos\theta\hat{k} - \cos\theta \sin\theta\hat{k}) &= \vec{0} \\ (4mgd \sin\theta - 2Fd \sin\theta \cos\theta)\hat{k} &= \vec{0} \end{aligned}$$

Dotting this equation with  $\hat{k}$  and assuming that  $\theta \neq 0$ , we get

$$2F \cos\theta = 4mg \quad (4.60)$$

Substituting eqn. (4.60) in eqn. (4.59) we get

$$\begin{aligned} k\delta &= mg + 2F \cos\theta = mg + 4mg = 5mg \\ \Rightarrow \delta &= \frac{5mg}{k} = \frac{5 \cdot 2 \text{ kg} \cdot 9.81 \text{ m/s}^2}{500 \text{ N/m}} = 0.196 \text{ m} \end{aligned}$$

- The equilibrium configuration is specified by the stretched length  $\ell$  of the spring (which specifies  $\theta$ ). Thus,

$$\ell = \ell_0 + \delta = 0.15 \text{ m} + 0.196 \text{ m} = 0.346 \text{ m}$$

Now, from  $\ell = 2d \cos\theta$ , we find that  $\theta = 30.12^\circ$ .

- The force in strut AB (or CD) is

$$F = 2mg / \cos\theta = 45.36 \text{ N}$$

- The force in the spring  $k\delta = 5mg$  as shown above and thus, it does not depend on  $\theta$ ! In fact, the angle  $\theta$  is determined by the relaxed length of the spring.

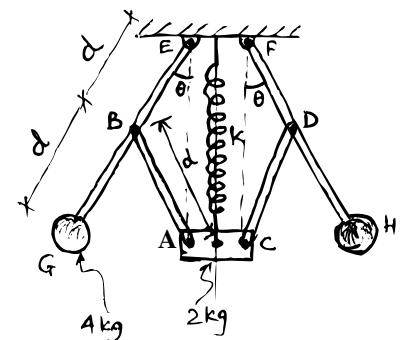


Figure 4.112: (Filename:sfig4.mech.gov)

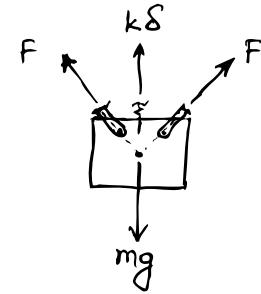


Figure 4.113: (Filename:sfig4.mech.gov.a)

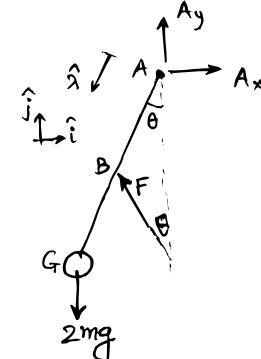


Figure 4.114: (Filename:sfig4.mech.gov.b)

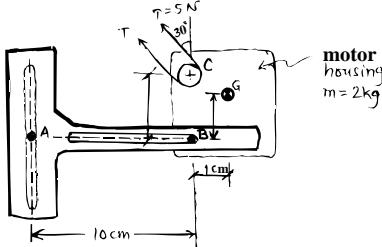


Figure 4.115: (Filename:sfig4.mech.motor)

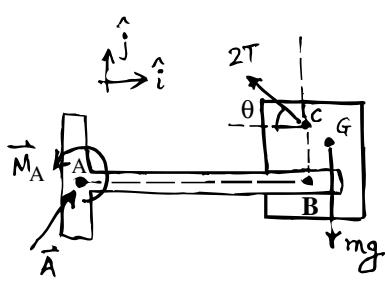


Figure 4.116: (Filename:sfig4.mech.motor.a)

**Solution** Although the mechanism looks complicated, the problem is straightforward. We cut the bolt at A and draw the free body diagram of the motor housing plus the slotted arm. Since the bolt, modeled as a welded joint, can take some torque, the unknowns at A are  $\vec{A} (= A_x \hat{i} + A_y \hat{j})$  and  $\vec{M}_A$ . The free body diagram is shown in Fig. 4.116. Note that we have replaced the tension at the two belt ends by a single equivalent tension  $2T$  acting at the center of the axle. Now taking moments about point A, we get

$$\vec{M}_A + \vec{r}_{C/A} \times 2\vec{T} + \vec{r}_{G/A} \times m\vec{g} = \vec{0}$$

where

$$\begin{aligned}\vec{r}_{C/A} \times 2\vec{T} &= (\ell \hat{i} + h \hat{j}) \times 2T(-\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= 2T(\ell \sin \theta + h \cos \theta) \hat{k} \\ \vec{r}_{G/A} \times m\vec{g} &= [(\ell + d)\hat{i} + (\text{anything})\hat{j}] \times (-mg \hat{j}) \\ &= -mg(\ell + d) \hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}\vec{M}_A &= -\vec{r}_{C/A} \times 2\vec{T} - \vec{r}_{G/A} \times m\vec{g} \\ &= -2T(\ell \sin \theta + h \cos \theta) \hat{k} + mg(\ell + d) \hat{k} \\ &= -2(5 \text{ N})(0.1 \text{ m} \cdot \sin 60^\circ + 0.04 \text{ m} \cdot \cos 60^\circ) \hat{k} \\ &\quad + 2 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot (0.1 + 0.01) \text{ m} \hat{k} \\ &= 1.092 \text{ N}\cdot\text{m} \hat{k}\end{aligned}$$

The reaction force  $\vec{A}$  can be determined from the force balance,  $\sum \vec{F} = \vec{0}$  as follows.

$$\begin{aligned}\vec{A} + 2\vec{T} + m\vec{g} &= \vec{0} \\ \Rightarrow \vec{A} &= -2\vec{T} - m\vec{g} \\ &= -10 \text{ N} \left( -\frac{1}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j} \right) - (-19.62 \text{ N} \hat{j}) \\ &= 5 \text{ N} \hat{i} + 10.96 \text{ N} \hat{j}\end{aligned}$$

$\vec{M}_A = 1.092 \text{ N}\cdot\text{m} \hat{k}$  and  $\vec{A} = 5 \text{ N} \hat{i} + 10.96 \text{ N} \hat{j}$

**SAMPLE 4.26 A gear train:** In the compound gear train shown in the figure, the various gear radii are:  $R_A = 10 \text{ cm}$ ,  $R_B = 4 \text{ cm}$ ,  $R_C = 8 \text{ cm}$  and  $R_D = 5 \text{ cm}$ . The input load  $F_i = 50 \text{ N}$ . Assuming the gears to be in static equilibrium find the machine load  $F_o$ .

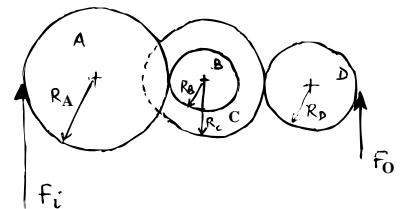


Figure 4.117: (Filename:sfig4.mech.gear)

**Solution** You may be tempted to think that a free body diagram of the entire gear train will do since we only need to find  $F_o$ . However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load  $F_o$  easily if we go gear by gear from the left to the right.

The free body diagram of gear A is shown in Fig. 4.118. Let  $F_1$  be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O,  $\sum \vec{M}_O = \vec{0}$ , we have

$$\begin{aligned} \vec{r}_M \times \vec{F}_i + \vec{r}_N \times \vec{F}_1 &= \vec{0} \\ -F_i R_A \hat{k} + F_1 R_A \hat{k} &= \vec{0} \\ \Rightarrow F_1 &= F_i \end{aligned}$$

Similarly, from the free body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$\begin{aligned} F_1 R_B \hat{k} + F_2 R_C \hat{k} &= \vec{0} \\ \Rightarrow F_2 &= \frac{R_B}{R_C} F_1 \\ &= \frac{R_B}{R_C} F_i \end{aligned}$$

Finally, from the free body diagram of the last gear D and the moment equilibrium about its center R, we get

$$\begin{aligned} -F_2 R_D \hat{k} + F_o R_D \hat{k} &= \vec{0} \\ \Rightarrow F_o &= F_2 \\ &= \frac{R_B}{R_C} F_i \\ &= \frac{4 \text{ cm}}{8 \text{ cm}} \cdot 50 \text{ N} = 25 \text{ N} \end{aligned}$$

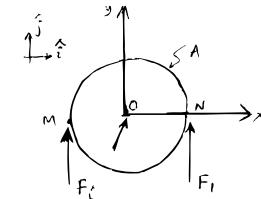


Figure 4.118: (Filename:sfig4.mech.gear.a)

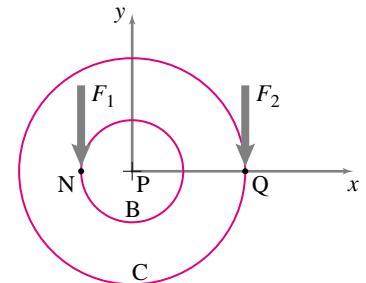


Figure 4.119: (Filename:sfig4.mech.gear.b)

$$F_o = 25 \text{ N}$$

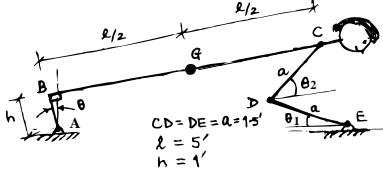


Figure 4.120: (Filename:sfig4.mech.pushup)

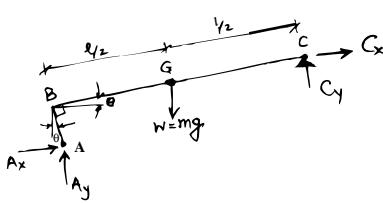


Figure 4.121: (Filename:sfig4.mech.pushup.a)

**SAMPLE 4.27 Push-up mechanics:** During push-ups the body, including the legs, usually moves as a single rigid unit; the ankle is almost locked, and the push-up is powered by the shoulder and the elbow muscles. A simple model of the body during push-ups is a four-bar linkage ABCDE shown in the figure. In this model, each link is a rigid rod, joint B is rigid (thus ABC can be taken as a single rigid rod), joints C, D, and E are hinges, but there is a motor at D that can supply torque. The weight of the person,  $W = 150 \text{ lbf}$ , acts through G. Find the torque at D for  $\theta_1 = 30^\circ$  and  $\theta_2 = 45^\circ$ .

**Solution** The free body diagram of part ABC of the mechanism is shown in Fig. 4.121. Writing moment balance equation about point A,  $\sum \vec{M}_A = \vec{0}$ , we get

$$\vec{r}_C \times \vec{C} + \vec{r}_G \times \vec{W} = \vec{0}$$

Let  $\vec{r}_C = r_{C_x} \hat{i} + r_{C_y} \hat{j}$  and  $\vec{r}_G = r_{G_x} \hat{i} + r_{G_y} \hat{j}$  for now (we can figure it out later). Then, the moment equation becomes

$$\begin{aligned} (r_{C_x} \hat{i} + r_{C_y} \hat{j}) \times (C_x \hat{i} + C_y \hat{j}) + (r_{G_x} \hat{i} + r_{G_y} \hat{j}) \times (-W \hat{j}) &= \vec{0} \\ [(C_y r_{C_x} - C_x r_{C_y}) \hat{k} - W r_{G_x} \hat{k}] &= \vec{0} \\ [ ] \cdot \hat{k} \Rightarrow C_y r_{C_x} - C_x r_{C_y} &= W r_{G_x} \end{aligned} \quad (4.61)$$

We now draw free body diagrams of the links CD and DE separately (Fig. 4.122) and write the moment and force balance equations for them.

For link CD, the force equilibrium  $\sum \vec{F} = \vec{0}$  gives

$$(-C_x + D_x) \hat{i} + (D_y - C_y) \hat{j} = \vec{0}$$

Dotting with  $\hat{i}$  and  $\hat{j}$  gives

$$\begin{aligned} D_x &= C_x \\ D_y &= C_y \end{aligned} \quad (4.62)$$

and the moment equilibrium about point D, gives

$$\begin{aligned} M \hat{k} - a(\cos \theta_2 \hat{i} + \sin \theta_2 \hat{j}) \times (-C_x \hat{i} - C_y \hat{j}) &= \vec{0} \\ M \hat{k} + (C_y a \cos \theta_2 - C_x a \sin \theta_2) \hat{k} &= \vec{0} \end{aligned} \quad (4.63)$$

Similarly, the force equilibrium for link DE requires that

$$\begin{aligned} E_x &= D_x \\ E_y &= D_y \end{aligned} \quad (4.64)$$

and the moment equilibrium of link DE about point E gives

$$-M + D_x a \sin \theta_1 + D_y a \cos \theta_1 = 0. \quad (4.65)$$

Now, from eqns. (4.62) and (4.65)

$$-M + C_x a \sin \theta_1 + C_y a \cos \theta_1 = 0 \quad (4.66)$$

Adding eqns. (4.63) and (4.66) and solving for  $C_x$  we get

$$C_x = \frac{\cos \theta_1 + \cos \theta_2}{\sin \theta_2 - \sin \theta_1} C_y$$

For simplicity, let

$$f(\theta_1, \theta_2) = \frac{\cos \theta_1 + \cos \theta_2}{\sin \theta_2 - \sin \theta_1}$$

so that

$$C_x = f(\theta_1, \theta_2)C_y \quad (4.67)$$

Now substituting eqn. (4.67) in (4.61) we get

$$C_y = \frac{r_{G_x}}{r_{C_x} - r_{C_y}f} W$$

Now substituting  $C_y$  and  $C_x$  into eqn. (4.66) we get

$$M = \frac{r_{G_x} a (\cos \theta_1 + f \sin \theta_1)}{r_{C_x} - r_{C_y} f} W$$

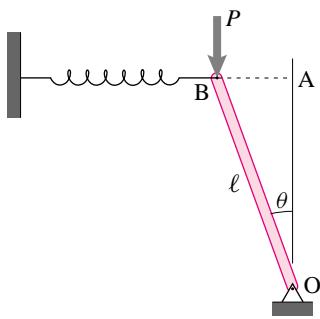
where

$$\begin{aligned} r_{G_x} &= (\ell/2) \cos \theta - h \sin \theta \\ r_{C_x} &= \ell \cos \theta - h \sin \theta \\ r_{C_y} &= \ell \sin \theta + h \cos \theta \end{aligned}$$

Now plugging all the given values:  $W = 160 \text{ lbf}$ ,  $\theta_1 = 30^\circ$ ,  $\theta_2 = 45^\circ$ ,  $\ell = 5 \text{ ft}$ ,  $h = 1 \text{ ft}$ ,  $a = 1.5 \text{ ft}$ , and, from simple geometry,  $\theta = 9.49^\circ$ ,

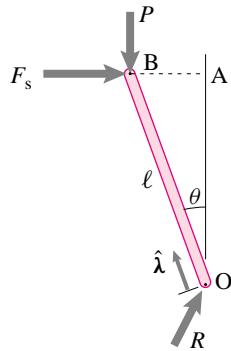
$$\begin{aligned} f &= 7.60 \\ r_{C_x} &= 4.77 \text{ ft}, \quad r_{C_y} = 1.81 \text{ ft}, \quad r_{G_x} = 2.30 \text{ ft} \\ \Rightarrow M &= -269.12 \text{ lb}\cdot\text{ft} \end{aligned}$$

$M = -269.12 \text{ lb}\cdot\text{ft}$



**SAMPLE 4.28 A spring and rod buckling model:** A simple model of sideways buckling of a rod can be constructed with a spring and a rod as shown in the figure. Assume the rod to be in static equilibrium at some angle  $\theta$  from the vertical. Find the angle  $\theta$  for a given vertical load  $P$ , spring stiffness  $k$ , and bar length  $\ell$ . Assume that the spring is relaxed when the rod is vertical.

Figure 4.123: (Filename:sfig4.mech.buckling)



**Solution** When the rod is displaced from its vertical position, the spring gets compressed or stretched depending on which side the rod tilts. The spring then exerts a force on the rod in the opposite direction of the tilt. The free body diagram of the rod with a counterclockwise tilt  $\theta$  is shown in Fig. 4.124. From the moment balance  $\sum \vec{M}_O = \vec{0}$  (about the bottom support point O of the rod), we have

$$\vec{r}_B \times \vec{P} + \vec{r}_B \times \vec{F}_s = \vec{0}$$

Noting that

$$\begin{aligned}\vec{r}_B &= \ell\hat{\lambda}, \\ \vec{P} &= -P\hat{j}, \\ \text{and } \vec{F}_s &= k(\vec{r}_A - \vec{r}_B) \\ &= k(\ell\hat{j} - \ell\hat{\lambda}),\end{aligned}$$

Figure 4.124: (Filename:sfig4.mech.buckling.a)

we get

$$\begin{aligned}\ell\hat{\lambda} \times (P\hat{j}) + \ell\hat{\lambda} \times k\ell(\hat{j} - \hat{\lambda}) &= \vec{0} \\ -P\ell(\hat{\lambda} \times \hat{j}) + k\ell^2(\hat{\lambda} \times \hat{j}) &= \vec{0}\end{aligned}$$

Dotting this equation with  $(\hat{\lambda} \times \hat{j})$  we get

$$\begin{aligned}-P\ell + k\ell^2 &= 0 \\ \Rightarrow P &= k\ell.\end{aligned}$$

Thus the equilibrium only requires that  $P$  be equal to  $k\ell$  and it is independent of  $\theta$ ! That is, the system will be in static equilibrium at any  $\theta$  as long as  $P = k\ell$ .

If  $P = k\ell$ , any  $\theta$  is an equilibrium position.

## 4.7 Hydrostatics

*Hydrostatics* is primarily concerned with finding the net force and moment of still water on a surface. The surfaces are typically the sides of a pool, dam, container, or pipe, or the outer surfaces of a floating object such as a boat or of a submerged object like a toilet bowl float, or the imagined surface that separates some water of interest from the other water. Although the hydrostatics of air helps explain the floating of hot air balloons, dirigibles, and chimney smoke; and the hydrostatics of oil is important for hydraulics (hydraulic breaks for example), often the fluid of concern for engineers is water, and we will use the word ‘water’ as an informal synonym for ‘fluid.’

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions:

- 1) The force of water on a surface is perpendicular to the surface; and
- 2) The density of water,  $\rho$  (pronounced ‘row’) is a constant (doesn’t vary with depth or pressure),

The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated: still water cannot carry any shear stress. For near-still water this constitutive assumption is abnormally good (in the world of constitutive assumptions), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress) much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) only causes water to increase its density about 4% from that of water at the surface. That water density does depend measurably on salinity and temperature is, however, important for some hydrostatic calculations, in particular for determining which water floats on which other water. Sometimes instead of talking about the mass per unit of volume  $\rho$  we will use the weight per unit volume  $\gamma = g\rho$  ('gammuh = gee row').

### Surface area $A$ , outward normal $\hat{n}$ , pressure $p$ , and force $\vec{F}$

We are going to be generalizing the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is  $A$  which we can think of as being the sum of the bits of area  $\Delta A$  that compose it:

$$A = \int dA.$$

Every bit of surface area has an *outer* normal  $\hat{n}$  that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure  $p$ , so that the force on a small bit of surface is

$$\Delta\vec{F} = p(\Delta A)(-\hat{n}),$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

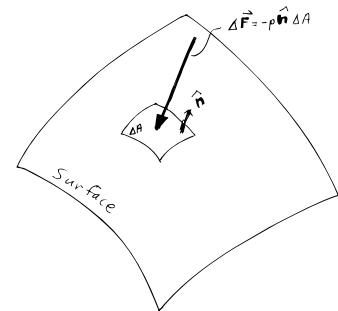


Figure 4.125: A bit of area  $\Delta A$  on a surface on which pressure  $p$  acts. The outward (into the water) normal of the surface is  $\hat{n}$  so the increment of force is  $\Delta\vec{F} = -p\hat{n}\Delta A$ .

(Filename: figure.deltaA)

$$\begin{aligned}\vec{F} &= \int d\vec{F} = - \int_A p \hat{n} dA \\ \vec{M}_C &= \int_A d\vec{M}_C = - \int_A \vec{r}_{/C} \times (p \hat{n}) dA\end{aligned}\quad (4.68)$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 4.68 and their role in equilibrium equations. In the rest of this section we consider a variety of important special cases.

## Water in equilibrium with itself

Before we worry about how water pushes on other things, let's first understand what it means for water to be in static equilibrium. These first important facts about hydrostatics follow from drawing free body diagrams of various chunks of water and assuming static equilibrium.

*Pressure doesn't depend on direction*

We assume that the pressure  $p$  does not vary in wild ways from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. Now if we draw a free body diagram of a little triangular prism of water the net forces on the prism must add to zero (see Fig. 4.126). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is  $a \cos \theta w$  and the pressure on the surface is  $p_x$ , the net force is  $a \cos \theta w p_x \hat{i}$ . Calculating similarly for the other surfaces:

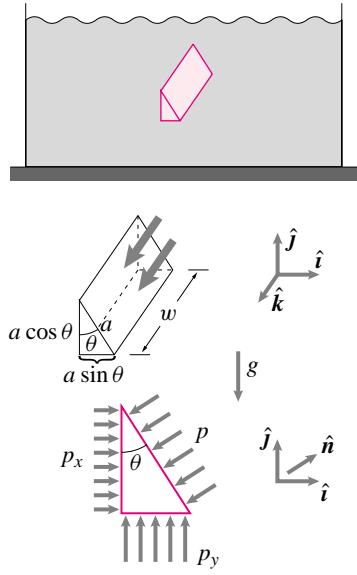


Figure 4.126: A small prism of water is isolated from some water in equilibrium. The free body diagram does not show the forces in the  $z$  direction.

(filename:tfigure.waterprism)

$$\begin{aligned}\vec{0} &= \sum \vec{F}_i \\ &= \underbrace{(a \cos \theta) w p_x \hat{i} + (a \sin \theta) w p_y \hat{j} - a w p \hat{n}}_{\text{pressure terms}} - \underbrace{\frac{a^2 \cos \theta \sin \theta w}{2} \rho g \hat{j}}_{\text{weight}} \\ &= aw \left( \cos \theta p_x \hat{i} + \sin \theta p_y \hat{j} - p (\cos \theta \hat{i} + \sin \theta \hat{j}) - \frac{a \cos \theta \sin \theta}{2} \rho g \hat{j} \right)\end{aligned}$$

If  $a$  is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by  $aw$  we get

$$\vec{0} = \cos \theta p_x \hat{i} + \sin \theta p_y \hat{j} - p (\cos \theta \hat{i} + \sin \theta \hat{j}).$$

Taking the dot product of both sides of this equation with  $\hat{i}$  and  $\hat{j}$  gives that  $p = p_x = p_y$ . Since  $\theta$  could be anything, force balance for the free body diagram of a small prism tells us that for a fluid in static equilibrium<sup>①</sup>

pressure is the same in every direction.

① That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction is also sometimes based on equilibrium of a small tetrahedron.

*Pressure doesn't vary with side to side position*

Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (Fig. 4.127a). The forces on the end caps at A and B are the only forces along the box. Therefor they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

pressure doesn't depend on horizontal position.

If we take the  $\hat{j}$  or  $y$  direction to be up, then we have

$$p(x, y, z) = p(y).$$

*Pressure increases linearly with depth*

Consider the vertically aligned box of Fig. 4.127b.

$$\begin{aligned} \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{j} &\Rightarrow \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} = 0 \\ &\Rightarrow p_{\text{bottom}} - p_{\text{top}} = \rho g h. \end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure  $p_a$  then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where  $h$  is the distance down from the surface,  $H$  is the depth to some reference point underwater and  $y$  is the distance up from that reference point (so that  $h = H - y$ ). Neglecting atmospheric pressure at the top surface we have the useful and easy to remember formula:

$$p = \gamma h. \quad (4.69)$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

*The buoyant force of water on water.*

In a place under water in a still swimming pool where there is nothing but water, imagine a chunk of water the shape of a sea monster. Now draw a free body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center of mass of the water monster. So, in static equilibrium:

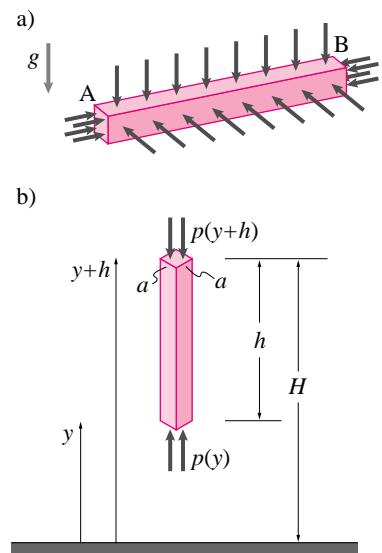


Figure 4.127: Free body diagram of a horizontally aligned box of water cut out of a bigger body of water.

(Filename: tfigure.waterbox)

The pressure forces acting on a surface enclosing a volume of water is equivalent to the negative weight passing through the center of mass of the water.

## The force of water on submerged and floating objects

① If there is no column of water from the point up to the surface it is still true that the pressure is  $\gamma h$ , as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 4.69<sup>①</sup>. So if you can find the resultant force on any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

The resultant of all pressure forces on a totally submerged object is minus the weight of the displaced water. The resultant acts at the centroid of the displaced volume:

$$\vec{F}_{\text{buoyancy}} = \gamma V \hat{k} \quad \text{acting at} \quad \vec{r} = \frac{\int \vec{r}/_0 dV}{V}.$$

The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 4.7 on page 201).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

## Displaced fluid

Sometimes people discuss Archimedes' principle in terms of the displaced fluid. A floating object in equilibrium displaces an amount of fluid with the same weight as the object; this is also the amount of volume of the floating object that is below the water level. On the other hand an object that is totally under water, for whatever reason (it is resting on the bottom, or it is being held underwater by a string, etc), occupies exactly as much space as it occupies. Putting these two ideas together one can remember that

A floating object displaces its weight, a submerged object displaces its volume.

## The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting  $\gamma = 0$  in the formulas above.

## The force of constant pressure on a flat surface

The net force of constant pressure on one flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned}\vec{F}_{\text{net}} &= \int_A -p \hat{n} dA \\ &= -p A \hat{n}.\end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C.

$$\begin{aligned}\vec{M}_{C,\text{net}} &= \int_A \vec{r}_{/C} \times (-p \hat{n} dA) \\ &= \underbrace{\left( \int_A \vec{r}_{/C} dA \right)}_0 \times (-p \hat{n}) \\ &= 0\end{aligned}$$

where the zero follows from the position of the center of mass relative to the center of mass being zero.

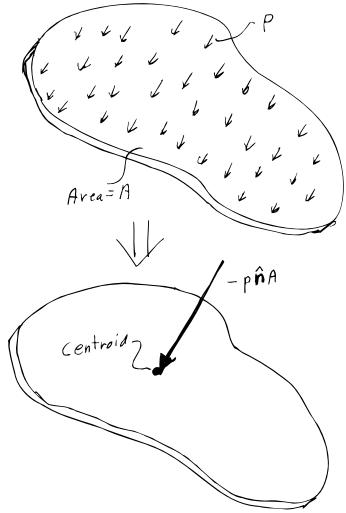


Figure 4.128: The resultant force from a constant pressure  $p$  on a flat plate is  $\vec{F} = -pA\hat{n}$  acting at the centroid of the plate.

(Filename:tfigure.centroidpressure)

## The force of water on a rectangular plate

Consider a rectangular plate with width into the page  $w$  and length  $\ell$ . Assume the water-side normal to the plate is  $\hat{n}$  and that the top edge of the plate is horizontal. Take  $\hat{j}$  to be the up direction with  $y$  being distance up from the bottom and the total depth of the water is  $H$ . Thus the area of the plate is  $A = \ell w$ . If the bottom and top of the plate are at  $y_1$  and  $y_2$  the net force on the plate can be found as:

$$\begin{aligned}\vec{F}_{\text{net}} &= - \int_A p \hat{n} dA \\ &= - \int_A \gamma(H - y)\hat{n} dA \\ &= -w \int_0^\ell \gamma(H - y(s))\hat{n} ds \\ &= -w \int_0^\ell \gamma(H - (y_1 + \hat{n} \cdot \hat{j} s))\hat{n} ds \\ &= -w\gamma(H\ell - y_1\ell - \hat{n} \cdot \hat{j} \ell^2/2)\hat{n} \\ &= -w\ell\gamma(H - (y_1 + \hat{n} \cdot \hat{j} \ell/2))\hat{n} \\ &= -w\ell\gamma(H - (y_1 + (y_2 - y_1)/2))\hat{n} \\ &= -w\ell(\gamma(H - y_1)/2 + \gamma(H - y_2)/2)\hat{n} \\ &= -w\ell \frac{p_1 + p_2}{2} \hat{n}. \\ &= -(area)(average\ pressure)(outwards\ normal\ direction)\end{aligned}$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the

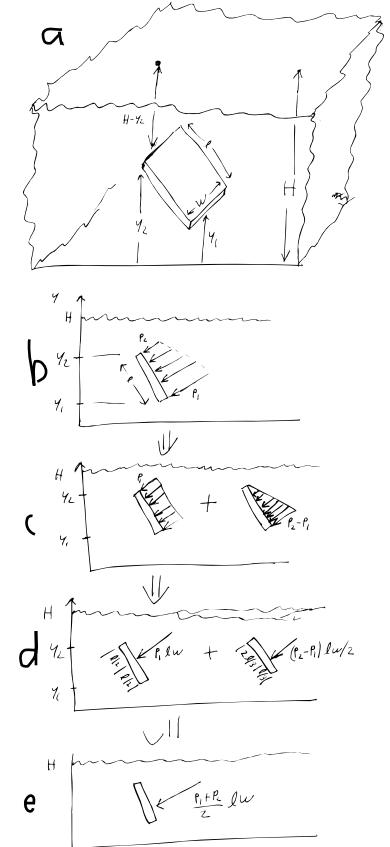


Figure 4.129: The resultant force from a constant depth-increasing pressure on a rectangular plate.

(Filename:tfigure.plateunder)

pressure of the top of the plate. The other varies linearly from zero at the top to  $\gamma(y_2 - y_1)$  at the bottom.

$$p = \gamma(H - y) = \underbrace{\gamma(H - y_2)}_{\text{Constant pressure, the pressure at the top edge.}} + \underbrace{\gamma(y_2 - y)}_{\text{Varies linearly from 0 at the top to } \gamma(y_2 - y_1) \text{ at the bottom.}}$$

The first corresponds to a force of  $w\ell\gamma(H - y_2)$  acting at the middle of the plate. The second corresponds to a force of  $w\ell\gamma \frac{y_2 - y_1}{2}$  acting a third of the way up from the bottom of the plate.

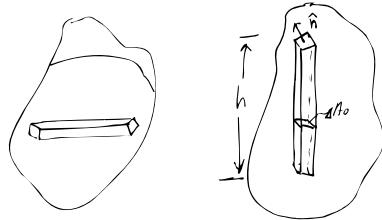
## 4.7 THEORY

### Adding forces to derive Archimedes' principle

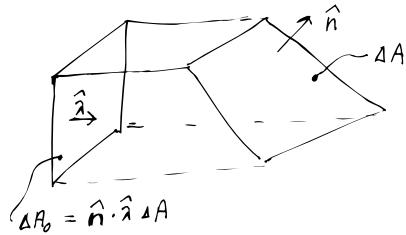
Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped submerged solid, say something potato shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

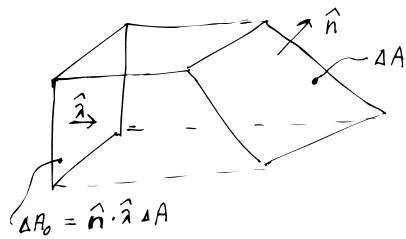
First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).



The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is  $\Delta A_0$  then the area of one of the prism caps is



$\Delta A = \Delta A_0 / (\hat{n} \cdot \hat{\lambda})$  where  $\hat{\lambda}$  is along the axis of the prism and  $\hat{n}$  is the outer unit normal to the end cap (Note  $\Delta A \geq \Delta A_0$  because  $\hat{n} \cdot \hat{\lambda} \leq 1$ ).



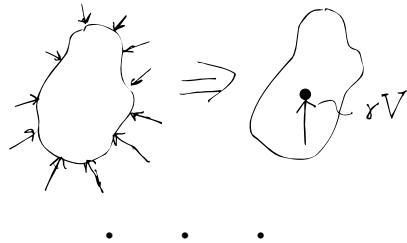
So the net force on the cap is  $-p\Delta A_0 \hat{n}/(\hat{n} \cdot \hat{\lambda})$ . The component of the force along the prism is  $[-p\Delta A_0 \hat{n}/(\hat{n} \cdot \hat{\lambda})] \cdot \hat{\lambda}$  which is  $-p\Delta A_0$ . An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus

the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$\begin{aligned}\text{net upwards force on vertical french fry} &= \Delta p \Delta A_0 \\ &= (\gamma h) \Delta A_0 \\ &= \gamma (h \Delta A_0) \\ &= \gamma \Delta V_0\end{aligned}$$

where  $\Delta V_0$  is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is  $\gamma V$ . The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)



#### A vector calculus derivation

Here is a derivation of Archimedes' principle, at least the net force part, using multi-variable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$\begin{aligned}\vec{F}_{\text{buoyancy}} &= - \int_A p \hat{n} dA \\ &= - \int_S p \hat{n} dS \\ &= - \int_S (H-z)\gamma \hat{n} dS \\ &= - \int_V \vec{\nabla}((H-z)\gamma) dV \\ &= - \int_V (-\hat{k})\gamma dV \\ &= \int_V \gamma dV \hat{k} \\ &= (\text{weight of displaced water}) \hat{k}.\end{aligned}$$

In this derivation we first changed from calling bits of surface area  $dA$  to  $dS$  because that is a common notation in calculus books. The depth from the surface, of a point with vertical component  $z$  from the bottom, is  $H - z$ . The  $\vec{\nabla}$  symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$\int_S (\text{any scalar}) \hat{n} dS = \int_V \vec{\nabla}(\text{the same scalar}) dV.$$

The gradient of  $(H-z)\gamma$  is  $-\hat{k}\gamma$  because  $H$  and  $\gamma$  are constants. Note, where we write  $\int_S$  some books would write  $\iint_S$ , and where we write  $\int_V$  some books would write  $\iiint_V$ .

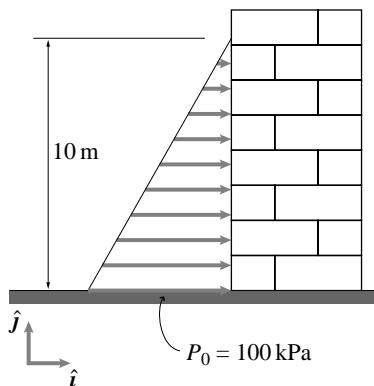


Figure 4.130: (Filename:sfig4.hydro.force1)

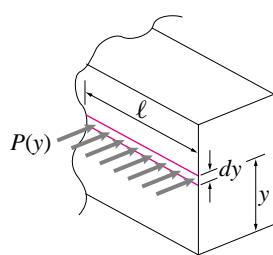
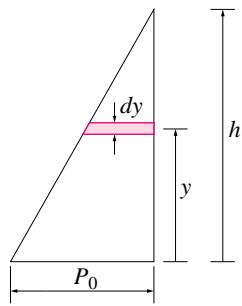


Figure 4.131: (Filename:sfig4.hydro.force1.a)

**SAMPLE 4.29 The force due to the hydrostatic pressure:** The hydrostatic pressure distribution on the face of a wall submerged in water up to a height \$h = 10 \text{ m}\$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be unit.

$$\begin{aligned} d\vec{F} &= p(y) \cdot y \cdot \ell \hat{i} \\ &= p_0 \frac{y}{h} \ell dy \hat{i} \end{aligned}$$

The net force due to the pressure distribution on the whole wall can now be found by integrating \$d\vec{F}\$ along the wall.

$$\begin{aligned} \vec{F} &= \int d\vec{F} \\ &= \int_0^h p_0 \frac{y}{h} \ell dy \hat{i} \\ &= \frac{p_0}{h} \ell \int_0^h y dy \hat{i} \\ &= \frac{p_0}{h} \ell \frac{h^2}{2} \hat{i} \\ &= \frac{1}{2} p_0 h \ell \hat{i} \\ &= \frac{1}{2} \cdot (100 \frac{\text{kN}}{\text{m}^2}) \cdot (10 \text{ m}) \cdot (1 \text{ m}) \hat{i} \\ &= 500 \text{ kN} \hat{i} \end{aligned}$$

$$\boxed{\vec{F} = 500 \text{ kN} \hat{i}}$$

**Alternatively,** the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length (\$\ell = 1 \text{ m}\$), i.e.,

$$\begin{aligned} \vec{F} &= (\frac{1}{2} \cdot h \cdot p_0 \hat{i}) \ell \\ &= \frac{1}{2} \cdot 10 \text{ m} \cdot 100 \frac{\text{kN}}{\text{m}^2} \cdot 1 \text{ m} \hat{i} \\ &= 500 \text{ kN} \end{aligned}$$

**SAMPLE 4.30** *The equivalent force due to hydrostatic pressure:* Find the net force and its location on each face of the dam due to the pressure distributions shown in the figure. Take unit length of the dam (into the page).

**Solution** We can determine the net force on each face of the dam by considering the given pressure distribution on one face at a time and finding the net force and its point of action.

On the left face of the dam we are given a trapezoidal pressure distribution. We break the given distribution into two parts — a triangular distribution given by ABE, and a rectangular distribution given by EBCD. We find the net force due to each distribution by finding the area of the distribution and multiplying by the unit length of the dam.

$$\begin{aligned}\vec{F}_1 &= (\text{area of ABE}) \cdot \ell \hat{i} = \frac{1}{2}(p_2 - p_1)h_l \ell \hat{i} \\ &= \frac{1}{2}(60\text{kPa} - 10\text{kPa}) \cdot 5\text{ m} \cdot 1\text{ m} \hat{i} \\ &= 125\text{ kN} \hat{i} \\ \vec{F}_2 &= (\text{area of EBCD}) \cdot \ell \hat{i} = p_1 h_l \ell \hat{i} \\ &= 10\text{kPa} \cdot 5\text{ m} \cdot 1\text{ m} \hat{i} \\ &= 50\text{ kN} \hat{i}\end{aligned}$$

The two forces computed above act through the centroids of the triangle ABE and the rectangle EBCD, respectively. The centroid are marked in Fig. 4.133. Now the net force on the left face is the vector sum of these two forces, *i.e.*,

$$\vec{F}_L = \vec{F}_1 + \vec{F}_2 = 175\text{ kN} \hat{i}$$

The net force  $\vec{F}_L$  acts through point G which is determined by the moment balance of the two forces  $\vec{F}_1$  and  $\vec{F}_2$  about point G:

$$\begin{aligned}\vec{r}_{G_1/G} \times \vec{F}_1 &= -\vec{r}_{G_2/G} \times \vec{F}_2 \\ F_1(h_G - \frac{h_l}{3}) \hat{k} &= -F_2(\frac{h_l}{2} - h_G)(-\hat{k}) \\ \Rightarrow h_G &= \frac{F_1 \frac{h_l}{3} + F_2 \frac{h_l}{2}}{F_1 + F_2} \\ &= \frac{125\text{ kN} \cdot 1.667\text{ m} + 50\text{ kN} \cdot 2.5\text{ m}}{175\text{ kN}} \\ &= 1.905\text{ m}\end{aligned}$$

Similarly, we compute the force on the right face of the dam by calculating the area of the triangular distribution shown in Fig. 4.134.

$$\begin{aligned}\vec{F}_R &= \frac{1}{2} p_0 \underbrace{(h_r / \sin \theta)}_d \underbrace{(-\sin \theta \hat{i} - \cos \theta \hat{j})}_{-\hat{n}} \\ &= \frac{1}{2} p_0 h_r (-\hat{i} - \tan \theta \hat{j}) \\ &= -20(\hat{i} + \sqrt{3}\hat{j})\text{ kN}\end{aligned}$$

and this force acts though the centroid of the triangle as shown in Fig. 4.134.

$$\boxed{\vec{F}_L = 175\text{ kN} \hat{i}, \text{ and } \vec{F}_R = -20(\hat{i} + \sqrt{3}\hat{j})\text{ kN}}$$

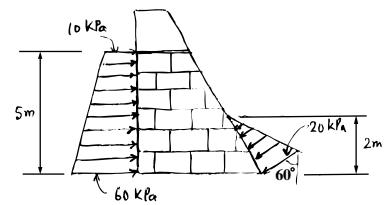


Figure 4.132: (Filename:sfig4.hydro.force2)

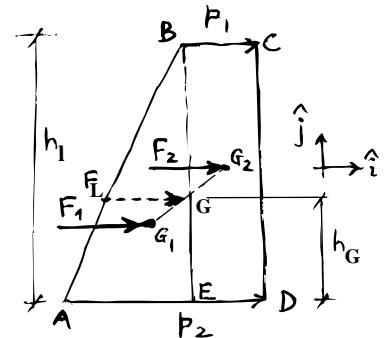


Figure 4.133: (Filename:sfig4.hydro.force2.a)

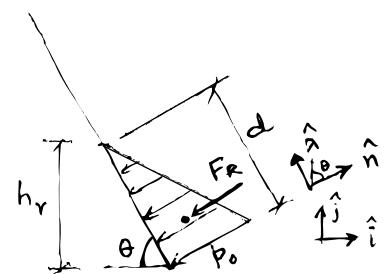


Figure 4.134: (Filename:sfig4.hydro.force2.b)

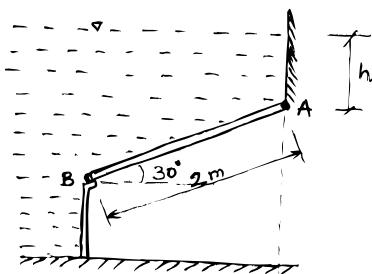


Figure 4.135: (Filename:sfig4.hydro.gate)

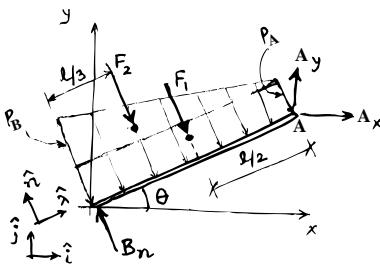


Figure 4.136: (Filename:sfig4.hydro.gate.a)

**SAMPLE 4.31 Forces on a submerged sluice gate:** A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is  $h$ . Ignoring the weight of the plate, find the forces on the hinge at A as a function of  $h$ . In particular, find the vertical pull on the hinge for  $h = 0$  and  $h = 2$  m.

**Solution** Let  $\gamma = \rho g$  be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is  $p_A = \gamma h$  and at point B is  $p_B = \gamma(h + \ell \sin \theta)$ . The pressure acts perpendicular to the plate and varies linearly from  $p_A$  at A to  $p_B$  at B. The free body diagram of the plate is shown in Fig. 4.136. Let  $\hat{\lambda}$  be a unit vector along BA and  $\hat{n}$  be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces  $F_1$  and  $F_2$  by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A) \frac{\ell}{2} = \frac{1}{2} \gamma \ell^2 \sin \theta$$

Now, we carry out moment balance about point A,  $\sum \vec{M}_A = \vec{0}$ , which gives

$$\begin{aligned} \vec{r}_{B/A} \times \vec{B} + \vec{r}_{D/A} \times \vec{F}_2 + \vec{r}_{C/A} \times \vec{F}_1 &= \vec{0} \\ -\ell \hat{\lambda} \times B_n \hat{n} - \frac{2\ell}{3} \hat{\lambda} \times (-F_1 \hat{n}) - \frac{\ell}{2} \hat{\lambda} \times (-F_2 \hat{n}) &= \vec{0} \\ -B_n \ell \hat{k} + F_1 \frac{2\ell}{3} \hat{k} + F_2 \frac{\ell}{2} \hat{k} &= \vec{0} \\ \Rightarrow B_n = \frac{2F_1}{3} + \frac{F_2}{2} &= \gamma \ell \left( \frac{2}{3}h + \frac{1}{4}\ell \sin \theta \right) \end{aligned}$$

and, from force balance,  $\sum \vec{F} = \vec{0}$ , we get

$$\begin{aligned} \vec{A} &= -B_n \hat{n} + F_1 \hat{n} + F_2 \hat{n} \\ &= \left( -\gamma \ell \left( \frac{2}{3}h + \frac{1}{4}\ell \sin \theta \right) + \gamma h \ell + \frac{1}{2} \gamma \ell^2 \sin \theta \right) \hat{n} \\ &= \left( \frac{1}{3} \gamma h \ell + \frac{1}{2} \gamma \ell^2 \sin \theta \right) \hat{n} = \gamma \ell \left( \frac{1}{3}h + \frac{1}{2}\ell \sin \theta \right) \hat{n} \end{aligned}$$

The force  $\vec{A}$  computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is  $-\vec{A}$  as shown in Fig. 4.137. From the expression for this force, we see that it varies linearly with  $h$ .

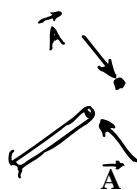


Figure 4.137: (Filename:sfig4.hydro.gate.b)

Let the vertical pull on the hinge be  $A_{\text{hinge}} y$ . Then

$$A_{\text{hinge}} y = -\vec{A} \cdot \hat{j} = -\gamma \ell \left( \frac{1}{3}h + \frac{1}{2}\ell \sin \theta \right) \hat{n} \cdot \hat{j} = \frac{1}{4} \gamma \ell \sin 2\theta + \left( \frac{1}{3} \gamma \ell \cos \theta \right) h$$

Now, substituting  $\gamma = 9.81 \text{ kN/m}^3$ ,  $\ell = 2 \text{ m}$ ,  $\theta = 30^\circ$ , the two specified values of  $h$ , and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$A_{\text{hinge}} y (h = 0) = 4.25 \text{ kN}, \quad A_{\text{hinge}} y (h = 2 \text{ m}) = 15.58 \text{ kN}$$

$A_{\text{hinge}} y  _{h=0} = 4.25 \text{ kN}, \quad A_{\text{hinge}} y  _{h=2 \text{ m}} = 15.58 \text{ kN}$
---

**SAMPLE 4.32 Tipping of a dam:** The cross section of a concrete dam is shown in the figure. Take the weight-density  $\gamma (= \rho g)$  of water to be  $10 \text{ kN/m}^3$  and that of concrete to be  $25 \text{ kN/m}^3$ . For the given design of the cross-section, find the ratio  $h/H$  that is safe enough for the dam to not tip over (about the downstream edge E).

**Solution** Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.<sup>①</sup> With this assumption, the free body diagram of the dam is shown in Fig. 4.139.

To compute all the forces acting on the dam, we assume the width  $w$  (into the paper) to be unit (*i.e.*,  $w = 1 \text{ m}$ ). Let  $\gamma_w$  and  $\gamma_c$  denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2} \gamma_w h \cdot h \cdot w = \frac{1}{2} \gamma_w h^2 w$$

This is the horizontal force (in the  $-\hat{i}$  direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes:

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w \cdot \gamma_c}_{\text{volume}} = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2} \cdot 3\alpha H \cdot 3\alpha H \tan \theta \cdot w \cdot \gamma_c}_{\text{volume}} = \frac{9}{2} \gamma_c \alpha^2 H^2 w \tan \theta \end{aligned}$$

Now we apply moment balance about point E,  $\sum \vec{M}_E = \vec{0}$ , which gives

$$\begin{aligned} \vec{r}_{G_1} \times \vec{W}_1 + \vec{r}_{G_2} \times \vec{W}_2 + \vec{r}_{G_3} \times \vec{F} &= \vec{0} \\ -(3\alpha H + \frac{1}{2}\alpha H)W_1 \hat{k} - \frac{2}{3}(3\alpha H)W_2 \hat{k} + \frac{h}{3}F \hat{k} &= \vec{0} \end{aligned}$$

Dotting this equation with  $\hat{k}$ , we get

$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H) \cdot \gamma_c \alpha H^2 w + \frac{2}{3}(3\alpha H) \cdot \frac{9}{2} \gamma_c \alpha^2 H^2 w \tan \theta \\ \frac{1}{2} \gamma_w \frac{h^3}{3} &= 9 \gamma_c \alpha^3 H^3 \tan \theta + \frac{7}{2} \gamma_c \alpha^2 H^3 \\ \Rightarrow \left( \frac{h}{H} \right)^3 &= \frac{\gamma_c}{\gamma_w} (54 \alpha^3 \tan \theta + 21 \alpha^2) \\ &= 2.5(54 \cdot 0.1^3 \cdot \sqrt{3} + 21 \cdot 0.1^2) = 0.7588 \\ \Rightarrow \frac{h}{H} &= 0.91 \end{aligned}$$

Thus, for the dam to not tip over,  $h \leq 0.91H$  or 91% of  $H$ .

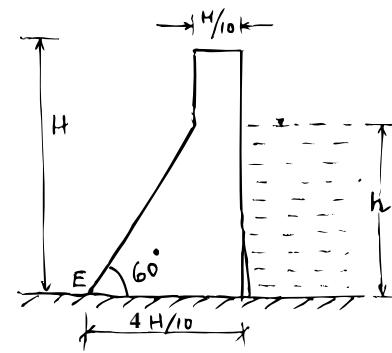


Figure 4.138: (Filename:sfig4.hydro.dam1)

<sup>①</sup> This assumption is valid only if water does not leak through the edge B to the bottom of the dam. If it does, there would be some force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

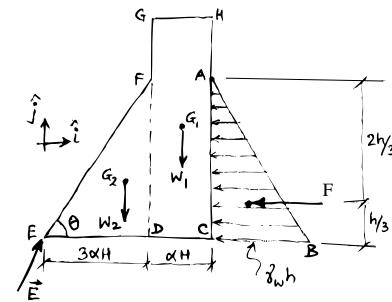


Figure 4.139: (Filename:sfig4.hydro.dam1.a)

$$\boxed{\frac{h}{H} \leq 0.91}$$

**SAMPLE 4.33 Dam design:** You are to design a dam of rectangular cross section ( $b \times H$ ), ensuring that the dam does not tip over even when the water level  $h$  reaches the top of the dam ( $h = H$ ). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

- The downstream bottom edge of the dam is plugged so that there is no leakage underneath.
- The downstream edge is not plugged and the water leaked under the dam bottom has full pressure across the bottom.

**Solution** Let  $\gamma_c$  and  $\gamma_w$  denote the weight densities of concrete and water, respectively. We are given that  $\gamma_c/\gamma_w = 3$ . Also, let  $b/H = \alpha$  so that  $b = \alpha H$ . Now we consider the two scenarios and carry out analysis to find appropriate cross-section of the dam. In the calculations below, we consider unit length (into the paper) of the dam.

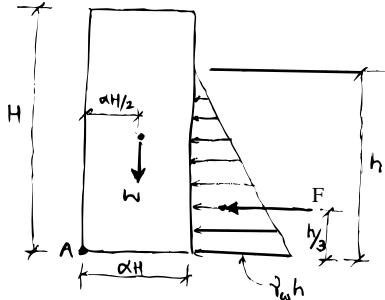


Figure 4.140: (Filename:sfig4.hydro.dam2.a)

- (a) *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free body diagram of the dam, considering critical tipping (just about to tip), is shown in Fig. 4.140 in which  $F$  is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are  $W = \gamma_c \alpha H^2$ , and  $F = (1/2)\gamma_w h^2$ . The moment balance about point A gives

$$\begin{aligned} F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\ \Rightarrow \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3 \end{aligned}$$

Considering the case of critical water level up to the height of the dam, i.e.,  $h/H = 1$ , and substituting  $\gamma_c/\gamma_w = 3$ , we get

$$\alpha^2 = 1/9 \Rightarrow \alpha = 1/3 = 0.333$$

Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$b/H = 0.33$$

- (b) *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, i.e.,  $p = \gamma_w h$ . The free body diagram diagram is shown in Fig. 4.141 where the known forces are  $W = \gamma_c \alpha H^2$ ,  $F = (1/2)\gamma_w h^2$ , and  $R = \gamma_w \alpha h H$ . Again, we carry out moment balance about point A to get

$$\begin{aligned} F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha H}{2} \\ \gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H) \alpha H \\ \alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)} \end{aligned}$$

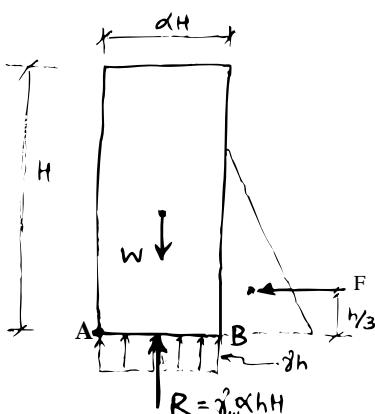
Once again, substituting the given values and  $h/H = 1$ , we get

$$\alpha^2 = 1/6 \Rightarrow \alpha = 0.408$$

Thus the width in this case needs to be at least 0.41 times the height  $H$ , slightly wider than the previous case.

$$b/H \geq 0.41$$

Figure 4.141: (Filename:sfig4.hydro.dam2.b)



## 4.8 Advanced statics

We now continue our study of statics, but with the goal of developing facility at some harder problems. One way that the material is expanded here is to take the three dimensionality of the world a little more seriously. Each subsection here corresponds to one of the six previous sections, namely, statics of one body, trusses, internal forces, springs, machines and mechanisms, and hydrostatics.

Primarily, the subject of 3D statics is the same as for 2D. However, generally one needs to take more care with vectors when working problems.

### Statics of one body in 3D

The consideration of statics of one body in 3D follows the same general principles as for 2D.

- Draw a free body diagram.
- Using the forces and moments shown, write the equilibrium equations
  - force balance (one 3D vector equation, three scalar equations), and
  - moment balance (one 3D vector equation, three scalar equations).

As for the case in 2D when one could use moment balance about 3 non-colinear points and not use force balance at all, in 3D one can use moment balance about 6 sufficiently different axes. If  $a$  is a distance, then one such set is, for example: the  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  axis through  $\vec{r} = \vec{0}$ , the  $\hat{j}$  axis through  $\vec{r} = a\hat{i}$ , the  $\hat{k}$  axis through  $\vec{r} = a\hat{j}$ , and the  $\hat{i}$  axis through  $\vec{r} = a\hat{k}$ . Other combinations of force balance and moment balance are also sufficient. One can test the sufficiency of the equations by seeing if they imply that, if a force at the origin and a couple are the only forces applied to a system, that they must be zero.

For 2D problems we used the phrase ‘moment about a point’ to be short for ‘moment about an axis in the  $z$  direction that passes through the point. In 3D moment about a point is a vector and moment about an axis is a scalar.

#### *Two- and three-force bodies*

The concepts of two-force and three-force bodies are identical in 3D. If there are only two forces applied to a body in equilibrium they must be equal and opposite and acting along the line connecting the points of application. If there are only three force applied to a body they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point.

#### *What does it mean for a problem to be ‘2D’?*

The world we live in is three dimensional, all the objects to which we wish to study mechanically are three dimensional, and if they are in equilibrium they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D,  $\sum F_x = 0$ ,  $\sum F_y = 0$ , and  $\sum \vec{M}_0 \cdot \hat{k} = 0$ . So, if when doing 2D mechanics, one just neglects the  $z$  component of any applied forces and the  $x$  and  $y$  components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations then 2D mechanics is really 3D mechanics, ignoring equations you don’t need.

- If the  $xy$  plane is a plane of symmetry for the object and any applied loading, then the three dimensional equilibrium equations not covered by the two dimensional equations, are automatically satisfied. For a car, say, the assumption of symmetry implies that the forces in the  $z$  direction will automatically add to zero, and the moments about the  $x$  and  $y$  axis will automatically be zero.
- If the object is thin and there are constraint forces holding it near the  $xy$  plane, and these constraint forces are not of interest, then 2D statics is also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper. The 2D equations describe the forces between the slippery glass plates.

## Trusses

The basic theory of trusses is the same in 3D as 2D. The method of joints is the primary basic approach. In ideal 3D truss theory the connections are ‘ball and socket’ not pins. That is the joints cannot carry any moments. For each joint the force balance equation can be reduced to three (rather than two for 2D trusses) scalar equations.

For the whole structure and for sections of the structure, the equilibrium equations can be reduced to two (rather than three for 2D trusses) scalar equations. The method of sections is less likely to be as useful a short-cut as in 2D because it is unlikely to find a section cut and equilibrium equation where only one bar force is unknown.

The counts for determinacy by matching the number of equations and number of unknowns change as follows. Instead of the 2D eqn. 4.28 from page 139 we have

$$3j = b + r \quad (4.70)$$

where  $j$  is the number of joints, including joints at reaction points,  $b$  is the number of bars, and  $r$  is the number of reaction components that shows on a free body diagram of the whole structure.

### *Example: A tripod*

A tripod is the simplest rigid 3D structure. With four joints ( $j = 4$ ), three bars ( $b = 3$ ), and nine unknown reaction components ( $r = 3 \times 3 = 9$ ), it exactly satisfies the equation  $3j = b + r$ , a check for determinacy of rigidity of 3D structures.

A tripod is the 3D equivalent of the two-bar truss shown in Fig. 4.35a on page 141. □

The check for determinacy of a floating (unattached) structure is

$$3j = b + 6. \quad (4.71)$$

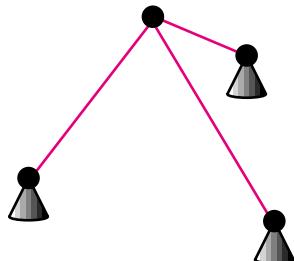


Figure 4.142: A tripod is the simplest rigid 3D truss.  
(Filename:tfigure.tripod)

There are various ways to think about the number six in the equation above. Assuming the structure is more than a point, six is the number of ways a structure can move in three dimensional space (three translations and three rotations), six is the number of equilibrium equations for the whole structure (one 3D vector moment, and one 3D vector force), and six is the number of constraints needed to hold a structure in place.

### Example: A tetrahedron

The simplest 3D rigid floating structure is a tetrahedron. With four joints ( $j = 4$ ) and six bars ( $b = 6$ ) it exactly satisfies the equation  $3j = b + 6$  which is a check for determinacy of 3D structures.

A tetrahedron is thus, in some sense the 3D equivalent of a triangle in 2D.  $\square$

## Internal forces

At a free body diagram cut on a long narrow structural piece in 2D there showed two force components, tension and shear, and one scalar moment. In 3D such a cut shows a force  $\vec{F}$  and a moment  $\vec{M}$  each with three components. If one picks a coordinate system with the  $x$  axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{F} \cdot \hat{i}.$$

The two other force components,  $F_x$  and  $F_y$ , are two components of shear. The net shear force is a vector in the plane orthogonal to  $\hat{i}$ .

The new concept, often called *torsion* is the component of  $\vec{M}$  along the axis:

$$\text{torsion} = M_x = \vec{M} \cdot \hat{i}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the  $\vec{M}$ , in the  $yz$  plane, is the bending moment. It has two components  $M_x$  and  $M_y$ .

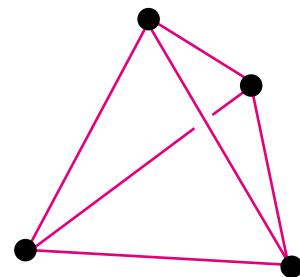


Figure 4.143: A tetrahedron is the simplest rigid truss in 3D that does not depend on grounding.

(Filename: tfigure.tetrahedron)

## Springs

Ideal springs are simple two force bodies, whether in 2D or 3D. The equation describing the force *on* end B of a spring, in terms of the relative positions of the ends  $\vec{r}_{AB}$ , the rest length of the spring  $\ell_0$ , and the spring constant  $k$  is still eqn. 4.46 from page 163, namely,

$$\vec{F}_B = k \cdot \underbrace{\left( |\vec{r}_{AB}| - \ell_0 \right)}_{\Delta\ell} \underbrace{\left( \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} \right)}_{\hat{\lambda}_{AB}}. \quad (4.72)$$

## Machines and structures

The approach to analysis of general machines and structures in 3D is the same as in 2D. One should draw a free body diagrams of the whole machine and of each of its parts, taking advantage of the principle of action and reaction. For each free body diagram the two vector equilibrium equations now lead to 6 scalar equations. Thus, for any but the simplest of 3D structures and machines one either tries to make a two dimensional model or one must resort to numerical solution.

## Hydrostatics

The basic results of hydrostatics are 3D results.

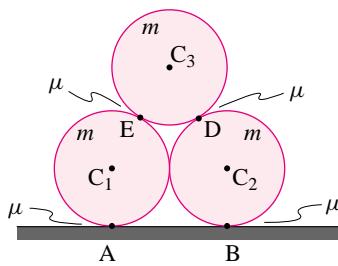


Figure 4.144: (Filename:sfig4.single.3balls)

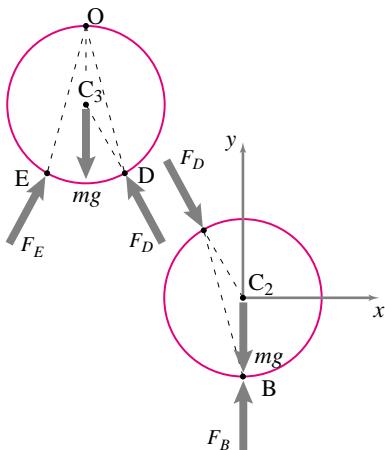


Figure 4.145: (Filename:sfig4.single.3balls.a)

**SAMPLE 4.34** *Can a stack of three balls be in static equilibrium?* Three identical spherical balls, each of mass  $m$  and radius  $R$ , are stacked such that the top ball rests on the lower two balls. The two balls at the bottom do not touch each other. Let the coefficient of friction at each contact surface be  $\mu$ . Find the minimum value of  $\mu$  so that the three balls are in static equilibrium.

**Solution** Let us assume that the three balls are in equilibrium. We can then find the forces required on each ball to maintain the equilibrium. If we can find a plausible value of the friction coefficient  $\mu$  from the required friction force on any of the balls, then we are done, otherwise our initial assumption of static equilibrium is wrong.

The free body diagrams of the upper ball and the lower right ball (why the right ball? No particular reason) are shown in Fig. 4.145. The contact forces,  $\vec{F}_E$  and  $\vec{F}_D$ , act on the upper ball at points E and D, respectively. Each contact force is the resultant of a tangential friction force and a normal force acting at the point of contact. From the free body diagrams, we see that each ball is a three-force-body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires that the two contact forces must intersect on the vertical line passing through the center of the ball (the line of action of the force of gravity). Now, if we consider the free body diagram of the lower right ball, we find that force  $\vec{F}_D$  has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force  $\vec{F}_D$ .

Let  $\alpha$  be the angle between the contact force  $\vec{F}_D$  and the normal to the ball surface at D. Now, from geometry,  $\angle C_3DO + \angle C_3OD + \angle OC_3D = 180^\circ$ . But,  $\alpha = \angle C_3DO = \angle C_3OD$ . Therefore,

$$\begin{aligned}\alpha &= \frac{1}{2}(180^\circ - \angle OC_3D) = \frac{1}{2}(\angle GC_3D) \\ &= \frac{1}{2}30^\circ = 15^\circ\end{aligned}$$

where  $\angle GC_3D = 30^\circ$  follows from the fact that  $C_1C_2C_3$  is an equilateral triangle and  $C_3G$  bisects  $\angle C_1C_3C_2$ .

Now, from Fig. 4.146, we see that

$$\tan \alpha = \frac{F_s}{N}$$

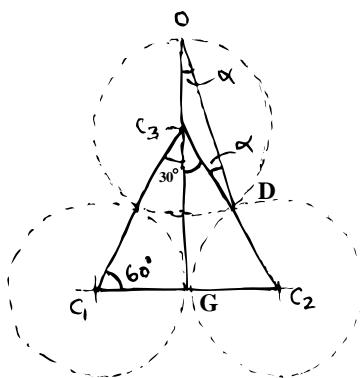
But, the force of friction  $F_s \leq \mu N$ . Therefore, it follows that

$$\mu \geq \tan \alpha = \tan 15^\circ = 0.27$$

Thus, the friction coefficient must be at least 0.27 if the three balls have to be in static equilibrium.

$$\boxed{\mu \geq 0.27}$$

Figure 4.146: (Filename:sfig4.single.3balls.b)



**SAMPLE 4.35 A simple 3-D truss:** The 3-D truss shown in the figure has 12 bars and 6 joints. Nine of the 12 bars that are either horizontal or vertical have length  $\ell = 1 \text{ m}$ . The truss is supported at A on a ball and socket joint, at B on a linear roller, and at C on a planar roller. The loads on the truss are  $\vec{F}_1 = -50 \text{ N} \hat{k}$ ,  $\vec{F}_2 = -60 \text{ N} \hat{k}$ , and  $\vec{F}_3 = 30 \text{ N} \hat{j}$ . Find all the support reactions and the force in the bar BC.

**Solution** The free body diagram of the entire structure is shown in Fig. 4.148. Let the support reactions at A, B, and C be  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ ,  $\vec{B} = B_x \hat{i} + B_z \hat{k}$ , and  $\vec{C} = C_z \hat{k}$ . Then the moment balance about point A,  $\sum \vec{M}_A = \vec{0}$  gives

$$\vec{r}_{B/A} \times \vec{B} + \vec{r}_{C/A} \times \vec{C} + \vec{r}_{E/A} \times \vec{F}_2 + \vec{r}_{F/A} \times \vec{F}_3 = \vec{0} \quad (4.73)$$

Note that  $\vec{F}_1$  passes through A and, therefore, produces no moment about A. Now we compute each term in the equation above.

$$\begin{aligned} \vec{r}_{B/A} \times \vec{B} &= \ell \hat{j} \times (B_x \hat{i} + B_z \hat{k}) &= -B_x \ell \hat{k} + B_z \ell \hat{i}, \\ \vec{r}_{C/A} \times \vec{C} &= \ell (\cos 60^\circ \hat{j} - \sin 60^\circ \hat{i}) \times C_z \hat{k} &= C_z \frac{\ell}{2} \hat{i} + C_z \frac{\sqrt{3}\ell}{2} \hat{j} \\ \vec{r}_{E/A} \times \vec{F}_2 &= (\ell \hat{j} + \ell \hat{k}) \times (-F_2 \hat{k}) &= -F_2 \ell \hat{i}, \\ \vec{r}_{F/A} \times \vec{F}_3 &= [\ell (\cos 60^\circ \hat{j} - \sin 60^\circ \hat{i}) + \ell \hat{k}] \times F_3 \hat{j} &= -F_3 \ell \hat{i} - F_3 \frac{\sqrt{3}\ell}{2} \hat{k} \end{aligned}$$

Substituting these products in eqn. (4.73), and dotting the resulting equation with  $\hat{j}$ ,  $\hat{k}$ , and  $\hat{i}$ , respectively, we get

$$\begin{aligned} C_z &= 0 \\ B_x &= -\frac{\sqrt{3}}{2} F_3 = -15\sqrt{3} \text{ N} \\ B_z &= -\frac{1}{2} C_z + F_2 + F_3 = 90 \text{ N} \end{aligned}$$

Thus,  $\vec{B} = B_x \hat{i} + B_z \hat{k} = -15\sqrt{3} \text{ N} \hat{i} + 90 \text{ N} \hat{k}$  and  $\vec{C} = \vec{0}$ . Now from the force balance,  $\sum \vec{F} = \vec{0}$ , we find  $\vec{A}$  as

$$\begin{aligned} \vec{A} &= -\vec{B} - \vec{C} - \vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\ &= -(-15\sqrt{3} \text{ N} \hat{i} + 90 \text{ N} \hat{k}) - (-50 \text{ N} \hat{k}) - (-60 \text{ N} \hat{k}) - (30 \text{ N} \hat{j}) \\ &= 15\sqrt{3} \text{ N} \hat{i} - 30 \text{ N} \hat{j} + 20 \text{ N} \hat{k} \end{aligned}$$

To find the force in bar BC, we draw a free body diagram of joint B (which connects BC) as shown in Fig. 4.149. Now, writing the force balance for the joint in the  $x$ -direction, i.e.,  $[\sum \vec{F} = \vec{0}] \cdot \hat{i}$ , gives

$$\begin{aligned} B_x + \vec{T}_{BC} \cdot \hat{i} &= 0 \\ \text{or } B_x + T_{BC} \sin 60^\circ &= 0 \\ \Rightarrow T_{BC} &= -\frac{B_x}{\sin 60^\circ} \\ &= -\frac{-15\sqrt{3} \text{ N}}{\sqrt{3}/2} = 30 \text{ N} \end{aligned}$$

Thus the force in bar BC is  $T_{BC} = 30 \text{ N}$  (tensile force).

$$\boxed{\vec{A} = 15\sqrt{3} \text{ N} \hat{i} - 30 \text{ N} \hat{j} + 20 \text{ N} \hat{k}, \vec{B} = -15\sqrt{3} \text{ N} \hat{i} + 90 \text{ N} \hat{k}, \vec{C} = \vec{0}, T_{BC} = 30 \text{ N}}$$

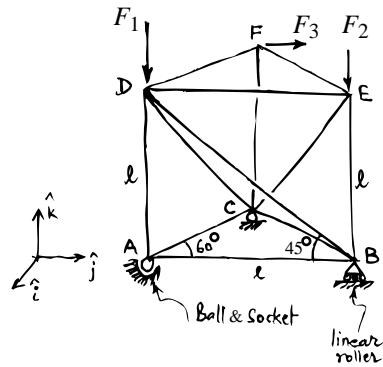


Figure 4.147: (Filename:sfig4.3d.truss1)

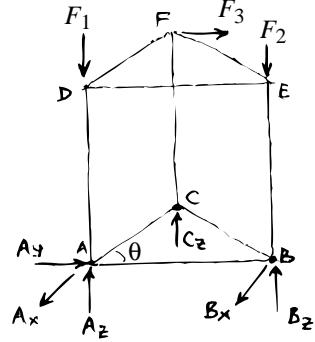


Figure 4.148: (Filename:sfig4.3d.truss1.a)

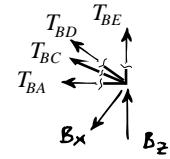


Figure 4.149: (Filename:sfig4.3d.truss1.b)

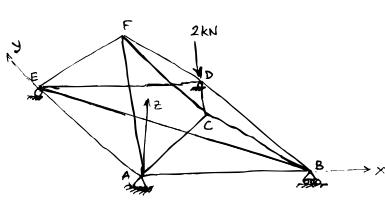


Figure 4.150: (Filename:sfig4.3d.truss2)

**SAMPLE 4.36 A 3-D truss solved on the computer:** The 3-D truss shown in the figure is fabricated with 12 bars. Bars 1–5 are of length  $\ell = 1$  m, bars 6–9 have length  $\ell/\sqrt{2} (\approx 0.71 \text{ m})$ , and bars 10–12 are cut to size to fit between the joints they connect. The truss is supported at A on a ball and socket, at B on a linear roller, and at C on a planar roller. A load  $F = 2 \text{ kN}$  is applied at D as shown. Write all equations required to solve for all bar forces and support reactions and solve the equations using a computer.

**Solution** There are 12 bars and 6 joints in the given truss. The unknowns are 12 bar forces and six support reactions (3 at A ( $A_x, A_y, A_z$ ), 2 at B ( $B_y, B_z$ ), and 1 at E ( $E_z$ )). Therefore, we need 18 independent equations to solve for all the unknowns. Since the force equilibrium at each joint gives one vector equation in 3-D, *i.e.*, three scalar equations, the 6 joints in the truss can generate the required number ( $6 \times 3 = 18$ ) of equations. Therefore, we go joint by joint, draw the free body diagram of the joint, write the force equilibrium equation, and extract the 3 scalar equations from each vector equation.

At each joint we use the following convention for force labels. At joint A, the force from bar AB is  $\vec{F}_{AB} = T_1 \hat{\lambda}_{AB}$  and at joint B, the force from the same bar AB is  $\vec{F}_{BA} = T_1 \hat{\lambda}_{BA} = T_1 (-\hat{\lambda}_{AB}) = -\vec{F}_{AB}$ . We switch from the letters to denote the bars in the force vectors to numbers in its scalar representation ( $T_1, T_2, \text{etc.}$ ) to facilitate computer solution.

Joint A:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{AB} + \vec{F}_{AC} + \vec{F}_{AF} + \vec{F}_{AE} + \vec{A} = \vec{0} \\ T_1 \hat{i} + \frac{T_6}{\sqrt{2}}(\hat{i} + \hat{k}) + \frac{T_{10}}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k}) + T_4 \hat{j} + A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = \vec{0} \quad (4.74)$$

Joint B:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{BA} + \vec{F}_{BC} + \vec{F}_{BD} + \vec{F}_{BE} + \vec{B} = \vec{0} \\ -T_1 \hat{i} + \frac{T_7}{\sqrt{2}}(-\hat{i} + \hat{k}) + T_2 \hat{j} + \frac{T_{12}}{\sqrt{2}}(-\hat{i} + \hat{j}) + B_y \hat{j} + B_z \hat{k} = \vec{0} \quad (4.75)$$

Joint C:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{CA} + \vec{F}_{CB} + \vec{F}_{CF} + \vec{F}_{CD} = \vec{0} \\ -\frac{T_6}{\sqrt{2}}(\hat{i} + \hat{k}) - \frac{T_7}{\sqrt{2}}(-\hat{i} + \hat{k}) + T_5 \hat{j} + \frac{T_{11}}{\sqrt{6}}(\hat{i} + 2\hat{j} - \hat{k}) = \vec{0} \quad (4.76)$$

Joint D:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} + \vec{F}_{DF} + \vec{F} = \vec{0} \\ -T_2 \hat{j} - \frac{T_{11}}{\sqrt{6}}(\hat{i} + 2\hat{j} - \hat{k}) - T_3 \hat{i} + \frac{T_9}{\sqrt{2}}(-\hat{i} + \hat{k}) - F \hat{k} = \vec{0} \quad (4.77)$$

Joint E:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{EA} + \vec{F}_{EB} + \vec{F}_{ED} + \vec{F}_{EF} + \vec{E} = \vec{0} \\ -T_4 \hat{j} + \frac{T_{12}}{\sqrt{2}}(\hat{i} - \hat{j}) + T_3 \hat{i} + \frac{T_8}{\sqrt{2}}(\hat{i} + \hat{k}) + E_z \hat{k} = \vec{0} \quad (4.78)$$

Joint F:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{F}_{FC} + \vec{F}_{FE} + \vec{F}_{FA} + \vec{F}_{FD} = \vec{0} \\ -T_5 \hat{j} - \frac{T_8}{\sqrt{2}}(\hat{i} + \hat{k}) - \frac{T_{10}}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k}) - \frac{T_9}{\sqrt{2}}(-\hat{i} + \hat{k}) = \vec{0} \quad (4.79)$$

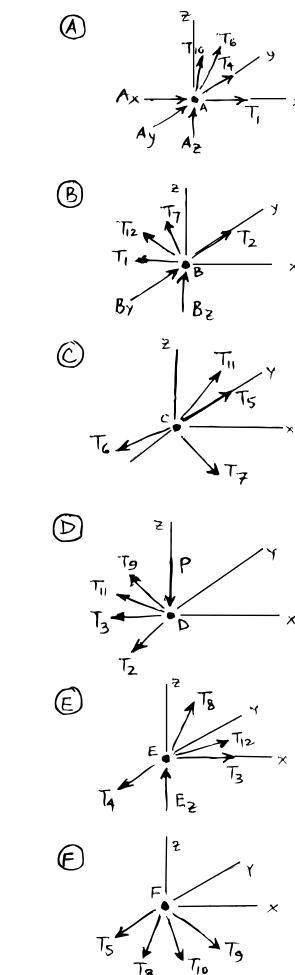


Figure 4.151: (Filename:sfig4.3d.truss2.a)

Now we can separate out 3 scalar equations from each of the vector equations from eqn. (4.74)–eqn. (4.79) by dotting them with  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ .

Eqn.	[Eqn.] $\cdot \hat{i}$	[Eqn.] $\cdot \hat{j}$	[Eqn.] $\cdot \hat{k}$
(1)	$T_1 + \frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + A_x = 0$ ,	$\frac{2}{\sqrt{6}}T_{10} + T_4 + A_y = 0$ ,	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + A_z = 0$
(2)	$-T_1 - \frac{1}{\sqrt{2}}T_7 - \frac{1}{\sqrt{2}}T_{12} = 0$ ,	$T_2 + \frac{1}{\sqrt{2}}T_{12} + B_y = 0$ ,	$\frac{1}{\sqrt{2}}T_7 + B_z = 0$
(3)	$-\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$ ,	$T_5 + \frac{2}{\sqrt{6}}T_{11} = 0$ ,	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$
(4)	$-\frac{1}{\sqrt{6}}T_{11} - T_3 - \frac{1}{\sqrt{2}}T_9 = 0$ ,	$-T_2 - \frac{2}{\sqrt{6}}T_{11} = 0$ ,	$\frac{1}{\sqrt{6}}T_{11} + \frac{1}{\sqrt{2}}T_9 = F$
(5)	$\frac{1}{\sqrt{2}}T_{12} + T_3 + \frac{1}{\sqrt{2}}T_8 = 0$ ,	$-T_4 - \frac{1}{\sqrt{2}}T_{12} = 0$ ,	$\frac{1}{\sqrt{2}}T_8 + E_z = 0$
(6)	$-\frac{1}{\sqrt{2}}T_8 - \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0$ ,	$-T_5 - \frac{2}{\sqrt{6}}T_{10} = 0$ ,	$\frac{1}{\sqrt{2}}T_8 + \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0$

Thus, we have 18 required equations for the 18 unknowns. Before we go to the computer, we need to do just one more little thing. We need to order the unknowns in some way in a one-dimensional array. So, let

$$x = [A_x \quad A_y \quad A_z \quad B_x \quad B_y \quad E_z \quad T_1 \dots T_{12}]$$

Thus  $x_1 = A_x$ ,  $x_2 = A_y$ ,  $\dots$ ,  $x_7 = T_1$ ,  $x_8 = T_2$ ,  $\dots$ ,  $x_{18} = T_{12}$ . Now we are ready to go to the computer, feed these equations, and get the solution. We enter each equation as part of a matrix [A] and a vector {b} such that [A]{x} = {b}. Here is the pseudocode:

```

sq2i = 1/sqrt(2)          % define a constant
sq6i = 1/sqrt(6)          % define another constant
F = 2                      % specify given load

A(1,[1 7 12 16]) = [1 1 sq2i sq6i]
A(2,[2 10 16]) = [1 1 2*sq6i]
.
.
A(18,[14 15 16]) = [sq2i sq2i sq6i]
b(12,1) = F
form A and b setting all other entries to zero
solve A*x = b for x

```

The solution obtained from the computer is the one-dimensional array x which after decoding according to our numbering scheme gives the following answer.

$$\begin{aligned}
 A_x = A_y = 0, \quad A_z &= -2 \text{ kN}, \quad B_y = 0, \quad B_z = 2 \text{ kN}, \quad E_z = 2 \text{ kN}, \\
 T_1 = T_3 &= -2 \text{ kN}, \quad T_2 = T_4 = T_5 = -4 \text{ kN}, \quad T_6 = 0, \\
 T_7 = T_8 &= -2.83 \text{ kN}, \quad T_9 = 0, \quad T_{10} = T_{11} = 4.9 \text{ kN}, \quad T_{12} = 5.66 \text{ kN},
 \end{aligned}$$

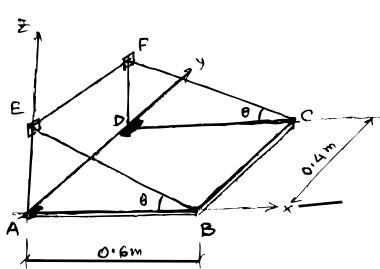


Figure 4.152: (Filename:sfig4.3d.plate)

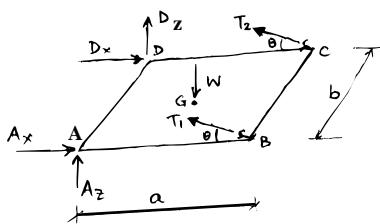


Figure 4.153: (Filename:sfig4.3d.plate.a)

**SAMPLE 4.37 An unsolvable problem?** A  $0.6 \text{ m} \times 0.4 \text{ m}$  uniform rectangular plate of mass  $m = 4 \text{ kg}$  is held horizontal by two strings  $BE$  and  $CF$  and linear hinges at  $A$  and  $D$  as shown in the figure. The plate is loaded uniformly with books of total mass  $6 \text{ kg}$ . If the maximum tension the strings can take is  $100 \text{ N}$ , how much more load can the plate take?

**Solution** The free body diagram of the plate is shown in Fig. 4.153. Note that we model the hinges at  $A$  and  $D$  with no resistance in the  $y$ -direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load  $\vec{W}$  acting vertically through point  $G$ .

The various forces acting on the plate are

$$\vec{W} = -W\hat{k}, \quad \vec{T}_1 = T_1\hat{\lambda}_{BE}, \quad \vec{T}_2 = T_2\hat{\lambda}_{CF}, \quad \vec{A} = A_x\hat{i} + A_z\hat{k}, \quad \vec{D} = D_x\hat{i} + D_z\hat{k}$$

Here  $\hat{\lambda}_{BE} = \hat{\lambda}_{CF} = -\cos\theta\hat{i} + \sin\theta\hat{k} = \hat{\lambda}$  (let). Now, we apply moment equilibrium about point  $A$ , i.e.,  $\sum \vec{M}_A = \vec{0}$ .

$$\vec{r}_B \times \vec{T}_1 + \vec{r}_C \times \vec{T}_2 + \vec{r}_G \times \vec{W} + \vec{r}_D \times \vec{D} = \vec{0} \quad (4.80)$$

where,

$$\begin{aligned} \vec{r}_B \times \vec{T}_1 &= a\hat{i} \times T_1\hat{\lambda} = -aT_1 \sin\theta\hat{j} \\ \vec{r}_C \times \vec{T}_2 &= (a\hat{i} + b\hat{j}) \times T_2\hat{\lambda} = T_2b \sin\theta\hat{i} - T_2a \sin\theta\hat{j} + T_2b \cos\theta\hat{k} \\ \vec{r}_G \times \vec{W} &= \frac{1}{2}(a\hat{i} + b\hat{j}) \times (-W\hat{k}) = -\frac{Wa}{2}\hat{i} + \frac{Wa}{2}\hat{j} \\ \vec{r}_D \times \vec{D} &= b\hat{j} \times (D_x\hat{i} + D_z\hat{k}) = D_zb\hat{i} - D_xb\hat{k} \end{aligned}$$

Substituting these products in eqn. (4.80) and dotting with  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ , we get

$$T_2 \sin\theta + D_z = \frac{W}{2} \quad (4.81)$$

$$T_2 \cos\theta - D_x = 0 \quad (4.82)$$

$$(T_1 + T_2) \sin\theta = \frac{W}{2} \quad (4.83)$$

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$\vec{A} + \vec{D} + \vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}$$

Again, substituting the forces in their component form and dotting with  $\hat{i}$  and  $\hat{k}$  (there are no  $\hat{j}$  components), we get

$$\begin{aligned} A_x + D_x - (T_1 + T_2) \cos\theta &= 0 \\ \Rightarrow A_x - T_1 \cos\theta &= 0 \end{aligned} \quad (4.84)$$

$$\begin{aligned} A_z + D_z + (T_1 + T_2) \sin\theta &= 0 \\ \Rightarrow A_z + T_1 \sin\theta &= \frac{W}{2} \end{aligned} \quad (4.85)$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (4.81) to (4.85)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the structure is symmetric about the

axis passing through G and parallel to  $x$ -axis. From this symmetry we conclude that  $T_1 = T_2$ . Then, from eqn. (4.84) we have

$$2T \sin \theta = \frac{W}{2} \Rightarrow T = \frac{W}{4 \sin \theta}$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T \sin \theta \\ \Rightarrow W_{\max} &= 4T_{\max} \sin \theta \\ &= 4(100 \text{ N}) \cdot \frac{1}{2} = 200 \text{ N} \end{aligned}$$

The total load as given is  $(6 + 4) \text{ kg} \cdot 9.81 \text{ m/s}^2 = 98.1 \text{ N} \approx 100 \text{ N}$ . Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (4.81), (4.82), (4.84), and (4.85).

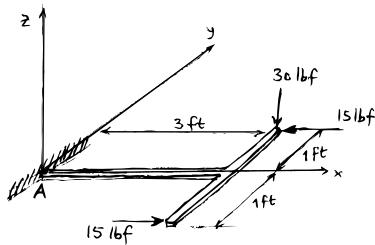
$$\begin{aligned} D_z = A_z &= \frac{W}{2} - T \sin \theta = \frac{W}{2} \\ D_x = A_x &= T \cos \theta = \frac{W}{4} \cot \theta \end{aligned}$$

$W_{\max} = 200 \text{ N}$

### Comments:

- (a) We got only five independent equations (instead of the usual 6) because the force equilibrium in the  $y$ -direction gives a zero identity ( $0 = 0$ ). There are no forces in the  $y$ -direction. The structure seems to be unstable in the  $y$ -direction — if you push a little, it will move. Remember, however, that it is so because we chose to model the hinges at A and D that way keeping in mind the only vertical loading. The actual hinges used on a bookshelf will not allow movement in the  $y$ -direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine  $A_y$  and  $D_y$  from equilibrium equations alone.
- (b) The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, which requires understanding of mechanics, makes them determinate and solvable.



**SAMPLE 4.38 3-D moment at the support:** A 'T' shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

Figure 4.154: (Filename:sfig4.intern.cant3D)

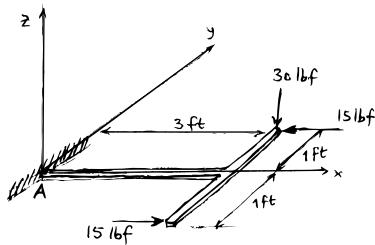


Figure 4.155: (Filename:sfig4.intern.cant3D)

**Solution** The free body diagram of the beam is shown in Fig. ???. Note that the forces acting on the beam can produce in-plane as well as out of plane moments. Therefore, we show the unknown reactions  $\vec{A}$  and  $\vec{M}_A$  as general 3-D vectors at A. The moment equilibrium about point A,  $\sum \vec{M}_A = \vec{0}$ , gives

$$\vec{M}_A + \vec{r}_{C/A} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_{D/A} \times \vec{F}_3 = \vec{0}$$

$$\begin{aligned} \Rightarrow \vec{M}_A &= (\vec{r}_{B/A} + \vec{r}_{C/B}) \times (\vec{F}_1 + \vec{F}_2) + (\vec{r}_{B/A} + \vec{r}_{D/B}) \times \vec{F}_3 \\ &= (\ell\hat{i} + a\hat{j}) \times (-F_1\hat{k} - F_2\hat{i}) + (\ell\hat{i} - a\hat{j}) \times F_3\hat{i} \end{aligned}$$

But  $F_3 = -F_2 = F$  (say). Therefore,

$$\begin{aligned} &= (\ell\hat{i} + a\hat{j}) \times (-F_1\hat{k} - F\hat{k}) + (\ell\hat{i} - a\hat{j}) \times F\hat{i} \\ &= F_1\ell\hat{j} - F_1a\hat{i} - 2Fa\hat{k} \\ &= 30 \text{ lbf} \cdot 3 \text{ ft}\hat{j} - 30 \text{ lbf} \cdot 1 \text{ ft}\hat{i} - 2(30 \text{ lbf} \cdot 1 \text{ ft})\hat{k} \\ &= (-30\hat{i} + 90\hat{j} - 60\hat{k}) \text{ lb}\cdot\text{ft} \end{aligned}$$

The force equilibrium,  $\sum \vec{F} = \vec{0}$ , gives

$$\begin{aligned} \vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\ &= -\vec{F}_1 - \vec{F} + \vec{F} \\ &= -(-F_1\hat{k}) = F_1\hat{k} \\ &= 30 \text{ lbf}\hat{k} \end{aligned}$$

$\boxed{\vec{A} = 30 \text{ lbf}\hat{k}, \text{ and } \vec{M}_A = (-30\hat{i} + 90\hat{j} - 60\hat{k}) \text{ lb}\cdot\text{ft}}$

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# 5 Dynamics of particles

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We now progress from statics to dynamics. Although we treated statics as an independent topic, statics is really a special case of dynamics. In statics we neglected the inertial terms (the terms involving acceleration times mass) in the linear and angular momentum balance equations. In dynamics these terms are of central interest. In statics all the forces and moments cancel each other. In dynamics the forces and moments add to cause the acceleration of mass. As the names imply, statics is generally concerned with things that don't move, or at least don't move much, whereas dynamics with things that move a lot. How to quantify what is 'still' (statics) vs 'moving' (dynamics) is itself a dynamics question.

A big part of learning dynamics is learning to keep track of motion, *kinematics*. In addition, kinematic analysis is also useful for work and energy methods in statics. We are going to develop our understanding of dynamics by considering progressively harder-to-understand motions.

This chapter is limited to the dynamics of particles. A *particle* is a system totally characterized by its position (as a function of time) and its (fixed) mass. Often one imagines that a particle is something small. But the particle idealization is used, for example, to describe a galaxy in the context of its motion in a cluster of galaxies. Rather, a particle is something whose spatial extent is neglected in the evaluation of mechanics equations. An object's spatial extent might be neglected because the object is small compared to other relevant distances, or because distortion and rotation happen to be of secondary interest.

In this chapter we further limit our study of the dynamics of particles to cases where the applied forces are given as a function of time or can be determined from the positions and velocities of the particles. The time-varying thrust from an engine might be thought of as a force given as a function of time. Gravity and springs cause forces which are functions of position. And the drag on a particle as it moves through

air or water can be modeled as a force depending on velocity. Discussion of geometric constraints, as for particles connected with strings or rods, where some of the forces depend on finding the accelerations, begins in chapter 6.

The most important equation in this chapter is linear momentum balance applied to one particle. If we start with the general form in the front cover, discussed in general terms in chapter 1, we get:

$$\begin{aligned}\sum \vec{F}_i &= \dot{\vec{L}} && \text{Linear momentum balance for any system} \\ &= \sum m_i \vec{a}_i && \text{for a system of particles} \\ &= m \vec{a} && \text{for one particle}\end{aligned}$$

If we define  $\vec{F}$  to be the net force on the particle ( $\vec{F} = \sum \vec{F}_i$ ) then we get

$$\vec{F} = m \vec{a} \quad (5.1)$$

which is sometimes called ‘Newton’s second law of motion.’ In his words,

*“Any change of motion is proportional to the force that acts, and it is made in the direction of the straight line in which that force is acting.”*

In modern language, explicitly including the role of mass, the net force on a particle is its mass times its acceleration. Intuitively people think of this law as saying force causes motion, and, more precisely, that force causes acceleration of mass. Actually, what causes what, *causality*, is just a philosophical question. The important fact is that when there is a net force there is acceleration of mass, and when there is acceleration of mass there is a net force. When a car crashes into a pole there is a big force and a big deceleration of the car. You could think of the force on the bumper as causing the car to slow down rapidly. Or you could think of the rapid car deceleration as necessitating a force. It is just a matter of personal taste because in both cases equation 5.1 applies.

*Acceleration is the second derivative of position*

What is the acceleration of a particle? Lets assume that  $\vec{r}(t)$  is the position of the particle as a function of time relative to some origin. Then its acceleration is

$$\begin{aligned}\vec{a} &\equiv \frac{d}{dt} \vec{v} = \frac{d}{dt} \left( \frac{d}{dt} \vec{r} \right) = \frac{d^2}{dt^2} \vec{r} \\ &= \ddot{\vec{v}} = \frac{d}{dt} (\dot{\vec{r}}) = \ddot{\vec{r}}\end{aligned}$$

where one or two dots over something is a short hand notation for the first or second time derivative.

*Newton’s laws are accurate in a Newtonian reference frame*

When the acceleration is calculated from position it is calculated using a particular coordinate system. A *reference frame* is, for our purposes at the moment, a coordinate system. The calculated acceleration of a particle depends on how the coordinate system itself is moving. So the simple equation

$$\vec{F} = m \vec{a}$$

has as many different interpretations as there are differently moving coordinate systems (and there are an infinite number of those). Sir Isaac was standing on earth measuring position relative to the ground when he noticed that his second law accurately described things like falling apples. So the equation  $\vec{F} = m\vec{a}$  is valid using coordinate systems that are fixed to the earth. Well, not quite. Isaac noticed that the motion of the planets around the sun only followed his law if the acceleration was calculated using a coordinate system that was still relative to ‘the fixed stars.’ With a fixed-star coordinate system you calculate *very* slightly different accelerations for things like falling apples than you do using a coordinate system that is stuck to the earth. And nowadays when astrophysicists try to figure out how the laws of mechanics explain the shapes of spiral galaxies they realize that none of the so-called ‘fixed stars’ are so totally fixed. They need even more care to pick a coordinate system where eqn. 5.1 is accurate.

Despite all this confusion, it is generally agreed that there exists some coordinate system for which Newton’s laws are incredibly accurate. Further, once you know one such coordinate system there are rules (which we will discuss in later chapters) to find many others. Any such reference frame is called a *Newtonian reference frame*. Sometimes people also call such a frame a *Fixed frame*, as in ‘fixed to the earth’ or ‘fixed to the stars’.

For most engineering purposes, not counting, for example, trajectory control of interplanetary missions, a coordinate system attached to the ground under your feet is good approximation to a Newtonian frame. Fortunately. Or else apples would fall differently. Newton might not have discovered his laws. And this book would be much shorter.

#### The organization of this chapter

In the first four sections of this chapter we give a thorough introduction to the one-dimensional mechanics of single particle. This is a review and deepening of material covered in freshman physics. These sections introduce you to the time-varying nature of dynamics without the complexity of vector geometry. The later sections concern dynamics with more particles or more spatial dimensions or both.

## 5.1 Force and motion in 1D

We now limit our attention to the special case where one particle moves on a given straight line. We postpone until Chapter 6 issues about what forces might be required to keep the particle on that line. For problems with motion in only one direction, the kinematics is particularly simple. Although we use vectors here because of their help with signs, they are really not needed.

### Position, velocity, and acceleration in one dimension

If, say, we call the direction of motion the  $\hat{i}$  direction, then we can call  $x$  the position of the particle we study (see figure 5.1). Even though we are neglecting the spatial extent of the particle, to be precise we can define  $x$  to be the  $x$  coordinate of the particle’s center. We can write the position  $\vec{r}$ , velocity  $\vec{v}$  and acceleration  $\vec{a}$  as

$$\vec{r} = x\hat{i} \quad \text{and} \quad \vec{v} = v\hat{i} = \frac{dx}{dt}\hat{i} = \dot{x}\hat{i} \quad \text{and} \quad \vec{a} = a\hat{i} = \frac{dv}{dt}\hat{i} = \frac{d^2x}{dt^2}\hat{i} = \ddot{x}\hat{i},$$

Figure 5.2 shows example graphs of  $x(t)$  and  $v(t)$  versus time. When we don’t use vector notation explicitly we will take  $v$  and  $a$  to be positive if they have the same direction as increasing  $x$  (or  $y$  or whatever coordinate describes position).

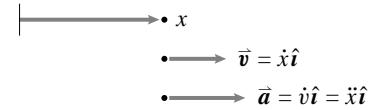


Figure 5.1: One-dimensional position, velocity, and acceleration in the  $x$  direction.  
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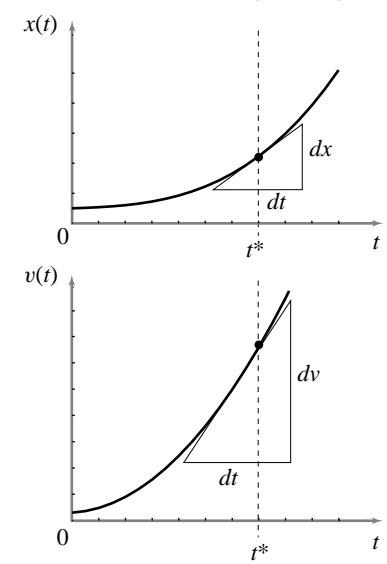


Figure 5.2: Graphs of  $x(t)$  and  $v(t) = \frac{dx}{dt}$  versus time. The slope of the position curve  $dx/dt$  at  $t^*$  is  $v(t^*)$ . And the slope of the velocity curve  $dv/dt$  at  $t^*$  is  $a(t^*)$ .  
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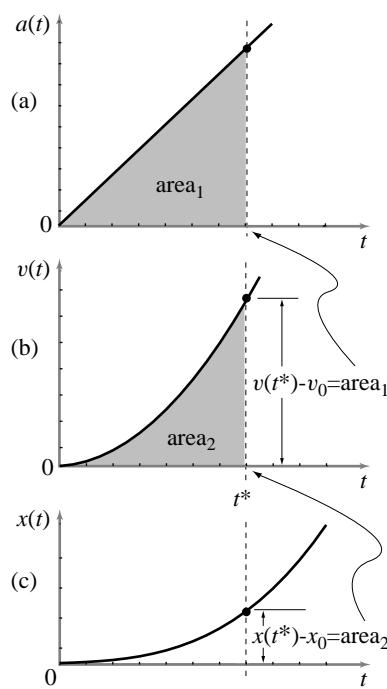


Figure 5.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle  $a(t)$ ; (b) is a graph of the particle velocity  $v(t)$  and the integral of  $a(t)$  from  $t_0 = 0$  to  $t^*$ , the shaded area under the acceleration curve; (c) is the position of the particle  $x(t)$  and the integral of  $v(t)$  from  $t_0 = 0$  to  $t^*$ , the shaded area under the velocity curve.

(Filename: figure3.4.1b)

① To cover the range of calculus problems you need to be a very good rider, however, able to ride frontwards, backwards, at zero speed and infinitely fast.

### Example: Position, velocity, and acceleration in one dimension

If position is given as  $x(t) = 3e^{4t/s}$  m then  $v(t) = dx/dt = 12e^{4t/s}$  (m/s) and  $a(t) = dv/dt = 48e^{4t/s}$  (m/s<sup>2</sup>). So at, say, time  $t = 2$  s the acceleration is  $a|_{t=2\text{s}} = 48e^{4 \cdot 2/\text{s}}$  (m/s<sup>2</sup>) =  $48 \cdot e^8$  m/s<sup>2</sup>  $\approx 1.43 \cdot 10^5$  m/s<sup>2</sup>.  $\square$

We can also, using the fundamental theorem of calculus, look at the integral versions of these relations between position, velocity, and acceleration (see Fig. 5.3).

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t v(\tau) d\tau && \text{with } x_0 = x(t_0), \text{ and} \\ v(t) &= v_0 + \int_{t_0}^t a(\tau) d\tau && \text{with } v_0 = v(t_0). \end{aligned}$$

With more informal notation, these equations can also be written as:

$$\begin{aligned} x &= \int v dt \\ v &= \int a dt. \end{aligned}$$

So one-dimensional kinematics includes almost all elementary calculus problems. You are given a function and you have to differentiate it or integrate it. To put it the other way around, almost every calculus question could be phrased as a question about your bicycle speedometer. On your bicycle speedometer (which includes an odometer) you can read your speed and distance travelled as functions of time. Given one of those two functions, find the other.<sup>①</sup> As of this writing, common bicycle computers don't have accelerometers. But acceleration as a function of time is also of interest. For example, if you are given the (scalar part of) velocity  $v(t)$  as a function of time and are asked to find the acceleration  $a(t)$  you have to differentiate. If instead you were asked to find the position  $x(t)$ , you would be asked to calculate an integral (see figure 5.3).

If acceleration is given as a function of time, then position is found by integrating twice.

### Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \quad \text{or, more explicitly} \quad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is one that contains ordinary derivatives (as opposed to partial differential equations which we will not use in this book).

### Example: Calculating a derivative solves an ODE

Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3 \text{ m}) \frac{(1 - \cos(\frac{\pi t}{5\text{s}}))}{2}$$

we can solve the differential equation  $v = \frac{dy}{dt}$  by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt} \left[ (3 \text{ m}) \frac{(1 - \cos(\frac{\pi t}{5 \text{ s}}))}{2} \right] = \frac{3\pi}{10} \sin\left(\frac{\pi t}{5 \text{ s}}\right) \text{ m/s}$$

(Note: this would be considered a harsh elevator because of the jump in the acceleration at the start and stop.)  $\square$

A little less trivial is the case when you want to find a function when you are given the derivative.

**Example: Integration solves a simple ODE**

Given that you start at home ( $x = 0$ ) and, over about 30 seconds, you accelerate towards a steady-state speed of 4 m/s according to the function

$$v(t) = 4(1 - e^{-t/(30 \text{ s})}) \text{ m/s}$$

and your whole ride lasts 1000 seconds (about 17 minutes). You can find how far you travel by solving the differential equation

$$\dot{x} = v(t) \quad \text{with the initial condition } x(0) = 0$$

which can be accomplished by integration. Say, after 1000 seconds

$$\begin{aligned} x(t = 1000 \text{ s}) &= \int_0^{1000 \text{ s}} v(t) dt = \int_0^{1000 \text{ s}} 4(1 - e^{-t/(30 \text{ s})})(\text{m/s}) dt \\ &= (4t + (120 \text{ s})e^{-t/(30 \text{ s})}) \Big|_0^{1000 \text{ s}} \text{ m/s} \\ &= ((4 \cdot 1000 \text{ s} + (120 \text{ s})e^{-100/3}) - (0 + (120 \text{ s})e^0)) \text{ m/s} \\ &= (4000 - 120 + 120e^{-100/3}) \text{ m} \\ &\approx 3880 \text{ m} \quad (\text{to within an angstrom or so}) \end{aligned}$$

The distance travelled is only 120 m less than would be travelled if the whole trip was travelled at a steady 4 m/s ( $4 \text{ m/s} \times 1000 \text{ s} = 4000 \text{ m}$ ).  $\square$

Unlike the integral above, many integrals cannot be evaluated by hand.

**Example: An ODE that leads to an intractable integral**

If you were told that the velocity as a function of time was

$$v(t) = \frac{4t}{t + e^{-t/(30 \text{ s})}} \text{ m}$$

you would, as for the previous example, be describing a bike trip where you started at zero speed and exponentially approached a steady speed of 4 m/s. Thus your position as a function of time should be similar. But what is it? Let's proceed as for the last example to solve the equation

$$\dot{x} = v(t) \quad \text{with the initial condition } x(0) = 0$$

and the given  $v(t)$ . We can set up the integral to get

$$\begin{aligned} x(t = 1000 \text{ s}) &= \int_0^{1000 \text{ s}} v(t) dt = \int_0^{1000 \text{ s}} \frac{4t}{t + e^{-t/(30 \text{ s})}} \text{ m} dt \\ &= \dots \end{aligned} \tag{5.2}$$

which is the kind of thing you have nightmares about seeing on an exam. This is an integral that you couldn't do if your life depended on it. No-one could. There is no formula for  $x(t)$  that solves the differential equation  $\dot{x} = v(t)$ , with the given  $v(t)$ , unless you regard eqn. 5.2 as a formula. In days of old they would say ‘the problem has been reduced to quadrature’ meaning that all that remained was to evaluate an integral, even if they didn’t know how to evaluate it. But you can always resort to numerical integration. One of many ways to evaluate the integral numerically is by the following pseudo code (note that the problem is formulated with consistent units so they can be dropped for the numerics).

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0 }
solve ODE with IC and evaluate at t=1000
```

The result is  $x \approx 3988$  m which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m.  $\square$

## The equations of dynamics

### *Linear momentum balance*

For a particle moving in the  $x$  direction the velocity and acceleration are  $\vec{v} = v\hat{i}$  and  $\vec{a} = a\hat{i}$ . Thus the linear momentum and its rate of change are

$$\begin{aligned}\vec{L} &\equiv \sum m_i \vec{v}_i = m\vec{v} = mv\hat{i}, \text{ and} \\ \dot{\vec{L}} &\equiv \sum m_i \vec{a}_i = m\vec{a} = ma\hat{i}.\end{aligned}$$

Thus the equation of linear momentum balance<sup>①</sup>, eqn. I from the front inside cover, or equation 5.1 reduces to:

$$F\hat{i} = ma\hat{i} \quad \text{or} \quad F = ma \quad (5.3)$$

where  $F$  is the net force to the right and  $a$  is the acceleration to the right.

Now the force could come from a spring, or a fluid or from your hand pushing the particle to the right or left. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t).$$

Special cases would be, say,

$$\begin{aligned}F &= f(x) = -kx && \text{for a linear spring,} \\ F &= f(v) = -cv && \text{for a linear viscous drag,} \\ F &= f(t) = F_0 \sin(\beta t) && \text{for an oscillating load, and} \\ F &= f(x, v, t) = -kx - cv + F_0 \sin(\beta t) && \text{for all three forces at once.}\end{aligned}$$

So all elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\frac{dv}{dt} = \underbrace{f(x, v, t)/m}_{a(t)} \quad (a)$$

$$\frac{dx}{dt} = v(t) \quad (b)$$

where the function  $f(x, v, t)$  is given and  $x(t)$  and  $v(t)$  are to be found.

*Example: viscous drag*

If the only applied force is a viscous drag,  $F = -cv$ , then linear momentum balance would be  $-cv = ma$  and Eqns. 5.4 are

$$\frac{dv}{dt} = -cv/m$$

$$\frac{dx}{dt} = v$$

where  $c$  and  $m$  are constants and  $x(t)$  and  $v(t)$  are yet to be determined functions of time. Because the force does nothing but slow the particle down there will be no motion unless the particle has some initial velocity. In general, one needs to specify the initial position and velocity in order to determine a solution. So we complete the problem statement by specifying the initial conditions that

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where  $x_0$  and  $v_0$  are given constants. Before worrying about how to solve such a set of equations, one should first know how to recognize a solution set. In this case the two functions

$$\begin{aligned} v(t) &= v_0 e^{-ct/m}, & \text{and} \\ x(t) &= x_0 + mv_0(1 - e^{-ct/m})/c \end{aligned}$$

solve the equations. You can check that the initial conditions are satisfied by evaluating the expressions at  $t = 0$ . To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see if an identity results.  $\square$

Just like the case of integration (or equivalently the solution for  $x$  of  $\dot{x} = v(t)$ ), one often cannot find formulas for the solutions of differential equations.

*Example: A dynamics problem with no pencil and paper solution*

Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants  $c$ ,  $d$ ,  $F_0$ ,  $\beta$ , and  $m$ ,

$$\frac{dv}{dt} = (c \sin(x/d) + F_0 \sin(\beta t)) / m$$

$$\frac{dx}{dt} = v$$

with initial conditions  $x(0) = 0$  and  $v(0) = 0$ .

There is no known formula for  $x(t)$  that solves this ODE.  $\square$

Just writing the ordinary differential equations and initial conditions is quite analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get to this point. You just ask your computer to finish up.

## Special methods and special cases in 1D mechanics

In some problems, the acceleration can be found as a function of *position* (as opposed to time) easily. In this case, one can find velocity as a function of position by the following formula (see Box 5.1):

$$(v(x))^2 = (v(x_0))^2 + 2 \int_{x_0}^x a(x^*) dx^*. \quad (5.5)$$

An especially simple case is constant acceleration. Then we get the following kinematics formulas which are greatly loved and hated in high school and freshman physics:

$$\begin{aligned} a = \text{const} &\Rightarrow x = x_0 + v_0 t + at^2/2 \\ a = \text{const} &\Rightarrow v = v_0 + at \\ a = \text{const} &\Rightarrow v = \pm\sqrt{v_0^2 + 2ax}. \end{aligned}$$

Some of these equations are also discussed in box 5.2 about the solution of the simplest ordinary differential equations on page 226.

### *Example: Ramping up the acceleration at the start*

If you get a car going by gradually depressing the ‘accelerator’ so that its acceleration increases linearly with time, we have

$$\begin{aligned} a &= ct && \text{(take } t = 0 \text{ at the start)} \\ \Rightarrow v(t) &= \int_0^t ad\tau + v_0 = \int_0^t c\tau d\tau = ct^2/2 && \text{(since } v_0 = 0\text{)} \\ \Rightarrow x(t) &= \int_0^t vd\tau + x_0 = \int_0^t (c\tau^2/2)d\tau = ct^3/6 && \text{(since } x_0 = 0\text{).} \end{aligned}$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop.  $\square$

## 5.1 THEORY

### Finding $v(x)$ from $a(x)$

Equation 5.5 for velocity as a function of position can be derived as [Derivation 2](#):  
follows. Two derivations are given.

Derivation 1:

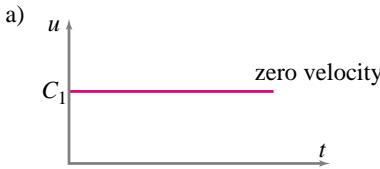
$$\begin{aligned}\frac{dv}{dt} &= a \\ \Rightarrow v \frac{dv}{dt} \frac{dt}{dx} &= a, \quad \text{since } \left(\frac{1}{v} = \frac{dt}{dx}\right) \\ \Rightarrow \left[ \frac{d}{dt} \left( \frac{1}{2} v^2 \right) \right] dt &= a dx, \\ &\quad \text{since } \left( \frac{d}{dt} \left( \frac{1}{2} v^2 \right) = v \frac{dv}{dt} \right) \\ \Rightarrow \frac{1}{2} v^2 - \frac{1}{2} v_0^2 &= \int_{x_0}^x a(x^*) dx^*.\end{aligned}$$

$$\begin{aligned}\frac{dv}{dt} &= a \\ \Rightarrow \frac{dv}{dx} \cdot \frac{dx}{dt} &= a, \\ \Rightarrow v dv &= a dx, \\ \Rightarrow \int_{v_0}^v v^* dv^* &= \int_{x_0}^x a(x^*) dx^*, \\ \Rightarrow \frac{1}{2} v^2 - \frac{1}{2} v_0^2 &= \int_{x_0}^x a(x^*) dx^*.\end{aligned}$$

## 5.2 The simplest ODEs, their solutions, and heuristic explanations

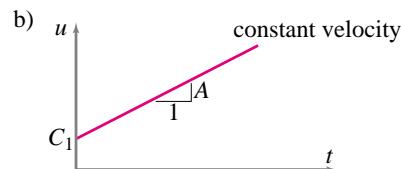
Sometimes differential equations you want to solve are simple enough that you might quickly find their solution. This table presents some of the simplest ODEs for  $u(t)$  and their general solution. Each of these solutions can be used to solve one or another simple mechanics problem. In order to make these simplest ordinary differential equations (ODE's) feel like more than just a group of symbols, we will try to make each of them intuitively plausible. For this purpose, we will interpret the variable  $u$  as the distance an object has moved to the right of its 'home', the origin at 0. The velocity of motion to the right is thus  $\dot{u}$  and its acceleration to the right is  $\ddot{u}$ . If  $\dot{u} < 0$  the particle is moving to the left. If  $\ddot{u} < 0$  the particle is accelerating to the left.

In all cases we assume that  $A$  and  $B$  are constants and that  $\lambda$  is a positive constant.  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  are arbitrary constants in the solutions that may be chosen to satisfy any initial conditions.



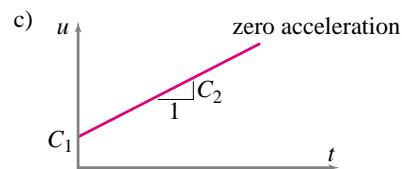
**a)** ODE:  $\dot{u} = 0 \Rightarrow$  Soln:  $u = C_1$ .

$\dot{u} = 0$  means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position  $C_1$ . Hence the general solution  $u = C_1$ , as can be found by direct integration.



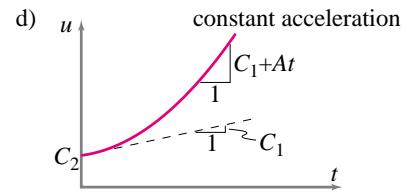
**b)** ODE:  $\dot{u} = A \Rightarrow$  Soln:  $u = At + C_1$ .

$\dot{u} = A$  means the object has constant speed. This equation describes the motion of a particle that starts with speed  $v_0 = A$  and because it has no force acting on it continues to move at constant speed. How far does it go in time  $t$ ? It goes  $v_0t$ . Where was it at time  $t = 0$ ? It could have been anywhere then, say  $C_1$ . So where is it at time  $t$ ? It's at its original position plus how far it has moved,  $u = v_0t + C_1$ , as can also be found by direct integration.



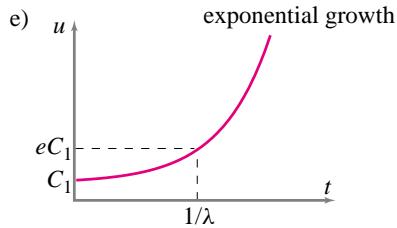
**c)** ODE:  $\ddot{u} = 0 \Rightarrow$  Soln:  $u = C_1t + C_2$ .

$\ddot{u} = 0$  means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say  $C_1$ . Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity:  $u = C_1t + C_2$ , where the constants  $C_1$  and  $C_2$  depend on the initial velocity and initial position. If you know that the position at  $t = 0$  is  $u_0$  and the velocity at  $t = 0$  is  $v_0$ , then the position is  $u = u_0 + v_0t$ .



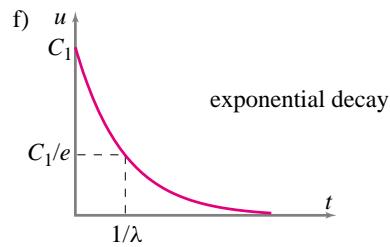
**d)** ODE:  $\ddot{u} = A \Rightarrow$  Soln:  $u = At^2/2 + C_1t + C_2$ .

This constant acceleration  $A$ , constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus  $At$  and the velocity is  $v = \dot{u} = v_0 + At$  (given that the velocity was  $v_0$  at  $t = 0$ ). Because the velocity is increasing constantly over time, the average velocity in a trip of length  $t$  occurs at  $t/2$  and is  $v_0 + At/2$ . The distance traveled is the average velocity times the time of travel so the distance of travel is  $t \cdot (v_0 + At/2) = v_0t + At^2/2$ . The position is the position at  $t = 0$ ,  $u_0$ , plus the distance traveled since time zero. So  $u = u_0 + v_0t + At^2/2 = C_2 + C_1t + At^2/2$ . This solution can also be found by direct integration.



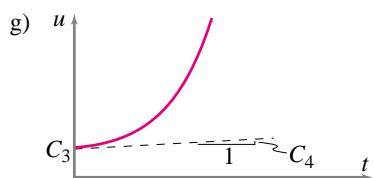
**e)** ODE:  $\dot{u} = \lambda u \Rightarrow$  Soln:  $u = C_1 e^{\lambda t}$ .

The displacement  $u$  grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the  $u$ , the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution  $u$  is an exponential function of time:  $u(t) = C_1 e^{\lambda t}$ , as can be found by separating variables or guessing.



**f)** ODE:  $\dot{u} = -\lambda u \Rightarrow$  Soln:  $u = C_1 e^{-\lambda t}$ .

The smaller  $u$  is, the more slowly it gets smaller.  $u$  gradually tapers towards nothing:  $u$  decays exponentially. The solution to the equation is:  $u(t) = C_1 e^{-\lambda t}$ . This expression is essentially the same equation as in (e) above.

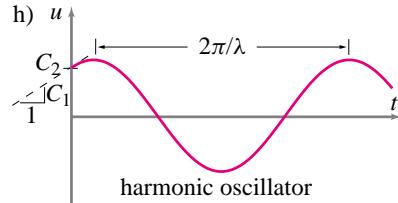


**g)** ODE:  $\ddot{u} = \lambda^2 u$

$$\Rightarrow \text{Soln: } u = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

$$\Rightarrow u = C_3 \cosh(\lambda t) + C_4 \sinh(\lambda t).$$

Note, sinh and cosh are just combinations of exponentials. For  $\ddot{u} = \lambda^2 u$ , the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because without the perfect initial condition the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.



**h)** ODE:  $\ddot{u} = -\lambda^2 u$  or  $\ddot{u} + \lambda^2 u = 0$   
 $\Rightarrow$  Soln:  $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$ .

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at  $u = 0$ . When  $u$  is positive,  $\ddot{u}$  is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where  $u = 0$ , it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is  $u = \sin(\lambda t)$ , that is, sinusoidal oscillations. The oscillations are faster if  $\lambda$  is bigger. Another solution is  $u = \cos(\lambda t)$ . The general solution is  $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$ . A plot of this function reveals a sine wave shape for any value of  $C_1$  or  $C_2$ , although the phase depends on the relative values of  $C_1$  and  $C_2$ . The equation  $\ddot{u} = -\lambda^2 u$  or  $\ddot{u} + \lambda^2 u = 0$  is called the ‘harmonic oscillator’ equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise).

**i)** There are a few other not-too-hard ODEs besides those listed in the box. For example, the general second order, constant coefficient ODE with sinusoidal forcing:  $A\ddot{u} + B\dot{u} + Cu = F \sin(Dt)$ . But the solution is a little more complicated and not quite so easily verified. So we save it for chapter 10 on vibrations. Most engineers, when confronted with an equation not on this list, will resort to a *numerical* computer solution.

**SAMPLE 5.1 Time derivatives:** The position of a particle varies with time as  $\vec{r}(t) = (C_1 t + C_2 t^2) \hat{i}$ , where  $C_1 = 4 \text{ m/s}$  and  $C_2 = 2 \text{ m/s}^2$ .

- Find the velocity and acceleration of the particle as functions of time.
- Sketch the position, velocity, and acceleration of the particle against time from  $t = 0$  to  $t = 5 \text{ s}$ .
- Find the position, velocity, and acceleration of the particle at  $t = 2 \text{ s}$ .

### Solution

- We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{r} = (C_1 t + C_2 t^2) \hat{i} = (4 \text{ m/s} t + 2 \text{ m/s}^2 t^2) \hat{i} \quad (5.6)$$

$$\begin{aligned} \vec{v} &\equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2) \hat{i} \\ &= (C_1 + C_2 t) \hat{i} = (4 \text{ m/s} + 2 \text{ m/s}^2 t) \hat{i} \end{aligned} \quad (5.7)$$

$$\begin{aligned} \vec{a} &\equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}(C_1 + C_2 t) \hat{i} \\ &= C_2 \hat{i} = (2 \text{ m/s}^2) \hat{i} \end{aligned} \quad (5.8)$$

$$\boxed{\vec{v} = (4 \text{ m/s} + 2 \text{ m/s}^2 t) \hat{i}, \vec{a} = (2 \text{ m/s}^2) \hat{i}}$$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

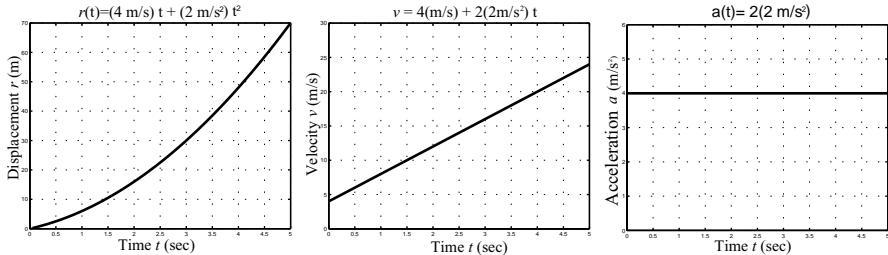


Figure 5.4: (Filename:fig5.1.vrplot)

- We plot eqns. (5.6, 5.7, and 5.8) against time by taking 100 points between  $t = 0$  and  $t = 5 \text{ s}$ , and evaluating  $\vec{r}$ ,  $\vec{v}$  and  $\vec{a}$  at those points. The plots are shown below.
- We can find the position, velocity, and acceleration at  $t = 2 \text{ s}$  by evaluating their expressions at the given time instant:

$$\begin{aligned} \vec{r}(t = 2 \text{ s}) &= [(4 \text{ m/s}) \cdot (2 \text{ s}) + (2 \text{ m/s}^2) \cdot (2 \text{ s})^2] \hat{i} \\ &= (16 \text{ m}) \hat{i} \\ \vec{v}(t = 2 \text{ s}) &= [(4 \text{ m/s}) + (2 \text{ m/s}^2) \cdot (2 \text{ s})] \hat{i} \\ &= (8 \text{ m/s}) \hat{i} \\ \vec{a}(t = 2 \text{ s}) &= (2 \text{ m/s}^2) \hat{i} = \vec{a}(\text{at all t}) \end{aligned}$$

$$\boxed{\text{At } t = 2 \text{ s}, \vec{r} = (16 \text{ m}) \hat{i}, \vec{v} = (8 \text{ m/s}) \hat{i}, \vec{a} = (2 \text{ m/s}^2) \hat{i}}$$

**SAMPLE 5.2** *Math review: Solving simple differential equations.* For the following differential equations, find the solution for the given initial conditions.

- (a)  $\frac{dv}{dt} = a$ ,  $v(t = 0) = v_0$ , where  $a$  is a constant.
- (b)  $\frac{d^2x}{dt^2} = a$ ,  $x(t = 0) = x_0$ ,  $\dot{x}(t = 0) = \dot{x}_0$ , where  $a$  is a constant.

### Solution

(a)

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a dt \\ \text{or} \quad \int dv &= \int a dt = a \int dt \\ \text{or} \quad v &= at + C, \quad \text{where } C \text{ is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution,  $v(t = 0) = v_0 = a \cdot 0 + C \Rightarrow C = v_0$ . Therefore,

$$v = at + v_0.$$

$$v = v_0 + at$$

Alternatively, we can use definite integrals:

$$\int_{v_0}^v dv = \int_0^t a dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

- (b) This is a second order differential equation in  $x$ . We can solve this equation by first writing it as a first order differential equation in  $v \equiv dx/dt$ , solving for  $v$  by integration, and then solving again for  $x$  in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \quad \int dv &= \int a dt \\ \Rightarrow v \equiv \dot{x} &= at + C_1 \quad (5.9) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at dt + \int C_1 dt \\ \text{or} \quad x &= \frac{1}{2}at^2 + C_1 t + C_2, \quad (5.10) \end{aligned}$$

where  $C_1$  and  $C_2$  are constants of integration. Substituting the initial condition for  $\dot{x}$  in Eqn. (5.9), we get

$$\dot{x}(t = 0) = \dot{x}_0 = a \cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for  $x$  in Eqn. (5.10), we get

$$x(t = 0) = x_0 = \frac{1}{2}a \cdot 0 + \dot{x}_0 \cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2$$

**SAMPLE 5.3 Constant speed motion:** A ship cruises at a constant speed of 15 knots per hour due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 knots straight ahead. At what time does the ship pass the next lighthouse?

**Solution** We are given the distance  $s$  and the speed of travel  $v$ . We need to find how long it takes to travel the given distance.

$$\begin{aligned} s &= vt \\ \Rightarrow t &= \frac{s}{v} = \frac{35 \text{ knots}}{15 \text{ knots/hour}} = 2.33 \text{ hrs.} \end{aligned}$$

Now, the time at  $t = 0$  is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

10 : 50 am

**SAMPLE 5.4 Constant velocity motion:** A particle travels with constant velocity  $\vec{v} = 5 \text{ m/s} \hat{i}$ . The initial position of the particle is  $\vec{r}_0 = 2 \text{ m} \hat{i} + 3 \text{ m} \hat{j}$ . Find the position of the particle at  $t = 3 \text{ s}$ .

**Solution** Here, we are given the velocity, *i.e.*, the time derivative of position:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = v_0 \hat{i}, \quad \text{where } v_0 = 5 \text{ m/s.}$$

We need to find  $\vec{r}$  at  $t = 3 \text{ s}$ , given that  $\vec{r}$  at  $t = 0$  is  $\vec{r}_0$ .

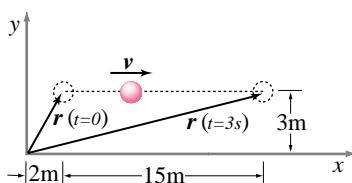


Figure 5.5: (Filename:fig5.1.new1)

$$\begin{aligned} d\vec{r} &= v_0 \hat{i} dt \\ \Rightarrow \int_{\vec{r}_0}^{\vec{r}(t)} d\vec{r} &= \int_0^t v_0 \hat{i} dt = v_0 \hat{i} \int_0^t dt \\ \vec{r}(t) - \vec{r}_0 &= v_0 \hat{i} t \\ \vec{r}(t) &= \vec{r}_0 + v_0 t \hat{i} \\ \vec{r}(3 \text{ s}) &= (2 \text{ m} \hat{i} + 3 \text{ m} \hat{j}) + (5 \text{ m/s}) \cdot (3 \text{ s}) \hat{i} \\ &= 17 \text{ m} \hat{i} + 3 \text{ m} \hat{j}. \end{aligned}$$

$\vec{r} = 17 \text{ m} \hat{i} + 3 \text{ m} \hat{j}$

**SAMPLE 5.5 Constant acceleration:** A 0.5 kg mass starts from rest and attains a speed of  $20 \text{ m/s} \hat{i}$  in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

**Solution** Here, we are given the initial velocity  $\vec{v}(0) = \vec{0}$  and the final velocity  $\vec{v}$  after  $t = 4 \text{ s}$ . We have to find the force acting on the mass. The net force on a particle is given by  $\vec{F} = m\vec{a}$ . Thus, we need to find the acceleration  $\vec{a}$  of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t$$

. Therefore, we can find the acceleration  $\vec{a}$  as

$$\begin{aligned}\vec{a} &= \frac{\vec{v}(t) - \vec{v}(0)}{t} \\ &= \frac{20 \text{ m/s} \hat{i} - \vec{0}}{4 \text{ s}} \\ &= 5 \text{ m/s}^2 \hat{i}.\end{aligned}$$

The force on the particle is

$$\vec{F} = m\vec{a} = (0.5 \text{ kg}) \cdot (5 \text{ m/s}^2 \hat{i}) = 2.5 \text{ N} \hat{i}.$$

$\boxed{\vec{F} = 2.5 \text{ N} \hat{i}}$

**SAMPLE 5.6 Time of travel for a given distance:** A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity  $g = 10 \text{ m/s}^2$ .

**Solution** The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let  $y$  denote the distance travelled by the ball. Let us measure  $y$  vertically downwards, starting from the height at which the ball starts falling (see Fig. 5.6). Under constant acceleration  $g$ , we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2} g t^2.$$

Note that at  $t = 0$ ,  $y_0 = 0$  and  $v_0 = 0$ . We are given that at some instant  $t$  (that we need to find)  $y = 30 \text{ m}$ . Thus,

$$\begin{aligned}y &= \frac{1}{2} g t^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30 \text{ m}}{10 \text{ m/s}^2}} = 2.45 \text{ s}\end{aligned}$$

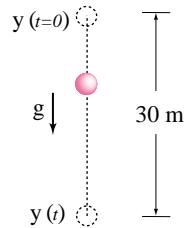


Figure 5.6: (Filename:sfig5.2.new3)

$\boxed{t = 2.45 \text{ s}}$

**SAMPLE 5.7 Numerical integration of ODE's:**

- Write the second order linear nonhomogeneous differential equation,  $\ddot{x} + c\dot{x} + kx = a_0 \sin \omega t$ , as a set of first order equations that can be used for numerical integration.
- Write the second order nonlinear homogeneous differential equation,  $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ , as a set of first order equations that can be used for numerical integration.
- Solve the nonlinear equation given in (b) by numerical integration taking  $c = 0.05$ ,  $k = 1$ ,  $x(0) = 0$ , and  $\dot{x}(0) = 0.1$ . Compare this solution with that of the linear equation in (a) by setting  $a_0 = 0$  and taking other values to be the same as for (b).

**Solution**

(a)

$$\begin{aligned} \text{If we let } \dot{x} &= y, \\ \text{then } \dot{y} &= \ddot{x} = -c\dot{x} - kx + a_0 \sin \omega t \\ &= -cy - kx + a_0 \sin \omega t \end{aligned}$$

$$\text{or } \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ a_0 \sin \omega t \end{Bmatrix}. \quad (5.11)$$

Equation (5.11) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc.*

(b)

$$\begin{aligned} \text{If } \dot{x} &= y \\ \text{then } \dot{y} &= \ddot{x} = c\dot{x} - kx^3 = -cy - kx^3 \\ \text{or } \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} &= \begin{Bmatrix} y \\ -cy - kx^3 \end{Bmatrix}. \end{aligned} \quad (5.12)$$

Equation (5.12) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 5.11 because of the nonlinearity in  $x$  and  $\dot{x}$ . It is, however, in an appropriate form for numerical integration.

- Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```

ODEs = {xdot = y, ydot = -c y - k x^3}
IC    = {x(0) = 0, y(0) = 0.1}
Set   k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)

```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in Fig. 5.7. A similar program used for the equation in (a) with  $a_0 = 0$  gives the plot shown in Fig. 5.8. The two plots show how a simple nonlinearity changes the response drastically.

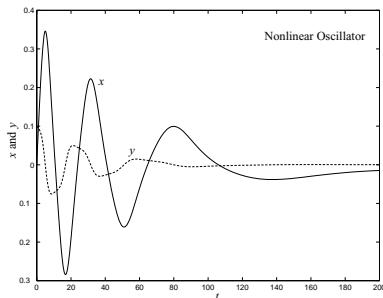


Figure 5.7: Numerical solution of the nonlinear ODE  $\ddot{x} + c\dot{x}^2 + kx^3 = 0$  with initial conditions  $x(0) = 0$  and  $\dot{x}(0) = 0.1$ .

(Filename:sfig5.1.nonlin)

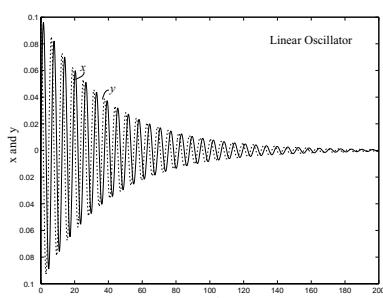


Figure 5.8: Numerical solution of the linear ODE  $\ddot{x} + c\dot{x}^2 + kx = 0$  with initial conditions  $x(0) = 0$  and  $\dot{x}(0) = 0.1$ .

(Filename:sfig5.1.lin)

## 5.2 Energy methods in 1D

Energy is an important concept in science and is even a kind of currency in human trade. But for us now, an energy equations is primarily a short-cut for solving some mechanics problems.

### The work-energy equation

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The kinetic energy of a particle is defined as

$$E_K = \frac{1}{2}m_{\text{tot}}v^2.$$

The power balance equation is thus, in rate form,

$$P = \frac{d}{dt} \left( \frac{1}{2}mv^2 \right),$$

where  $P = Fv$  is the power of the applied force  $F$ . Integrating in time we get, using that  $v = dx/dt$ ,

$$\begin{aligned} \Rightarrow \quad \int Fv dt &= \int \frac{d}{dt} \left( \frac{1}{2}mv^2 \right) dt \\ \Rightarrow \quad \int F dx &= \Delta \left( \frac{1}{2}mv^2 \right). \end{aligned} \quad (5.13)$$

$$\Rightarrow \quad W = \Delta E_K$$

The integral  $W = \int F dx$  is called the work. The derivations above, from the general equations to the particle equations, are the opposite of historical. As Box 5.1 on page 225 shows, in this case the work-energy equation can be derived from the momentum-balance equation. In fact it is this one-dimensional mechanical case that first led to the discovery of energy as a concept. But now that we know that  $F = ma$  implies that work is change in kinetic energy, we can use the result without deriving it every time.

### Conservation of energy

One of the most useful intuitive concepts for simple mechanics problems is conservation of energy. So far we know that the work of a force on a particle gives its change of energy (eqn. 5.13). But some forces come from a source that has associated with it a potential energy. If, for example, the force to the right on a particle is a function of  $x$  (and not, say, of  $\dot{x}$ ) then we have a *force field*. In one dimension we can define a new function of  $x$  that we will call  $E_P(x)$  as the integral of the force with respect to  $x$ :

$$E_P(x) = - \int_0^x F(x') dx' = -( \text{Work done by the force in moving from 0 to } x ) \quad (5.14)$$

Note also, by the fundamental theorem of calculus, that given  $E_P(x)$  we can find  $F(x)$  as

$$F(x) = - \frac{dE_P(x)}{dx}.$$

Now let's consider the work done by the force on the particle when the particle moves from point  $x_1$  to  $x_2$ . It is

$$\text{Work done by force from } x_1 \text{ to } x_2 = -(E_{P2} - E_{P1}) = -\Delta E_P.$$

That is, the decrease in  $E_P$  is the amount of work that the force does. Or, in other words,  $E_P$  represents a potential to do work. Because work causes an increase in kinetic energy,  $E_P$  is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 5.13 to find that

$$-\Delta E_P = \Delta E_K \quad \Rightarrow \quad 0 = \Delta \underbrace{(E_P + E_K)}_{E_T}.$$

The *total energy*  $E_T$  doesn't change ( $\Delta E_T = 0$ ) and thus is a constant. In other words,

as a particle moves in the presence of a force field with a potential energy, the total energy  $E_T = E_K + E_P$  is constant.

This fact goes by the name of *conservation of energy*.

#### Example: Falling ball

Consider the ball in the free body diagram 5.9. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_P = - \int_0^y (-mg) dy' = mgy = mgh.$$

For vertical motion

$$E_K = \frac{1}{2}m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_P + E_K = mgy + m\dot{y}^2$$

which you can also derive directly from  $m\ddot{y} = -mg$ . Alternatively, we could start with conservation of energy and differentiate to get

$$\begin{aligned} E_T = \text{constant} \quad \Rightarrow \quad 0 &= \frac{d}{dt} E_T \\ &= \frac{d}{dt} (E_P + E_K) \\ &= \frac{d}{dt} (mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg \end{aligned}$$

where we had to assume (and this is just a technical point) that  $\dot{y} \neq 0$  in one of the cancellations. Thus, for this problem, energy balance can be used to derive linear-momentum balance.

We could also start with the power-balance equation,

power of gravity force = rate of change in particle's kinetic energy

$$\begin{aligned} P &= \frac{d}{dt} (E_K) \\ \vec{F} \cdot \vec{v} &= \frac{d}{dt} (E_K) \\ (-mg\hat{j}) \cdot (\dot{y}\hat{j}) &= \frac{d}{dt} \left[ \frac{1}{2}mv^2 \right] \end{aligned}$$

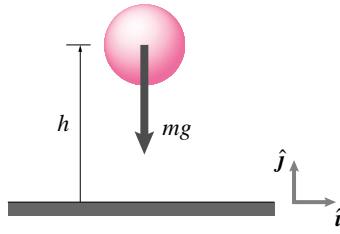


Figure 5.9: Free body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

(Filename:tfigure1.falling.ball)

$$\begin{aligned}-mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}(v^2) \\ -mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}(y^2) \\ m\ddot{y} &= -mg,\end{aligned}$$

and again get the same result. Thus, for one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others.  $\square$

### 5.3 THEORY

#### *Derivation of the work energy equation*

Because  $F = ma$ , all our kinematics calculations above turn into dynamics calculations by making the substitution  $F/m$  every place that  $a$  appears. Equation 5.5, for example, becomes

$$(v(x))^2 = (v(x_0))^2 + \frac{2}{m} \int_{x_0}^x F(x^*) dx^*.$$

In box 5.1 on page 225 we found that

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \int_{x_0}^x a(x^*) dx^*.$$

If we multiply both sides of all equations in the above derivation by  $m$  and substitute  $F$  for  $ma$  the derivation above shows that

$$\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}_{\Delta E_K} = \underbrace{\int_0^x \widetilde{F(x^*)} dx^*}_{\text{work done by a force}}$$

For straight-line motion with a force in only one direction on a particle, we have no heat flow, dissipation, or internal energy to fuss over so that the energy equation (III) from the inside front cover has been derived.

Alternatively, if you can remember the work-energy equation ('The positive work of a force on a particle is the positive change in kinetic energy'), you can use it to recall the related kinematics equation. For example, if  $F$  and  $a$  are constant, and  $x$  is the total displacement,

$$\begin{aligned}\Delta(\frac{1}{2}mv^2) &= F\Delta x, \text{ and} \\ \Rightarrow \Delta(\frac{1}{2}v^2) &= a\Delta x.\end{aligned}$$

**SAMPLE 5.8** How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

### Solution

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= Pdt \\ W_{12} &= \int_{t_0}^{t_1} Pdt = P(t_1 - t_0) = P\Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_K)_2 - (E_K)_1 \\ &= \frac{1}{2}m(v_2^2 - v_1^2) \\ &= \frac{800 \text{ kg}[(60 \text{ mph})^2 - 0]}{2} \\ &= \frac{1}{2} \cdot 800 \text{ kg} \left( 60 \frac{\text{mi}}{\text{hr}} \cdot \frac{1.61 \times 10^3 \text{ m}}{1 \text{ mi}} \cdot \frac{1 \text{ hr}}{3600 \text{ s}} \right)^2 \\ &= 288.01 \times 10^3 \text{ kg} \cdot \text{m} \cdot \text{m/s}^2 \\ &= 288 \text{ KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288 \times 10^3 \text{ J}}{40 \times 745.7 \text{ W}} = 9.66 \text{ s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$\Delta t = 9.66 \text{ s}$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time  $t_0$ . In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_K + \dot{E}_{\text{int}} + \dot{E}_P$$

and set  $-(\dot{E}_{\text{int}} + \dot{E}_P)$  = ‘the engine power’ where the engine power is from the decrease in gasoline potential energy ( $-\dot{E}_P$  is positive) less the increase in ‘heat’ ( $\dot{E}_{\text{int}}$ ) from engine inefficiencies.

**SAMPLE 5.9** *Energy of a mass-spring system.* A mass  $m = 2\text{ kg}$  is attached to a spring with spring constant  $k = 2\text{ kN/m}$ . The relaxed (unstretched) length of the spring is  $\ell = 40\text{ cm}$ . The mass is pulled up and released from rest at position A shown in Fig. 5.10. The mass falls by a distance  $h = 10\text{ cm}$  before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

**Solution** The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k(\text{stretch})^2 = \frac{1}{2}kh^2 \\ \text{Energy of the mass} &= E_K + E_P = \frac{1}{2}m\underbrace{v_A^2}_0 + mg(\ell + h) = mg(\ell + h).\end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell + h).$$

Let the speed of the mass at position B be  $v_B$ . When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k(\text{stretch})^2 = 0 \\ \text{Energy of the mass} &= E_K + E_P = \frac{1}{2}mv_B^2 + mg\ell,\end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned}0 &= \Delta E \\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell + h) - \frac{1}{2}mv_B^2 - mg\ell \\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh \\ \Rightarrow v_B^2 &= kh^2/m + 2gh \\ \Rightarrow |v_B| &= \left(kh^2/m + 2gh\right)^{1/2} \\ &= \left((2000\text{ N/m} \cdot (0.1\text{ m})^2/2\text{ kg}) + 2 \cdot 9.81\text{ m/s}^2 \cdot 0.1\text{ m}\right)^{1/2} \\ &= 3.46\text{ m/s}.\end{aligned}$$

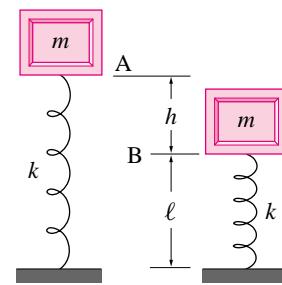


Figure 5.10: (Filename:sfig2.6.1)

$$|v_B| = 3.46\text{ m/s}$$

**SAMPLE 5.10 Which is the best bicycle helmet?** Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

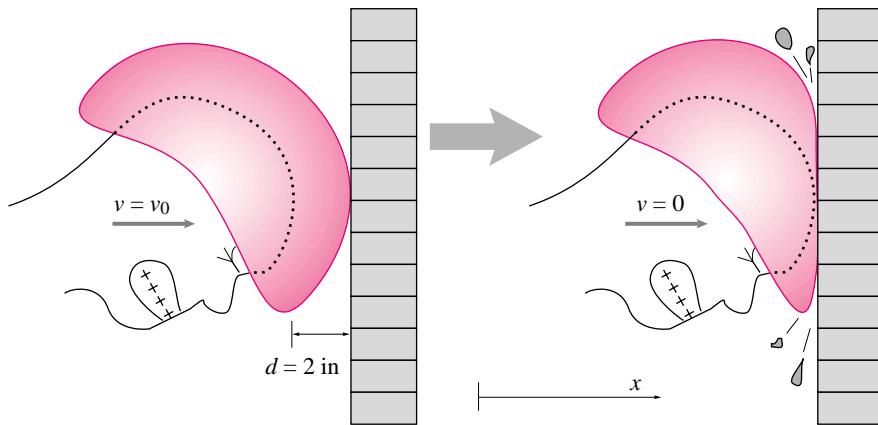


Figure 5.11: (Filename:sfig3.2.ouchie)

**Solution 1 – Kinematics method 1:** We are given the initial speed of  $V_0$ , a final speed of 0, and a constant acceleration  $a$  (which is negative) over a given distance of travel  $d$ . If we call  $t_c$  the time when the helmet is fully crushed,

$$\begin{aligned}
 v(t) &= v_0 + \int_0^{t_c} a(t') dt' \\
 &= v_0 + at_c \\
 0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \\
 x(t) &= x_0 + \int_0^{t_c} v(t') dt' \\
 &= 0 + \int_0^{t_c} (v_0 + at) dt \\
 d = x(t_c) &= 0 + v_0 t_c + at_c^2/2 \\
 d &= v_0 \left( \frac{-v_0}{a} \right) + a \left( \frac{v_0}{a} \right)^2 / 2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \quad (\text{using (5.15)}) \\
 \Rightarrow a &= \frac{-v_0^2}{2d} \\
 &= \frac{-(25 \text{ mph})^2}{2 \cdot (2 \text{ in})} \\
 &= \frac{-25^2}{4} \cdot \frac{\text{mi}^2}{\text{hr}^2 \cdot \text{in}} \cdot \underbrace{\left( \frac{5280 \text{ ft}}{\text{mi}} \right)^2}_{1} \cdot \underbrace{\left( \frac{1 \text{ hr}}{3600 \text{ s}} \right)^2}_{1} \cdot \underbrace{\left( \frac{12 \text{ in}}{\text{ft}} \right)}_{1} \cdot \underbrace{\left( \frac{1 \text{ g}}{32.2 \text{ ft/s}^2} \right)}_{1} \\
 &= \frac{-25}{4} \cdot \frac{5280^2}{3600^2} \cdot 12 \cdot \frac{1}{32.2} g \\
 a &= -125g
 \end{aligned} \tag{5.15}$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

**Solution 2 – Kinematics method 2:**

$$\begin{aligned}
 \frac{dv}{dt} &= a \Rightarrow dv = adt \\
 \Rightarrow vdv &= avdt \Rightarrow vdv = a\frac{dx}{dt}dt \\
 \Rightarrow vdv &= adx \\
 \Rightarrow \int vdv &= \int adx \\
 \Rightarrow \Delta \frac{v^2}{2} &= ax \quad (\text{since } a = \text{constant}) \\
 \Rightarrow 0 - \frac{v_0^2}{2} &= ad \Rightarrow a = \frac{-v_0^2}{2d} \quad (\text{as before})
 \end{aligned}$$

**Solution 3 – Quote formulas:**

$$\begin{aligned}
 "v" &= \sqrt{2ad} \\
 \Rightarrow a &= \frac{v^2}{2d} \quad \text{which is right if you know how to interpret it!}
 \end{aligned}$$

**Solution 4 – Work-Energy:**

Constant acceleration  $\Rightarrow$  constant force

$$\begin{aligned}
 \text{Work in} &= \Delta E_K \\
 -Fd &= 0 - \frac{mv_0^2}{2} \\
 F &= \frac{mv_0^2}{2d} \\
 \text{But } \vec{F} &= m\vec{a} \Rightarrow -F\hat{i} = -ma\hat{i} \\
 &\Rightarrow a = \frac{-F}{m} \\
 \text{So } a &= \frac{-v_0^2}{2d} \quad (\text{again})
 \end{aligned}$$

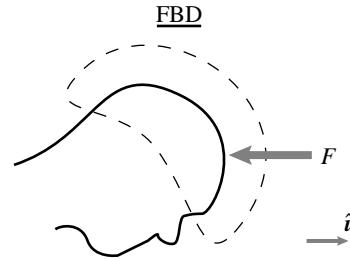


Figure 5.12:  $F$  is the force of the helmet on the moving head.

(Filename:sfig3.2.ouchie.fbd)

Assuming a head mass of 8 lbm, the force on the head during impact is

$$\boxed{|F| = \frac{mv_0^2}{2d} = ma = 8 \text{ lbm} \cdot 125g}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 in thickness that could make the deceleration for this collision less than 125g or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of 250g or greater are often fatal. Even 125g usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

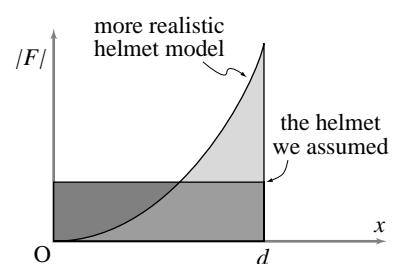


Figure 5.13: (Filename:sfig3.2.ouchie.graph)

## 5.3 The harmonic oscillator

Most engineering materials are nearly elastic under working conditions. And, of course, all real things have mass. These ingredients, elasticity and mass, are what make vibration possible. Even structures which are fairly rigid will vibrate if encouraged to do so by the shaking of a rotating motor, the rough rolling of a truck, or the ground motion of an earthquake. The vibrations of a moving structure can also excite oscillations in flowing air which can in turn excite the structure further. This mutual excitement of fluids and solids is the cause of the vibrations in a clarinet reed, and may have been the source of the wild oscillations in the famous collapse of the Tacoma Narrows bridge. Mechanical vibrations are not only the source of most music but also of most annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendula or quartz crystals are used to measure time. Vibrations can cause a machine to go out of control, or a buildings to collapse. So the study of vibrations, for better or for worse, is not surprisingly one of the most common applications of dynamics.

When an engineer attempts to understand the oscillatory motion of a machine or structure, she undertakes a *vibration* analysis. A vibration analysis is a study of the motions that are associated with vibrations. Study of motion is what dynamics is all about, so vibration analysis is just a part of dynamics.

A vibration analysis could mean the making of a dynamical model of the structure one is studying, writing equations of motion using the momentum balance or energy equations and then looking at the solution of these equations. But, in practice, the motions associated with vibrations have features which are common to a wide class of structures and machines. For this reason, a special vocabulary and special methods of approach have been developed for vibration analysis. For example, one can usefully discuss *resonance*, *normal modes*, and *frequency response*, concepts which we will soon discuss, without ever writing down any equations of motion. We will first approach these concepts within the framework of the differential equations of motion and their solutions. But after the concepts have been learned, we can use them without necessarily referring directly to the governing differential equations.

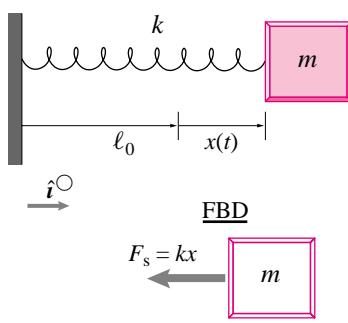


Figure 5.14: A spring mass system.

(Filename:tfigure3.MS)

The unforced oscillations of a spring and mass is the basic model for all vibrating systems.

So it is worth knowing well.

We start with a free body diagram of a mass which is cut from a spring in an extended state, as shown in figure 5.14. The mass slides on a frictionless surface. The spring is relaxed at  $x = 0$ . The spring is thus stretched from  $\ell_0$  to  $\ell_0 + \Delta\ell$ , a stretch of  $\Delta\ell = x$ . The free body diagram at the bottom shows the force on the mass. Gravity is neglected.

Linear momentum balance in the  $x$  direction ( $\{\sum \vec{F} = \dot{\vec{L}}\} \cdot \hat{i}$ ) gives:

$$\begin{aligned}\sum F_x &= \dot{L}_x \\ -kx &= m\ddot{x}.\end{aligned}$$

Rearranging this equation we get one of the most famous and useful differential equations of all time:

$$\ddot{x} + \frac{k}{m}x = 0. \quad (5.16)$$

This equation appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable  $x$  and the parameter combination  $k/m$  are replaced by other physical quantities. In an electrical circuit, for example,  $x$  might represent a voltage and the term corresponding to  $k/m$  might be  $1/LC$ , where  $C$  is a capacitance and  $L$  an inductance. But even in dynamics the equation appears with other physical quantities besides  $k/m$  multiplying the  $x$ , and  $x$  itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written as

$$\ddot{x} + \lambda^2 x = 0. \quad (5.17)$$

The constant in front of the  $x$  is called  $\lambda^2$  instead of just, say,  $\lambda$  ('lambda')<sup>①</sup>, for two reasons:

- (a) This convention shows that  $\lambda^2$  is positive,
- (b) In the solution we need the square root of this coefficient, so it is convenient to have  $\sqrt{\lambda^2} = \lambda$ .

For the spring-block system,  $\lambda^2$  is  $k/m$  and in other problems  $\lambda^2$  is some other combination of physical quantities.

<sup>①</sup>**Caution:** Most books use  $p^2$  or  $\omega^2$  in the place we have put  $\lambda^2$ . Using  $\omega$  ('omega') can lead to confusion because we will later use  $\omega$  for angular velocity. If one is studying vibrations of a rotating shaft then there would be two very different  $\omega$ 's in the problem. One, the coefficient of a differential equation and, the other, the angular velocity. To add to the confusion, this coincidence of notation is not accidental. Simple harmonic oscillations and circular motion have a deep connection. Despite this deep connection, the  $\omega$  in the differential equation is not the same thing as the  $\omega$  describing angular motion of a physical object. We avoid this confusion by using  $\lambda$  instead of  $\omega$ . Note that this  $\lambda$  is unrelated to the unit vector  $\hat{\lambda}$  that we use in some problems.

## Solution of the harmonic oscillator differential equation

Finding solutions to the harmonic oscillator differential equation 5.17 from first principles is a topic for a math class. Here we content ourselves with remembering its general solution, namely

$$\begin{aligned} x(t) &= A \cos(\lambda t) + B \sin(\lambda t), \\ \text{or } x(t) &= C_1 \cos(\lambda t) + C_2 \sin(\lambda t). \end{aligned} \quad (5.18)$$

This sum of sine waves<sup>②</sup> is a solution of differential equation 5.17 for any values of the constants  $A$  (or  $C_1$ ) and  $B$  (or  $C_2$ ).

<sup>②</sup> A cosine function is also a sine wave.

What does it mean to say " $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$  solves the equation:  $\ddot{u} = -\lambda^2 u$ ?" The solution is a function that has the property that its second derivative is the same as minus the original function multiplied by the constant  $\lambda^2$ . That is, the function  $u(t) = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$  has the property that its second derivative is the original function multiplied by  $-\lambda^2$ . You need not take this property on faith.

To check if a function is a solution, plug it into the differential equation and see if an identity is obtained.

Is this equality correct for the proposed  $u(t)$ ?

$$\frac{d^2}{dt^2} u = -\lambda^2 u$$

$$\frac{d^2}{dt^2} \underbrace{[C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]}_{u(t)} \stackrel{?}{=} -\lambda^2 \underbrace{[C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]}_{u(t)}$$

$$\frac{d}{dt} \left( \frac{d}{dt} [C_1 \sin(\lambda t) + C_2 \cos(\lambda t)] \right) \stackrel{?}{=} -\lambda^2 [C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]$$

$$\frac{d}{dt} [C_1 \lambda \cos(\lambda t) - C_2 \lambda \sin(\lambda t)] \stackrel{?}{=} -\lambda^2 [C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]$$

$$\underbrace{-C_1 \lambda^2 \sin(\lambda t) - C_2 \lambda^2 \cos(\lambda t)}_{\ddot{u}} \stackrel{\checkmark}{=} -\lambda^2 \underbrace{[C_1 \sin(\lambda t) + C_2 \cos(\lambda t)]}_{u(t)}$$

The equation  $\ddot{u} = -\lambda^2 u$  does hold with the given  $u(t)$

This calculation verifies that, no matter what the constants  $C_1$  and  $C_2$ , the proposed solution satisfies the given differential equation.

Although we have checked the solution, we have not proved its uniqueness. That is, there might be other solutions to the differential equation. There are not. We leave discussion of uniqueness to your math classes.

## Interpreting the solution of the harmonic oscillator equation

The solution above means that if we built a system like that shown in figure 5.14 and watched how the mass moved, it would move (approximately) so that  $x(t) = A \cos(\lambda t) + B \sin(\lambda t)$ , as shown in the graph in figure 5.15.

This back and forth motion is called *vibration*. One might think that vibrations are fast oscillations. But in mechanics anything that oscillates is a vibration. For example, the slow rocking of a ship might be called a vibration.

### Angular frequency, period, and frequency

Three related measures of the rate of oscillation are angular frequency, period, and frequency. The simplest of these is *angular frequency*  $\lambda = \sqrt{(k/m)}$ , sometimes called *circular frequency*. The period  $T$  is the amount of time that it takes to complete one oscillation. One oscillation of both the *sine* function and the *cosine* function occurs when the argument of the function advances by  $2\pi$ , that is when

$$\lambda T = 2\pi, \quad \text{so} \quad T = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{(k/m)}},$$

formulas often memorized in elementary physics courses. The *natural frequency*  $f$  is the reciprocal of the period

$$f = \frac{1}{T} = \frac{\lambda}{2\pi} = \frac{\sqrt{(k/m)}}{2\pi}.$$

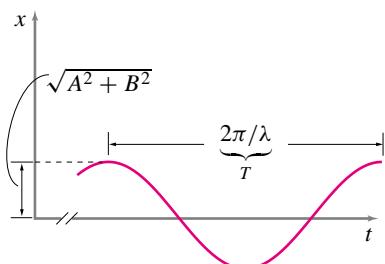


Figure 5.15: Position versus time for an undamped, unforced harmonic oscillator.  $x$  is the position of the mass,  $t$  is time.

(Filename:tfigure12.sinewave)

Typically, natural frequency  $f$  is measured in cycles per second or *Hertz* and the angular frequency  $\lambda$  in radians per second. Mechanical vibrations can have frequencies from millions of cycles per second, for the vibrations of a microscopic quartz timing crystal, to thousandths of a cycle per second (i.e. thousands of seconds per cycle), say, for the free vibrations of the whole earth.

The amplitude of the sine wave that results from the addition of the *sine* function and the *cosine* function is given by the square root of the sum of the squares of the two amplitudes. That is, the amplitude of the resulting sine wave is  $\sqrt{A^2 + B^2}$ . Another way of describing this sum is through the trigonometric identity:

$$A \cos(\lambda t) + B \sin(\lambda t) = R \cos(\lambda t - \phi), \quad (5.19)$$

where  $R = \sqrt{A^2 + B^2}$  and  $\tan \phi = B/A$ . So,

the only possible motion of a spring and mass is a sinusoidal oscillation which can be thought of either as the sum of a *cosine* function and a *sine* function or as a single *cosine* function with *phase shift*  $\phi$ .

## What are the constants $A$ and $B$ in the solution?

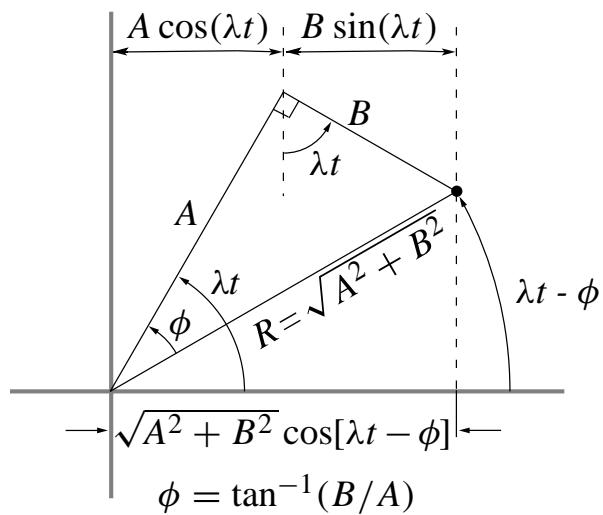
The general motion of the harmonic oscillator, equation 5.18, has the constants  $A$  and  $B$  which could have any value. Or, equivalently, the amplitude  $R$  and phase  $\phi$  in equation 5.19 could be anything. They are determined by the way motion is started, the *initial conditions*. Two special initial conditions are worth getting a feel for: release from rest and initial velocity with no spring stretch.

### 5.4 THEORY

*Visualization of  $A \cos(\lambda t) + B \sin(\lambda t) = R \cos(\lambda t - \phi)$*

Here is a demonstration that the sum of a *cosine* function and a *sine* function is a new sine wave. By sine wave we mean a function whose shape is the same as the sine function, though it may be displaced along the time axis. First, consider the line segment  $A$  spinning in circles about the origin at rate  $\lambda$ ; that is, the angle the segment makes with the positive  $x$  axis is  $\lambda t$ . The projection of that segment onto the  $x$  axis is  $A \cos(\lambda t)$ . Now consider the segment labeled  $B$  in the figure, glued at a right angle to  $A$ . The length of its projection on the  $x$ -axis is  $B \sin(\lambda t)$ . So, the sum of these two projections is  $A \cos(\lambda t) + B \sin(\lambda t)$ . The two segments  $A$  and  $B$  make up a right triangle with diagonal  $R = \sqrt{A^2 + B^2}$ .

The projection or ‘shadow’ of  $R$  on the  $x$  axis is the same as the sum of the shadows of  $A$  and  $B$ . The angle it makes with the  $x$  axis is  $\lambda t - \phi$  where one can see from the triangle drawn that  $\phi = \arctan(B/A)$ . So, by adding the shadow lengths, we see  $A \cos(\lambda t) + B \sin(\lambda t) = \sqrt{A^2 + B^2} \cos(\lambda t - \phi)$ .



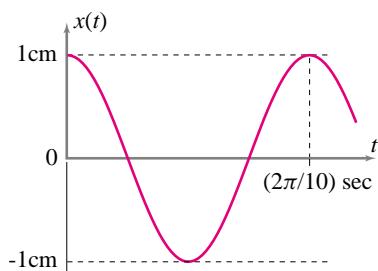


Figure 5.16: The position of a mass as a function of time if  $k = 50 \text{ N/m}$ ,  $m = 0.5 \text{ kg}$ ,  $x(0) = 1 \text{ cm}$  and  $v(0) = 0$ .

(Filename:tfigure12.cosine)

**① Caution:** It is tempting, but wrong, to evaluate  $x(t)$  at  $t = 0$  and then differentiate to get  $v(0)$ . This procedure is wrong because  $x(0)$  is just a number, differentiating it would always give zero, even when the initial velocity is not zero.

### Release from rest

The simplest motion to consider is when the spring is stretched a given amount and the mass is *released from rest*, meaning the initial velocity of the mass is zero. For example, if the mass in figure 5.14 is 0.5 kg, the spring constant is  $k = 50 \text{ N/m}$ , and the initial displacement is 2 cm, we find the motion by looking at the general solution

$$x(t) = A \cos(\sqrt{(k/m)} t) + B \sin(\sqrt{(k/m)} t).$$

At  $t = 0$ , this general solution has to agree with the initial condition that the displacement is 1 cm, so

$$x(0) = A \underbrace{\cos(0)}_1 + B \underbrace{\sin(0)}_0 = A \quad \Rightarrow \quad A = 2 \text{ cm}.$$

The initial velocity must also match, so first we find the velocity by differentiating the position to get

$$v(t) = \dot{x}(t) = -A\sqrt{(k/m)} \sin(\sqrt{(k/m)} t) + B\sqrt{(k/m)} \cos(\sqrt{(k/m)} t).$$

Now, we evaluate this expression at  $t = 0$  and set it equal to the given initial velocity which in this case was zero: ①

$$v(0) = -A\sqrt{(k/m)} \underbrace{\sin(0)}_0 + B\sqrt{(k/m)} \underbrace{\cos(0)}_1 = B\sqrt{(k/m)} \quad \Rightarrow \quad B = 0.$$

Substituting in the values for  $k = 5 \text{ N/m}$  and  $m = 0.5 \text{ kg}$ , we get

$$x(t) = 2 \cos \left( \sqrt{\frac{0.5 \text{ kg}}{50 \text{ N/m}}} t \right) \text{ cm} = 2 \cos(0.1t / \text{s}) \text{ cm}$$

which is plotted in figure 5.16.

### Initial velocity with no spring stretch

Another simple case is when the spring has no initial stretch but the mass has some initial velocity. Such might be the case just after a resting mass is hit by a hammer. Using the same 0.5 kg mass and  $k = 50 \text{ N/m}$  spring, we now consider an initial displacement of zero but an initial velocity of 10 cm/s. We can find the motion for this case from the general solution by the same procedure we just used. We get

$$x(t) = B \sin(\sqrt{(k/m)} t)$$

with  $B\sqrt{(k/m)} = 10 \text{ cm/s} \quad \Rightarrow \quad B = 1 \text{ cm}$ . The resulting motion,  $x(t) = (1 \text{ cm}) \cdot \sin(\frac{0.1t}{s})$ , is shown in figure 5.17.

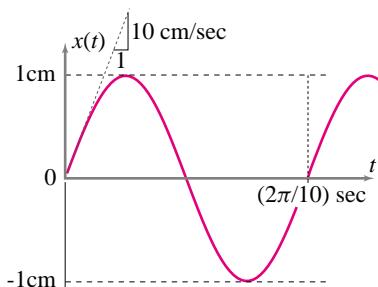


Figure 5.17: The position of a mass as a function of time if  $k = 50 \text{ N/m}$ ,  $m = 0.5 \text{ kg}$ ,  $x(0) = 0$  and  $v(0) = 10 \text{ cm/s}$ .

(Filename:tfigure12.sine)

## Work, energy, and the harmonic oscillator

Various energy concepts give another viewpoint for looking at the harmonic oscillator. We can derive energy balance from momentum balance. Or, if we already trust energy balance, we can use it instead of momentum balance to derive the governing differential equation. Energy balance can be used as a check of a solution. Energy accounting gives an extra intuitive way to think about what happens in an oscillator.

### The work of a spring

Associated with the force of a spring on a mass is a potential energy. Because the force of a spring on a mass is  $-kx$ , and the work of a force on a mass is  $\int_0^x F(x')dx'$  we find the potential for work, measured from the relaxed state  $x = 0$ , on the mass to be

$$E_P = - \int_0^x F(x') dx' - \int_0^x -kx' dx' = \frac{1}{2}kx^2.$$

### Conservation of energy

Because there is no damping or dissipation, the total mechanical energy of the harmonic oscillator is constant in time. That is, the sum of the kinetic energy  $E_K = \frac{1}{2}mv^2$  and the potential energy  $E_P = \frac{1}{2}k(\Delta L)^2$  is constant.

$$E_T = E_K + E_P = \text{constant}.$$

As the oscillation progresses, energy is exchanged back and forth between kinetic and potential energy. At the extremes in the displacement, the spring is most stretched, the potential energy is at a maximum, and the kinetic energy is zero. When the mass passes through the center position the spring is relaxed, so the potential energy is at a minimum (zero), the mass is at its maximum speed, and the kinetic energy reaches its maximum value.

Although energy conservation is a basic principle, this is a case where it can be derived, or more easily, checked. Using the special case where the motion starts from rest (i.e.,  $x(t) = A \cos(\sqrt{k/m} t)$ ), we can make sure that the total energy really is constant.

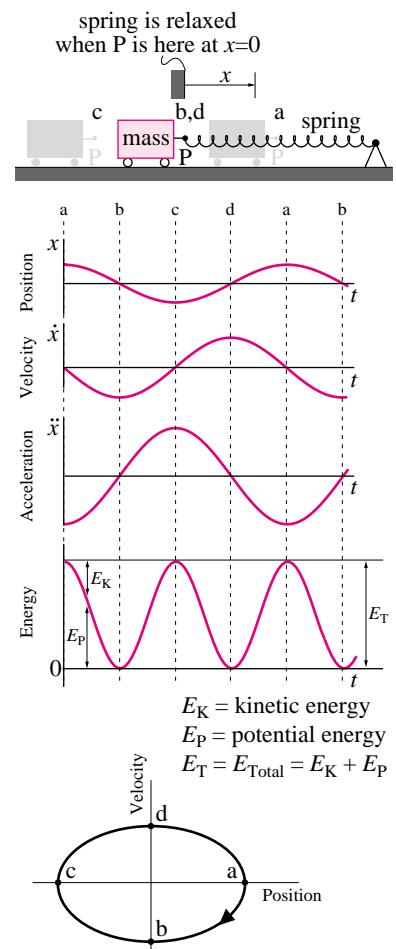
$$\begin{aligned} E_T &= E_K + E_P \\ &= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\ &= \frac{1}{2}k(A \cos(\sqrt{k/m} t))^2 + \frac{1}{2}m(A \sqrt{k/m} \sin(\sqrt{k/m} t))^2 \\ &= \frac{1}{2}kA^2 \underbrace{\{\cos^2(\sqrt{k/m} t) + \sin^2(\sqrt{k/m} t)\}}_1 \\ &= \frac{1}{2}kA^2 = \text{initial energy in spring} \end{aligned}$$

which does not change with time.

### Using energy to derive the oscillator equation

Rather than just checking the energy balance, we could use the energy balance to help us find the equations of motion. As for all one-degree-of-freedom systems, the equations of motion can be derived by taking the time derivative of the energy balance equation. Starting from  $E_T = \text{constant}$ , we get

$$\begin{aligned} 0 &= \frac{d}{dt} E_T \\ &= \frac{d}{dt} (E_P + E_K) \\ &= \frac{d}{dt} \left( \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \right) \\ &= kx \underbrace{\dot{x}}_v + mv \underbrace{\ddot{x}}_a \end{aligned}$$



Cross plot or phase plane portrait

Figure 5.18: Various plots of the motion of the harmonic oscillator. Points a,b,c,d show what is happening at different parts of the motion. The spring is relaxed at  $x = 0$ . Some things to note are the following: The acceleration curve is proportional to the negative of the displacement curve. The displacement is at a maximum or minimum when the velocity is zero. The velocity is at a maximum or minimum when the displacement is zero. The kinetic and potential energy fluctuate at twice the frequency as the position. The motion is an ellipse in the cross plot of velocity vs. position.

(Filename: tfigure12.oscpplots)

$$= kx\dot{y} + m\dot{y} \underbrace{a}_{\ddot{x}}$$

$$0 = kx + m\ddot{x}$$

which is the differential equation for the harmonic oscillator. (A technical defect of this derivation is that it does not apply at the instant when  $v = 0$ .)

Power balance can also be used as a starting point to find the harmonic oscillator equation. Referring to the FBD in figure 5.14, the equation of energy balance for the block during its motion after release is:

$\overbrace{P}^{\text{Power in}}$	$=$	$\overbrace{\dot{E}_K}^{\text{Rate of change of internal energy}}$
-----------------------------------	-----	--

$$\vec{F}_{spring} \cdot \vec{v}_A = \frac{d}{dt} \left( \frac{1}{2} m \vec{v}_A \cdot \vec{v}_A \right)$$

$$-kx_A \hat{i} \cdot \dot{x}_A \hat{i} = \frac{d}{dt} \left( \frac{1}{2} m \dot{x}_A^2 \right)$$

$$-kx_A \dot{x}_A = m \dot{x}_A \ddot{x}_A$$

Dividing both sides by  $\dot{x}_A$  (assuming it is not zero), we again get

$$-kx_A = m\ddot{x}_A \quad \text{or} \quad m\ddot{x}_A + kx_A = 0,$$

the familiar equation of motion for a spring-mass system.

We can now talk through a cycle of oscillation in terms of work and energy. Let's assume the block is released from rest at  $x = x_A > 0$ .

After the mass is released, the mass begins to move to the left and the spring does positive work on the mass since the motion and the force are in the same direction. After the block passes through the rest point  $x = O$ , it does work on the spring until it comes to rest at its left extreme. The spring then commences to do work on the block again as the block gains kinetic energy in its rightward motion. The block then passes through the rest position and does work on the spring until its kinetic energy is all used up and it is back in its rest position.

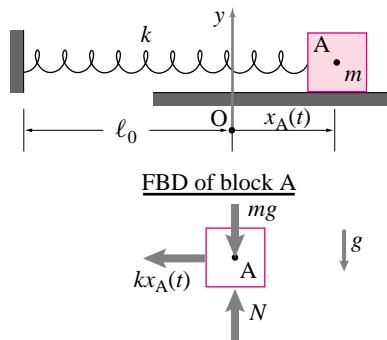


Figure 5.19: (Filename:t.ex.2.7.1)

#### A spring-mass system with gravity

When a mass is attached to a spring but gravity also acts one has to take some care to get things right (see fig. 5.20). Once a good free body diagram is drawn using well defined coordinates, all else follows easily.

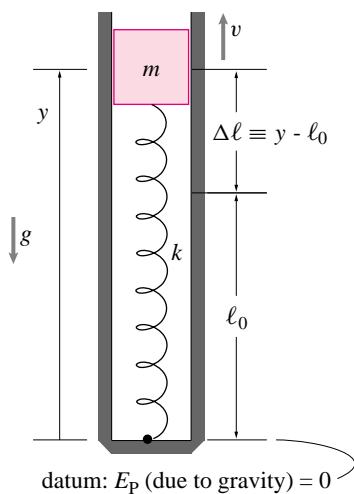


Figure 5.20: Spring and mass with gravity.  
(Filename:t.ex.2.6.1)



**SAMPLE 5.11** *Math review: Solution of a second order ODE:* Solve the equation:

$$\ddot{x} + k^2x = 0, \quad \text{with initial conditions } x(0) = x_0, \quad \dot{x}(0) = u_0. \quad (5.20)$$

**Solution** Let us guess a solution. We need a function  $x(t)$  whose second derivative is equal to  $-k^2$  times the function itself. We know at least two such functions: *sine* and *cosine*. To check, let

$$\begin{aligned} x(t) &= \sin kt \\ \Rightarrow \ddot{x} &= -k^2 \sin kt = -k^2 x. \end{aligned}$$

Similarly, let

$$\begin{aligned} x(t) &= \cos kt \\ \Rightarrow \ddot{x} &= -k^2 \cos kt = -k^2 x. \end{aligned}$$

Thus both functions satisfy the equation. Because Eqn. (5.20) is a linear differential equation, a linear combination of the two solutions will also satisfy it. Therefore, let

$$x(t) = A \sin kt + B \cos kt. \quad (5.21)$$

Substituting in Eqn. (5.20), we get

$$\ddot{x} + k^2x = -Ak^2 \sin kt - Bk^2 \cos kt + k^2(A \sin kt + B \cos kt) = 0,$$

which shows that the solution in Eqn. (5.21) satisfies the given differential equation. Now we evaluate the two constants  $A$  and  $B$  using the given initial conditions.

$$\begin{aligned} x(0) = x_0 &= A \cdot 0 + B \cdot 1 \\ \Rightarrow B &= x_0 \\ \dot{x}(0) = u_0 &= (Ak \cos kt - Bk \sin kt)|_{t=0} \\ &= Ak \cdot 1 - Bk \cdot 0 \\ \Rightarrow A &= \frac{u_0}{k}. \end{aligned}$$

Therefore, the solution is

$$x(t) = \frac{u_0}{k} \sin kt + x_0 \cos kt.$$

$x(t) = \frac{u_0}{k} \sin kt + x_0 \cos kt$

**Alternatively**, you could also guess  $x(t) = e^{rt}$ , plug it into the given equation, and find that you must have  $r = \pm ik$  satisfy the equation. Now take a linear combination of the two solutions, say  $x(t) = A e^{ikt} + B e^{-ikt}$ , and find the constants  $A$  and  $B$  from the given initial conditions.

**SAMPLE 5.12** A block of mass  $m = 20 \text{ kg}$  is attached to two identical springs each with spring constant  $k = 1 \text{ kN/m}$ . The block slides on a horizontal surface without any friction.

- Find the equation of motion of the block.
- What is the oscillation frequency of the block?
- How much time does the block take to go back and forth 10 times?

### Solution

- (a) The free body diagram of the block is shown in Figure 5.22. The linear momentum balance,  $\sum \vec{F} = m\vec{a}$ , for the block gives

$$-2kx\hat{i} + (N - mg)\hat{j} = m\vec{a}$$

Dotting both sides with  $\hat{i}$  we have,

$$-2kx = ma_x = m\ddot{x} \quad (5.22)$$

$$\text{or } m\ddot{x} + 2kx = 0 \quad (5.23)$$

$$\text{or } \ddot{x} + \frac{2k}{m}x = 0. \quad (5.24)$$

$$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$

- (b) Comparing Eqn. (5.24) with the standard harmonic oscillator equation,  $\ddot{x} + \lambda^2 x = 0$ , where  $\lambda$  is the oscillation frequency, we get

$$\begin{aligned} \lambda^2 &= \frac{2k}{m} \\ \Rightarrow \lambda &= \sqrt{\frac{2k}{m}} \\ &= \sqrt{\frac{2 \cdot (1 \text{ kN/m})}{20 \text{ kg}}} \\ &= 10 \text{ rad/s}. \end{aligned}$$

$$\boxed{\lambda = 10 \text{ rad/s}}$$

- (c) Time period of oscillation  $T = \frac{2\pi}{\lambda} = \frac{2\pi}{10 \text{ rad/s}} = \frac{\pi}{5} \text{ s}$ . Since the time period represents the time the mass takes to go back and forth just once, the time it takes to go back and forth 10 times (*i.e.*, to complete 10 cycles of motion) is

$$t = 10T = 10 \cdot \frac{\pi}{5} \text{ s} = 2\pi \text{ s.}$$

$$\boxed{t = 2\pi \text{ s}}$$

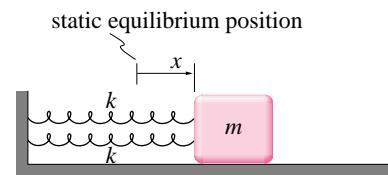


Figure 5.21: (Filename:sfig10.1.1.1)

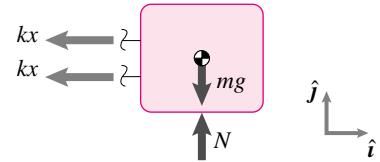


Figure 5.22: (Filename:sfig10.1.1.1a)

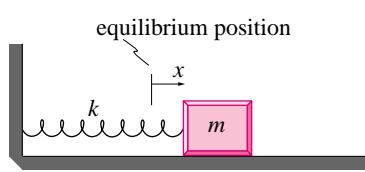


Figure 5.23: (Filename:sfig10.1.3)

**SAMPLE 5.13** A spring-mass system executes simple harmonic motion:  $x(t) = A \cos(\lambda t - \phi)$ . The system starts with initial conditions  $x(0) = 25 \text{ mm}$  and  $\dot{x}(0) = 160 \text{ mm/s}$  and oscillates at the rate of 2 cycles/sec.

- Find the time period of oscillation and the oscillation frequency  $\lambda$ .
- Find the amplitude of oscillation  $A$  and the phase angle  $\phi$ .
- Find the displacement, velocity, and acceleration of the mass at  $t = 1.5 \text{ s}$ .
- Find the maximum speed and acceleration of the system.
- Draw an accurate plot of displacement *vs.* time of the system and label all relevant quantities. What does  $\phi$  signify in this plot?

### Solution

- (a) We are given  $f = 2 \text{ Hz}$ . Therefore, the time period of oscillation is

$$T = \frac{1}{f} = \frac{1}{2 \text{ Hz}} = 0.5 \text{ s},$$

and the oscillation frequency  $\lambda = 2\pi f = 4\pi \text{ rad/s}$ .

$T = 0.5 \text{ s}, \quad \lambda = 4\pi \text{ rad/s.}$

- (b) The displacement  $x(t)$  of the mass is given by

$$x(t) = A \cos(\lambda t - \phi).$$

Therefore the velocity (actually the speed) is

$$\dot{x}(t) = -A\lambda \sin(\lambda t - \phi)$$

At  $t = 0$ , we have

$$x(0) = A \cos(-\phi) = A \cos \phi \quad (5.25)$$

$$\dot{x}(0) = -A\lambda \sin(-\phi) = A\lambda \sin \phi \quad (5.26)$$

By squaring Eqn (5.25) and adding it to the square of [Eqn (5.26) divided by  $\lambda$ ], we get

$$\begin{aligned} A^2 \cos^2 \phi + \frac{A^2 \lambda^2 \sin^2 \phi}{\lambda^2} &= A^2 = x^2(0) + \frac{\dot{x}^2(0)}{\lambda^2} \\ \Rightarrow A &= \sqrt{(25 \text{ mm})^2 + \frac{(160 \text{ mm/s})^2}{(4\pi \text{ rad/s})^2}} \\ &= 28.06 \text{ mm}. \end{aligned}$$

Substituting the value of  $A$  in Eqn (5.25), we get

$$\begin{aligned} \phi &= \cos^{-1} \frac{x(0)}{A} \\ &= \cos^{-1} \frac{25 \text{ mm}}{28.06 \text{ mm}} \\ &= 0.471 \text{ rad} \approx 27^\circ. \end{aligned}$$

$A = 28.06 \text{ mm.} \quad \phi = 0.471 \text{ rad.}$

- (c) The displacement, velocity, and acceleration of the mass at any time  $t$  can now be calculated as follows

$$\begin{aligned}x(t) &= A \cos(\lambda t - \phi) \\ \Rightarrow x(1.5 \text{ s}) &= 28.06 \text{ mm} \cdot \cos(6\pi - 0.471) \\ &= 25 \text{ mm}.\end{aligned}$$

$$\begin{aligned}\dot{x}(t) &= -A\lambda \sin(\lambda t - \phi) \\ \Rightarrow \dot{x}(1.5 \text{ s}) &= 28.06 \text{ mm} \cdot (4\pi \text{ rad/s}) \cdot \sin(6\pi - 0.471) \\ &= 160 \text{ mm/s}.\end{aligned}$$

$$\begin{aligned}\ddot{x}(t) &= -A\lambda^2 \cos(\lambda t - \phi) \\ \Rightarrow \ddot{x}(1.5 \text{ s}) &= 28.06 \text{ mm} \cdot (4\pi \text{ rad/s})^2 \cdot \cos(6\pi - 0.471) \\ &= -3.95 \times 10^3 \text{ mm/s}^2 \\ &= -3.95 \text{ m/s}^2.\end{aligned}$$

①

$$x(1.5 \text{ s}) = 25 \text{ mm}, \quad \dot{x}(1.5 \text{ s}) = 160 \text{ mm/s}, \quad \ddot{x}(1.5 \text{ s}) = -3.93 \text{ m/s}^2.$$

- (d) Maximum speed:

$$|\dot{x}_{\max}| = A\lambda = (28.06 \text{ mm}) \cdot (4\pi \text{ rad/s}) = 0.35 \text{ m/s}.$$

Maximum acceleration:

$$|\ddot{x}_{\max}| = A\lambda^2 = (28.06 \text{ mm}) \cdot (4\pi \text{ rad/s})^2 = 4.43 \text{ m/s}^2.$$

$$|\dot{x}_{\max}| = 0.35 \text{ m/s}, \quad |\ddot{x}_{\max}| = 4.43 \text{ m/s}^2.$$

- (e) The plot of  $x(t)$  versus  $t$  is shown in Fig. 5.24. The phase angle  $\phi$  represents the shift in  $\cos(\lambda t)$  to the right by an amount  $\frac{\phi}{\lambda}$ .

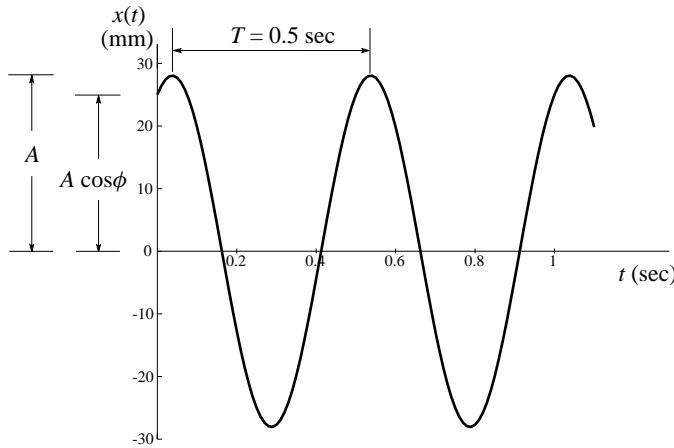


Figure 5.24: (Filename:fig10.1.3a)

① We can find the displacement and velocity at  $t = 1.5 \text{ s}$  without any differentiation. Note that the system completes 2 cycles in 1 second, implying that it will complete 3 cycles in 1.5 seconds. Therefore, at  $t = 1.5 \text{ s}$ , it has the same displacement and velocity as it had at  $t = 0 \text{ s}$ .

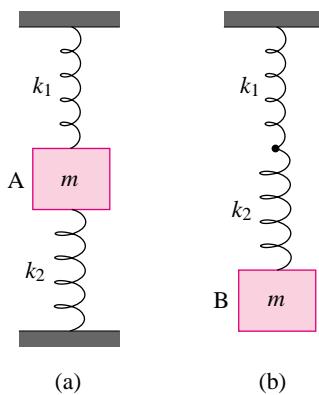
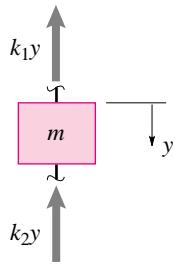
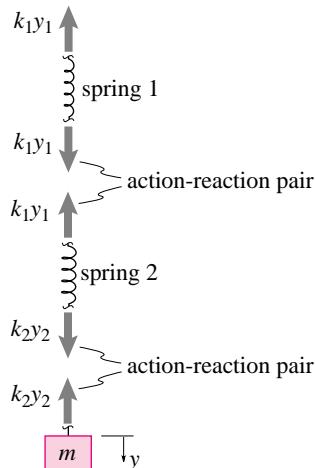


Figure 5.25: (Filename:sfig3.4.2)

Figure 5.26: Free body diagram of the mass.  
(Filename:sfig3.4.2a)Figure 5.27: Free body diagrams  
(Filename:sfig3.4.2b)

**SAMPLE 5.14 Springs in series versus springs in parallel:** Two massless springs with spring constants  $k_1$  and  $k_2$  are attached to mass A in parallel (although they look superficially as if they are in series) as shown in Fig. 5.25. An identical pair of springs is attached to mass B in series. Taking  $m_A = m_B = m$ , find and compare the natural frequencies of the two systems. Ignore gravity.

**Solution** Let us pull each mass downwards by a small vertical distance  $y$  and then release. Measuring  $y$  to be positive downwards, we can derive the equations of motion for each mass by writing the Balance of Linear Momentum for each as follows.

- **Mass A:** The free body diagram of mass A is shown in Fig. 5.26. As the mass is displaced downwards by  $y$ , spring 1 gets stretched by  $y$  whereas spring 2 gets compressed by  $y$ . Therefore, the forces applied by the two springs,  $k_1y$  and  $k_2y$ , are in the same direction. The LMB of mass A in the vertical direction gives:

$$\begin{aligned} \sum F &= ma_y \\ \text{or } -k_1y - k_2y &= m\ddot{y} \\ \text{or } \ddot{y} + \left(\frac{k_1 + k_2}{m}\right)y &= 0. \end{aligned}$$

Let the natural frequency of this system be  $\omega_p$ . Comparing with the standard simple harmonic equation  $\ddot{x} + \lambda^2 x = 0$  we get the natural frequency ( $\lambda$ ) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \quad (5.27)$$

$$\boxed{\omega_p = \sqrt{\frac{k_1+k_2}{m}}}$$

- **Mass B:** The free body diagram of mass B and the two springs is shown in Fig. 5.27. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be  $y_1$  and in spring 2 be  $y_2$ .  $y_1$  and  $y_2$  are unknown, of course, but we know that

$$y_1 + y_2 = y \quad (5.28)$$

Now, using the free body diagram of spring 2 and then writing linear momentum balance we get,

$$\begin{aligned} k_2y_2 - k_1y_1 &= \underbrace{m}_0 a = 0 \\ y_1 &= \frac{k_2}{k_1}y_2 \end{aligned} \quad (5.29)$$

Solving (5.28) and (5.29) we get

$$y_2 = \frac{k_1}{k_1 + k_2}y.$$

Now, linear momentum balance of mass B in the vertical direction gives:

$$\begin{aligned} -k_2y_2 &= ma_y = m\ddot{y} \\ \text{or } m\ddot{y} + k_2 \underbrace{\frac{k_1}{k_1 + k_2}y}_{y_2} &= 0 \\ \text{or } \ddot{y} + \frac{k_1k_2}{m(k_1 + k_2)}y &= 0. \end{aligned} \quad (5.30)$$

Let the natural frequency of this system be denoted by  $\omega_s$ . Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}. \quad (5.31)$$

$$\boxed{\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}}$$

From (5.27) and (5.31)

$$\frac{\omega_p}{\omega_s} = \frac{k_1 + k_2}{\sqrt{k_1 k_2}}.$$

Let  $k_1 = k_2 = k$ . Then,  $\omega_p/\omega_s = 2$ , i.e., the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the forces add and in the other they don't and each spring is stretched less. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B.

#### Comments:

- (a) Although the springs attached to mass A do not visually seem to be in parallel, from mechanics point of view they are parallel. You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* but the *same force*.
- (b) When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, i.e., the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by  $k_{pe}$  and  $k_{se}$  respectively. By comparing eqns. (5.27) and (5.31) with the expression for natural frequency of a simple spring mass system, we see that

$$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

These expressions can be easily extended for any arbitrary number of springs, say,  $N$  springs:

$$k_{pe} = k_1 + k_2 + \dots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_N}.$$

**SAMPLE 5.15** Figure 5.28 shows two responses obtained from experiments on two spring-mass systems. For each system

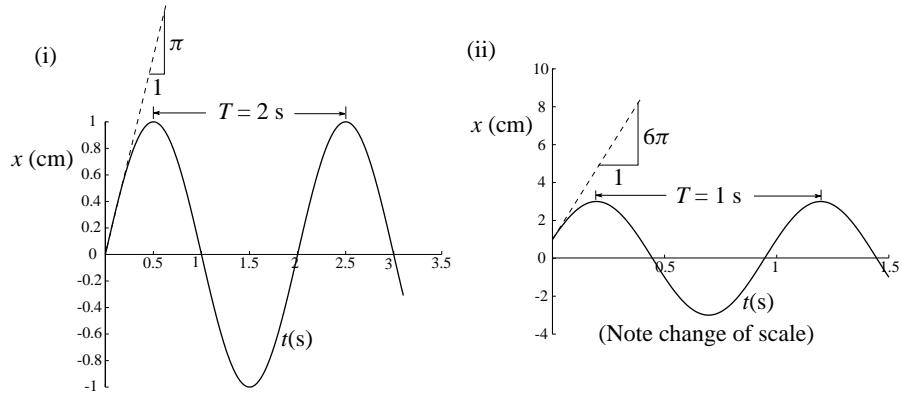


Figure 5.28: (Filename:fig10.1.4)

- (a) Find the natural frequency.
- (b) Find the initial conditions.

### Solution

- (a) **Natural frequency:** By definition, the natural frequency  $f$  is the number of cycles the system completes in one second. From the given responses we see that:

**Case(i):** the system completes  $\frac{1}{2}$  a cycle in 1 s.

$$\Rightarrow f = \frac{1}{2} \text{ Hz.}$$

**Case(ii):** the system completes 1 cycle in 1 s.

$$\Rightarrow f = 1 \text{ Hz.}$$

It is usually hard to measure the fraction of cycle occurring in a short time. It is easier to first find the time period, *i.e.*, the time taken to complete 1 cycle.

① Then the natural frequency can be found by the formula  $f = \frac{1}{T}$ . From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 5.28 we find that for

**Case (i):**

$$f = \frac{1}{T} = \frac{1}{2 \text{ s}} = \frac{1}{2} \text{ Hz,}$$

**Case (ii):**

$$f = \frac{1}{T} = \frac{1}{1 \text{ s}} = 1 \text{ Hz}$$

$\boxed{\text{case (i) } f = \frac{1}{2} \text{ Hz.} \quad \text{case (ii) } f = 1 \text{ Hz.}}$

- (b) **Initial conditions:** Now we are to find the displacement and velocity at  $t = 0$  s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at  $t = 0$  from the plots:

**Case (i):**  $x(0) = 0$ .

**Case (ii):**  $x(0) = 1 \text{ cm}$ .

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at  $t = 0$ .

**Case (i):**  $\dot{x}(0) = \frac{dx}{dt}(t = 0) = \frac{\pi \text{ cm}}{1 \text{ s}} = 3.14 \text{ cm/s}$ .

**Case (ii):**  $\dot{x}(0) = \frac{dx}{dt}(t = 0) = \frac{6\pi \text{ cm}}{1 \text{ s}} = 18.85 \text{ cm/s}$ .

Thus the initial conditions are

Case (i)  $x(0) = 0, \dot{x}(0) = 3.14 \text{ cm/s}$ . Case (ii)  $x(0) = 1 \text{ cm}, \dot{x}(0) = 18.85 \text{ cm/s}$ .

**Comments:** Estimating the speed from the initial slope of the displacement curve at  $t = 0$  is not a very good method because it is hard to draw an accurate tangent to the curve at  $t = 0$ . A slightly different line but still seemingly tangential to the curve at  $t = 0$  can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

**Case(i):** We know that  $x(0) = 0$ . Therefore the entire energy at  $t = 0$  is the kinetic energy  $= \frac{1}{2}mv_0^2$ . At  $t = 0.5 \text{ s}$  we note that the displacement is maximum, *i.e.*, the speed is zero. Therefore, the entire energy is potential energy  $= \frac{1}{2}kx^2$ , where  $x = x(t = 0.5 \text{ s}) = 1 \text{ cm}$ .

Now, from the conservation of energy:

$$\begin{aligned}\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5 \text{ s}})^2 \\ \Rightarrow v_0 &= \sqrt{\frac{k}{m}} \cdot (x_{t=0.5 \text{ s}}) \\ &= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda} \cdot (1 \text{ cm}) \\ &= 2\pi f \cdot (1 \text{ cm}) \\ &= 2\pi \cdot \frac{1}{2} \text{ Hz} \cdot 1 \text{ cm} \\ &= 3.14 \text{ cm/s.}\end{aligned}$$

Similar calculations can be done for the second system.

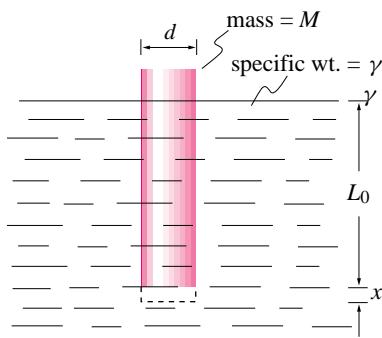


Figure 5.29: (Filename:sfig3.4.1)

**SAMPLE 5.16** *Simple harmonic motion of a buoy.* A cylinder of cross sectional area  $A$  and mass  $M$  is in static equilibrium inside a fluid of specific weight  $\gamma$  when  $L_o$  length of the cylinder is submerged in the fluid. From this position, the cylinder is pushed down vertically by a small amount  $x$  and let go. Assume that the only forces acting on the cylinder are gravity and the buoyant force and assume that the buoy's motion is purely vertical. Derive the equation of motion of the cylinder using Linear Momentum Balance. What is the period of oscillation of the cylinder?

**Solution** The free body diagram of the cylinder is shown in Fig. 5.30 where  $F_B$  represents the buoyant force. Before the cylinder is pushed down by  $x$ , the linear momentum balance of the cylinder gives

$$F_B - Mg = M \underbrace{\frac{a}{0}}_0 = 0 \Rightarrow F_B = Mg$$

Now  $F_B = (\text{volume of the displaced fluid}) \cdot (\text{its specific weight}) = AL_o\gamma$ . Thus,

$$AL_o\gamma = Mg. \quad (5.32)$$

Now, when the cylinder is pushed down by an amount  $x$ ,

$$F'_B = \text{new buoyant force} = (L_o + x)A\gamma.$$

Therefore, from LMB we get

$$\begin{aligned} F'_B - Mg &= -M\ddot{x} \\ \text{or } (L_o + x)A\gamma - Mg &= -M\ddot{x} \\ &\quad = 0 \text{ from (5.32).} \\ \text{or } M\ddot{x} + A\gamma x &= \overbrace{-AL_o\gamma + Mg}^{\text{from (5.32)}} \\ \text{or } M\ddot{x} + A\gamma x &= 0 \\ \text{or } \ddot{x} + \frac{A\gamma}{M}x &= 0. \end{aligned}$$

$$\boxed{\ddot{x} + \frac{A\gamma}{M}x = 0}$$

Comparing this equation with the standard simple harmonic equation (*e.g.*, eqn.(g), in the box on ODE's on page 226).

$$\text{The circular frequency } \lambda = \sqrt{\frac{A\gamma}{M}},$$

$$\text{Therefore, the period of oscillation } T = \frac{2\pi}{\lambda} = 2\pi \sqrt{\frac{M}{A\gamma}}$$

$$\boxed{T = 2\pi \sqrt{\frac{M}{A\gamma}}}$$

**Comments:** Note this calculation neglects the fluid mechanics. The common way of making a correction is to use 'added mass' to account for fluid that moves more-or-less with the cylinder. The added mass is usually something like one-half the mass of the fluid with volume equal to that of the cylinder. Another way to see the error is to realize that the pressure used in this calculation assumes fluid statics when in fact the fluid is moving.

## 5.4 More on vibrations: damping

The mother of all vibrating machines is the simple harmonic oscillator from the previous section. With varying degrees of approximation, car suspensions, violin strings, buildings responding to earthquakes, earthquake faults themselves, and vibrating machines are modeled as mass-spring-dashpot systems. Almost all of the concepts in vibration theory are based on concepts associated with the behavior of the harmonic oscillator. The harmonic oscillator has no friction or inelastic deformation so that mechanical energy is conserved. Such vibrations will, once started, persist forever even with no pushing, pumping, or energy supply of any kind. Total lack of friction does not describe any real system perfectly, but it is a useful approximation if one is trying to understand the oscillations of a system and not the decay of those oscillations.

But for any real system the oscillations will decay in time due to friction. We would now like to study this decay.

### Damping

The simplest system to study is the *damped harmonic oscillator* and the motions that are of interest are *damped* oscillations.

Again the simplest model, and also the prototype of all models, is a spring and mass system. But now we add a component called a *damper* or *dashpot*, shown in figure 5.31. The dashpot provides resistance to motion by drawing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the opening that is related to the speed of the fluid flowing through. Ideally, this viscous resistance produces linear damping, meaning that the force is exactly proportional to the velocity. In a physical dashpot nonlinearities are introduced from the fluid flow and from friction between the piston and the cylinder. Also, dashpots that use air as a working fluid may have compressibility that introduces non-negligible springiness to the system in addition to that of any metallic springs.

Adding a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in figure 5.32. Figure 5.33 is a free body diagram of the mass. It has two forces acting on it, neglecting gravity:

$$\begin{aligned} F_s &= kx && \text{is the spring force, assuming a linear spring, and} \\ F_d &= c dx/dt = c\dot{x} && \text{is the dashpot force assuming a linear dashpot.} \end{aligned}$$

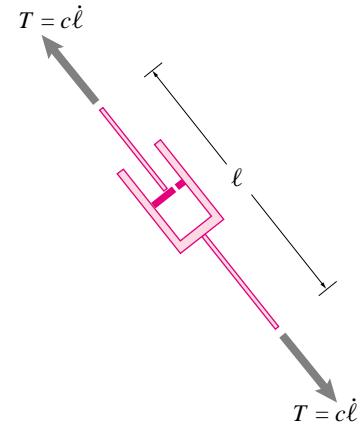
The system is a one degree of freedom system because a single coordinate  $x$  is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \quad (5.33)$$

Assuming a linear spring and a linear dashpot this expression becomes

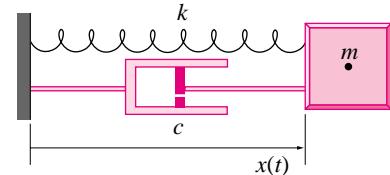
$$m\ddot{x} + c\dot{x} + kx = 0. \quad (5.34)$$

We have taken care with the signs of the various terms. You should check that you can derive equation 5.34 without introducing any sign errors. ①



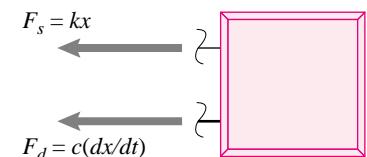
**Figure 5.31: A damper or dashpot.** The symbol shown represents a device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for  $\dot{L} > 0$  and  $\dot{L} < 0$ .

(Filename:tfigure12.dashpot)



**Figure 5.32: A mass spring dashpot system, or damped harmonic oscillator.**

(Filename:tfigure12.MSD)



**Figure 5.33: Free body diagram of the mass spring dashpot system.**

(Filename:tfigure12.MSDFBD)

①**Caution:** When push comes to shove, so to speak, many students have trouble deriving equations like 5.34 without getting sign errors from figures like 5.32.

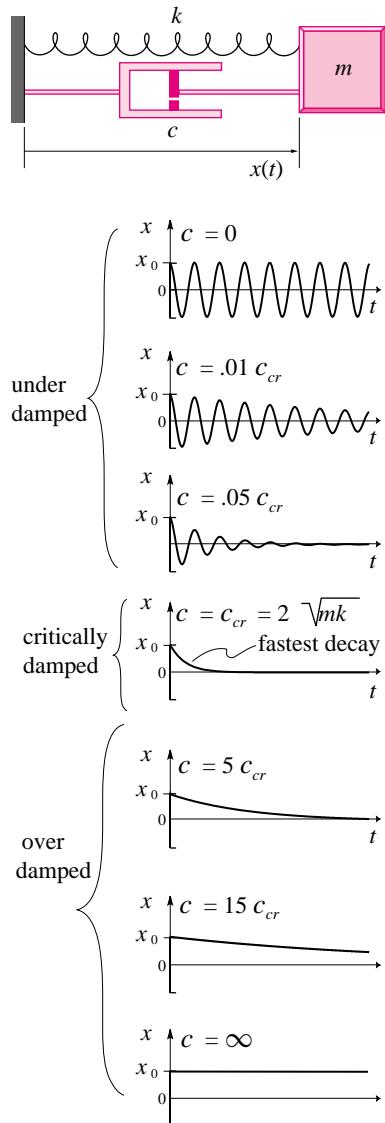


Figure 5.34: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at  $x = x_0$ . In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of  $c \rightarrow \infty$ . The fastest relaxation occurs for critical damping.

(Filename:tfigure12.damping)

## Solution of the damped-oscillator equations

The governing equation 5.34 has a solution which depends on the values of the constants. There are cases where one wants to consider negative springs or negative dashpots, but for the purposes of understanding classical vibration theory we can assume that  $m$ ,  $c$ , and  $k$  are all positive. Even with this restriction the solution depends on the relative values of  $m$ ,  $c$ , and  $k$ . You can learn all about these solutions in any book that introduces ordinary differential equations; most freshman calculus books have such a discussion.

The three solutions are categorized as follows:

- *Under-damped*:  $c^2 < 4mk$ . In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

$$x(t) = e^{(-\frac{c}{2m})t} [A \cos(\lambda_d t) + B \sin(\lambda_d t)], \quad (5.35)$$

where  $\lambda_d$  is the damped natural frequency and is given by  $\lambda_d = \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}$ .

- *Critically damped*:  $c^2 = 4mk$ . In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

$$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \quad (5.36)$$

- *Over-damped*:  $c^2 > 4mk$ . Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{(-\frac{c}{2m} + \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}})t} + Be^{(-\frac{c}{2m} - \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}})t}. \quad (5.37)$$

The solution 5.37 actually includes equations 5.36 and 5.35 as special cases. To interpret equation 5.37 as the general solution you need to know the relation between complex exponentials and trigonometric functions for the cases when the argument of the square root term is negative.

For a given mass and spring we can imagine the damping as a variable to adjust. A system which has small damping (small  $c$ ) is *under-damped* and does not come to equilibrium quickly because oscillations persist for a long time. A system which has a lot of damping (big  $c$ ) is *over-damped* does not come to equilibrium quickly because the dashpot holds it away from equilibrium. A system which is *critically-damped* comes to equilibrium most quickly. In many cases, the purpose of damping is to purge motions after disturbance from equilibrium. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished by picking  $c = \sqrt{4km}$  and achieving critical damping. This damping design is commonly employed.

## Measurement of damping: logarithmic decrement method

In the under-damped case, the viscous damping constant  $c$  may be determined experimentally by measuring the rate of decay of unforced oscillations. This decay can be quantified using the logarithmic decrement. The logarithmic decrement is the natural logarithm of the ratio of any two successive amplitudes. The larger the damping, the greater will be the rate of decay of oscillations and the bigger the logarithmic decrement:

$$\text{logarithmic decrement} \equiv D = \ln\left(\frac{x_n}{x_{n+1}}\right) \quad (5.38)$$

where  $x_n$  and  $x_{n+1}$  are the heights of two successive peaks in the decaying oscillation pictured in figure 5.35. Because of the exponential envelope that this curve has,  $x_n = (\text{const.})e^{-(\frac{c}{2m})t_1}$  and  $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})t_1+T}$ .

$$D = \ln[(e^{-(\frac{c}{2m})t_1})/(e^{-(\frac{c}{2m})t_1+T})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where  $T$  is the period of oscillation. Thus, the damping constant  $c$  can be measured by measuring the logarithmic decrement  $D$  and the period of oscillation  $T$  as

$$c = \frac{2mD}{T}.$$

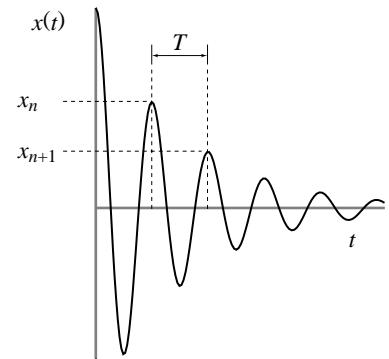


Figure 5.35: The logarithmic decrement method.  $D = \ln(x_n/x_{n+1})$

(Filename:tfigure12.decrement)

## Summary of equations for the unforced harmonic oscillator

- $\ddot{x} + \frac{k}{m}x = 0$ , mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$ , harmonic oscillator equation
- $x(t) = A \cos(\lambda t) + B \sin(\lambda t)$ , general solution to harmonic oscillator equation
- $x(t) = R \cos(\lambda t - \phi)$ , amplitude-phase version of solution to harmonic oscillator solution,  $R = \sqrt{A^2 + B^2}$ ,  $\phi = \tan^{-1}(\frac{B}{A})$
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$ , mass-spring-dashpot equation (see equations 5.35-5.37 for solutions)
- $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ , logarithmic decrement.  $c = \frac{2mD}{T}$ .

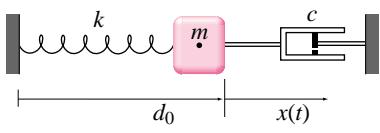


Figure 5.36: Spring-mass dashpot.  
(Filename:sfig10.2.1)

**SAMPLE 5.17** A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 5.36. The spring constant  $k = 1000 \text{ N/m}$  and a damping rate  $c = 50 \text{ N}\cdot\text{s/m}$ . When the block is at a distance  $d_0$  from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

- the equation of motion of the block.
- the position of the block at  $t = 2 \text{ s}$ .

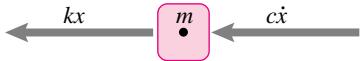


Figure 5.37: free body diagram of system at instant  $t$  goes here!  
(Filename:sfig10.2.1a)

### Solution

(a) Let  $x$  be the position of the block, measured positive to the right of the static equilibrium position, at some time  $t$ . Let  $\dot{x}$  be the corresponding speed. The free body diagram of the block at the instant  $t$  is shown in Figure 5.37. Since the motion is only horizontal, we can write the linear momentum balance in the  $x$ -direction ( $\sum F_x = m a_x$ ):

$$\begin{aligned} \underbrace{-kx - c\dot{x}}_{\sum F_x} &= m \underbrace{\ddot{x}}_{a_x} \\ \text{or } \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x &= 0 \end{aligned} \quad (5.39)$$

which is the desired equation of motion of the block.

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.$$

(b) To find the position and velocity of the block at any time  $t$  we need to solve Eqn (5.39). Since the solution depends on the relative values of  $m$ ,  $k$ , and  $c$ , we first compute  $c^2$  and compare with the *critical value*  $4mk$ .

$$\begin{aligned} c^2 &= 2500(\text{N}\cdot\text{s/m})^2 \\ \text{and } 4mk &= 4 \cdot 10 \text{ kg} \cdot 1000 \text{ N/m} = 4000(\text{N}\cdot\text{s/m})^2. \\ \Rightarrow c^2 &< 4mk. \end{aligned}$$

Therefore, the system is underdamped and we may write the general solution as

$$x(t) = e^{-\frac{c}{2m}t} [A \cos \lambda_D t + B \sin \lambda_D t] \quad (5.40)$$

where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682 \text{ rad/s.}$$

Substituting the initial conditions  $x(0) = 0.5 \text{ m}$  and  $\dot{x}(0) = 0 \text{ m/s}$  in Eqn (5.40) (we need to differentiate Eqn (5.40) first to substitute  $\dot{x}(0)$ ), we get

$$\begin{aligned} x(0) &= 0.5 \text{ m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m} \cdot A + \lambda_D \cdot B \\ \Rightarrow B &= \frac{A c}{2m \lambda_D} = \frac{(0.5 \text{ m}) \cdot (50 \text{ N}\cdot\text{s/m})}{2 \cdot (10 \text{ kg}) \cdot (9.682 \text{ rad/s})} = 0.13 \text{ m}. \end{aligned}$$

Thus, the solution is

$$x(t) = e^{(-2.5 \frac{1}{s})t} [0.50 \cos(9.68 \text{ rad/s} t) + 0.13 \sin(9.68 \text{ rad/s} t)] \text{ m.}$$

Substituting  $t = 2 \text{ s}$  in the above expression we get  $x(2 \text{ s}) = 0.003 \text{ m}$ .

$$x(2 \text{ s}) = 0.003 \text{ m.}$$

**SAMPLE 5.18** A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be  $x(t) = Ae^{-\xi\lambda t} \sin(\lambda_D t)$  where  $A = 0.3 \text{ m}$ ,  $\xi \equiv \text{damping ratio} = 0.02$ ,  $\lambda \equiv \text{undamped circular frequency} = 1 \text{ rad/s}$ , and  $\lambda_D \equiv \text{damped circular frequency} = \lambda\sqrt{1 - \xi^2} \approx \lambda$ .

- Find an expression for the ratio of energies of the system at the  $(n + 1)$ th displacement peak and the  $n$ th displacement peak.
- What percent of energy available at the first peak is lost after 5 cycles?

### Solution

- (a) We are given that

$$x(t) = Ae^{-\xi\lambda t} \sin(\lambda_D t).$$

The structure attains its first displacement peak when  $\sin \lambda_D t$  is maximum, i.e.,  $\lambda_D t = \frac{\pi}{2}$ , or  $t = \frac{\pi}{2\lambda_D}$ . At this instant,

$$x(t) = Ae^{-\xi\lambda\cdot\frac{\pi}{2\lambda_D}} = Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} = (0.3 \text{ m}) \cdot e^{-0.0314} = 0.29 \text{ m}.$$

Let  $x_n$  and  $x_{n+1}$  be the values of the displacement at the  $n$ th and the  $(n + 1)$ th peak, respectively. Since  $x_n$  and  $x_{n+1}$  are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \quad (5.41)$$

Let  $t_n$  be the time at which the  $n$ th peak displacement  $x_n$  is attained, i.e.,

$$x_n = Ae^{-\xi\lambda t_n} \quad (5.42)$$

Since  $x_{n+1}$  is the next peak displacement, it must occur at  $t = t_n + T_D$  where  $T_D$  is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n + T_D)} \quad (5.43)$$

From Eqns (5.41), (5.42), and (5.43)

$$\frac{E_{n+1}}{E_n} = \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n + T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} = e^{-2\xi\lambda T_D}.$$

$$\boxed{\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.}$$

- (b) Noting that  $T_D = \frac{2\pi}{\lambda_D}$  and  $\lambda_D = \lambda\sqrt{1 - \xi^2}$ , we get

$$E_{n+1} = E_n e^{-2\xi\lambda\cdot\frac{2\pi}{\lambda\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \Rightarrow E_{n+1} = e^{-4\pi\xi} E_n.$$

Applying this equation recursively for  $n = n - 1, n - 2, \dots, 1, 0$ , we get

$$E_n = e^{-4\pi\xi} \cdot E_{n-1} = e^{-4\pi\xi} \cdot (e^{-4\pi\xi} \cdot E_{n-2}) = (e^{-4\pi\xi})^3 \cdot E_{n-3} \dots = (e^{-4\pi\xi})^n \cdot E_0.$$

Now we use this equation to find the percentage of energy of the first peak ( $n = 0$ ) lost after 5 cycles ( $n = 5$ ):

$$\Delta E_5 = \frac{E_0 - E_5}{E_0} \times 100 = \left(1 - e^{-4\pi\xi \cdot 5}\right) \times 100 = 71.5\%.$$

$$\boxed{\Delta E_5 = 71.5\%.}$$

**SAMPLE 5.19 A SDOF spring-mass model from given data:** The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.

**Data:**

peak number $n$	0	1	2	3	4	5	6
time ( s )	0.0000	0.6279	1.2558	1.8837	2.5116	3.1395	3.7674
peak disp. ( m )	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443

**Solution** Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period  $T_D$  from which we can determine the damped circular frequency  $\lambda_D$ . From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \dots = 0.6279 \text{ s}$$

Therefore,

$$\begin{aligned} T_D &= 0.6279 \text{ s.} \\ \Rightarrow \quad \lambda_D &= \frac{2\pi}{T_D} = 10 \text{ rad/s.} \end{aligned} \quad (5.44)$$

Now we make a table for the logarithmic decrement of the peak displacements:

peak disp. $x_n$ ( m )	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443
$\frac{x_n}{x_{n+1}}$	1.0658	1.0648	1.0647	1.0645	1.0637	1.0627	
$\ln\left(\frac{x_n}{x_{n+1}}\right)$	0.0637	0.0628	0.0627	0.0624	0.0618	0.0608	

① Theoretically, all of these values should be the same, but it is rarely the case in practice. When  $x_n$ 's are measured from an experimental setup, the values of  $D$  may vary even more.

Thus, we get several values of the logarithmic decrement  $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$  ①.

We take the average value of  $D$ :

$$D = \bar{D} = 0.0624. \quad (5.45)$$

Let the equivalent single degree of freedom model have mass  $m$ , spring stiffness  $k$ , and damping rate  $c$ . Then

$$\lambda_D = \lambda \sqrt{1 - \xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (5.44),

$$\frac{k}{m} = \lambda^2 = 100(\text{rad/s})^2, \quad (5.46)$$

and, since  $D = \frac{cT_D}{2m}$ , from Eqn (5.45) we get

$$\begin{aligned} c &= \frac{2mD}{T_D} \\ &= \frac{2m(0.0624)}{0.6279 \text{ s}} \\ &= (0.1988 \frac{1}{\text{s}})m. \end{aligned} \quad (5.47)$$

Equations (5.46) and (5.47) have three unknowns:  $k$ ,  $m$ , and  $c$ . We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass  $m = 5 \text{ kg}$ . Then

$$\begin{aligned} k &= (100 \frac{1}{\text{s}^2}) \cdot (5 \text{ kg}) \\ &= 500 \text{ N/m}, \end{aligned}$$

and

$$\begin{aligned} c &= (0.1988 \frac{1}{\text{s}}) \cdot (5 \text{ kg}) \\ &= 0.99 \text{ N}\cdot\text{s/m}. \end{aligned}$$

$m = 5 \text{ kg}$ ,
$k = 500 \text{ N/m}$ ,
$c = 0.99 \text{ N}\cdot\text{s/m}$ .

Of course, we could choose many other sets of values for  $m$ ,  $k$ , and  $c$  which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine  $k$  and  $c$  uniquely from the given response.

## 5.5 Forced oscillations and resonance

If the world of oscillators was as we have described them so far, there wouldn't be much to talk about. The undamped oscillators would be oscillating away and the damped oscillators (all the real ones) would be all damped out. The reason vibrations exist is because they are somehow excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The simplest example of a ‘forced’ harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. A picture of such a system is shown in figure 5.38. The governing equation for a forced harmonic oscillator is:

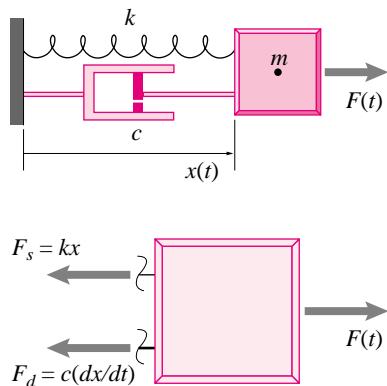


Figure 5.38: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force  $F(t)$  from some external source.

(Filename: tfigure12.MSDforced)

① The best approximation of a function with a sum of sine waves is a Fourier series, a topic we discuss no further here.

$$m\ddot{x} + c\dot{x} + kx = F(t). \quad (5.48)$$

When  $F(t) = 0$  there is no forcing and the governing equation reduces to that of the un-forced harmonic oscillator, eqn. (5.34). There are two special forcings of common interest:

- Constant force, and
- Sinusoidal forcing.

Constant force idealizes situations where the force doesn’t vary much as due say, to gravity, a steady wind, or sliding friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by vibrating machine parts or earthquakes. Sums of sine waves can accurately approximate any force that varies with time ①.

### Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored. If  $F = \text{constant}$ , then the general solution of equation 5.48 for  $x(t)$  is the same as the unforced case but with a constant added. The constant is  $F/k$ . The usual way of accommodating this case is to describe a new equilibrium point at  $x = F/k$  and to pick a new deflection variable that is zero at that point. If we pick a new variable  $w$  and define it as  $w = x - F/k$ , the amount of motion away from equilibrium, then, substituting into equation 5.48 the forced oscillator equation becomes

$$m\ddot{w} + c\dot{w} + kw = 0, \quad (5.49)$$

which is the unforced oscillator equation. The case of constant forcing reduces to the case of no forcing if one merely changes what one calls the equilibrium point to be the place where the mass is in equilibrium, taking account of the constant applied force.

$$x(t) = \underbrace{A \cos(\lambda t) + B \sin(\lambda t)}_{x_h} + \underbrace{F/k}_{x_p}$$

An alternative approach is to use *superposition*. Here we say  $x(t) = x_h(t) + x_p(t)$  where  $x_h(t)$  satisfies  $m\ddot{x} + c\dot{x} + kx = 0$  and  $x_p(t)$  is any solution of  $m\ddot{x} + c\dot{x} + kx = F$ . Such a solution is  $x_p = F/k$  if  $F(t)$  is constant. So the net solution is  $F/k$  plus a solution to the ‘homogeneous’ equation 5.49.

## Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that  $F(t) = F_0 \cos(pt)$  where  $F_0$  is the amplitude of forcing and  $p$  is the angular frequency of the forcing.

The general solution of equation 5.48 is given by the sum of two parts. One is the general solution of equation 5.34,  $x_h(t)$ , and the other is *any* solution of equation 5.48,  $x_p(t)$ . The solution  $x_h(t)$  of the damped oscillator equation 5.34 is called the ‘homogeneous’ or ‘complementary’ solution. Any solution  $x_p(t)$  of the forced oscillator equation 5.48 is called a ‘particular’ solution.

We already know the solution  $x_h(t)$  of the undamped governing differential equation 5.34. This solution is equation 5.35, 5.36, or 5.37, depending on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 5.48. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case if  $F(t) = F_0 \cos(pt)$  we make the guess that

$$x_p(t) = A \cos(pt) + B \sin(pt). \quad (5.50)$$

If we plug this guess into the forced oscillator equation (5.48), we find, after much tedious algebra, that we do in fact have a solution if

$$A = \frac{\frac{F_0}{k} \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)}{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{\frac{k}{m}} \right) + \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)^2},$$

$$\text{and } B = \frac{\frac{F_0}{k} \left( \frac{cp}{k} \right)}{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{\frac{k}{m}} \right) + \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)^2}.$$

So the response to the cosine-wave forcing is the sum of a sine wave and a cosine wave.

$$x_p(t) = \underbrace{\left( \frac{\frac{F_0}{k} \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)}{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{\frac{k}{m}} \right) + \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)^2} \right)}_A \cos(pt) + \underbrace{\left( \frac{\frac{F_0}{k} \left( \frac{cp}{k} \right)}{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{\frac{k}{m}} \right) + \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)^2} \right)}_B \sin(pt)$$

Alternatively sum of sine waves can be written as a cosine wave that has been shifted in phase as

$$x_p(t) = C \cos(pt - \phi),$$

$$\text{where } C = \sqrt{(A^2 + B^2)} = \frac{\frac{F_0}{k}}{\sqrt{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{\frac{k}{m}} \right) + \left( 1 - \frac{p^2}{\left(\frac{k}{m}\right)} \right)^2}}, \quad (5.51)$$

$$\text{and } \phi = \tan^{-1} \left( \frac{B}{A} \right) = \tan^{-1} \left( \frac{\left( \frac{c^2}{km} \right) \left( \frac{p^2}{k} \right)}{\left( 1 - \frac{p^2}{\left( \frac{k}{m} \right)} \right)} \right). \quad (5.52)$$

The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \quad (5.53)$$

## Uses of resonance

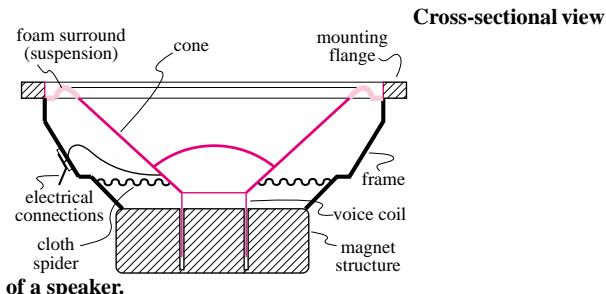
Though resonance is often a problem, it is also often of engineering use. Nuclear Magnetic Resonance imaging is used for medical diagnosis. The resonance of quartz crystals is used to time most watches now-a-days. In the old days, the resonant excitation of a clock pendulum was used to keep time. Self excited resonance is what makes musical instruments have such clear pitches.

## Frequency response

One way to characterize a structures sensitivity to oscillatory loads is by a *frequency response* curve. The frequency response curve might be found by a physical experiment or from a calculation based on a simplified model of the structure. The curve somewhat describes the answer to the following question about a structure:

*How does the size of the motion of a structure depend on the frequency and amplitude of an applied sinusoidal forcing?*

### 5.5 A Loudspeaker cone is a forced oscillator.



of a speaker.

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, it forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This rapid oscillation

of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as ‘added mass.’ The ‘spring’ and ‘dashpot’ effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \quad (5.54)$$

where  $i(t)$  is the electrical current flow through the coil in amps, and  $\alpha$  is the electro-mechanical coupling coefficient, in force per unit current.

Here is how the method works. First, you must apply a sinusoidal force, say  $F = F_0 \cos(pt)$ , to the structure at a physical point of interest. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let's assume you measure the displacement of some point on the structure  $\delta$ .

If the structure is linear and has some damping, the eventual motion of the structure will be a sinusoidal oscillation. In particular, you will measure that

$$\delta = C \cdot \cos(pt - \phi). \quad (5.55)$$

where  $C$  and  $\phi$  have been defined previously in equation 5.51. If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force  $C/F_0$  is independent of the size of the force  $F_0$ . Let's define:

$$R = \frac{C}{F_0} \quad (5.56)$$

That is, the response variable  $R$  is the ratio of the amplitude of the displacement sine

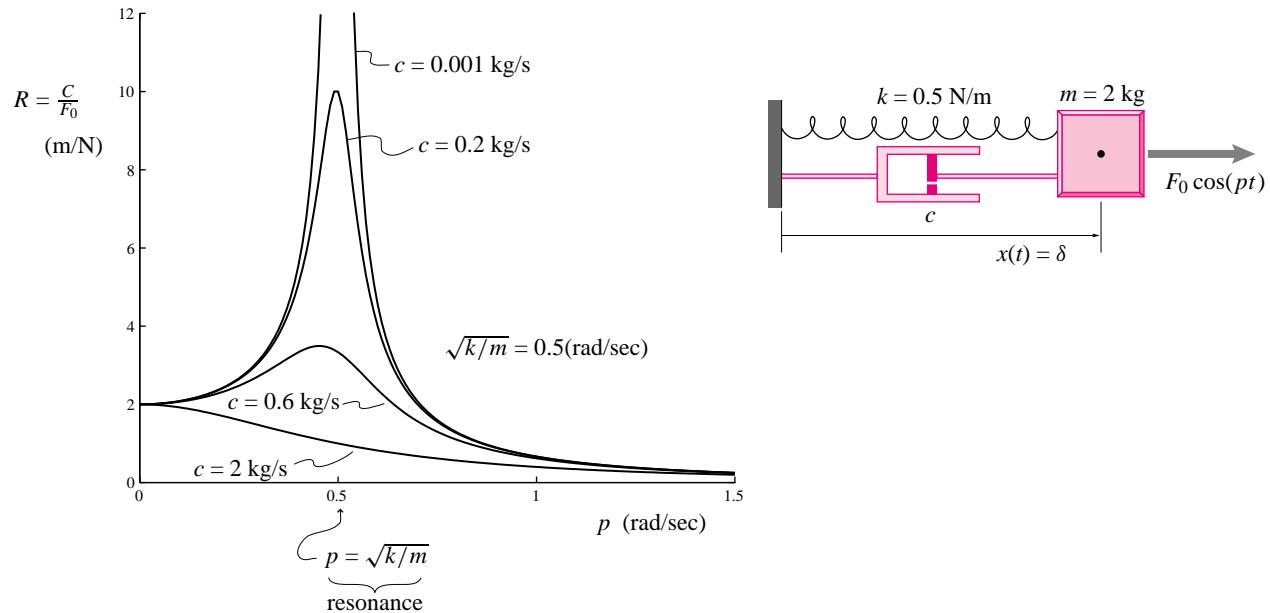


Figure 5.39: (filename:tfigure12.ampl.vs.freq)

wave to the amplitude of the forcing sine wave.

Now, this experiment can be repeated for different values of the angular forcing frequency  $p$ . The ratio of the vibrating displacement  $\delta$  to that of the applied forcing  $F_0$  will depend on  $p$ . The structure has different sensitivities to forcing at different frequencies. So the response ratio amplitude  $R$  depends on  $p$ . The function  $R = R(p)$  is called the frequency response. A plot of the amplitude ratio  $R$  versus the driving frequency  $p$  is shown in figure 5.39 for various values of the damping coefficient  $c$ . Numerical values are shown for definiteness although the plot could be shown as dimensionless.

#### Experimental measurement

To measure the frequency response function experimentally, one can apply forcing at a whole range of forcing frequencies. Another approach is to apply a sudden,

'impulsive', force and look at the response. This second method is equivalent, it turns out, as you may learn in the context of Laplace transforms or Fourier analysis.

Why does one want to know the frequency response? The answer is because it is one way to think about structural response. A car suspension may never be tested on a sinusoidal road. But knowing how the suspension would respond to sine wave shaped roads of all possible wave lengths somehow characterizes the car's response to roads with any kind of bumpiness.

*Example: Resonance of a building*

A mildly damped structure has a natural frequency of 17 hz and is forced at 17 hz. Because the frequency response function has a peak at 17 hz, resonance, the structures motions will be very large.  $\square$

**SAMPLE 5.20** *Particular solution:* Find a particular solution of the forced oscillator equation  $\ddot{x} + \lambda^2 x = F(t)$  where

- (a)  $F(t) = mg$  (a constant),
- (b)  $F(t) = At$ ,
- (c)  $F(t) = C \sin(pt)$ .

**Solution** The given differential equation is a second order linear ordinary differential equation with a non-zero right hand side. A particular solution of this equation must satisfy the entire equation. For such equations, we guess a particular solution to have the *same functional form* as the right hand side (the forcing function) and plug it into the equation to see if our guess works. We can usually determine the values of any unknown, assumed constants so that the assumed solution satisfies the equation. Let us see how it works here.

- (a) The forcing function is a constant,  $mg$ . So, let us assume the particular solution to be a constant, *i.e.*, let  $x_p = C$ . Plugging it into the equation, we have

$$\underbrace{\dot{C}}_0 + \lambda^2 C = mg \Rightarrow C = mg/\lambda^2 \Rightarrow x_p = mg/\lambda^2$$

$$x_p = mg/\lambda^2$$

- (b) The forcing function is linear in  $t$ . So, let us assume a linear function as a particular solution,  $x_p = \alpha t$  where  $\alpha$  is a constant. Now, noting that  $\dot{x}_p = \alpha$   $\Rightarrow$   $\ddot{x}_p = 0$ , and plugging back into the differential equation, we get

$$\lambda^2 \alpha t = At \Rightarrow \alpha = A/\lambda^2 \Rightarrow x_p(t) = (A/\lambda^2)t.$$

$$x_p(t) = (A/\lambda^2)t$$

- (c) The forcing function is a harmonic function. So, let  $x_p = \beta \sin(pt)$  where  $\beta$  is a constant to be determined later. Now, plugging  $x_p$  into the differential equation and noting that  $\ddot{x}_p = -\beta \omega^2 \sin(pt)$ , we get

$$(-\beta p^2 + \beta \lambda^2) \sin(pt) = C \sin(pt) \Rightarrow \beta = \frac{C}{\lambda^2 - p^2}.$$

Thus the particular solution in this case is

$$x_p(t) = \frac{C}{\lambda^2 - p^2} \sin(pt).$$

$$x_p(t) = \frac{C}{\lambda^2 - p^2} \sin(pt)$$

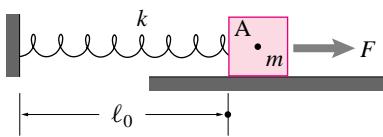


Figure 5.40: (Filename:sfig5.5.forcedosc)

**SAMPLE 5.21 Response to a constant force:** A constant force  $F = 50 \text{ N}$  acts on a mass-spring system as shown in the figure. Let  $m = 5 \text{ kg}$  and  $k = 10 \text{kN/m}$ .

- Write the equation of motion of the system.
- If the system starts from the initial displacement  $x_0 = 0.01 \text{ m}$  with zero velocity, find the displacement of the mass as a function of time.
- Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

### Solution

(a) The free body diagram of the mass is shown in Fig. 5.41 at a displacement  $x$  (assumed positive to the right). Applying linear momentum balance in the  $x$ -direction, i.e.,  $(\sum \vec{F} = m\vec{a}) \cdot \hat{i}$ , we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow m\ddot{x} + kx &= F \end{aligned} \quad (5.57)$$

which is the equation of motion of the system.

(b) The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution  $x_h$  which is the solution of the unforced system (eqn. (5.57) with  $F = 0$ ), and a particular solution  $x_p$  that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \quad (5.58)$$

Now, let us find  $x_h(t)$  and  $x_p(t)$ .

**Homogeneous solution:**  $x_h(t)$  has to satisfy  $m\ddot{x} + kx = 0$ . Let  $\lambda = \sqrt{k/m}$ .

Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A \sin(\lambda t) + B \cos(\lambda t)$$

where  $A$  and  $B$  are constants to be determined later from initial conditions.

**Particular solution:**  $x_p$  must satisfy eqn. (5.57). Since the nonhomogeneous part of the equation is a constant ( $F$ ), we guess that  $x_p$  must be a constant too (of the same form as  $F$ ). Let  $x_p = C$ . Now we substitute  $x_p = C$ ,  $\Rightarrow \ddot{x}_p = 0$  in eqn. (5.57) to determine  $C$ .

$$kC = F \Rightarrow C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting  $x_h$  and  $x_p$  in eqn. (5.58), we get

$$x(t) = A \sin(\lambda t) + B \cos(\lambda t) + F/k. \quad (5.59)$$

Now we use the given initial conditions to determine  $A$  and  $B$ .

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \quad (\text{given}) \Rightarrow B = x_0 - F/k \\ \dot{x}(t) &= A\lambda \cos(\lambda t) - B\lambda \sin(\lambda t) \\ \Rightarrow \dot{x}(t=0) &= A = 0 \quad (\text{given}) \Rightarrow A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k) \cos(\lambda t) + F/k.$$

- (c) Let us plug the given numerical values,  $k = 10 \text{ kN/m}$ ,  $m = 5 \text{ kg}$ ,  $\Rightarrow \lambda = \sqrt{k/m} = 44.72 \text{ rad/s}$ ,  $F = 50 \text{ N}$  and  $x_0 = 0.01 \text{ m}$  in eqn. (5.60). The displacement is now given as

$$x(t) = -(0.04 \text{ m}) \cos(44.72 \cdot t) + .05 \text{ m}.$$

This response is plotted in Fig. 5.42 against time. Note that the oscillations of the mass are about a non-zero mean value,  $x_{eq} = 0.04 \text{ m}$ . A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by  $mg/k$  to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length  $\ell_0$  of the spring but at  $\ell_0 + mg/k$ . Any oscillations of the mass will be about this new equilibrium.

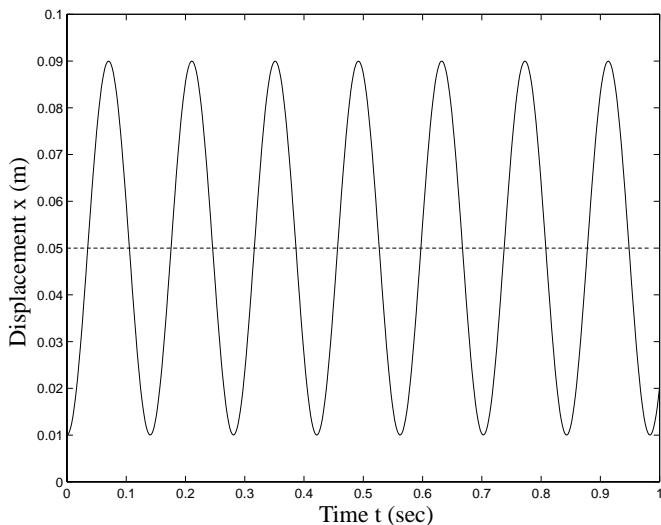


Figure 5.42: Displacement of the mass as a function of time. Note that the mass oscillates about a nonzero value of  $x$ .

(Filename:sfig5.5.forcedosc.b)

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by  $90^\circ$ . The new static equilibrium is at  $x_{eq} = F/k$  and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let  $\tilde{x} = x - F/k$ . Then, eqn. (5.60) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where  $\tilde{x}_0 = x_0 - F/k$  is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

□

**SAMPLE 5.22 Damping and forced response:** When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing  $F(t) = F_0 \sin(pt)$ , then the response of the system is given by

$$x(t) = C \cos(pt - \phi)$$

where  $C = \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}$ ,  $\phi = \tan^{-1} \frac{2\xi r}{1-r^2}$ ,  $r = \frac{p}{\lambda}$ ,  $\lambda = \sqrt{k/m}$  and  $\xi$  is the damping ratio.

- (a) For  $r \ll 1$ , i.e., the forcing frequency  $p$  much smaller than the natural frequency  $\lambda$ , how does the damping ratio  $\xi$  affect the response amplitude  $C$  and the phase  $\phi$ ?
- (b) For  $r \gg 1$ , i.e., the forcing frequency  $p$  much larger than the natural frequency  $\lambda$ , how does the damping ratio  $\xi$  affect the response amplitude  $C$  and the phase  $\phi$ ?

### Solution

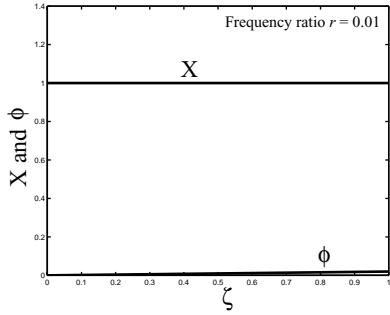
- (a) If the frequency ratio  $r \ll 1$ , then  $r^2$  will be even smaller; so we can ignore  $r^2$  terms with respect to 1 in the expressions for  $C$  and  $\phi$ . Thus, for  $r \ll 1$ ,

$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\xi r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio  $\xi$ , and the phase also remains constant at zero. As an example, we use the full expressions for  $C$  and  $\phi$  for plotting them against  $\xi$  for  $r = 0.01$  in Fig. 5.43

For  $r \ll 1$ ,  $C \approx F_0/k$ , and  $\phi \approx 0$

Figure 5.43: (Filename:sfig5.5.smallr)



- (b) If  $r \gg 1$ , then the denominator in the expression for  $C$ ,  $4\xi^2 r^2 + (1-r^2)^2 \approx r^4$  (because we can ignore all other terms with respect to  $r^4$ ). Similarly, we can ignore 1 with respect to  $r^2$  in the expression for  $\phi$ . Thus, for  $r \gg 1$ ,

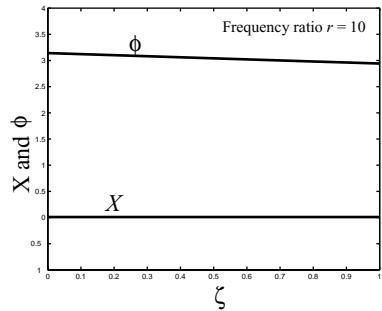
$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} C \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1} \frac{2\xi r}{-r^2} \approx \tan^{-1} \frac{2\xi}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with  $\xi$ . This is also evident from Fig. 5.44 where we plot  $C$  and  $\phi$  using their full expressions for  $r = 10$ . The slight variation in  $\phi$  around  $\pi$  goes away as we take higher values of  $r$ .

For  $r \gg 1$ ,  $C \approx 0$ , and  $\phi \approx \pi$

Figure 5.44: (Filename:sfig5.5.bigr)

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.



**SAMPLE 5.23 Energetics of resonance:** Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

**Solution** Let us consider the damped harmonic oscillator shown in Fig. 5.45 with  $F(t) = F_0 \sin(pt)$ . The equation of motion of the system is  $m\ddot{x} + c\dot{x} + kx = F_0 \sin(pt)$  and the response of the system may be expressed as  $X \sin(pt - \phi)$  where  $X = (F_0/k)/\sqrt{(2\zeta r)^2 + (1-r^2)^2}$  and  $\phi = \tan^{-1}(2\zeta r/(1-r^2))$ , with  $r = p/\lambda$ ,  $\lambda = \sqrt{k/m}$  and  $\zeta = c/\sqrt{2km}$ .

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{onecycle}} F dx$$

But,  $x = X \sin(pt - \phi) \Rightarrow dx = Xp \cos(pt - \phi) dt$ . Therefore,

$$\begin{aligned} W &= \int_0^{2\pi/\lambda} F_0 \sin(pt) \cdot Xp \cos(pt - \phi) dt \\ &= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt) \cos(pt - \phi) dt \\ &= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt)(\cos(pt) \cos \phi + \sin(pt) \sin \phi) dt \\ &= F_0 Xp \left[ \cos \phi \cdot \frac{1}{2} \int_0^{2\pi/\lambda} \sin(2pt) dt + \sin \phi \cdot \frac{1}{2} \int_0^{2\pi/\lambda} (1 - \cos(2pt)) dt \right] \\ &= \frac{F_0 Xp}{2} \left[ \cos \phi \left( -\frac{\cos(2pt)}{2p} \right) \Big|_0^{2\pi/\lambda} + \sin \phi \left( t - \frac{\sin(2pt)}{2p} \right) \Big|_0^{2\pi/\lambda} \right] \\ &= \frac{F_0 Xp}{2} \left[ \frac{\cos \phi}{2p} (-1 + 1) + \frac{2\pi}{p} \sin \phi + 0 \right] \\ &= \frac{F_0 Xp}{2} \cdot \frac{2\pi}{p} \sin \phi \\ &= F_0 \pi X \sin \phi \end{aligned}$$

Although the expression obtained above for  $W$  looks simple, we must substitute for  $X$  and  $\phi$  to see the dependence of  $W$  on the damping ratio  $\zeta$  and the frequency ratio  $r$ .

$$W = \frac{F_0^2 \pi}{\pi \sqrt{(2\zeta r)^2 + (1-r^2)^2}} \cdot \sin \left( \tan^{-1} \frac{2\zeta r}{1-r^2} \right) \quad (5.60)$$

Unfortunately, this expression is too complicated to see the dependence of  $W$  on  $\zeta$  and  $r$ . However, we know that for small  $r (< 1)$ ,  $\phi \approx 0$  and for large  $r (> 1)$ ,  $\phi \approx \pi$ , implying that  $W$  is almost zero in both these cases. On the other hand, for  $r$  close to one, that is, close to resonance,  $\phi \approx \pi/2 \Rightarrow \sin \phi \approx 1$ , but the response amplitude  $X$  is large (for small  $\zeta$ ), which makes  $W$  to be big near the resonance. Figure 5.46 shows a plot of  $W$  against  $r$ , using eqn. (5.60), for different values of  $\zeta$ . It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

$$W = F_0 \pi X \sin \phi$$

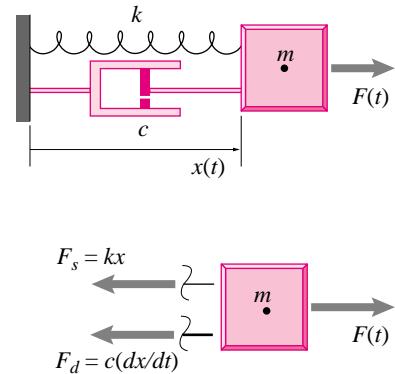


Figure 5.45: (Filename:sfig5.5.reswork)

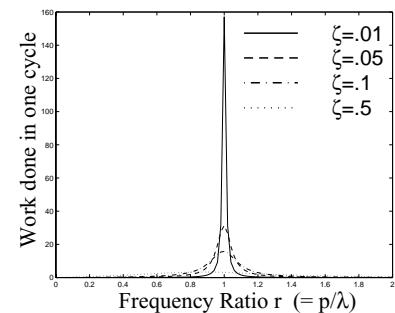


Figure 5.46: (Filename:sfig5.5.reswork.plot)

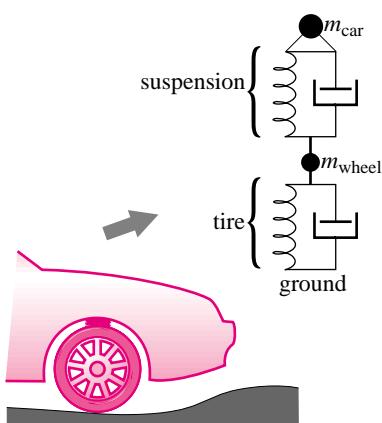


Figure 5.47: (Filename:tefig3.4.car.susp)

## 5.6 Coupled motions in 1D

Many important engineering systems have parts that move independently. A simple dynamic model using a single particle is not adequate. So here, still using one dimensional mechanics, we consider systems that can be modelled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but not the same amount at the same time.

**Example: Car suspension.**

A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car.  $\square$

The simplest way of dealing with the coupled motion of two or more particles is to write  $\vec{F} = m\vec{a}$  for each particle and use the forces on the free body diagrams to evaluate the forces. Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

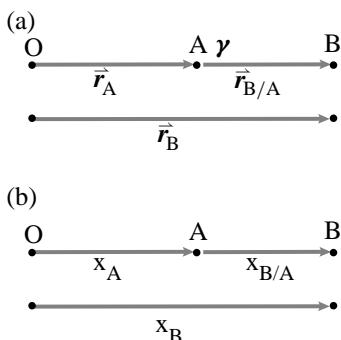


Figure 5.48: The relative position of points A and B in one dimension.  
(Filename:tefigure.relepos1D)

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \text{ and } \vec{a}_{B/A} = \vec{a}_B - \vec{a}_A.$$

In one dimension, the relative position diagram of Fig. 2.5 on page 11 becomes Fig. 5.48.  $\vec{r} = x\hat{i}$ ,  $\vec{v} = v\hat{i}$ , and  $\vec{a} = a\hat{i}$ . So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A, \text{ and} \\ a_{B/A} &\equiv a_B - a_A. \end{aligned}$$

**Example: Two masses connected by a spring.**

Consider the two masses on a frictionless support (Fig. 5.49). Assume the spring is unstretched when  $x_1 = x_2 = 0$ . After drawing free body diagrams of the two masses we can write  $\vec{F} = m\vec{a}$  for each mass:

$$\begin{aligned} \text{mass 1: } \vec{F}_1 &= m\vec{a}_1 \Rightarrow T\hat{i} = m_1\ddot{x}_1\hat{i} \\ \text{mass 2: } \vec{F}_2 &= m\vec{a}_2 \Rightarrow -T\hat{i} = m_2\ddot{x}_2\hat{i} \end{aligned} \quad (5.61)$$

The stretch of the spring is

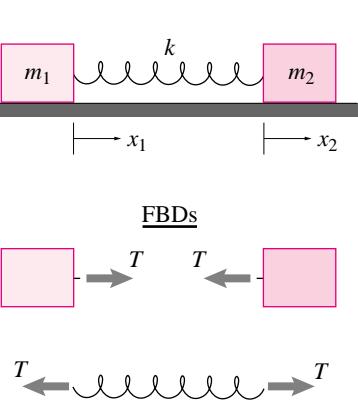


Figure 5.49: (Filename:tefig3.4.mass.springer)

$$\Delta\ell = x_2 - x_1$$

$$\text{so } T = k\Delta\ell = k(x_2 - x_1). \quad (5.62)$$

Combining (5.61) and (5.62) we get

$$\begin{aligned}\ddot{x}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\frac{1}{m_2}\right) (-k(x_2 - x_1))\end{aligned}\quad (5.63)$$

Note: Take care with signs when setting up this type of problem. You can check for example that if  $x_2 > x_1$ , mass 1 accelerates to the right ( $\ddot{x}_1 > 0$ ) and mass 2 accelerates to the left ( $\ddot{x}_2 < 0$ ).  $\square$

The differential equations that result from writing  $\vec{F} = m\vec{a}$  for the separate particles are coupled second-order equations. They are often solved by writing them as a system of first-order equations.

**Example: Writing second-order ODEs as first-order ODEs.**

Refer again to Fig. 5.49 If we define  $v_1 = \dot{x}_1$  and  $v_2 = \dot{x}_2$  we can rewrite equation 5.63 as

$$\begin{aligned}\dot{x}_1 &= v_1 \\ \dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \left(\frac{1}{m_2}\right) (-k)(x_2 - x_1)\end{aligned}$$

or, defining  $z_1 = x_1$ ,  $z_2 = v_1$ ,  $z_3 = x_2$ ,  $z_4 = v_2$ , we get

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\left(\frac{k}{m_1}\right) z_1 + \left(\frac{k}{m_1}\right) z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \frac{k}{m_2} z_1 - \frac{k}{m_2} z_3.\end{aligned}$$

$\square$

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

**Example: computer solution**

If we take, in consistent units,  $m_1 = 1$ ,  $k = 1$ ,  $m_2 = 1$ ,  $x_1(0) = 0$ ,  $x_2(0) = 0$ ,  $v_1(0) = 1$ , and  $v_2(0) = 0$ , we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer.:

```
ODEs = {z1dot = z2
        z2dot = -z1 + z3
        z3dot = z4
        z4dot = z1 - z3}
ICs = {z1(0)=0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in Fig. 5.50.  $\square$

As the samples show, the same methods work for problems involving connections with dashpots.

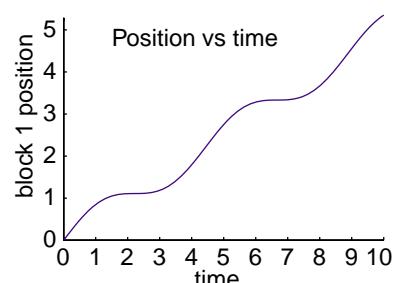


Figure 5.50: Plot of the position of the left mass vs. time.

(Filename: tfig.coupledmasses)

## Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center of mass. For a collection of particles in one dimension the center of mass is

$$x_{CM} = \frac{\sum x_i m_i}{m_{tot}}, \quad (5.64)$$

where  $m_{tot} = \sum m_i$  is the total mass of the system. The velocity and acceleration of the center of mass are found by differentiation to be

$$v_{CM} = \frac{\sum v_i m_i}{m_{tot}} \quad \text{and} \quad a_{CM} = \frac{\sum a_i m_i}{m_{tot}}. \quad (5.65)$$

If we imagine a system of interconnected masses and add the  $\vec{F} = m\vec{a}$  equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\text{external}} = \sum a_i m_i = m_{tot} a_{CM}. \quad (5.66)$$

Thus, the center of mass of a system that may be deforming wildly, obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in any number of dimensions.

## 5.6 THEORY

### *What saith Newton about collisions?*

Page 25 of Newton Principia, Motte's translation revised, by Florian Cajori (Univ. of CA press, 1947) He discusses collisions of spheres as measured in pendulum experiments. He takes account of air friction. He has already discussed momentum conservation.

"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes bodies to return one from the other with a relative velocity, which is in a given

ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendula's bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity to which they met, as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. "

**SAMPLE 5.24** For the given quantities and initial conditions, find  $x_1(t)$ . Assume the spring is unstretched when  $x_1 = x_2$ .

$$\begin{aligned} m_1 &= 1 \text{ kg}, & m_2 &= 2 \text{ kg}, & k &= 3 \text{ N/m}, & c &= 5 \text{ N/(m/s)} \\ x_1(0) &= 1 \text{ m}, & \dot{x}_1(0) &= 0, & x_2(0) &= 2 \text{ m}, & \dot{x}_2(0) &= 0. \end{aligned}$$

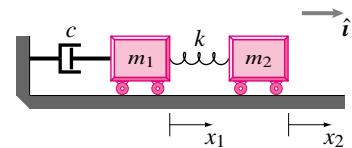


Figure 5.51: (Filename:sfig3.4.unstr.ini)

### Solution

#### FBDs

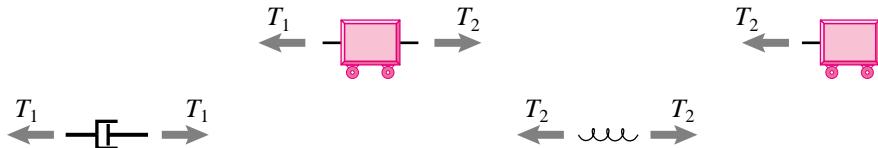


Figure 5.52: (Filename:sfig3.4.unstr.ini.fbd)

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \quad T_2 = k(x_2 - x_1). \quad (5.67)$$

### LMB

$$\begin{aligned} \sum \vec{F} &= m\vec{a} \\ \text{mass 1: } -T_1\hat{i} + T_2\hat{i} &= m_1\ddot{x}_1\hat{i} \\ \text{mass 2: } -T_2\hat{i} &= m_2\ddot{x}_2\hat{i}. \end{aligned} \quad (5.68)$$

Applying the constitutive laws (5.67) to the momentum balance equations (5.68) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining  $z_1 = x_1$ ,  $z_2 = \dot{x}_1$ ,  $z_3 = x_2$ ,  $z_4 = \dot{x}_2$  gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1 \text{ m}, \quad z_2(0) = 0, \quad z_3(0) = 2 \text{ m}, \quad z_4(0) = 0.$$

We are now set for numerical solution.

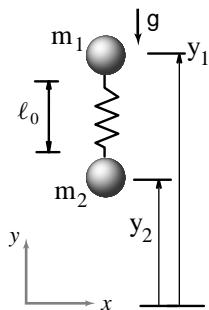


Figure 5.53: (Filename:fig5.6.hopper)

**SAMPLE 5.25** Flight of a toy hopper. A hopper model is made of two masses  $m_1 = 0.4 \text{ kg}$  and  $m_2 = 1 \text{ kg}$ , and a spring with stiffness  $k = 100 \text{ N/m}$  as shown in Fig. ???. The unstretched length of the spring is  $\ell_0 = 1 \text{ m}$ . The model is released from rest from the configuration shown in the figure with  $y_1 = 25.5 \text{ m}$  and  $y_2 = 24 \text{ m}$ .

- Find and plot  $y_1(t)$  and  $y_2(t)$  for  $t = 0$  to  $2 \text{ s}$ .
- Plot the motion of  $m_1$  and  $m_2$  with respect to the center of mass of the hopper during the same time interval.
- Plot the motion of the center of mass of the hopper from the solution obtained for  $y_1(t)$  and  $y_2(t)$  and compare it with analytical values obtained by integrating the center of mass motion directly.

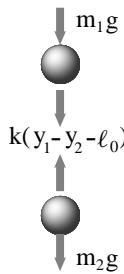


Figure 5.54: Free body diagram of the two masses  $m_1$  and  $m_2$

(Filename:fig5.6.hopper.a)

**Solution** The free body diagrams of the two masses are shown in Fig. 5.54. From the linear momentum balance in the  $y$  direction, we can write the equations of motion at once.

$$\begin{aligned} m_1 \ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\ \Rightarrow \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \end{aligned} \quad (5.69)$$

$$\begin{aligned} m_2 \ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\ \Rightarrow \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g \end{aligned} \quad (5.70)$$

- The equations of motion obtained above are coupled linear differential equations of second order. We can solve for  $y_1(t)$  and  $y_2(t)$  by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.

Letting  $\dot{y}_1 = v_1$  and  $\dot{y}_2 = v_2$ , we get

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\ \dot{y}_2 &= v_2 \\ \dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g \end{aligned}$$

Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```

ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-10) - g,
        y2dot = v2,
        v2dot = k/m2*(y1-y2-10) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set k=100, m1=0.4, m2=1, 10=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
    
```

The solution obtained thus is shown in Fig. 5.55.

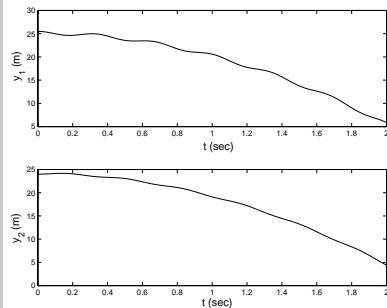


Figure 5.55: Numerically obtained solutions  $y_1(t)$  and  $y_2(t)$

(Filename:fig5.6.hopper.b)

- (b) We can find the motion of  $m_1$  and  $m_2$  with respect to the center of mass by subtraction the motion of the center of mass,  $y_{cm}$  from  $y_1$  and  $y_2$ . Since,

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \quad (5.71)$$

we get,

$$\begin{aligned} y_{1/cm} &= y_1 - y_{cm} = \frac{m_2}{m_1 + m_2} (y_1 - y_2) \\ y_{2/cm} &= y_2 - y_{cm} = -\frac{m_1}{m_1 + m_2} (y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in Fig. 5.56. We note that the motions of  $m_1$  and  $m_2$ , as seen by an observer sitting at the center of mass, are simple harmonic oscillations.

- (c) We can find the center of mass motion  $y_{cm}(t)$  from  $y_1$  and  $y_2$  by using eqn. (5.71). The solution obtained thus is shown as a solid line in Fig. 5.58. We can also solve for the center of mass motion analytically by first writing the equation of motion of the center of mass and then integrating it analytically. The free body diagram of the hopper as a single system is shown in Fig. 5.57. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1 + m_2) \ddot{y}_{cm} &= -m_1 g - m_2 g \\ \Rightarrow \ddot{y}_{cm} &= -g. \end{aligned}$$

We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{cm}(t) = y_{cm}(0) + \dot{y}_{cm}(0)t - \frac{1}{2}gt^2$$

Noting that  $y_{cm}(0) = 24.43$  m (from eqn. (5.71)), and  $\dot{y}_{cm}(0) = 0$  (the system is released from rest), we get

$$y_{cm}(t) = 24.43 \text{ m} - \frac{1}{2} \cdot 9.81 \text{ m/s}^2 \cdot t^2.$$

The values obtained for the center of mass position from the above expression are shown in Fig. 5.58 by small circles.

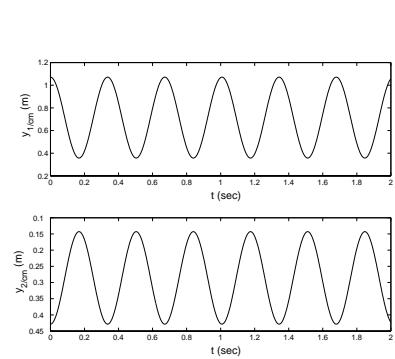


Figure 5.56: Numerically obtained solutions  $y_{1/cm}(t)$  and  $y_{2/cm}(t)$ .

(Filename:sfig5.6.hopper.c)

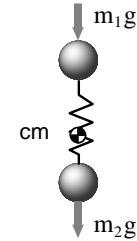


Figure 5.57: Free body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system

(Filename:sfig5.6.hopper.e)

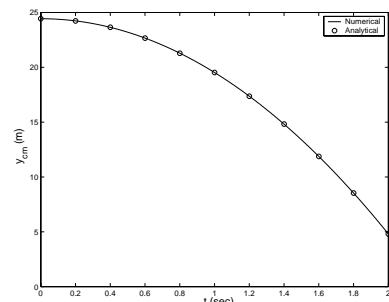


Figure 5.58: Numerically obtained solution for the position of the center of mass,  $y_{cm}(t)$ .

(Filename:sfig5.6.hopper.d)

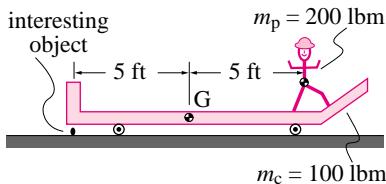


Figure 5.59: Mr. P spots an interesting object.

(Filename:sfig2.6.5)

**SAMPLE 5.26 Conservation of linear momentum.** Mr. P with mass  $m_p = 200 \text{ lbm}$  is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

**Solution** From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart moves to the right. Here, we want to find how far the cart moves.

Consider the cart and Mr. P together to be the system of interest. The free body diagram of the system is shown in Fig. 5.60(a). From the diagram it is clear that

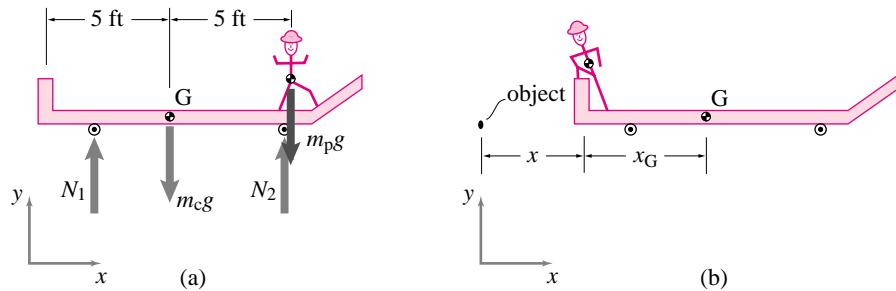


Figure 5.60: (a) Free body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance  $x$  when Mr. P reaches the other end.

(Filename:sfig2.6.5a)

there are no external forces in the  $x$ -direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \Rightarrow L_x = \text{constant}$$

that is, the linear momentum of the system in the  $x$ -direction is ‘conserved’. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \Rightarrow (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center of mass is always zero, the center of mass does not change its horizontal position. Now let  $x_{cm}$  and  $x'_{cm}$  be the  $x$ -coordinates of the center of mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 5.60(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200 \text{ lbm} \cdot 10 \text{ ft}}{300 \text{ lbm}} = 6\frac{2}{3} \text{ ft.} \end{aligned}$$

6.67 ft

[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

## 5.7 Time derivative of a vector: position, velocity and acceleration

So far in this chapter we have only considered things that move in a straight line. Of course we are interested also in things that move on more complicated paths. What are the paths of a hit baseball, a satellite, or a crashing plane? We now need to think about vector-valued functions of time. For example, the vectors linear momentum  $\vec{L}$  and angular momentum  $\vec{H}$  have a central place in the basic mechanics governing equations. Evaluation of these terms depends, in turn, on understanding the relation between position  $\vec{r}$ , its rate of change velocity  $\vec{v}$ , and between velocity  $\vec{v}$  and its rate of change the acceleration  $\vec{a}$ .

What do we mean by the rate of change of a vector? The rate of change of any quantity, including vectors, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ① The notation for the rate of change of a vector  $\vec{r}$  is

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}.$$

Or, in the short hand ‘dot’ notation invented by Newton for just this purpose,  $\vec{v} = \dot{\vec{r}}$ . The expression for the derivative of a vector  $\frac{d\vec{r}}{dt}$  or  $\dot{\vec{r}}$  has the same definition as the derivative of a scalar that one learns in elementary calculus. That is,

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}.$$

Vector differentiation is also sometimes needed for the calculation of the rate of change of linear momentum  $\vec{L}$  and rate of change of angular momentum  $\vec{H}_C$  for use in the momentum balance equations.

### Cartesian coordinates

The most primitive way to understand the motion of a system is to understand the motion of each of its parts using cartesian coordinates. That is each bit of mass in a system has a location  $\vec{r}$ , relative to the origin of a ‘good’ reference frame as shown in figure 5.61, which can be written as:

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k} \quad \text{or} \quad \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}.$$

So velocity and acceleration are simply described by derivatives of  $\vec{r}$ . Since the base vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are constant, differentiation to get velocity and acceleration is simple:

$$\vec{v} = \dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k} \quad \text{and} \quad \vec{a} = \ddot{x} \hat{i} + \ddot{y} \hat{j} + \ddot{z} \hat{k}.$$

So if  $x$ ,  $y$ , and  $z$  are known functions of time for every particle in the system, we can evaluate the rate of change of linear and angular momentum just by differentiating the functions twice to get the acceleration and then summing (or integrating) to get  $\vec{L}$  and  $\vec{H}$ .

The idea is illustrated in figure 5.62. Let’s assume

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

or

$$\vec{r}(t) = r_x(t) \hat{i} + r_y(t) \hat{j} + r_z(t) \hat{k}.$$

① Strictly speaking these words describe the average rate of change over the small time interval. Only in the mathematical limit of vanishing time intervals is this ratio not just approximately the rate of change, but exactly the rate of change.

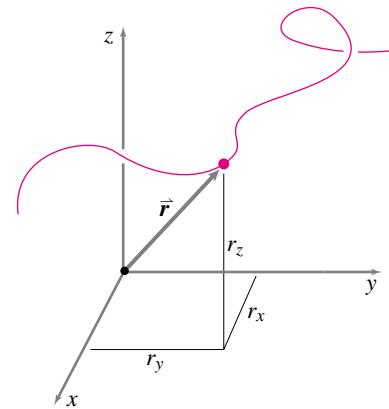


Figure 5.61: Cartesian coordinates

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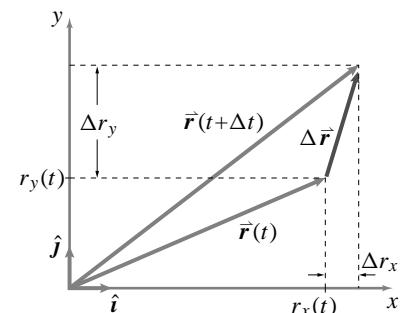
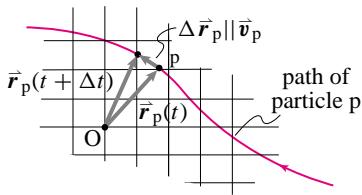


Figure 5.62: Change of position in  $\Delta t$  broken into components in 2-D.  $\Delta \vec{r}$  is  $\vec{r}(t + \Delta t) - \vec{r}(t)$ .  $\Delta \vec{r}$  has components  $\Delta r_x$  and  $\Delta r_y$ . So  $\Delta \vec{r} = \Delta r_x \hat{i} + \Delta r_y \hat{j}$ . In the limit as  $\Delta t$  goes to zero,  $\vec{r}$  is the ratio of  $\Delta \vec{r}$  to  $\Delta t$ .

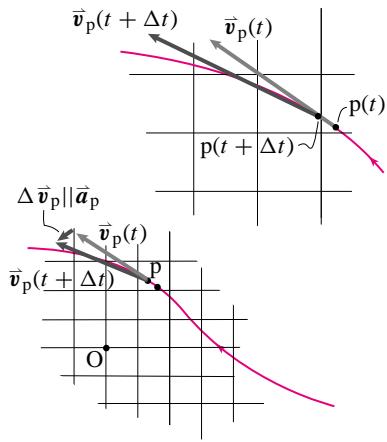
(Filename:tfigure2.g)

① **Caution:** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components.

(a) **Position**



(b) **Velocity** is tangent to the path. It is approximately in the direction of  $\Delta \vec{r}$ .



(c) **Acceleration** is generally *not* tangent to the path. It is approximately in the direction of  $\Delta \vec{v}$ .

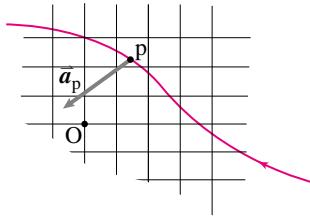


Figure 5.63: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times:  $t$  and  $t + \Delta t$ . The velocity at time  $t$  is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

(Filename:figure2.2)

Now we apply the definition of derivative and find

$$\begin{aligned}\dot{\vec{r}}(t) &= \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \\ &= \frac{\overbrace{(\vec{r}_{x(t+\Delta t)}\hat{i} + \vec{r}_{y(t+\Delta t)}\hat{j} + \vec{r}_{z(t+\Delta t)}\hat{k})}^{\vec{r}(t+\Delta t)} - \overbrace{(\vec{r}_{x(t)}\hat{i} + \vec{r}_{y(t)}\hat{j} + \vec{r}_{z(t)}\hat{k})}^{\vec{r}(t)}}{\Delta t} \\ &= \frac{r_x(t + \Delta t) - r_x(t)}{\Delta t} \hat{i} + \frac{r_y(t + \Delta t) - r_y(t)}{\Delta t} \hat{j} + \frac{r_z(t + \Delta t) - r_z(t)}{\Delta t} \hat{k} \\ &= \dot{r}_x(t)\hat{i} + \dot{r}_y(t)\hat{j} + \dot{r}_z(t)\hat{k}.\end{aligned}$$

We have found the palatable result that the components of the velocity vector are the time derivatives of the components of the position vector ①. Vector differentiation is done to find the velocity and acceleration of particles or parts of bodies. The curve in figure 5.63 shows a particle P's path, that is, its position at a sequence of times. The position vector  $\vec{r}_{P/O}$  is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity  $\vec{v}$  at time  $t$  is the rate of change of position at that time,  $\vec{v} \equiv \dot{\vec{r}}$ .

**Example: Given position as a function of time, find the velocity.**

Given that the position of a point is:

$$\vec{r}(t) = C_1 \cos(\omega t)\hat{i} + C_2 \sin(\omega t)\hat{j}$$

with  $C_1 = 4 \text{ m}$ ,  $C_2 = 2 \text{ m}$  and  $\omega = 10 \text{ rad/s}$ . What is the velocity (a vector) at  $t = 3 \text{ s}$ ?

First we note that the components of  $\vec{r}(t)$  have been given implicitly as

$$r_x(t) = C_1 \cos(\omega t) \quad \text{and} \quad r_y(t) = C_2 \sin(\omega t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{v}(t) = \dot{\vec{r}} = -C_1 \omega \sin(\omega t)\hat{i} + C_2 \omega \cos(\omega t)\hat{j}$$

Now we evaluate this expression with the given values of  $C_1 = 4 \text{ m}$ ,  $C_2 = 2 \text{ m}$ ,  $\omega = 10 \text{ rad/s}$  and  $t = 3 \text{ s}$  to get the velocity at 3 s as:

$$\begin{aligned}\vec{v}(3 \text{ s}) &= -(4 \text{ m})(10/\text{s}) \sin((10/\text{s})(3 \text{ s}))\hat{i} + \\ &\quad (2 \text{ m})(10/\text{s}) \cos((10/\text{s})(3 \text{ s}))\hat{j} \quad (5.72)\end{aligned}$$

$$= (-40 \sin(30)\hat{i} + 20 \cos(30)\hat{j}) \text{ m/s} \quad (5.73)$$

$$= (39.5\hat{i} + 3.09\hat{j}) \text{ m/s} \quad (5.74)$$

Note that the last line is calculated using the angle as measured in radians, not degrees.  $\square$

## Product rule

We know three ways to multiply vectors. You can multiply a vector by a scalar, take the dot product of two vectors, and take the cross product of two vectors. Because these forms all show up in dynamics we need to know a method for differentiating,

The method is simple. All three kinds of vector multiplication obey the product rule of differentiation that you learned in freshmen calculus.

$$\begin{aligned}\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}} \\ \frac{d}{dt}(\vec{A} \cdot \vec{B}) &= \dot{\vec{A}} \cdot \vec{B} + \vec{A} \cdot \dot{\vec{B}} \\ \frac{d}{dt}(\vec{A} \times \vec{B}) &= \dot{\vec{A}} \times \vec{B} + \vec{A} \times \dot{\vec{B}}.\end{aligned}$$

The proofs of these identities is nearly an exact copy of the proof used for scalar multiplication.

**Example: Derivative of a vector of constant length.**

Assume

$$\frac{d}{dt}|\vec{C}| = 0$$

so

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C} \cdot \vec{C}) = 0.$$

Using the product rule above, we get

$$\frac{d}{dt}(\vec{C} \cdot \vec{C}) = \vec{C} \cdot \dot{\vec{C}} + \dot{\vec{C}} \cdot \vec{C} = 2\vec{C} \cdot \dot{\vec{C}} = 0$$

so

$$\vec{C} \cdot \dot{\vec{C}} = 0 \Rightarrow \vec{C} \perp \dot{\vec{C}}.$$

The rate of change of a vector of constant length is perpendicular to that vector. This observation is a useful fact to remember about time varying unit vectors, a special case of time varying constant length vectors.  $\square$

## The motion quantities

The various quantities that show up in the equations of dynamics are defined on the inside cover. To calculate any of them you must multiply some combination of position, velocity and acceleration by mass.

### Rate of change of a vector depends on frame

We just explained that the time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ . But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

*The rate of change of a given vector is different if calculated in different reference frames.*

*When applying the laws of mechanics, we must be sure that when we differentiate vectors we do so with respect to a Newtonian frame.*

Because most often we use the “fixed” ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script  $\mathcal{F}$ , for fixed. So, when being careful with notation we will write the velocity of point B as

$$\overset{\mathcal{F}}{\dot{\vec{r}}}_{B/O}$$

### **Non-Newtonian frames**

It is useful to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. These non-Newtonian frames will be discussed in chapter 9. Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are useful help with the understanding of the motion and forces of systems composed of objects with complex relative motion.

**SAMPLE 5.27 Velocity and acceleration:** The position vector of a particle is given as a function of time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2) \hat{i} + C_4 t \hat{j}$$

where  $C_1 = 1 \text{ m}$ ,  $C_2 = 3 \text{ m/s}$ ,  $C_3 = 1 \text{ m/s}^2$ , and  $C_4 = 2 \text{ m/s}$ .

- (a) Find the position, velocity, and acceleration of the particle at  $t = 2 \text{ s}$ .
- (b) Find the change in the position of the particle between  $t = 2 \text{ s}$  and  $t = 3 \text{ s}$ .

**Solution** We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2) \hat{i} + C_4 t \hat{j}.$$

Therefore,

$$\begin{aligned}\vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t) \hat{i} + C_4 \hat{j} \\ \vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3 \hat{i}.\end{aligned}$$

- (a) Substituting the given values of the constants and  $t = 2 \text{ s}$  in the equations above we get,

$$\begin{aligned}\vec{r}(t = 2 \text{ s}) &= (1 \text{ m} + 3 \frac{\text{m}}{\text{s}} \cdot 2 \text{ s} + 1 \frac{\text{m}}{\text{s}^2} \cdot 4 \text{ s}^2) \hat{i} + (2 \frac{\text{m}}{\text{s}} \cdot 2 \text{ s}) \hat{j} \\ &= 11 \text{ m} \hat{i} + 4 \text{ m} \hat{j} \\ \vec{v}(t = 2 \text{ s}) &= (3 \frac{\text{m}}{\text{s}} + 2 \cdot 1 \frac{\text{m}}{\text{s}^2} \cdot 2 \text{ s}) \hat{i} + (2 \frac{\text{m}}{\text{s}}) \hat{j} \\ &= 7 \text{ m/s} \hat{i} + 2 \text{ m/s} \hat{j} \\ \vec{a}(t = 2 \text{ s}) &= (2 \cdot 1 \frac{\text{m}}{\text{s}^2}) \hat{i} = 2 \text{ m/s}^2 \hat{i}.\end{aligned}$$

$$\boxed{\vec{r} = (11\hat{i} + 4\hat{j}) \text{ m}, \quad \vec{v} = (7\hat{i} + 2\hat{j}) \text{ m/s}, \quad \vec{a} = 2 \text{ m/s}^2 \hat{i}}$$

- (b) The change in the position of the particle between the two time instants is,

$$\Delta \vec{r} = \vec{r}(t = 3 \text{ s}) - \vec{r}(t = 2 \text{ s})$$

We already have  $\vec{r}$  at  $t = 2 \text{ s}$ . We need to calculate  $\vec{r}$  at  $t = 3 \text{ s}$ .

$$\begin{aligned}\vec{r}(t = 3 \text{ s}) &= (1 \text{ m} + 3 \frac{\text{m}}{\text{s}} \cdot 3 \text{ s} + 1 \frac{\text{m}}{\text{s}^2} \cdot 9 \text{ s}^2) \hat{i} + (2 \frac{\text{m}}{\text{s}} \cdot 3 \text{ s}) \hat{j} \\ &= 19 \text{ m} \hat{i} + 6 \text{ m} \hat{j}\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta \vec{r} &= (19 \text{ m} \hat{i} + 6 \text{ m} \hat{j}) - (11 \text{ m} \hat{i} + 4 \text{ m} \hat{j}) \\ &= 8 \text{ m} \hat{i} + 2 \text{ m} \hat{j}.\end{aligned}$$

$$\boxed{\Delta \vec{r} = 8 \text{ m} \hat{i} + 2 \text{ m} \hat{j}}$$

**SAMPLE 5.28** Find velocity and acceleration from position. Given that the position of a particle is

$$\vec{r} = A \cos(\omega t) \hat{i} + B \sin(\omega t) \hat{j} + Ct \hat{k},$$

with  $A$ ,  $B$ ,  $C$ , and  $\omega$  constants, find

- (a) the velocity as a function of time,
- (b) the acceleration as a function of time.

### Solution

- (a) The velocity:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt}[A \cos(\omega t) \hat{i} + B \sin(\omega t) \hat{j} + Ct \hat{k}] \\ &= -A\omega \sin(\omega t) \hat{i} + B\omega \cos(\omega t) \hat{j} + C\hat{k}\end{aligned}$$

$$\boxed{\vec{v} = -A\omega \sin(\omega t) \hat{i} + B\omega \cos(\omega t) \hat{j} + C\hat{k}}$$

- (b) The acceleration:

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}[-A\omega \sin(\omega t) \hat{i} + B\omega \cos(\omega t) \hat{j}] \\ &= -A\omega^2 \cos(\omega t) \hat{i} - B\omega^2 \sin(\omega t) \hat{j}\end{aligned}$$

$$\boxed{\vec{a} = -A\omega^2 \cos(\omega t) \hat{i} - B\omega^2 \sin(\omega t) \hat{j}}$$

**Note:** The path is an elliptical helix with axis in the  $z$  direction. The component of velocity in the  $z$  direction is constant so the acceleration is entirely in the  $xy$  plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

**SAMPLE 5.29** Find a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal.

**Solution** First, we recognize that if there is a solution, it must be in three dimensions since there cannot be three mutually orthogonal vectors in a plane. Next we make a sketch (see figure 5.64). We see that if we can make the acceleration orthogonal to the velocity we can put the origin on the line defined by their common normal.

One of many solutions is to take

$$\vec{r}(t) = A\hat{j} - Bt\hat{i} + Ct^2\hat{k}$$

which is illustrated in figure 5.65. We can verify this guess as follows:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = -B\hat{i} + 2Ct\hat{k} \\ \vec{a} &= \frac{d\vec{v}}{dt} = 2C\hat{k}.\end{aligned}$$

So, at  $t = 0$ ,

$$\vec{r} = A\hat{j}, \quad \vec{v} = -B\hat{i}, \quad \text{and } \vec{a} = 2C\hat{k}.$$

Because the dot products between the vectors above are:  $\vec{r} \cdot \vec{v} = 0$ ,  $\vec{v} \cdot \vec{a} = 0$ , and  $\vec{r} \cdot \vec{a} = 0$ , these vectors are mutually orthogonal. So, for the path shown the position, velocity, and acceleration are mutually orthogonal at  $t = 0$  as desired. (**aside:** Why is there a  $-B$  in this solution? Answer: no reason, the solution could have been given with  $+B$  as well.)

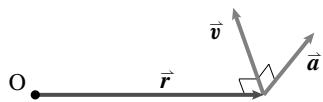


Figure 5.64: (Filename:sfig3.1.three.orthos)

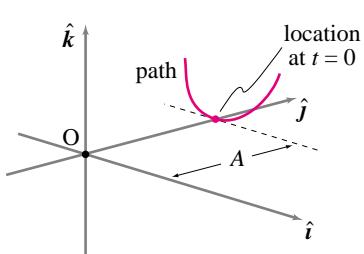


Figure 5.65: (Filename:sfig3.1.three.orthos.a)

**SAMPLE 5.30** Assume the expression for velocity  $\vec{v} (= \frac{d\vec{r}}{dt})$  of a particle is given:  $\vec{v} = v_0\hat{i} - gt\hat{j}$ . Find the expressions for the  $x$  and  $y$  coordinates of the particle at a general time  $t$ , if the initial coordinates at  $t = 0$  are  $(x_0, y_0)$ .

**Solution** The position vector of the particle at any time  $t$  is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}.$$

We are given that

$$\vec{r}(t = 0) = x_0\hat{i} + y_0\hat{j}.$$

Now

$$\begin{aligned}\vec{v} &\equiv \frac{d\vec{r}}{dt} = v_0\hat{i} - gt\hat{j} \\ \text{or} \quad dx\hat{i} + dy\hat{j} &= (v_0\hat{i} - gt\hat{j}) dt.\end{aligned}$$

Dotting both sides of this equation with  $\hat{i}$  and  $\hat{j}$ , we get

$$\begin{aligned}dx &= v_0 dt \\ \Rightarrow \int_{x_0}^x dx &= v_0 \int_0^t dt \\ \Rightarrow x &= x_0 + v_0 t,\end{aligned}$$

and

$$\begin{aligned}dy &= -gt dt \\ \Rightarrow \int_{y_0}^y dy &= -g \int_0^t t dt \\ \Rightarrow y &= y_0 - \frac{1}{2}gt^2.\end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0 t)\hat{i} + (y_0 - \frac{1}{2}gt^2)\hat{j}$$

and the  $(x, y)$  coordinates are

$$\begin{aligned}x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2.\end{aligned}$$

$$(x_0 + v_0 t, y_0 - \frac{1}{2}gt^2)$$

**SAMPLE 5.31** *The path of a particle.* A particle moves in the  $xy$  plane such that its coordinates are given by  $x(t) = at$  and  $y(t) = bt^2$ , where  $a = 2 \text{ m/s}$  and  $b = 0.5 \text{ m/s}^2$ .

- Find the velocity and acceleration of the particle at  $t = 3 \text{ s}$ .
- Show that the path of the particle is neither a straight-line nor a circle.

### Solution

- This problem is straightforward. We are given the position of the particle as a function of time  $t$ . We can find the velocity and acceleration by differentiating the position with respect to time:

$$\begin{aligned}\vec{r}(t) &= \underbrace{at}_{x} \hat{i} + \underbrace{bt^2}_{y} \hat{j} \\ \vec{v} &= \frac{d\vec{r}}{dt} = a\hat{i} + 2bt\hat{j} \\ &= 2 \text{ m/s} \hat{i} + 2 \cdot 0.5 \text{ m/s}^2 \cdot 3 \text{ s} \hat{j} \\ &= (2\hat{i} + 3\hat{j}) \text{ m/s} \\ \vec{a} &= \frac{d\vec{v}}{dt} = 2b\hat{j} \\ &= 1 \text{ m/s}^2 \hat{j}.\end{aligned}$$

$$\boxed{\vec{v} = (2\hat{i} + 3\hat{j}) \text{ m/s}, \vec{a} = 1 \text{ m/s}^2 \hat{j}}$$

- There are many ways to show that the path of the particle is neither a straight line nor a circle. One of the easiest ways is graphical. Calculate the position of the particle at various times and plot a curve through the positions. This curve is the path of the particle. Using a computer, for example, we can plot the path as follows:

```
a = 2,      b = 0.5          % specify constants
t = [0 4 8 ... 20]        % take 6 points from 0-20 sec.
x = a*t,      y = b*t^2    % calculate coordinates
plot x vs y                % plot the particle path
```

A plot so generated is shown in Fig. 5.66. Clearly, the path is neither a straight line nor a circle.

Another way to find the path of the particle is to find an explicit equation of the path. We find this equation by eliminating  $t$  from the expressions for  $x$  and  $y$  and thus relating  $x$  to  $y$ :

$$x = at \Rightarrow t = \frac{x}{a}.$$

Substituting  $x/a$  for  $t$  in  $y = bt^2$ , we get

$$y = \frac{b}{a^2}x^2$$

which is clearly not the equation of a straight line or a circle. In fact, it is the equation of a parabola, i.e., the path of the particle is parabolic.

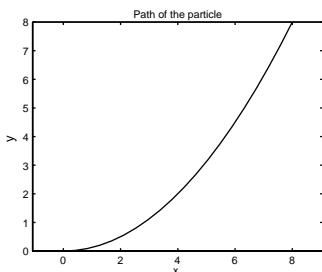


Figure 5.66: The path of the particle generated by computing its coordinates at various points in time.

(Filename:sfig6.1.1a.M)

## 5.8 Spatial dynamics of a particle

One of Isaac Newton's interests was the motion of the planets around the sun. By applying his equation  $\vec{F} = m\vec{a}$ , his law of gravitation, his calculus, and his inimitable geometric reasoning he learned much about the motions of celestial bodies. After learning the material in this section you will know enough to reproduce many of Newton's calculations. You don't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same equations to find motions that Newton could never find, say the trajectory of projectile with a realistic model of air friction. In this chapter, the main approach we take to celestial mechanics and related topics is as follows: ①

- (a) draw a free body diagram of each particle,  $n$  free body diagrams if there are  $n$  particles,
- (b) find the forces on each particle in terms of their positions and velocities and any other external forces (for example, these forces could involve spring, dashpot, gravity, or air friction terms),
- (c) write the linear momentum balance equations for each particle, that is write  $\vec{F} = m\vec{a}$  once for each particle. That is, write  $n$  vector equations.
- (d) break each vector equation into components to make 2 or 3 scalar equations for each vector equation, in 2 or 3 dimensions, respectively.
- (e) write the  $2n$  or  $3n$  equations in first order form. You now have  $4n$  or  $6n$  first order ordinary differential equations in 2 or 3 dimensions, respectively.
- (f) write these first order equations in standard form, with all the time derivatives on the left hand side.
- (g) feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
- (h) plot some aspect(s) of the solution and
  - (i) use the solution to help you find errors in your formulation, and
  - (ii) interpret the solution so that it makes sense to you and increases your understanding of the system of study.

① Eventually you may develop analytic skills which will allow you to shortcut this brute-force numerical approach, at least for some simple problems. For hard problems, even the greatest analytic geniuses resort to methods like those prescribed here.

We can use this approach if the forces on all of the point masses composing the system can be found in terms of their positions, velocities, and the present time. In this section we will just look at the motion of a single particle with forces coming, say, from gravity, springs, dashpots and air drag.

Some problems are of the *instantaneous dynamics* type. That is, they use the equations of dynamics but do not involve tracking motion in time.

**Example: Knowing the forces find the acceleration.**

Say you know the forces on a particle at some instant in time, say  $\vec{F}_1$  and  $\vec{F}_2$ , and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

□

Even some problems involving motion are simple and you can determine most all you want to know with pencil and paper.

**Example: Parabolic trajectory of a projectile**

If we assume a constant gravitational field, neglect air drag, and take the  $y$  direction as up the only force acting on a projectile is  $\vec{F} = -mg\hat{j}$ . Thus the “equations of motion” (linear momentum balance) are

$$-mg\hat{j} = m\vec{a}.$$

If we take the dot product of this equation with  $\hat{i}$  and  $\hat{j}$  (take  $x$  and  $y$  components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{r} = (A + Bt)\hat{i} + (C + Dt - gt^2/2)\hat{j}$$

which is a parametric description of all possible trajectories. By making plots or simple algebra you might convince yourself that these trajectories are parabolas for all possible  $A$ ,  $B$ ,  $C$ , and  $D$ . That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.  $\square$

Some problems are hard and necessitate computer solution.

**Example: Trajectory with quadratic air drag.**

For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is  $\vec{F} = -mg\hat{j} - Cv^2(\vec{v}/v)$ , where  $\vec{v}/v$  is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{j} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with  $\hat{i}$  and  $\hat{j}$  we get

$$\ddot{x} = -(C/m) \left( \sqrt{\dot{x}^2 + \dot{y}^2} \right) \dot{x} \quad \text{and} \quad \ddot{y} = -(C/m) \left( \sqrt{\dot{x}^2 + \dot{y}^2} \right) \dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations and can be solved numerically.  $\square$

Some special problems turn out to be easy, though you might not realize it at first glance.

**Example: Zero-length spring**

Imagine a massless spring whose unstretched length is zero (see chapter 2 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin:  $\vec{F} = -k\vec{r}$ . Thus linear momentum balance yields

$$-k\vec{r} = m\vec{a}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the  $x$  direction and one in the  $y$  direction. The general solution is

$$\vec{r} = \left( A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t \right) \hat{i} + \left( C \cos \sqrt{\frac{k}{m}} t + D \sin \sqrt{\frac{k}{m}} t \right) \hat{j}$$

which is always an ellipse (special cases of which are a circle and a straight line).  $\square$

Some problems are within the reach of advanced analytic methods, but can also be solved with a computer.

**Example: Path of the earth around the sun.**

Assume the sun is big and unmoving with mass  $M$  and the earth has mass  $m$ . Take the origin to be at the sun. The force on the earth is  $\vec{F} = -(mMG/r^2)(\vec{r}/r)$  where  $\vec{r}/r$  is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$\frac{mMGr}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it. But the solution is beyond this course. On the other hand the equation of motion is easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does.

$\square$

## The work-energy equation

Energy balance is one of the basic governing equations. For a single particle with no stored internal energy, the energy balance equation is

$$P = \frac{d}{dt} E_K \quad (\text{III}d)$$

Before getting into the technical definitions of the terms, let's first summarize the most basic of the energy equations in words.

The power  $P$  of *all* the external forces acting on a particle is the rate of change of its kinetic energy  $\dot{E}_K$ .

From other physics texts and courses, you know energy principles help you solve a variety of simple problems, both in mechanics and other parts of physics. In many engineering applications, one can determine useful things about the motion of a machine or object by thinking about its energy and change of energy.

For particles and rigid bodies that interact in the simple ways we consider in this book, the energy equations can be derived from the momentum balance equations. They follow logically.

However, in practice, one uses the various work-energy relations as if they were independent. Sometimes energy equations can be used in place of, or as a check of, momentum balance equations.

## Neglecting the right hand side

The right hand side of the energy equations is the rate of change of kinetic energy. This term is not zero if the speeds of the various mass points change non-negligibly. But, for negligible motion, we neglect all terms that involve motion, in this case  $v_i$  and  $\dot{v}_i$ . Thus, we assume that

$$\dot{E}_K = 0.$$

Thus, for better or worse, equation 5.8 reduces to

$$P = 0. \quad (5.75)$$

The net power into the system is zero. Equation 5.75 is useful for system that can be modeled as having constant (or zero) kinetic energy.

### *The power into a system $P$*

In mechanics, the sources of power are applied forces. The power of an applied force  $\vec{F}$  acting on a particle is

$$P = \vec{F} \cdot \vec{v},$$

where  $\vec{v}$  is the velocity of the point of the material body being acted on by the force.

If many forces are applied, then

$$P = \sum \vec{F}_i \cdot \vec{v}.$$

## The work of a force $\vec{F}$ : $W_{12}$

Previously in Physics, and more recently in one dimensional mechanics, you learned that

*Work is force times distance.*

This is actually a special case of the formula

$$P = \vec{F} \cdot \vec{v}.$$

How is that? If  $\vec{F}$  is constant and parallel to the displacement  $\Delta\vec{x}$ , then

$$\begin{aligned} \text{Work} &= \int \dot{W} dt = \int P dt = \int \underbrace{\vec{F} \cdot \vec{v}}_{d\vec{x}} dt = \int \vec{F} \cdot d\vec{x} = \vec{F} \cdot \int d\vec{x} \\ &= \vec{F} \cdot \Delta\vec{x} = F\Delta x = \text{Force} \cdot \text{distance}. \end{aligned}$$

Or,

$$dW = \dot{W} dt = P dt = \vec{F} \cdot \vec{v} dt = \vec{F} \cdot d\vec{x} \text{ (or } \vec{F} \cdot d\vec{r}).$$

Being a little more precise about notation, we can write that the work of a force acting on a particle or body in moving from state 1 to state 2 is

$$W_{12} \equiv \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \quad (5.76)$$

where the path of integration is the path of the material point at which the force is applied.

## Potential energy of a force

Some forces have the property that the work they do is independent of the path followed by the material point (or pair of points between which the force acts). If the work of a force is path independent in this way, then a potential energy can be defined so that the work done by the force is the decrease in the Potential Energy  $E_P$ :  $W_{12} = E_{P1} - E_{P2}$ . The common examples are listed below:

- **linear spring:**  $E_P = (1/2)k(\text{stretch})^2$ .
- **gravity near earth's surface:**  $E_P = mgh$
- **gravity between spheres or points:**  $E_P = -MmG/r$
- **constant force  $\vec{F}$  acting on a point:**  $E_P = -\vec{F} \cdot \vec{r}$

In the cases of the spring and gravity between spheres, the change in potential energy is the net work done by the spring or gravity on the pair of objects between which the force acts. If both ends of a spring are moving, the net work of the spring on the two objects to which it is connected is the decrease in potential energy of the spring.

There is a possible source of confusion in our using the same symbol  $E_P$  to represent the potential work of an external force and for internal potential energy. In practice, however, they are used identically, so we use the same symbol for both. The potential energy in a stretched spring is the same whether it is the cause of force on a system or it is internal to the system.

## Forces that do no work

Fortunately for the evaluation of power and work one often encounters forces that do no work or forces that come in pairs where the pair of forces does no net work.

The net work done by the interaction force between body  $A$  and body  $B$  is zero if the force on body  $A$  dotted with the relative velocity of  $A$  and  $B$  is zero. Examples are:

- frictionless sliding,
- the force caused by a magnetic field on a moving charged particle.

## Summary

To find the motion of a particle you draw a free body diagram, write the linear momentum balance equation and then solve it, most often on the computer. The power and energy equations can sometimes be used to check your solution and to determine special features of the solution, and in special case.

## 5.7 THEORY

### *Angular momentum and energy of a point mass*

For a point-mass particle, we can derive the angular momentum equation (II) and the energy equation (III) from linear momentum balance in a snap.

For a single particle we have  $\vec{F} = m\vec{a}$ . Taking the cross product of both sides with the position relative to a point  $C$  gives:

$$\vec{r}_{/C} \times \vec{F} = \vec{r}_{/C} \times (m\vec{a}).$$

For a single point-mass particle the angular momentum equation is a direct un-refutable consequence of the linear momentum balance equation.

The power equation is found with a shade more difficulty. We take the equation  $\vec{F} = m\vec{a}$  and dot both sides with the velocity  $\vec{v}$  of the particle:

$$\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v}. \quad (5.77)$$

Evaluating  $\vec{v} \cdot \vec{a}$  is most easily done with the benefit of hindsight. So we cheat and look at the time derivative of the speed squared:

$$\frac{d}{dt} \left( \frac{1}{2} v^2 \right) = \frac{1}{2} \frac{d}{dt} (\vec{v} \cdot \vec{v})$$

$$\begin{aligned} &= \frac{1}{2} (\hat{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \hat{\vec{v}}) \\ &= \vec{v} \cdot \hat{\vec{v}} \\ &= \vec{v} \cdot \vec{a} \end{aligned}$$

Applying this result to equation 5.77 we get

$$\underbrace{\vec{F} \cdot \vec{v}}_P = \frac{d}{dt} \underbrace{\left( \frac{1}{2} mv^2 \right)}_{E_K},$$

the energy (or power balance) equation for a particle.

So for one particle angular momentum balance and power balance (eqns. II and III on the inside cover) follow directly from  $\vec{F} = m\vec{a}$ .

**SAMPLE 5.32** Find  $\vec{L}$ ,  $\dot{\vec{L}}$ ,  $\vec{H}_C$ ,  $\dot{\vec{H}}_C$ ,  $E_K$ ,  $\dot{E}_K$  for a given particle P with mass  $m_P = 1 \text{ kg}$ , given position, velocity, acceleration, and a point C. Specifically, we are given  $\vec{r}_P = (\hat{i} + \hat{j} + \hat{k}) \text{ m}$ ,  $\vec{v}_P = 3 \text{ m/s}(\hat{i} + \hat{j})$ ,  $\vec{a}_P = 2 \text{ m/s}^2(\hat{i} - \hat{j} - \hat{k})$ , and  $\vec{r}_C = (2\hat{i} + \hat{k}) \text{ m}$ .

**Solution** Since  $\vec{r}_P = (\hat{i} + \hat{j} + \hat{k}) \text{ m}$  and  $\vec{r}_C = (2\hat{i} + \hat{k}) \text{ m}$ ,

$$\vec{r}_{P/C} = \vec{r}_P - \vec{r}_C = (-\hat{i} + \hat{j}) \text{ m}.$$

So we have the motion quantities

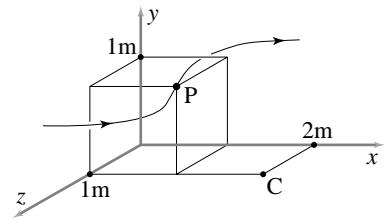


Figure 5.67: (Filename:sfig1.1.DH1)

$$\begin{aligned}\vec{L} &= m\vec{v}_P \\ &= (1 \text{ kg}) \cdot [(3 \text{ m/s})(\hat{i} + \hat{j})] \\ &= 3(\hat{i} + \hat{j}) \frac{\text{kg} \cdot \text{m}}{\text{s}} \\ &= 3 \text{ N} \cdot \text{s}(\hat{i} + \hat{j})\end{aligned}$$

$$\begin{aligned}\dot{\vec{L}} &= m\vec{a}_P \\ &= (1 \text{ kg})[(2 \text{ m/s}^2)(\hat{i} - \hat{j} - \hat{k})] \\ &= 2(\hat{i} - \hat{j} - \hat{k}) \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \\ &= 2 \text{ N}(\hat{i} - \hat{j} - \hat{k})\end{aligned}$$

$$\begin{aligned}\vec{H}_C &= \vec{r}_{P/C} \times m\vec{v}_P \\ &= [(-\hat{i} + \hat{j}) \text{ m}] \times [(1 \text{ kg})3 \text{ m/s}(\hat{i} + \hat{j})] \\ &= -6 \frac{\text{kg} \cdot \text{m}^2}{\text{s}} \hat{k}\end{aligned}\tag{5.78}$$

$$\begin{aligned}\dot{\vec{H}}_C &= \vec{r}_{P/C} \times m\vec{a} \\ &= [(-\hat{i} + \hat{j}) \text{ m}] \times [(1 \text{ kg})2 \text{ m/s}^2(\hat{i} - \hat{j} - \hat{k})] \\ &= -2 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2}(\hat{i} + \hat{j})\end{aligned}$$

$$\begin{aligned}E_K &= \frac{1}{2}m|\vec{v}_P|^2 \\ &= \frac{1}{2}(1 \text{ kg})(3\sqrt{2} \text{ m/s})^2 \\ &= 9 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \\ &= 9 \text{ N} \cdot \text{m}\end{aligned}$$

$$\begin{aligned}\dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}m\vec{v}_P \cdot \vec{v}_P\right) \\ &= \frac{m}{2}[\vec{v}_P \cdot \dot{\vec{v}}_P + \dot{\vec{v}}_P \cdot \vec{v}_P] \\ &= m\vec{v}_P \cdot \vec{a}_P \\ &= 1 \text{ kg}[(3 \text{ m/s})(\hat{i} + \hat{j})] \cdot [(2 \text{ m/s}^2)(\hat{i} - \hat{j} - \hat{k})] \\ &= 0.\end{aligned}$$

Note:  $\frac{d}{dt}\left(\frac{1}{2}v^2\right) \neq |\vec{v}| |\vec{a}|$ .

**SAMPLE 5.33** *Linear momentum: direct application of formula.* A 2 kg block is moving with a velocity  $\vec{v}(t) = u_0 e^{-ct} \hat{i} + v_0 \hat{j}$ , where  $u_0 = 5 \text{ m/s}$ ,  $v_0 = 10 \text{ m/s}$ , and  $c = 0.5/\text{s}$ .

- Find the linear momentum  $\vec{L}$  and its rate of change  $\dot{\vec{L}}$  at  $t = 5 \text{ s}$ .
- What is the net change in linear momentum of the block from  $t = 0 \text{ s}$  to  $t = 5 \text{ s}$ ?

**Solution** Since  $\vec{L} = m\vec{v}$  and  $\dot{\vec{L}} = \frac{d\vec{L}}{dt}$ ; for the given block we have

(a)

$$\begin{aligned}\vec{L}(t) &= m(u_0 e^{-ct} \hat{i} + v_0 \hat{j}) \\ \dot{\vec{L}}(t) &= m(-u_0 c e^{-ct} \hat{i}).\end{aligned}$$

Substituting the given values,  $m = 2 \text{ kg}$ ,  $u_0 = 5 \text{ m/s}$ ,  $v_0 = 10 \text{ m/s}$ ,  $c = 0.5/\text{s}$  and  $t = 5 \text{ s}$ , we get

$$\begin{aligned}\vec{L}(5 \text{ s}) &= 2 \text{ kg}(5 \text{ m/s} \cdot e^{-2.5} \hat{i} + 10 \text{ m/s} \hat{j}) \\ &= (0.82 \hat{i} + 20 \hat{j}) \text{ kg} \cdot \text{m/s} \\ \dot{\vec{L}}(5 \text{ s}) &= 2 \text{ kg}(-5 \text{ m/s} \cdot 0.5/\text{s} \cdot e^{-2.5} \hat{i}) \\ &= -0.41 \text{ kg} \cdot \text{m/s}^2 \hat{i} = -0.41 \text{ N} \hat{i}.\end{aligned}$$

$$\boxed{\vec{L} = (0.82 \hat{i} + 20 \hat{j}) \text{ kg} \cdot \text{m/s}, \quad \dot{\vec{L}} = -0.41 \text{ N} \hat{i}}$$

(b)

$$\begin{aligned}\vec{L}(0 \text{ s}) &= 2 \text{ kg}(5 \text{ m/s} \cdot e^0 \hat{i} + 10 \text{ m/s} \hat{j}) \\ &= (10 \hat{i} + 20 \hat{j}) \text{ kg} \cdot \text{m/s}.\end{aligned}$$

Therefore, the net change in the linear momentum in  $t = 0 \text{ s}$  to  $t = 5 \text{ s}$  is,

$$\begin{aligned}\Delta \vec{L} &= \vec{L}(5 \text{ s}) - \vec{L}(0 \text{ s}) \\ &= (0.82 \hat{i} + 20 \hat{j}) \text{ kg} \cdot \text{m/s} - (10 \hat{i} + 20 \hat{j}) \text{ kg} \cdot \text{m/s} \\ &= -9.18 \text{ kg} \cdot \text{m/s} \hat{i}.\end{aligned}$$

Note that the net change is only in  $x$ -direction. This result makes sense because the  $y$ -component of  $\vec{L}$  is constant and therefore,  $y$ -component of  $\dot{\vec{L}}$  is zero.

$$\boxed{\Delta \vec{L} = -9.18 \text{ kg} \cdot \text{m/s} \hat{i}}$$

**SAMPLE 5.34** *Angular momentum: direct application of the formula.* The position of a particle of mass  $m = 0.5 \text{ kg}$  is  $\vec{r}(t) = \ell \sin(\omega t)\hat{i} + h\hat{j}$ ; where  $\omega = 2 \text{ rad/s}$ ,  $h = 2 \text{ m}$ ,  $\ell = 2 \text{ m}$ , and  $\vec{r}$  is measured from the origin.

- Find the angular momentum  $\vec{H}_O$  of the particle about the origin at  $t = 0 \text{ s}$  and  $t = 5 \text{ s}$ .
- Find the rate of change of angular momentum  $\dot{\vec{H}}$  about the origin at  $t = 0 \text{ s}$  and  $t = 5 \text{ s}$ .

**Solution** since  $\vec{H}_O = \vec{r}_{/O} \times m\vec{v}$  and  $\dot{\vec{H}}_O = \vec{r}_{/O} \times m\vec{a}$ , we need to find  $\vec{r}$ ,  $\vec{v}$  and  $\vec{a}$  to compute  $\vec{H}_O$  and  $\dot{\vec{H}}_O$ . Now,

$$\begin{aligned}\vec{r}(t) &= \ell \sin(\omega t)\hat{i} + h\hat{j} \\ \Rightarrow \vec{v}(t) &= \dot{\vec{r}}(t) = \ell \omega \cos(\omega t)\hat{i} + 0\hat{j} \\ \Rightarrow \vec{a}(t) &= \ddot{\vec{r}}(t) = -\ell \omega^2 \sin(\omega t)\hat{i}\end{aligned}$$

- (a) Since the position is measured from the origin,

$$\vec{r}_{/O} = \ell \sin(\omega t)\hat{i} + h\hat{j}.$$

Therefore,

$$\begin{aligned}\vec{H}_O &= \vec{r}_{/O} \times m\vec{v} = (\ell \sin(\omega t)\hat{i} + h\hat{j}) \times m(\ell \omega \cos(\omega t)\hat{i}) \\ &= m\ell^2 \omega \sin(\omega t) \cos(\omega t)(\hat{i} \times \hat{i}) + m\ell \omega h \cos(\omega t)(\hat{j} \times \hat{i}) \\ &= -m\ell \omega h \cos(\omega t)\hat{k}.\end{aligned}$$

Now we can substitute the desired values:

$$\begin{aligned}\vec{H}_O(0 \text{ s}) &= -(0.5 \text{ kg}) \cdot (2 \text{ rad/s}) \cdot (2 \text{ m}) \cdot (2 \text{ m}) \cdot \cos(0)\hat{k} \\ &= -4 \text{ kg} \cdot \text{m}^2/\text{s}\hat{k} \\ \vec{H}_O(5 \text{ s}) &= -(4 \text{ kg} \cdot \text{m}^2/\text{s}) \cdot \cos(2 \text{ rad/s} \cdot 5 \text{ s}) = 3.36 \text{ kg} \cdot \text{m}^2/\text{s}\hat{k}.\end{aligned}$$

$\vec{H}_O(0 \text{ s}) = -4 \text{ kg} \cdot \text{m}^2/\text{s}\hat{k}, \quad \vec{H}_O(5 \text{ s}) = 3.36 \text{ kg} \cdot \text{m}^2/\text{s}\hat{k}$

(b)

$$\begin{aligned}\dot{\vec{H}}_O &= \vec{r}_{/O} \times m\vec{a} \\ &= (\ell \sin(\omega t)\hat{i} + h\hat{j}) \times m(-\ell \omega^2 \sin(\omega t)\hat{i}) \\ &= ml\omega^2 h \sin(\omega t)\hat{k}\end{aligned}$$

Substituting the values of constants and the time, we get  $\dot{\vec{H}}_O(0 \text{ s}) = \vec{0}$ , and

$$\begin{aligned}\dot{\vec{H}}_O(5 \text{ s}) &= (0.5 \text{ kg}) \cdot (2 \text{ m}) \cdot (2 \text{ rad/s})^2 \cdot (2 \text{ m}) \cdot \sin(2 \text{ rad/s} \cdot 5 \text{ s})\hat{k} \\ &= -4.35 \text{ (kg} \cdot \text{m/s}^2) \cdot \text{m}\hat{k} = -4.35 \text{ N} \cdot \text{m}\hat{k}.\end{aligned}$$

$\dot{\vec{H}}_O(0 \text{ s}) = \vec{0}, \quad \dot{\vec{H}}_O(5 \text{ s}) = -4.35 \text{ N} \cdot \text{m}\hat{k}$

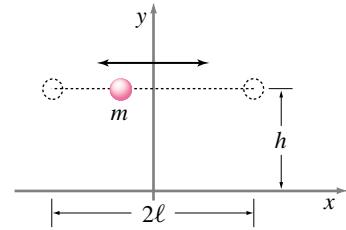


Figure 5.68: (Filename:sfig3.2.direct.appl)

**Comments:** Note that both  $\vec{H}$  and  $\dot{\vec{H}}$  point out of the plane, in the  $\hat{k}$  direction.  $\vec{H}$  and  $\dot{\vec{H}}$  are always in the  $\hat{k}$  direction for all motions in the  $xy$ -plane for all masses in the  $xy$ -plane (provided, of course, that the reference point about which  $\vec{H}$  and  $\dot{\vec{H}}$  are calculated also lies in the  $xy$  plane).

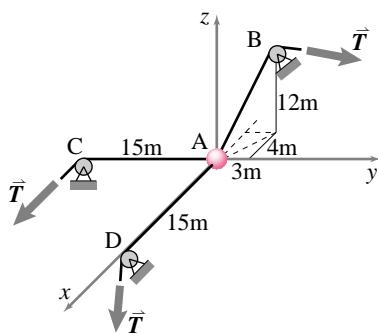


Figure 5.69: A ball in 3-D

(Filename:sfig2.1.10a)

**SAMPLE 5.35** Acceleration of a point mass in 3-D. A ball of mass  $m = 13 \text{ kg}$  is being pulled by three strings as shown in Fig. 5.69. The tension in each string is  $T = 13 \text{ N}$ . Find the acceleration of the ball.

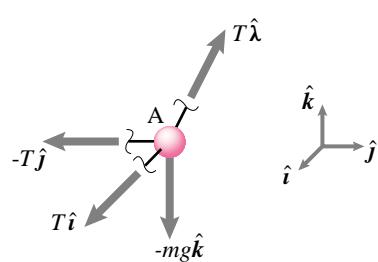


Figure 5.70: FBD of the ball

(Filename:sfig2.1.10b)

**Solution** The forces acting on the body are shown in the free body diagram in Fig.5.70. From geometry:

$$\begin{aligned}\hat{\lambda} &= \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{-4\hat{i} + 3\hat{j} + 12\hat{k}}{\sqrt{4^2 + 3^2 + 12^2}} \\ &= \frac{-4\hat{i} + 3\hat{j} + 12\hat{k}}{13}.\end{aligned}$$

Balance of linear momentum for the ball:

$$\sum \vec{F} = m\vec{a} \quad (5.79)$$

$$\begin{aligned}\sum \vec{F} &= T\hat{i} - T\hat{j} + T\hat{\lambda} - mg\hat{k} \\ &= T\left(\hat{i} - \hat{j} + \frac{-4\hat{i} + 3\hat{j} + 12\hat{k}}{13}\right) - mg\hat{k} \\ &= \frac{T}{13}(9\hat{i} + 10\hat{j} + 12\hat{k}) - mg\hat{k}.\end{aligned}$$

Substituting  $\sum \vec{F}$  in eqn. (5.79):

$$\vec{a} = \frac{T}{13m}(9\hat{i} + 10\hat{j} + 12\hat{k}) - g\hat{k}.$$

Now plugging in the given values:  $T = 13 \text{ N}$ ,  $m = 13 \text{ kg}$ , and  $g = 10 \text{ m/s}^2$ , we get

$$\begin{aligned}\vec{a} &= \frac{13 \text{ N}}{13 \cdot 13 \text{ kg}}(9\hat{i} - 10\hat{j} + 12\hat{k}) - 10 \text{ m/s}^2\hat{k} \\ &= (0.69\hat{i} - 0.77\hat{j} - 9.08\hat{k}) \text{ m/s}^2.\end{aligned}$$

$\boxed{\vec{a} = (0.69\hat{i} - 0.77\hat{j} - 9.08\hat{k}) \text{ m/s}^2}$



**SAMPLE 5.36** *Trajectory of a food-bag.* In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of  $g$ , the gravitational acceleration, to be  $10 \text{ m/s}^2$ .

**Solution** You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the  $x$  and  $y$  directions, realizing that there is no force in the  $x$  direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free body diagram of the bag during its free flight is shown in Fig. 5.71. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{a} = -mg\hat{j}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag  $\vec{r}(0) = h\hat{j} = 80 \text{ m}\hat{j}$ . The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{v}(0) \equiv \dot{\vec{r}}(0) = v_x\hat{i} = 200 \text{ km/h}\hat{i}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{r}} = -g\hat{j}$$

with two initial conditions:

$$\vec{r}(0) = h\hat{j} \quad \text{and} \quad \dot{\vec{r}}(0) = v_x\hat{i}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors  $\hat{i}$  and  $\hat{j}$  do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{r}} &\equiv \frac{d\dot{\vec{r}}}{dt} = -g\hat{j} \\ \int d\dot{\vec{r}} &= -\hat{j} \int g dt \\ \text{or} \quad \dot{\vec{r}} &= -gt\hat{j} + \vec{c}_1 \end{aligned} \tag{5.80}$$

and integrating once again, we get

$$\begin{aligned} \vec{r} &= \int (-gt\hat{j} + \vec{c}_1) dt \\ &= -\frac{1}{2}gt^2\hat{j} + \vec{c}_1 t + \vec{c}_2 \end{aligned} \tag{5.81}$$

where  $\vec{c}_1$  and  $\vec{c}_2$  are constants of integration and are vector quantities. Now substituting the initial conditions in eq6.1.2.1 and eq6.1.2.2 we get

$$\begin{aligned} \dot{\vec{r}}(0) &= v_x\hat{i} = \vec{c}_1, \quad \text{and} \\ \vec{r}(0) &= h\hat{j} = \vec{c}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{r}(t) &= -\frac{1}{2}gt^2\hat{j} + v_x t \hat{i} + h\hat{j} \\ &= v_x t \hat{i} + (h - \frac{1}{2}gt^2)\hat{j}. \end{aligned}$$

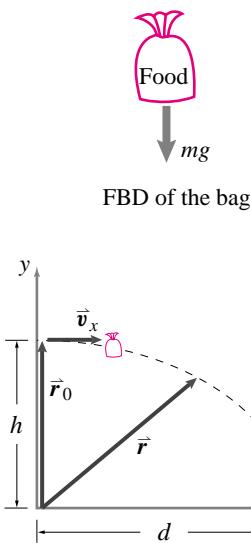


Figure 5.71: Free body diagram of the bag and the geometry of its motion.

(Filename:sfig6.1.2a)

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the  $x$ -component of  $\vec{r}$ , i.e.,  $v_x t$ . But we do not know  $t$ . However, when the bag hits the ground, its position vector has no  $y$ -component, i.e., we can write  $\vec{r} = d\hat{i} + 0\hat{j}$  where  $d$  is the distance we are interested in. Now equating the components of  $\vec{r}$  with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2} g t^2.$$

Solving for  $t$  from the second equation and substituting in the first equation we get

$$d = v_x \sqrt{\frac{2h}{g}} = \frac{200 \text{ km}}{3600 \text{ s}} \cdot \sqrt{\frac{2 \cdot 80 \text{ m}}{10 \text{ m/s}^2}} = \frac{2}{9} \text{ km} \approx 222 \text{ m}.$$

$d = 222 \text{ m}$

**Comments:** Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too “*mathy*” or intimidating to you then follow the usual scalar way of doing this problem.

*The scalar method:*

From the linear momentum balance,  $-mg\hat{j} = m\vec{a}$ , writing the acceleration as  $\vec{a} = a_x\hat{i} + a_y\hat{j}$  and equating the  $x$  and  $y$  components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3,  $x = x_0 + v_0 t + \frac{1}{2} a t^2$ , in both  $x$  and  $y$  directions, we get

$$\begin{aligned} d &= \overbrace{x_0}^0 + v_x t + \frac{1}{2} \overbrace{a_x}^0 t^2 \\ &= v_x t \\ 0 &= \overbrace{y_0}^h + \overbrace{v_y}^0 t + \frac{1}{2} \overbrace{a_y}^{-g} t^2 \\ &= h - \frac{1}{2} g t^2 \\ \Rightarrow t &= \sqrt{\frac{2h}{g}}. \end{aligned}$$

Substituting for  $t$  in the equation for  $d$  we get

$$d = v_x \sqrt{\frac{2h}{g}} = \frac{200 \text{ km}}{3600 \text{ s}} \cdot \sqrt{\frac{2 \cdot 80 \text{ m}}{10 \text{ m/s}^2}} = \frac{2}{9} \text{ km} \approx 222 \text{ m}.$$

as above.

**SAMPLE 5.37** *Projectile motion with air drag.* A projectile is fired into the air at an initial angle  $\theta_0$  and with initial speed  $v_0$ . The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be  $k$ . Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

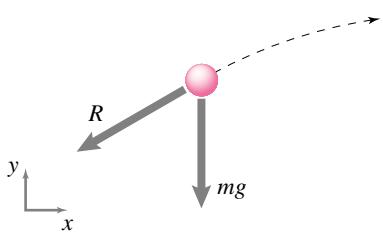


Figure 5.72: FBD of the projectile.

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**Solution** The free body diagram of the projectile is shown in the figure at some constant  $t$  during motion. At the instant shown, let the velocity of the projectile be  $\vec{v} = v\hat{e}_t$ , where

$$\hat{e}_t = \cos \theta \hat{i} + \sin \theta \hat{j}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or } -kv^2\hat{e}_t - mg\hat{j} &= m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \end{aligned} \quad (5.82)$$

Noting that  $v = |\vec{v}| = |\dot{x}\hat{i} + \dot{y}\hat{j}| = \sqrt{\dot{x}^2 + \dot{y}^2}$ , and dotting both sides of equation 5.82 with  $\hat{i}$  and  $\hat{j}$  we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t \cdot \hat{i}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t \cdot \hat{j}) - mg &= m\ddot{y} \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)\cos \theta \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)\sin \theta \end{aligned}$$

Note that  $\theta$  changes with time. We can express  $\theta$  in terms of  $\dot{x}$  and  $\dot{y}$  because  $\theta$  is the slope of the trajectory:

$$\begin{aligned} \theta &= \tan^{-1} \frac{dy}{dx} = \tan^{-1} \frac{dy/dt}{dx/dt} = \tan^{-1} \frac{\dot{y}}{\dot{x}} \quad (\text{i.e., } \tan \theta = \frac{\dot{y}}{\dot{x}}) \\ \Rightarrow \quad \cos \theta &= \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad \text{and} \quad \sin \theta = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}. \end{aligned}$$

Substituting these expression in to the equations for  $\ddot{x}$  and  $\ddot{y}$  we get

$$\boxed{\ddot{x} = -\frac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\frac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

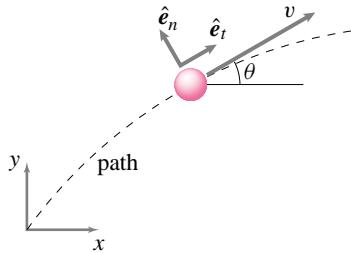


Figure 5.73: (Filename:sfig6.4.DH2)

**SAMPLE 5.38** *Cartoon mechanics: The cannon.* It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

- 1) Students don't have trouble with dynamics! They love the subject.
- 2) Nowadays many cartoons are made using 'correct' mechanics, and
- 3) the cartoons are sometimes more accurate than the pedagogues anyway.

**Problem:** What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

**Solution** The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$\begin{aligned}\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\ &= cv^2 \cdot \left( \frac{-\vec{v}}{|\vec{v}|} \right) \\ &= -c|\vec{v}|\vec{v} \\ &= -c\sqrt{\dot{x}^2 + \dot{y}^2}(\dot{x}\hat{i} + \dot{y}\hat{j})\end{aligned}$$

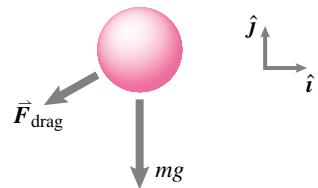


Figure 5.74: (Filename:sfig3.5.cart.cannon)

So LMB gives

$$\begin{gathered}\sum \vec{F} = \dot{\vec{L}} \\ \left\{ -mg\hat{j} - c\sqrt{\dot{x}^2 + \dot{y}^2}(\dot{x}\hat{i} + \dot{y}\hat{j}) = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \right\}\end{gathered}$$

$$\begin{aligned}\{\} \cdot \hat{i} \Rightarrow \ddot{x} &= \left[ -c\sqrt{\dot{x}^2 + \dot{y}^2}\dot{x}/m \right] \\ \ddot{y} &= -c\sqrt{\dot{x}^2 + \dot{y}^2}\dot{y}/m - g\end{aligned}$$

Solving these equations numerically with reasonable values <sup>①</sup> of  $\dot{x}_0$ ,  $\dot{y}_0$ ,  $m$  and  $c$  gives

<sup>①</sup> To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

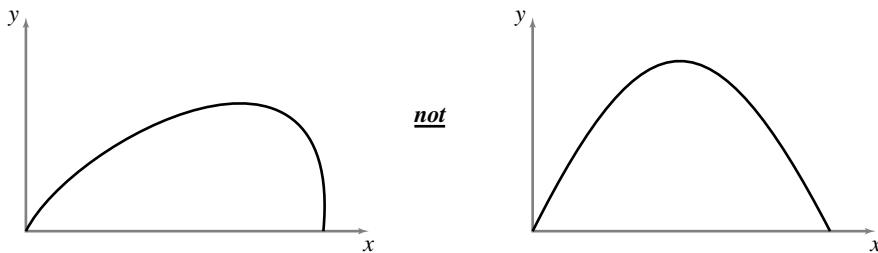


Figure 5.75: (Filename:sfig3.5.cart.cannon.graph)

which is closer to a cartoon's triangle than to a naive physicist's parabola.

## 5.9 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation  $\vec{F} = m\vec{a}$  and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

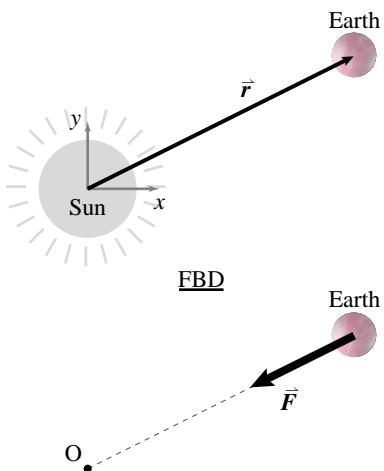


Figure 5.76: The earth moving around a fixed sun. The attraction force  $\vec{F}$  is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

(Filename:tfigure.earthfixedsun)

① Soon after Newton, Cavendish found  $G$  in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on radius.

If we define the position of the particle as  $\vec{r}$  with magnitude  $r$ , linear momentum balance for central-force motion is

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow \vec{F} &= m\vec{a} \\ \Rightarrow F(r) \left( \frac{-\vec{r}}{r} \right) &= m\ddot{\vec{r}}, \end{aligned} \quad (5.83)$$

where  $-\vec{r}/r$  is a unit vector pointed toward the origin and  $F(r)$  is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of eqn. (5.83). We start with the most historically important example.

### Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (Fig. 5.76). Thus we have ①

$$F = \frac{Gm_e m_s}{r^2}$$

where  $m_e$  and  $m_s$  are the masses of the earth and sun,  $r$  is the distance between the earth and sun. 'Big G' is a universal constant  $G \approx 6.67 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2$ . What

is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned}\vec{F} &= \left( \frac{Gm_e m_s}{r^2} \right) \left( \frac{-\vec{r}}{|\vec{r}|} \right) \\ \Rightarrow \vec{F} &= -Gm_e m_s \left( \frac{\vec{r}}{r^3} \right) \\ \Rightarrow \vec{F} &= -Gm_e m_s \left( \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}} \right)\end{aligned}\quad (5.84)$$

where we have used that  $\vec{r} = x\hat{i} + y\hat{j}$ ,  $r = |\vec{r}| = \sqrt{x^2 + y^2}$ , and  $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$ . Now we can write the linear momentum balance equation for the earth in great detail.

$$\vec{F} = m\vec{a} \Rightarrow -Gm_e m_s \left( \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}} \right) = m_e (\ddot{x}\hat{i} + \ddot{y}\hat{j}) \quad (5.85)$$

Taking the dot product of equation 5.85 with  $\hat{i}$  and  $\hat{j}$  successively (*i.e.*, taking  $x$  and  $y$  components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-Gm_s x}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-Gm_s y}{(x^2 + y^2)^{3/2}}. \quad (5.86)$$

This pair of coupled second order differential equations describes the motion of the earth.<sup>①</sup> Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations eqn. (5.86) in the form of a set of coupled first order ordinary differential equations. If we define  $z_1 = x$ ,  $z_2 = \dot{x}$ ,  $z_3 = y$ , and  $z_4 = \dot{y}$ . We can now write equations 5.86 as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -Gm_s z_1 / (z_1^2 + z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -Gm_s z_3 / (z_1^2 + z_3^2)^{3/2}.\end{aligned}\quad (5.87)$$

To actually solve these numerically we need a value for  $Gm_s$  and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

<sup>①</sup> Note that  $G$  appears in the product  $Gm_s$ . Newton didn't know the value of big  $G$ , but he could do a lot of figuring without it. All he needed was the product  $Gm_s$  which he could find from the period and radius of the earth's orbit. The entanglement of  $G$  with the mass of the sun is why some people call Cavendish's measurement of big  $G$ , "weighing the sun". From Newton's calculation of  $Gm_s$  and Cavendish's measurement of  $G$  you can find  $m_s$ . Naturally, the real history is a bit more complicated. Cavendish presented his result as weighing the earth.

## Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency  $\omega$  and radius  $r_o$  obeys the parametric equation

$$\begin{aligned}\vec{r} &= r_o (\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}) \\ \text{differentiating twice} \Rightarrow \ddot{\vec{r}} &= -\omega^2 r_o (\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}) \\ &= -\omega^2 \vec{r}.\end{aligned}\quad (5.88)$$

Comparing eqn. (5.88) with eqn. (5.83) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where  $m = m_e$  we have  $F(r) = Gm_s m_e / r^2$  so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \quad \Rightarrow \quad T = 2\pi \sqrt{\frac{r^3}{Gm_s}} \quad (5.89)$$

because angular frequency is inversely proportional to the period ( $\omega = 2\pi/T$ ). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

## Conservation of energy

Any force of the form

$$\vec{F} = -F(r) \frac{\vec{r}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_P = \int F(r) dr.$$

For the case of gravitational attraction, the potential energy is

$$E_P = \frac{-Gm_s m_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned} \text{Constant} &= E_K + E_P \\ &= \frac{1}{2}mv^2 + \frac{-Gm_s m_e}{r} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_s m_e}{\sqrt{x^2 + y^2}}. \end{aligned} \quad (5.90)$$

If that constant is bigger than zero than the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than "escape velocity" and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

## Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for  $m_s$  and the mass of the satellite for  $m_e$ . Thus we can write the governing equation eqn. (5.85) as

$$-GMm \left( \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}} \right) = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \quad (5.91)$$

where now  $M$  is the mass of the earth and  $m$  is the mass of the satellite. At the surface of the earth  $r = R$ , the earth's radius, and  $GM/R^2 = g$  so we can rewrite the governing equation for rockets and the like as

$$-gR^2 \left( \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}} \right) = (\ddot{x}\hat{i} + \ddot{y}\hat{j}). \quad (5.92)$$

## Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned}\vec{F} &= m\vec{a} \\ -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{i} + y\hat{j}) &= m(\ddot{x}\hat{i} + \ddot{y}\hat{j}).\end{aligned}\quad (5.93)$$

Dotting both sides with  $\hat{i}$  and  $\hat{j}$  we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{and} \quad \ddot{y} + \frac{k}{m}y = 0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x = A \cos(\lambda t) + B \sin(\lambda t) \quad \text{and} \quad y = C \cos(\lambda t) + D \sin(\lambda t) \quad (5.94)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are arbitrary constants which are determined by initial conditions. For all  $A$ ,  $B$ ,  $C$ , and  $D$  eqn. (5.94) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

## Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation eqn. (5.83) and take the cross product of both sides with  $\vec{r}$  we get the following.

$$\begin{aligned}\vec{F} &= m\vec{a} \\ \Rightarrow F(r)\left(\frac{-\vec{r}}{r}\right) &= m\ddot{\vec{r}} \\ \Rightarrow \vec{r} \times \left(F(r)\left(\frac{-\vec{r}}{r}\right)\right) &= \vec{r} \times (m\ddot{\vec{r}}) \\ \Rightarrow \vec{0} &= \frac{d}{dt}(m\vec{r} \times \dot{\vec{r}}) \quad (\text{because } \dot{\vec{r}} \times \ddot{\vec{r}} = \vec{0}) \\ \Rightarrow \text{constant} &= m\vec{r} \times \dot{\vec{r}}.\end{aligned}\quad (5.95)$$

But this last quantity is exactly the rate at which area is swept out by a moving particle. Thus Kepler's third law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from  $\vec{F} = m\vec{a}$ .

**SAMPLE 5.39 Circular orbits of planets:** Refer to eqn. (5.86) in the text that governs the motion of planets around a fixed sun.

- Let  $x = A \cos(\lambda t)$  and  $y = A \sin(\lambda t)$ . Show that  $x$  and  $y$  satisfy the equations of planetary motion and that they describe a circular orbit.
- Show that the solution assumed in (a) satisfies Kepler's third law by showing that the orbital period  $T = 2\pi/\lambda$  is proportional to the  $3/2$  power of the size of the orbit (which can be characterized by its radius).

### Solution

- (a) The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} &= \frac{-Gm_s}{(x^2 + y^2)^{3/2}} = \frac{\ddot{y}}{y} \\ \Rightarrow \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \quad (5.96)$$

Now,

$$\begin{aligned} x &= A \cos(\lambda t) \Rightarrow \ddot{x} = -\lambda^2 A \cos(\lambda t) \\ y &= A \sin(\lambda t) \Rightarrow \ddot{y} = -\lambda^2 A \sin(\lambda t) \end{aligned}$$

Substituting these values in eqn. (5.96), we get

$$-\lambda^2 A^2 \cos(\lambda t) \cdot \sin(\lambda t) + \lambda^2 A \sin(\lambda t) \cdot \cos(\lambda t) \stackrel{?}{=} 0$$

Thus the assumed form of  $x$  and  $y$  satisfy the governing equations of planetary motion, i.e.,  $x(t) = A \cos(\lambda t)$  and  $y(t) = A \sin(\lambda t)$  form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = \cos^2(\lambda t) + \sin^2(\lambda t) = 1,$$

i.e.,  $x$  and  $y$  satisfy the equation of a circle of radius  $A$ . Thus, the assumed solution gives a circular orbit.

- (b) Substituting  $x = A \cos(\lambda t)$  in eqn. (5.86), and noting that square of the radius of the orbit is  $r^2 = x^2 + y^2 = A^2$ , we get

$$\begin{aligned} -\lambda^2 A \cos(\lambda t) &= -Gm_s \frac{A \cos(\lambda t)}{r^3} \\ \Rightarrow \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or } \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow T^2 &= \frac{4\pi^2}{Gm_s} A^3 \\ \text{or } T &= KA^{3/2} \end{aligned}$$

where  $K = 2\pi/\sqrt{Gm_s}$  is a constant. Thus the orbital period  $T$  is proportional to the  $3/2$  power of the radius, or the size, of the circular orbit.

Of, course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates,  $x$  and  $y$ .

□



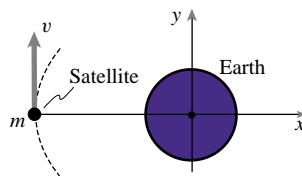


Figure 5.77: (Filename:sfig5.9.satorbit)

**SAMPLE 5.40 Numerical computation of satellite orbits:** The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point (perigee) on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 km/s. The radius of the earth is 6400 km and the acceleration due to gravity  $g = 9.8 \text{ m/s}^2$ .

- Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.
- From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.
- Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

### Solution

- The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2x}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2y}{(x^2 + y^2)^{3/2}},$$

where  $g$  is the acceleration due to gravity and  $R$  is the radius of the earth (see eqn. (5.91) in text). From the given data at perigee, the initial conditions are

$$x(0) = -7500 \text{ km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500 \text{ m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -gR^2z_1/(z_1^2 + z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -gR^2z_3/(z_1^2 + z_3^2)^{3/2}.\end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z_1(0) = -7.5 \times 10^6 \text{ m}, \quad z_2(0) = 0, \quad z_3(0) = 0, \quad z_4(0) = 9500 \text{ m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```

ODEs = {z1dot=z2, z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
        z3dot=z4, z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
IC   = {z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
Set g = 9.81, R = 6.4E06
Solve ODEs with IC for t=0 to t=4E04
Plot z1 vs z3

```

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in Fig. 5.77 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptic. From the computer output, we find the following data for the apogee.

$$x = 4.0049 \times 10^7 \text{ m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791 \times 10^3 \text{ m/s}$$

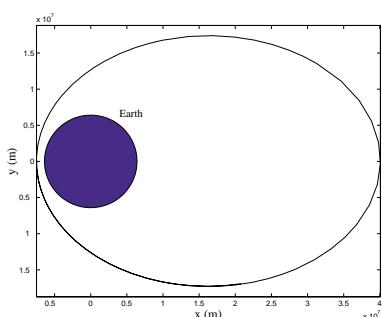


Figure 5.78: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

(Filename:sfig5.9.satorbit.a)

(b) The expressions for energy  $E$  and angular momentum  $H$  for a satellite are,

$$\begin{aligned} E &= E_K + E_P = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{GMm}{r} \\ \vec{H}_o &= \vec{r} \times m\vec{v} = (x\hat{i} + y\hat{j}) \times m(\dot{x}\hat{i} + \dot{y}\hat{j}) = m(x\dot{y} - y\dot{x})\hat{k} \end{aligned}$$

At both apogee and perigee,  $y = 0$  and the velocity (which is tangential) is in the  $y$  direction, i.e.,  $\dot{x} = 0$ . Therefore, the expressions for energy and angular momentum become simpler:

$$E = \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|} \quad \text{and} \quad H = mx\dot{y}$$

Let  $E_1$  and  $H_1$  be the energy and the angular momentum of the satellite at the perigee, respectively, and  $E_2$  and  $H_2$  be the respective quantities at the apogee. Then, from the given data,

$$\begin{aligned} E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} &= \frac{1}{2}2000 \text{ kg} \cdot (9500 \text{ m/s})^2 - \frac{9.81 \text{ m/s}^2 \cdot (6.4 \times 10^6 \text{ m})^2}{7.5 \times 10^6 \text{ m}} \\ &= -1.6901 \times 10^{10} \text{ Joules} \\ H_1 = mx_1\dot{y}_1 &= 2000 \text{ kg} \cdot (-7.5 \times 10^6 \text{ m}) \cdot (9500 \text{ m/s}) \\ &= -1.4250 \times 10^{14} \text{ N}\cdot\text{m}\cdot\text{s} \\ E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} &= \frac{1}{2}2000 \text{ kg} \cdot (-1779 \text{ m/s})^2 - \frac{9.81 \text{ m/s}^2 \cdot (6.4 \times 10^6 \text{ m})^2}{4.0049 \times 10^7 \text{ m}} \\ &= -1.6901 \times 10^{10} \text{ Joules} \\ H_2 = mx_2\dot{y}_2 &= 2000 \text{ kg} \cdot (4.0049 \times 10^7 \text{ m}) \cdot (-1779 \text{ m/s}) \\ &= -1.4250 \times 10^{14} \text{ N}\cdot\text{m}\cdot\text{s} \end{aligned}$$

Clearly, the energy and the angular momentum are conserved.

(c) From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be  $3.6335 \times 10^4 \text{ s} = 10.09 \text{ hrs}$ . Now let us compare this result with the analytical value of the orbital period.

Let  $A$  be the semimajor axis of the elliptic orbit. Then the square of the orbital time period  $T$  is given by

$$T^2 = \frac{4\pi^2 A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$\begin{aligned} 2A &= |x_1| + |x_2| = 7.5 \times 10^6 \text{ m} + 4.0049 \times 10^7 \text{ m} \\ &= 4.7549 \times 10^7 \text{ m} \\ \Rightarrow A &= 2.3774 \times 10^7 \text{ m} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2 \cdot (2.3774 \times 10^7 \text{ m})^3}{9.81 \text{ m/s}^2 \cdot (6.4 \times 10^6 \text{ m})^2}} \\ &= 3.6335 \times 10^4 \text{ s} \end{aligned}$$

which is the same value as obtained from numerical solution.

$$T = 3.6335 \times 10^4 \text{ s} = 10.09 \text{ hrs}$$

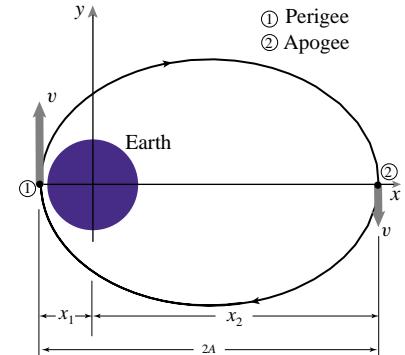


Figure 5.79: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

(Filename:fig5.9.sorbit.b)

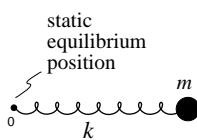


Figure 5.80: (Filename:sfig5.9.zerospring)

① No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. For example, let a spring be fixed to the ground and let its free end pass through a hole in a horizontal table. Let the relaxed length of the spring be exactly up to the top of the hole. Now, if the spring is pulled further and tied to a mass that is constrained to move on the horizontal table, then the spring behaves like a zero-length spring for the planar motion of the mass around the hole in the table. This is because the length of the position vector of the mass is exactly the stretch in the spring.

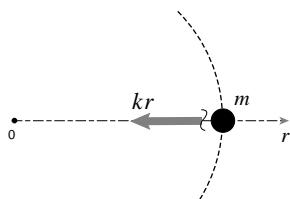


Figure 5.81: Free body diagram of the mass.  
(Filename:sfig5.9.zerospring.a)

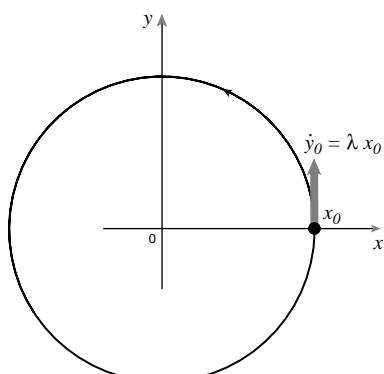


Figure 5.82: Circular trajectory of the mass.  
(Filename:sfig5.9.zerospring.b)

**SAMPLE 5.41 Zero-length spring and central force motion:** A zero-length spring ①(the relaxed length is zero) is tied to a mass  $m = 1 \text{ kg}$  on one end and fixed on the other end. The spring stiffness is  $k = 1 \text{ N/m}$ .

- Find appropriate initial conditions for the mass so that its trajectory is a straight line along the  $y$ -axis.
- Find appropriate initial conditions for the mass so that its trajectory is a circle.
- Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
- Let  $\vec{r}(0) = 0.5 \hat{m}$  and  $\dot{\vec{r}}(0) = (0.5\hat{i} + 0.6\hat{j}) \text{ m/s}$ . Describe the motion of the mass by plotting its trajectory for 12 s.

**Solution** Let the position of the mass be  $\vec{r}$  at some instant  $t$ . Since the relaxed length of the spring is zero, the stretch in the spring is  $|\vec{r}|$  and the spring force on the mass is  $-k\vec{r}$ . Then the equation of motion of the mass is

$$\begin{aligned} -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{i} + y\hat{j}) &= m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \\ \Rightarrow \ddot{x} + \frac{k}{m}x &= 0 \quad \text{and} \quad \ddot{y} + \frac{k}{m}y = 0. \end{aligned}$$

Thus the equations of motion are decoupled in the  $x$  and  $y$  directions. The solutions, as discussed in the text (see eqn. (5.94)), are

$$x = A \cos(\lambda t) + B \sin(\lambda t) \quad \text{and} \quad y = C \cos(\lambda t) + D \sin(\lambda t) \quad (5.97)$$

where the constants  $A, B, C, D$  are determined from initial conditions. Let us take the most general initial conditions  $x(0) = x_0, \dot{x}(0) = \dot{x}_0, y(0) = y_0$ , and  $\dot{y}(0) = \dot{y}_0$ . By substituting these values in  $x$  and  $y$  equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$x = x_0 \cos(\lambda t) + \dot{x}_0/\lambda \sin(\lambda t) \quad \text{and} \quad y = y_0 \cos(\lambda t) + \dot{y}_0/\lambda \sin(\lambda t) \quad (5.98)$$

- For a straight line motion along the  $y$ -axis, we should have the  $x$ -component of motion identically zero. We can, therefore, set  $x_0 = 0, \dot{x}_0 = 0$  and take any value for  $y_0$  and  $\dot{y}_0$  to give

$$x(t) = 0 \quad \text{and} \quad y(t) = y_0 \cos(\lambda t) + \dot{y}_0/\lambda \sin(\lambda t).$$

- For a circular trajectory, we must pick initial conditions such that we get  $x^2 + y^2 = (\text{a constant})^2$ . We can easily achieve this by choosing, say,  $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$ , and  $\dot{y}(0) = x_0\lambda$ . Substituting these values in eqn. (5.98), we get

$$x^2 + y^2 = x_0^2 \cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2 \sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius  $x_0$ . Note that the initial position of the mass for this orbit is  $\vec{r}(0) = x_0\hat{i}$ , and the initial velocity is  $(\vec{v}(0) = x_0\lambda\hat{j})$ , i.e., the velocity is normal to the position vector ( $\vec{r} \cdot \vec{v} = 0$ ), and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact, it must be exactly equal to the product of the distance from the center and the orbital frequency  $\lambda$ .

- (c) In order to have elliptic orbits, the initial conditions should be selected such that  $x$  and  $y$  satisfy the equation of an ellipse. By examining the solutions in eqn. (5.98), we see that if we set  $\dot{x}_0 = 0$  and  $y_0 = 0$  and let the other two initial conditions have any value,  $x_0$  and  $\dot{y}_0$ , we get

$$\begin{aligned} x(t) &= x_0 \cos(\lambda t) & \text{and} & \quad y(t) = (\dot{y}_0/\lambda) \sin(\lambda t) \\ \Rightarrow \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}/\lambda)^2} &= \cos^2(\lambda t) + \sin^2(\lambda t) = 1 \end{aligned}$$

which is the equation of an ellipse with semimajor axis  $x_0$  and semiminor axis  $\dot{y}/\lambda$ . Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting  $x_0 = 0$  and  $\dot{y}_0 = 0$ , and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector (either  $\vec{r}(0) = x_0 \hat{i}$ ,  $\dot{\vec{r}}(0) = \dot{y}_0 \hat{j}$  or  $\vec{r}(0) = y_0 \hat{j}$ ,  $\dot{\vec{r}}(0) = \dot{x}_0 \hat{i}$ , or more generally,  $\vec{r}(0) = r_0 \hat{\lambda}$ ,  $\dot{\vec{r}}(0) = v \hat{n}$  where  $\hat{\lambda}$  is a unit vector along the position vector of the mass and  $\hat{n}$  is normal to  $\hat{\lambda}$ ).

Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep  $x_0$  fixed and vary  $\dot{y}_0$  we can get different elliptic orbits, including a circular one, based on the same major axis. Taking  $x_0 = 1$  m, we show different orbits obtained for the mass by varying  $\dot{y}_0$  in Fig. 5.83

- (d) By substituting the given initial values  $x_0 = 0.5$  m,  $\dot{x}(0) = 0.5$  m/s,  $y(0) = 0$  and  $\dot{y} = 0.6$  m/s in eqn. (5.98) and noting that  $\lambda \equiv \sqrt{k/m} = \sqrt{(1 \text{ N/m})/(1 \text{ kg})} = (1/\text{s})$ , we get

$$\begin{aligned} x(t) &= (0.5 \text{ m}) \cdot \cos\left(\frac{1}{s} \cdot t\right) + \left(\frac{0.5 \text{ m/s}}{s}\right) \cdot \sin\left(\frac{1}{s} \cdot t\right) \\ y(t) &= \left(\frac{0.6 \text{ m/s}}{s}\right) \cdot \sin\left(\frac{1}{s} \cdot t\right) \end{aligned}$$

The functions  $x(t)$  and  $y(t)$  do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between  $x$  and  $y$  that we can recognize. Instead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate  $x$  and  $y$  at, say, 100 values of  $t$  between 0 and 10 s and then plot  $x$  vs  $y$ .

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in Fig. 5.84.

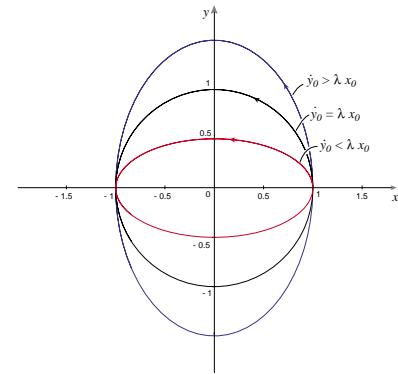


Figure 5.83: Elliptic orbits of the mass obtained from the initial conditions  $x_0 = 1$  m,  $\dot{x}_0 = 0$ ,  $y_0 = 0$ , and various values of  $\dot{y}_0$ .

(Filename:sfig5.9.zerospring.c)

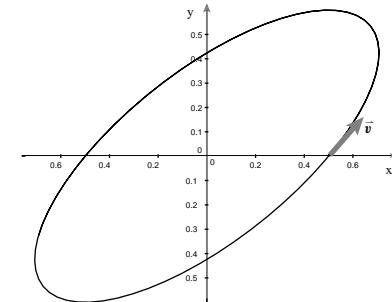


Figure 5.84: The orbit of the mass obtained from the initial conditions  $x_0 = 0.5$  m,  $\dot{x}_0 = 0.5$  m/s,  $y_0 = 0$ , and  $\dot{y}_0 = 0.6$  m/s.

(Filename:sfig5.9.zerospring.d)



## 5.10 Coupled motions of particles in space

In the previous two sections you have seen that once you know the forces on a particle, or on how those forces vary with position, velocity and time, you can easily set up the equations of motion. That is, the linear momentum balance equation for a particle

$$\vec{F} = m\vec{a}$$

with initial conditions gives a well defined mathematical problem whose solution is the motion of the particle. The solution may be hard or impossible to find with pencil and paper, but can generally be found in a straightforward way with numerical integration.

Now we generalize this simple point of view to two, three or more particles. Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. Then that means we can write the governing equations for the system of particles like this:

$$\begin{aligned}\vec{a}_1 &= \frac{1}{m_1} \vec{F}_1 \\ \vec{a}_2 &= \frac{1}{m_2} \vec{F}_2 \\ \vec{a}_3 &= \frac{1}{m_3} \vec{F}_3 \\ &\text{etc.}\end{aligned}\quad (5.99)$$

where  $\vec{F}_1, \vec{F}_2$  etc. are the total of the forces on the corresponding particles. If the force on each particle comes from well-understood air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, etc., then all the forces on all the particles are known given the positions and velocities of the particles. Thus eqn. (5.99) can be written as a system of first order differential equations in standard form ready for computer simulation. Given accurate initial conditions and a good computer and the motions of all the particles can be found accurately.

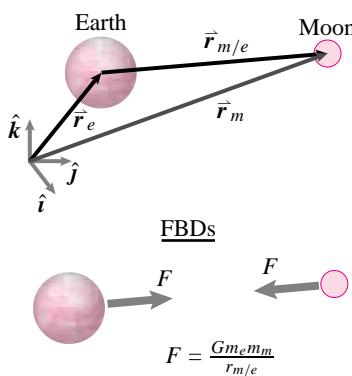


Figure 5.85: The earth and moon. Position is measured relative to some inertial point  $C$ .

(Filename:tfigure.earthmoon)

### Example: Coupled motion of the earth and moon in three dimensions.

Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a central-force problem is that we now need to look at the 'absolute' positions of the sun and moon ( $\vec{r}_e$  and  $\vec{r}_m$ ), and the 'relative' position, say  $\vec{r}_{m/e} \equiv \vec{r}_m - \vec{r}_e$  (Fig. 5.85).

The linear momentum balance equations are now

$$m_e \ddot{\vec{r}}_e = \frac{-Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \quad (5.100)$$

$$m_m \ddot{\vec{r}}_m = \frac{+Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \quad (5.101)$$

which, when broken into  $x$ ,  $y$ , and  $z$  components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12  $z$  variables:  $z_1 = x_e, z_2 = \dot{x}_e, z_3 = \ddot{x}_e, z_4 = y_e, z_5 = \dot{y}_e, z_6 = \ddot{y}_e, z_7 = z_e, z_8 = \dot{z}_e, z_9 = \ddot{z}_e, z_{10} = \dot{r}_e, z_{11} = \ddot{r}_e, z_{12} = \dot{r}_{m/e}$ .

After one finds solutions with appropriate initial conditions one can see if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

- (a) that the line between the earth and moon always lies on one fixed plane,
- (b) the center of mass moves at constant speed on a straight line,
- (c) relative to the center of mass both the earth and moon travel on paths that are conic sections
- (d) the energy of the system is constant,
- (e) and that the angular momentum of the system about the center of mass is a constant.

□

If we could think of all materials as made of atoms, and of all the atoms moving in deterministic ways governed by Newton's laws and known force laws, and we knew the initial positions and velocities accurately enough, then we could accurately predict the motions of all things for all time.

To put it in other words, given a simple atomic view of the world and a big computer, we could end a course on dynamics here. You know how to use  $\vec{F} = m\vec{a}$  for each atom, so you can simulate anything made of atoms. Now there are some serious catches here, so before proceeding we name some of them:

- there are no computers big enough to accurately integrate Newton's laws for the  $10^{23}$  or so atoms needed to describe macroscopic objects;
- the laws of interaction between atoms are not simple and are not that well known;
- the state of the world (the positions and velocities of all the atoms is not that well known);
- the solutions of the equations are often unstable in that a very small error in the initial conditions propagates into a large error in the calculations;
- the world is not deterministic, quantum mechanics says that you *cannot* know the state of the world perfectly; and
- massive simulations, even if accurate, are not always the best way to understand how things work.

Despite these limitations on the ultimate utility of the approach, in this section we look at the nature of systems of interacting particles. In particular we look at the momentum, angular momentum, and energy of a system of particles.

## Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

One of our three basic dynamics equations is linear momentum balance:

$$\sum \vec{F} = \dot{\vec{L}}.$$

The first quantity of interest in this section is the linear momentum  $\vec{L}$ <sup>①</sup>, whose derivative,  $\dot{\vec{L}}$ , with respect to a Newtonian frame is so important. Linear momentum is a measure of the translational motion of a system.

$$\underbrace{\vec{L}}_{\text{linear momentum}} \equiv \underbrace{\sum_{\substack{\text{summed over} \\ \text{all the mass particles}}} m_i \vec{v}_i}_{\text{ }} = m_{\text{tot}} \vec{v}_{cm} \quad (5.102)$$

<sup>①</sup> In Isaac Newton's language: '*The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly*'. In other words, Newton's dynamics equations for a particle were based on the product of  $\vec{v}$  and  $m$ . This quantity,  $m\vec{v}$ , is now called  $\vec{L}$ , the linear momentum of a particle.

**Example: Center of Mass position, velocity, and acceleration**

A particle of mass  $m_A = 3 \text{ kg}$  and another point of mass  $m_B = 2 \text{ kg}$  have positions, respectively,

$$\vec{r}_A(t) = \left[ 3\hat{i} + 5\left(\frac{t}{\text{s}}\right)\hat{j} \right] \text{ m}, \text{ and } \vec{r}_B(t) = \left[ 6\left(\frac{t^2}{\text{s}^2}\right)\hat{i} - 4\hat{j} \right] \text{ m}$$

due to forces that we do not discuss here. The position of the center of mass of the system of particles, according to equation ?? on page ??, is

$$\begin{aligned} \vec{r}_{cm}(t) &= \frac{\sum m_i \vec{r}_i}{m_{\text{tot}} = 5 \text{ kg}} \\ \vec{r}_{cm}(t) &= \left[ \left( \frac{9}{5} + \frac{12}{5} \left( \frac{t^2}{\text{s}^2} \right) \right) \hat{i} + \left( 3 \left( \frac{t}{\text{s}} \right) - \frac{8}{5} \right) \hat{j} \right] \text{ m}. \end{aligned}$$

- ① That is, particle A travels on the line  $x = 3 \text{ m}$  with constant speed  $\dot{r}_{A_y} = 5 \text{ m/s}$  and particle B travels on the line  $y = -4 \text{ m}$  at changing speed  $\dot{r}_{B_x} = 12t \text{ (m/s}^2\text{)}$ .

- ② Some books use the symbol  $\vec{P}$  for linear momentum. Because  $\vec{P}$  is often used to mean force or impulse and  $P$  for power we use  $\vec{L}$  for linear momentum.

To get the velocity and acceleration of the center of mass, we differentiate the position of the center of mass once and twice, respectively, to get<sup>①</sup>

$$\vec{v}_{cm}(t) = \dot{\vec{r}}_{cm}(t) = \left[ \frac{24}{5} \left( \frac{t}{\text{s}^2} \right) \hat{i} + \frac{3}{\text{s}} \hat{j} \right] \text{ m} = \left[ \frac{24}{5} \left( \frac{t}{\text{s}} \right) \hat{i} + 3\hat{j} \right] \text{ m/s}$$

and

$$\vec{a}_{cm}(t) = \ddot{\vec{r}}_{cm}(t) = \ddot{\vec{r}}_{cm}(t) = \left[ \frac{24}{5} \left( \frac{1}{\text{s}^2} \right) \hat{i} \right] \text{ m} = \left( \frac{24}{5} \right) \text{ m/s}^2 \hat{i}.$$

In this example, the center of mass turns out to have constant acceleration in the  $x$ -direction.  $\square$

The second part of equation 5.102 follows from the definition of the center of mass (see box 5.8 on page 316).<sup>②</sup> The total linear momentum of a system is the same as that of a particle that is located at the center of mass and which has mass equal to that of the whole system. The linear momentum is also given by

$$\vec{L} = \frac{d}{dt} (m_{\text{tot}} \vec{r}_{cm}).$$

## 5.8 Velocity and acceleration of the center of mass of a system of particles

The average position of mass in a system is at a point called the center of mass. The position of the center of mass is

$$\vec{r}_{cm} = \frac{\sum \vec{r}_i m_i}{m_{\text{tot}}}.$$

Multiplying through by  $m_{\text{tot}}$ , we get

$$\vec{r}_{cm} m_{\text{tot}} = \sum \vec{r}_i m_i.$$

By taking the time derivatives of the equation above, we get

$$\begin{aligned} \vec{v}_{cm} m_{\text{tot}} &= \sum \vec{v}_i m_i \quad \text{and} \\ \vec{a}_{cm} m_{\text{tot}} &= \sum \vec{a}_i m_i. \end{aligned}$$

for the velocity and acceleration of the center of mass. The results above are useful for simplifying various momenta and energy expressions. Note, for example, that

$$\begin{aligned} \vec{L} &= \sum \vec{v}_i m_i = \vec{r}_{cm} m_{\text{tot}} \\ \vec{L} &= \sum \vec{a}_i m_i = \vec{a}_{cm} m_{\text{tot}}. \end{aligned}$$

We only consider systems of fixed mass,  $\frac{d}{dt}(m_{\text{tot}}) = 0$ . Thus, for a fixed mass system, the linear momentum of the system is equal to the total mass of the system times the derivative of the center of mass position.

Finally, since the sum defining linear momentum can be grouped any which way (the associative rule of addition) the linear momentum can be found by dividing the system into parts and using the mass of those parts and the center of mass motion of those parts. That is, the sum  $\sum m_i \vec{v}_i$  can be interpreted as the sum over the center of mass velocities and masses of the various subsystems, say the parts of a machine.

### Example: System Momentum

See figure 5.86 for a schematic example of the total momentum of system being made of the sum of the momenta of its two parts.  $\square$

The reasoning for this allowed subdivision is similar to that for the center of mass in box ?? on page ??.

The quantity  $\dot{\vec{L}}$  figures a little more directly in our presentation of dynamics than just plain  $\vec{L}$  <sup>①</sup>. The rate of change of linear momentum,  $\dot{\vec{L}}$ , is

$$\begin{aligned}\dot{\vec{L}} &= \frac{d}{dt} \vec{L} \\ &= \frac{d}{dt} \sum m_i \vec{v}_i \\ &= m_{\text{tot}} \frac{d \vec{v}_{\text{cm}}}{dt} \\ \dot{\vec{L}} &= m_{\text{tot}} \vec{a}_{\text{cm}}\end{aligned}$$

The last three equations could be thought of as the *definition* of  $\dot{\vec{L}}$ . That  $\dot{\vec{L}}$  turns out to be  $\frac{d}{dt}(\vec{L})$  is, then, a derived result. Again, using the definition of center of mass,

the total rate of change of linear momentum is the same as that of a particle that is located at the center of mass which has mass equal to that of the whole system.

The rate of change of linear momentum is also given by

$$\dot{\vec{L}} = \frac{d}{dt}(m_{\text{tot}} \vec{v}_{\text{cm}}) = \frac{d^2}{dt^2}(m_{\text{tot}} \vec{r}_{\text{cm}}).$$

The momentum  $\vec{L}$  and its rate of change  $\dot{\vec{L}}$  can be expressed in terms of the total mass of a system and the motion of the center of mass. This simplification holds for any system, however complex, and any motion, however contorted and wild.

## Angular momentum $\vec{H}$ and its rate of change $\dot{\vec{H}}$

After linear momentum balance, the second basic mechanics principle is angular momentum balance:

$$\sum \vec{M}_C = \dot{\vec{H}}_C,$$

where C is any point, preferably one that is fixed in a Newtonian frame. If you choose your point C to be a moving point you may have the confusing problem that the quantity we would like to call  $\dot{\vec{H}}_C$  is not the time derivative of  $\vec{H}_C$ . The first quantity

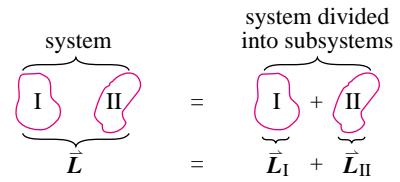


Figure 5.86: System composed of two parts. The momentum of the whole is the sum of the momenta of the two parts.

(Filename:tfigure3.2.2)

<sup>①</sup> No slight of Sir Isaac is intended.

of interest in this sub-section is the angular momentum with respect to some point C,  $\vec{H}_C$ , whose rate of change  $\dot{\vec{H}}_C = d\vec{H}_C/dt$  is so important.

$$\underbrace{\vec{H}_C}_{\text{angular momentum.}} \equiv \underbrace{\sum_{\substack{\text{summed over all} \\ \text{the mass particles}}} \vec{r}_{i/C} \times m_i \vec{v}_i}_{}$$

A useful theorem about angular momentum is the following (see box 5.9 on page 319), applicable to all systems

$$\vec{H}_C = \underbrace{\widehat{\vec{r}_{cm/C}} \times \vec{v}_{cm} m_{tot}}_{\text{angular momentum due to center of mass motion}} + \underbrace{\sum \underbrace{\vec{r}_{i/cm} \times \vec{v}_{i/cm}}_{\text{position of } m_i \text{ relative to the center of mass } \vec{r}_{i/cm} \equiv \vec{r}_i - \vec{r}_{cm}} m_i}_{\text{angular momentum relative to the center of mass}}. \quad (5.103)$$

A system of particles is shown in figure 5.87. The angular momentum of any system is the same as that of a particle at its center of mass *plus* the angular momentum associated with motion relative to the center of mass.

The angular momentum about point C is a measure of the average rotation rate of the system about point C. Angular momentum is not so intuitive as linear momentum for a number of reasons:

- First, recall that linear momentum is the derivative of the total mass times the center of mass position. Unfortunately, in general,

*angular momentum is not the derivative of anything.*

- Second, the angular momentum of a given system at a given time depends on the reference point C. So there is not one single quantity that is *the* angular momentum. For different points  $C_1, C_2$ , etc., the same system has different angular momentums.
- Finally, calculation of angular momentum involves a vector cross product and many beginning dynamics students are intimidated by vector cross products.

Despite these confusions, the concept of angular momentum allows the solution of many practical problems and eventually becomes somewhat intuitive.

Actually, it is  $\dot{\vec{H}}_C$  which is the more fundamental quantity.  $\dot{\vec{H}}_C$  is what you use in the equation of motion. You can find  $\dot{\vec{H}}_C$  from  $\vec{H}_C$  as shown in the box on page 320. But, in general,

$$\dot{\vec{H}}_C \equiv \sum \vec{r}_{i/C} \times (m_i \vec{a}_i).$$

A useful theorem about rate of change of angular momentum is the following (see box 5.9 on page 319), applicable to all systems:

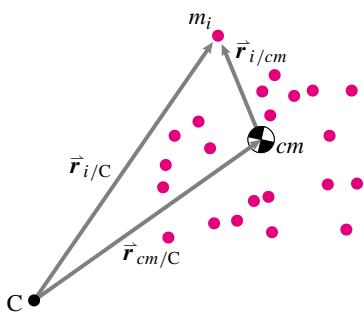
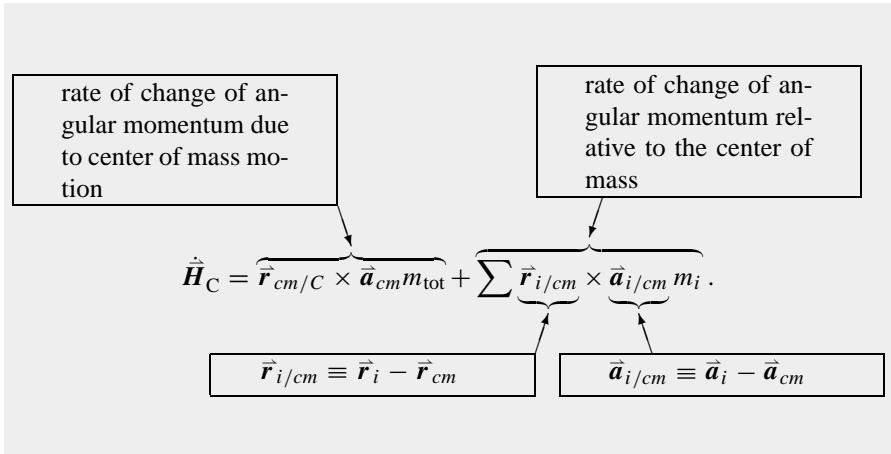


Figure 5.87: A system of particles showing its center of mass and the  $i_{th}$  particle of mass  $m_i$ . The  $i_{th}$  particle has position  $\vec{r}_{i/cm}$  with respect to the center of mass. The center of mass has position  $\vec{r}_{cm/C}$  with respect to the point C  
(Filename:tfigure3.ang.mom.bal)



This expression is completely analogous to equation 5.103 on page 318 and is derived in a manner nearly identical to that shown in box 5.9 on page 319. The rate of change of angular momentum of any system is the same as that of a particle at its center of mass *plus* the rate of change of angular momentum associated with motion relative

## 5.9 THEORY Simplifying $\vec{H}_C$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_C = \sum \vec{r}_{i/C} \times m_i \vec{v}_i.$$

If we rewrite  $\vec{v}_i$  as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned} \vec{H}_C &= \sum (\vec{r}_{cm} + \vec{r}_{i/cm}) \times [\vec{v}_{cm} + \vec{v}_{i/cm}] m_i \\ &= \sum \vec{r}_{cm} \times \vec{v}_{cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i \\ &\quad + \sum \vec{r}_{cm} \times \vec{v}_{i/cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{cm} m_i \\ &= \vec{r}_{cm} \times \vec{v}_{cm} m_{tot} + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i \\ &\quad + \vec{r}_{cm} \times \underbrace{\left[ \sum \vec{v}_{i/cm} m_i \right]}_{\vec{0}} + \underbrace{\left[ \sum \vec{r}_{i/cm} m_i \right]}_{\vec{0}} \times \vec{v}_{cm} \end{aligned}$$

So,

$$\vec{H}_C = \underbrace{\vec{r}_{cm} \times \vec{v}_{cm} m_{tot}}_{\text{contribution of center of mass motion}} + \underbrace{\sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i}_{\text{contribution of motion relative to center of mass}}.$$

The reason  $\sum \vec{r}_{i/cm} m_i = \vec{0}$  is somewhat intuitive. It is what you would calculate if you were looking for the center of mass relative to the center of mass. More formally,

$$\begin{aligned} \sum \vec{r}_{i/cm} m_i &= \sum (\vec{r}_i - \vec{r}_{cm}) m_i \\ &= \underbrace{\sum \vec{r}_i m_i}_{m_{tot} \vec{r}_{cm}} - m_{tot} \vec{r}_{cm} \\ &= \vec{0}. \end{aligned}$$

Similarly,  $\sum \vec{v}_{i/cm} m_i = \vec{0}$  because it is what you would calculate if you were looking for the velocity of the center of mass relative to the center of mass.

The central result of this box is that

angular momentum of any system is that due to motion of the center of mass *plus* motion *relative to* the center of mass.

to the center of mass. A special point for any system is, as we have mentioned, the center of mass. In the above equations for angular momentum we could take C to be a fixed point in space that happens to coincide with the center of mass. In this case we would most naturally define  $\vec{H}_{cm} = \int \vec{r}_{cm} \times \vec{v} dm$  with  $\vec{v}$  being the absolute velocity. But we have the following theorem:

$$\vec{H}_{cm} = \int \vec{r}_{cm} \times \vec{v} dm = \int \vec{r}_{cm} \times \vec{v}_{cm} dm$$

where  $\vec{r}_{cm} = \vec{r} - \vec{r}_{cm}$  and  $\vec{v}_{cm} = \vec{v} - \vec{v}_{cm}$ . Similarly,

$$\dot{\vec{H}}_{cm} = \int \vec{r}_{cm} \times \vec{a} dm = \int \vec{r}_{cm} \times \vec{a}_{cm} dm.$$

with  $\vec{a}_{cm} = \vec{a} - \vec{a}_{cm}$ . That is,

*the angular momentum and rate of change of angular momentum relative to the center of mass, defined in terms of the velocity and acceleration relative to the center of mass, are the same as the angular momentum and the rate of change of angular momentum defined in terms of a fixed point in space that coincides with the center of mass.*

The angular momentum relative to the center of mass  $\vec{H}_{cm}$  can be calculated with all positions and velocities calculated relative to the center of mass. Similarly, the rate of change of angular momentum relative to the center of mass  $\dot{\vec{H}}_{cm}$  can be calculated with all positions and *accelerations* calculated relative to the center of mass.

Combining the results above we get the often used result:

$$\sum \vec{M}_{i/cm} = \dot{\vec{H}}_{cm} \quad (5.104)$$

This formula is the version of angular momentum balance that many people think of as being basic. In this equation,  $\dot{\vec{H}}_{cm}$  can be found using either the absolute acceleration

## 5.10 Relation between $\frac{d}{dt}\vec{H}_C$ and $\vec{H}_C$

The expression for  $\dot{\vec{H}}_C$  follows from that for  $\vec{H}_C$  but requires a few steps of algebra to show. Like the rate of change of linear momentum,  $\dot{\vec{L}}$ , the derivative of  $\vec{L}$ , the derivative of angular momentum must be taken with respect to a Newtonian frame in order to be useful in momentum balance equations. Note that since we assumed that C is a point fixed in a Newtonian frame that  $\frac{d}{dt}\vec{r}_{i/C} = \vec{v}_{i/C} = \vec{v}_i$ .

Starting with the definition of  $\vec{H}_C$ , we can calculate as follows:

$$= \sum \underbrace{\vec{v}_i \times (m_i \vec{v}_i)}_{\frac{d}{dt}\vec{r}_{i/C}} + \vec{r}_{i/C} \times (m_i \frac{d}{dt} \vec{v}_i)$$

$$\dot{\vec{H}}_C = \sum \vec{r}_{i/C} \times (m_i \vec{a}_i),$$

$$\begin{aligned} \dot{\vec{H}}_C &= \frac{d}{dt} \vec{H}_C \\ &= \frac{d}{dt} \sum \vec{r}_{i/C} \times (m_i \vec{v}_i) \\ &= \sum \frac{d}{dt} \vec{r}_{i/C} \times (m_i \vec{v}_i) + \vec{r}_{i/C} \times (m_i \frac{d}{dt} \vec{v}_i) \end{aligned}$$

We have used the fact that the product rule of differentiation works for cross products between vector-valued functions of time. This final formula,  $\dot{\vec{H}}_C = \sum \vec{r}_{i/C} \times (m_i \vec{a}_i)$ , or its integral form,  $\vec{H}_C = \int \vec{r}_{i/C} \times \vec{a}_i dm$  are always applicable. They can be simplified in many special cases which we will discuss in this chapter and those that follow.

$\vec{a}$  or the acceleration relative to the center of mass,  $\vec{a}_{/cm}$ . The same  $\hat{\vec{H}}_{cm}$  is found both ways. In this book, we do not give equation 5.104 quite such central status as equations III where the reference point can be any point  $C$  not just the center of mass.

## Kinetic energy $E_K$

The equation of mechanical energy balance (III) is:

$$P = \dot{E}_K + \dot{E}_P + \dot{E}_{int}.$$

For discrete systems, the kinetic energy is calculated as

$$\frac{1}{2} \sum m_i v_i^2$$

and its rate of change as

$$\frac{d}{dt} \left[ \frac{1}{2} \sum m_i v_i^2 \right].$$

There is also a general result about the kinetic energy that takes advantage of the center of mass. The kinetic energy for any system in any motion can be decomposed into the sum of two terms. One is associated with the motion of the center of mass of the system and the other is associated with motion relative to the center of mass. Namely,

$$\begin{aligned} E_K &= \underbrace{\frac{1}{2} m_{tot} v_{cm}^2}_{\text{kinetic energy due to center of mass motion}} + \underbrace{\frac{1}{2} \sum m_i v_{i/cm}^2}_{\text{kinetic energy relative to the center of mass}}, \\ &= \frac{1}{2} m_{tot} v_{cm}^2 + E_{K/cm} \end{aligned}$$

## 5.11 Using $\hat{\vec{H}}_O$ and $\hat{\vec{H}}_O$ to find $\hat{\vec{H}}_C$ and $\hat{\vec{H}}_C$

You can find the angular momentum  $\hat{\vec{H}}_C$  relative to a fixed point  $C$  if you know the angular momentum  $\hat{\vec{H}}_O$  relative to some other fixed point  $O$  and also know the linear momentum of the system  $\vec{L}$  (which does not depend on the reference point). The result is:

$$\hat{\vec{H}}_C = \hat{\vec{H}}_O + \vec{r}_{O/C} \times \vec{L}.$$

The formula is similar to the formula for the effective moment of a system of forces that you learned in statics:  $\vec{M}_C = \vec{M}_O + \vec{r}_{O/C} \times \vec{F}_{tot}$ . Similarly, for the rate of change of angular momentum we have:

$$\hat{\vec{H}}_C = \hat{\vec{H}}_O + \vec{r}_{O/C} \times \vec{L}$$

So once you have found  $\vec{L}$  and also  $\hat{\vec{H}}_O$  with respect to some point  $O$  you can easily calculate the right hand sides of the momentum balance equations using any point  $C$  that you like.

where

$$\begin{aligned} E_{K/cm} &= \frac{1}{2} \sum m_i v_{i/cm}^2 && \text{for discrete systems, and} \\ &= \frac{1}{2} \int (v_{/cm})^2 dm && \text{for continuous systems.} \end{aligned}$$

The results above can be verified by direct expansion of the basic definitions of  $E_K$  and the center of mass. To repeat,

*the kinetic energy of a system is the same as the kinetic energy of a particle with the system's mass at the center of mass plus kinetic energy due to motion relative to the center of mass.*

In this chapter, all particles in the system are assumed to have the same velocity so that they all have the same velocity as the center of mass. Thus,  $\vec{v}_{i/cm} = \vec{0}$  for all particles, and for straight line motion,

$$E_K = \frac{1}{2} m_{\text{tot}} v_{cm}^2.$$

### Summary on general results about $\vec{L}$ , $\dot{\vec{L}}$ , $\vec{H}_C$ , $\dot{\vec{H}}_C$ , $E_K$ , and center of mass

$m_{\text{tot}} \vec{r}_{cm}$	$= \sum \vec{r}_i m_i$	for all systems
$m_{\text{tot}} \vec{v}_{cm}$	$= \sum \vec{v}_i m_i$	for all systems
$m_{\text{tot}} \vec{a}_{cm}$	$= \sum \vec{a}_i m_i$	for all systems
$\vec{L}$	$= \sum m_i \vec{v}_i = m_{\text{tot}} \vec{v}_{cm}$	for all systems
$\dot{\vec{L}}$	$= \sum m_i \vec{a}_i = m_{\text{tot}} \vec{a}_{cm}$	for all systems
$\vec{H}_C$	$= \sum \vec{r}_{i/C} \times (m_i \vec{v}_i)$	for all systems
	$= \vec{r}_{cm/C} \times m_{\text{tot}} \vec{v}_{cm} + \sum \vec{r}_{i/cm} \times (\vec{v}_{i/cm} m_i)$	for all systems
	$= \vec{H}_O + \vec{r}_{O/C} \times \vec{L}$	for all systems
$\dot{\vec{H}}_C$	$= \sum \vec{r}_{i/C} \times (m_i \vec{a}_i)$	for all systems
	$= \vec{r}_{cm/C} \times m_{\text{tot}} \vec{a}_{cm} + \sum \vec{r}_{i/cm} \times (\vec{a}_{i/cm} m_i)$	for all systems
	$= \dot{\vec{H}}_O + \vec{r}_{O/C} \times \dot{\vec{L}}$	for all systems
$E_K$	$= \frac{1}{2} \sum m_i v_i^2$	for all systems
	$= \frac{1}{2} m_{\text{tot}} v_{cm}^2 + \frac{1}{2} \sum m_i v_{i/cm}^2$	for all systems
$\dot{E}_K$	$= \sum m_i v_i \dot{v}_i$	for all systems
	$= m_{\text{tot}} v_{cm} \dot{v}_{cm} + \sum m_i v_{i/cm} \dot{v}_{i/cm}$	for all systems

## 5.12 THEORY

### Deriving system momentum balance from the particle equations.

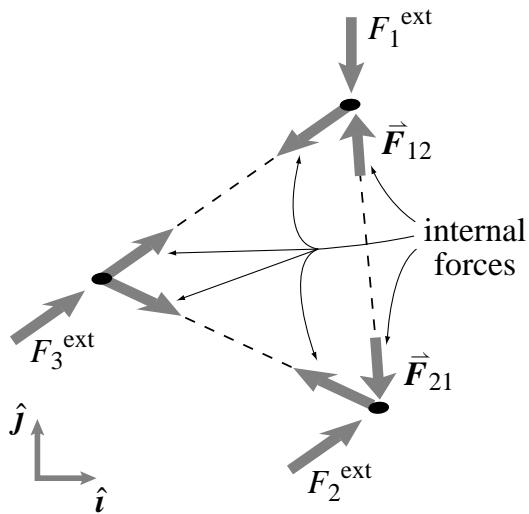
In the front cover you see that we have linear and angular momentum balance equations that apply to arbitrary systems. Another approach to mechanics is to use the equation

$$\vec{F} = m\vec{a}$$

for every particle in the system and then *derive* the system linear and angular momentum balance equations. This derivation depends on the following assumptions

- (a) All bodies and systems are composed of point masses.
- (b) These point masses interact in a pair-wise manner. For every pair of point masses A and B the interaction force is equal and opposite and along the line connecting the point masses.

We then look at any system, which we now assume is a system of point masses, and apply  $\vec{F} = m\vec{a}$  to every point mass and add the equations for all point masses in the system. For each point mass we can break the total force into two parts: 1) the interaction forces between the point mass and other point masses in the system, these forces are ‘internal’ forces ( $\vec{F}^{int}$ ), and 2) the forces acting on the system from the outside, the ‘external’ forces. The situation is shown for a three particle system below.



### System linear momentum balance

Now let's take the equation  $\sum \vec{F} = m\vec{a}$  for each particle and add over all the particles.

$$\underbrace{\sum_{\text{all particles}} \left[ \sum_{\text{each particle}} \vec{F} \right]}_{\text{The sum of all forces on the system, internal and external}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

Since all the internal forces come in cancelling pairs we can rewrite

this equation as:

$$\underbrace{\sum_{\text{all external forces}} \vec{F}^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

That is, we have derived equation I in the front cover from  $\vec{F} = m\vec{a}$  for a point mass by assuming the system is composed of point masses with pair-wise equal and opposite forces.

### System angular momentum balance

For any particle we can take the equation

$$\sum_{\text{forces on particle } i} \vec{F} = m_i \vec{a}_i$$

and take the cross product of both sides with the position of the particle relative to some point C:

$$\vec{r}_{i/C} \times \left[ \sum_{\text{forces on particle } i} \vec{F} \right] = \vec{r}_{i/C} \times [m_i \vec{a}_i].$$

Now we can add this equation up over all the particles to get

$$\underbrace{\sum_{\text{particles}} \left\{ \vec{r}_{i/C} \times \left[ \sum_{\text{on particle } i} \vec{F} \right] \right\}}_{r/C \times \vec{F} \text{ added up for all forces on the system, internal and external}} = \sum_{\text{particles}} \{ \vec{r}_{i/C} \times [m_i \vec{a}_i] \}$$

But, by our pair-wise assumption, for every internal force there is an equal and opposite force with the same line of action. So all the internal forces drop out of this sum and we have:

$$\underbrace{\sum_{\text{all external forces}} \vec{r}_{i/C} \times \vec{F}_i^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} \vec{r}_{i/C} \times m_i \vec{a}_i.$$

This equation is equation II, the system angular momentum balance equation (assuming we do not allow the application of any pure moments).

The derivations above are classic and are found in essentially all mechanics books. However, some people feel it is fine to take the system linear momentum balance and angular momentum balance equations as postulates and not make the subject of mechanics depend on the unrealistic view of so-simply interacting point masses.

### 5.13 A preview of rigid body simplifications and advanced kinematics

We have formulas for the motion quantities  $\vec{L}$ ,  $\dot{\vec{L}}$ ,  $\vec{H}_C$ , and  $\dot{\vec{H}}_C$  and  $E_K$  in terms of the positions, velocities, and accelerations of all of the mass bits in a system. Most often in this book we deal with the mechanics of *rigid bodies*, objects with negligible deformation. This assumed simplification means that the relative motions of the  $10^{23}$  or so atoms in a body are highly restricted. In fact, if one knows these five vectors:

- $\vec{r}_{cm}$ , the position of the center of mass,
- $\vec{v}_{cm}$ , the velocity of the center of mass,
- $\vec{a}_{cm}$ , the acceleration of the center of mass
- $\vec{\omega}$ , the *angular velocity* of the body, and
- $\vec{\alpha}$ , the *angular acceleration* of the body,

then one can find the position, velocity, and acceleration of every point on the body in terms of its position relative to the center of mass,  $\vec{r}_{/cm} = \vec{r} - \vec{r}_{cm}$ .

We will save the derivations for later since we have not yet discussed the concepts of angular velocity  $\vec{\omega}$  and angular acceleration  $\vec{\alpha}$ .

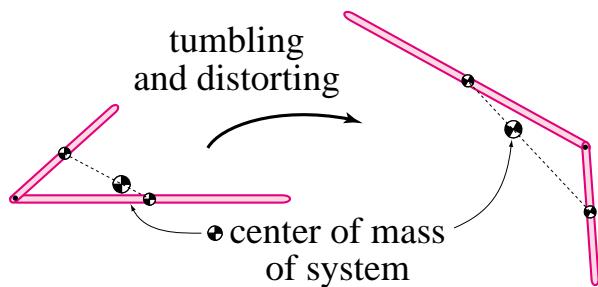
We will also use a new quantity  $[I^{cm}]$ , the *moment of inertia matrix*. For 2-D problems,  $[I^{cm}]$  is just a number. For 3-D problems,  $[I^{cm}]$  is a matrix; hence, the square brackets [ ], our notation for a matrix.

As intimidating as these new concepts may appear now, they lead to a *vast* simplification over the alternative — summing over  $10^{23}$  particles or so.

Note that the formulas for linear momentum  $\vec{L}$  and rate of change of linear momentum  $\dot{\vec{L}}$  do not really look any simpler for a rigid body than the general case.

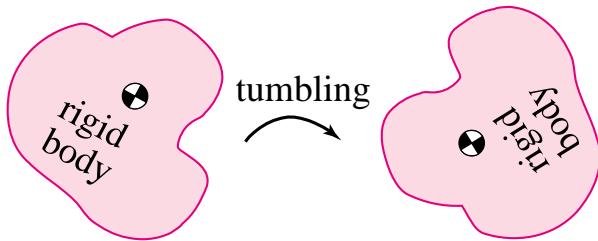
$$\begin{aligned}\vec{L} &= m_{\text{tot}} \vec{v}_{cm} \\ \dot{\vec{L}} &= m_{\text{tot}} \vec{a}_{cm}\end{aligned}$$

But, they are actually simpler in the following sense. For a general system, when we write  $\vec{v}_{cm}$ , we are talking about an abstract point that moves in a different way than any point on the system. For example, consider the linked arms below, tumbling in space.

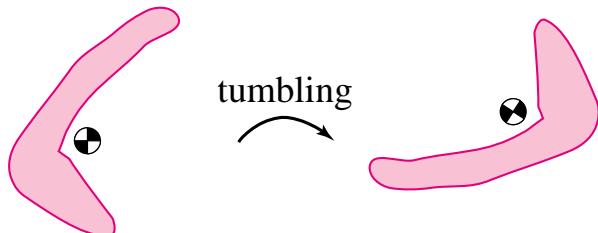


The center of mass is not even on any point in the system and, although it represents the average position in the system, it does not move with any point on the system.

On the other hand, for a *rigid body*, the center of mass is fixed relative to the body as the body moves,



even if the center of mass is not on the body, such as for this 'L-shaped' object.



In this case, the center of mass is not literally *on* the body. It is fixed with respect to the body, however. If you were rigidly attached to the body and fixed your gaze on the location of the center of mass, it would not waver in your view as the body, with you attached, tumbled wildly. In this sense the center of mass is fixed "on" a rigid body even if not on the body at all.

**SAMPLE 5.42** *Location of the center of mass.* A structure is made up of three point masses,  $m_1 = 1 \text{ kg}$ ,  $m_2 = 2 \text{ kg}$  and  $m_3 = 3 \text{ kg}$ . At the moment of interest, the coordinates of the three masses are  $(1.25 \text{ m}, 3 \text{ m})$ ,  $(2 \text{ m}, 2 \text{ m})$ , and  $(0.75 \text{ m}, 0.5 \text{ m})$ , respectively. At the same instant, the velocities of the three masses are  $2 \text{ m/s} \hat{i}$ ,  $2 \text{ m/s} (\hat{i} - 1.5 \hat{j})$  and  $1 \text{ m/s} \hat{j}$ , respectively.

- Find the coordinates of the center of mass of the structure.
- Find the velocity of the center of mass.

### Solution

- Let  $(\bar{x}, \bar{y})$  be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\ &= \frac{1 \text{ kg} \cdot 1.25 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.75 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}} \\ &= \frac{7.25 \text{ kg} \cdot \text{m}}{6 \text{ kg}} = 1.25 \text{ m}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} &= \frac{\sum m_i y_i}{\sum m_i} \\ &= \frac{1 \text{ kg} \cdot 3 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.5 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}} \\ &= \frac{8.55 \text{ kg} \cdot \text{m}}{6 \text{ kg}} = 1.42 \text{ m}.\end{aligned}$$

Thus the center of mass is located at the coordinates  $(1.25 \text{ m}, 1.42 \text{ m})$ .

(1.25 m, 1.42 m)

- For a system of particles, the linear momentum

$$\begin{aligned}\vec{L} &= \sum m_i \vec{v}_i = m_{\text{tot}} \vec{v}_{cm} \\ \Rightarrow \vec{v}_{cm} &= \frac{\sum m_i \vec{v}_i}{m_{\text{tot}}} \\ &= \frac{1 \text{ kg} \cdot (2 \text{ m/s} \hat{i}) + 2 \text{ kg} \cdot (2 \hat{i} - 3 \hat{j}) \text{ m/s} + 3 \text{ kg} \cdot (1 \text{ m/s} \hat{j})}{6 \text{ kg}} \\ &= \frac{(6 \hat{i} - 3 \hat{j}) \text{ kg} \cdot \text{m/s}}{6 \text{ kg}} \\ &= 1 \text{ m/s} \hat{i} + 0.5 \text{ m/s} \hat{j}.\end{aligned}$$

$\vec{v}_{cm} = 1 \text{ m/s} \hat{i} + 0.5 \text{ m/s} \hat{j}$

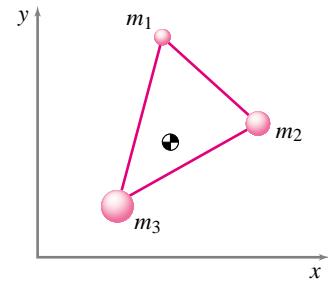


Figure 5.88: (filename:sfig2.4.2)

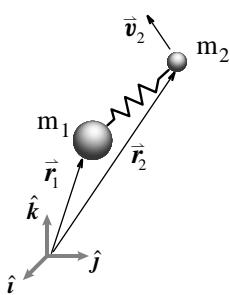
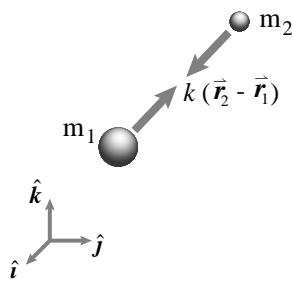


Figure 5.89: (Filename:sfig5.10.hopper)

**SAMPLE 5.43** *A spring-mass system in space.* A spring-mass system consists of two masses,  $m_1 = 10\text{ kg}$  and  $m_2 = 1\text{ kg}$ , and a weak spring with stiffness  $k = 1\text{ N/m}$ . The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the moment of observation, i.e., at  $t = 0$ ,  $\vec{r}_1 = \vec{0}$ ,  $\vec{r}_2 = 1\text{ m}(\hat{i} + \hat{j} + \hat{k})$ ,  $\dot{\vec{r}}_1 = \vec{0}$ , and  $\dot{\vec{r}}_2 = \sqrt{6}\text{ m/s}(-\hat{i} + \hat{j})$ . Track the motion of the system for the next 20 seconds. In particular,

- Plot the trajectory of the two masses in space.
- Plot the trajectory of the center of mass of the system.
- Plot the trajectory of the two masses as seen by an observer sitting at the center of mass.
- Compute and plot the total energy of the system and show that it remains constant during the entire motion.

**Solution** The free body diagrams of the two masses are shown in Fig. 5.90. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

Figure 5.90: Free body diagram of the two masses.  
(Filename:sfig5.10.hopper.a)

$$\begin{aligned} m_1 \ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\ m_2 \ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1) \end{aligned}$$

Let  $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$  and  $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ . Substituting above and dotting the two equations with  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ , we get

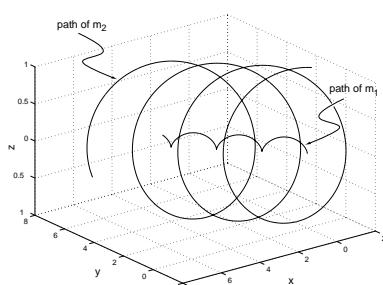
$$\begin{aligned} \ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\ \ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\ \ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1) \end{aligned}$$

Thus we get six second order coupled linear ODEs as equations of motion.

- To plot the trajectory of the two masses, we need to solve for  $\vec{r}_1(t)$  and  $\vec{r}_2(t)$ , i.e., for  $x_1(t)$ ,  $y_1(t)$ ,  $z_1(t)$ , and  $x_2(t)$ ,  $y_2(t)$ ,  $z_2(t)$ . We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
       uldot = k/m1*(x2-x1),
       y1dot = v1,
       v1dot = k/m1*(y2-y1),
       z1dot = w1,
       w1dot = k/m1*(z2-z1),
       x2dot = u2,
       u2dot = -k/m2*(x2-x1),
       y2dot = v2,
       v2dot = -k/m2*(y2-y1),
       z2dot = w2,
       w2dot = -k/m2*(z2-z1) }
```

IC = {x1(0)=0, y1(0)=0, z1(0)=0,

Figure 5.91: 3-D trajectory of  $m_1$  and  $m_2$  plotted from numerical solution of the equations of motion.  
(Filename:sfig5.10.hopper.b)

```

u1(0)=0, v1(0)=0, w1(0)=0,
x2(0)=1, y2(0)=1, z2(0)=1,
u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}

```

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in Fig. 5.91. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

- (b) We can find the trajectory of the center of mass using the following relationships.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{cm} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center of mass of the system has zero acceleration. Therefore, we expect the center of mass to move on a straight path with constant velocity. The center of mass coordinates  $x_{cm}$ ,  $y_{cm}$ , and  $z_{cm}$  are plotted against time in Fig. 5.92 which show that the center of mass moves on a straight line in a plane parallel to the  $xy$ -plane ( $z$  is constant). This is expected since the initial velocity of the center of has no  $z$ -component:

$$\begin{aligned}\vec{v}_{cm} &= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \\ &= \frac{m_1 \cdot \vec{0} + 1 \text{ kg} \cdot \sqrt{6} \text{ m/s}(-\hat{i} + \hat{j})}{10 \text{ kg} + 1 \text{ kg}} \\ &= 0.22 \text{ m/s}(-\hat{i} + \hat{j}).\end{aligned}$$

- (c) The trajectory of the two masses with respect to the center of mass can be easily obtained by the following relationships.

$$\begin{aligned}x_{1/cm} &= x_1 - x_{cm}, & y_{1/cm} &= y_1 - y_{cm}, & z_{1/cm} &= z_1 - z_{cm} \\ x_{2/cm} &= x_2 - x_{cm}, & y_{2/cm} &= y_2 - y_{cm}, & z_{2/cm} &= z_2 - z_{cm}\end{aligned}$$

The trajectories thus obtained are shown in Fig. 5.92. It is clear that the two masses have closed orbits with respect to the center of mass. These closed orbits are actually conic sections as we would expect in a central force problem.

- (d) We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned}(E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\ (E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\ E_p &= \frac{1}{2} k [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\ E_{total} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p\end{aligned}$$

The energies so calculated are plotted in Fig. 5.93. It is clear from the plot that the total energy remains constant during the entire motion.

□

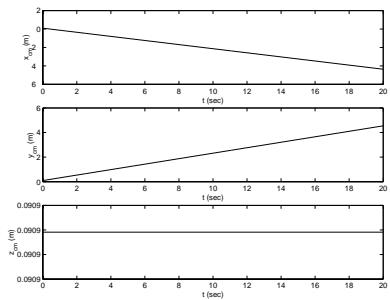


Figure 5.92: The center of mass coordinates  $x_{cm}(t)$ ,  $y_{cm}(t)$ , and  $z_{cm}(t)$ . The center of mass moves on a straight line in a plane parallel to the  $xy$ -plane.

(Filename:sfig5.10.hopper.c)

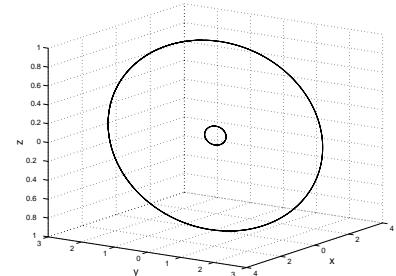


Figure 5.93: The paths of  $m_1$  and  $m_2$  as seen from the center of mass. The two masses are on closed orbits with respect to the center of mass.

(Filename:sfig5.10.hopper.d)

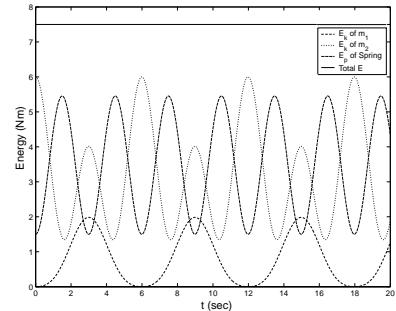


Figure 5.94: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

(Filename:sfig5.10.hopper.e)



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# 6 Constrained straight line motion

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In the previous chapter you learned that it is straightforward to write the equations of motion for a particle, or for a collection of a few particles, if you have a model for the forces on the particles in terms of their positions, velocities, and time. After writing  $\vec{F} = m\vec{a}$  for each particle, finding unknown forces and accelerations with given velocities is then a matter of solving linear algebraic equations. And finding the motions is a matter of solving the resulting differential equations which, if often too complicated for analytic solution, are straightforward to set up for numerical solution.

If every object is made of particles which interact with known force laws, then we can solve all dynamics problems using the methods of chapter 5. But the particle view of the world has two major shortcomings:

- if the particles in question are, say, atoms, then solving a typical engineering scale dynamics problem involves writing  $6 \cdot 10^{23}$  (Avagadro's number) or so coupled differential equations (even the smallest micro-electronic-machines have thousands of atoms). Solving so many equations is more than we can generally ask of our computers.
- Sometimes, often actually, the simplest model of mechanical interaction is not a law for force as a function of position, velocity and time, but just a geometric restriction on the relative positions or velocities of points. The reason's for this geometric, instead of force-based, approach are two-fold:
  - Sometimes the minute details of the motion are not of interest and therefore not worth tracking (*e.g.*, the vibrations of a solid, or relative motions of atoms in a solid are not of interest), and
  - Often one does not know an accurate force law (*e.g.*, at the microscopic level one does not know the details of atomic interactions; or, at the machine level, one may not know exactly the relations between the small play in an axle and the force on the axle, even though one knows that the

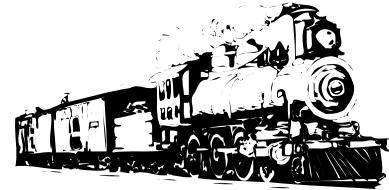


Figure 6.1: A train running on straight level track is in straight-line motion, neglecting, of course, the wheel rotation, the bouncing, the moving engine parts, and the wandering eyes of the passengers.

(Filename: tfigure3.0.train)

axle restricts the relative motion of a train with its wheels and the ground).

So, much mechanical modeling involves assumptions about the geometry of the motions, or *kinematic constraint*.

The utility of free body diagrams, the principle of action and reaction, the linear and angular momentum balance equations, and the balance of energy apply to all systems, no matter how they are or are not constrained. But, if objects are constrained the methods in mechanics have a slightly different flavor. It is easiest if we start with systems that have simple constraints and that move in simple ways. In this short chapter, we will discuss the mechanics of things where every point in the body has the same velocity and acceleration as every other point (so called *parallel motion*) and furthermore where every point moves in a straight line.

#### Example: Train on Straight Level Tracks

Consider a train on straight level tracks. If we focus on the body of the train, we can approximate the motion as parallel straight-line motion. All parts move the same amount, with the same velocities and accelerations in the same fixed direction.  $\square$

We start with 1-D mechanics and constraint with string and pulleys, and then move on to rigid bodies.

## 6.1 1-D constrained motion and pulleys

The kinematic constraints we consider here are those imposed by connection with bars or ropes. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as an inextensible connection. This idealization of zero deformation is a simplification. But it is a simplification that requires special treatment. It is the simplest example of a kinematic constraint.

Figure 6.2 shows a schematic of one car pulling another. One-dimensional free body diagrams are also shown. The force  $F$  is the force transmitted from the road to the front car through the tires. The tension  $T$  is the tension in the connecting chain. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1 \ddot{x}_1 \quad \text{and} \quad F - T = m_2 \ddot{x}_2. \quad (6.1)$$

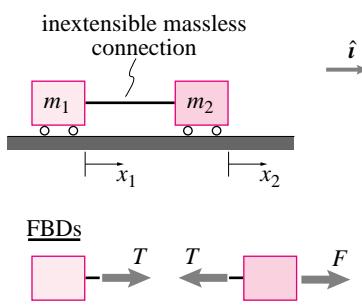


Figure 6.2: A schematic of one car pulling another, or of a boat pulling a barge.

(Filename:tfigure.boatpullsbarge)

But these equations are exactly the same as we would have if the cars were connected by a spring, a dashpot, or any idealized-as-massless connector. And all these systems have different motions. We need our equations to somehow indicate that the two particles are not allowed to move independently. We need something to replace the constitutive law that we would have used for a spring or dashpot.

### Kinematic constraint: two approaches

In the simplest example below we two ways of dealing with kinematic constraints:

- (a) Use separate free body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or
- (b) do something clever to avoid having to find the constraint forces.

*Finding the constraint force with the accelerations*

The geometric (or kinematic) restriction that two masses must move in lock-step is

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \quad (6.2)$$

If we take  $F$  and the two masses as given, equations 6.1 and 6.2 are three equations for the unknowns  $\ddot{x}_1$ ,  $\ddot{x}_2$ , and  $T$ . In matrix form, we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

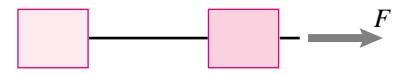
We can solve these equations to find  $\ddot{x}_1$ ,  $\ddot{x}_2$ , and  $T$  in terms of  $F$ .

*Finessing the finding of the constraint force*

On the other hand, if all we are interested in is the acceleration of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free body diagram of the entire system as in figure 6.3. If we just call the acceleration of the system  $\ddot{x}$  we have, from linear momentum balance, that

$$F = (m_1 + m_2)\ddot{x},$$

which is one equation in one unknown.



**Figure 6.3:** A free body diagram of the whole system. Note that the unknown tension (constraint) force does not show.

(Filename: tfigure.twocarstogether)

*Two particles connected by an inextensible rod make up the simplest rigid body*

A generalization of the 1D inextensible-cable constraint example above is the rigid-body constraint where not just two, but many particles are assumed to keep constant distance from one another, and in two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics is as the difficulty of dealing with kinematic constraints.

## Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Objects connected by inextensible ropes around ideal pulleys are also an example of kinematic constraint.

### Constant length and constant tension

Problems with pulleys are solved by using two facts about idealized string. First, ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant (not varying in time).

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + \dots = \text{constant} \quad (6.3)$$

① See figure 4.4 on page 114 and the related text which shows why  $T_1 = T_2$  for one pulley idealized as frictionless and massless.

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string<sup>①</sup>. The tension on one side of a pulley is the same as the tension on the other side. And this can carry on if a rope is wrapped around several pulleys.

$$T_1 = T_2 = T_3 \dots \quad (6.4)$$

We use the trivial pulley example in figure 6.4 to show how to analyze the relative motion of various points in a pulley system.

#### Example: Length of string calculation

Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv \text{constant}. \quad (6.5)$$

The portion of string wrapped around the pulley contacts half of the pulley so that its length is half the pulley circumference,  $\pi r$ . Even if  $x_A$  and  $x_B$  change in time and different portions of string wrap around the pulley, the length of string touching the pulley is always  $\pi r$ .

We can now formally deduce the intuitively obvious relations between the velocities and accelerations of points A and B. Differentiating equation 6.5 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 &= \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A &= -\dot{x}_B \\ \Rightarrow \ddot{x}_A &= -\ddot{x}_B \end{aligned} \quad (6.6)$$

When point A is displaced to the right by an amount  $\Delta x_A$ , the point B is displaced exactly the same amount but to the left; that is,  $\Delta x_A = -\Delta x_B$ . Note that in order to derive the kinematic relations 6.6 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant-in-time quantities (the pulley half-circumference and the string length) get ‘killed’ in the process of differentiation.  $\square$

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations in which cases the pulley contact length drops out of the equations anyway.

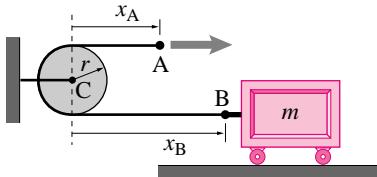


Figure 6.4: One mass, one pulley, and one string

(Filename:tfigure3.pulleyex)

## The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in Fig. 6.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in Fig. 6.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively. The details of the calculations are a homework problem.

In Fig. 6.5b we pull one direction and the mass accelerates the other way. This illustrates one use of a pulley, to redirect an applied force. The force on the mass has magnitude  $|\vec{F}|$  and there is no mechanical advantage.

Fig. 6.5c shows the most classic use of a pulley. A free body diagram of the pulley at C will show you that the tension in rope AC is  $2|\vec{F}|$  and we have thus doubled the force acting on the mass. However, counting string length and displacement you will see that point A moves only half the distance that point B moves. Thus the force at B is multiplied by two to give the force at A and the displacement at B is divided by two to give the displacement at A. This is a general property of ideal transmissions, from levers to pulleys to gear boxes:

If force is amplified then motion is equally attenuated.

This result is most solidly understood using energy balance. The power of the force at B goes entirely into the mass. On the other hand if we cut the string AB, the same amount of power must be applied to the mass (it gains the same energy). Thus the product of the tension and velocity at A must equal the product of the tension and velocity at B.

$$T_A v_A = T_B v_B$$

Fig. 6.5d shows the opposite use of a pulley. A free body diagram of the pulley shows that the tension in AB is  $\frac{1}{2}|\vec{F}|$ . Thus the force is attenuated by a factor of 2. A kinematic analysis reveals that the motion of A is twice that of B. Thus, as expected from energy considerations, the motion is amplified when the force is attenuated.

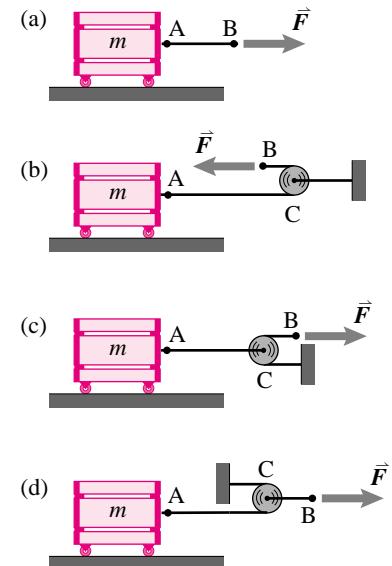


Figure 6.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that multiplies force and attenuates motion, and (d) a pulley system that attenuates force and amplifies motion.

(Filename:tfigure.pulley1)

## Effective mass

Of concern for design of machines that people work with is the feel of the machine. One aspect of feel is the effective mass. The *effective mass* is defined by the response of a point when a force is applied.

$$m_{\text{eff}} = \frac{|\vec{F}_B|}{|\vec{a}_B|}.$$

For the case of Fig. 6.5a and Fig. 6.5b the effective mass of point B is the mass of the block,  $m$ . For the case of Fig. 6.5c the block has twice the twice  $|\vec{F}|$  acting on it and point B has twice the acceleration of point A, so the effective mass of point B is  $m/4$ . For the case of Fig. 6.5d, the mass only has half  $|\vec{F}|$  acting on it and point B only has half the acceleration of point A, so the effective mass is  $4m$ .

These special cases exemplify the general rule:

The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems Fig. 6.5c and Fig. 6.5d which might look so similar, actually differ by a factor of 16 ( $= 2^2 \cdot 2^2$ ). With a given  $F$  and  $m$  point B in Fig. 6.5c has 16 times the acceleration of point B in Fig. 6.5d.

**SAMPLE 6.1** Find the motion of two cars. One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is  $F$ . The second car rolls frictionlessly. Find the acceleration of the system two ways:

- (a) using separate free body diagrams,
- (b) using a system free body diagram.

### Solution

- (a) From linear momentum balance of the two cars, we get

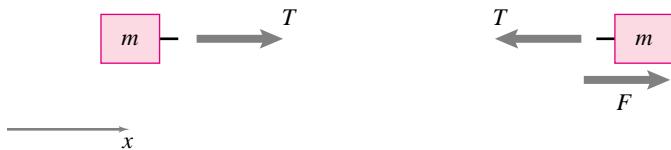


Figure 6.6: (Filename:sfig4.1.twocars.fbda)

$$m\ddot{x}_1 = T \quad (6.7)$$

$$F - T = m\ddot{x}_2 \quad (6.8)$$

The kinematic constraint of towing (the cars move together, *i.e.*, no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \quad (6.9)$$

Solving eqns. (6.7), (6.8), and (6.9) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

- (b) From linear momentum balance of the two cars as one system, we get

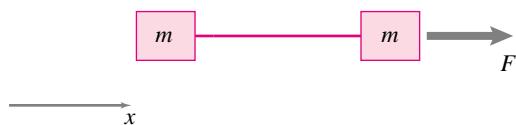


Figure 6.7: (Filename:sfig4.1.twocars.fbdb)

$$\begin{aligned} m\ddot{x} + m\ddot{x} &= F \\ \ddot{x} &= F/2m \end{aligned}$$

$\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m$

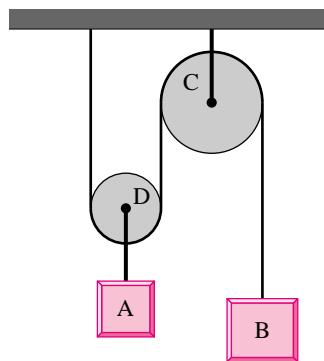


Figure 6.8: (Filename:sfig3.3.DH1)

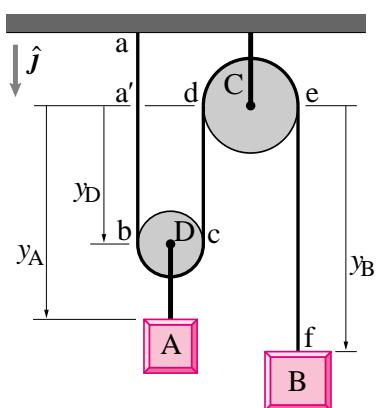


Figure 6.9: (Filename:sfig3.3.DH2)

① We have done an elaborate calculation of  $\ell_{tot}$  here. Usually, the constant lengths over the pulleys and some constant segments such as  $aa'$  are ignored in calculating  $\ell_{tot}$ . These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

**SAMPLE 6.2 Pulley kinematics.** For the masses and ideal-massless pulleys shown in figure 6.8, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

**Solution** Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let  $y_A$  and  $y_B$  be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{r}_A = y_A \hat{j} \quad \text{and} \quad \vec{r}_B = y_B \hat{j}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned}\vec{v}_A &= \dot{y}_A \hat{j}, & \vec{v}_B &= \dot{y}_B \hat{j}, \\ \vec{a}_A &= \ddot{y}_A \hat{j}, & \vec{a}_B &= \ddot{y}_B \hat{j}.\end{aligned}$$

Since all quantities are in the same direction ( $\hat{j}$ ), we can drop  $\hat{j}$  from our calculations and just do scalar calculations. We are asked to relate  $\ddot{y}_A$  to  $\ddot{y}_B$ .

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string  $\ell_{tot}$  is: ①

$$\begin{aligned}\ell_{tot} &= ab + bc + cd + de + ef = \text{constant} \\ \text{where } ab &= \underbrace{aa'}_{\text{constant}} + \underbrace{a'b}_{(=cd=y_D)} \\ bc &= \text{string over the pulley D} = \text{constant} \\ de &= \text{string over the pulley C} = \text{constant} \\ ef &= y_B \\ &\qquad\qquad\qquad (\text{aa}' + \text{bc} + \text{de}) \\ \text{thus } \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(aa' + bc + de)}.\end{aligned}$$

Taking the time derivative on both sides, we get

0 because  $\ell_{tot}$  does not change with time

$$\underbrace{\frac{d}{dt}(\ell_{tot})}_{0} = 2\dot{y}_D + \dot{y}_B \Rightarrow \dot{y}_D = -\frac{1}{2}\dot{y}_B \quad (6.10)$$

$$\Rightarrow \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \quad (6.11)$$

$$\begin{aligned}\text{But } y_D &= y_A - AD \text{ and } AD = \text{constant} \\ \Rightarrow \dot{y}_D &= \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A.\end{aligned}$$

Thus, substituting  $\dot{y}_A$  and  $\ddot{y}_A$  for  $\dot{y}_D$  and  $\ddot{y}_D$  in (6.10) and (6.11) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$\ddot{y}_A = -\frac{1}{2}\ddot{y}_B$



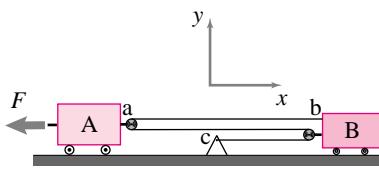


Figure 6.10: A two-mass pulley system.

(Filename:sfig3.3.1)

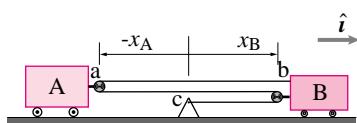


Figure 6.11: Pulley kinematics. Note that the distance from c to a is minus the x coordinate of a.

(Filename:sfig3.3.1b)

**SAMPLE 6.3** A two-mass pulley system. The two masses shown in Fig. 6.10 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that  $F = 130\text{ N}$ ,  $m_A = m_B = m = 40\text{ kg}$ , find the acceleration of the two blocks using:

- linear momentum balance (LMB) and
- energy balance.

**Solution**

- Using Linear Momentum Balance: The free body diagrams of the two

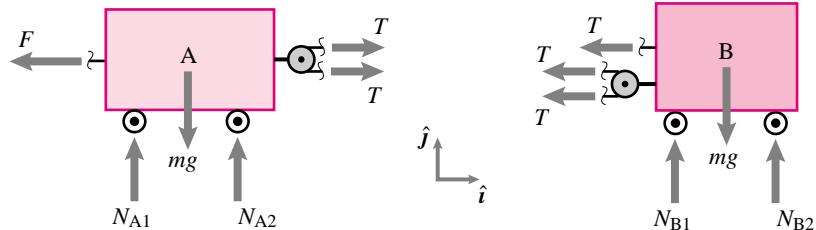


Figure 6.12: (Filename:sfig3.3.1a)

masses A and B are shown in Fig. 6.12 above. Linear momentum balance for mass A gives (assuming  $\vec{a}_A = a_A \hat{i}$  and  $\vec{a}_B = a_B \hat{i}$ ):

$$\begin{aligned} (2T - F)\hat{i} + (N_{A1} + N_{A2} - mg)\hat{j} &= m\vec{a}_A = -ma_A \hat{i} \\ (\text{dotting with } \hat{j}) \Rightarrow 2N_A &= mg \\ \text{and} \quad 2T - F &= ma_A \end{aligned} \quad (6.12)$$

Similarly, linear momentum balance for mass B gives:

$$\begin{aligned} -3T\hat{i} + (N_{B1} + N_{B2} - mg)\hat{j} &= m\vec{a}_B = ma_B \hat{i} \\ \Rightarrow 2N_B &= mg \\ \text{and} \quad -3T &= ma_B. \end{aligned} \quad (6.13)$$

① You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

From (6.12) and (6.13) we have three unknowns:  $T$ ,  $a_A$ ,  $a_B$ , but only 2 equations!. We need an extra equation to solve for the three unknowns. ①

We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. Using the fixed point C as the origin of our  $xy$  coordinate system we can write

$$\begin{aligned} \ell_{tot} &\equiv \text{length of the string connecting A and B} \\ &= 3x_B + 2(-x_A) \\ \Rightarrow \overbrace{\ell_{tot}}^0 &= 3\dot{x}_B + 2(-\dot{x}_A) \\ \Rightarrow \dot{x}_B &= -\frac{2}{3}(-\dot{x}_A) \Rightarrow \ddot{x}_B = -\frac{2}{3}(-\ddot{x}_A) \end{aligned} \quad (6.14)$$

Since

$$\vec{v}_A = v_A \hat{i} = -(-\dot{x}_A) \hat{i},$$

$$\begin{aligned}\vec{a}_A &= a_A \hat{i} = \dot{x}_A \hat{i}, \\ \vec{v}_B &= v_B \hat{i} = \dot{x}_B \hat{i}, \text{ and} \\ \vec{a}_B &= a_B \hat{i} = \ddot{x}_B \hat{i},\end{aligned}$$

we get

$$a_B = \frac{2}{3}a_A. \quad (6.15)$$

Substituting (6.15) into (6.13), we get

$$9T = -2m_B a_A. \quad (6.16)$$

Now solving (6.12) and (6.16) for  $T$ , we get

$$T = \frac{2F}{13} = \frac{2 \cdot 130 \text{ N}}{13} = 20 \text{ N}.$$

Therefore,

$$\begin{aligned}a_A &= -\frac{9T}{2m} = -\frac{9 \cdot 20 \text{ N}}{2 \cdot 40 \text{ kg}} = -2.25 \text{ m/s}^2 \\ a_B &= \frac{2}{3}a_A = -1.5 \text{ m/s}^2\end{aligned}$$

$$\boxed{\vec{a}_A = -2.25 \text{ m/s}^2 \hat{i}, \vec{a}_B = -1.5 \text{ m/s}^2 \hat{i}.}$$

(b) **Using Power Balance (III):** We have,

$$P = \dot{E}_K.$$

The power balance equation becomes

$$\sum \vec{F} \cdot \vec{v} = m a_A v_A + m_B a_B v_B.$$

Because the force at A is the only force that does work on the system, when we apply power balance to the whole system (using  $\vec{F} = -F \hat{i}$ ), we get

$$\begin{aligned}-F v_A &= m_A v_A a_A + m_B v_B a_B \\ \text{or } F &= -ma_A - m \frac{v_B}{v_A} a_B \\ &= -a_A (m + m \frac{v_B}{v_A} \frac{a_B}{a_A}).\end{aligned}$$

Substituting  $a_B = 2/3a_A$  and  $v_B = 2/3v_A$  from Eqn. (6.15),

$$a_A = \frac{-F}{m + \frac{4}{9}m} = \frac{-130 \text{ N}}{40 \text{ kg}(1 + \frac{4}{9})} = -2.25 \text{ m/s}^2,$$

and since  $a_B = 2/3a_A$ ,

$$a_B = -1.5 \text{ m/s}^2,$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25 \text{ m/s}^2 \hat{i}, a_B = -1.5 \text{ m/s}^2 \hat{i}}$$

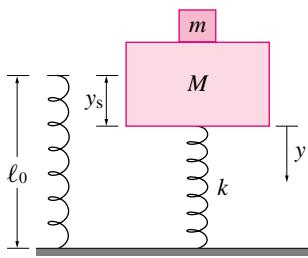
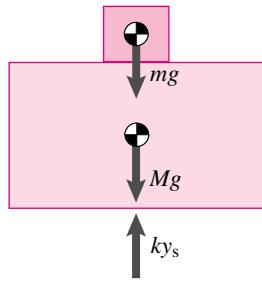
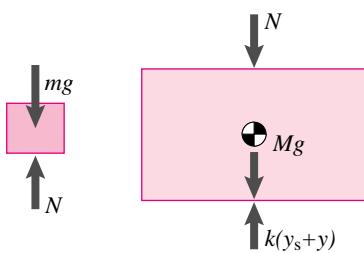


Figure 6.13: (Filename:sfig10.1.5)

Figure 6.14: Free body diagram of the two masses as one system when in static equilibrium (this special case could be skipped as it follows from the free body diagram below).  
(Filename:sfig10.1.5a)Figure 6.15: Free body diagrams of the individual masses.  
(Filename:sfig10.1.5b)

**SAMPLE 6.4** In static equilibrium the spring in Fig. 6.13 is compressed by  $y_s$  from its unstretched length  $\ell_0$ . Now, the spring is compressed by an additional amount  $y_0$  and released with no initial velocity.

- Find the force on the top mass  $m$  exerted by the lower mass  $M$ .
- When does this force become minimum? Can this force become zero?
- Can the force on  $m$  due to  $M$  ever be negative?

### Solution

- The free body diagram of the two masses is shown in Figure 6.14 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad ky_s = (m + M)g. \quad (6.17)$$

The free body diagrams of the two masses at an arbitrary position  $y$  during motion are given in Figure 6.15. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass  $m$  we get

$$mg - N = m\ddot{y}. \quad (6.18)$$

We are interested in finding the normal force  $N$ . Clearly, we need to find  $\ddot{y}$  to calculate  $N$ . Now, from linear momentum balance for mass  $M$  we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \quad (6.19)$$

Adding Eqn (6.18) with Eqn (6.19) we get

$$(m + M)g - ky - ky_s = (m + M)\ddot{y}.$$

But  $ky_s = (m + M)g$  from Eqn 6.17. Therefore, the equation of motion of the system is

$$\begin{aligned} -ky &= (m + M)\ddot{y} \\ \text{or } \ddot{y} + \frac{k}{(m + M)}y &= 0. \end{aligned} \quad (6.20)$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) = A \sin \lambda t + B \cos \lambda t \quad (6.21)$$

where  $\lambda = \sqrt{\frac{k}{m+M}}$  and the constants  $A$  and  $B$  are to be determined from the initial conditions. From Eqn (6.21) we obtain

$$\dot{y}(t) = A\lambda \cos \lambda t - B\lambda \sin \lambda t. \quad (6.22)$$

Substituting the given initial conditions  $y(0) = y_0$  and  $\dot{y}(0) = 0$  in Eqns (6.21) and (6.22), respectively, we get

$$\begin{aligned} y(0) &= y_0 = B \\ \dot{y}(0) &= 0 = A\lambda \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$y(t) = y_0 \cos \lambda t. \quad (6.23)$$

Now we can find the acceleration by differentiating Eqn (6.23) twice :

$$\ddot{y} = -y_0 \lambda^2 \cos \lambda t.$$

Substituting this expression in Eqn (6.18) we get the force applied by mass  $M$  on the smaller mass  $m$ :

$$\begin{aligned} mg - N &= m \overbrace{(-y_0\lambda^2 \cos \lambda t)}^{\ddot{y}} \\ \Rightarrow N &= mg + my_0\lambda^2 \cos \lambda t \\ &= m(g + y_0\lambda^2 \cos \lambda t) \end{aligned} \quad (6.24)$$

$$N = m(g + y_0\lambda^2 \cos \lambda t)$$

- (b) Since  $\cos \lambda t$  varies between  $\pm 1$ , the value of the force  $N$  varies between  $mg \pm y_0\lambda^2$ . Clearly,  $N$  attains its minimum value when  $\cos \lambda t = -1$ , i.e., when  $\lambda t = \pi$ . This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{min} = m(g - y_0\lambda^2)$$

If  $y_0$ , the initial displacement from the static equilibrium position, is chosen such that  $y_0 = \frac{g}{\lambda^2}$ , then  $N = 0$  when  $\cos \lambda t = -1$ , i.e., at the topmost point in the vertical motion. This condition means that the two masses momentarily lose contact with each other when they are about to begin their downward motion.  $\triangleleft$

- (c) From Eqn (6.24) we can get a negative value of  $N$  when  $\cos \lambda t = -1$  and  $y_0\lambda^2 > g$ . However, a negative value for  $N$  is nonsense unless the blocks are glued. Without glue the bigger mass  $M$  cannot apply a negative compression on  $m$ , i.e., it cannot “suck”  $m$ . When  $y_0\lambda^2 > g$  then  $N$  becomes zero before  $\cos \lambda t$  decreases to  $-1$ . That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between  $m$  and  $M$ . Eqn (6.24) is inapplicable when  $N \leq 0$ .  $\triangleleft$

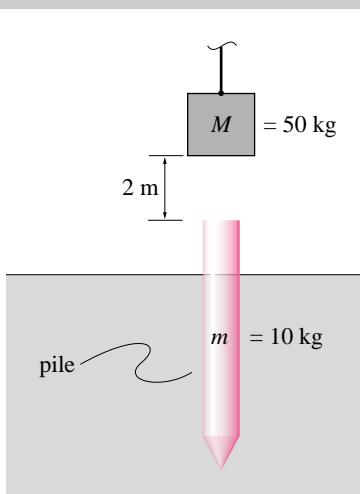
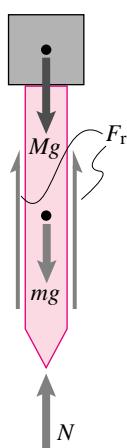


Figure 6.16: (Filename:sfig3.5.DH1)

Figure 6.17: Free body diagram of the hammer and pile system.  $F_r$  is the total resistance of the ground.  
(Filename:sfig3.5.DH2)

**SAMPLE 6.5 Driving a pile into the ground.** A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the  $n$ th blow the pile is driven into the ground by an additional 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.*, the two stick together), find the average resistance of the soil to penetration of the pile.

**Solution** Let  $F_r$  be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free body diagram of the pile and hammer system, we have

$$\sum \vec{F} = -mg\hat{j} - Mg\hat{j} + N\hat{j} + F_r\hat{j}.$$

But  $N$  is the normal reaction of the ground, which from static equilibrium, must be equal to  $mg + Mg$ . Thus,

$$\sum \vec{F} = Fr\hat{j}.$$

Therefore, from linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ),

$$\vec{a} = \frac{Fr}{M+m}\hat{j}.$$

Now we need to find the acceleration from given conditions. Let  $v$  be the speed of the hammer just before impact and  $V$  be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page 89). The impact force is internal to the system. Therefore, during impact,  $\sum \vec{F} = \vec{0}$  which implies that linear momentum is conserved. Thus

$$\begin{aligned} -Mv\hat{j} &= -(m+M)V\hat{j} \\ \Rightarrow V &= \left(\frac{M}{m+M}\right)v = \frac{50\text{ kg}}{60\text{ kg}}v = \frac{5}{6}v. \end{aligned}$$

The hammer speed  $v$  can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = \sqrt{2 \cdot (9.81 \text{ m/s}^2) \cdot (2 \text{ m})} = 6.26 \text{ m/s} \Rightarrow V = \frac{5}{6}v = 5.22 \text{ m/s.}$$

The pile and the hammer travel a distance of  $s = 5 \text{ cm}$  under the deceleration  $a$ . The initial speed  $V = 5.22 \text{ m/s}$  and the final speed = 0. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 = v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad (\text{Note that } a \text{ is negative}) \\ \Rightarrow a &= \frac{V^2}{2s} = \frac{(5.22 \text{ m/s})^2}{2 \times 0.05 \text{ m}} = 272.48 \text{ m/s}^2. \end{aligned}$$

Thus  $\vec{a} = 272.48 \text{ m/s}^2 \hat{j}$ . Therefore,

$$F_r = (m+M)a = (60 \text{ kg}) \cdot (272.48 \text{ m/s}^2) = 1.635 \times 10^4 \text{ N}$$

$$F_r \approx 16.35 \text{ kN}$$

## 6.2 2-D and 3-D forces even though the motion is straight

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

### Example: Piston in a cylinder.

Consider a piston sliding vertically in a cylinder. For now neglect the spatial extent of the cylinder. Let's assume a coefficient of friction  $\mu$  between the piston and the cylinder wall and that the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 4.1b. That is, the force transmitted to the piston from the connecting rod is along the connecting rod. The free body diagram of the piston (with a bit of the connecting rod) is shown in figure 6.18. We have assumed that the piston is moving up so the friction force is directed down, resisting the motion. Linear momentum balance for this system is:

$$\begin{aligned}\sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{i} - \mu N\hat{j} + T\hat{\lambda}_{rod} &= m_{piston} a\hat{j}.\end{aligned}$$

If we assume that the acceleration  $a\hat{j}$  of the piston is known, as is its mass  $m_{piston}$ , the coefficient of friction  $\mu$ , and the orientation of the connecting rod  $\hat{\lambda}_{rod}$ , then we can solve for the rod tension  $T$  and the normal reaction  $N$ . Note that even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation.  $\square$

The kinematically simple 1-D motions we assume in this chapter simplify the evaluation of the right hand sides of the vector momentum balance equations. But they are still vector equations.

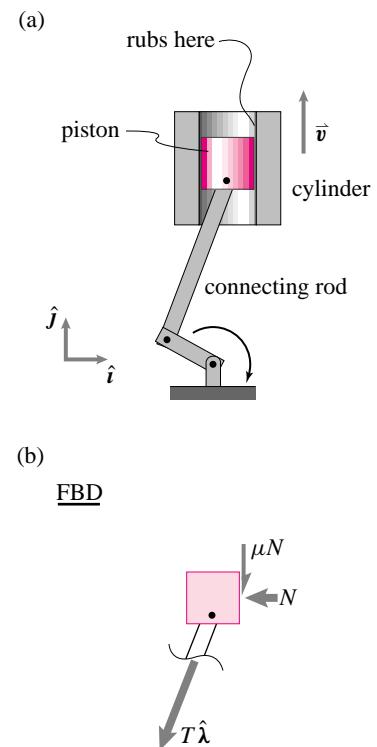


Figure 6.18: (a) shows a piston in a cylinder. (b) shows a free body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction  $\mu$  between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

(Filename:tfigure3.1)

## Highly constrained bodies

This chapter is about rigid bodies that do not rotate. Most objects will not agree to be the topic of such discussion without being forced into doing so. In general, one expects bodies to rotate or move along a curved path. To keep an object that is subject to various forces from rotating or curving takes some constraint. The object needs to be rigid and held by wires, rods, rails, hinges, welds, etc. that keep it from spinning, keeping it in parallel motion. We do not mean to imply that the presence of constraint always is associated with disallowance of rotation — constraints could even cause rotation. But to keep a rigid object in straight-line motion usually does require some kind of constraint.

Of common interest for constrained structures is making sure that static and dynamic loads do not cause failure of the constraints. For example, suppose a truck hauls a very heavy load that is held down by chains tightened by come-alongs. When the truck accelerates, what is the tension in the chains, and will it exceed the strength limit of the chains so that they might break?

In this chapter, we assume all points of a system or body are moving in a straight line with the same velocity and acceleration. Let's consider a set of points in the system of interest. Let's call them  $A$  to  $G$ , or generically,  $P$ . For convenience we

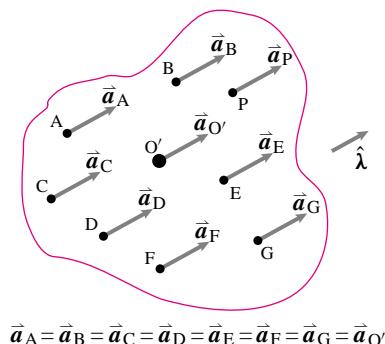


Figure 6.19: Parallel motion: all points on the body have the same acceleration  $\vec{a} = a\hat{\lambda}$ . For straight-line motion:  $\hat{\lambda}(t) = \text{constant}$  in time and  $\vec{v} = v\hat{\lambda}$ .

(Filename:tfigure3.1a)

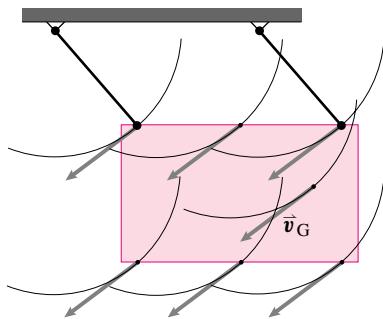


Figure 6.20: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.

(Filename:tfigure3.swing)

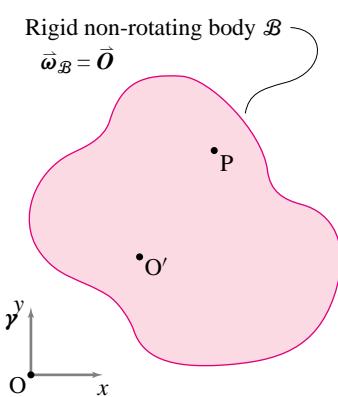


Figure 6.21: A non-rotating body  $\mathcal{B}$  with points  $O'$  and  $P$ .

(Filename:tfigure3.2.1)

distinguish a reference point  $O'$ .  $O'$  may be the center of mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that  $\vec{a}_P = \vec{a}_{O'}$ , and  $\vec{v}_P = \vec{v}_{O'}$  (Fig. 6.19). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that  $\vec{v}_A = \dots = \vec{v}_P = \vec{v}_{O'}$ .

A special case of parallel motion is straight-line motion.

*a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.*

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction  $\hat{\lambda}$ , then we have

$$\vec{v}(t) = v(t)\hat{\lambda}, \quad \vec{a}(t) = a(t)\hat{\lambda} \quad \text{and} \quad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\lambda}$$

for every point in the system.  $\vec{r}_0$  is the position of a point at time 0 and  $s$  is the distance the point moves in the  $\hat{\lambda}$  direction. Every point in the system has the same  $s$ ,  $v$ ,  $a$ , and  $\hat{\lambda}$  as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train in figure 6.1.

#### Example: Parallel swing is not straight-line motion

The swing shown does not rotate — all points on the swing have the same velocity. The motion of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed in Chapter 7. □

A special way of analyzing straight-line motion is with one-dimensional mechanics as we did in the previous chapter. For one-dimensional mechanics, we assume that, in addition to the restricted kinematics, everything of interest mechanically happens in the  $\hat{\lambda}$  direction, often taken to be the  $x$  direction. That is, we ignore *all* torques and angular momenta, and only consider the  $\hat{\lambda}$  components of the forces (*i.e.*,  $\vec{F} \cdot \hat{\lambda}$ ) and linear momentum ( $\vec{L} \cdot \hat{\lambda}$ ). For example, if  $\hat{\lambda}$  is in the  $\hat{i}$  direction, the components would be  $F_x$  and  $L_x$ .

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center of mass allows some useful simplifications of the general expressions for  $\vec{L}$ ,  $\vec{L}$ ,  $\vec{H}_C$ ,  $\vec{H}_C$  and  $E_K$ .

#### Velocity of a point

The velocity of any point  $P$  on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 6.21).

$$\vec{v}_P = \vec{v}_{O'}$$

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector  $\vec{\omega}$ . For now all you need to know is that  $\vec{\omega} = \vec{0}$  when something is not rotating. In 5b in Table II, if you set  $\vec{\omega}_{\mathcal{B}} = \vec{0}$  and  $\vec{v}_{P/\mathcal{B}} = \vec{0}$  it says that  $\vec{v}_P = \vec{r}_{O'/O}$  or in shorthand,  $\vec{v}_P = \vec{v}_{O'}$ , as we have written above.

### Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

### Angular momentum and its rate of change, $\vec{H}_C$ and $\dot{\vec{H}}_C$ for straight-line motion

For the motions in this chapter, where  $\vec{a}_i = \vec{a}_{cm}$  and thus  $\vec{a}_{i/cm} = \vec{0}$ , angular momentum considerations are simplified, as explained in Box 6.2 on 345<sup>①</sup>. But for straight-line motion (and for parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center of mass:<sup>②</sup>

$$\begin{aligned}\vec{H}_C &\equiv \sum(\vec{r}_{i/C} \times m_i \vec{v}_i) = \vec{r}_{cm/C} \times (m_{\text{total}} \vec{v}_{cm}), \\ \dot{\vec{H}}_C &\equiv \sum(\vec{r}_{i/C} \times m_i \vec{a}_i) = \vec{r}_{cm/C} \times (m_{\text{total}} \vec{a}_{cm}).\end{aligned}$$

### Approach

To study systems in straight-line motion (as always) we:

- draw a free body diagram, showing the appropriate forces and couples at places where connections are ‘cut’,
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and
- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

#### Example: Plate on a cart

A uniform rectangular plate  $ABCD$  of mass  $m$  is supported by a light rigid rod  $DE$  and a hinge joint at point  $B$ . The dimensions are as shown. The cart has acceleration  $a_x \hat{i}$  due to a force  $F \hat{i}$  and the constraints of the

<sup>①</sup> Calculating rate of change of angular momentum will get more difficult as the book progresses. For a rigid body  $\mathcal{B}$  in more general motion, the calculation of rate of change of angular momentum involves the angular velocity  $\vec{\omega}_{\mathcal{B}}$ , its rate of change  $\dot{\vec{\omega}}_{\mathcal{B}}$ , and the moment of inertia matrix  $[I_{cm}^{\mathcal{B}}]$ . If you look in the back of the book at Table I, entries 6c and 6d, you will see formulas that reduce to the formulas below if you assume no rotation and thus use  $\vec{\omega} = \vec{0}$  and  $\dot{\vec{\omega}} = \vec{0}$ .

But rate of change of linear momentum is simple, at least in concept, in this chapter, as well as in the rest of this book, where

$$\dot{\vec{L}} = m_{tot} \vec{a}_{cm}$$

always applies.

<sup>②</sup> **Caution:** Unfortunately, the special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

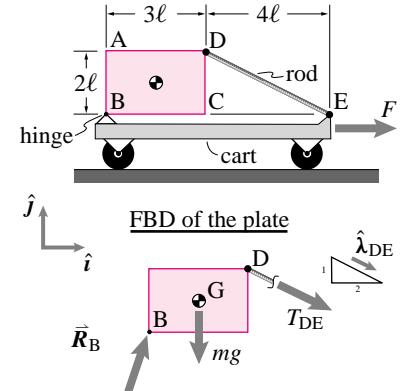


Figure 6.22: Uniform plate supported by a hinge and a rod on an accelerating cart.

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## 6.1 THEORY

### Calculation of $\vec{H}_C$ and $\dot{\vec{H}}_C$ for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of  $\vec{H}_C$  as follows:

$$\begin{aligned}\vec{H}_C &\equiv \sum \vec{r}_{i/C} \times m_i \vec{v}_i \text{ (definition)} \\ &= \sum \vec{r}_{i/C} \times m_i \vec{v}_{cm} \quad (\text{since, } \vec{v}_i = \vec{v}_{cm})\end{aligned}$$

$$\begin{aligned}&= \left( \sum \vec{r}_{i/C} m_i \right) \times \vec{v}_{cm}, \\ &= \vec{r}_{cm/C} \times (m_{\text{tot}} \vec{v}_{cm}), \\ &\quad (\text{since, } \sum \vec{r}_{i/C} m_i \equiv m_{\text{tot}} \vec{r}_{cm/C}).\end{aligned}$$

The derivation that  $\dot{\vec{H}}_C = \vec{r}_{cm/C} \times (m \vec{a}_{cm})$  follows from  $\dot{\vec{H}}_C \equiv \sum \vec{r}_{i/C} \times m_i \vec{a}_i$  by the same reasoning.

wheels. Referring to the free body diagram in figure 6.22 and writing angular momentum balance for the plate about point  $B$ , we can get an equation for the tension in the rod  $T_{DE}$  in terms of  $m$  and  $a_x$ :

$$\begin{aligned}\sum \vec{M}_{/B} &= \dot{\vec{H}}_{/B} \\ \left\{ \vec{r}_{D/B} \times (T_{DE} \hat{\lambda}_{DE}) + \vec{r}_{G/B} \times (-mg \hat{j}) \right. &= \left. \vec{r}_{G/B} \times (ma_x \hat{i}) \right\} \\ \{ \} \cdot \hat{k} \Rightarrow T_{DE} &= \frac{\sqrt{5}}{7} m(a_x - \frac{3}{2}g).\end{aligned}$$

□

Summarizing note:

angular momentum balance is important even when there is no rotation.

## Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car due to the suspension and tire deformation, can be neglected. The free body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system. Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

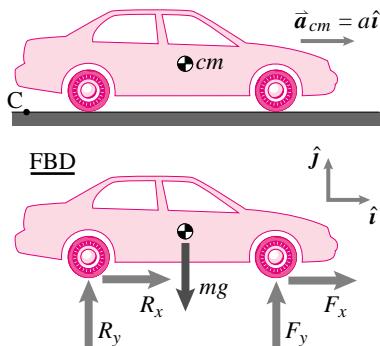


Figure 6.23: A four-wheel drive car accelerating but not tipping. See fig. 6.24 on page 347 for more about FBDs involving wheel contact.

(Filename:figure3.4wd.car)

### Example: Car skidding

Consider the accelerating four-wheel drive car in figure 6.23. The motion quantities for the car are  $\dot{\vec{L}} = m_{car} \vec{a}_{car}$  and  $\dot{\vec{H}}_C = \vec{r}_{cm/C} \times \vec{a}_{car} m_{car}$ . We could calculate angular momentum balance relative to the car's center of mass in which case  $\sum \vec{M}_{cm} = \dot{\vec{H}}_{cm} = \vec{0}$  (because the position of the center of mass relative to the center of mass is  $\vec{0}$ ). □

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

### Wheels

The function of wheels is to allow easy sliding-like (pseudo-sliding) motion between objects, at least in the direction they are pointed. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free body diagrams will be as in figure 3.20 on page 94 and *not* like the one in figure ?? on page ???. With the ideal wheel approximation, all of the various cases for a car traveling to the right are shown with partial free body diagrams of a wheel in figure 6.24. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see pages 90-92).

### 3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details in 3D, where vector methods must be used carefully, is more involved than in 2D. As for statics, three dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

#### *Angular momentum balance about an axis*

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\lambda} \cdot \left\{ \sum \vec{M}_{/C} = \vec{H}_{/C} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling an axle. The equation above applies for any point C and any vector  $\hat{\lambda}$ . If you choose C and  $\hat{\lambda}$  judiciously many terms in your equations may drop out.

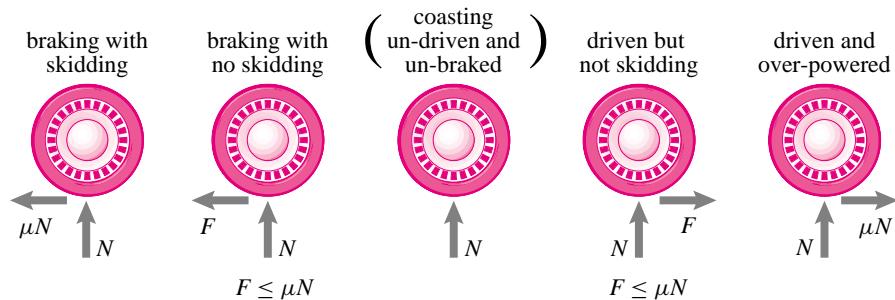


Figure 6.24: Partial free body diagrams of wheel in a braking or accelerating car that is pointed and moving to the right. The force of the ground on the tire is shown. But the forces of the axle, gravity, and brakes on the wheel are not shown. An ideal point-contact wheel is assumed.

(Filename: tfigure3.2.car.breaking)

**SAMPLE 6.6 Force in braking.** A front-wheel-drive car of mass  $m = 1200 \text{ kg}$  is cruising at  $v = 60 \text{ mph}$  on a straight road when the driver slams on the brake. The car slows down to  $20 \text{ mph}$  in  $4 \text{ s}$  while maintaining its straight path. What is the average force (average in time) applied on the car during braking?

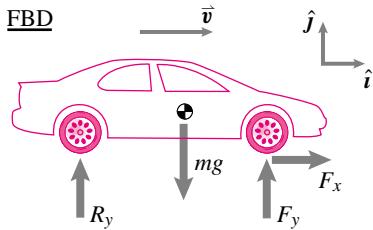


Figure 6.25: Free body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed  $F_x$  to the right. The algebra in this problem will tell us that  $F_x < 0$ .

(Filename:sfig2.6.3a)

**Solution** Let us assume that we have an  $xy$  coordinate system in which the car is traveling along the  $x$ -axis during the entire time under consideration. Then, the velocity of the car before braking,  $\vec{v}_1$ , and after braking,  $\vec{v}_2$ , are

$$\begin{aligned}\vec{v}_1 &= v_1 \hat{i} &= 60 \text{ mph } \hat{i} \\ \vec{v}_2 &= v_2 \hat{i} &= 20 \text{ mph } \hat{i}.\end{aligned}$$

The linear impulse during braking is  $\vec{F}_{ave} \Delta t$  where  $\vec{F} \equiv F_x \hat{i}$  (see free body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F} \Delta t = \vec{L}_2 - \vec{L}_1,$$

where  $\vec{L}_1$  and  $\vec{L}_2$  are linear momenta of the car before and after braking, respectively, and  $\vec{F}$  is the average applied force. Therefore,

$$\begin{aligned}\vec{F} &= \frac{1}{\Delta t} (\vec{L}_2 - \vec{L}_1) \\ &= \frac{m}{\Delta t} (\vec{v}_2 - \vec{v}_1) \\ &= \frac{1200 \text{ kg}}{4 \text{ s}} (20 - 60) \text{ mph} \hat{i} \\ &= -12000 \frac{\text{kg}}{\text{s}} \cdot \frac{\text{mi}}{\text{hr}} \cdot \frac{1600 \text{ m}}{1 \text{ mi}} \cdot \frac{1 \text{ hr}}{3600 \text{ s}} \hat{i} \\ &= -\frac{16,000}{3} \text{ kg} \cdot \text{m/s}^2 \hat{i} \\ &= -5.33 \text{ kN} \hat{i}.\end{aligned}$$

Thus

$$\begin{aligned}F_x \hat{i} &= -5.33 \text{ kN} \hat{i} \\ \Rightarrow F_x &= -5.33 \text{ kN}.\end{aligned}$$

$F_x = -5.33 \text{ kN}$

**SAMPLE 6.7 Sliding to a stop.** A block of mass  $m = 2.5 \text{ kg}$  slides down a frictionless incline from a  $5 \text{ m}$  height. The block encounters a frictional bed AB of length  $1 \text{ m}$  on the ground. If the speed of the block is  $9 \text{ m/s}$  at point B, find the coefficient of friction between the block and the frictional surface AB.

**Solution** We divide the problem in two parts: We first find the speed of the block as it reaches point A using conservation of energy for its motion on the inclined surface, and then use the work-energy principle to find the speed at B. Let the ground level be the datum for potential energy and let  $v$  be the speed at A. For the motion on the incline;

$$\begin{aligned} (E_K)_1 + (E_P)_1 &= (E_K)_2 + (E_P)_2 \\ 0 + mgh &= \frac{1}{2}mv^2 + 0 \\ \Rightarrow v &= \sqrt{2gh} \\ &= \sqrt{2 \cdot 9.81 \text{ m/s}^2 \cdot 5 \text{ m}} \\ &= 9.90 \text{ m/s}. \end{aligned}$$

Now, as the block slides on the surface AB, a force of friction  $= \mu N = \mu mg$  (since  $N = mg$ , from linear momentum balance in the vertical direction) opposes the motion (see Fig. 8.57). Work done by this force on the block is

$$\begin{aligned} W &= \vec{F} \cdot \Delta \vec{r} \\ &= -\mu mg \hat{i} \cdot (1 \text{ m}) \hat{i} \\ &= -\mu mg (1 \text{ m}). \end{aligned}$$

From the work-energy relationship (e.g., see the inside cover) we have,

$$\begin{aligned} W &= \Delta E_K = (E_K)_2 - (E_K)_1 \\ \Rightarrow (E_K)_2 &= (E_K)_1 + W \\ \frac{1}{2}mv_B^2 &= \frac{1}{2}mv^2 - \mu mg(1 \text{ m}) \\ -\mu mg(1 \text{ m}) &= \frac{1}{2}m(v_B^2 - v^2) \\ \Rightarrow \mu &= \frac{1}{2g(1 \text{ m})}(v_B^2 - v^2) \\ &= \frac{(9.90 \text{ m/s})^2 - (9 \text{ m/s})^2}{2 \cdot 9.81 \text{ m/s}^2 \cdot 1 \text{ m}} \\ &= 0.87 \end{aligned}$$

$$\boxed{\mu = 0.87}$$

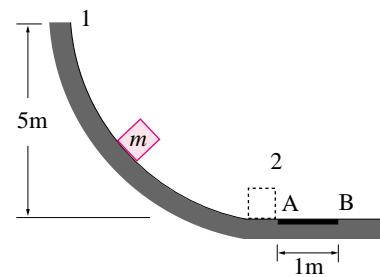


Figure 6.26: (Filename:sfig2.9.1a)

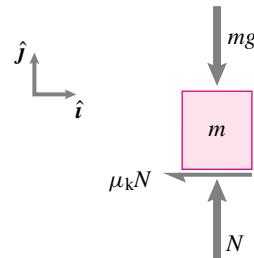


Figure 6.27: Free body diagram of the block on the frictional surface.

(Filename:sfig2.9.1b)

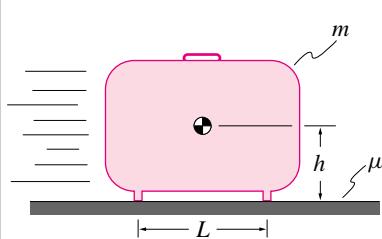


Figure 6.28: A suitcase in motion.

(Filename:fig3.5.1)

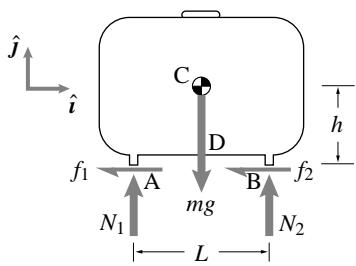


Figure 6.29: FBD of the suitcase.

(Filename:fig3.5.1a)

**SAMPLE 6.8** *A suitcase skidding on frictional ground.* A suitcase of mass  $m$  is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is  $\mu$ , find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

**Solution** As usual, the first thing we do is draw a free body diagram of the suitcase. The FBD is shown in Fig. 6.29. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{i} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{i}. \quad (6.25)$$

Now we write the balance of linear momentum for the suitcase:

$$\begin{aligned} \sum \vec{F} &= m \vec{a} \\ \Rightarrow -(F_1 + F_2) \hat{i} + (N_1 + N_2 - mg) \hat{j} &= ma \hat{i} \end{aligned} \quad (6.26)$$

where  $\vec{a} = a \hat{i}$  is the unknown acceleration. Dotting eqn. (6.26) with  $\hat{i}$  and  $\hat{j}$  and substituting for  $F_1$  and  $F_2$  from eqn. (6.25) we get

$$-\mu(N_1 + N_2) = ma \quad (6.27)$$

$$N_1 + N_2 = mg \quad (6.28)$$

Equations (6.27) and (6.28) represent 2 scalar equations in three unknowns  $N_1$ ,  $N_2$  and  $a$ . Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_A = \dot{\vec{H}}_A$$

$$\begin{aligned} \sum \vec{M}_A &= \vec{r}_{B/A} \times N_2 \hat{j} + \vec{r}_{D/A} \times (-mg) \hat{j} \\ &= L \hat{i} \times N_2 \hat{j} + \frac{L}{2} \hat{i} \times (-mg) \hat{j} \\ &= (LN_2 - mg \frac{L}{2}) \hat{k} \end{aligned} \quad (6.29)$$

and

$$\dot{\vec{H}}_A = \vec{r}_{cm/A} \times m \vec{a} \quad (6.30)$$

$$\begin{aligned} &= (\frac{L}{2} \hat{i} + h \hat{j}) \times ma \hat{i} \\ &= -mah \hat{k} \end{aligned} \quad (6.31)$$

Equating (6.29) and (6.31) and dotting both sides with  $\hat{k}$  we get the following third scalar equation:

$$LN_2 - mg \frac{L}{2} = -mah. \quad (6.32)$$

Solving eqns. (6.27) and (6.28) for  $a$  we get

$$a = -\mu g$$

and substituting this value of  $a$  in eqn. (6.32) we get

$$\begin{aligned} N_2 &= \frac{m\mu gh + mgL/2}{L} \\ &= mg \left( \frac{1}{2} + \frac{h}{L} \mu \right). \end{aligned}$$

Substituting the value of  $N_2$  in either of the equations (6.27) or (6.28) we get

$$N_1 = mg \left( \frac{1}{2} - \frac{h}{L} \mu \right).$$

$$N_1 = mg \left( \frac{1}{2} - \frac{h}{L} \mu \right), \quad N_2 = mg \left( \frac{1}{2} + \frac{h}{L} \mu \right), \quad f_1 = \mu N_1, \quad f_2 = \mu N_2.$$

**Discussion:** From the expressions for  $N_1$  and  $N_2$  we see that

- (a)  $N_1 = N_2 = \frac{1}{2}mg$  if  $\mu = 0$  because without friction there is no deceleration.  
The problem becomes equivalent to a statics problem.
- (b)  $N_1 = N_2 \approx \frac{1}{2}mg$  if  $L \gg h$ . In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take  $L = 20h$  and calculate moment about the center of mass to convince yourself.

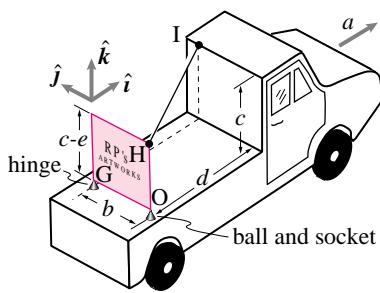


Figure 6.30: An accelerating board in 3-D  
(Filename:sfig3.5.2)

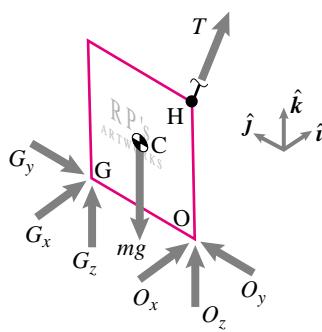


Figure 6.31: FBD of the board  
(Filename:sfig3.5.2a)

**SAMPLE 6.9 Uniform acceleration of a board in 3-D.** A uniform sign-board of mass  $m = 20 \text{ kg}$  sits in the back of an accelerating flatbed truck. The board is supported with ball-and-socket joint at  $O$  and a hinge at  $G$ . A light rod from  $H$  to  $I$  keeps the board from falling over. The truck is on level ground and has forward acceleration  $\vec{a} = 0.6 \text{ m/s}^2 \hat{i}$ . The relevant dimensions are  $b = 1.5 \text{ m}$ ,  $c = 1.5 \text{ m}$ ,  $d = 3 \text{ m}$ ,  $e = 0.5 \text{ m}$ . There is gravity ( $g = 10 \text{ m/s}^2$ ).

- Draw a free body diagram of the board.
- Set up equations to solve for all the unknown forces shown on the FBD.
- Use the balance of angular momentum about an axis to find the tension in the rod.

### Solution

- The free body diagram of the board is shown in Fig. 6.31.
- Linear momentum balance for the board:

$$\sum \vec{F} = m\vec{a}, \quad \text{or}$$

$$(G_x + O_x)\hat{i} + (G_y + O_y)\hat{j} + (G_z + O_z - mg)\hat{k} + T\hat{\lambda}_{HI} = ma\hat{i} \quad (6.33)$$

where

$$\hat{\lambda}_{HI} = \frac{d\hat{i} + b\hat{j} + e\hat{k}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{i} + b\hat{j} + e\hat{k}}{\ell},$$

and where  $\ell$  is the length of the rod  $HI$ .

Dotting eqn. (6.33) with  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \quad (6.34)$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \quad (6.35)$$

$$G_z + O_z + T\frac{e}{\ell} = mg \quad (6.36)$$

Angular momentum balance about point  $G$ :

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$$\begin{aligned} \sum \vec{M}_{/G} &= \vec{r}_{C/G} \times (-mg\hat{k}) + \vec{r}_{O/G} \times (O_x\hat{i} + O_z\hat{k}) + \vec{r}_{H/G} \times T\hat{\lambda}_{HI} \\ &= \left(-\frac{b}{2}\hat{j} + \frac{c-e}{2}\hat{k}\right) \times (-mg\hat{k}) - b\hat{j} \times (O_x\hat{i} + O_z\hat{k}) \\ &\quad + [-b\hat{j} + (c-e)\hat{k}] \times \frac{T}{\ell}(d\hat{i} + b\hat{j} + e\hat{k}) \\ &= \left(\frac{b}{2}mg - bO_y - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{i} \\ &\quad + (c-e)d\frac{T}{\ell}\hat{j} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{k} \end{aligned} \quad (6.37)$$

and

$$\begin{aligned} \dot{\vec{H}}_{/G} &= \vec{r}_{C/G} \times ma\hat{i} \\ &= \left(-\frac{b}{2}\hat{j} + \frac{c-e}{2}\hat{k}\right) \times ma\hat{i} \\ &= \frac{b}{2}mak + \frac{c-e}{2}maj\hat{j}. \end{aligned} \quad (6.38)$$

Equating (6.37) and (6.38) and dotting both sides with  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$  and  $\hat{\mathbf{k}}$  we get the following three additional scalar equations:

$$G_y + O_y + \frac{c}{\ell}T = \frac{1}{2}mg \quad (6.39)$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \quad (6.40)$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \quad (6.41)$$

Now we have seven scalar equations for six unknowns —  $O_x$ ,  $G_y + O_y$ ,  $O_z$ ,  $G_x$ ,  $G_z$ , and  $T$ . Note, however, that  $G_y$  and  $O_y$  appear as the sum  $G_y + O_y$ . That is, they cannot be found independently. This mathematical problem corresponds to the physical circumstance that the supports at points  $O$  and  $G$  could be squeezing the plate along the line  $OG$  with, say,  $O_y = 1000 \text{ lbf}$  and  $G_y = -1000 \text{ lbf}$ . To make problems like this tractable, people often make assumptions like, ‘Assume  $G_y = 0$ ’. Fortunately, no one asked us to find  $O_y$  or  $G_y$  and we can find the tension in the wire  $HI$  without adding assumptions about the pre-stress in the structure.

- (c) Balance of angular momentum about axis OG gives:

$$\begin{aligned} \hat{\lambda}_{OG} \cdot \sum \vec{M}_{/G} &= \hat{\lambda}_{OG} \cdot \hat{\vec{H}}_{/G} \\ &= \hat{\lambda}_{OG} \cdot (\vec{r}_{C/G} \times ma\hat{\mathbf{i}}). \end{aligned} \quad (6.42)$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned} \hat{\lambda}_{OG} \cdot \sum \vec{M}_{/G} &= \hat{\mathbf{j}} \cdot (\vec{r}_{H/G} \times T\hat{\lambda}_{HI}) \\ &= T \frac{d(c-e)}{\ell}. \end{aligned} \quad (6.43)$$

$$\begin{aligned} \hat{\lambda}_{OG} \cdot (\vec{r}_{C/G} \times ma\hat{\mathbf{i}}) &= \hat{\mathbf{j}} \cdot \left[ \left( \frac{b}{2}\hat{\mathbf{j}} + \frac{c-e}{2}\hat{\mathbf{k}} \right) \times ma\hat{\mathbf{i}} \right] \\ &= ma \frac{(c-e)}{2}. \end{aligned} \quad (6.44)$$

Equating (6.43) and (6.44), we get

$$\begin{aligned} T &= \frac{ma\ell}{2d} \\ &= \frac{20 \text{ kg} \cdot 0.5 \text{ m/s}^2 \cdot 3.39 \text{ m}}{2 \cdot 3 \text{ m}} \\ &= 6.78 \text{ N}. \end{aligned}$$

$T_{HI} = 6.78 \text{ N}$

**SAMPLE 6.10 Computer solution of algebraic equations.** In the previous sample problem (Sample 6.9), six equations were obtained to solve for the six unknown forces (assuming  $G_y = 0$ ). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

### Solution

- (a) The six scalar equations — (6.34), (6.35), (6.36), (6.39), (6.40), and (6.41) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \quad (6.45)$$

The above equation can be written, in matrix notation, as

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

where  $\mathbf{A}$  is the coefficient matrix,  $\mathbf{x}$  is the vector of the unknown forces, and  $\mathbf{b}$  is the vector on the right hand side of the equation. Now we are ready to solve the system of equations on the computer.

- (b) We use the following pseudo-code to solve the above matrix equation. ①

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),
```

```
A = [1 0 0 1 0 d/l
      0 1 0 0 0 b/l
      0 0 1 0 1 e/l
      0 1 0 0 0 c/l
      0 0 0 0 0 d/l
      1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'
```

{Solve  $\mathbf{A} \mathbf{x} = \mathbf{b}$  for  $\mathbf{x}$ }

```
x = % this is the computer output
    0
   -3.0000
   97.0000
    6.0000
  102.0000
   6.7823
```

The solution obtained from the computer means:

$$O_x = 0, O_y = -3 \text{ N}, O_z = 97 \text{ N}, G_x = 6 \text{ N}, G_z = 102 \text{ N}, T = 6.78 \text{ N}.$$

We now hand-check the solution by substituting the values obtained in, say, Eqns. (6.35) and (6.40). Before we substitute the values of forces, we need to calculate the length  $\ell$ .

$$\begin{aligned}\ell &= \sqrt{d^2 + b^2 + e^2} \\ &= 3.3912 \text{ m.}\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Eqn. (6.35): } O_y + T \frac{b}{\ell} &= -3 \text{ N} + 6.78 \text{ N} \cdot \frac{1.5 \text{ m}}{3.3912 \text{ m}} \\ &\stackrel{\checkmark}{=} 0,\end{aligned}$$

$$\begin{aligned}\text{Eqn. (6.40): } \frac{d}{\ell} T - \frac{1}{2} ma &= \frac{3 \text{ m}}{3.3912 \text{ m}} 6.78 \text{ N} - \frac{1}{2} 20 \text{ kg} 0.6 \text{ m/s}^2 \\ &\stackrel{\checkmark}{=} 0.\end{aligned}$$

Thus, the computer solution agrees with our equations.

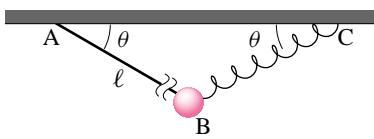


Figure 6.32: A small ball of mass  $m$  is supported by a string and a zero length (in relaxed position) spring. The string is suddenly cut.

(Filename:sfig6.4.2)

**SAMPLE 6.11** *Zero length springs do interesting things.* A small ball of mass  $m$  is supported by a string AB of length  $\ell$  and a spring BC with spring stiffness  $k$ . The spring is relaxed when the mass is at C (BC is a zero length spring). The spring and the string make the same angle  $\theta$  with the horizontal in the static equilibrium of the mass. At this position, the string is suddenly cut near the mass point B. Find the resulting motion of the mass.

**Solution** The Free Body Diagrams of the mass are shown in Fig. 6.33(a) and (b) before and after the string is cut. Since the stretch in the spring, in the static equilibrium position of the mass, is equal to the length of the string,  $\vec{F}_b = k\ell\hat{\lambda}_{BC}$ . linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the mass in the static position gives

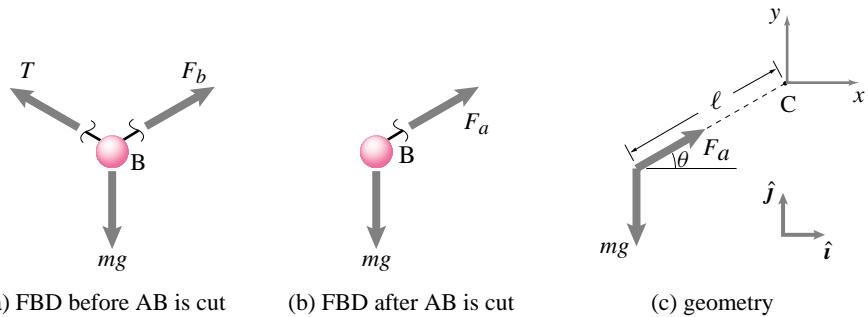


Figure 6.33: Free Body Diagram of the ball (a) before the string is cut, (b) sometime after the string is cut, and (c) the geometry at the moment of interest.

(Filename:sfig6.4.2a)

$$(F_b \cos \theta_o - T \cos \theta_o)\hat{i} + (F_b \sin \theta_o + T \sin \theta_o - mg)\hat{j} = \vec{0}.$$

The  $x$  and  $y$  components of this equation give

$$\begin{aligned} F_b &= T, \\ (F_b + T) \sin \theta_o &= mg. \end{aligned}$$

Substituting  $F_b = T$  in the second equation and replacing  $F_b$  by  $k\ell$  we get

$$\begin{aligned} 2k\ell \sin \theta_o &= mg \\ \text{or} \quad \theta_o &= \sin^{-1} \frac{mg}{2k\ell}. \end{aligned} \quad (6.46)$$

After the string is cut, let the mass be at some general angular position  $\theta$ . Let the stretch in the spring at this position be  $\ell$ . Then, the Linear Momentum Balance for the mass may be written as

$$\sum \vec{F} = m\vec{a}$$

where (refer to Fig. 6.33(c))

$$\begin{aligned} \sum \vec{F} &= F_a \cos \theta \hat{i} + (F_a \sin \theta - mg) \hat{j} \\ &= k\ell \cos \theta \hat{i} + (k\ell \sin \theta - mg) \hat{j} \end{aligned}$$

But,  $\ell \cos \theta = -x$  and  $\ell \sin \theta = -y$ . Therefore,

$$\begin{aligned} \sum \vec{F} &= -kx \hat{i} + (-ky - mg) \hat{j}, \\ \vec{a} &= \ddot{x} \hat{i} + \ddot{y} \hat{j}. \end{aligned}$$

Now, substituting the expressions for  $\sum \vec{F}$  and  $\vec{a}$  in the Linear Momentum Balance equation and dotting both sides with  $\hat{i}$  and  $\hat{j}$  we get

$$(\sum \vec{F} = m\vec{a}) \cdot \hat{i} \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \quad (6.47)$$

$$(\sum \vec{F} = m\vec{a}) \cdot \hat{j} \Rightarrow \ddot{y} + \frac{k}{m}y = -g. \quad (6.48)$$

Unbelievable!! Two such nasty looking nonlinear, coupled equations (??) and (??) in polar coordinates become so simple, friendly looking linear, uncoupled equations (6.47) and (6.48) in cartesian coordinates. We can now write the solutions of these second order ODE's:

$$\begin{aligned} x(t) &= A \sin(\lambda t) + B \cos(\lambda t), \\ y(t) &= C \sin(\lambda t) + D \cos(\lambda t) - \frac{mg}{k}, \end{aligned}$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are constants and  $\lambda = \sqrt{k/m}$ . We need initial conditions to evaluate the constants  $A$ ,  $B$ ,  $C$  and  $D$ . Since the mass starts at  $t = 0$  from the rest position when  $\theta = \theta_o$ ,

$$\begin{aligned} x(0) &= -\ell \cos \theta_o \quad \text{and} \quad \dot{x}(0) = 0, \\ y(0) &= -\ell \sin \theta_o \quad \text{and} \quad \dot{y}(0) = 0. \end{aligned}$$

Substituting these initial conditions in the solutions above, we get

$$\begin{aligned} x(t) &= -(\ell \cos \theta_o) \cos(\sqrt{(k/m)}t), \\ y(t) &= -(\ell \sin \theta_o - \frac{mg}{k}) \cos(\sqrt{(k/m)}t) - \frac{mg}{k}. \end{aligned}$$

From these equations, we can relate  $x$  and  $y$  by eliminating the cosine term, *i.e.*,

$$\begin{aligned} \frac{x(t)}{\ell \cos \theta_o} &= \frac{y(t) + (mg/k)}{\ell \sin \theta_o - (mg/k)} \\ \text{or} \quad y(t) &= \frac{\ell \sin \theta_o - (mg/k)}{\ell \cos \theta_o} x(t) - \frac{mg}{k} \end{aligned}$$

which is the equation of a straight line passing through the vertical equilibrium position  $y = -mg/k$ . Thus the mass moves along a straight line! ①

① By choosing appropriate initial conditions, you can show that there are other straight line motions (for example, just horizontal or vertical motions) and motions on elliptic paths.



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# 7 Circular motion

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When a rigid object moves, it translates and rotates. In general, the points on the body move on complicated paths. When considering the unconstrained motions of particles in chapter 5, such as the motion of a thrown ball, we observed such curved paths. Things which are constrained to translate in straight lines were covered in the previous chapter. Now we would like to consider motion that may be constrained to a curved path. More specifically, this chapter concerns mechanics when particles move on the archetypal curved path, a circle. Circular motion deserves special attention because

- the most common connection between moving parts on a machine is with a bearing (or hinge or axle) (Fig. 7.1), if the axle on one part is fixed then all points on the part move in circles;
- circular motion is the simplest case of curved-path motion;
- circular motion provides a simple way to introduce time-varying base vectors;
- in some sense, that you will appreciate with hind-sight, circular motion includes all of the conceptual ingredients of more general curved motions;
- at least in 2 dimensions, the only way two particles on one rigid body can move relative to each other is by circular motion (no matter how the body is moving); and
- circular motion is the simplest case with which to introduce two important rigid body concepts:
  - angular velocity, and
  - moment of inertia.

Because of some mixture of simplicity and natural applicability, useful calculations can be made for many things by approximating their motion as one for which

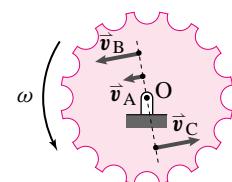


Figure 7.1: All the points on a gear move in circles, assuming the axle is stationary.  
(Filename:tfigure4.1a)

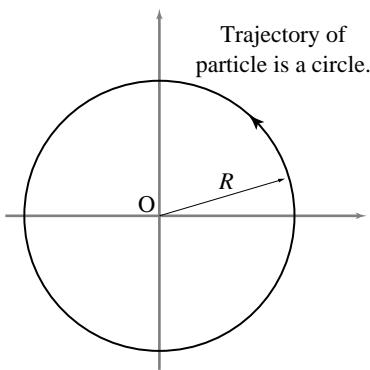


Figure 7.2: Trajectory of particle for circular motion.

(Filename:tfigure1.i)

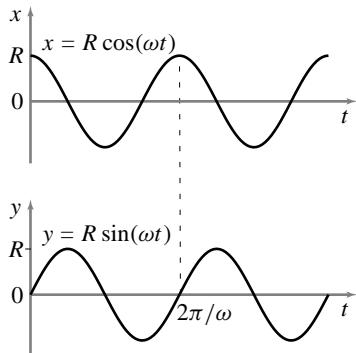


Figure 7.3: Plots of  $x$  versus  $t$  and  $y$  versus  $t$  for a particle going in a circle of radius  $R$  at constant rate. Both  $x$  and  $y$  vary as sinusoidal functions of time:  $x = R \cos(\omega t)$  and  $y = R \sin(\omega t)$ .

(Filename:tfigure1.h)

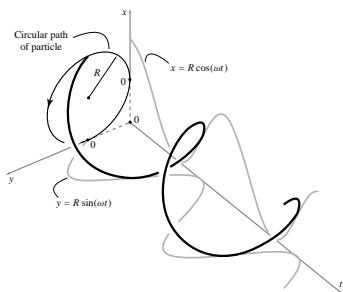


Figure 7.4: Plot of  $x$  and  $y$  versus time for a particle going in circles.  $x$  versus  $t$  is the cosine curve,  $y$  versus  $t$  is the sine curve. Together they make up the helical curve in three-dimensional space.

(Filename:tfigure1.j)

all particles are going in circles. For example, a jet engine's turbine blade, a car engine's crank shaft, a car's wheel, a windmill's propeller, the earth spinning about its axis, points on a clock pendulum, a bicycle's approximately circular path when going around a corner, and a spinning satellite might all be reasonably approximated by the assumption of circular motion.

This chapter concerns only motion in two dimensions. The next chapter discusses circular motion in a three-dimensional context. The first two sections consider the kinematics and mechanics of a single particle going in circles. The later sections concern the kinematics and mechanics of rigid bodies.

For the systems in this chapter, we have, as always,

$$\text{linear momentum balance, } \sum \vec{F}_i = \dot{\vec{L}}$$

$$\text{angular momentum balance, } \sum \vec{M}_{i/C} = \dot{\vec{H}}_C,$$

$$\text{and power balance: } P = \dot{E}_K + \dot{E}_P + \dot{E}_{\text{int}}.$$

The left hand sides of the momentum equations are found using the forces and couples shown on the free body diagram of the system of interest; the right sides are evaluated in terms of motion of the system. Because you already know how to work with forces and moments, the primary new skill in this chapter is to learn how to evaluate  $\dot{\vec{L}}$ ,  $\dot{\vec{H}}_C$ , and  $\dot{E}_K$  for a rotating particle or rigid body. Towards this end, but also useful for other purposes, you need to know the velocity and accelerations of points on a rigid body in circular motion.

## 7.1 Kinematics of a particle in planar circular motion

Consider a particle on the  $xy$  plane going in circles around the origin at a constant rate. One way of representing this situation is with the equation:

$$\vec{r} = R \cos(\omega t) \hat{i} + R \sin(\omega t) \hat{j},$$

with  $R$  and  $\omega$  constants. Another way is with the pair of equations:

$$x = R \cos(\omega t) \quad \text{and} \quad y = R \sin(\omega t).$$

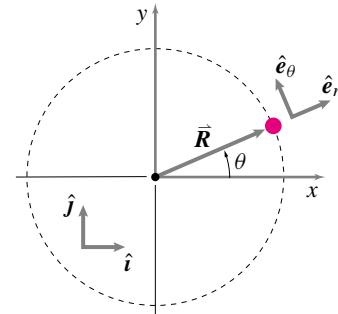
How do we represent this motion graphically? One way is to plot the particle trajectory, that is, the path of the particle. Figure 7.2 shows a circle of radius  $R$  drawn on the  $xy$  plane. Note that this plot doesn't show the speed the particle moves in circles. That is, a particle moving in circles slowly and another moving quickly would both have the same plotted trajectory.

Another approach is to plot the functions  $x(t)$  and  $y(t)$  as in Fig. 7.3. This figure shows how  $x$  and  $y$  vary in time but does not directly convey that the particle is going in circles. How do you make these plots? Using a calculator or computer you can evaluate  $x$  and  $y$  for a range of values of  $t$ . Then, using pencil and paper, a plotting calculator, or a computer, plot  $x$  vs  $t$ ,  $y$  vs  $t$ , and  $y$  vs  $x$ .

If one wishes to see both the trajectory and the time history of both variables one can make a 3-D plot of  $xy$  position versus time (Fig. 7.4). The shadows of this curve (a helix) on the three coordinate planes are the three graphs just discussed. How you make such a graph with a computer depends on the software you use.

Finally, rather than representing time as a spatial coordinate, one can use time directly by making an animated movie on a computer screen showing a particle on

the  $xy$  plane as it moves. Move your finger around in circles on the table. That's it. These days, the solutions of complex dynamics problems are often presented with computer animations.



## The velocity and acceleration of a point going in circles: polar coordinates

Let's redraw Fig. 7.3 but introduce unit base vectors  $\hat{e}_R$  and  $\hat{e}_\theta$  in the direction of the position vector  $\vec{R}$  and perpendicular to  $\vec{R}$ . At any instant in time, the radial unit vector  $\hat{e}_R$  is directed from the center of the circle towards the point of interest and the transverse vector  $\hat{e}_\theta$ , perpendicular to  $\hat{e}_R$ , is tangent to the circle at that point. As the particle goes around, its  $\hat{e}_R$  and  $\hat{e}_\theta$  unit vectors change. Note also, that two different particles both going in circles with the same center at the same rate each have their own  $\hat{e}_R$  and  $\hat{e}_\theta$  vectors. We will make frequent use of the polar coordinate unit vectors  $\hat{e}_R$  and  $\hat{e}_\theta$ .

Here is one of many possible ways to derive the polar-coordinate expressions for velocity and acceleration. First, observe that the position of the particle is (see figure 7.5)

$$\vec{R} = R\hat{e}_R. \quad (7.1)$$

That is, the position vector is the distance from the origin times a unit vector in the direction of the particle's position. Given the position, it is just a matter of *careful* differentiation to find velocity and acceleration. First, velocity is the time derivative of position, so

$$\vec{v} = \frac{d}{dt}\vec{R} = \frac{d}{dt}(R\hat{e}_R) = \underbrace{\dot{R}}_0 \hat{e}_R + R\dot{\hat{e}}_R.$$

Because a circle has constant radius  $R$ ,  $\dot{R}$  is zero. But what is  $\dot{\hat{e}}_R$ , the rate of change of  $\hat{e}_R$  with respect to time?

One way to find  $\dot{\hat{e}}_R$  uses the geometry of figure 7.6 and the informal calculus of finite differences (represented by  $\Delta$ ).  $\Delta\hat{e}_R$  is evidently (about) in the direction  $\hat{e}_\theta$  and has magnitude  $\Delta\theta$  so  $\Delta\hat{e}_R \approx (\Delta\theta)\hat{e}_\theta$ . Dividing by  $\Delta t$ , we have  $\Delta\hat{e}_R/\Delta t \approx (\Delta\theta/\Delta t)\hat{e}_\theta$ . So, using this sloppy calculus, we get  $\dot{\hat{e}}_R = \dot{\theta}\hat{e}_\theta$ . Similarly, we could get  $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R$ .

Alternatively, we can be a little less geometric and a little more algebraic, and use the decomposition of  $\hat{e}_R$  and  $\hat{e}_\theta$  into cartesian coordinates. These decompositions are found by looking at the projections of  $\hat{e}_R$  and  $\hat{e}_\theta$  in the  $x$  and  $y$ -directions (see figure 7.7).

$$\begin{aligned} \hat{e}_R &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ \hat{e}_\theta &= -\sin\theta\hat{i} + \cos\theta\hat{j} \end{aligned} \quad (7.2)$$

Figure 7.5: A particle going in circles. The position vector of the particle relative to the center of the circle is  $\vec{R}$ . It makes an angle  $\theta$  measured counter-clockwise from the positive  $x$ -axis. The unit vectors  $\hat{e}_R$  and  $\hat{e}_\theta$  are shown in the radial and tangential directions, the directions of increasing  $R$  and increasing  $\theta$ .

(Filename:tfigure4.4)

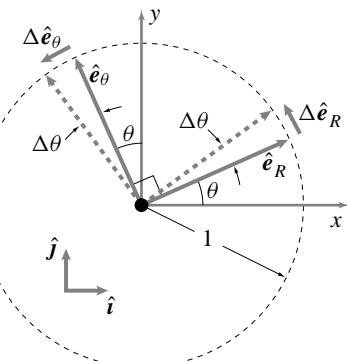


Figure 7.6: A close up view of the unit vectors  $\hat{e}_R$  and  $\hat{e}_\theta$ . They make an angle  $\theta$  with the positive  $x$  and  $y$ -axis, respectively. As the particle advances an amount  $\Delta\theta$  both  $\hat{e}_R$  and  $\hat{e}_\theta$  change. In particular, for small  $\Delta\theta$ ,  $\Delta\hat{e}_R$  is approximately in the  $\hat{e}_\theta$  direction and  $\Delta\hat{e}_\theta$  is approximately in the  $-\hat{e}_R$  direction.

(Filename:tfigure4.5)

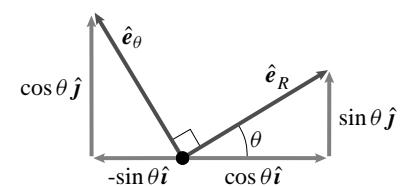


Figure 7.7: Projections of  $\hat{e}_R$  and  $\hat{e}_\theta$  in the  $x$  and  $y$  directions

(Filename:tfigure4.5a)

So to find  $\dot{\hat{e}}_R$  we just differentiate, taking into account that  $\theta$  is changing with time but that the unit vectors  $\hat{i}$  and  $\hat{j}$  are fixed (so they don't change with time).

$$\begin{aligned}\dot{\hat{e}}_R &= \frac{d}{dt}(\cos \theta \hat{i} + \sin \theta \hat{j}) = -\dot{\theta} \sin \theta \hat{i} + \dot{\theta} \cos \theta \hat{j} = \dot{\theta} \hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \frac{d}{dt}(-\sin \theta \hat{i} + \cos \theta \hat{j}) = -\dot{\theta} \hat{e}_R\end{aligned}$$

We had to use the chain rule, that is

$$\frac{d \sin \theta(t)}{dt} = \frac{d \sin \theta}{d\theta} \frac{d\theta(t)}{dt} = \dot{\theta} \cos \theta.$$

Now, two different ways, we know

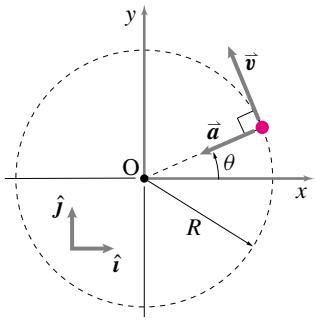


Figure 7.8: The directions of velocity  $\vec{v}$  and acceleration  $\vec{a}$  are shown for a particle going in circles at *constant* rate. The velocity is tangent to the circle and the acceleration is directed towards the center of the circle.

(Filename:tfigure4.6)

$$\dot{\hat{e}}_R = \dot{\theta} \hat{e}_\theta \quad \text{and} \quad \dot{\hat{e}}_\theta = -\dot{\theta} \hat{e}_R \quad (7.3)$$

so we can find  $\vec{v}$ ,

$$\vec{v} = \dot{\vec{R}} = R \dot{\hat{e}}_R = R \dot{\theta} \hat{e}_\theta. \quad (7.4)$$

Similarly we can find  $\ddot{\vec{R}}$  by differentiating once again,

$$\vec{a} = \ddot{\vec{R}} = \dot{\vec{v}} = \frac{d}{dt}(R \dot{\hat{e}}_\theta) = \underbrace{\dot{R} \dot{\hat{e}}_\theta}_{\vec{0}} + R \ddot{\hat{e}}_\theta + R \dot{\theta} \dot{\hat{e}}_\theta \quad (7.5)$$

The first term on the right hand side is zero because  $\dot{R}$  is 0 for circular motion. The third term is evaluated using the formula we just found for the rate of change of  $\hat{e}_\theta$ :  $\dot{\hat{e}}_\theta = -\dot{\theta} \hat{e}_R$ . So, using that  $\vec{R} = R \hat{e}_R$ ,

$$\vec{a} = -\dot{\theta}^2 \vec{R} + R \ddot{\theta} \hat{e}_\theta. \quad (7.6)$$

The velocity  $\vec{v}$  and acceleration  $\vec{a}$  are shown for a particle going in circles at constant rate in figure 7.8.

#### *Example: A person standing on the earth's equator*

A person standing on the equator has velocity

$$\begin{aligned}\vec{v} &= \dot{\theta} R \hat{e}_\theta \approx \left( \frac{2\pi \text{ rad}}{24 \text{ hr}} \right) 4000 \text{ mi} \hat{e}_\theta \\ &\approx 1050 \text{ mph} \hat{e}_\theta \approx 1535 \text{ ft/s} \hat{e}_\theta\end{aligned}$$

and acceleration

$$\begin{aligned}\vec{a} &= -\dot{\theta}^2 R \hat{e}_R \approx -\left( \frac{2\pi \text{ rad}}{24 \text{ hr}} \right)^2 4000 \text{ mi} \hat{e}_R \\ &\approx -274 \text{ mi/hr}^2 \hat{e}_R \approx -0.11 \text{ ft/s}^2 \hat{e}_R.\end{aligned}$$

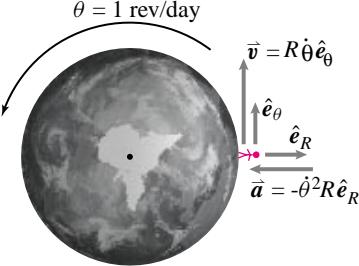


Figure 7.9: (Filename:tfigure4.1.example1)

The velocity of a person standing on the equator, due to the earth's rotation, is about 1000 mph tangent to the earth. Her acceleration is about  $0.11 \text{ ft/s}^2$  towards the center of the earth, about  $1/300$  of  $g$ , the inwards acceleration of a person in frictionless free-fall.  $\square$

## Another derivation of the velocity and acceleration formulas

We now repeat the derivation for velocity and acceleration, but more concisely. The position of the particle is  $\vec{R} = R\hat{e}_R$ . Recall that the rates of change of the polar base vectors are  $\dot{\hat{e}}_R = \dot{\theta}\hat{e}_\theta$  and  $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R$ . We find the velocity by differentiating the position with respect to time, keeping  $R$  constant.

$$\begin{aligned}\vec{v} &= \frac{d}{dt}\vec{R} = \frac{d}{dt}(R\hat{e}_R) = \underbrace{\dot{R}}_0 \hat{e}_R + R\dot{\hat{e}}_R \\ &= \dot{\vec{R}} = R\dot{\hat{e}}_R = R\dot{\theta}\hat{e}_\theta\end{aligned}$$

We find the acceleration  $\vec{a}$  by differentiating again,

$$\begin{aligned}\vec{a} &= \ddot{\vec{R}} = \ddot{\vec{v}} = \frac{d}{dt}(R\dot{\theta}\hat{e}_\theta) \\ &= (\dot{R}\dot{\theta}\hat{e}_\theta) + (R\ddot{\theta}\hat{e}_\theta) + (R\dot{\theta}\underbrace{\dot{\hat{e}}_\theta}_{-\dot{\theta}\hat{e}_R}) \\ &= -(\dot{\theta})^2 R\hat{e}_R + \ddot{\theta}R\hat{e}_\theta = -(v^2/R)\hat{e}_R + \dot{v}\hat{e}_\theta.\end{aligned}$$

Thus, the formulas for velocity and acceleration of a point undergoing variable rate circular motion in 2-D are:

$$\begin{aligned}\vec{v} &= R\dot{\theta}\hat{e}_\theta \\ \vec{a} &= -\frac{v^2}{R}\hat{e}_R + \dot{v}\hat{e}_\theta,\end{aligned}$$

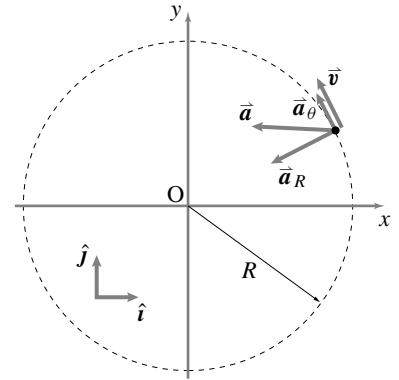


Figure 7.10: The directions of velocity  $\vec{v}$  and acceleration  $\vec{a}$  are shown for a particle going in circles at variable rate. The velocity is tangent to the circle and the acceleration is the sum of two components: one directed towards the center of the circle and one tangent to the circle.

(Filename: tfigure5.6)

where  $\dot{v}$  is the rate of change of tangential speed<sup>①</sup>.

The rotation  $\theta$  can vary with  $t$  arbitrarily, depending on the problem at hand.

For uniform rotational acceleration,  $\frac{d}{dt}\omega = \alpha = \text{constant}$ , the following formulas are useful for some elementary problems:

$$\omega(t) = \omega_0 + \alpha t, \text{ and} \quad (7.7)$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \quad (7.8)$$

You can also write the above formulas in terms of  $\dot{\theta}$ ,  $\ddot{\theta}$ , etc., by simply substituting  $\dot{\theta}$  for  $\omega$  and  $\ddot{\theta}$  for  $\alpha$  (see samples).

<sup>①</sup>**Caution:** Note that the rate of change of speed is not the magnitude of the acceleration:  $\dot{v} \neq |\vec{a}|$  or in other words:  $\frac{d}{dt}|\vec{v}| \neq |\frac{d}{dt}\vec{v}|$ . Consider the case of a car driving in circles at constant rate. Its rate of change of speed is zero, yet it has an acceleration.

## The motion quantities

We can use our results for velocity and acceleration to better evaluate the momenta and energy quantities. These results will allow us to do mechanics problems associated with circular motion. For one particle in circular motion.

$$\begin{aligned}
 \vec{L} &= \vec{v}m & = R\dot{\theta}\hat{e}_\theta m, \\
 \dot{\vec{L}} &= \vec{a}m & = (-\dot{\theta}^2\vec{R} + R\ddot{\theta}\hat{e}_\theta)m, \\
 \vec{H}_O &= \vec{r}_{/0} \times \vec{v}m & = R^2\dot{\theta}m\hat{k}, \\
 \dot{\vec{H}}_O &= \vec{r}_{/0} \times \vec{a}m & = R^2\ddot{\theta}m\hat{k}, \\
 E_K &= \frac{1}{2}v^2m & = \frac{1}{2}R^2\dot{\theta}^2m, & \text{and} \\
 \dot{E}_K &= \vec{v} \cdot \vec{a}m & = mR^2\dot{\theta}\ddot{\theta}
 \end{aligned}$$

We have used the fact that  $\hat{e}_R \times \hat{e}_\theta = \hat{k}$  which can be verified with the right hand rule definition of the cross product or using the Cartesian representation of the polar base vectors.

**SAMPLE 7.1** *The velocity vector.* A particle executes circular motion in the  $xy$  plane with constant speed  $v = 5 \text{ m/s}$ . At  $t = 0$  the particle is at  $\theta = 0$ . Given that the radius of the circular orbit is 2.5 m, find the velocity of the particle at  $t = 2 \text{ sec}$ .

**Solution** It is given that

$$\begin{aligned} R &= 2.5 \text{ m} \\ v &= \text{constant} = 5 \text{ m/s} \\ \theta_{(t=0)} &= 0. \end{aligned}$$

The velocity of a particle in constant-rate circular motion is:

$$\begin{aligned} \vec{v} &= R\dot{\theta}\hat{e}_\theta \\ \text{where } \hat{e}_\theta &= -\sin\theta\hat{i} + \cos\theta\hat{j}. \end{aligned}$$

Since  $R$  is constant and  $v = |\vec{v}| = R\dot{\theta}$  is constant,

$$\dot{\theta} = \frac{v}{R} = \frac{5 \text{ m/s}}{2.5 \text{ m}} = 2 \frac{\text{rad}}{\text{s}}$$

is also constant.

$$\text{Thus } \vec{v}_{(t=2 \text{ s})} = \underbrace{R\dot{\theta}}_v \hat{e}_\theta |_{t=2 \text{ s}} = 5 \text{ m/s } \hat{e}_{\theta(t=2 \text{ s})}.$$

Clearly, we need to find  $\hat{e}_\theta$  at  $t = 2 \text{ sec}$ .

$$\begin{aligned} \text{Now } \dot{\theta} &\equiv \frac{d\theta}{dt} = 2 \text{ rad/s} \\ \Rightarrow \int_0^\theta d\theta &= \int_0^{2 \text{ s}} 2 \text{ rad/s } dt \\ \Rightarrow \theta &= (2 \text{ rad/s}) t \Big|_0^{2 \text{ s}} \\ &= 2 \text{ rad/s} \cdot 2 \text{ s} \\ &= 4 \text{ rad}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{e}_\theta &= -\sin 4\hat{i} + \cos 4\hat{j} \\ &= 0.76\hat{i} - 0.65\hat{j}, \end{aligned}$$

and

$$\begin{aligned} \vec{v}_{(2 \text{ s})} &= 5 \text{ m/s}(0.76\hat{i} - 0.65\hat{j}) \\ &= (3.78\hat{i} - 3.27\hat{j}) \text{ m/s}. \end{aligned}$$

$$\boxed{\vec{v} = (3.78\hat{i} - 3.27\hat{j}) \text{ m/s}}$$

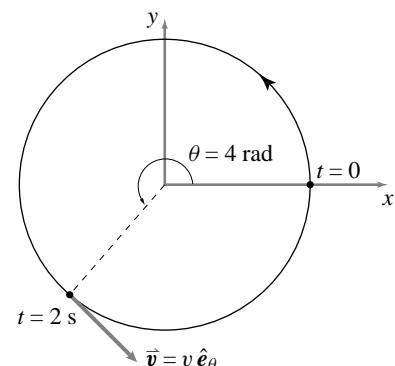


Figure 7.11: (Filename:sfig4.1.DH)

**SAMPLE 7.2 Basic kinematics:** A point mass executes circular motion with angular acceleration  $\ddot{\theta} = 5 \text{ rad/s}^2$ . The radius of the circular path is 0.25 m. If the mass starts from rest at  $\theta = 0^\circ$ , find and draw

- the velocity of the mass at  $\theta = 0^\circ, 30^\circ, 90^\circ$ , and  $210^\circ$ ,
- the acceleration of the mass at  $\theta = 0^\circ, 30^\circ, 90^\circ$ , and  $210^\circ$ .

**Solution** We are given,  $\ddot{\theta} = 5 \text{ rad/s}^2$ , and  $R = 0.25 \text{ m}$ .

- The velocity  $\vec{v}$  in circular (constant or non-constant rate) motion is given by:

$$\vec{v} = R\dot{\theta}\hat{e}_\theta.$$

So, to find the velocity at different positions we need  $\dot{\theta}$  at those positions. Here the angular acceleration is constant, i.e.,  $\ddot{\theta} = 5 \text{ rad/s}^2$ . Therefore, we can use the formula ①

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\ddot{\theta}\theta$$

to find the angular speed  $\dot{\theta}$  at various  $\theta$ 's. But  $\dot{\theta}_0 = 0$  (mass starts from rest), therefore  $\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$ . Now we make a table for computing the velocities at different positions:

Position ( $\theta$ )	$\theta$ in radians	$\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$	$\vec{v} = R\dot{\theta}\hat{e}_\theta$
$0^\circ$	0	0 rad/s	$\vec{0}$
$30^\circ$	$\pi/6$	$\sqrt{10\pi/6} = 2.29 \text{ rad/s}$	$0.57 \text{ m/s} \hat{e}_\theta$
$90^\circ$	$\pi/2$	$\sqrt{10\pi/2} = 3.96 \text{ rad/s}$	$0.99 \text{ m/s} \hat{e}_\theta$
$210^\circ$	$7\pi/6$	$\sqrt{70\pi/6} = 2.29 \text{ rad/s}$	$1.51 \text{ m/s} \hat{e}_\theta$

The computed velocities are shown in Fig. 7.12.

- The acceleration of the mass is given by

$$\begin{aligned} \vec{a} &= \overbrace{a_R \hat{e}_R}^{\text{radial}} + \overbrace{a_\theta \hat{e}_\theta}^{\text{tangential}} \\ &= -R\dot{\theta}^2 \hat{e}_R + R\ddot{\theta} \hat{e}_\theta. \end{aligned}$$

Since  $\ddot{\theta}$  is constant, the tangential component of the acceleration is constant at all positions. We have already calculated  $\dot{\theta}$  at various positions, so we can easily calculate the radial (also called the normal) component of the acceleration. Thus we can find the acceleration. For example, at  $\theta = 30^\circ$ ,

$$\begin{aligned} \vec{a} &= -R\dot{\theta}^2 \hat{e}_R + R\ddot{\theta} \hat{e}_\theta \\ &= -0.25 \text{ m} \cdot \frac{10\pi}{6} \frac{1}{\text{s}^2} \hat{e}_R + 0.25 \text{ m} \cdot 5 \frac{1}{\text{s}^2} \hat{e}_\theta \\ &= -1.31 \text{ m/s}^2 \hat{e}_R + 1.25 \text{ m/s}^2 \hat{e}_\theta. \end{aligned}$$

Similarly, we find the acceleration of the mass at other positions by substituting the values of  $R$ ,  $\ddot{\theta}$  and  $\dot{\theta}$  in the formula and tabulate the results in the table below.

① We use this formula because we need  $\dot{\theta}$  at different values of  $\theta$ . In elementary physics books, the same formula is usually written as

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

where  $\alpha$  is the constant angular acceleration.

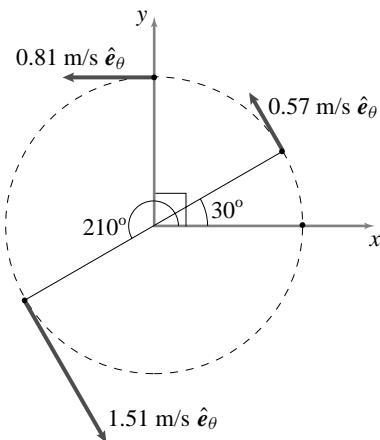


Figure 7.12: Velocity of the mass at  $\theta = 0^\circ, 30^\circ, 90^\circ$ , and  $210^\circ$ .

(Filename:sfig5.1.1a)

Position ( $\theta$ )	$a_r = -R\dot{\theta}^2$	$a_\theta = R\ddot{\theta}$	$\vec{a} = a_r \hat{e}_R + a_\theta \hat{e}_\theta$
0°	0	1.25 m/s <sup>2</sup>	1.25 m/s <sup>2</sup> $\hat{e}_\theta$
30°	-1.31 m/s <sup>2</sup>	1.25 m/s <sup>2</sup>	(-1.31 $\hat{e}_R$ + 1.25 $\hat{e}_\theta$ ) m/s <sup>2</sup>
90°	-3.93 m/s <sup>2</sup>	1.25 m/s <sup>2</sup>	(-3.93 $\hat{e}_R$ + 1.25 $\hat{e}_\theta$ ) m/s <sup>2</sup>
210°	-9.16 m/s <sup>2</sup>	1.25 m/s <sup>2</sup>	(-9.16 $\hat{e}_R$ + 1.25 $\hat{e}_\theta$ ) m/s <sup>2</sup>

The accelerations computed are shown in Fig. 7.13. The acceleration vector as well as its tangential and radial components are shown in the figure at each position.

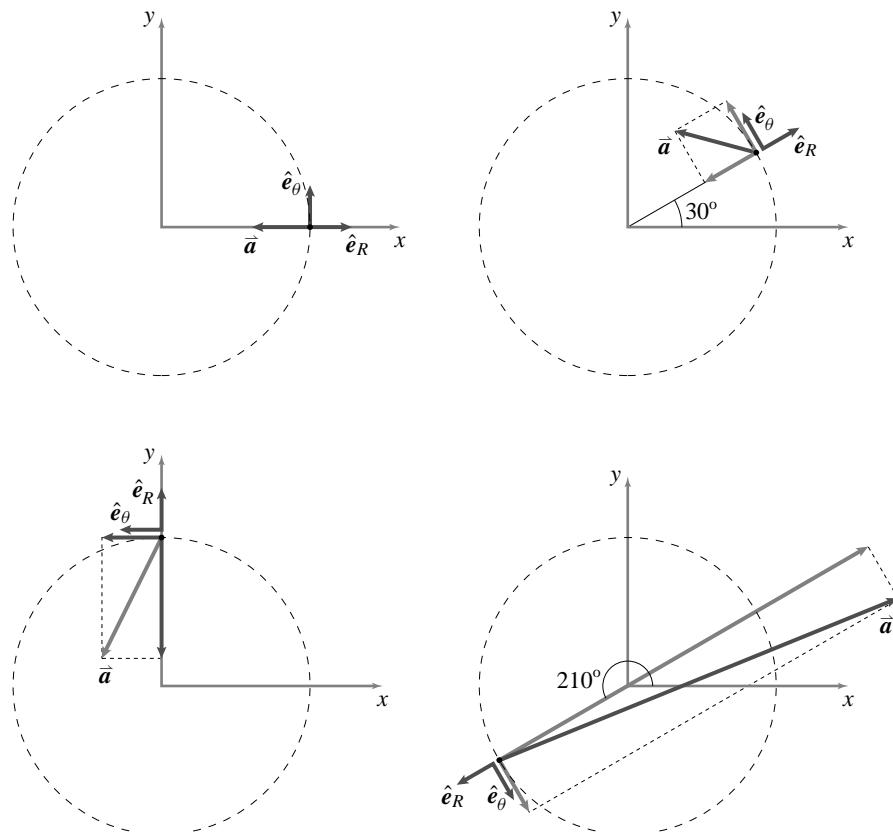


Figure 7.13: Acceleration of the mass at  $\theta = 0^\circ$ ,  $30^\circ$ ,  $90^\circ$ , and  $210^\circ$ . The radial and tangential components are shown with grey arrows. As the angular velocity increases, the radial component of the acceleration increases; therefore, the total acceleration vector leans more and more towards the radial direction.

(Filename:sfig5.1.1b)

**SAMPLE 7.3** In an experiment, the magnitude of angular deceleration of a spinning ball is found to be proportional to its angular speed  $\omega$  (ie.,  $\dot{\omega} \propto -\omega$ ). Assume that the proportionality constant is  $k$  and find an expression for  $\omega$  as a function of  $t$ , given that  $\omega(t = 0) = \omega_0$ .

**Solution** The equation given is:

$$\dot{\omega} = \frac{d\omega}{dt} = -k\omega. \quad (7.9)$$

Let us guess a solution of the exponential form with arbitrary constants and plug into Eqn. (7.9) to check if our solution works. Let  $\omega(t) = C_1 e^{C_2 t}$ . Substituting in Eqn. (7.9), we get

$$\begin{aligned} C_1 C_2 e^{C_2 t} &= -k C_1 e^{C_2 t} \\ \Rightarrow C_2 &= -k, \\ \text{also, } \omega(0) &= \omega_0 = C_1 e^{C_2 \cdot 0} \\ \Rightarrow C_1 &= \omega_0. \end{aligned}$$

Therefore,

$$\omega(t) = \omega_0 e^{-kt}. \quad (7.10)$$

**Alternatively,**

$$\begin{aligned} \frac{d\omega}{\omega} &= -k dt \\ \text{or} \quad \int_{\omega_0}^{\omega(t)} \frac{d\omega}{\omega} &= - \int_0^t k dt \\ \Rightarrow \ln \omega \Big|_{\omega_0}^{\omega(t)} &= -kt \\ \Rightarrow \ln \omega(t) - \ln \omega_0 &= -kt \\ \Rightarrow \ln \left( \frac{\omega(t)}{\omega_0} \right) &= -kt \\ \Rightarrow \frac{\omega(t)}{\omega_0} &= e^{-kt}. \end{aligned}$$

Therefore,

$$(7.11)$$

$$\omega(t) = \omega_0 e^{-kt},$$

which is the same solution as equation (7.10).

$$\boxed{\omega(t) = \omega_0 e^{-kt}}$$

**SAMPLE 7.4** Using kinematic formulae: The spinning wheel of a stationary exercise bike is brought to rest from 100 rpm by applying brakes over a period of 5 seconds.

- Find the average angular deceleration of the wheel.
- Find the number of revolutions it makes during the braking.

**Solution** We are given,

$$\dot{\theta}_0 = 100 \text{ rpm}, \quad \dot{\theta}_{\text{final}} = 0, \quad \text{and} \quad t = 5 \text{ s}.$$

- Let  $\alpha$  be the average (constant) deceleration. Then

$$\dot{\theta}_{\text{final}} = \dot{\theta}_0 - \alpha t.$$

Therefore,

$$\begin{aligned} \alpha &= \frac{\dot{\theta}_0 - \dot{\theta}_{\text{final}}}{t} \\ &= \frac{100 \text{ rpm} - 0 \text{ rpm}}{5 \text{ s}} \\ &= \frac{100 \text{ rev}}{60 \text{ s}} \cdot \frac{1}{5 \text{ s}} \\ &= 0.33 \frac{\text{rev}}{\text{s}^2}. \end{aligned}$$

$$\boxed{\alpha = 0.33 \frac{\text{rev}}{\text{s}^2}}$$

- To find the number of revolutions made during the braking period, we use the formula

$$\theta(t) = \underbrace{\theta_0}_{0} + \dot{\theta}_0 t + \frac{1}{2}(-\alpha)t^2 = \dot{\theta}_0 t - \frac{1}{2}\alpha t^2.$$

Substituting the known values, we get

$$\begin{aligned} \theta &= \frac{100 \text{ rev}}{60 \text{ s}} \cdot 5 \text{ s} - \frac{1}{2} 0.33 \frac{\text{rev}}{\text{s}^2} \cdot 25 \text{ s}^2 \\ &= 8.33 \text{ rev} - 4.12 \text{ rev} \\ &= 4.21 \text{ rev}. \end{aligned}$$

$$\boxed{\theta = 4.21 \text{ rev}}$$

### Comments:

- Note the negative sign used in both the formulae above. Since  $\alpha$  is deceleration, that is, a negative acceleration, we have used negative sign with  $\alpha$  in the formulae.
- Note that it is not always necessary to convert rpm in rad/s. Here we changed rpm to rev/s because time was given in seconds.

**SAMPLE 7.5 Non-constant acceleration:** A particle of mass 500 grams executes circular motion with radius  $R = 100\text{ cm}$  and angular acceleration  $\ddot{\theta}(t) = c \sin \beta t$ , where  $c = 2\text{ rad/s}^2$  and  $\beta = 2\text{ rad/s}$ .

- Find the position of the particle after 10 seconds if the particle starts from rest, that is,  $\theta(0) = 0$ .
- How much kinetic energy does the particle have at the position found above?

### Solution

- (a) We are given  $\ddot{\theta}(t) = c \sin \beta t$ ,  $\dot{\theta}(0) = 0$  and  $\theta(0) = 0$ . We have to find  $\theta(10\text{ s})$ . Basically, we have to solve a second order differential equation with given initial conditions.

$$\begin{aligned}\ddot{\theta} &\equiv \frac{d}{dt}(\dot{\theta}) = c \sin \beta t \\ \Rightarrow \int_{\dot{\theta}_0=0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t c \sin \beta \tau d\tau \\ \dot{\theta}(t) &= -\frac{c}{\beta} \cos \beta \tau \Big|_0^t = \frac{c}{\beta} (1 - \cos \beta t).\end{aligned}$$

Thus, we get the expression for the angular speed  $\dot{\theta}(t)$ . We can solve for the position  $\theta(t)$  by integrating once more:

$$\begin{aligned}\dot{\theta} &\equiv \frac{d}{dt}(\theta) = \frac{c}{\beta} (1 - \cos \beta t) \\ \Rightarrow \int_{\theta_0=0}^{\theta(t)} d\theta &= \int_0^t \frac{c}{\beta} (1 - \cos \beta \tau) d\tau \\ \theta(t) &= \frac{c}{\beta} \left[ \tau - \frac{\sin \beta \tau}{\beta} \right]_0^t \\ &= \frac{c}{\beta^2} (\beta t - \sin \beta t).\end{aligned}$$

Now substituting  $t = 10\text{ s}$  in the last expression along with the values of other constants, we get

$$\begin{aligned}\theta(10\text{ s}) &= \frac{2\text{ rad/s}^2}{(2\text{ rad/s})^2} [2\text{ rad/s} \cdot 10\text{ s} - \sin(2\text{ rad/s} \cdot 10\text{ s})] \\ &= 9.54\text{ rad}.\end{aligned}$$

$$\boxed{\theta = 9.54\text{ rad}}$$

- (b) The kinetic energy of the particle is given by

$$\begin{aligned}E_K &= \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\theta})^2 \\ &= \frac{1}{2}mR^2 \underbrace{\left[ \frac{c}{\beta} (1 - \cos \beta t) \right]^2}_{\dot{\theta}(t)} \\ &= \frac{1}{2} 0.5\text{ kg} \cdot 1\text{ m}^2 \cdot \left[ \frac{2\text{ rad/s}^2}{2\text{ rad/s}} \cdot (1 - \cos(20)) \right]^2 \\ &= 0.086\text{ kg} \cdot \text{m}^2 \cdot \text{s}^2 = 0.086\text{ Joule}.\end{aligned}$$

$$\boxed{E_K = 0.086\text{ J}}$$

## 7.2 Dynamics of a particle in circular motion

The simplest examples of circular motion concern the motion of a particle constrained by a massless connection to be a fixed distance from a support point.

### Example: Rock spinning on a string

Neglecting gravity, we can now deal with the familiar problem of a point mass being held in constant circular-rate motion by a massless string or rod. Linear momentum balance for the mass gives:

$$\begin{aligned}\sum \vec{F}_i &= \hat{\vec{L}} \\ \Rightarrow -T\hat{e}_R &= m\vec{a} \\ \{-T\hat{e}_R &= m(-\dot{\theta}^2\ell\hat{e}_R)\} \\ \{ \} \cdot \hat{e}_R \Rightarrow & T = \dot{\theta}^2\ell m = (v^2/\ell)m\end{aligned}$$

The force required to keep a mass in constant rate circular motion is  $mv^2/\ell$  (sometimes remembered as  $mv^2/R$ ).  $\square$

### The simple pendulum

As a child's swing, the inside of a grandfather clock, a hypnotist's device, or a gallows, the motion of a simple pendulum is a clear image to all of us. Galileo studied the simple pendulum and it is a topic in freshman physics. Now a days the pendulum is popular as an example of "chaos"; if you push a pendulum periodically its motions can be wild. Pendula are useful as models of many phenomena from the swing of leg joints in walking to the tipping of a chimney in an earthquake. Pendula also serve as a simple example for many concepts in mechanics.

For starters, we consider a 2-D pendulum of fixed length with no forcing other than gravity. All mass is concentrated at a point. The tension in the pendulum rod acts along the length since it is a massless two-force body. Of primary interest is the motion of the pendulum. First we find governing differential equations. Here are two ways to get the equation of motion.

#### Method One: linear momentum balance in cartesian coordinates

The equation of linear momentum balance is

$$\sum \vec{F} = \overset{m\vec{a}}{\underset{\hat{\vec{L}}}{\curvearrowright}}$$

Evaluating the left side (using the free body diagram) and right side (using the kinematics of circular motion), we get

$$-T\hat{e}_R + (-mg)\hat{j} = m[\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R] \quad (7.12)$$

From the picture (or recalling) we see that  $\hat{e}_R = \cos\theta\hat{i} + \sin\theta\hat{j}$  and  $\hat{e}_\theta = \cos\theta\hat{j} - \sin\theta\hat{i}$ . So, upon substitution into the equation above, we get

$$-T(\cos\theta\hat{i} + \sin\theta\hat{j}) + mg\hat{i} = m[\ell\ddot{\theta}(\cos\theta\hat{j} - \sin\theta\hat{i}) - \ell\dot{\theta}^2(\cos\theta\hat{i} + \sin\theta\hat{j})]$$

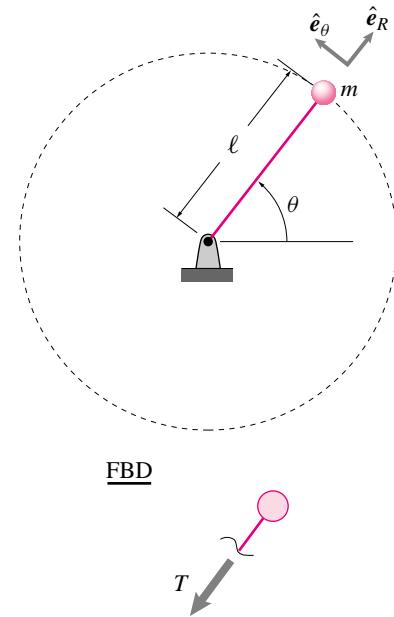


Figure 7.14: Point mass spinning in circles. Sketch of system and a free body diagram.  
(Filename:tfigure4.1.rockandstring)

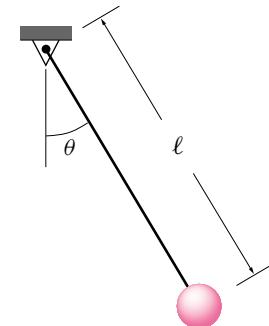


Figure 7.15: The simple pendulum.  
(Filename:tfigure5.spend)

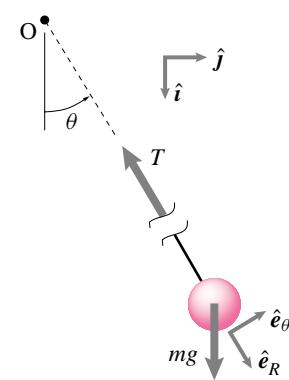


Figure 7.16: Free body diagram of the simple pendulum.  
(Filename:tfigure5.spend.fbd)

Breaking this equation into its  $x$  and  $y$  components (by dotting both sides with  $\hat{i}$  and  $\hat{j}$ , respectively) gives

$$-T \cos \theta + mg = -m\ell (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \text{ and} \quad (7.13)$$

$$-T \sin \theta = m\ell (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \quad (7.14)$$

① As always when seeking equations of motion, we think of the rates and velocities as knowns. Thus we take  $\dot{\theta}$  as known. But how do we know it? We don't, but at any instant in time we can find it as the integral of  $\ddot{\theta}$ . More simply, regarding  $\dot{\theta}$  as known helps us write a set of differential equations in a form suitable for seeking a solution (analytically or by computer integration).

which are two simultaneous equations that we can solve for the two unknowns<sup>①</sup>  $T$  and  $\ddot{\theta}$  to get

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta \quad (7.15)$$

$$T = m[\ell\dot{\theta}^2 + g \cos \theta]. \quad (7.16)$$

### Method 2: linear momentum balance in polar coordinates

A more direct way to get the equation of motion is to take eqn. (7.12) and dot both sides with  $\hat{e}_\theta$  to get

$$\begin{aligned} -T \underbrace{\hat{e}_R \cdot \hat{e}_\theta}_{0} + -mg \underbrace{\hat{j} \cdot \hat{e}_\theta}_{\sin \theta} &= m\ell\ddot{\theta} \underbrace{\hat{e}_\theta \cdot \hat{e}_\theta}_{1} - \ell\dot{\theta}^2 \underbrace{\hat{e}_R \cdot \hat{e}_\theta}_{0} \\ \Rightarrow -mg \sin \theta &= m\ell\ddot{\theta} \\ \text{so } \ddot{\theta} &= -\frac{g}{\ell} \sin \theta. \end{aligned}$$

### Method Three: angular momentum balance

Using angular momentum balance, we can 'kill' the tension term at the start. Taking angular momentum balance about the point  $O$ , we get

$$\begin{aligned} \sum \vec{M}_O &= \dot{\vec{H}}_O \\ -mg\ell \sin \theta \hat{k} &= \vec{r}_{/O} \times \vec{a} m \\ \ell \hat{e}_R & \qquad \qquad \qquad \ell \ddot{\theta} \hat{e}_\theta - \ell \dot{\theta}^2 \hat{e}_R \\ -mg\ell \sin \theta \hat{k} &= m\ell^2 \ddot{\theta} \hat{k} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{\ell} \sin \theta \end{aligned}$$

since  $\hat{e}_R \times \hat{e}_R = 0$  and  $\hat{e}_R \times \hat{e}_\theta = \hat{k}$ . So, the governing equation for a simple pendulum is

$$\boxed{\ddot{\theta} = -\frac{g}{\ell} \sin \theta}$$

### Small angle approximation (linearization)

For small angles,  $\sin \theta \approx \theta$ , so we have

$$\ddot{\theta} = -\frac{g}{\ell} \theta$$

for small oscillations. This equation describes a harmonic oscillator with  $\frac{g}{\ell}$  replacing the  $\sqrt{\frac{k}{m}}$  coefficient in a spring-mass system.

### The inverted pendulum

A pendulum with the mass-end up is called an inverted pendulum. By methods just like we used for the regular pendulum, we find the equation of motion to be

$$\ddot{\theta} = \frac{g}{\ell} \sin \theta$$

which, for small  $\theta$ , is well approximated by

$$\ddot{\theta} = \frac{g}{\ell} \theta.$$

As opposed to the simple pendulum, which has oscillatory solutions, this differential equation has exponential solutions ( $\theta = C_1 e^{gt/\ell} + C_2 e^{-gt/\ell}$ ), one term of which has exponential growth, indicating the inherent *instability* of the inverted pendulum. That is it has tendency to fall over when slightly disturbed from the vertical position<sup>①</sup>.

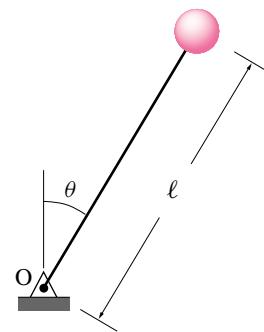


Figure 7.17: The inverted pendulum

(Filename:tfigure5.spend.inv)

<sup>①</sup> After the pendulum falls a ways, say past 30 degrees from vertical, the exponential solution is not an accurate description, but the actual motion (as viewed by and experiment, a computer simulation, or the exact elliptic integral solution of the equations) shows that the pendulum keeps falling.

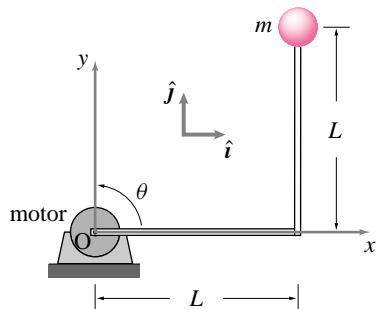


Figure 7.18: The motor rotates the structure at a constant angular speed in the counterclockwise direction.

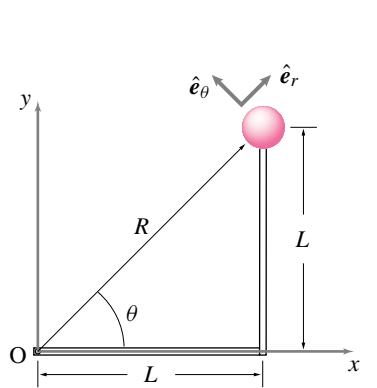


Figure 7.19: The ball follows a circular path of radius  $R$ . The position, velocity, and acceleration of the ball can be expressed in terms of the polar basis vectors  $\hat{e}_R$  and  $\hat{e}_\theta$ .

**SAMPLE 7.6** *Circular motion in 2-D.* Two bars, each of negligible mass and length  $L = 3$  ft, are welded together at right angles to form an ‘L’ shaped structure. The structure supports a 3.2 lbf ( $= mg$ ) ball at one end and is connected to a motor on the other end (see Fig. 7.18). The motor rotates the structure in the vertical plane at a constant rate  $\dot{\theta} = 10 \text{ rad/s}$  in the counter-clockwise direction. Take  $g = 32 \text{ ft/s}^2$ . At the instant shown in Fig. 7.18, find

- the velocity of the ball,
- the acceleration of the ball, and
- the net force and moment applied by the motor and the support at O on the structure.

**Solution** The motor rotates the structure at a constant rate. Therefore, the ball is going in circles with angular velocity  $\vec{\omega} = \dot{\theta}\hat{k} = 10 \text{ rad/s}\hat{k}$ . The radius of the circle is  $R = \sqrt{L^2 + L^2} = L\sqrt{2}$ . Since the motion is in the  $xy$  plane, we use the following formulae to find the velocity  $\vec{v}$  and acceleration  $\vec{a}$ .

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta,$$

where  $\hat{e}_R$  and  $\hat{e}_\theta$  are the polar basis vectors shown in Fig. 7.19. In Fig. 7.19, we note that  $\theta = 45^\circ$ . Therefore,

$$\begin{aligned}\hat{e}_R &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ &= \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}), \\ \hat{e}_\theta &= -\sin\theta\hat{i} + \cos\theta\hat{j} \\ &= \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}).\end{aligned}$$

Since  $R = L\sqrt{2} = 3\sqrt{2}$  ft is constant,  $\dot{R} = 0$  and  $\ddot{R} = 0$ . Thus,

- the velocity of the ball is

$$\begin{aligned}\vec{v} &= R\dot{\theta}\hat{e}_\theta \\ &= 3\sqrt{2} \text{ ft} \cdot 10 \text{ rad/s} \hat{e}_\theta \\ &= 30\sqrt{2} \text{ ft/s} \cdot \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}) \\ &= 30 \text{ ft/s}(-\hat{i} + \hat{j}).\end{aligned}$$

$\vec{v} = 30 \text{ ft/s}(-\hat{i} + \hat{j})$

- The acceleration of the ball is

$$\begin{aligned}\vec{a} &= -R\dot{\theta}^2\hat{e}_R \\ &= -3\sqrt{2} \text{ ft} \cdot (10 \text{ rad/s})^2 \hat{e}_R \\ &= -300\sqrt{2} \text{ ft/s}^2 \cdot \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) \\ &= -300 \text{ ft/s}^2(\hat{i} + \hat{j}).\end{aligned}$$

$\vec{a} = -300 \text{ ft/s}^2(\hat{i} + \hat{j})$

- (c) Let the net force and the moment applied by the motor-support system be  $\vec{F}$  and  $\vec{M}$  as shown in Fig. 7.20. From the linear momentum balance for the structure,

$$\begin{aligned}
 \sum \vec{F} &= m\vec{a} \\
 \vec{F} - mg\hat{j} &= m\vec{a} \\
 \Rightarrow \vec{F} &= m\vec{a} + mg\hat{j} \\
 &= \frac{\cancel{m}}{32 \text{ ft/s}^2} \left( \frac{3.2 \text{ lbf}}{\cancel{32 \text{ ft/s}^2}} (-300\sqrt{2} \text{ ft/s}^2) \hat{e}_R + \frac{mg}{\cancel{3.2 \text{ lbf}}} \hat{j} \right) \\
 &= -30\sqrt{2} \text{ lbf} \hat{e}_R + 3.2 \text{ lbf} \hat{j} \\
 &= -30\sqrt{2} \text{ lbf} \frac{1}{\sqrt{2}} (\hat{i} + \hat{j}) + 3.2 \text{ lbf} \hat{j} \\
 &= -30 \text{ lbf} \hat{i} - 26.8 \text{ lbf} \hat{j}.
 \end{aligned}$$

Similarly, from the angular momentum balance for the structure,

$$\begin{aligned}
 \sum \vec{M}_O &= \dot{\vec{H}}_O, \\
 \text{where } \sum \vec{M}_O &= \vec{M} + \vec{r}_{/O} \times mg(-\hat{j}) \\
 &= \vec{M} + \underbrace{R\hat{e}_R \times mg(-\hat{j})}_{L(\hat{i}+\hat{j})} \\
 &= \vec{M} - mgL\hat{k}, \\
 \text{and } \dot{\vec{H}}_O &= \vec{r}_{/O} \times m\vec{a} \\
 &= R\hat{e}_R \times m(-R\dot{\theta}^2 \hat{e}_R) \\
 &= -mR^2\dot{\theta}^2 \underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} \\
 &= \vec{0}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \vec{M} &= mgL\hat{k} \\
 &= \underbrace{3.2 \text{ lbf}}_{mg} \cdot \underbrace{3 \text{ ft}}_L \hat{k} \\
 &= 9.6 \text{ lbf} \cdot \text{ft} \hat{k}.
 \end{aligned}$$

$$\boxed{\vec{F} = -30 \text{ lbf} \hat{i} - 26.8 \text{ lbf} \hat{j}, \quad \vec{M} = 9.6 \text{ lbf} \cdot \text{ft} \hat{k}}$$

Note: If there was no gravity, the moment applied by the motor would be zero.

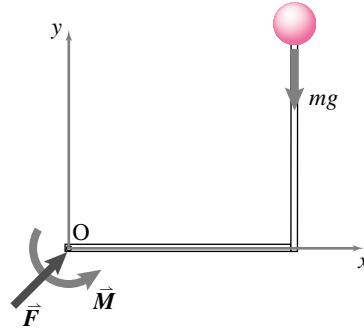


Figure 7.20: Free body diagram of the structure.  
(Filename:sfig4.1.1b)

**SAMPLE 7.7** A 50 gm point mass executes circular motion with angular acceleration  $\ddot{\theta} = 2 \text{ rad/s}^2$ . The radius of the circular path is 200 cm. If the mass starts from rest at  $t = 0$ , find

- Its angular momentum  $\vec{H}$  about the center at  $t = 5 \text{ s}$ .
- Its rate of change of angular momentum  $\dot{\vec{H}}$  about the center.

### Solution

- From the definition of angular momentum,

$$\begin{aligned}\vec{H}_O &= \vec{r}_{/0} \times m\vec{v} \\ &= R\hat{\mathbf{e}}_R \times m\dot{\theta}R\hat{\mathbf{e}}_\theta \\ &= mR^2\dot{\theta}(\hat{\mathbf{e}}_R \times \hat{\mathbf{e}}_\theta) \\ &= mR^2\dot{\theta}\hat{\mathbf{k}}\end{aligned}$$

On the right hand side of this equation, the only unknown is  $\dot{\theta}$ . Thus to find  $\vec{H}_O$  at  $t = 5 \text{ s}$ , we need to find  $\dot{\theta}$  at  $t = 5 \text{ s}$ . Now,

$$\begin{aligned}\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\ d\dot{\theta} &= \ddot{\theta} dt \\ \int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t \ddot{\theta} dt \\ \dot{\theta}(t) - \dot{\theta}_0 &= \ddot{\theta}(t_t - t_0) \\ \dot{\theta} &= \dot{\theta}_0 + \ddot{\theta}(t - t_0)\end{aligned}$$

Writing  $\alpha$  for  $\ddot{\theta}$  and substituting  $t_0 = 0$  in the above expression, we get  $\dot{\theta}(t) = \dot{\theta}_0 + \alpha t$ , which is the angular speed version of the linear speed formula  $v(t) = v_0 + at$ . <sup>①</sup> Substituting  $t = 5 \text{ s}$ ,  $\dot{\theta}_0 = 0$ , and  $\alpha = 2 \text{ rad/s}^2$  we get  $\dot{\theta} = 2 \text{ rad/s}^2 \cdot 5 \text{ s} = 10 \text{ rad/s}$ . Therefore,

$$\begin{aligned}\vec{H}_O &= 0.05 \text{ kg} \cdot (0.2 \text{ m})^2 \cdot 10 \text{ rad/s} \hat{\mathbf{k}} \\ &= 0.02 \text{ kg} \cdot \text{m}^2/\text{s} = 0.02 \text{ N} \cdot \text{m} \cdot \text{s}.\end{aligned}$$

$$\boxed{\vec{H}_O = 0.02 \text{ N} \cdot \text{m} \cdot \text{s}.}$$

- Similarly, we can calculate the rate of change of angular momentum:

$$\begin{aligned}\dot{\vec{H}}_O &= \vec{r}_{/0} \times m\vec{a} \\ &= R\hat{\mathbf{e}}_R \times m(R\ddot{\theta}\hat{\mathbf{e}}_\theta - \dot{\theta}^2 R\hat{\mathbf{e}}_R) \\ &= mR^2\ddot{\theta}(\hat{\mathbf{e}}_R \times \hat{\mathbf{e}}_\theta) \\ &= mR^2\ddot{\theta}\hat{\mathbf{k}} \\ &= 0.02 \text{ kg} \cdot (0.2 \text{ m})^2 \cdot 2 \text{ rad/s}^2 \hat{\mathbf{k}} \\ &= 0.004 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 0.004 \text{ N} \cdot \text{m}\end{aligned}$$

$$\boxed{\dot{\vec{H}}_O = 0.004 \text{ N} \cdot \text{m}}$$

<sup>①</sup> Be warned that these formulae are valid only for constant acceleration.

**SAMPLE 7.8** *The simple pendulum.* A simple pendulum swings about its vertical equilibrium position (2-D motion) with maximum amplitude  $\theta_{max} = 10^\circ$ . Find

- the magnitude of the maximum angular acceleration,
- the maximum tension in the string.

### Solution

- The equation of motion of the pendulum is given by (see equation 7.15 of text):

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta.$$

We are given that  $|\theta| \leq \theta_{max}$ . For  $\theta_{max} = 10^\circ = 0.1745 \text{ rad}$ ,  $\sin \theta_{max} = 0.1736$ . Thus we see that  $\sin \theta \approx \theta$  even when  $\theta$  is maximum. Therefore, we can safely use linear approximation (although we could solve this problem without it); i.e.,

$$\ddot{\theta} = -\frac{g}{\ell} \theta.$$

Clearly,  $|\ddot{\theta}|$  is maximum when  $\theta$  is maximum. Thus,

$$|\dot{\theta}|_{max} = \frac{g}{\ell} \theta_{max} = \frac{9.81 \text{ m/s}^2}{1 \text{ m}} \cdot (0.1745 \text{ rad}) = 1.71 \text{ rad/s}^2.$$

$$|\dot{\theta}|_{max} = 1.71 \text{ rad/s}^2$$

- The tension in the string is given by (see equation 7.16 of text):

$$T = m(\ell \dot{\theta}^2 + g \cos \theta).$$

This time, we will not make the small angle assumption. We can find  $T_{max}$  and where it is maximum as follows using conservation of energy. Let the position of maximum amplitude be position 1. and the position at any  $\theta$  be position 2. At its maximum amplitude, the mass comes to rest and switches directions; thus, its angular velocity and, hence, its kinetic energy is zero there. Using conservation of energy, we have

$$\begin{aligned} E_{K1} + E_{P1} &= E_{K2} + E_{P2} \\ 0 + mg\ell(1 - \cos \theta_{max}) &= \frac{1}{2}m(\ell \dot{\theta})^2 + mg\ell(1 - \cos \theta). \end{aligned} \quad (7.17)$$

and solving for  $\dot{\theta}$ ,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos \theta - \cos(\theta_{max}))}.$$

Therefore, the tension at any  $\theta$  is

$$T = m(\ell \dot{\theta}^2 + g \cos \theta) = mg(3 \cos \theta - 2 \cos(\theta_{max})).$$

To find the maximum value of the tension  $T$ , we set its derivative with respect to  $\theta$  equal to zero and find that, for  $0 \leq \theta \leq \theta_{max}$ ,  $T$  is maximum when  $\theta = 0$ , or

$$T_{max} = mg(3 \cos(0) - 2 \cos(\theta_{max})) = 0.2 \text{ kg} \cdot 9.81 \text{ m/s}^2 (3 - 1.97) = 2.02 \text{ N}.$$

The maximum tension corresponds to maximum speed which occurs at the bottom of the swing where all of the potential energy is converted to kinetic energy.

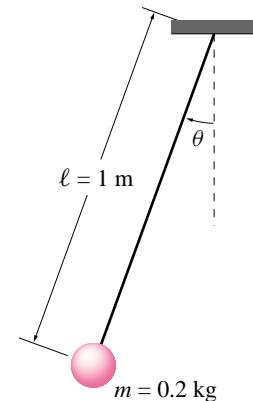


Figure 7.21: (Filename:fig5.5.DH1)

$$T_{max} = 2.02 \text{ N}$$

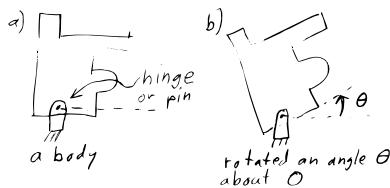


Figure 7.22: a) A body, b) rotated counterclockwise an angle  $\theta$  about O.

(Filename:tfigure.circmot2D)

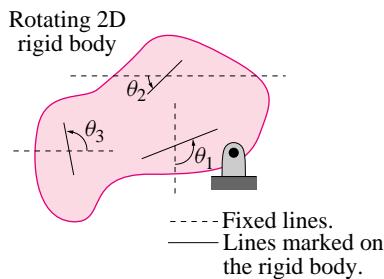


Figure 7.23: **Rotation of lines on a rotating rigid body.** Some real or imagined lines marked on the rigid body are shown. They make the angles  $\theta_1, \theta_2, \theta_3, \dots$  with respect to various fixed lines which do not rotate. As the body rotates, each of these angles increases by the same amount.

(Filename:tfigure4.2Domega)

## 7.3 Kinematics of a rigid body in planar circular motion

The most common non-rigid attachment in machine design is a hinge or pin connection (Fig. 7.22), or something well modeled as a pin. In this chapter on circular motion we study machine parts hinged to structures which do not move. If we take the hinge axis to be the  $z$  axis fixed at O, then the hinge's job is to make the part's only possible motion to be rotation about O. As usual in this book, we think of the part itself as rigid. Thus to study dynamics of a hinged part we need to understand the position, velocity and acceleration of points on a rigid body which rotates. This section discusses the geometry and algebra of rotation, of rotation rate which we will call the angular velocity, and of rate of change of the angular velocity.

The rest of the book rests heavily on the material in this section.

### Rotation of a rigid body counterclockwise by $\theta$

We start by imagining the object in some configuration which we call the *reference configuration* or *reference state*. Often the reference state is one where prominent features of the object are aligned with the vertical or horizontal direction or with prominent features of another nearby part. The reference state may or may not be the start of the motion of interest. We measure an object's rotation relative to the reference state, as in Fig. 7.22 where a body is shown and shown again, rotated. For definiteness, rotation is the change, relative to the reference state, in the counterclockwise angle  $\theta$  of a reference line marked in the body relative to a fixed line outside. Which reference line? Fortunately,

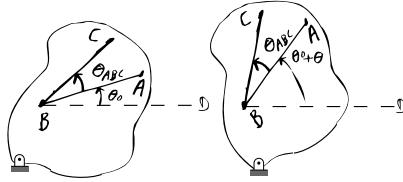
*All real or imagined lines marked on a rotating rigid body rotate by the same angle.*

### 7.1 THEORY

#### Rotation is uniquely defined for a rigid body (2D)

We show here the intuitively clear result that, starting at a reference orientation, all lines marked on a rigid body rotate by the same angle  $\theta$ .

Because the body is rigid, the act of rotation preserves all distances between pairs of points. That's a geometric definition of the word *rigid*. Thus, by the "similar triangle theorem" (side-side-side) of elementary geometry, all relative angles between marked line segments are preserved by the rotation. Consider a pair of line segments with each segment defined by two points on the body. First, extend the segments to their point of intersection. Such a pair of lines is shown before and after rotation here with their intersection at B.



Initially BA makes an angle  $\theta_0$  with a horizontal reference line. BC then makes an angle of  $\theta_{ABC} + \theta_0$ . After rotation we measure the angle to the line BD (displaced in a parallel manner). BA now makes an angle of  $\theta_0 + \theta$  where  $\theta$  is the angle of rotation of the body. By the addition of angles in the rotated configuration line BC now makes an angle of  $\theta_{ABC} + \theta_0 + \theta$  which is, because angle  $\theta_{ABC}$  is unchanged, also an increase by  $\theta$  in the angle made by BC with the horizontal reference line. Both lines rotate by the same angle  $\theta$ .

We could use one of these lines and compare with an arbitrary third line and show that those have equal rotation also, and so on for any lines of interest in the body. So all lines rotate by the same angle  $\theta$ . The demonstration for a pair of parallel lines is easy, they stay parallel so always make a common angle with any reference line.

Thus, all lines marked on a rigid body rotate by the same angle  $\theta$  and the concept of a body's rotation from a given reference state is uniquely defined.

(See box 7.1). Thus, once we have decided on a reference configuration, we can measure the rotation of the body, and of all lines marked on the body, with a single number, the *rotation angle*  $\theta$ <sup>①</sup>.

① In three dimensions the situation is more complicated. Rotation of a rigid body is also well defined, but its representation is more complicated than a single number  $\theta$ .

## Rotated coordinates and base vectors $\hat{i}'$ and $\hat{j}'$

Often it is convenient to pick two orthogonal lines on a body and give them distinguished status as *body fixed* rotating coordinate axes  $x'$  and  $y'$ . The algebra we will develop is most simple if these axes are chosen to be parallel with a fixed  $x$  and  $y$  axes when  $\theta = 0$  in the reference configuration.

We will follow a point P at  $\vec{r}_P$ . With this rotating coordinate axes  $x'$  and  $y'$  are associated rotating base vectors  $\hat{i}'$  and  $\hat{j}'$  (Fig. 11.9). The position coordinates of P, in the rotating coordinates, are  $[\vec{r}]_{x'y'} = [x', y']$ , which we sometimes write as  $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ .

### Example: A particle on the $x'$ axis

If a particle of interest is fixed on the  $x'$ -axis at position  $x' = 3$  cm, then we have.

$$\vec{r}_P = 3 \text{ cm} \hat{i}'$$

for all time, even as the body rotates.  $\square$

For a general point P fixed to a body rotating about O it is always true that

$$\vec{r}_P = x' \hat{i}' + y' \hat{j}', \quad (7.18)$$

$$(7.19)$$

with the  $x'$  and  $y'$  values not changing as  $\theta$  increases. Obviously point P moves, and the axes move, but the particle's coordinates  $x'$  and  $y'$  do not change. The change in motion is expressed in eqn. (7.20) by the base vectors changing as the body rotates. Thus we could write more explicitly that

$$\vec{r}_P = x' \hat{i}'(\theta) + y' \hat{j}'(\theta). \quad (7.20)$$

In particular, just like for polar base vectors (see eqn. (7.3) on page 361) we can express the rotating base vectors in terms of the fixed base vectors and  $\theta$ .

$$\begin{aligned} \hat{i}' &= \cos \theta \hat{i} + \sin \theta \hat{j}, \\ \hat{j}' &= -\sin \theta \hat{i} + \cos \theta \hat{j}. \end{aligned} \quad (7.21)$$

One also sometimes wants to know the fixed basis vectors in terms of the rotating vectors,

$$\begin{aligned} \hat{i} &= \cos \theta \hat{i}' - \sin \theta \hat{j}' \\ \hat{j} &= \sin \theta \hat{i}' + \cos \theta \hat{j}'. \end{aligned} \quad (7.22)$$

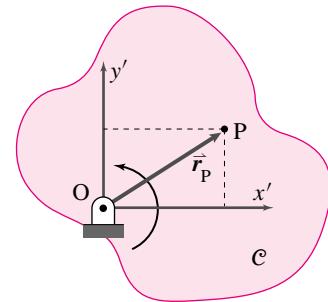


Figure 7.24: A rotating rigid body  $C$  with rotating coordinates  $x'y'$  rigidly attached.

(Filename:tfigure4.intro.rot.frames)

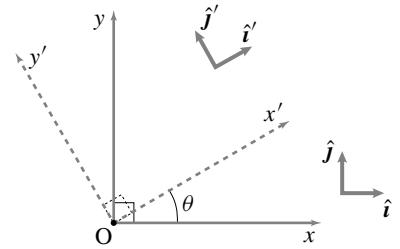


Figure 7.25: Fixed coordinate axes and rotating coordinate axes.

(Filename:tfigure4.1.rot.coord)

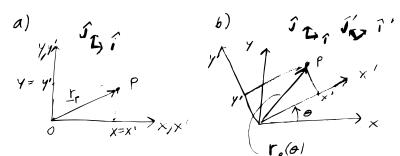


Figure 7.26: The  $x'$  and  $y'$  coordinates of a point fixed on a rotating body stay constant while the base vectors  $\hat{i}'$  and  $\hat{j}'$  change while they rotate with the body.

(Filename:tfigure.rotatepbytheta)

You should review the material in section 2.2 to see how these formulae can be derived with dot products.

We will use the phrase *reference frame* or just *frame* to mean “a coordinate system attached to a rigid body”. One could imagine that the coordinate grid is like a metal framework that rotates with the body. We would refer to a calculation based on the rotating coordinates in Fig. 11.9 as “in the frame  $C$ ” or “using the  $x'y'$  frame” or “in the  $\hat{i}'\hat{j}'$  frame<sup>①</sup>.

<sup>①</sup> Advanced aside. Sometimes a *reference frame* is defined as the set of all coordinate systems that could be attached to a rigid body. Two coordinate systems, even if rotated with respect to each other, then represent the same frame so long as they rotate together at the time of interest. Some of the results we will develop only depend on this definition of frame, that the coordinates are glued to the body, and not on their orientation on the body.

In computer calculations it is often best to manipulate lists and arrays of numbers and not geometric vectors. Thus we like to keep track of coordinates of vectors. Lets look at a point whose coordinates we know in the reference configuration:  $[\vec{r}_P^{\text{ref}}]_{xy}$ . Taking the body axes and fixed axes coincide in the reference configuration, the body coordinates of a point  $[\vec{r}_P]_{x'y'}$  are equal to the space fixed coordinates of the point in the reference configuration  $[\vec{r}_P^{\text{ref}}]_{xy}$ . We can think of the point as equivalently defined either way,

$$[\vec{r}_P]_{x'y'} = [\vec{r}_P^{\text{ref}}]_{xy}$$

## Coordinate representation of rotations using $[R]$

Here is a question we often need to answer, especially in computer animation: What are the fixed basis coordinates of a point with coordinates  $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ ? Here is one way to find the answer:

$$\begin{aligned} \vec{r}_P &= x'\hat{i}' + y'\hat{j}' \\ &= x'(\cos\theta\hat{i} + \sin\theta\hat{j}) + y'(-\sin\theta\hat{i} + \cos\theta\hat{j}) \\ &= \underbrace{((\cos\theta)x' - (\sin\theta)y')}_{x}\hat{i} + \underbrace{((\sin\theta)x' + (\cos\theta)y')}_{y}\hat{j} \end{aligned} \quad (7.23)$$

so we can pull out the  $x$  and  $y$  coordinates compactly as,

$$[\vec{r}_P]_{xy} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta x' + \sin\theta(-y') \\ \sin\theta x' + \cos\theta(y') \end{bmatrix}. \quad (7.24)$$

But this can, in turn be written in matrix notation as

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \\ [\vec{r}_P]_{xy} &= [R][\vec{r}_P]_{x'y'}, \quad \text{or} \\ [\vec{r}_P]_{xy} &= [R]\left[\vec{r}_P^{\text{ref}}\right]_{xy}, \end{aligned} \quad (7.25)$$

The matrix  $[R]$  or  $[R(\theta)]$  is the *rotation matrix* for counterclockwise rotations by  $\theta$ . If you know the coordinates of a point on a body before rotation, you can find its coordinates after rotation by multiplying the coordinate column vector by the matrix  $[R]$ . A feature of eqn. (7.25) is that the same matrix  $[R]$  prescribes the coordinate

change for every different point on the body. Thus for points called 1, 2 and 3 we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [R] \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [R] \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = [R] \begin{bmatrix} x'_3 \\ y'_3 \end{bmatrix}.$$

A more compact way to write a matrix times a list of column vectors is to arrange the column vectors one next to the other in a matrix. By multiplying this matrix by  $[R]$  we get a new matrix whose columns are the new coordinates of various points. For example,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [R] \begin{bmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \end{bmatrix}. \quad (7.26)$$

Eqn. 7.26 is useful for computer animation of rotating things in video games (and in dynamics simulations too) where points 1,2, and 3 are vertices of the polygonal drawing of some object.

#### Example: Rotate a picture

If a simple picture of a house is drawn by connecting the six points (Fig. 7.27a) with the first point at  $(x, y) = (1, 2)$ , the second at  $(x, y) = (3, 2)$ , etc., and the sixth point on top of the first, we have,

$$[xy \text{ points BEFORE}] \equiv \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix}.$$

After a  $30^\circ$  counter-clockwise rotation about O, the coordinates of the house, in a coordinate system that rotates with the house, are unchanged (Fig. 7.27b). But in the fixed (non-rotating, Newtonian) coordinate system the new coordinates of the rotated house points are,

$$\begin{aligned} [xy \text{ points AFTER}] &= [R][xy \text{ points BEFORE}] = [R][x'y' \text{ points}] \\ &= \begin{bmatrix} \sqrt{3}/2 & -.5 \\ .5 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix} \\ &\approx \begin{bmatrix} -0.1 & 1.6 & 0.6 & -0.8 & -1.1 & -0.1 \\ 2.2 & 3.2 & 5.0 & 5.3 & 4.0 & 2.2 \end{bmatrix} \end{aligned}$$

as shown in Fig. 7.27c.  $\square$

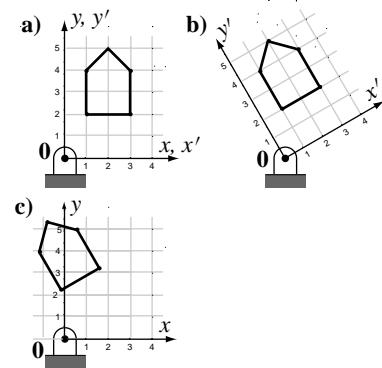


Figure 7.27: a) A house is drawn by connecting lines between 6 points, b) the house and coordinate system are rotated, thus its coordinates in the rotating system do not change c) But the coordinates in the original system do change  
(Filename:tfigure.rotatedhouse)

## Angular velocity of a rigid body: $\vec{\omega}$

Thus far we have talked about rotation, but not how it varies in time. Dynamics is about motion, velocities and accelerations, so we need to think about rotation rates and their rate of change.

In 2D, a rigid body's net rotation is most simply measured by the change that a line marked on the body (any line) makes with a fixed line (any fixed line). We have called this net change of angle  $\theta$ . Thus, the simplest measure of rotation rate is  $\dot{\theta} \equiv \frac{d\theta}{dt}$ . Because all marked lines rotate the same amount they all have the same rates of change, so  $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \text{etc.}$  So the concept of rotation rate of a rigid body, just like the concept of rotation, transcends the concept of rotation rate of this or that line. So we give it a special symbol  $\omega$  (omega),

For all lines marked on a rigid body,

$$\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dots = \dot{\theta}. \quad (7.27)$$

For calculation purposes in 2D, and necessarily in 3D, we think of angular velocity as a vector. Its direction is the axis of the rotation which is  $\hat{k}$  for bodies in the  $xy$  plane. Its scalar part is  $\omega$ . So, the *angular velocity vector* is

$$\vec{\omega} \equiv \omega \hat{k} \quad (7.28)$$

with  $\omega$  as defined in eqn. (7.27).

### Rate of change of $\hat{i}'$ , $\hat{j}'$

Our first use of the angular velocity vector  $\vec{\omega}$  is to calculate the rate of change of rotating unit base vectors. We can find the rate of change of, say,  $\hat{i}'$ , by taking the time derivative of the first of eqn. (7.21), and using the chain rule while recognizing that  $\theta = \theta(t)$ . We can also make an analogy with polar coordinates (page 361), where we think of  $\hat{e}_R$  as like  $\hat{i}'$  and  $\hat{e}_\theta$  as like  $\hat{j}'$ . We found there that  $\dot{\hat{e}}_R = \dot{\theta} \hat{e}_\theta$  and  $\dot{\hat{e}}_\theta = -\dot{\theta} \hat{e}_R$ . Either way,

$$\begin{aligned} \dot{\hat{i}'} &= \dot{\theta} \hat{j}' & \text{or} & \quad \dot{\hat{i}'} = \vec{\omega} \times \hat{i}' & \text{and} \\ \dot{\hat{j}'} &= -\dot{\theta} \hat{i}' & \text{or} & \quad \dot{\hat{j}'} = \vec{\omega} \times \hat{j}' \end{aligned} \quad (7.29)$$

because  $\hat{j}' = \hat{k}' \times \hat{i}'$  and  $\hat{i}' = -\hat{k}' \times \hat{j}'$ . Depending on the tastes of your lecturer, you may find eqn. (7.29) one of the most used equations from this point onward<sup>①</sup>.

### The fixed Newtonian reference frame $\mathcal{F}$

All of mechanics depends on the laws of mechanics. A frame in which Newton's laws are accurate is called a *Newtonian frame*. In engineering practice the frames we use as approximations of a Newtonian frame often seem, loosely speaking, somehow still. So we sometimes call such a frame *the fixed frame* and label it with a script capital  $\mathcal{F}$ . When we talk about velocity and acceleration of mass points, for use in the equations of mechanics, we are always talking about the velocity and acceleration relative to a  $\mathcal{F}$ ixed, or equivalently, Newtonian frame.

Assume  $x$  and  $y$  are the coordinates of a vector  $\vec{r}_P$  and  $\mathcal{F}$  is a fixed frame with fixed axis (with associated constant base vectors  $\hat{i}$  and  $\hat{j}$ ). When we write  $\dot{\vec{r}}_P$  we mean  $\dot{x}\hat{i} + \dot{y}\hat{j}$ . But we could be more explicit (and notationally ornate) and write

$$\frac{\mathcal{F} d \vec{r}_P}{dt} \equiv \dot{\vec{r}}_P \quad \text{by which we mean } \dot{x}\hat{i} + \dot{y}\hat{j}.$$

The  $\mathcal{F}$  in front of the time derivative (or in front of the dot) means that when we calculate a derivative we hold the base vectors of  $\mathcal{F}$  constant. This is no surprise, because for  $\mathcal{F}$  the base vectors *are* constant. In general, however, when taking a derivative in a given frame you

- write vectors in terms of base vectors stuck to the frame, and
- only differentiate the components.

We will avoid the ornate notation of labeling frames when we can. If you don't see any script capital letters floating around in front of derivatives, you can assume that we are taking derivatives relative to a fixed Newtonian frame.

## Velocity of a point fixed on a rigid body

Lets call some rotating body  $\mathcal{B}$  (script capital B) to which is glued a coordinate system  $x'y'$  with base vectors  $\hat{i}'$  and  $\hat{j}'$ . Consider a point P at  $\vec{r}_P$  that is glued to the body. That is, the  $x'$  and  $y'$  coordinates of  $\vec{r}_P$  do not change in time. Using the new frame notation we can write

$$\frac{\mathcal{B}d\vec{r}_P}{dt} \equiv \dot{\vec{r}}_P = \dot{x}'\hat{i}' + \dot{y}'\hat{j}' = \vec{0}.$$

That is, relative to a moving frame, the velocity of a point glued to the frame is zero (no surprise).

We would like to know the velocity of such a point in the fixed frame. We just take the derivative, using the differentiation rules we have developed.

$$\begin{aligned} \vec{r}_P &= x'\hat{i}' + y'\hat{j}' \\ \Rightarrow \vec{v}_P = \dot{\vec{r}}_P &= \frac{d}{dt}(x'\hat{i}' + y'\hat{j}') = x'\dot{\hat{i}}' + y'\dot{\hat{j}}' = x'(\vec{\omega} \times \hat{i}') + y'(\vec{\omega} \times \hat{j}') \\ &= \vec{\omega} \times (x'\hat{i}' + y'\hat{j}') \end{aligned}$$

where  $\vec{r}_P$  is the simple way to write  $\frac{\mathcal{F}d\vec{r}_P}{dt}$ . Thus,

$$\vec{v}_P = \vec{\omega} \times \vec{r}_P \quad (7.30)$$

We can rewrite eqn. (7.30) in a minimalist or elaborate notation as

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r} \quad \text{or} \\ \frac{\mathcal{F}d\vec{r}_P}{dt} &= \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O}. \end{aligned}$$

In the first case you have to use common sense to know what point you are talking about, that you are interested in the velocity of the same point and that it is on a body rotating with absolute angular velocity  $\vec{\omega}$ . In the second case everything is laid out clearly (which is why it looks so confusing). On the left side of the equation it says that we are interested in how point P moves relative to, not just any frame, but the fixed frame  $\mathcal{F}$ . On the right side we make clear that the rotation rate we are looking at is that of body  $\mathcal{B}$  relative to  $\mathcal{F}$  and not some other relative rotation. We further make clear that the formula only makes sense if the position of the point P is measured relative to a point which doesn't move, namely O.

What we have just found largely duplicates what we already learned in section 7.1 for points moving in circles. The slight generalization is that the same angular velocity  $\vec{\omega}$  can be used to calculate the velocities of multiple points on one rigid body. The key idea remains: the velocity of a point going in circles is tangent to the circle it is going around and with magnitude proportional both to distance from the center and the angular rate of rotation (Fig. 7.28a).

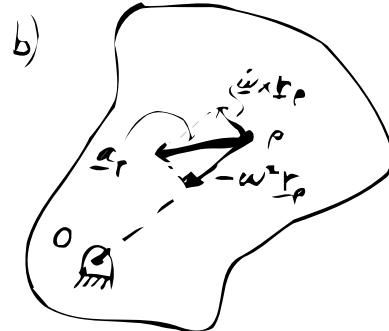
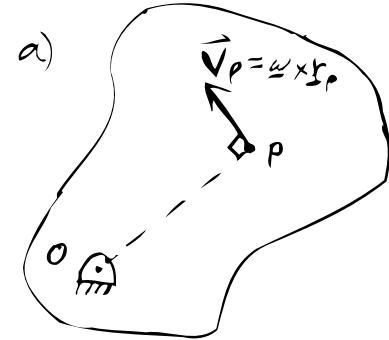


Figure 7.28: Velocity and acceleration of two points on a rigid body rotating about O.  
(Filename:tfigure.velandaccelofp)

## Acceleration of a point on a rotating rigid body

Let's again consider a point with position

$$\vec{r}_P = x' \hat{i}' + y' \hat{j}'.$$

Relative to the frame  $\mathcal{B}$  to which a point is attached, its acceleration is zero (again no surprise). But what is its acceleration in the fixed frame? We find this by writing the position vector and then differentiating twice, repeatedly using the product rule and eqn. (7.29).

Leaving off the ornate pre-super-script  $\mathcal{F}$  for simplicity, we have

$$\begin{aligned} \vec{a}_P &= \dot{\vec{v}}_P &= \frac{d}{dt} \left( \frac{d}{dt} (x' \hat{i}' + y' \hat{j}') \right) \\ &= \frac{d}{dt} (x' (\vec{\omega} \times \hat{i}') + y' (\vec{\omega} \times \hat{j}')). \end{aligned} \quad (7.31)$$

To continue we need to use the product rule of differentiation for the cross product of two time dependent vectors like this:

$$\begin{aligned} \frac{d}{dt} (\vec{\omega} \times \hat{i}') &= \dot{\vec{\omega}} \times \hat{i}' + \vec{\omega} \times \dot{\hat{i}'} &= \dot{\vec{\omega}} \times \hat{i}' + \vec{\omega} \times (\vec{\omega} \times \hat{i}'), \\ \frac{d}{dt} (\vec{\omega} \times \hat{j}') &= \dot{\vec{\omega}} \times \hat{j}' + \vec{\omega} \times \dot{\hat{j}'} &= \dot{\vec{\omega}} \times \hat{j}' + \vec{\omega} \times (\vec{\omega} \times \hat{j}'). \end{aligned} \quad (7.32)$$

Substituting back into eqn. (7.31) we get

$$\begin{aligned} \vec{a}_P &= (x' (\dot{\vec{\omega}} \times \hat{i}' + \vec{\omega} \times (\vec{\omega} \times \hat{i}')) + y' (\dot{\vec{\omega}} \times \hat{j}' + \vec{\omega} \times (\vec{\omega} \times \hat{j}'))) \\ &= \dot{\vec{\omega}} \times (x' \hat{i}' + y' \hat{j}') + \vec{\omega} \times (\vec{\omega} \times (x' \hat{i}' + y' \hat{j}')) \\ &= \dot{\vec{\omega}} \times \vec{r}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P) \end{aligned} \quad (7.33)$$

which is hardly intuitive at a glance<sup>①</sup>. Recalling that in 2D  $\vec{\omega} = \omega \hat{k}$  we can use either the right hand rule or manipulation of unit vectors to rewrite eqn. (7.33) as

$$\vec{a}_P = \dot{\omega} \hat{k} \times \vec{r}_P - \omega^2 \vec{r}_P \quad (7.34)$$

where  $\omega = \dot{\theta}$  and  $\dot{\omega} = \ddot{\theta}$  and  $\theta$  is the counterclockwise rotation of any line marked on the body relative to any fixed line.

Thus, as we found in section 7.1 for a particle going in circles, the acceleration can be written as the sum of two terms, a tangential acceleration  $\dot{\omega} \hat{k} \times \vec{r}_P$  due to increasing tangential speed, and a centrally directed (centripetal) acceleration  $-\omega^2 \vec{r}_P$  due to the direction of the velocity continuously changing towards the center (see Fig. 7.28b). The generalization we have made in this section is that the same  $\vec{\omega}$  can be used to calculate the acceleration for all the different points on one rotating body. A second brief derivation of the acceleration eqn. (7.34) goes like this (using minimalist notation):

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt} (\vec{\omega} \times \vec{r}) = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{r}.$$

## Relative motion of points on a rigid body

As you well know by now, the position of point B relative to point A is  $\vec{r}_{B/A} \equiv \vec{r}_B - \vec{r}_A$ . Similarly the relative velocity and acceleration of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad \text{and} \quad \vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A \quad (7.35)$$

So, the relative velocity (as calculated relative to a fixed frame) of two points glued to one spinning rigid body  $\mathcal{B}$  is given by

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad (7.36)$$

$$= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} \quad (7.37)$$

$$= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) \quad (7.38)$$

$$= \vec{\omega} \times \vec{r}_{B/A}, \quad (7.39)$$

$$(7.40)$$

where point  $O$  is the point in the Newtonian frame on the fixed axis of rotation and  $\vec{\omega} = \vec{\omega}_{\mathcal{B}}$  is the angular velocity of  $\mathcal{B}$ . Clearly, since points A and B are fixed on  $\mathcal{B}$  their velocities and hence their relative velocity as observed in a reference frame fixed to  $\mathcal{B}$  is  $\vec{0}$ . But, point A has some absolute velocity that is different from the absolute velocity of point B. So they have a relative velocity as seen in the fixed frame. And it is what you would expect if B was just going in circles around A. Similarly, the relative acceleration of two points glued to one rigid body spinning at constant rate is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = -\omega^2 \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \quad (7.41)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point  $O$  fixed on the axis. These kinematics results, 11.13 and 11.14, are useful for calculating angular momentum relative to the center of mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

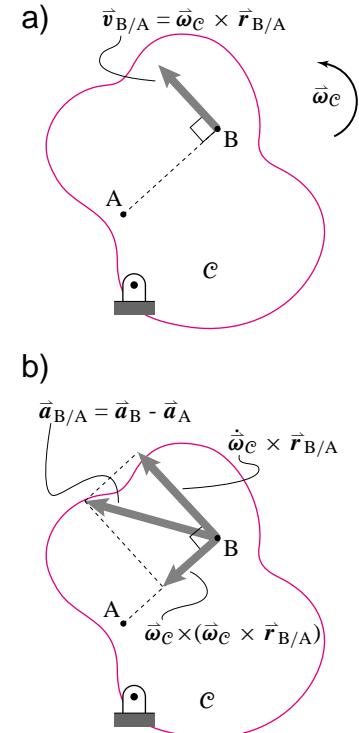


Figure 7.29: Relative velocity and acceleration of two points A and B on the same body  $\mathcal{C}$ .

(Filename:tfigure5.vel.accel.rel)

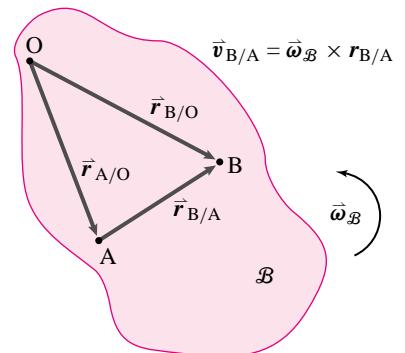


Figure 7.30: The acceleration of B relative to A if they are both on the same rotating rigid body.

(Filename:tfigure4.vel.accel.rel)

## Another definition of $\vec{\omega}$

For two points on one rigid body we have that

$$\dot{\vec{r}}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \quad (7.42)$$

This last equation (11.15) is perhaps the most fundamental equation for those desiring a deeper understanding of rotation. In three dimensions, unless one uses matrix representations of rotation, equation (11.15) is *the* defining equation for the angular velocity  $\vec{\omega}$  of a rigid body.

## Calculating relative velocity directly, using rotating frames

A coordinate system  $x'y'$  to a rotating rigid body  $\mathcal{C}$ , defines a reference frame  $\mathcal{C}$  (Fig. 11.9). Recall, the base vectors in this frame change in time by

$$\frac{d}{dt} \hat{\mathbf{i}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{i}}' \quad \text{and} \quad \frac{d}{dt} \hat{\mathbf{j}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{j}}'.$$

If we now write the relative position of B to A in terms of  $\hat{\mathbf{i}}'$  and  $\hat{\mathbf{j}}'$ , we have

$$\vec{\mathbf{r}}_{B/A} = x' \hat{\mathbf{i}}' + y' \hat{\mathbf{j}}'.$$

Since the coordinates  $x'$  and  $y'$  rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned} \frac{d}{dt}(\vec{\mathbf{r}}_{B/A}) &= \frac{d}{dt}(x' \hat{\mathbf{i}}' + y' \hat{\mathbf{j}}') \\ &= \underbrace{\dot{x}' \hat{\mathbf{i}}'}_0 + x' \frac{d}{dt} \hat{\mathbf{i}}' + \underbrace{\dot{y}' \hat{\mathbf{j}}'}_0 + y' \frac{d}{dt} \hat{\mathbf{j}}' \\ &= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{i}}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{j}}') \\ &= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x' \hat{\mathbf{i}}' + y' \hat{\mathbf{j}}')}_{{\vec{\mathbf{r}}}_{B/A}} \\ &= \vec{\omega}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}. \end{aligned}$$

We could similarly calculate  $\vec{\mathbf{a}}_{B/A}$  by taking another derivative to get

$$\vec{\mathbf{a}}_{B/A} = \vec{\omega}_{\mathcal{C}} \times (\vec{\omega}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}) + \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}.$$

The concept of measuring velocities and accelerations relative to a rotating frame will be of central interest chapters 10 and 11.

### 7.2 Plato's discussion of spinning in circles

*In discussion of an object maintaining contradictory attributes simultaneously ...*

“Socrates: Now let's have a more precise agreement so that we won't have any grounds for dispute as we proceed. If someone were to say of a human being standing still, but moving his hands and head, that the same man at the same time stands still and moves, I don't suppose we'd claim that it should be said like that, but rather that one part of him stands still and another moves. Isn't that so?

Glaucon: Yes it is.

Socrates: Then if the man who says this should become still more charming and make the subtle point that tops as wholes stand still and move at the same time when the peg is fixed in the same place

and they spin, or that anything else going around in a circle on the same spot does this too, we wouldn't accept it because it's not with respect to the same part of themselves that such things are at the same time both at rest and in motion. But we'd say that they have in them both a straight and a circumference; and with respect to the straight they stand still since they don't lean in any direction –while with respect to the circumference they move in a circle; and when the straight inclines to the right the left, forward, or backward at the same time that it's spinning, then in no way does it stand still.

Glaucon: And we'd be right.”

This chapter is about things that are still with respect to their own parts (they do not distort) but in which the points do move in circles.

**SAMPLE 7.9** A uniform bar AB of length  $\ell = 50 \text{ cm}$  rotates counterclockwise about point A with constant angular speed  $\omega$ . At the instant shown in Fig 7.31 the linear speed  $v_C$  of the center of mass C is  $7.5 \text{ cm/s}$ .

- What is the angular speed of the bar?
- What is the angular velocity of the bar?
- What is the linear velocity of end B?
- By what angles do the angular positions of points C and B change in 2 seconds?

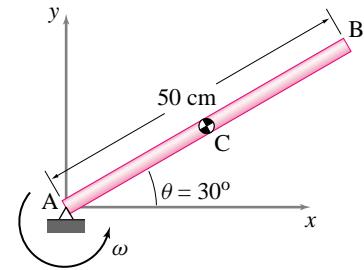


Figure 7.31: (Filename:sfig4.4.1)

**Solution** Let the angular velocity of the bar be  $\vec{\omega} = \dot{\theta}\hat{k}$ .

- Angular speed of the bar =  $\dot{\theta}$ . The linear speed of point C is  $v_C = 7.5 \text{ cm/s}$ . Now,

$$\begin{aligned} v_C &= \dot{\theta} r_C \\ \Rightarrow \dot{\theta} &= \frac{v_C}{r_C} = \frac{7.5 \text{ cm/s}}{25 \text{ cm}} = 0.3 \text{ rad/s}. \end{aligned}$$

$$\boxed{\dot{\theta} = 0.3 \text{ rad/s}}$$

- The angular velocity of the bar is  $\vec{\omega} = \dot{\theta}\hat{k} = 0.3 \text{ rad/s}\hat{k}$ .

$$\boxed{\vec{\omega} = 0.3 \text{ rad/s}\hat{k}}$$

(c)

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B = \dot{\theta}\hat{k} \times \ell(\cos\theta\hat{i} + \sin\theta\hat{j}) \\ &= \dot{\theta}\ell(\cos\theta\hat{j} - \sin\theta\hat{i}) \\ &= 0.3 \text{ rad/s} \cdot 50 \text{ cm} \left( \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i} \right) \\ &= 15 \text{ cm/s} \left( \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i} \right). \end{aligned}$$

$$\boxed{\vec{v}_B = 15 \text{ cm/s} \left( \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i} \right)}$$

We can also write  $\vec{v}_B = 15 \text{ cm/s} \hat{e}_\theta$  where  $\hat{e}_\theta = \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i}$ .

- Let  $\theta_1$  be the position of point C at some time  $t_1$  and  $\theta_2$  be the position at time  $t_2$ . We want to find  $\Delta\theta = \theta_2 - \theta_1$  for  $t_2 - t_1 = 2 \text{ s}$ .

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} = \text{constant} = 0.3 \text{ rad/s}. \\ \Rightarrow \frac{d\theta}{dt} &= (0.3 \text{ rad/s})dt. \\ \Rightarrow \int_{\theta_1}^{\theta_2} d\theta &= \int_{t_1}^{t_2} (0.3 \text{ rad/s})dt. \\ \Rightarrow \theta_2 - \theta_1 &= 0.3 \text{ rad/s}(t_2 - t_1) \\ \text{or } \Delta\theta &= 0.3 \frac{\text{rad}}{\text{s}} \cdot 2 \cancel{\text{s}} = 0.6 \text{ rad}. \end{aligned}$$

The change in position of point B is the same as that of point C. In fact, all points on AB undergo the same change in angular position because AB is a rigid body.

$$\boxed{\Delta\theta_C = \Delta\theta_B = 0.6 \text{ rad}}$$

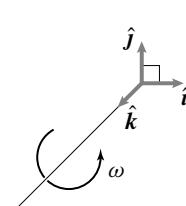
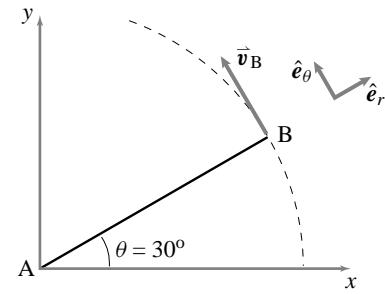


Figure 7.32: (Filename:sfig4.4.1a)



$$\begin{aligned} \hat{e}_r &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ \hat{e}_\theta &= -\sin\theta\hat{i} + \cos\theta\hat{j} \\ \vec{v}_B &= |\vec{v}_B|\hat{e}_\theta \end{aligned}$$

(Filename:sfig4.4.1b)

**SAMPLE 7.10** A flywheel of diameter 2 ft is made of cast iron. To avoid extremely high stresses and cracks it is recommended that the peripheral speed not exceed 6000 to 7000 ft/min. What is the corresponding rpm rating for the wheel?

### Solution

$$\text{Diameter of the wheel} = 2 \text{ ft.}$$

$$\Rightarrow \text{radius of wheel} = 1 \text{ ft.}$$

Now,

$$\begin{aligned} v &= \omega r \\ \Rightarrow \omega &= \frac{v}{r} = \frac{6000 \text{ ft/min}}{1 \text{ ft}} \\ &= 6000 \frac{\text{rad}}{\text{min}} \cdot \frac{1 \text{ rev}}{2\pi \text{ rad}} \\ &= 955 \text{ rpm.} \end{aligned}$$

Similarly, corresponding to  $v = 7000 \text{ ft/min}$

$$\begin{aligned} \omega &= \frac{7000 \text{ ft/min}}{1 \text{ ft}} \\ &= 7000 \frac{\text{rad}}{\text{min}} \cdot \frac{1 \text{ rev}}{2\pi \text{ rad}} \\ &= 1114 \text{ rpm.} \end{aligned}$$

Thus the rpm rating of the wheel should read 955 – 1114 rpm.

$$\boxed{\omega = 955 \text{ to } 1114 \text{ rpm.}}$$

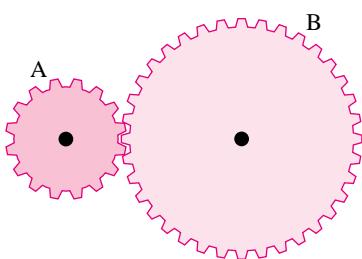


Figure 7.34: (Filename:sfig4.4.3)

**SAMPLE 7.11** Two gears A and B have the diameter ratio of 1:2. Gear A drives gear B. If the output at gear B is required to be 150 rpm, what should be the angular speed of the driving gear? Assume no slip at the contact point.

**Solution** Let C and C' be the points of contact on gear A and B respectively at some instant  $t$ . Since there is no relative slip between C and C', both points must have the same linear velocity at instant  $t$ . If the velocities are the same, then the linear speeds must also be the same. Thus

$$\begin{aligned} v_C &= v_{C'} \\ \Rightarrow \omega_A r_A &= \omega_B r_B \\ \Rightarrow \omega_A &= \omega_B \cdot \frac{r_B}{r_A} \\ &= \omega_B \cdot \frac{2r}{r} = 2\omega_B \\ &= (2) \cdot (150 \text{ rpm}) \\ &= 300 \text{ rpm.} \end{aligned}$$

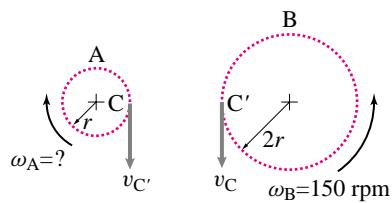


Figure 7.35: (Filename:sfig4.4.3a)

$$\boxed{\omega_A = 300 \text{ rpm}}$$

**SAMPLE 7.12** A uniform rigid rod AB of length  $\ell = 0.6 \text{ m}$  is connected to two rigid links OA and OB. The assembly rotates at a constant rate about point O in the  $xy$  plane. At the instant shown, when rod AB is vertical, the velocities of points A and B are  $\vec{v}_A = -4.64 \text{ m/s} \hat{j} - 1.87 \text{ m/s} \hat{i}$ , and  $\vec{v}_B = 1.87 \text{ m/s} \hat{i} - 4.64 \text{ m/s} \hat{j}$ . Find the angular velocity of bar AB. What is the length R of the links?

**Solution** Let the angular velocity of the rod AB be  $\vec{\omega} = \omega \hat{k}$ . ① Since we are given the velocities of two points on the rod we can use the relative velocity formula to find  $\vec{\omega}$ :

$$\begin{aligned}\vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} = \vec{v}_B - \vec{v}_A \\ \text{or } \underbrace{\vec{\omega} \hat{k} \times \ell \hat{j}}_{\vec{\omega}} \times \underbrace{\vec{r}_{B/A}}_{\ell} &= (1.87 \hat{i} - 4.64 \hat{j}) \text{ m/s} - (-4.64 \hat{j} - 1.87 \hat{i}) \text{ m/s} \\ \text{or } \omega \ell (-\hat{i}) &= (1.87 \hat{i} + 1.87 \hat{i}) \text{ m/s} - (4.64 \hat{j} - 4.64 \hat{j}) \text{ m/s} \\ &= 3.74 \hat{i} \text{ m/s} \\ \Rightarrow \quad \omega &= -\frac{3.74 \text{ m/s}}{\ell} \\ &= -\frac{3.74}{0.6} \text{ rad/s} \\ &= -6.233 \text{ rad/s}\end{aligned}\quad (7.43)$$

Thus,  $\vec{\omega} = -6.233 \text{ rad/s} \hat{k}$ .

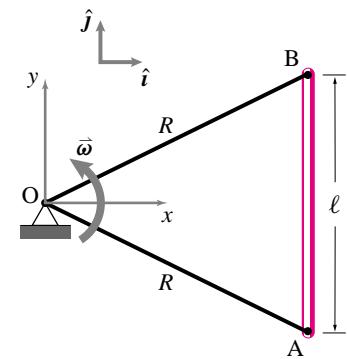


Figure 7.36: (Filename:sfig4.4.4)

① We know that the rod rotates about the  $z$ -axis but we do not know the sense of the rotation i.e.,  $+\hat{k}$  or  $-\hat{k}$ . Here we have assumed that  $\vec{\omega}$  is in the positive  $\hat{k}$  direction, although just by sketching  $\vec{v}_A$  we can easily see that  $\vec{\omega}$  must be in the  $-\hat{k}$  direction.

Let  $\theta$  be the angle between link OA and the horizontal axis. Now,

$$\begin{aligned}\vec{v}_A &= \vec{\omega} \times \vec{r}_A = \omega \hat{k} \times \underbrace{R(\cos \theta \hat{i} - \sin \theta \hat{j})}_{\vec{r}_A} \\ \text{or } (-4.64 \hat{j} - 1.87 \hat{i}) \text{ m/s} &= \omega R(\cos \theta \hat{j} + \sin \theta \hat{i})\end{aligned}$$

Dotting both sides of the equation with  $\hat{i}$  and  $\hat{j}$  we get

$$-1.87 \text{ m/s} = \omega R \sin \theta \quad (7.44)$$

$$-4.64 \text{ m/s} = \omega R \cos \theta \quad (7.45)$$

Squaring and adding Eqns (7.44) and (7.45) together we get

$$\begin{aligned}\omega^2 R^2 &= (-4.64 \text{ m/s})^2 + (-1.87 \text{ m/s})^2 \\ &= 25.026 \text{ m}^2/\text{s}^2 \\ \Rightarrow \quad R^2 &= \frac{25.026 \text{ m}^2/\text{s}^2}{(-6.23 \text{ rad/s})^2} \\ &= 0.645 \text{ m}^2 \\ \Rightarrow \quad R &= 0.8 \text{ m}\end{aligned}$$

$$R = 0.8 \text{ m}$$

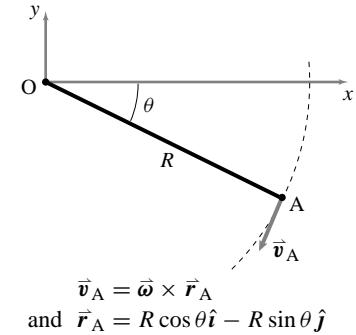


Figure 7.37: (Filename:sfig4.4a)

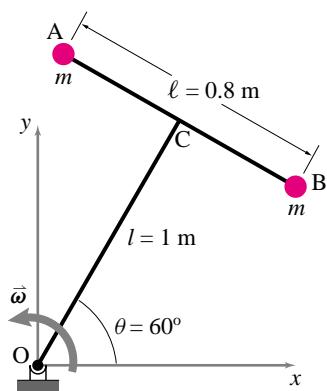


Figure 7.38: (Filename:sfig4.3.1)

**SAMPLE 7.13** A dumbbell AB, made of two equal masses and a rigid rod AB of negligible mass, is welded to a rigid arm OC, also of negligible mass, such that OC is perpendicular to AB. Arm OC rotates about O at a constant angular velocity  $\vec{\omega} = 10 \text{ rad/s} \hat{k}$ . At the instant when  $\theta = 60^\circ$ , find the relative velocity of B with respect to A.

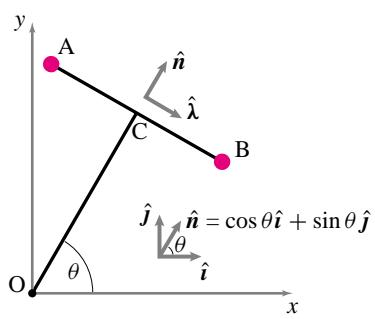


Figure 7.39: (Filename:sfig4.3.1a)

① The vector  $\vec{r}_{B/A}$  may also be expressed directly in terms of unit vectors  $\hat{i}$  and  $\hat{j}$ , but it involves a little bit more geometry. Note how assuming  $\hat{\lambda}$  and  $\hat{n}$  in the directions shown makes calculations easier and cleaner.

Substituting  $\vec{\omega}$  and  $\vec{r}_{B/A}$  in Eqn (7.46) we get

$$\begin{aligned}\vec{v}_{B/A} &= \vec{\omega} \hat{k} \times \ell \hat{\lambda} \\ &= \omega \ell \underbrace{(\hat{k} \times \hat{\lambda})}_{\hat{n}} \\ &= \omega \ell \hat{n} \\ &= \omega \ell (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= 10 \text{ rad/s} (0.8 \text{ m}) \cdot \left( \frac{1}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j} \right) \\ &= 4 \text{ m/s} (\hat{i} + \sqrt{3} \hat{j})\end{aligned}$$

$$\boxed{\vec{v}_{B/A} = 4 \text{ m/s} (\hat{i} + \sqrt{3} \hat{j})}$$

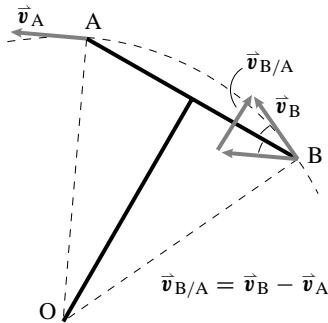


Figure 7.40: (Filename:sfig4.3.1b)

**Comments:**  $\vec{v}_{B/A}$  can also be obtained by adding vectors  $\vec{v}_B$  and  $-\vec{v}_A$  geometrically. Since A and B execute circular motion with the same radius  $R = OA = OB$ , the magnitudes of  $\vec{v}_B$  and  $\vec{v}_A$  are the same ( $= \omega R$ ) and since the velocity in circular motion is tangential to the circular path,  $\vec{v}_A \perp OA$  and  $\vec{v}_B \perp OB$ . Then moving  $\vec{v}_A$  to point B, we can easily find  $\vec{v}_B - \vec{v}_A = \vec{v}_{B/A}$ . Its direction is found to be perpendicular to AB, i.e., along OC.

**SAMPLE 7.14** For the same problem and geometry as in Sample 7.13, find the acceleration of point B relative to point A.

**Solution** Since points A and B are on the same rigid body AB which is rotating at a constant rate  $\omega = 10 \text{ rad/s}$ , the relative acceleration of B is:

$$\begin{aligned}\vec{a}_{B/A} &= \vec{a}_B - \vec{a}_A = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \\ &= \omega \hat{k} \times (\omega \hat{k} \times \ell \hat{\lambda}) \\ &= \omega \hat{k} \times \omega \ell \hat{n} \quad (\text{since } \hat{k} \times \hat{\lambda} = \hat{n}) \\ &= \omega^2 \ell (\hat{k} \times \hat{n}) \\ &= \omega^2 \ell (-\hat{\lambda})\end{aligned}$$

① Now we need to express  $\hat{\lambda}$  in terms of known basis vectors  $\hat{i}$  and  $\hat{j}$ . If you are good with geometry, then by knowing that  $\hat{\lambda} \perp \hat{n}$  and  $\hat{n} = \cos \theta \hat{i} + \sin \theta \hat{j}$  you can immediately write

$$\hat{\lambda} = \sin \theta \hat{i} - \cos \theta \hat{j} \quad (\text{so that } \hat{\lambda} \cdot \hat{n} = 0).$$

Or you may draw a big and clear picture of  $\hat{\lambda}$ ,  $\hat{n}$ ,  $\hat{i}$  and  $\hat{j}$  and label the angles as shown in Fig 7.41. Then, it is easy to see that

$$\hat{\lambda} = \sin \theta \hat{i} - \cos \theta \hat{j}.$$

Substituting for  $\hat{\lambda}$  in the expression for  $\vec{a}_{B/A}$ , we get

$$\begin{aligned}\vec{a}_{B/A} &= -\omega^2 \ell (\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= -100 \frac{\text{rad}^2}{\text{s}^2} \cdot \left[ 0.8 \text{ m} \left( \frac{\sqrt{3}}{2} \hat{i} - \frac{1}{2} \hat{j} \right) \right] \\ &= -40 \text{ m/s}^2 (\sqrt{3} \hat{i} - \hat{j})\end{aligned}$$

$$\boxed{\vec{a}_{B/A} = -40 \text{ m/s}^2 (\sqrt{3} \hat{i} - \hat{j})}$$

**Comments:** We could also find  $\vec{a}_{B/A}$  using geometry and geometric addition of vectors. Since A and B are going in circles about O at constant speed, their accelerations are centripetal accelerations. Thus,  $\vec{a}_A$  points along AO and  $\vec{a}_B$  points along BO. Also  $|\vec{a}_A| = |\vec{a}_B| = \omega^2 (OA)$ . Now adding  $-\vec{a}_A$  to  $\vec{a}_B$  we get  $\vec{a}_{B/A}$  which is seen to be along BA.

① If a body rotates in a plane, i.e.,  $\vec{\omega} = \omega \hat{k}$ , then  $\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 \vec{r}$ . Using this fact we can immediately write  $\vec{a}_{B/A} = -\omega^2 \vec{r}_{B/A} = -\omega^2 \ell \hat{\lambda}$ .

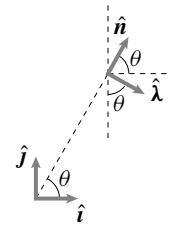


Figure 7.41: The geometry of vectors  $\hat{i}$  and  $\hat{n}$ .  
 $\hat{\lambda} = -\cos \theta \hat{j} + \sin \theta \hat{i}$

(Filename:sfig4.3.2)

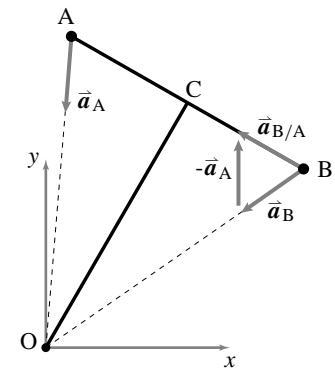


Figure 7.42: (Filename:sfig4.3.2a)

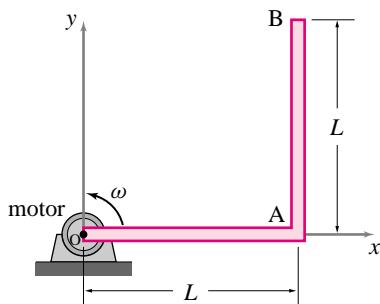


Figure 7.43: An 'L' shaped bar rotates at speed  $\omega$  about point O.

(Filename:fig5.3.1a)

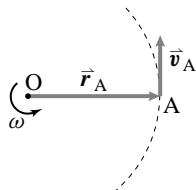


Figure 7.44:  $\vec{v}_A = \vec{\omega} \times \vec{r}_A$  is tangential to the circular path of point A.

(Filename:fig5.3.1b)

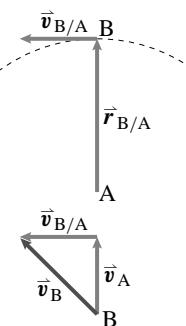


Figure 7.45:  $\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$  and  $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ .

(Filename:fig5.3.1c)

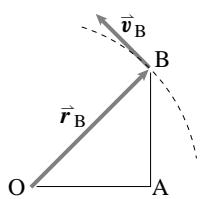


Figure 7.46:  $\vec{v}_B = \vec{\omega} \times \vec{r}_B$ .

(Filename:fig5.3.1d)

**SAMPLE 7.15** Test the velocity formula on something you know. The motor at O in Fig. 7.43 rotates the 'L' shaped bar OAB in counterclockwise direction at an angular speed which increases at  $\dot{\omega} = 2.5 \text{ rad/s}^2$ . At the instant shown, the angular speed  $\omega = 4.5 \text{ rad/s}$ . Each arm of the bar is of length  $L = 2 \text{ ft}$ .

- Find the velocity of point A.
- Find the relative velocity  $\vec{v}_{B/A} (= \vec{\omega} \times \vec{r}_{B/A})$  and use the result to find the absolute velocity of point B ( $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ ).
- Find the velocity of point B directly. Check the answer obtained in part (b) against the new answer.

### Solution

- As the bar rotates, every point on the bar goes in circles centered at point O. Therefore, we can easily find the velocity of any point on the bar using circular motion formula  $\vec{v} = \vec{\omega} \times \vec{r}$ . Thus,

$$\begin{aligned}\vec{v}_A &= \vec{\omega} \times \vec{r}_A = \omega \hat{k} \times L \hat{i} = \omega L \hat{j} \\ &= 4.5 \text{ rad/s} \cdot 2 \text{ ft} \hat{j} = 9 \text{ ft/s} \hat{j}.\end{aligned}$$

The velocity vector  $\vec{v}_A$  is shown in Fig. 7.44.

$$\boxed{\vec{v}_A = 9 \text{ ft/s} \hat{j}}$$

- Point B and A are on the same rigid body. Therefore, with respect to point A, point B goes in circles about A. Hence the relative velocity of B with respect to A is

$$\begin{aligned}\vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\ &= \omega \hat{k} \times L \hat{j} = -\omega L \hat{i} \\ &= -4.5 \text{ rad/s} \cdot 2 \text{ ft} \hat{i} = -9 \text{ ft/s} \hat{i}.\end{aligned}$$

and       $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$

$$\vec{v}_B = 9 \text{ ft/s}(-\hat{i} + \hat{j}).$$

These velocities are shown in Fig. 7.45.

$$\boxed{\vec{v}_{B/A} = -9 \text{ ft/s} \hat{i}, \vec{v}_B = 9 \text{ ft/s}(-\hat{i} + \hat{j})}$$

- Since point B goes in circles of radius OB about point O, we can find its velocity directly using circular motion formula:

$$\begin{aligned}\vec{v}_B &= \vec{\omega} \times \vec{r}_B \\ &= \omega \hat{k} \times (L \hat{i} + L \hat{j}) = \omega L(\hat{j} - \hat{i}) \\ &= 9 \text{ ft/s}(-\hat{i} + \hat{j}).\end{aligned}$$

The velocity vector is shown in Fig. 7.46. Of course this velocity is the same velocity as obtained in part (b) above.

$$\boxed{\vec{v}_B = 9 \text{ ft/s}(-\hat{i} + \hat{j})}$$

**Note:** Nothing in this sample uses  $\dot{\omega}$ !

**SAMPLE 7.16** Test the acceleration formula on something you know. Consider the 'L' shaped bar of Sample 7.15 again. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s<sup>2</sup>.

- Find the acceleration of point A.
- Find the relative acceleration  $\vec{a}_{B/A}$  of point B with respect to point A and use the result to find the absolute acceleration of point B ( $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ ).
- Find the acceleration of point B directly and verify the result obtained in (ii).

**Solution** We are given:

$$\vec{\omega} = \omega \hat{k} = 4 \text{ rad/s} \hat{k}, \quad \text{and} \quad \dot{\omega} = -\dot{\omega} \hat{k} = -2 \text{ rad/s}^2 \hat{k}.$$

- Point A is going in circles of radius L. Hence,

$$\begin{aligned} \vec{a}_A &= \dot{\omega} \times \vec{r}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r}_A) = \dot{\omega} \times \vec{r}_A - \omega^2 \vec{r}_A \\ &= -\dot{\omega} \hat{k} \times L \hat{i} - \omega^2 L \hat{i} = -\dot{\omega} L \hat{j} - \omega^2 L \hat{i} \\ &= -2 \text{ rad/s} \cdot 2 \text{ ft} \hat{j} - (4 \text{ rad/s})^2 \cdot 2 \text{ ft} \hat{i} \\ &= -(4 \hat{j} + 32 \hat{i}) \text{ ft/s}^2. \end{aligned}$$

$$\boxed{\vec{a}_A = -(4 \hat{j} + 32 \hat{i}) \text{ ft/s}^2}$$

- The relative acceleration of point B with respect to point A is found by considering the motion of B with respect to A. Since both the points are on the same rigid body, point B executes circular motion with respect to point A. Therefore,

$$\begin{aligned} \vec{a}_{B/A} &= \dot{\omega} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) = \dot{\omega} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \\ &= -\dot{\omega} \hat{k} \times L \hat{j} - \omega^2 L \hat{j} \\ &= \dot{\omega} L \hat{i} - \omega^2 L \hat{j} = 2 \text{ rad/s}^2 \cdot 2 \text{ ft} \hat{i} - (4 \text{ rad/s})^2 \cdot 2 \text{ ft} \hat{j} \\ &= (4 \hat{i} - 32 \hat{j}) \text{ ft/s}^2, \end{aligned}$$

and

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = (-28 \hat{i} - 36 \hat{j}) \text{ ft/s}^2.$$

$$\boxed{\vec{a}_B = -(28 \hat{i} + 36 \hat{j}) \text{ ft/s}^2}$$

- Since point B is going in circles of radius OB about point O, we can find the acceleration of B as follows.

$$\begin{aligned} \vec{a}_B &= \dot{\omega} \times \vec{r}_B + \vec{\omega} \times (\vec{\omega} \times \vec{r}_B) \\ &= \dot{\omega} \times \vec{r}_B - \omega^2 \vec{r}_B \\ &= -\dot{\omega} \hat{k} \times (L \hat{i} + L \hat{j}) - \omega^2 (L \hat{i} + L \hat{j}) \\ &= (-\dot{\omega} L - \omega^2 L) \hat{j} + (\dot{\omega} L - \omega^2 L) \hat{i} \\ &= (-4 - 32) \text{ ft/s}^2 \hat{j} + (4 - 32) \text{ ft/s}^2 \hat{i} \\ &= (-36 \hat{j} - 28 \hat{i}) \text{ ft/s}^2. \end{aligned}$$

This acceleration is, naturally again, the same acceleration as found in (ii) above.

$$\boxed{\vec{a}_B = -(28 \hat{i} + 36 \hat{j}) \text{ ft/s}^2}$$

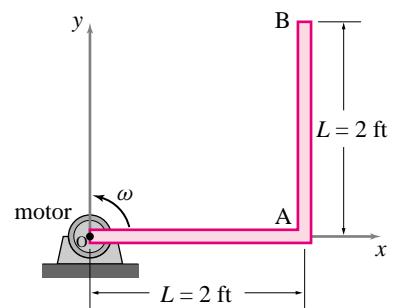


Figure 7.47: The 'L' shaped bar is rotating counterclockwise and is slowing down.  
(Filename:sfig5.3.2a)

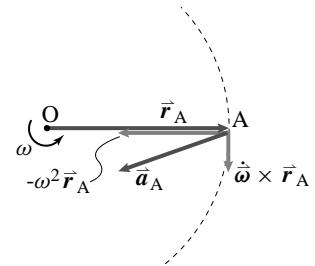


Figure 7.48: (Filename:sfig5.3.2b)

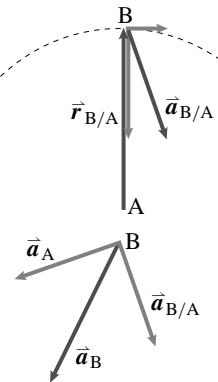


Figure 7.49: (Filename:sfig5.3.2c)

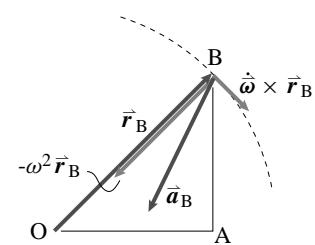


Figure 7.50: (Filename:sfig5.3.2d)

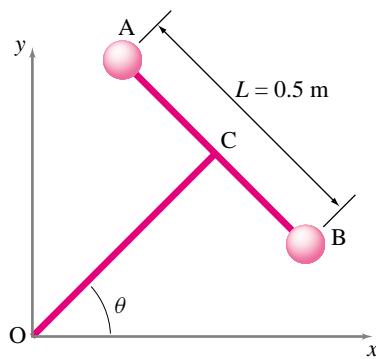


Figure 7.51: Relative velocity and acceleration:

(Filename:fig5.3.3)

**SAMPLE 7.17** The dumbbell AB shown in the figure rotates counterclockwise about point O with angular acceleration  $3 \text{ rad/s}^2$ . Bar AB is perpendicular to bar OC. At the instant of interest,  $\theta = 45^\circ$  and the angular speed is  $2 \text{ rad/s}$ .

- Find the velocity of point B relative to point A. Will this relative velocity be different if the dumbbell were rotating at a *constant* rate of  $2 \text{ rad/s}$ ?
- Without calculations, draw a vector approximately representing the acceleration of B relative to A.
- Find the acceleration of point B relative to A. What can you say about the direction of this vector as the motion progresses in time?

### Solution

- Velocity of B relative to A:**

$$\begin{aligned}\vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\ &= \dot{\theta} \hat{k} \times L(\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= \dot{\theta} L(\sin \theta \hat{j} + \cos \theta \hat{i}) \\ &= 2 \text{ rad/s} \cdot 0.5 \text{ m}(\sin 45^\circ \hat{j} + \cos 45^\circ \hat{i}) \\ &= 0.707 \text{ m/s}(\hat{i} + \hat{j}).\end{aligned}$$

Thus the relative velocity is perpendicular to AB, that is, parallel to OC.

No, the relative velocity will not be any different at the instant of interest if the dumbbell were rotating at constant rate. As is evident from the formula, the relative velocity only depends on  $\vec{\omega}$  and  $\vec{r}_{B/A}$ , and not on  $\dot{\vec{\omega}}$ . Therefore,  $\vec{v}_{B/A}$  will be the same if at the instant of interest,  $\vec{\omega}$  and  $\vec{r}_{B/A}$  are the same.

- Relative acceleration vector:** The velocity and acceleration of some point B on a rigid body *relative* to some other point A on the same body is the same as the velocity and acceleration of B if the body is considered to rotate about point A with the same angular velocity and acceleration as given. Therefore, to find the relative velocity and acceleration of B, we take A to be the center of rotation and draw the circular path of B, and then draw the velocity and acceleration vectors of B.

Since we know that the acceleration of a point under circular motion has tangential ( $\dot{\vec{\omega}} \times \vec{r}$  or  $\dot{\theta} R \hat{e}_R$  in 2-D) and radial or centripetal ( $\vec{\omega} \times (\vec{\omega} \times \vec{r})$  or  $-\dot{\theta}^2 R \hat{e}_R$  in 2-D) components, the total acceleration being the vector sum of these components, we draw an approximate acceleration vector of point B as shown in Fig. 7.52.

- Acceleration of B relative to A:**

$$\begin{aligned}\vec{a}_{B/A} &= \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \\ &= \ddot{\theta} \hat{k} \times L \hat{e}_R + \dot{\theta} \hat{k} \times (\dot{\theta} \hat{k} \times L \hat{e}_R) \\ &= L \ddot{\theta} \hat{e}_R - L \dot{\theta}^2 \hat{e}_R \\ &= 0.5 \text{ m} \cdot 3 \text{ rad/s}^2(\cos 45^\circ \hat{i} + \sin 45^\circ \hat{j}) \\ &\quad - 0.5 \text{ m} \cdot (2 \text{ rad/s})^2(\sin 45^\circ \hat{i} - \cos 45^\circ \hat{j}) \\ &= 1.061 \text{ m/s}^2(\hat{i} + \hat{j}) - 1.414 \text{ m/s}^2(\hat{i} - \hat{j}) \\ &= (-0.353 \hat{i} + 2.474 \hat{j}) \text{ m/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_{B/A} = (-0.353 \hat{i} + 2.474 \hat{j}) \text{ m/s}^2}$$

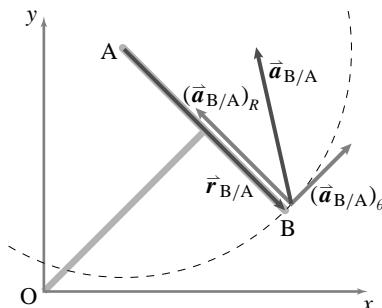


Figure 7.52: To draw the relative acceleration of B,  $\vec{a}_{B/A}$ , consider point B going in circles about point A.

(Filename:fig5.3.3a)

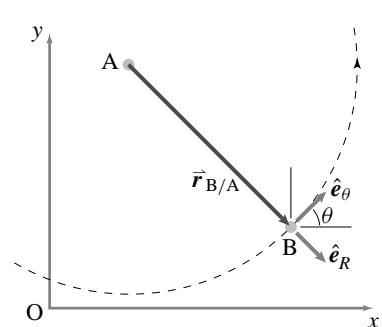


Figure 7.53: The geometry of  $\hat{e}_R$  and  $\hat{e}_\theta$  for the imagined motion of B about A.  $\hat{e}_R = \sin \theta \hat{i} - \cos \theta \hat{j}$  and  $\hat{e}_\theta = \cos \theta \hat{i} + \sin \theta \hat{j}$ .

(Filename:fig5.3.3b)

## 7.4 Dynamics of a rigid body in planar circular motion

Our goal here is to evaluate the terms in the momentum, angular momentum, and energy balance equations for a planar body that is rotating about one point, like a part held in place by a hinge or bearing. The evaluation of forces and moments for use in the momentum and angular momentum equations is the same in statics as in the most complex dynamics, there is nothing new or special about circular motion. What we need to work out are the terms that quantify the motion of mass.

### Mechanics and the motion quantities

If we can calculate the velocity and acceleration of every point in a system, we can evaluate all the momentum and energy terms in the equations of motion (inside cover), namely:  $\vec{L}$ ,  $\dot{\vec{L}}$ ,  $\vec{H}_C$ ,  $\dot{\vec{H}}_C$ ,  $E_K$  and  $\dot{E}_K$  for any reference point C of our choosing. For rotational motion these calculations are a little more complex than the special case of straight-line motion in chapter 6, where all points in a system had the same acceleration as each other.

For circular motion of a rigid body, we just well-learned in the previous section that the velocities and accelerations are

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}), \\ &= \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{r}\end{aligned}$$

where  $\vec{\omega}$  is the angular velocity of the body relative to a fixed frame and  $\vec{r}$  is the position of a point relative to the axis of rotation. These relations apply to every point on a rotating rigid body.

#### Example: Spinning disk

The round flat uniform disk in figure 11.18 is in the  $xy$  plane spinning at the constant rate  $\vec{\omega} = \omega \hat{k}$  about its center. It has mass  $m_{\text{tot}}$  and radius  $R_0$ . What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{F}_i = \dot{\vec{L}} = m_{\text{tot}} \vec{a}_{cm} = \vec{0},$$

Which we could also have calculated by evaluating the integral  $\dot{\vec{L}} \equiv \int \vec{a} dm$  instead of using the general result that  $\dot{\vec{L}} = m_{\text{tot}} \vec{a}_{cm}$ . From angular momentum balance we have:

$$\begin{aligned}\sum \vec{M}_{i/O} &= \dot{\vec{H}}_{/O} \\ \Rightarrow \vec{M} &= \int \vec{r}_{/O} \times \vec{a} dm \\ &= \int_0^{R_0} \int_0^{2\pi} (R \hat{e}_R) \times (-R \omega^2 \hat{e}_R) \underbrace{\frac{m_{\text{tot}}}{\pi R_O^2} \widetilde{R d\theta dR}}_{dm} \\ &= \int \int \vec{0} d\theta dR \\ &= \vec{0}.\end{aligned}$$

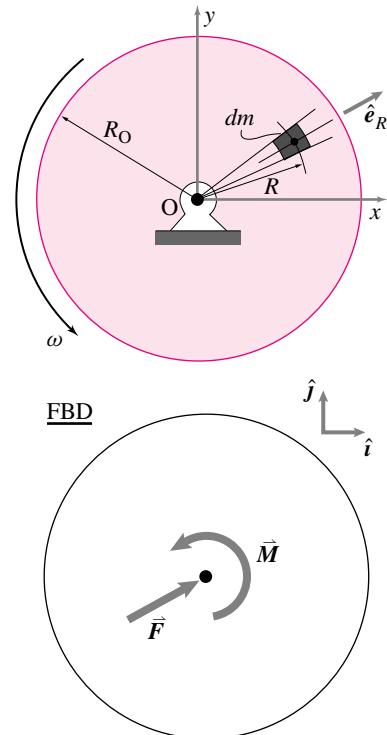


Figure 7.54: A uniform disk turned by a motor at a constant rate.  
(Filename:tfigure4.3.motordisk)

So the net force and moment needed are  $\vec{F} = \vec{0}$  and  $\vec{M} = \vec{0}$ . Like a particle that moves at constant velocity with no force, a uniform disk rotates at constant rate with no torque (at least in 2D).  $\square$

We'd now like to consider the most general case that the subject of the section allows, an arbitrarily shaped 2D rigid body with arbitrary  $\omega$  and  $\dot{\omega}$ .

## Linear momentum

For any system in any motion we know, as we have often used, that

$$\vec{L} = m_{\text{tot}} \vec{v}_{\text{cm}} \quad \text{and} \quad \dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}.$$

For a rigid body, the center of mass is a particular point G that is fixed relative to the body. So the velocity and acceleration of that point can be expressed the same way as for any other point. So, for a body in planar rotational motion about 0

$$\vec{L} = m_{\text{tot}} \vec{\omega} \times \vec{r}_{G/0}$$

and

$$\dot{\vec{L}} = m_{\text{tot}} (\dot{\vec{\omega}} \times \vec{r}_{G/0} + -\omega^2 \vec{r}_{G/0}).$$

If the center of mass is at 0 the momentum and its rate of change are zero. But if the center of mass is off the axis of rotation, there must be a net force on the object with a component parallel to  $\vec{r}_{0/G}$  (if  $\omega \neq 0$ ) and a component orthogonal to  $\vec{r}_{0/G}$  (if  $\dot{\omega} \neq 0$ ). This net force need not be applied at 0 or G or any other special place on the object.

## Angular momentum: $\vec{H}_O$ and $\dot{\vec{H}}_O$

The angular momentum itself is easy enough to calculate,

$$\begin{aligned} \vec{H}_O &= \int_{\text{all mass}} \vec{r} \times \vec{v} dm & \text{(a)} \\ &= \int \vec{r} \times (\vec{\omega} \times \vec{r}) dm & \text{(b)} \\ &= \omega \hat{k} \int r^2 dm & \text{(c)} \\ \Rightarrow H_0 &= \omega \int r^2 dm. & \text{(d)} \end{aligned} \tag{7.47}$$

Here eqn. (7.47)c is the vector equation. But since both sides are in the  $\hat{k}$  direction we can dot both sides with  $\hat{k}$  to get the scalar moment equation eqn. (7.47)d, taking both  $M_{\text{net}}$  and  $\omega$  as positive when counterclockwise.

To get the all important angular momentum balance equation for this system we could easily differentiate eqn. (7.47), taking note that the derivative is being taken

relative to a fixed frame. More reliably, we use the general expression for  $\dot{\vec{H}}_0$  to write the angular momentum balance equation as follows.

$$\begin{aligned}
 \text{Net moment}_0 &= \text{rate of change of angular momentum}_0 & (a) \\
 \vec{M}_{\text{net}} &= \dot{\vec{H}}_0 & (b) \\
 &= \int_{\text{all mass}} \vec{r} \times \vec{a} dm & (c) \\
 &= \int \vec{r} \times (-\omega^2 \vec{r} + \dot{\omega} \hat{k} \times \vec{r}) dm & (d) \\
 &= \int \vec{r} \times (\dot{\omega} \hat{k} \times \vec{r}) dm & (e) \quad (7.48) \\
 &= \int \vec{r} \times (\dot{\omega} \hat{k} \times \vec{r}) dm & (f) \\
 \vec{M}_{\text{net}} &= \dot{\omega} \hat{k} \int r^2 dm & (g) \\
 \Rightarrow M_{\text{net}} &= \dot{\omega} \int r^2 dm & (h)
 \end{aligned}$$

We get from eqn. (7.48)f to eqn. (7.48)g by noting that  $\vec{r}$  is perpendicular to  $\hat{k}$ . Thus, using the right hand rule twice we get  $\vec{r} \times (\hat{k} \times \vec{r}) = r^2 \hat{k}$ .

Eqn. 7.48g and eqn. (7.48)h are the vector and scalar versions of the angular momentum balance equation for rotation of a planar body about 0.

## Power and Energy

Although we could treat distributed forces similarly, lets assume that there are a set of point forces applied. And, to be contrary, lets assume the mass is continuously distributed (the derivation for rigidly connected point masses would be similar). The power balance equation for one rotating rigid body is (discussed below):

$$\begin{aligned}
 \text{Net power in} &= \text{rate of change of kinetic energy} & (a) \\
 P &= \dot{E}_K & (b) \\
 \sum_{\text{all applied forces}} \vec{F}_i \cdot \vec{v}_i &= \frac{d}{dt} \int_{\text{all mass}} \frac{1}{2} v^2 dm & (c) \\
 \sum \vec{F}_i \cdot (\vec{\omega} \times \vec{r}_i) &= \frac{d}{dt} \int \frac{1}{2} (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) dm & (d) \\
 \sum \vec{\omega} \cdot (\vec{r}_i \times \vec{F}_i) &= \frac{d}{dt} \int \frac{1}{2} \omega^2 r^2 dm & (e) \quad (7.49) \\
 \vec{\omega} \cdot \sum (\vec{r}_i \times \vec{F}_i) &= \frac{d}{dt} \left( \frac{1}{2} \omega^2 \right) \int r^2 dm & (f) \\
 \vec{\omega} \cdot \sum \vec{M}_i &= \dot{\omega} \omega \int r^2 dm & (g) \\
 \vec{\omega} \cdot \vec{M}_{\text{tot}} &= \underbrace{\dot{\omega} \cdot \left( \vec{\omega} \int r^2 dm \right)}_{\dot{\vec{H}}_0} & (h)
 \end{aligned}$$

When not noted clearly, positions and moments are relative to the hinge at 0. Derivation 7.49 is two derivations in one. The left side about power and the right side about kinetic energy. Lets discuss one at a time.

On the left side of eqn. (7.49) we note in (c) that the power of each force is the dot product of the force with the velocity of the point it touches. In (d) we use what we know about the velocities of points on rotating rigid bodies. In (e) we use the vector identity  $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A}$  from chapter 2. In (f) we note that  $\vec{\omega}$  is common to all points so factors out of the sum. In (g) we note that  $\vec{r} \times \vec{F}_i$  is the moment of the force

about pt O. And in (g) we sum the moments of the forces. So we find that The power of a set of forces acting on a rigid body is the product of their net moment(about O) and the body angular velocity,

$$P = \vec{\omega} \cdot \vec{M}_{\text{tot}}. \quad (7.50)$$

On the right side of eqn. (7.49) we note in (c) that the kinetic energy is the sum of the kinetic energy of the mass increments. In (d) we use what we know about the velocities of these bits of mass, given that they are on a common rotating body. In (e) we use that the magnitude of the cross product of orthogonal vectors is the product of the magnitudes ( $|\vec{A} \times \vec{B}| = AB$ ) and that the dot product of a vector with itself is its magnitude squared ( $\vec{A} \cdot \vec{A} = A^2$ ). In (f) we factor out  $\omega^2$  because it is common to all the mass increments and note that the remaining integral is constant in time for a rigid body. In (g) we carry out the derivative. In (h) we de-simplify the result from (g) in order to show a more general form that we will find later in 3D mechanics. Eqn. (h) follows from (g) because  $\vec{\omega}$  is parallel to  $\hat{\omega}$  for 2D rotations.

Note that we started here with the basic power balance equation from the front inside cover. Instead, we could have derived power balance from our angular momentum balance expression (see box 7.4 on 398).

### 7.3 THEORY

#### *The relation between angular momentum balance and power balance*

For this system, angular momentum balance can be derived from power balance and vice versa. Thus neither is essentially more fundamental than the other and both are reliable. First we can derive power balance from angular momentum balance as follows:

$$\begin{aligned} \vec{M}_{\text{net}} &= \dot{\omega} \hat{k} \int r^2 dm \\ \vec{\omega} \cdot \vec{M}_{\text{net}} &= \vec{\omega} \cdot \left( \dot{\omega} \hat{k} \int r^2 dm \right). \\ P &= \dot{E}_K \end{aligned} \quad (7.51)$$

That is, when we dot both sides of the angular momentum equation with  $\vec{\omega}$  we get on the left side a term which we recognize as the power of the forces and on the right side a term which is the rate of change of kinetic energy.

The opposite derivation starts with the power balance Fig. 7.49(g)

$$\begin{aligned} \vec{\omega} \cdot \sum \vec{M}_i &= \dot{\omega} \omega \int r^2 dm \quad (g) \\ \Rightarrow \omega \left( \hat{k} \cdot \sum \vec{M}_i \right) &= \dot{\omega} \omega \int r^2 dm \quad (7.52) \\ \Rightarrow \left( \hat{k} \cdot \sum \vec{M}_i \right) &= \dot{\omega} \int r^2 dm \end{aligned}$$

and, assuming  $\omega \neq 0$ , divide by  $\omega$  to get the angular momentum equation for planar rotational motion.

**SAMPLE 7.18** A rod going in circles at constant rate. A uniform rod of mass  $m$  and length  $\ell$  is connected to a motor at end O. A ball of mass  $m$  is attached to the rod at end B. The motor turns the rod in counterclockwise direction at a constant angular speed  $\omega$ . There is gravity pointing in the  $-\hat{j}$  direction. Find the torque applied by the motor (i) at the instant shown and (ii) when  $\theta = 0^\circ, 90^\circ, 180^\circ$ . How does the torque change if the angular speed is doubled?

**Solution** The FBD of the rod and ball system is shown in Fig. 7.56(a). Since the system is undergoing circular motion at a constant speed, the acceleration of the ball as well as every point on the rod is just radial (pointing towards the center of rotation O) and is given by  $\vec{a} = -\omega^2 r \hat{\lambda}$  where  $r$  is the radial distance from the center O to the point of interest and  $\hat{\lambda}$  is a unit vector along OB pointing away from O (Fig. 7.56(b)).

Angular Momentum Balance about point O gives

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\begin{aligned} \sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{j}) + \vec{r}_{B/O} \times (-mg\hat{j}) + M\hat{k} \\ &= -\frac{\ell}{2} \cos \theta mg\hat{k} - \ell \cos \theta mg\hat{k} + M\hat{k} \\ &= (M - \frac{3\ell}{2} mg \cos \theta)\hat{k} \end{aligned} \quad (7.53)$$

$$\begin{aligned} \dot{\vec{H}}_O &= \overbrace{\vec{r}_{B/O} \times m\vec{a}_B}^{\dot{\vec{H}}_{ball/O}} + \overbrace{\int_m \vec{r}_{dm/O} \times \vec{a}_{dm} dm}^{\dot{\vec{H}}_{rod/O}} \\ &= \ell\hat{\lambda} \times (-m\omega^2 \ell\hat{\lambda}) + \int_m \overbrace{l\hat{\lambda}}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 l\hat{\lambda})}^{\vec{a}_{dm}} dm \\ &= \vec{0} \quad (\text{since } \hat{\lambda} \times \hat{\lambda} = \vec{0}) \end{aligned} \quad (7.54)$$

(i) Equating (7.53) and (7.54) we get

$$M = \frac{3}{2}mg\ell \cos \theta.$$

$$M = \frac{3}{2}mg\ell \cos \theta$$

(ii) Substituting the given values of  $\theta$  in the above expression we get

$$M(\theta = 0^\circ) = \frac{3}{2}mg\ell, \quad M(\theta = 90^\circ) = 0 \quad M(\theta = 180^\circ) = -\frac{3}{2}mg\ell$$

$$M(0^\circ) = \frac{3}{2}mg\ell, \quad M(90^\circ) = 0 \quad M(180^\circ) = -\frac{3}{2}mg\ell$$

The values obtained above make sense (at least qualitatively). To make the rod and the ball go up from the  $0^\circ$  position, the motor has to apply some torque in the counterclockwise direction. In the  $90^\circ$  position no torque is required for the *dynamic balance*. In  $180^\circ$  position the system is accelerating downwards under gravity; therefore, the motor has to apply a clockwise torque to make the system maintain a uniform speed.

It is clear from the expression of the torque that it does not depend on the value of the angular speed  $\omega$ ! Therefore, the torque will not change if the speed is doubled. In fact, as long as the speed remains constant at any value, the only torque required to maintain the motion is the torque to counteract the moments at O due to gravity.

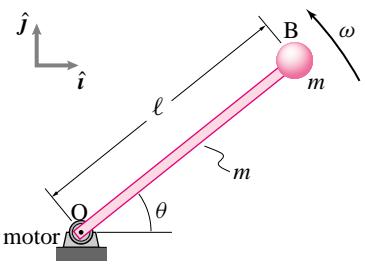
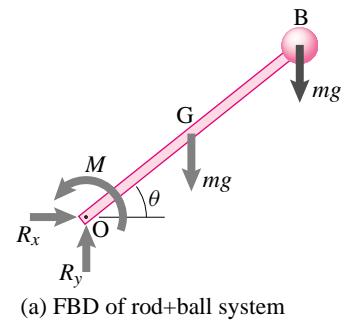
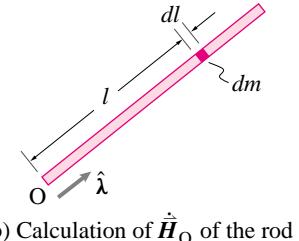


Figure 7.55: A rod goes in circles at a constant rate.  
(Filename:sfig4.5.5)



(a) FBD of rod+ball system



(b) Calculation of  $\dot{\vec{H}}_O$  of the rod

Figure 7.56: A rod goes in circles at a constant rate.  
(Filename:sfig4.5.5a)

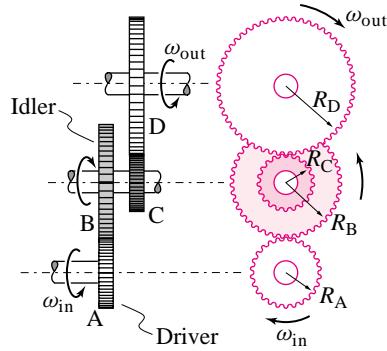


Figure 7.57: A compound gear train.

(Filename:sfig4.5.7)

**SAMPLE 7.19** A *compound gear train*. When the gear of an input shaft, often called the *driver* or the *pinion*, is directly meshed in with the gear of an output shaft, the motion of the output shaft is opposite to that of the input shaft. To get the output motion in the same direction as that of the input motion, an *idler* gear is used. If the idler shaft has more than one gear in mesh, then the gear train is called a *compound gear train*.

In the gear train shown in Fig. 7.57, the input shaft is rotating at 2000 rpm and the input torque is 200 N·m. The efficiency (defined as the ratio of output power to input power) of the train is 0.96 and the various radii of the gears are:  $R_A = 5 \text{ cm}$ ,  $R_B = 8 \text{ cm}$ ,  $R_C = 4 \text{ cm}$ , and  $R_D = 10 \text{ cm}$ . Find

- the input power  $P_{in}$  and the output power  $P_{out}$ ,
- the output speed  $\omega_{out}$ , and
- the output torque.

### Solution

- The power:

$$\begin{aligned} P_{in} &= M_{in}\omega_{in} = 200 \text{ N}\cdot\text{m} \cdot 2000 \text{ rpm} \\ &= 400000 \text{ N}\cdot\text{m} \cdot \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi}{1 \text{ rev}} \cdot \frac{1 \text{ min}}{60 \text{ s}} \\ &= 41887.9 \text{ N m/s} \approx 42 \text{ kW}. \\ \Rightarrow P_{out} &= \text{efficiency} \cdot P_{in} = 0.96 \cdot 42 \text{ kW} \approx 40 \text{ kW} \end{aligned}$$

$$P_{in} = 42 \text{ kW}, \quad P_{out} = 40 \text{ kW}$$

- The angular speed of meshing gears can be easily calculated by realizing that the linear speed of the point of contact has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus,

$$\begin{aligned} v_P &= \omega_{in} R_A = \omega_B R_B \\ \Rightarrow \omega_B &= \omega_{in} \cdot \frac{R_A}{R_B} \\ \text{and } v_R &= \omega_C R_C = \omega_{out} R_D \\ \Rightarrow \omega_{out} &= \omega_C \cdot \frac{R_C}{R_D} \\ \text{But } \omega_C &= \omega_B \\ \Rightarrow \omega_{out} &= \omega_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\ &= 2000 \text{ rpm} \cdot \frac{5}{8} \cdot \frac{4}{10} = 500 \text{ rpm}. \end{aligned}$$

$$\omega_{out} = 500 \text{ rpm}$$

- The output torque,

$$\begin{aligned} M_{out} &= \frac{P_{out}}{\omega_{out}} = \frac{40 \text{ kW}}{500 \text{ rpm}} \\ &= \frac{40}{500} \cdot 1000 \frac{\text{N}\cdot\text{m}}{\text{s}} \cdot \frac{\text{min}}{\text{rev}} \cdot \frac{1 \text{ rev}}{2\pi} \cdot \frac{60 \text{ s}}{1 \text{ min}} \\ &= 764 \text{ N}\cdot\text{m}. \end{aligned}$$

$$M_{out} = 764 \text{ N}\cdot\text{m}$$

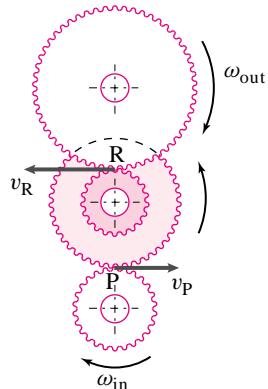


Figure 7.58: (Filename:sfig4.5.7a)

**SAMPLE 7.20** At the onset of motion: A  $2' \times 4'$  rectangular plate of mass 20 lbm is pivoted at one of its corners as shown in the figure. The plate is released from rest in the position shown. Find the force on the support immediately after release.

**Solution** The free body diagram of the plate is shown in Fig. 7.60. The force  $\vec{F}$  applied on the plate by the support is unknown.

The linear momentum balance for the plate gives

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_G \\ \vec{F} - mg\hat{j} &= m(\ddot{\theta}r_{G/O}\hat{e}_\theta - \dot{\theta}^2 R\hat{e}_R) \\ &= m\ddot{\theta}r_{G/O}\hat{e}_\theta \quad (\text{since } \dot{\theta} = 0 \text{ at } t = 0)\end{aligned}\quad (7.55)$$

Thus to find  $\vec{F}$  we need to find  $\ddot{\theta}$ .

The angular momentum balance for the plate about the fixed support point O gives

$$\begin{aligned}\vec{M}_O &= \dot{\vec{H}}_O \\ \vec{r}_{G/O} \times mg(-\hat{j}) &= I_{zz/O} \ddot{\theta} \hat{k} \\ (\underbrace{\frac{a}{2}\hat{i} - \frac{b}{2}\hat{j}}_{\vec{r}_{G/O}}) \times mg(-\hat{j}) &= \underbrace{\left[ \frac{m(a^2 + b^2)}{12} + m\left(\frac{a^2}{4} + \frac{b^2}{4}\right) \right] \ddot{\theta} \hat{k}}_{\text{parallel axis theorem}} \\ -mg\frac{a}{2}\hat{k} &= \frac{m(a^2 + b^2)}{3}\ddot{\theta}\hat{k} \\ \Rightarrow \ddot{\theta} &= -\frac{3g a}{2(a^2 + b^2)} \\ &= -\frac{3 \cdot 32.2 \text{ ft/s}^2 \cdot 4 \text{ ft}}{2(16 + 4) \text{ ft}^2} \\ &= -9.66 \text{ rad/s}^2.\end{aligned}$$

Substituting this value of  $\ddot{\theta}$  in eqn. (7.55), we get

$$\begin{aligned}\vec{F} &= mg\hat{j} + m\ddot{\theta}r_{G/O}\hat{e}_\theta \\ &= 20 \text{ lbm} \cdot 32.2 \text{ ft/s}^2 \hat{j} + 20 \text{ lbm} \cdot (-9.66 \text{ rad/s}^2) \cdot (\underbrace{\sqrt{2^2 + 1^2} \text{ ft}}_{r_{G/O}}) \underbrace{\frac{(2\hat{i} + 4\hat{j})}{\sqrt{20}}}_{\hat{e}_\theta} \\ &= 20 \text{ lbf} \hat{j} - \frac{20 \cdot 9.66}{32.2}(1\hat{i} + 2\hat{j}) \text{ lbf} \quad (\text{since } 1 \text{ lbm} \cdot \text{ft/s}^2 = \frac{1}{32.2} \text{ lbf}) \\ &= (-3\hat{i} + 14\hat{j}) \text{ lbf.}\end{aligned}$$

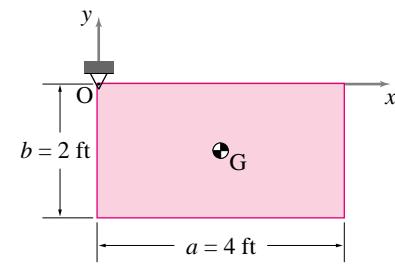
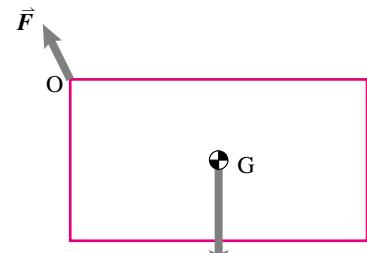
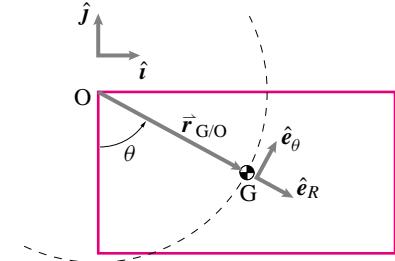


Figure 7.59: A rectangular plate is released from rest from the position shown.  
(Filename:sfig5.4.3)



(a) Free body diagram



(b) Geometry of motion

Figure 7.60: (a) The free body diagram of the plate. (b) The geometry of motion. From the given dimensions,  $\hat{e}_R = \frac{a\hat{i} - b\hat{j}}{\sqrt{a^2 + b^2}}$  and  $\hat{e}_\theta = \frac{b\hat{i} + a\hat{j}}{\sqrt{a^2 + b^2}}$ .  
(Filename:sfig5.4.3a)

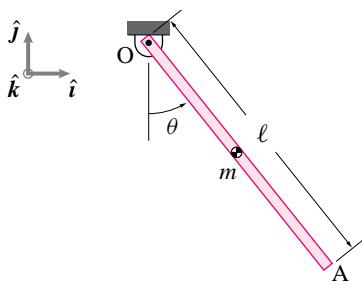


Figure 7.61: A uniform rod swings in the plane about its pinned end O.

(Filename:sfig5.4.1a)

**SAMPLE 7.21** *The swinging stick.* A uniform bar of mass  $m$  and length  $\ell$  is pinned at one of its ends O. The bar is displaced from its vertical position by an angle  $\theta$  and released (Fig. 7.61).

- Find the equation of motion using momentum balance.
- Find the reaction at O as a function of  $(\theta, \dot{\theta}, g, m, \ell)$ .

**Solution** First we draw a simple sketch of the given problem showing relevant geometry (Fig. 7.61(a)), and then a free body diagram of the bar (Fig. 7.61(b)).

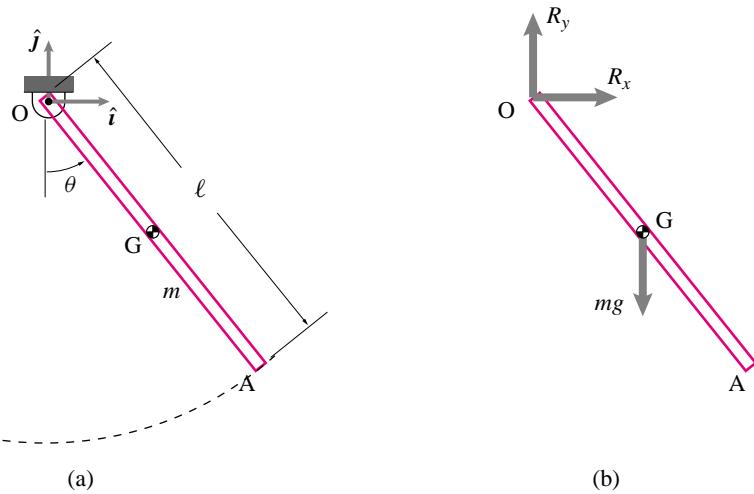


Figure 7.62: (a) A line sketch of the swinging rod and (b) free body diagram of the rod.

(Filename:sfig5.4.1b)

We should note for future reference that

$$\begin{aligned}\vec{\omega} &= \omega \hat{k} = \dot{\theta} \hat{k} \\ \vec{\dot{\omega}} &= \dot{\omega} \hat{k} = \ddot{\theta} \hat{k}\end{aligned}$$

- Equation of motion using momentum balance:** We can write angular momentum balance about point O as

$$\sum \vec{M}_O = \dot{\vec{H}}_O.$$

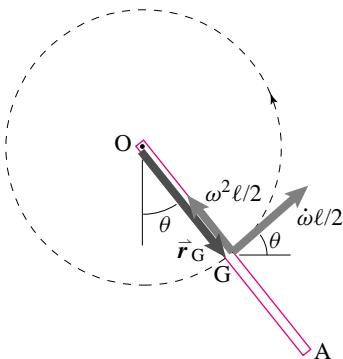
Let us now calculate both sides of this equation:

$$\begin{aligned}\sum \vec{M}_O &= \vec{r}_{G/O} \times mg(-\hat{j}) \\ &= \frac{\ell}{2}(\sin \theta \hat{i} - \cos \theta \hat{j}) \times mg(-\hat{j}) \\ &= -\frac{\ell}{2}mg \sin \theta \hat{k}. \quad (7.56)\end{aligned}$$

$$\begin{aligned}\dot{\vec{H}}_O &= I_{zz/G} \vec{\dot{\omega}} + \vec{r}_G \times m \vec{a}_G \\ &= \frac{m\ell^2}{12} \dot{\omega} \hat{k} + \vec{r}_G \times m \overbrace{(\dot{\omega} \hat{k} \times \vec{r}_G - \omega^2 \vec{r}_G)}^{\vec{a}_G} \\ &= \frac{m\ell^2}{12} \dot{\omega} \hat{k} + \frac{m\ell^2}{4} \dot{\omega} \hat{k}, = \frac{m\ell^2}{3} \dot{\omega} \hat{k} \quad (7.57)\end{aligned}$$

Figure 7.63: Radial and tangential components of  $\vec{a}_G$ . Since the radial component is parallel to  $\vec{r}_G$ ,  $\vec{r}_G \times \vec{a}_G = \frac{\ell^2}{4} \dot{\omega} \hat{k}$ .

(Filename:sfig5.4.1c)



where the last step,  $\vec{r}_G \times m\vec{a}_G = \frac{m\ell^2}{4}\dot{\omega}\hat{k}$ , should be clear from Fig. 7.63.  
Equating (7.56) and (7.57) we get

$$\begin{aligned} -\frac{\ell}{2}\eta g \sin \theta &= \eta \frac{\ell^2}{3}\dot{\omega} \\ \text{or } \dot{\omega} + \frac{3g}{2\ell} \sin \theta &= 0 \\ \text{or } \ddot{\theta} + \frac{3g}{2\ell} \sin \theta &= 0. \end{aligned} \quad (7.58)$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell} \sin \theta = 0}$$

(b) **Reaction at O:** Using linear momentum balance

$$\begin{aligned} \sum \vec{F} &= m \vec{a}_G, \\ \text{where } \sum \vec{F} &= R_x \hat{i} + (R_y - mg) \hat{j}, \\ \text{and } \vec{a}_G &= \frac{\ell}{2} \dot{\omega} (\cos \theta \hat{i} + \sin \theta \hat{j}) + \frac{\ell}{2} \omega^2 (-\sin \theta \hat{i} + \cos \theta \hat{j}) \\ &= \frac{\ell}{2} [(\dot{\omega} \cos \theta - \omega^2 \sin \theta) \hat{i} + (\dot{\omega} \sin \theta + \omega^2 \cos \theta) \hat{j}]. \end{aligned}$$

Dotting both sides of  $\sum \vec{F} = m \vec{a}_G$  with  $\hat{i}$  and  $\hat{j}$  and rearranging, we get

$$\begin{aligned} R_x &= m \frac{\ell}{2} (\dot{\omega} \cos \theta - \omega^2 \sin \theta) \\ &\equiv m \frac{\ell}{2} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta), \\ R_y &= mg + m \frac{\ell}{2} (\dot{\omega} \sin \theta + \omega^2 \cos \theta) \\ &\equiv mg + m \frac{\ell}{2} (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta). \end{aligned}$$

Now substituting the expression for  $\ddot{\theta}$  from (7.58) in  $R_x$  and  $R_y$ , we get

$$R_x = -m \sin \theta \left( \frac{3}{4}g \cos \theta + \frac{\ell}{2} \dot{\theta}^2 \right), \quad (7.59)$$

$$R_y = mg \left( 1 - \frac{3}{4} \sin^2 \theta \right) + m \frac{\ell}{2} \dot{\theta}^2 \cos \theta. \quad (7.60)$$

$$\boxed{\vec{R} = -m \left( \frac{3}{4}g \cos \theta + \frac{\ell}{2} \dot{\theta}^2 \right) \sin \theta \hat{i} + [mg \left( 1 - \frac{3}{4} \sin^2 \theta \right) + m \frac{\ell}{2} \dot{\theta}^2 \cos \theta] \hat{j}}$$

**Check:** We can check the reaction force in the special case when the rod does not swing but just hangs from point O. The forces on the bar in this case have to satisfy static equilibrium. Therefore, the reaction at O must be equal to  $mg$  and directed vertically upwards. Plugging  $\theta = 0$  and  $\dot{\theta} = 0$  (no motion) in Eqn. (7.59) and (7.60) we get  $R_x = 0$  and  $R_y = mg$ , the values we expect.

**SAMPLE 7.22** *The swinging stick: energy balance.* Consider the same swinging stick as in Sample 7.21. The stick is, again, displaced from its vertical position by an angle  $\theta$  and released (See Fig. 7.61).

- Find the equation of motion using energy balance.
- What is  $\dot{\theta}$  at  $\theta = 0$  if  $\theta(t = 0) = \pi/2$ ?
- Find the period of small oscillations about  $\theta = 0$ .

### Solution

(a) **Equation of motion using energy balance:** We use the power equation,  $\dot{E}_K = P$ , to derive the equation of motion of the bar.

$$E_K = \frac{1}{2} I_{zz/G} \omega^2 + \frac{1}{2} m v_G^2$$

where  $\frac{1}{2} I_{zz/G} \omega^2$  = kinetic energy of the bar due to rotation about the  $z$ -axis passing through the mass center G,

and  $\frac{1}{2} m v_G^2$  = kinetic energy of the bar due to translation of the mass center.

But  $v_G = \omega r_G = \omega \frac{\ell}{2}$ . Therefore,

$$E_K = \frac{1}{2} \frac{m\ell^2}{12} \omega^2 + \frac{1}{2} m \omega^2 \frac{\ell^2}{4} = \frac{1}{6} m \ell^2 \omega^2,$$

and

$$\dot{E}_K = \frac{d}{dt} \left( \frac{1}{6} m \ell^2 \omega^2 \right) = \frac{1}{3} m \ell^2 \omega \dot{\omega} = \frac{1}{3} m \ell^2 \dot{\theta} \ddot{\theta}.$$

**Calculation of power ( $P$ ):** There are only two forces acting on the bar, the reaction force,  $\vec{R} (= R_x \hat{i} + R_y \hat{j})$  and the force due to gravity,  $-mg \hat{j}$ . Since the support point O does not move, no work is done by  $\vec{R}$ . Therefore,

$$\begin{aligned} W &= \text{Work done by gravity force in moving from } G' \text{ to } G \\ &= -mgh \end{aligned}$$

Note that the negative sign stands for the work done *against* gravity. Now,

$$h = OG' - OG'' = \frac{\ell}{2} - \frac{\ell}{2} \cos \theta = \frac{\ell}{2}(1 - \cos \theta).$$

Therefore,

$$W = -mg \frac{\ell}{2}(1 - \cos \theta)$$

$$\text{and } P = \dot{W} = \frac{dW}{dt} = -mg \frac{\ell}{2} \sin \theta \dot{\theta}.$$

Equating  $\dot{E}_K$  and  $P$  we get

$$-mg \frac{\ell}{2} \sin \theta \dot{\theta} = \frac{1}{3} m \ell^2 \dot{\theta} \ddot{\theta}$$

$$\text{or } \ddot{\theta} + \frac{3g}{2\ell} \sin \theta = 0.$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell} \sin \theta = 0}$$

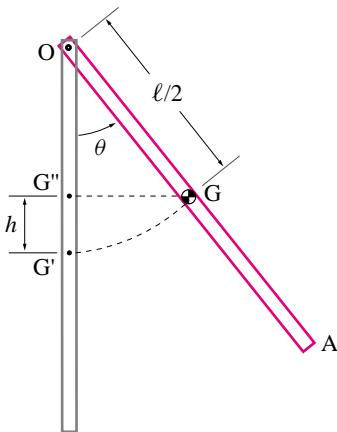


Figure 7.64: Work done by the force of gravity in moving from  $G'$  to  $G$   $\int \vec{F} \cdot d\vec{r} = -mg \hat{j} \cdot h \hat{j} = -mgh$ .

(Filename:sfig5.4.2a)

This equation is, of course, the same as we obtained using balance of angular momentum in Sample 7.21.

- (b) **Find  $\omega$  at  $\theta = 0$ :** We are given that at  $t = 0$ ,  $\theta = \pi/2$  and  $\dot{\theta} \equiv \omega = 0$  (released from rest). This position is (1) shown in Fig. 7.65. In position (2)  $\theta = 0$ , i.e., the rod is vertical. Since there are no dissipative forces, the total energy of the system remains constant. Therefore, taking datum for potential energy as shown in Fig. 7.65, we may write

$$\begin{aligned} \underbrace{E_{K1}}_0 + V_1 &= E_{K2} + \underbrace{V_2}_0 \\ \text{or } mg \frac{\ell}{2} &= \frac{1}{2} I_{zz/G} \omega^2 + \frac{1}{2} m v_G^2 \\ &= \frac{1}{2} \frac{m\ell^2}{12} \omega^2 + \frac{1}{2} m \overbrace{\frac{\ell}{2}}^{v_G} \omega^2 \\ &= \frac{1}{6} m\ell^2 \omega^2 \\ \Rightarrow \quad \omega &= \pm \sqrt{\frac{3g}{\ell}} \end{aligned}$$

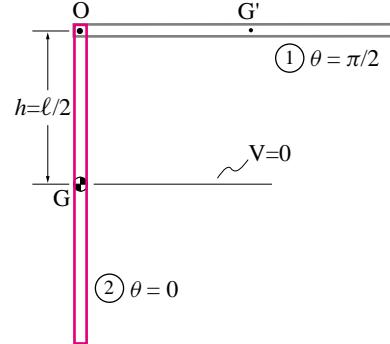


Figure 7.65: The total energy between positions (1) and (2) is constant.

(Filename:sfig5.4.2b)

$$\boxed{\omega = \pm \sqrt{\frac{3g}{\ell}}}$$

- (c) **Period of small oscillations:** The equation of motion is

$$\ddot{\theta} + \frac{3g}{2\ell} \sin \theta = 0.$$

For small  $\theta$ ,  $\sin \theta \approx \theta$

$$\begin{aligned} \Rightarrow \quad \ddot{\theta} + \frac{3g}{2\ell} \theta &= 0 & (7.61) \\ \text{or} \quad \ddot{\theta} + \lambda^2 \theta &= 0 \\ \text{where} \quad \lambda^2 &= \frac{3g}{2\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{the circular frequency} &= \lambda = \sqrt{\frac{3g}{2\ell}}, \\ \text{and the time period } T &= \frac{2\pi}{\lambda} = 2\pi \sqrt{\frac{2\ell}{3g}}. \end{aligned}$$

$$\boxed{T = 2\pi \sqrt{\frac{2\ell}{3g}}}$$

[Say for  $g = 9.81 \text{ m/s}^2$ ,  $\ell = 1 \text{ m}$  we get  $\frac{T}{4} = \frac{\pi}{2} \sqrt{\frac{2}{3} \frac{1}{9.81}} \text{ s} = 0.4097 \text{ s}$ ]

**SAMPLE 7.23** *The swinging stick: numerical solution of the equation of motion.* For the swinging stick considered in Samples 7.21 or 7.22, find the time that the rod takes to fall from  $\theta = \pi/2$  to  $\theta = 0$  if it is released from rest at  $\theta = \pi/2$ ?

**Solution**  $\pi/2$  is a big value of  $\theta$  – big in that we cannot assume  $\sin \theta \approx \theta$  (obviously  $1 \neq 1.5708$ ). Therefore we may not use the linearized equation (7.61) to solve for  $t$  explicitly. We have to solve the full nonlinear equation (7.58) to find the required time. Unfortunately, we cannot get a *closed form* solution of this equation using mathematical skills you have at this level. Therefore, we resort to numerical integration of this equation.

Here, we show how to do this integration and find the required time using the numerical solution. We assume that we have some numerical ODE solver, say `odesolver`, available to us that will give us the numerical solution given appropriate input.

The first step in numerical integration is to set up the given differential equation of second or higher order as a set of first order ordinary differential equations. To do so for Eqn. (7.58), we introduce  $\omega$  as a new variable and write

$$\dot{\theta} = \omega \quad (7.62)$$

$$\dot{\omega} = -\frac{3g}{2\ell} \sin \theta \quad (7.63)$$

Thus, the second order ODE (7.58) has been rewritten as a set of two first order ODE's (7.62) and (7.63). We may write these first order equations in vector form by assuming  $\mathbf{z} = [\theta \ \omega]^T$ . That is,

$$\begin{aligned} \mathbf{z} &= \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} \theta \\ \omega \end{Bmatrix} \\ \Rightarrow \quad \dot{\mathbf{z}} &= \begin{Bmatrix} \dot{\theta} \\ \dot{\omega} \end{Bmatrix} = \begin{Bmatrix} \theta \\ -\frac{3g}{2\ell} \sin z_1 \end{Bmatrix} \end{aligned}$$

To use any numerical integrator, we usually need to write a small program which will compute and return the value of  $\mathbf{z}$  as output if  $t$  and  $\mathbf{z}$  are supplied as input. Here is such a program written in pseudo-code, for our equations.

```

g = 9.81           % define constants
L = 1

ODES = { z1dot = z2
        z2dot = -3*g/(2*L) * sin(z1) }
ICS = { z1zero = pi/2
        z2zero = 0 }

solve ODES with ICS until t = 4.

plot(t,z)          % plots t vs. theta
                    % and t vs. omega together
xlabel('t'), ylabel('theta and omega') % label axes

```

The results obtained from the numerical solution are shown in Fig. 7.66.

The problem of finding the time taken by the bar to fall from  $\theta = \pi/2$  to  $\theta = 0$  numerically is nontrivial. It is called a *boundary value problem*. We have only illustrated how to solve *initial value problems*. However, we can get fairly good estimate of the time just from the solution obtained. We first plot  $\theta$  against time  $t$

to get the graph shown in Fig. 7.67. We find the values of  $t$  and the corresponding values of  $\theta$  that bracket  $\theta = 0$ . Now, we can use linear interpolation to find the value of  $t$  at  $\theta = 0$ . Proceeding this way, we get  $t = 0.4833$  (seconds), a little more than we get from the linear ODE in sample 7.22 of 0.40975. Additionally, we can get by interpolation that at  $\theta = 0$

$$\omega = -5.4246 \text{ rad/s.}$$

How does this result compare with the analytical value of  $\omega$  from sample 7.22 (which did not depend on the small angle approximates)? Well, we found that

$$\omega = -\sqrt{\frac{3g}{\ell}} = -\sqrt{\frac{3 \cdot 9.81 \text{ m/s}^2}{1 \text{ m}}} = -5.4249 \text{ s}^{-1}.$$

Thus, we get a fairly accurate value from numerical integration!

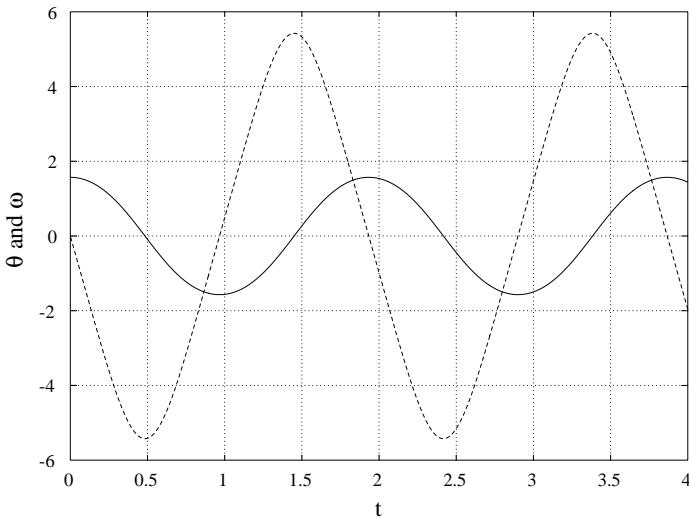


Figure 7.66: Numerical solution is shown by plotting  $\theta$  and  $\omega$  against time. (Filename:sfig5.4.4a)

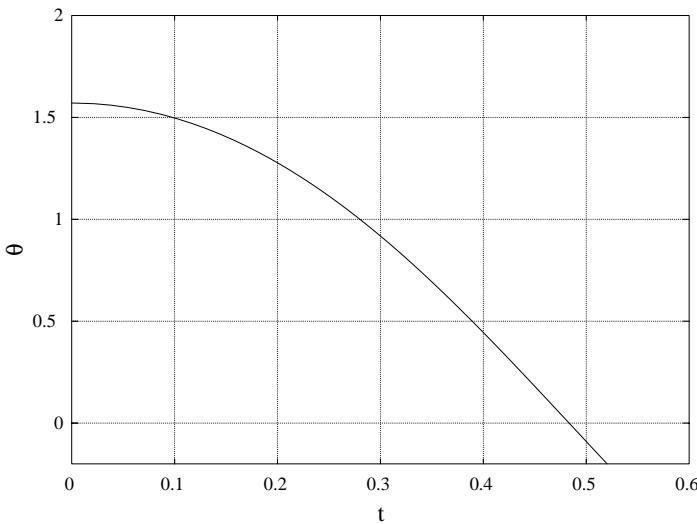


Figure 7.67: Graphic output of the plot command (Filename:sfig5.4.4b)

**SAMPLE 7.24** *The swinging stick with a destabilizing torque.* Consider the swinging stick of Sample 7.21 once again.

- Find the equation of motion of the stick, if a torque  $\vec{M} = M\hat{k}$  is applied at end O and a force  $\vec{F} = F\hat{i}$  is applied at the other end A.
- Take  $F = 0$  and  $M = C\theta$ . For  $C = 0$  you get the equation of free oscillations obtained in Sample 7.21 or 7.22. For small  $C$ , does the period of the pendulum increase or decrease?
- What happens if  $C$  is big?

### Solution

- A free body diagram of the bar is shown in Fig. 7.68. Once again, we can use  $\sum \vec{M}_O = \dot{\vec{H}}_O$  to derive the equation of motion as in Sample 7.21. We calculated  $\sum \vec{M}_O$  and  $\dot{\vec{H}}_O$  in Sample 7.21. Calculation of  $\dot{\vec{H}}_O$  remains the same in the present problem. We only need to recalculate  $\sum \vec{M}_O$ .

$$\begin{aligned}\sum \vec{M}_O &= M\hat{k} + \vec{r}_{G/O} \times mg(-\hat{j}) + \vec{r}_{A/O} \times \vec{F} \\ &= M\hat{k} - \frac{\ell}{2}mg \sin \theta \hat{k} + F\ell \cos \theta \hat{k} \\ &= (M + F\ell \cos \theta - \frac{\ell}{2}mg \sin \theta)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_O = m\ddot{\theta} \frac{\ell^2}{3} \hat{k} \quad (\text{see Sample 7.21})$$

Therefore, from  $\sum \vec{M}_O = \dot{\vec{H}}_O$

$$\begin{aligned}M + F\ell \cos \theta - \frac{\ell}{2}mg \sin \theta &= m\ddot{\theta} \frac{\ell^2}{3} \\ \Rightarrow \quad \ddot{\theta} + \frac{3g}{2\ell} \sin \theta - \frac{3F}{m\ell} \cos \theta - \frac{3M}{m\ell^2} &= 0.\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell} \sin \theta - \frac{3F}{m\ell} \cos \theta - \frac{3M}{m\ell^2} = 0}$$

- Now, setting  $F = 0$  and  $M = C\theta$  we get

$$\ddot{\theta} + \frac{3g}{2\ell} \sin \theta - \frac{3C\theta}{m\ell^2} = 0 \quad (7.64)$$

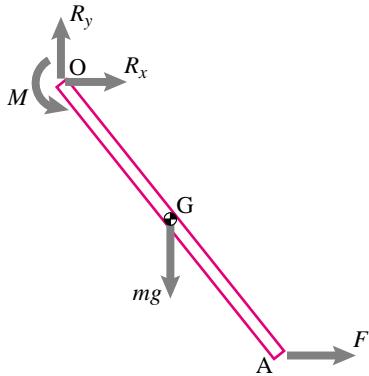


Figure 7.68: Free body diagram of the bar with applied torque  $\vec{M}$  and force  $\vec{F}$

(Filename:sfig5.4.5a)

**Numerical Solution:** We can numerically integrate (7.64) just as in the previous Sample to find  $\theta(t)$ . Here is the pseudo-code that can be used for this purpose.

```

g = 9.81, L = 1           % specify parameters
m = 1, C = 4

ODES = { thetadot = omega
        omegadot = -(3*g/(2*L)) * sin(theta)
                    + 3*C/(m*L^2) * theta      }
ICS = { thetazero = pi/20
        omegazero = 0      }
solve ODES with ICS until t = 10

```

Using this pseudo-code, we find the response of the pendulum. Figure 7.69 shows different responses for various values of  $C$ . Note that for  $C = 0$ , it is the same case as unforced bar pendulum considered above. From Fig. 7.69 it is clear that the bar has periodic motion for small  $C$ , with the period of motion increasing with increasing values of  $C$ . It makes sense if you look at Eqn. (7.64) carefully. Gravity acts as a restoring force while the applied torque acts as a destabilizing force. Thus, with the resistance of the applied torque, the stick swings more sluggishly making its period of oscillation bigger.

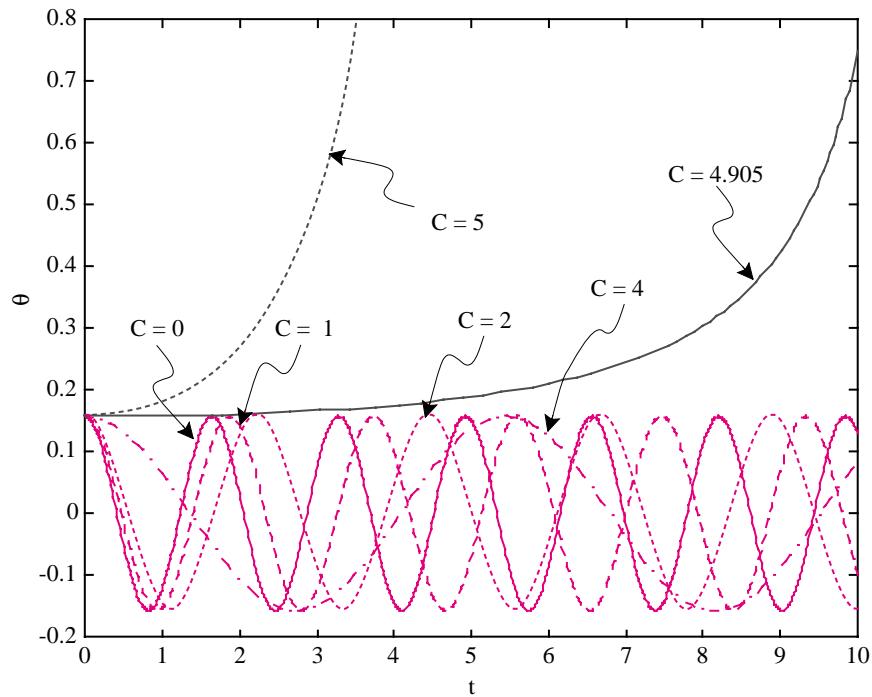


Figure 7.69:  $\theta(t)$  with applied torque  $M = C\theta$  for  $C = 0, 1, 2, 4, 4.905, 5$ . Note that for small  $C$  the motion is periodic but for large  $C$  ( $C \geq 4.4$ ) the motion becomes aperiodic.

(Filename:sfig5.4.5b)

- (c) From Fig. 7.69, we see that at about  $C \approx 4.9$  the stability of the system changes completely.  $\theta(t)$  is not periodic anymore. It keeps on increasing at faster and faster rate, that is, the bar makes complete loops about point O with ever increasing speed. Does it make physical sense? Yes, it does. As the value of  $C$  is increased beyond a certain value (can you guess the value?), the applied torque overcomes any restoring torque due to gravity. Consequently, the bar is forced to rotate continuously in the direction of the applied force.

## 7.5 Polar moment of inertia: $I_{zz}^{\text{cm}}$ and $I_{zz}^O$

We know how to find the velocity and acceleration of every bit of mass on a 2-D rigid body as it spins about a fixed axis. So, as explained in the previous section, it is just a matter of doing integrals or sums to calculate the various motion quantities (momenta, energy) of interest. As the body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut. A big sum (over all atoms, say), or a difficult integral is reduced to a simple multiplication.

① In fact the moment of inertia for a given object depends on what reference point is used. Most commonly when people say ‘the’ moment of inertia they mean to use the center of mass as the reference point. For clarity this moment of inertia matrix is often notated as  $[I^{\text{cm}}]$  in this book. If a different reference point, say point  $O$  is used, the matrix is notated as  $[I^O]$ .

The moment of inertia matrix  $[I]$ <sup>①</sup> is defined to simplify the expressions for the angular momentum, the rate of change of angular momentum, and the energy of a rigid body. For study of the analysis of flat objects in planar motion only one component of the matrix  $[I]$  is relevant, it is  $I_{zz}$ , called just  $I$  or  $J$  in elementary physics courses. Here are the results. A flat object spinning with  $\vec{\omega} = \omega \hat{k}$  in the  $xy$  plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia  $I_{zz}^{\text{cm}}$  or just ‘ $I$ ’ so that:

$$\vec{H}_{\text{cm}} = I \omega \hat{k} \quad (7.65)$$

$$\dot{\vec{H}}_{\text{cm}} = I \dot{\omega} \hat{k} \quad (7.66)$$

$$E_{K/cm} = \frac{1}{2} \omega^2 I. \quad (7.67)$$

### The moments of inertia in 2-D : $[I^{\text{cm}}]$ and $[I^O]$ .

We start by looking at the scalar  $I$  which is just the  $zz$  or 33 component of the matrix  $[I]$  that we will study later. The definition of  $I^{\text{cm}}$  is

$$\begin{aligned} I^{\text{cm}} &\equiv = \int \underbrace{x^2 + y^2}_{r^2} dm \\ &= \int \int r^2 \underbrace{\left( \frac{m_{\text{tot}}}{A} \right)}_{dm} dA \quad \text{for a uniform planar object} \end{aligned}$$

The mass per unit area.

where  $x$  and  $y$  are the distances of the mass in the  $x$  and  $y$  direction measured from an origin, and  $r$  is the direct distance from that origin. If that origin is at the center of mass then we are calculating  $I^{\text{cm}}$ , if the origin is at a point labeled C or O then we are calculating  $I^C$  or  $I^O$ .

The term  $I_{zz}$  is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the  $I_{xx}$  and  $I_{yy}$  terms which have little utility in planar dynamics (but are all important when calculating the stiffness of beams!).

What, physically, is the moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to  $I$  in proportion to the square of its distance from the reference point. Note from, say, eqn. (7.48) on page 397 that  $I$  is just the quantity we need to do mechanics problems.

### Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia, is the *radius of gyration*,  $r_{gyr}$ . The radius of gyration is sometimes called  $k$  but we save  $k$  for stiffness. The radius of gyration is defined as:

$$r_{gyr} \equiv \sqrt{I/m} \quad \Rightarrow \quad r_{gyr}^2 m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same  $I$  and the same mass as the given object.

### Other reference points

For the most part it is  $I^{cm}$  which is of primary interest. Other reference points are useful

- (a) if the rigid body is hinged at a fixed point  $O$  then a slight short cut in calculation of angular momentum and energy terms can be had; and
- (b) if one wants to calculate the moment of inertia of a composite body about its center of mass it is useful to first find the moment of inertia of each of its parts about that point. But the center of mass of the composite is usually not the center of mass of any of the separate parts.

The box 11.2 on page 666 shows the calculation of  $I$  for a number of simple 2 dimensional objects.

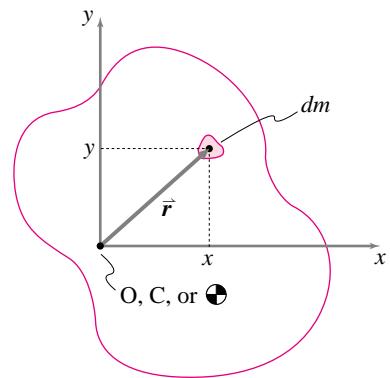


Figure 7.70: A general planar body.

(Filename:figure4.4.Defof1)

## The parallel axis theorem for planar objects

The planar parallel axis theorem is the equation

$$I_{zz}^C = I_{zz}^{cm} + m_{\text{tot}} \underbrace{r_{cm/C}^2}_{d^2}.$$

In this equation  $d = r_{cm/C}$  is the distance from the center of mass to a line parallel to the  $z$ -axis which passes through point  $C$ . See box 7.4 on page 413 for a derivation of the parallel axis theorem for planar objects.

Note that  $I_{zz}^C \geq I_{zz}^{cm}$ , always.

One can calculate the moment of inertia of a composite body about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of  $m_i$  is  $(x_i, y_i)$  relative to a fixed origin, and the moment of inertia of that part about its center of mass is  $I_i$ . We can then find the moment of inertia of the composite  $I_{tot}$  about its center of mass  $(x_{cm}, y_{cm})$  by the following sequence of calculations:

- (1)  $m_{\text{tot}} = \sum m_i$
- (2)  $x_{cm} = \left[ \sum x_i m_i \right] / m_{\text{tot}}$
- (3)  $y_{cm} = \left[ \sum y_i m_i \right] / m_{\text{tot}}$
- (4)  $d_i^2 = (x_i - x_{cm})^2 + (y_i - y_{cm})^2$
- (5)  $I_{tot} = \sum [I_i + m_i d_i^2]$ .

Of course if you are mathematically inclined you can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculation in about this order in any case. As presented here this sequence of steps lends itself naturally to computer calculation with a spread sheet or any program that deals easily with arrays of numbers.

The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find polar mass moment of inertia of a strange cutout shape.

### **The perpendicular axis theorem for planar rigid bodies**

The perpendicular axis theorem for planar objects is the equation

$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 7.4 on page 413. It gives the ‘polar’ inertia  $I_{zz}$  in terms of the inertias  $I_{xx}$  and  $I_{yy}$ . Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

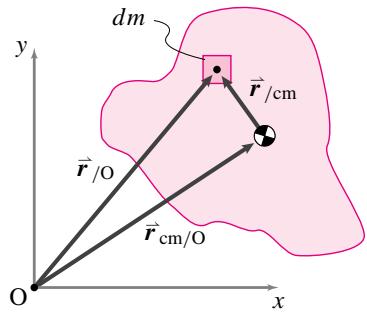
## 7.4 THEORY

### The 2-D parallel axis theorem and the perpendicular axis theorem

Sometimes, one wants to know the moment of inertia relative to the center of mass and, sometimes, relative to some other point  $O$ , if the object is held at a hinge joint at  $O$ . There is a simple relation between these two moments of inertia known as *the parallel axis theorem*.

### 2-D parallel axis theorem

For the two-dimensional mechanics of two-dimensional objects, our only concern is  $I_{zz}^O$  and  $I_{zz}^{cm}$  and *not* the full moment of inertia matrix. In this case,  $I_{zz}^O = \int r_{/O}^2 dm$  and  $I_{zz}^{cm} = \int r_{/cm}^2 dm$ . Now, let's prove the theorem in two dimensions referring to the figure.



$$\begin{aligned}
 I_{zz}^O &= \int r_{/O}^2 dm \\
 &= \int (x_{/O}^2 + y_{/O}^2) dm \\
 &= \int [(\underbrace{x_{cm/O} + x_{/cm}}_{x/O})^2 + (\underbrace{y_{cm/O} + y_{/cm}}_{y/O})^2] dm \\
 &= \int [(x_{cm/O}^2 + 2x_{cm/O}x_{/cm} + x_{/cm}^2) + \\
 &\quad (y_{cm/O}^2 + 2y_{cm/O}y_{/cm} + y_{/cm}^2)] dm \\
 &= (x_{cm/O}^2 + y_{cm/O}^2) \underbrace{\int dm}_{m} + 2x_{cm/O} \underbrace{\int x_{/cm} dm}_{0} + \\
 &\quad 2y_{cm/O} \underbrace{\int y_{/cm} dm}_{0} + \int (x_{/cm}^2 + y_{/cm}^2) dm \\
 &= r_{cm/O}^2 m + \underbrace{\int (x_{/cm}^2 + y_{/cm}^2) dm}_{I_{zz}^{cm}} \\
 &= I_{zz}^{cm} + \underbrace{r_{cm/O}^2 m}_{d^2}
 \end{aligned}$$

The cancellation  $\int y_{/cm} dm = \int x_{/cm} dm = 0$  comes from the definition of center of mass.

Sometimes, people write the parallel axis theorem more simply as

$$I^O = I^{cm} + md^2 \quad \text{or} \quad J_O = J_{cm} + md^2$$

using the symbol  $J$  to mean  $I_{zz}$ . One thing to note about the parallel axis theorem is that the moment of inertia about any point  $O$  is always *greater* than the moment of inertia about the center of mass. For a given object, the minimum moment of inertia is about the center of mass.

Why the name *parallel axis theorem*? We use the name because the two  $I$ 's calculated are the moments of inertia about two parallel axes (both in the  $z$  direction) through the two points  $cm$  and  $O$ .

One way to think about the theorem is the following. The moment of inertia of an object about a point  $O$  not at the center of mass is the same as that of the object about the *cm plus* that of a point mass located at the center of mass. If the distance from  $O$  to the *cm* is larger than the outer radius of the object, then the  $d^2 m$  term is larger than  $I_{zz}^{cm}$ . The distance of equality of the two terms is the radius of gyration,  $r_{gyr}$ .

### Perpendicular axis theorem (applies to planar objects only)

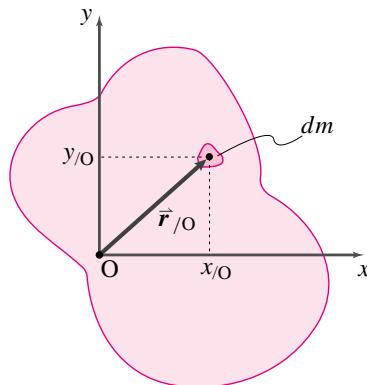
For planar objects,

$$\begin{aligned}
 I_{zz}^O &= \int |\vec{r}|^2 dm \\
 &= \int (x_{/O}^2 + y_{/O}^2) dm \\
 &= \int x_{/O}^2 dm + \int y_{/O}^2 dm \\
 &= I_{yy}^O + I_{xx}^O
 \end{aligned}$$

Similarly,

$$I_{zz}^{cm} = I_{xx}^{cm} + I_{yy}^{cm}.$$

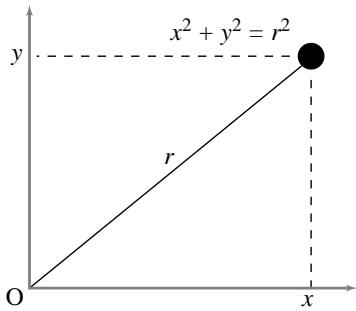
That is, the moment of inertia about the  $z$ -axis is the sum of the inertias about the two perpendicular axes  $x$  and  $y$ . Note that the objects must be planar ( $z = 0$  everywhere) or the theorem would not be true. For example,  $I_{xx}^O = \int (y_{/O}^2 + z_{/O}^2) dm \neq \int y_{/O}^2 dm$  for a three-dimensional object.



## 7.5 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

### One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I_{zz}^o = \int r_{i/o}^2 dm$$

reduces to the sum

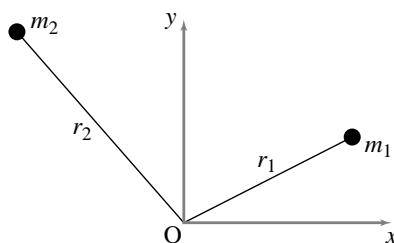
$$I_{zz}^o = \sum r_{i/o}^2 m_i$$

which reduces to one term if there is only one mass,

$$I_{zz}^o = r^2 m = (x^2 + y^2)m.$$

So, if  $x = 3$  in,  $y = 4$  in, and  $m = 0.1$  lbm, then  $I_{zz}^o = 2.5$  lbm in<sup>2</sup>. Note that, in this case,  $I_{zz}^{cm} = 0$  since the radius from the center of mass to the center of mass is zero.

### Two point masses

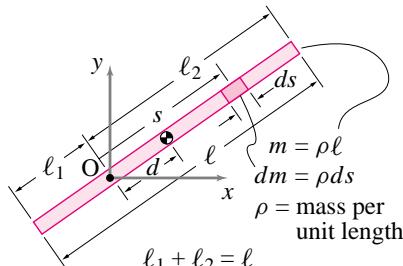


In this case, the sum that defines  $I_{zz}^o$  reduces to two terms, so

$$I_{zz}^o = \sum r_{i/o}^2 m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if  $r_1 = r_2 = r$ , then  $I_{zz}^o = m_{tot}r^2$ .

### A thin uniform rod



Consider a thin rod with uniform mass density,  $\rho$ , per unit length, and length  $\ell$ . We calculate  $I_{zz}^o$  as

$$\begin{aligned} I_{zz}^o &= \int r^2 \underbrace{dm}_{\rho ds} \\ &= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\ &= \frac{1}{3} \rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad (\text{since } \rho \equiv \text{const.}) \\ &= \frac{1}{3} \rho (\ell_1^3 + \ell_2^3). \end{aligned}$$

If either  $\ell_1 = 0$  or  $\ell_2 = 0$ , then this expression reduces to  $I_{zz}^o = \frac{1}{3} m \ell^2$ . If  $\ell_1 = \ell_2$ , then O is at the center of mass and

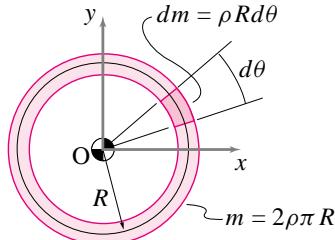
$$I_{zz}^o = I_{zz}^{cm} = \frac{1}{3} \rho \left( \left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3 \right) = \frac{m \ell^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for  $I_{zz}^o$  can be arranged as follows:

$$\begin{aligned} I_{zz}^o &= \frac{1}{3} \rho (\ell_1^3 + \ell_2^3) \\ &= \underbrace{\rho (\ell_1 + \ell_2)}_m \left( \underbrace{\frac{\ell_2 - \ell_1}{2}}_d \right)^2 + \underbrace{\rho \frac{(\ell_1 + \ell_2)^3}{12}}_{m \ell^2 / 12} \\ &= m d^2 + m \frac{\ell^2}{12} \\ &= m d^2 + I_{zz}^{cm} \end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center of mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

### A uniform hoop



For a hoop of uniform mass density,  $\rho$ , per unit length, we might consider all of the points to have the same radius  $R$ . So,

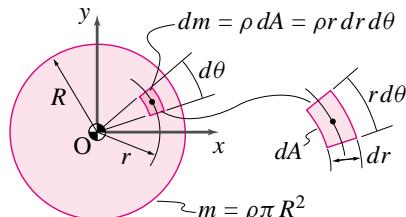
$$I_{zz}^o = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I_{zz}^o &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi \rho R^3 = \underbrace{(2\pi \rho R)}_m R^2 = m R^2. \end{aligned}$$

This  $I_{zz}^o$  is the same as for a single point mass  $m$  at a distance  $R$  from the origin  $O$ . It is also the same as for two point masses if they both are a distance  $R$  from the origin. For the hoop, however,  $O$  is at the center of mass so  $I_{zz}^o = I_{zz}^{cm}$  which is not the case for a single point mass.

### A uniform disk

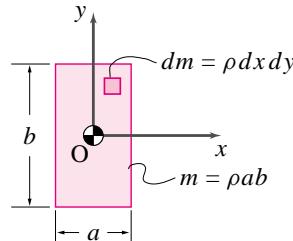


Assume the disk has uniform mass density,  $\rho$ , per unit area. For a uniform disk centered at the origin, the center of mass is at the origin so

$$\begin{aligned} I_{zz}^o = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r dr d\theta \\ &= \int_0^R 2\pi \rho r^3 dr \\ &= 2\pi \rho \frac{r^4}{4} \Big|_0^R = \pi \rho \frac{R^4}{2} = (\pi \rho R^2) \frac{R^2}{2} \\ &= m \frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

### Uniform rectangular plate



For the special case that the center of the plate is at point  $O$ , the center of mass of mass is also at  $O$  and  $I_{zz}^o = I_{zz}^{cm}$ .

$$\begin{aligned} I_{zz}^o = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) \rho dx dy \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho \left( \frac{x^3}{3} + xy^2 \right) \Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho \left( \frac{x^3 y}{3} + \frac{xy^3}{3} \right) \Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} \Big|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho \left( \frac{a^3 b}{12} + \frac{ab^3}{12} \right) \\ &= \frac{m}{12} (a^2 + b^2). \end{aligned}$$

Note that  $\int r^2 dm = \int x^2 dm + \int y^2 dm$  for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle,  $\int y^2 dm = \rho \int y^2 dA$ . But the integral  $y^2 dA$  is just the term often used for  $I$ , the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You may recall that  $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$  for a rectangle. Similarly,  $\int x^2 dA = \frac{Aa^2}{12}$ . So, the polar moment of inertia  $J = I_{zz}^o = m \frac{1}{12}(a^2 + b^2)$  can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

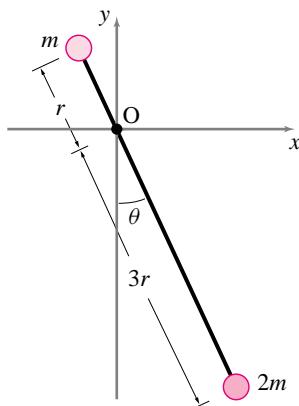


Figure 7.71: (Filename:sfig4.5.1)

**SAMPLE 7.25** A pendulum is made up of two unequal point masses  $m$  and  $2m$  connected by a massless rigid rod of length  $4r$ . The pendulum is pivoted at distance  $r$  along the rod from the small mass.

- Find the moment of inertia  $I_{zz}^O$  of the pendulum.
- If you had to put the total mass  $3m$  at one end of the bar and still have the same  $I_{zz}^O$  as in (a), at what distance from point O should you put the mass? (This distance is known as the radius of gyration).

**Solution** Here we have two point masses. Therefore, the integral formula for  $I_{zz}^O$  ( $I_{zz}^O = \int_m r_{i/O}^2 dm$ ) gets replaced by a summation over the two masses:

$$\begin{aligned} I_{zz}^O &= \sum_{i=1}^2 m_i r_{i/O}^2 \\ &= m_1 r_{1/O}^2 + m_2 r_{2/O}^2 \end{aligned}$$

- (a) For the pendulum,  $m_1 = m$ ,  $m_2 = 2m$ ,  $r_{1/O} = r$ ,  $r_{2/O} = 3r$ .

$$\begin{aligned} I_{zz}^O &= mr^2 + 2m(3r)^2 \\ &= 19mr^2 \end{aligned}$$

$$I_{zz}^O = 19mr^2$$

- (b) For the equivalent simple pendulum of mass  $3m$ , let the length of the massless rod (i.e., the distance of the mass from O) be  $r_{gyr}$ .

$$(I_{zz}^O)_{\text{simple}} = (3m) \cdot r_{gyr}^2$$

Now we need  $(I_{zz}^O)_{\text{simple}} = I_{zz}^O$  (from part (a))

$$\begin{aligned} \Rightarrow 3mr_{gyr}^2 &= 19mr^2 \\ \Rightarrow r_{gyr} &= \sqrt{\frac{19}{3}}r \\ &= 2.52r \end{aligned}$$

Thus the radius of gyration  $r_{gyr}$  of the given pendulum is  $r_{gyr} = 2.52r$ .

$$r_{gyr} = 2.52r$$

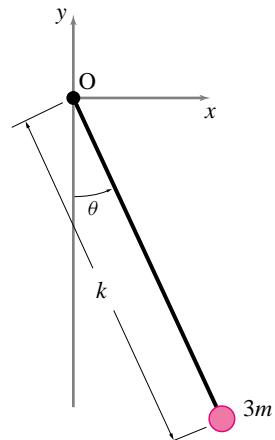


Figure 7.72: (Filename:sfig4.5.1a)

**SAMPLE 7.26** A uniform rigid rod AB of mass  $M = 2 \text{ kg}$  and length  $3\ell = 1.5 \text{ m}$  swings about the z-axis passing through the pivot point O.

- Find the moment of inertia  $I_{zz}^O$  of the bar using the fundamental definition  $I_{zz}^O = \int_m r_{/O}^2 dm$ .
- Find  $I_{zz}^O$  using the parallel axis theorem given that  $I_{zz}^{\text{cm}} = \frac{1}{12}m\ell^2$  where  $m = \text{total mass}$ , and  $\ell = \text{total length of the rod}$ . (You can find  $I_{zz}^{\text{cm}}$  for many commonly encountered objects in the table on the inside backcover of the text).

### Solution

- Since we need to carry out the integral,  $I_{zz}^O = \int_m r_{/O}^2 dm$ , to find  $I_{zz}^O$ , let us consider an infinitesimal length segment  $d\ell'$  of the bar at distance  $\ell'$  from the pivot point O. (see Figure 7.74). Let the mass of the infinitesimal segment be  $dm$ .

Now the mass of the segment may be written as

$$\begin{aligned} dm &= (\text{mass per unit length of the bar}) \cdot (\text{length of the segment}) \\ &= \frac{M}{3\ell} d\ell' \quad \left( \text{Note: } \frac{\text{mass}}{\text{unit length}} = \frac{\text{total mass}}{\text{total length}} \right). \end{aligned}$$

We also note that the distance of the segment from point O,  $r_{/O} = \ell$ . Substituting the values found above for  $r_{/O}$  and  $dm$  in the formula we get

$$\begin{aligned} I_{zz}^O &= \int_{-\ell}^{2\ell} (\ell')^2 \underbrace{\frac{M}{3\ell} d\ell'}_{r_{/O}^2 dm} \\ &= \frac{M}{3\ell} \int_{-\ell}^{2\ell} (\ell')^2 d\ell' = \frac{M}{3\ell} \left[ \frac{\ell'^3}{3} \right]_{-\ell}^{2\ell} \\ &= \frac{M}{3\ell} \left[ \frac{8\ell^3}{3} - \left( -\frac{\ell^3}{3} \right) \right] = M\ell^2 \\ &= 2 \text{ kg} \cdot (0.5 \text{ m})^2 \\ &= 0.5 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$I_{zz}^O = 0.5 \text{ kg} \cdot \text{m}^2$$

- The parallel axis theorem states that

$$I_{zz}^O = I_{zz}^{\text{cm}} + Mr_{O/\text{cm}}^2.$$

Since the rod is uniform, its center of mass is at its geometric center, i.e., at distance  $\frac{3\ell}{2}$  from either end. From the Fig 7.75 we can see that

$$r_{O/\text{cm}} = AG - AO = \frac{3\ell}{2} - \ell = \frac{\ell}{2}$$

$$\begin{aligned} \text{Therefore, } I_{zz}^O &= \underbrace{\frac{1}{12}M(3\ell)^2}_{I_{zz}^{\text{cm}}} + M\left(\frac{\ell}{2}\right)^2 \\ &= \frac{9}{12}M\ell^2 + M\frac{\ell^2}{4} = M\ell^2 \\ &= 0.5 \text{ kg} \cdot \text{m}^2 \quad (\text{same as in (a), of course}) \end{aligned}$$

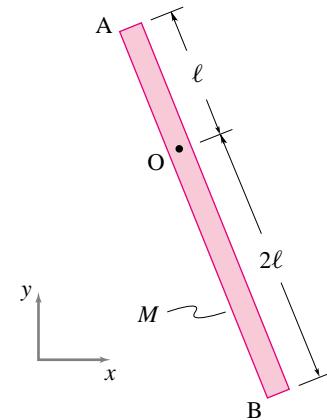


Figure 7.73: (Filename:sfig4.5.2)

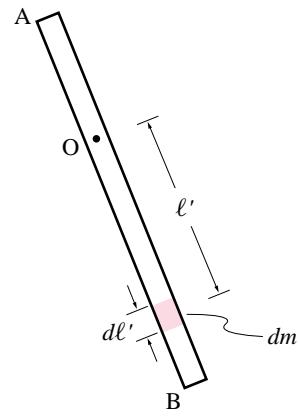


Figure 7.74: (Filename:sfig4.5.2a)

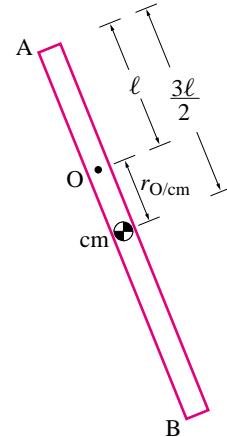


Figure 7.75: (Filename:sfig4.5.2b)

$$I_{zz}^O = 0.5 \text{ kg} \cdot \text{m}^2$$

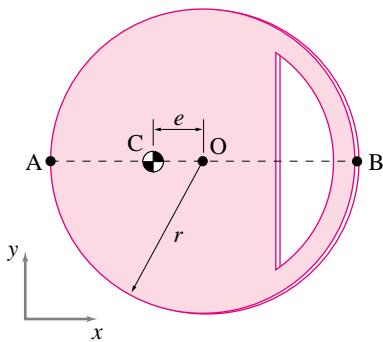


Figure 7.76: (Filename:sfig4.5.3)

**SAMPLE 7.27** A uniform rigid wheel of radius  $r = 1 \text{ ft}$  is made eccentric by cutting out a portion of the wheel. The center of mass of the eccentric wheel is at C, a distance  $e = \frac{r}{3}$  from the geometric center O. The mass of the wheel (after deducting the cut-out) is 3.2 lbm. The moment of inertia of the wheel about point O,  $I_{zz}^O$ , is  $1.8 \text{ lbm}\cdot\text{ft}^2$ . We are interested in the moment of inertia  $I_{zz}$  of the wheel about points A and B on the perimeter.

- Without any calculations, guess which point, A or B, gives a higher moment of inertia. Why?
- Calculate  $I_{zz}^C$ ,  $I_{zz}^A$  and  $I_{zz}^B$  and compare with the guess in (a).

### Solution

- The moment of inertia  $I_{zz}^B$  should be higher. Moment of inertia  $I_{zz}$  measures the geometric distribution of mass about the z-axis. But the distance of the mass from the axis counts more than the mass itself ( $I_{zz} = \int_m r^2 dm$ ). The distance  $r_{O/C}$  of the mass appears as a quadratic term in  $I_{zz}^O$ . The total mass is the same whether we take the moment of inertia about point A or about point B. However, the distribution of mass is not the same about the two points. Due to the cut-out being closer to point B there are more "dm's" at greater distances from point B than from point A. So, we guess that

$$I_{zz}^B > I_{zz}^A$$

- If we know the moment of inertia  $I_{zz}^C$  (about the center of mass) of the wheel, we can use the parallel axis theorem to find  $I_{zz}^A$  and  $I_{zz}^B$ . In the problem, we are given  $I_{zz}^O$ . But,

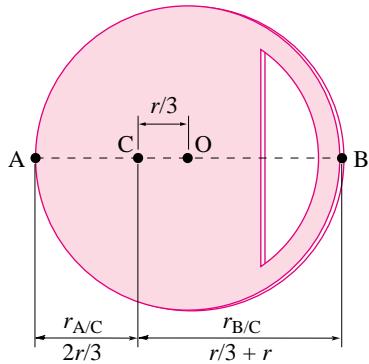


Figure 7.77: (Filename:sfig4.5.3a)

$$I_{zz}^O = I_{zz}^C + Mr_{O/C}^2 \quad (\text{parallel axis theorem})$$

$$\Rightarrow I_{zz}^C = I_{zz}^O - Mr_{O/C}^2$$

$$= 1.8 \text{ lbm}\cdot\text{ft}^2 - 3.2 \text{ lbm} \underbrace{\left(\frac{1 \text{ ft}}{3}\right)^2}_{r_{O/C}=e=\frac{r}{3}}$$

$$= 1.44 \text{ lbm}\cdot\text{ft}^2$$

$$\text{Now, } I_{zz}^A = I_{zz}^C + Mr_{A/C}^2 = I_{zz}^C + M \left(\frac{2r}{3}\right)^2$$

$$= 1.44 \text{ lbm}\cdot\text{ft}^2 + 3.2 \text{ lbm} \left(\frac{2 \text{ ft}}{3}\right)^2$$

$$= 2.86 \text{ lbm}\cdot\text{ft}^2$$

$$\text{and } I_{zz}^B = I_{zz}^C + Mr_{B/C}^2 = I_{zz}^C + M \left(r + \frac{r}{3}\right)^2$$

$$= 1.44 \text{ lbm}\cdot\text{ft}^2 + 3.2 \text{ lbm} \left(1 \text{ ft} + \frac{1 \text{ ft}}{3}\right)^2$$

$$= 7.13 \text{ lbm}\cdot\text{ft}^2$$

$$I_{zz}^C = 1.44 \text{ lbm}\cdot\text{ft}^2, I_{zz}^A = 2.86 \text{ lbm}\cdot\text{ft}^2, I_{zz}^B = 7.13 \text{ lbm}\cdot\text{ft}^2$$

Clearly,  $I_{zz}^B > I_{zz}^A$ , as guessed in (a).

**SAMPLE 7.28** A sphere or a point? A uniform solid sphere of mass  $m$  and radius  $r$  is attached to a massless rigid rod of length  $\ell$ . The sphere swings in the  $xy$  plane. Find the error in calculating  $I_{zz}^O$  as a function of  $r/l$  if the sphere is treated as a point mass concentrated at the center of mass of the sphere.

**Solution** The exact moment of inertia of the sphere about point O can be calculated using parallel axis theorem:

$$\begin{aligned} I_{zz}^O &= I_{zz}^{cm} + ml^2 \\ &= \underbrace{\frac{2}{5}mr^2}_{\text{See table 4.10}} + ml^2. \end{aligned}$$

of the text.

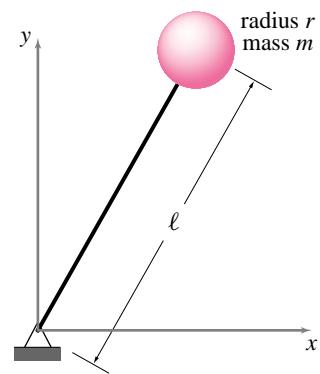


Figure 7.78: (Filename:sfig4.5.4)

If we treat the sphere as a point mass, the moment of inertia  $\tilde{I}_{zz}^O$  is

$$\tilde{I}_{zz}^O = ml^2.$$

Therefore, the relative error in  $I_{zz}^O$  is

$$\begin{aligned} \text{error} &= \frac{I_{zz}^O - \tilde{I}_{zz}^O}{I_{zz}^O} \\ &= \frac{\frac{2}{5}mr^2 + ml^2 - ml^2}{\frac{2}{5}mr^2 + ml^2} \\ &= \frac{\frac{2}{5}r^2}{\frac{2}{5}\frac{r^2}{l^2} + 1} \end{aligned}$$

From the above expression we see that for  $r \ll l$  the error is very small. From the graph of error in Fig. 7.79 we see that even for  $r = l/5$ , the error in  $I_{zz}^O$  due to approximating the sphere as a point mass is less than 2%.

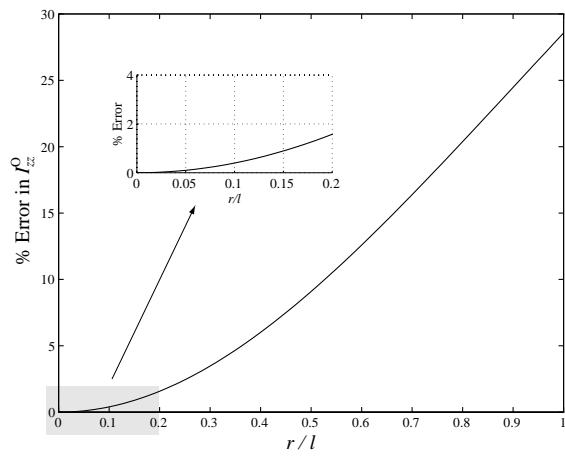


Figure 7.79: Relative error in  $I_{zz}^O$  of the sphere as a function of  $r/l$ . (Filename:sfig4.5.4a)

## 7.6 Using $I_{zz}^{\text{cm}}$ and $I_{zz}^O$ in 2-D circular motion dynamics

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from chapter 1.1. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. As you recall from chapter 3.6, the linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center of mass of the system of interest. One wishes for something so simple for the calculation of angular momentum.

It turns out that we are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification. For general motion of non-rigid bodies there is no simplification to be had. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to evaluate  $I \equiv I_{zz}^{\text{cm}}$  or  $[I^{\text{cm}}]$  but once this calculation is done, one need not work with the integrals while worrying about the dynamics. At this point we will assume that you are comfortable calculating and looking-up moments of inertia. We proceed to use it for the purposes of studying mechanics. For constant rate rotation, we can calculate the velocity and acceleration of various points on a rigid body using  $\vec{v} = \vec{\omega} \times \vec{r}$  and  $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ . So we can calculate the various motion quantities of interest: linear momentum  $\vec{L}$ , rate of change of linear momentum  $\dot{\vec{L}}$ , angular momentum  $\vec{H}$ , rate of change of angular momentum  $\dot{\vec{H}}$ , and kinetic energy  $E_K$ .

Consider a two-dimensional rigid body like that shown in figure 7.80. Now let us consider the various motion quantities in turn. First the linear momentum  $\vec{L}$ . The linear momentum of any system in any motion is  $\vec{L} = \vec{v}_{\text{cm}} m_{\text{tot}}$ . So, for a rigid body spinning at constant rate  $\omega$  about point O (using  $\vec{\omega} = \omega \hat{k}$ ):

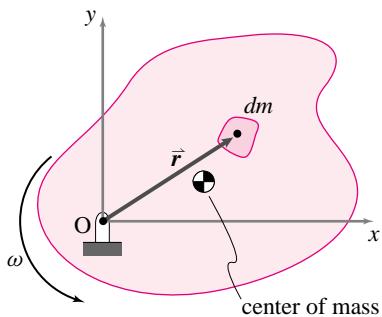


Figure 7.80: A two-dimensional body is rotating around the point O at constant rate  $\omega$ . A differential bit of mass  $dm$  is shown. The center of mass is also shown.

(Filename:tfigure4.2Dinertia)

$$\vec{L} = \vec{v}_{\text{cm}} m_{\text{tot}} = \vec{\omega} \times \vec{r}_{\text{cm}/o} m_{\text{tot}}.$$

Similarly, for any system, we can calculate the rate of change of linear momentum  $\dot{\vec{L}}$  as  $\dot{\vec{L}} = \vec{a}_{\text{cm}} m_{\text{tot}}$ . So, for a rigid body spinning at constant rate,

$$\dot{\vec{L}} = \vec{a}_{\text{cm}} m_{\text{tot}} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{\text{cm}/o}) m_{\text{tot}}.$$

That is, the linear momentum is correctly calculated for this special motion, as it is for all motions, by thinking of the body as a point mass at the center of mass.

Unlike the calculation of linear momentum, the angular momentum turns out to be something different than would be calculated by using a point mass at the center of mass. You can remember this important fact by looking at the case when the rotation is about the center of mass (point O coincides with the center of mass). In this case one can intuitively see that the angular momentum of a rigid body is not zero even

though the center of mass is not moving. Here's the calculation just to be sure:

$$\begin{aligned}
 \vec{H}_O &= \int \vec{r}_{/O} \times \vec{v} dm && \text{(by definition of } \vec{H}_O\text{)} \\
 &= \int \vec{r}_{/O} \times (\vec{\omega} \times \vec{r}_{/O}) dm && \text{(using } \vec{v} = \vec{\omega} \times \vec{r}\text{)} \\
 &= \int (x_{/O}\hat{i} + y_{/O}\hat{j}) \times [(\omega\hat{k}) \times (x_{/O}\hat{i} + y_{/O}\hat{j})] dm && \text{(substituting } \vec{r}_{/O} \text{ and } \vec{\omega}\text{)} \\
 &= \{ \int (x_{/O}^2 + y_{/O}^2) dm \} \omega\hat{k} && \text{(doing cross products)} \\
 &= \{ \int r_{/O}^2 dm \} \omega\hat{k} \\
 &= I_{zz}^O \omega\hat{k} \\
 &= \underbrace{I_{zz}^O}_{\boxed{I_{zz}^O \text{ is the 'polar' moment of inertia.}}} \omega\hat{k}
 \end{aligned}$$

We have defined the 'polar' moment of inertia as  $I_{zz}^o = \int r_{/O}^2 dm$ . In order to calculate  $I_{zz}^o$  for a specific body, assuming uniform mass distribution for example, one must convert the differential quantity of mass  $dm$  into a differential of geometric quantities. For a line or curve,  $dm = \rho d\ell$ ; for a plate or surface,  $dm = \rho dA$ , and for a 3-D region,  $dm = \rho dV$ .  $d\ell$ ,  $dA$ , and  $dV$  are differential line, area, and volume elements, respectively. In each case,  $\rho$  is the mass density per unit length, per unit area, or per unit volume, respectively. To avoid clutter, we do not define a different symbol for the density in each geometric case. The differential elements must be further defined depending on the coordinate systems chosen for the calculation; *e.g.*, for rectangular coordinates,  $dA = dx dy$  or, for polar coordinates,  $dA = r dr d\theta$ .

Since  $\vec{H}$  and  $\vec{\omega}$  always point in the  $\hat{k}$  direction for two dimensional problems people often just think of angular momentum as a scalar and write the equation above simply as ' $H = I\omega$ ', the form usually seen in elementary physics courses.

The derivation above has a feature that one might not notice at first sight. The quantity called  $I_{zz}^o$  does not depend on the rotation of the body. That is, the value of the integral does not change with time, so  $I_{zz}^o$  is a constant. So, perhaps unsurprisingly, a two-dimensional body spinning about the  $z$ -axis through  $O$  has constant angular momentum about  $O$  if it spins at a constant rate. ①

$$\vec{H}_O = \vec{0}.$$

Now, of course we could find this result about constant rate motion of 2-D bodies somewhat more cumbersomely by plugging in the general formula for rate of change of angular momentum as follows:

$$\begin{aligned}
 \dot{\vec{H}}_O &= \int \vec{r}_{/O} \times \vec{a} dm \\
 &= \int \vec{r}_{/O} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}_{/O})) dm \\
 &= \int (x_{/O}\hat{i} + y_{/O}\hat{j}) \times [\omega\hat{k} \times (\omega\hat{k} \times (x_{/O}\hat{i} + y_{/O}\hat{j}))] dm \\
 &= \vec{0}.
 \end{aligned} \tag{7.68}$$

Finally, we can calculate the kinetic energy by adding up  $\frac{1}{2}m_i v_i^2$  for all the bits of mass on a 2-D body spinning about the  $z$ -axis:

$$E_K = \int \frac{1}{2}v^2 dm = \int \frac{1}{2}(\omega r)^2 dm = \frac{1}{2}\omega^2 \int r^2 dm = \frac{1}{2}I_{zz}^o\omega^2. \tag{7.69}$$

If we accept the formulae presented for rigid bodies in the box at the end of chapter 7, we can find all of the motion quantities by setting  $\vec{\omega} = \omega\hat{k}$  and  $\vec{\alpha} = \vec{0}$ .

① Note that the angular momentum about some other point than O will not be constant unless the center of mass does not accelerate (*i.e.*, is at point O).

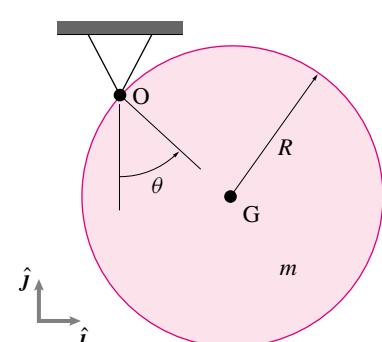


Figure 7.81: (Filename:tfigure5.3.pend.disk)

**Example: Pendulum disk**

For the disk shown in figure 7.81, we can calculate the rate of change of angular momentum about point  $O$  as

$$\begin{aligned}\dot{\vec{H}}_O &= \vec{r}_{G/O} \times m\vec{a}_{cm} + I_{zz}^{cm}\alpha\hat{k} \\ &= R^2m\ddot{\theta}\hat{k} + I_{zz}^{cm}\dot{\theta}\hat{k} \\ &= (I_{zz}^{cm} + R^2m)\ddot{\theta}\hat{k}.\end{aligned}$$

Alternatively, we could calculate directly

$$\begin{aligned}\dot{\vec{H}}_O &= I_{zz}^O\alpha\hat{k} \\ &= \underbrace{(I_{zz}^{cm} + R^2m)}_{\text{by the parallel axis theorem}}\ddot{\theta}\hat{k}.\end{aligned}$$

□

But you are cautioned against falling into the common misconception that the formula  $M = I\alpha$  applies in three dimensions by just thinking of the scalars as vectors and matrices. That is, the formula

$$\dot{\vec{H}}_O = [I^O] \cdot \underbrace{\dot{\vec{\omega}}}_{\vec{\alpha}} \quad (7.70)$$

is only correct when  $\vec{\omega}$  is zero or when  $\vec{\omega}$  is an eigen vector of  $[I_O]$ . To repeat, the equation

$$\sum \text{Moments about } O = [I^O] \cdot \vec{\alpha} \quad (7.71)$$

is generally wrong, it only applies if there is some known reason to neglect  $\vec{\omega} \times \vec{H}_0$ . For example,  $\vec{\omega} \times \vec{H}_0$  can be neglected when rotation is about a principal axis as for planar bodies rotating in the plane. The term  $\vec{\omega} \times \vec{H}_0$  can also be neglected at the start or stop of motion, that is when  $\vec{\omega} = \vec{0}$ .

The equation for linear momentum balance is the same as always, we just need to calculate the acceleration of the center of mass of the spinning body.

$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm} = m_{tot} [\vec{\omega} \times (\vec{\omega} \times \vec{r}_{cm/O}) + \dot{\vec{\omega}} \times \vec{r}_{cm/O}] \quad (7.72)$$

Finally, the kinetic energy for a planar rigid body rotating in the plane is:

$$E_K = \frac{1}{2}\vec{\omega} \cdot ([I^{cm}] \cdot \vec{\omega}) + \frac{1}{2}m \underbrace{v_{cm}^2}_{\vec{v}_{cm} = \vec{\omega} \times \vec{r}_{cm/O}}.$$



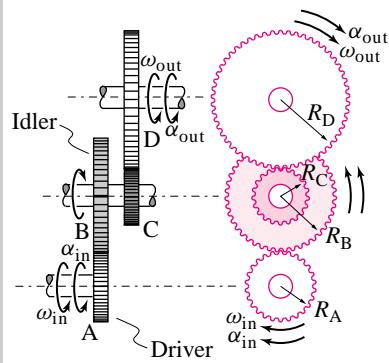


Figure 7.82: An accelerating compound gear train.

(Filename:fig5.6.1)

**SAMPLE 7.29** *An accelerating gear train.* In the gear train shown in Fig. 7.82, the torque at the input shaft is  $M_{in} = 200 \text{ N}\cdot\text{m}$  and the angular acceleration is  $\alpha_{in} = 50 \text{ rad/s}^2$ . The radii of the various gears are:  $R_A = 5 \text{ cm}$ ,  $R_B = 8 \text{ cm}$ ,  $R_C = 4 \text{ cm}$ , and  $R_D = 10 \text{ cm}$  and the moments of inertia about the shaft axis passing through their respective centers are:  $I_A = 0.1 \text{ kg m}^2$ ,  $I_{BC} = 5I_A$ ,  $I_D = 4I_A$ . Find the output torque  $M_{out}$  of the gear train.

**Solution** Since the difference between the input power and the output power is used in accelerating the gears, we may write

$$P_{in} - P_{out} = \dot{E}_K$$

Let  $M_{out}$  be the output torque of the gear train. Then,

$$P_{in} - P_{out} = M_{in} \omega_{in} - M_{out} \omega_{out}. \quad (7.73)$$

Now,

$$\dot{E}_K = \frac{d}{dt}(E_K) \quad (7.74)$$

$$\begin{aligned} &= \frac{d}{dt}(\frac{1}{2}I_A \omega_{in}^2 + \frac{1}{2}I_{BC} \omega_{BC}^2 + \frac{1}{2}I_D \omega_{out}^2) \\ &= I_A \omega_{in} \dot{\omega}_{in} + I_{BC} \omega_{BC} \dot{\omega}_{BC} + I_D \omega_{out} \dot{\omega}_{out} \\ &= I_A \omega_{in} \alpha_{in} + 5I_A \omega_{BC} \alpha_{BC} + 4I_A \omega_{out} \alpha_{out}. \end{aligned} \quad (7.75)$$

The different  $\omega$ 's and the  $\alpha$ 's can be related by realizing that the linear speed or the tangential acceleration of the point of contact between any two meshing gears has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus, using the linear speed and tangential acceleration calculations for points P and R in Fig. 7.83, we find

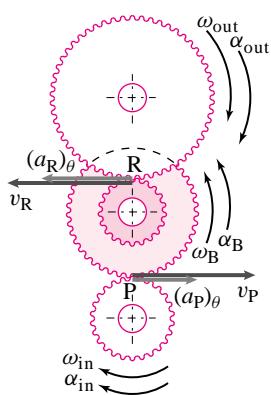


Figure 7.83: The velocity or acceleration of the point of contact between two meshing gears has to be the same irrespective of which meshing gear's geometry and motion is used to compute them.

(Filename:fig5.6.1a)

$$\begin{aligned} v_P &= \omega_{in} R_A = \omega_B R_B \\ \Rightarrow \omega_B &= \omega_{in} \cdot \frac{R_A}{R_B} \\ (a_P)_\theta &= \alpha_{in} R_A = \alpha_B R_B \\ \Rightarrow \alpha_B &= \alpha_{in} \cdot \frac{R_A}{R_B}. \end{aligned}$$

Similarly,

$$\begin{aligned} v_R &= \omega_C R_C = \omega_{out} R_D \\ \Rightarrow \omega_{out} &= \omega_C \cdot \frac{R_C}{R_D} \\ (a_R)_\theta &= \alpha_C R_C = \alpha_{out} R_D \\ \Rightarrow \alpha_{out} &= \alpha_C \cdot \frac{R_C}{R_D}. \end{aligned}$$

But

$$\begin{aligned} \omega_C &= \omega_B = \omega_{BC} \\ \Rightarrow \omega_{out} &= \omega_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \end{aligned}$$

and

$$\begin{aligned}\alpha_C &= \alpha_B = \alpha_{BC} \\ \Rightarrow \alpha_{out} &= \alpha_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}.\end{aligned}$$

Substituting these expressions for  $\omega_{out}$ ,  $\alpha_{out}$ ,  $\omega_{BC}$  and  $\alpha_{BC}$  in equations (7.73) and (7.75), we get

$$\begin{aligned}P_{in} - P_{out} &= M_{in} \omega_{in} - M_{out} \omega_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\ &= \omega_{in} \left( M_{in} - M_{out} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right). \\ \dot{E}_K &= I_A \left[ \omega_{in} \alpha_{in} + 5\omega_{in} \alpha_{in} \left( \frac{R_A}{R_B} \right)^2 + 4\omega_{in} \alpha_{in} \left( \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right] \\ &= I_A \omega_{in} \left[ \alpha_{in} + 5\alpha_{in} \left( \frac{R_A}{R_B} \right)^2 + 4\alpha_{in} \left( \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right].\end{aligned}$$

Now equating the two quantities,  $P_{in} - P_{out}$  and  $\dot{E}_K$ , and canceling  $\omega_{in}$  from both sides, we obtain

$$\begin{aligned}M_{out} \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} &= M_{in} - I_A \alpha_{in} \left[ 1 + 5\left(\frac{R_A}{R_B}\right)^2 + 4\left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right)^2 \right] \\ M_{out} \frac{5}{8} \cdot \frac{4}{10} &= 200 \text{ N}\cdot\text{m} - 5 \text{ kg m}^2 \cdot \text{rad/s}^2 \left[ 1 + 5\left(\frac{5}{8}\right)^2 + 4\left(\frac{5}{8} \cdot \frac{4}{10}\right)^2 \right] \\ M_{out} &= 735.94 \text{ N}\cdot\text{m} \\ &\approx 736 \text{ N}\cdot\text{m}.\end{aligned}$$

$M_{out} = 736 \text{ N}\cdot\text{m}$

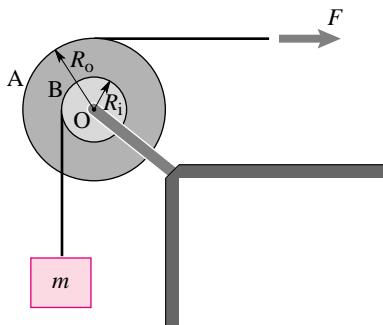


Figure 7.84: Two drums with strings wrapped around are used to pull up a mass  $m$ .

(Filename:sfig5.6.2)

**SAMPLE 7.30 Drums used as pulleys.** Two drums, A and B of radii  $R_o = 200$  mm and  $R_i = 100$  mm are welded together. The combined mass of the drums is  $M = 20$  kg and the combined moment of inertia about the  $z$ -axis passing through their common center O is  $I_{zz/O} = 1.6$  kg m $^2$ . A string attached to and wrapped around drum B supports a mass  $m = 2$  kg. The string wrapped around drum A is pulled with a force  $F = 20$  N as shown in Fig. 7.84. Assume there is no slip between the strings and the drums. Find

- (a) the angular acceleration of the drums,
- (b) the tension in the string supporting mass  $m$ , and
- (c) the acceleration of mass  $m$ .

**Solution** The free body diagram of the drums and the mass are shown in Fig. 7.85 separately where  $T$  is the tension in the string supporting mass  $m$  and  $O_x$  and  $O_y$  are the support reactions at O. Since the drums can only rotate about the  $z$ -axis, let

$$\vec{\omega} = \omega \hat{k} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega} \hat{k}.$$

Now, let us do angular momentum balance about the center of rotation O:

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\begin{aligned} \sum \vec{M}_O &= TR_i \hat{k} - FR_o \hat{k} \\ &= (TR_i - FR_o) \hat{k}. \end{aligned}$$

Since the motion is restricted to the  $xy$ -plane (*i.e.*, 2-D motion), the rate of change of angular momentum  $\dot{\vec{H}}_O$  may be computed as

$$\begin{aligned} \dot{\vec{H}}_O &= I_{zz/cm} \dot{\omega} \hat{k} + \vec{r}_{cm/O} \times \vec{a}_{cm} M_{total} \\ &= I_{zz/O} \dot{\omega} \hat{k} + \underbrace{\vec{r}_{O/O}}_0 \times \underbrace{\vec{a}_{cm}}_0 M_{total} \\ &= I_{zz/O} \dot{\omega} \hat{k}. \end{aligned}$$

Setting  $\sum \vec{M}_O = \dot{\vec{H}}_O$  we get

$$TR_i - FR_o = I_{zz/O} \dot{\omega}. \quad (7.76)$$

Now, let us write linear momentum balance,  $\sum \vec{F} = m \vec{a}$ , for mass  $m$ :

$$\underbrace{(T - mg) \hat{j}}_{\sum \vec{F}} = m \vec{a}.$$

Do we know anything about acceleration  $\vec{a}$  of the mass? Yes, we know its direction ( $\pm \hat{j}$ ) and we also know that it has to be the same as the tangential acceleration ( $\vec{a}_D$ ) $_\theta$  of point D on drum B (why?). Thus,

$$\begin{aligned} \vec{a} &= (\vec{a}_D)_\theta \\ &= \dot{\omega} \hat{k} \times (-R_i \hat{i}) \\ &= -\dot{\omega} R_i \hat{j}. \end{aligned} \quad (7.77)$$

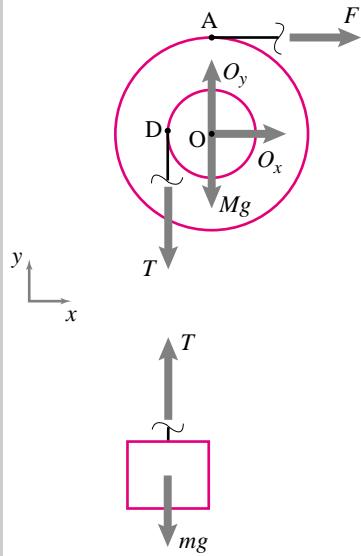


Figure 7.85: Free body diagram of the drums and the mass  $m$ .  $T$  is the tension in the string supporting mass  $m$  and  $O_x$  and  $O_y$  are the reactions of the support at O.

(Filename:sfig5.6.2a)

Therefore,

$$T - mg = -m\dot{\omega}R_i. \quad (7.78)$$

- (a) **Calculation of  $\dot{\omega}$ :** We now have two equations, (7.76) and (7.78), and two unknowns,  $\dot{\omega}$  and  $T$ . Subtracting  $R_i$  times Eqn.(7.78) from Eqn. (7.76) we get

$$\begin{aligned} -FR_o + mgR_i &= (I_{zz/O} + mR_i^2)\dot{\omega} \\ \Rightarrow \dot{\omega} &= \frac{-FR_o + mgR_i}{(I_{zz/O} + mR_i^2)} \\ &= \frac{-20 \text{ N} \cdot 0.2 \text{ m} + 2 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 0.1 \text{ m}}{1.6 \text{ kg m}^2 + 2 \text{ kg} \cdot (0.1 \text{ m})^2} \\ &= \frac{-2.038 \text{ kg m}^2/\text{s}^2}{1.62 \text{ kg m}^2} \\ &= -1.258 \frac{1}{\text{s}^2} \end{aligned}$$

$$\dot{\vec{\omega}} = -1.26 \text{ rad/s}^2 \hat{k}$$

- (b) **Calculation of tension T:** From equation (7.78):

$$\begin{aligned} T &= mg - m\dot{\omega}R_i \\ &= 2 \text{ kg} \cdot 9.81 \text{ m/s}^2 - 2 \text{ kg} \cdot (-1.26 \text{ s}^{-2}) \cdot 0.1 \text{ m} \\ &= 19.87 \text{ N} \end{aligned}$$

$$T = 19.87 \text{ N}$$

- (c) **Calculation of acceleration of the mass:** Since the acceleration of the mass is the same as the tangential acceleration of point D on the drum, we get (from eqn. (7.77))

$$\begin{aligned} \vec{a} &= (\vec{a}_D)_\theta = -\dot{\omega}R_i \hat{j} \\ &= -(-1.26 \text{ s}^{-2}) \cdot 0.1 \text{ m} \\ &= 0.126 \text{ m/s}^2 \hat{j} \end{aligned}$$

$$\vec{a} = 0.13 \text{ m/s}^2 \hat{j}$$

**Comments:** It is important to understand why the acceleration of the mass is the same as the tangential acceleration of point D on the drum. We have assumed (as is common practice) that the string is massless and inextensible. Therefore each point of the string supporting the mass must have the same linear displacement, velocity, and acceleration as the mass. Now think about the point on the string which is momentarily in contact with point D of the drum. Since there is no relative slip between the drum and the string, the two points must have the same vertical acceleration. This vertical acceleration for point D on the drum is the tangential acceleration  $(\vec{a}_D)_\theta$ .

**SAMPLE 7.31 Energy Accounting:** Consider the pulley problem of Sample 7.30 again.

- What percentage of the input energy (work done by the applied force  $F$ ) is used in raising the mass by 1 m?
- Where does the rest of the energy go? Provide an energy-balance sheet.

### Solution

- Let  $W_i$  and  $W_h$  be the input energy and the energy used in raising the mass by 1 m, respectively. Then the percentage of energy used in raising the mass is

$$\% \text{ of input energy used} = \frac{W_h}{W_i} \times 100.$$

Thus we need to calculate  $W_i$  and  $W_h$  to find the answer.  $W_i$  is the work done by the force  $F$  on the system during the interval in which the mass moves up by 1 m. Let  $s$  be the displacement of the force  $F$  during this interval. Since the displacement is in the same direction as the force (we know it is from Sample 7.30), the input-energy is

$$W_i = F s.$$

So to find  $W_i$  we need to find  $s$ .

For the mass to move up by 1 m the inner drum B must rotate by an angle  $\theta$  where

$$1 \text{ m} = \theta R_i \quad \Rightarrow \quad \theta = \frac{1 \text{ m}}{0.1 \text{ m}} = 10 \text{ rad.}$$

Since the two drums, A and B, are welded together, drum A must rotate by  $\theta$  as well. Therefore the displacement of force  $F$  is

$$s = \theta R_o = 10 \text{ rad} \cdot 0.2 \text{ m} = 2 \text{ m},$$

and the energy input is

$$W_i = F s = 20 \text{ N} \cdot 2 \text{ m} = 40 \text{ J}.$$

Now, the work done in raising the mass by 1 m is

$$W_h = mgh = 2 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 1 \text{ m} = 19.62 \text{ J}.$$

Therefore, the percentage of input-energy used in raising the mass

$$= \frac{19.62 \text{ N}\cdot\text{m}}{40} \times 100 = 49.05\% \approx 49\%.$$

- The rest of the energy ( $= 51\%$ ) goes in accelerating the mass and the pulley. Let us find out how much energy goes into each of these activities. Since the initial state of the system from which we begin energy accounting is not prescribed (that is, we are not given the height of the mass from which it is to be raised 1 m, nor do we know the velocities of the mass or the pulley at that initial height), let us assume that at the initial state, the angular speed of the pulley is  $\omega_o$  and the linear speed of the mass is  $v_o$ . At the end of raising the mass by 1 m from this state, let the angular speed of the pulley be  $\omega_f$  and the linear speed of the mass be  $v_f$ . Then, the energy used in accelerating the pulley

is

$$\begin{aligned}
 (\Delta E_K)_{\text{pulley}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
 &= \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_o^2 \\
 &= \frac{1}{2}I\underbrace{(\omega_f^2 - \omega_o^2)}_{\substack{\text{assuming constant acceleration, } \omega_f^2 = \omega_o^2 + 2\alpha\theta, \text{ or} \\ \omega_f^2 - \omega_o^2 = 2\alpha\theta.}} \\
 &= I\alpha\theta \quad (\text{from Sample 7.34, } \alpha = 1.258 \text{ rad/s}^2.) \\
 &= 1.6 \text{ kg m}^2 \cdot 1.258 \text{ rad/s}^2 \cdot 10 \text{ rad} \\
 &= 20.13 \text{ N}\cdot\text{m} = 20.13 \text{ J}.
 \end{aligned}$$

Similarly, the energy used in accelerating the mass is

$$\begin{aligned}
 (\Delta E_K)_{\text{mass}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
 &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 \\
 &= \frac{1}{2}m\underbrace{(v_f^2 - v_o^2)}_{2ah} \\
 &= mah \\
 &= 2 \text{ kg} \cdot 0.126 \text{ m/s}^2 \cdot 1 \text{ m} \\
 &= 0.25 \text{ J}.
 \end{aligned}$$

We can calculate the percentage of input energy used in these activities to get a better idea of energy allocation. Here is the summary table:

Activities	Energy Spent	
	in Joule	as % of input energy
In raising the mass by 1 m	19.62	49.05%
In accelerating the mass	0.25	0.62 %
In accelerating the pulley	20.13	50.33 %
<b>Total</b>	40.00	100 %

So, what would you change in the set-up so that more of the input energy is used in raising the mass? Think about what aspects of the motion would change due to your proposed design.

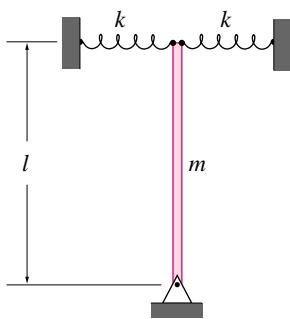


Figure 7.86: (Filename:sfig10.1.2)

**SAMPLE 7.32** A uniform rigid bar of mass  $m = 2 \text{ kg}$  and length  $L = 1 \text{ m}$  is pinned at one end and connected to two springs, each with spring constant  $k$ , at the other end. The bar is tweaked slightly from its vertical position. It then oscillates about its original position. The bar is timed for 20 full oscillations which take 12.5 seconds. Ignore gravity.

- Find the equation of motion of the rod.
- Find the spring constant  $k$ .
- What should be the spring constant of a torsional spring if the bar is attached to one at the bottom and has the same oscillating motion characteristics?

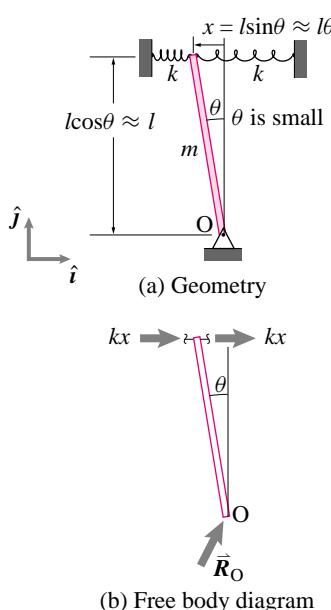


Figure 7.87: (Filename:sfig10.1.2a)

### Solution

- Refer to the free body diagram in figure 7.87. Angular momentum balance for the rod about point O gives

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\begin{aligned} \text{where } \vec{M}_O &= -2k \overset{l \sin \theta}{\curvearrowright} \cdot l \cos \theta \hat{k} \\ &= -2kl^2 \sin \theta \cos \theta \hat{k}, \\ \text{and } \dot{\vec{H}}_O &= I_{zz}^O \ddot{\theta} \hat{k} = \underbrace{\frac{1}{3} ml^2}_{I_{zz}^O} \ddot{\theta} \hat{k}. \end{aligned}$$

Thus

$$\frac{1}{3} ml^2 \ddot{\theta} = -2kl^2 \sin \theta \cos \theta.$$

However, for small  $\theta$ ,  $\cos \theta \approx 1$  and  $\sin \theta \approx \theta$ ,

$$\Rightarrow \ddot{\theta} + \frac{6kl^2}{ml^2} \theta = 0. \quad (7.79)$$

$$\boxed{\ddot{\theta} + \frac{6k}{m} \theta = 0}$$

- Comparing Eqn. (7.79) with the standard harmonic oscillator equation  $\ddot{x} + \lambda^2 x = 0$ , we get

$$\begin{aligned} \text{angular frequency } \lambda &= \sqrt{\frac{6k}{m}}, \\ \text{and the time period } T &= \frac{2\pi}{\lambda} \\ &= 2\pi \sqrt{\frac{m}{6k}}. \end{aligned}$$

From the measured time for 20 oscillations, the time period (time for one oscillation) is

$$T = \frac{12.5}{20} \text{ s} = 0.625 \text{ s}$$

Now equating the measured  $T$  with the derived expression for  $T$  we get

$$\begin{aligned} 2\pi\sqrt{\frac{m}{6k}} &= 0.625 \text{ s} \\ \Rightarrow k &= 4\pi^2 \cdot \frac{m}{6(0.625 \text{ s})^2} \\ &= \frac{4\pi^2 \cdot 2 \text{ kg}}{6(0.625 \text{ s})^2} \\ &= 33.7 \text{ N/m}. \end{aligned}$$

$$k = 33.7 \text{ N/m}$$

- (c) If the two linear springs are to be replaced by a torsional spring at the bottom, we can find the spring constant of the torsional spring by comparison. Let  $k_{\text{tor}}$  be the spring constant of the torsional spring. Then, as shown in the free body diagram (see figure 7.88), the restoring torque applied by the spring at an angular displacement  $\theta$  is  $k_{\text{tor}}\theta$ . Now, writing the angular momentum balance about point O, we get

$$\begin{aligned} \sum \vec{M}_O &= \dot{\vec{H}}_O \\ -k_{\text{tor}}\theta \hat{k} &= I_{zz}^O(\ddot{\theta}\hat{k}) \\ \Rightarrow \ddot{\theta} + \frac{k_{\text{tor}}}{I_{zz}^O}\theta &= 0. \end{aligned}$$

Comparing with the standard harmonic equation, we find the angular frequency

$$\lambda = \sqrt{\frac{k_{\text{tor}}}{I_{zz}^O}} = \sqrt{\frac{k_{\text{tor}}}{\frac{1}{3}ml^2}}$$

If this system has to have the same period of oscillation as the first system, the two angular frequencies must be equal, i.e.,

$$\begin{aligned} \sqrt{\frac{k_{\text{tor}}}{\frac{1}{3}ml^2}} &= \sqrt{\frac{6k}{m}} \\ \Rightarrow k_{\text{tor}} &= 6k \cdot \frac{1}{3}l^2 = 2kl^2 \\ &= 2 \cdot (33.7 \text{ N/m}) \cdot (1 \text{ m})^2 \\ &= 67.4 \text{ N}\cdot\text{m} \end{aligned}$$

$$k_{\text{tor}} = 67.4 \text{ N}\cdot\text{m}$$

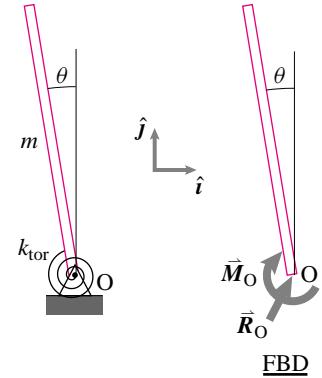


Figure 7.88: (Filename:sfig10.1.2b)

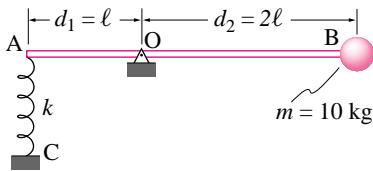


Figure 7.89: (Filename:fig3.4.3)

**SAMPLE 7.33** Hey Mom, look, I can seesaw by myself. A kid, modelled as a point mass with  $m = 10 \text{ kg}$ , is sitting at end B of a rigid rod AB of negligible mass. The rod is supported by a spring at end A and a pin at point O. The system is in static equilibrium when the rod is horizontal. Someone pushes the kid vertically downwards by a small distance  $y$  and lets go. Given that  $AB = 3 \text{ m}$ ,  $AC = 0.5 \text{ m}$ ,  $k = 1 \text{ kN/m}$ ; find

- the unstretched (relaxed) length of the spring,
- the equation of motion (a differential equation relating the position of the mass to its acceleration) of the system, and
- the natural frequency of the system.

If the rod is pinned at the midpoint instead of at O, what is the natural frequency of the system? How does the new natural frequency compare with that of a mass  $m$  simply suspended by a spring with the same spring constant?

### Solution

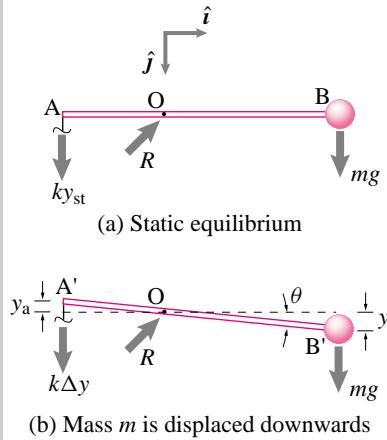


Figure 7.90: Free body diagrams

(Filename:fig3.4.3a)

① Here, we are considering a very small  $y$  so that we can ignore the arc the point mass B moves on and take its motion to be just vertical (i.e.,  $\sin \theta \approx \theta$  for small  $\theta$ ).

$$\begin{aligned} \sum \vec{M}_{/o} &= \dot{\vec{H}}_{/o} = \vec{0} \quad (\text{no motion}) \\ \Rightarrow (ky_{st})d_1 - (mg)d_2 &= 0 \\ \Rightarrow y_{st} &= \frac{mg}{k} \cdot \frac{d_2}{d_1} \\ &= \frac{10 \text{ kg} \cdot 9.8 \text{ m/s}^2 \cdot 2\ell}{1000 \text{ N/m} \cdot \ell} = 0.196 \text{ m} \\ \text{Therefore, } \ell_0 &= AC - y_{st} \\ &= 0.5 \text{ m} - 0.196 \text{ m} = 0.304 \text{ m.} \end{aligned}$$

$$\ell_0 = 30.4 \text{ cm}$$

- (b) **Equation of motion:** As point B gets displaced downwards by a distance  $y$ , point A moves up by a proportionate distance  $y_a$ . From geometry, ①

$$\begin{aligned} y &\approx d_2 \theta \quad \Rightarrow \quad \theta = \frac{y}{d_2} \\ y_a &\approx d_1 \theta = \frac{d_1}{d_2} y \end{aligned}$$

Therefore, the total stretch in the spring, in this position,

$$\Delta y = y_a + y_{st} = \frac{d_1}{d_2} y + \frac{d_2}{d_1} \frac{mg}{k}$$

Now, Angular Momentum Balance about point O gives:

$$\begin{aligned} \sum \vec{M}_{/o} &= \dot{\vec{H}}_{/o} \\ \sum \vec{M}_{/o} &= \vec{r}_B \times mg \hat{j} + \vec{r}_A \times k \Delta y \hat{j} \\ &= (d_2 mg - d_1 k \Delta y) \hat{k} \end{aligned} \tag{7.80}$$

$$\dot{\vec{H}}_{/o} = \vec{r}_B \times m \vec{a} = \vec{r}_B \times m \ddot{y} \hat{j} \tag{7.81}$$

$$= d_2 m \ddot{y} \hat{k} \tag{7.82}$$

Equating (7.80) and (7.82) we get

$$\begin{aligned} d_2mg - d_1k\Delta y &= d_2m\ddot{y} \\ \text{or } d_2mg - d_1k\left(\frac{d_1}{d_2}y + \frac{d_2mg}{d_1k}\right) &= d_2m\ddot{y} \\ \text{or } d_2mg - k\frac{d_1^2}{d_2}y - d_2mg &= d_2m\ddot{y} \\ \text{or } \ddot{y} + \frac{k}{m}\frac{d_1^2}{d_2^2}y &= 0 \end{aligned}$$

$$\boxed{\ddot{y} + \frac{k}{m}\frac{d_1^2}{d_2^2}y = 0}$$

- (c) **The natural frequency of the system:** We may also write the previous equation as

$$\ddot{y} + \lambda y = 0 \quad \text{where} \quad \lambda = \frac{k}{m}\frac{d_1^2}{d_2^2}. \quad (7.83)$$

Substituting  $d_1 = \ell$  and  $d_2 = 2\ell$  in the expression for  $\lambda$  we get the natural frequency of the system

$$\sqrt{\lambda} = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{1000 \text{ N/m}}{10 \text{ kg}}} = 5 \text{ s}^{-1}$$

$$\boxed{\sqrt{\lambda} = 5 \text{ s}^{-1}}$$

- (d) **Comparison with a simple spring mass system:** When  $d_1 = d_2$ , the equation

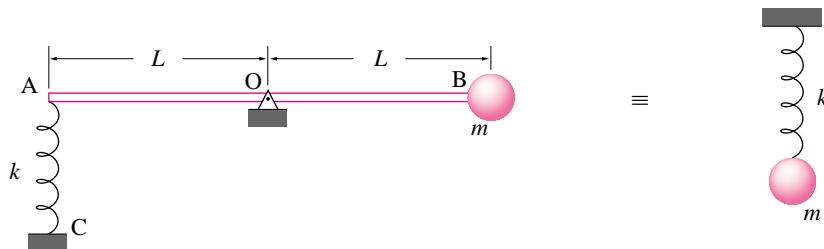


Figure 7.91: (Filename:sfig3.4.3b)

of motion (7.83) becomes

$$\ddot{y} + \frac{k}{m}y = 0$$

and the natural frequency of the system is simply

$$\sqrt{\lambda} = \sqrt{\frac{k}{m}}$$

which corresponds to the natural frequency of a simple spring mass system shown in Fig. 7.91.

In our system (with  $d_1 = d_2$ ) any vertical displacement of the mass at B induces an equal amount of stretch or compression in the spring which is exactly the case in the simple spring-mass system. Therefore, the two systems are mechanically equivalent. Such equivalences are widely used in modeling complex physical systems with simpler mechanical models.

**SAMPLE 7.34 Energy method:** Consider the pulley problem of Sample 7.30 again. Use energy method to

- find the angular acceleration of the pulley, and
- the acceleration of the mass.

**Solution** In energy method we use speeds, not velocities. Therefore, we have to be careful in our thinking about the direction of motion. In the present problem, let us assume that the pulley rotates and accelerates clockwise. Consequently, the mass moves up against gravity.

- The energy equation we want to use is

$$P = \dot{E}_K.$$

The power  $P$  is given by  $P = \sum \vec{F}_i \cdot \vec{v}_i$  where the sum is carried out over all external forces. For the mass and pulley system the external forces that do work are ①  $F$  and  $mg$ . Therefore,

$$\begin{aligned} P &= \vec{F} \cdot \vec{v}_A + m\vec{g} \cdot \vec{v}_m \\ &= F\hat{i} \cdot v_A\hat{i} + (-mg\hat{j}) \cdot \underbrace{v_D\hat{j}}_{\vec{v}_m} \\ &= Fv_A - mgv_D. \end{aligned}$$

① There are other external forces on the system: the reaction force of the support point O and the weight of the pulley—both forces acting at point O. But, since point O is stationary, these forces do no work.

The rate of change of kinetic energy is

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt} \left( \underbrace{\frac{1}{2}mv_D^2}_{\text{K.E. of the mass}} + \underbrace{\frac{1}{2}I_{zz/O}\omega^2}_{\text{K.E. of the pulley}} \right) \\ &= m v_D \dot{v}_D + I_{zz/O} \omega \dot{\omega}. \end{aligned}$$

Now equating the power and the rate of change of kinetic energy, we get

$$Fv_A - mgv_D = m v_D \dot{v}_D + I_{zz/O} \omega \dot{\omega}$$

From kinematics,  $v_A = \omega R_o$ ,  $v_D = \omega R_i$  and  $\dot{v}_D \equiv (a_D)_\theta = \dot{\omega}R_i$ . Substituting these values in the above equation, we get

$$\begin{aligned} \omega(FR_o - mgR_i) &= \omega \dot{\omega}(mR_i^2 + I_{zz/O}) \\ \Rightarrow \dot{\omega} &= \frac{FR_o - mgR_i}{(I_{zz/O} + mR_i^2)} \\ &= \frac{20\text{ N} \cdot 0.2\text{ m} - 2\text{ kg} \cdot 9.81\text{ m/s}^2 \cdot 0.1\text{ m}}{1.6\text{ kg m}^2 + 2\text{ kg} \cdot (0.1\text{ m})^2} \\ &= 1.258 \frac{1}{\text{s}^2} \quad (\text{same as the answer before.}) \end{aligned}$$

Since the sign of  $\dot{\omega}$  is positive, our initial assumption of clockwise acceleration of the pulley is correct.

$$\dot{\omega} = 1.26 \text{ rad/s}^2$$

(b) From kinematics,

$$a_m = (a_D)_\theta = \dot{\omega}R_i = 0.126 \text{ m/s}^2.$$

$$a_m = 0.13 \text{ m/s}^2$$

**SAMPLE 7.35** A flywheel of diameter 2 ft spins about the axis passing through its center and perpendicular to the plane of the wheel at 1000 rpm. The wheel weighs 20 lbf. Assuming the wheel to be a thin, uniform disk, find its kinetic energy.

**Solution** The kinetic energy of a 2-D rigid body spinning at speed  $\omega$  about the  $z$ -axis passing through its mass center is

$$E_K = \frac{1}{2} I_{zz}^{\text{cm}} \omega^2$$

where  $I_{zz}^{\text{cm}}$  is the mass moment of inertia about the  $z$ -axis. For the flywheel,

$$\begin{aligned} I_{zz}^{\text{cm}} &= \frac{1}{2} m R^2 \quad (\text{from table IV at the back of the book}) \\ &= \frac{1}{2} \frac{W}{g} R^2 \quad (\text{where } W \text{ is the weight of the wheel}) \\ &= \frac{1}{2} \cdot \underbrace{\left( \frac{20 \text{ lbf}}{g} \right)}_{20 \text{ lbm}} \cdot (1 \text{ ft})^2 = 10 \text{ lbm} \cdot \text{ft}^2 \end{aligned}$$

The angular speed of the wheel is

$$\begin{aligned} \omega &= 1000 \text{ rpm} \\ &= 1000 \cdot \frac{2\pi}{60} \text{ rad/s} \\ &= 104.72 \text{ rad/s.} \end{aligned}$$

Therefore the kinetic energy of the wheel is

$$\begin{aligned} E_K &= \frac{1}{2} \cdot (10 \text{ lbm} \cdot \text{ft}^2) \cdot (104.72 \text{ rad/s})^2 \\ &= 5.483 \times 10^4 \text{ lbm} \cdot \text{ft}^2 / \text{s}^2 \\ &= \frac{5.483 \times 10^4}{32.2} \text{ lbf} \cdot \text{ft} \\ &= 1.702 \times 10^3 \text{ ft} \cdot \text{lbf.} \end{aligned}$$

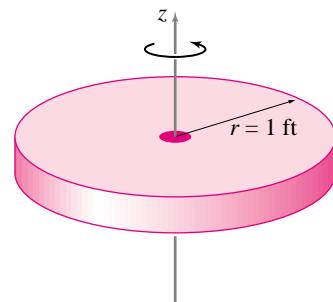


Figure 7.92: (Filename:sfig7.4.1)

$1.702 \times 10^3 \text{ ft} \cdot \text{lbf.}$



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# 8

# General planar motion of a single rigid body

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Many parts of practical machines and structures move in ways that can be idealized as straight-line or parallel motion (Chapter 6) or circular motion (Chapters 7 and 11). But often an engineer must analyze parts with more general motions, like a plane in unsteady flight or a connecting rod in a car engine. Of course, the same basic laws of mechanics still apply but keeping track of the motion is a bit more difficult. To keep things reasonably simple we only consider 2-D motions at this point.

The chapter starts with the kinematic description of motion and then progresses to the mechanics of these motions. Almost throughout this chapter, we will use two modeling approximations:

- The objects are planar, or symmetric with respect to a plane; and
- They have planar motions in that plane.

A *planar object* is one where the whole body is flat and all its matter is confined to one plane, say the  $xy$  plane. This is a palatable approximation for a flat piece cut out of sheet metal. For more substantial real objects, like a full car, the approximation seems at a glance to be terrible. But it turns out that so long as the *motion* is planar and the car is reasonably idealized as symmetrical (left to right) that the further idealization that the car is squished into a plane does not introduce any more approximation. Thus, even in this 3-D world, it is fruitful to do 2-D analysis of the type you will learn in this chapter.

A *planar motion* is one where the velocities of all points are in the same constant plane, say a fixed  $xy$  plane, at all times and where points with, say, the same  $z$  coordinate have the same velocity. Note that the positions of the points do not have to be in same plane for a planar motion. Each point stays in a plane, but different points can be in different planes, with each plane parallel to the others.

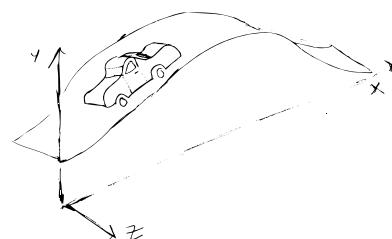


Figure 8.1: Planar motion of a 3D car.  
(Filename: tfigure.2D3Dcar)

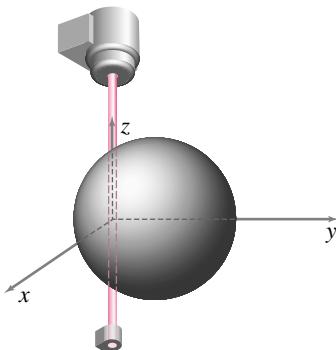


Figure 8.2: Planar motion of a skewed sphere.

(Filename:tfigure.ball)

#### **Example: A car going over a hill**

Assume the road is straight in map view, say in the  $x$  direction. Assume the whole width of the road has the same hump. Although the car is clearly not planar, the car motion is probably close to planar, with the velocities of all points in the car in the  $xy$  plane (see Fig. 8.1)  $\square$

#### **Example: Skewered sphere**

A sphere skewered and rotating about a fixed axes in the  $\hat{k}$  direction has a planar motion (see Fig. 8.2). The points on the object do not all lie in a common plane. But all of the velocities are orthogonal to  $\hat{k}$  and thus in the  $xy$  plane. This problem does fit in with the methods of this chapter. The symmetry of the sphere with respect to the  $xy$  plane makes it so that the three-dimensional mass distribution does not invalidate the two-dimensional analysis.  $\square$

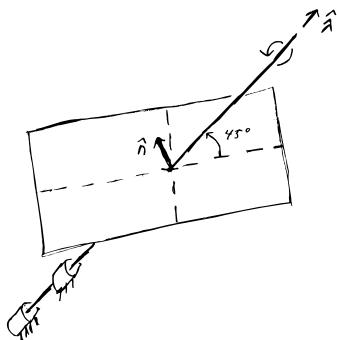


Figure 8.3: Planar motion of a planar object. But the plane of the motion is *not* the plane of the object.

(Filename:tfigure.crookedplate)

#### **Example: Skewered plate**

A flat rectangular plate with normal  $\hat{n}$  has a fixed axis of rotation in the direction  $\hat{\lambda}$  that makes a  $45^\circ$  to  $\hat{n}$  (see Fig. 8.3). This is a planar object (a plane normal to  $\hat{n}$ ) in planar motion (all velocities are in the plane normal to  $\hat{\lambda}$ ). But the plane of motion is not the plane of the mass distribution, the object is not symmetric with respect to a motion plane, o this example does not fit into the discussion of this chapter<sup>①</sup>.  $\square$

No real object is exactly planar and no real motion is exactly a planar motion. But many objects are relatively flat and thin or symmetrical and many motions are approximately planar motions. Thus many, if not most, simple engineering analysis assume planar motion. For bodies that are approximately symmetric about the  $xy$  plane of motion (such as a car, if the asymmetrically placed driver's mass etc. is neglected), there is no loss in doing a two-dimensional planar rather than full three dimensional analysis.

<sup>①</sup> Actually, a two-dimensional analysis of the plate in this example we would be legitimate in this sense. Project all the plates mass into the plane normal to the  $\hat{\lambda}$  direction. The projections of the forces on this plane would be correctly predicted, but three dimensional effects, like those associated with dynamic imbalance, would be lost in this projection.

## **8.1 Kinematics of planar rigid-body motion**

We start our study of planar motion with the kinematic question: How do points on a rigid body move? Lets review the two reasons to ask this question. First, velocities and accelerations of mass points are needed to apply the momentum-balance equations. Second, formulas for positions, velocities and accelerations of points are useful to understand mechanisms, machines where various parts (each one usually idealized as a rigid body) are connected to each other with hinges and bearings of one type of another.

The central observation in all rigid body kinematics, not just planar motion, is that

all pairs of points on a moving rigid body maintain constant distance from each other.

In this section you will learn how to use this restriction to calculate positions, velocities and accelerations of all points on a rigid body given only limited information. This goal is achieved by putting together the ideas from Chapter 5 (arbitrary motion of one particle), Chapter 6 (parallel motion), and chapter 7 (circular motion of a rigid body in a plane).

## Displacement and rotation

When a planar body (read, say, machine part)  $\mathcal{B}$  moves from one configuration in the plane it has a displacement and a rotation. For definiteness, we start in some reference position \*. We mark a reference point on the body that, in the reference configuration, coincides with a fixed reference point, say  $O$ . We also mark a (directed) line on the body that, in the reference configuration, coincides with a fixed reference line, say the positive  $x$  axis. The body never has to pass through this reference position, however. For example, the position of a plane flying from New York to Mumbai is measured relative to a point in the Gulf of Guinea 1000 miles west of Gabon,<sup>①</sup> even though the plane never goes there (nor does anyone want it to).

We could measure rotation by measuring the rotations of any lines that connected any pair of points fixed to the body. For each line we keep track of the angle that line makes with a line fixed in space, say the positive  $x$  or  $y$  axis. Its simplest to stick to the convention that counter-clockwise rotations are positive (Fig. 8.4). The angles  $\theta_1, \theta_2 \dots$ , all change with time and are all different from each other. But all the angles change the same amount, just like in section 7.3. We can pick any one line we like for definiteness and measure the body rotation by the rotation of that line. So

The net motion of a rigid planar body is described by *translation*, the vector displacement of a reference point from a reference position  $\vec{r}_{o'/o} = \vec{r}_{oo'}$ , and a *rotation*  $\theta$  of the body from the reference orientation.

That is, the general planar motion of a rigid body is the general motion of a point plus circular motion about that point.

<sup>①</sup> That's the location of  $O^0$  longitude and  $O^0$  latitude.

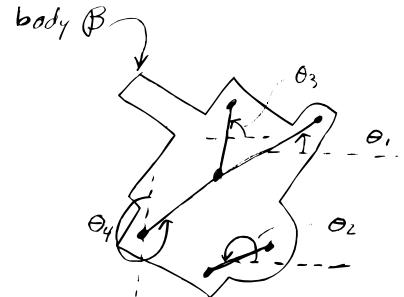


Figure 8.4: Rotation of body  $\mathcal{B}$  is measured by the rotation of real or imagined lines marked on the body. The lines make different angles:  $\theta_1 \neq \theta_2, \theta_2 \neq \theta_3$  etc, but  $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dots$ . Angular velocity is defined as  $\vec{\omega} = \omega \hat{k}$  with  $\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dots$

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## The position of a point on a moving rigid body.

Let's denote the reference configuration with a star (\*). Given that  $P$  on the body is at  $\vec{r}_{P/0}^*$  in the reference configuration, where is it (What is  $\vec{r}_{P/0}$ ?) after the body has been displaced by  $\vec{r}_{o'/0}$  and rotated an angle  $\theta$ ? An easy way to treat this is to write the new position of  $P$  as (see Fig. 8.5)

$$\vec{r}_{P/0} = \vec{r}_{o'/0} + \vec{r}_{P/o'}.$$

This is the *base-independent* or *direct vector* representation of the position of  $P$ . The formula is correct no matter what base vectors are used to represent the vectors in the formula. The vector  $\vec{r}_{o'/0}$  describes *translation*, that's half the story. The other term  $\vec{r}_{P/o'}$  we find by rotating  $\vec{r}_{P/0}^*$  as we did in Section 7.3. Thus, we can describe the coordinates of a point as,

$$[\vec{r}_{P/0}]_{xy} = \underbrace{[\vec{r}_{o'/0}]_{xy}}_{\text{displacement}} + \underbrace{[R(\theta)] [\vec{r}_{P/o'}]_{x'y'}}_{\text{rotation}} \quad (8.1)$$

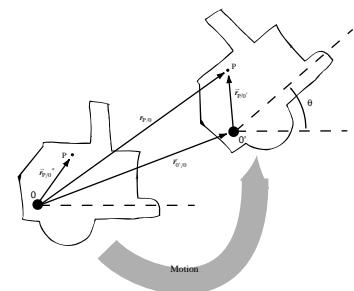


Figure 8.5: The displacement and rotation of a planar body relative to a reference configuration.

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or, writing out all the components of the vectors and matrices,

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_{P/0}^* \\ y_{P/0}^* \end{bmatrix}. \quad (8.2)$$

As the motion progresses the displacement  $\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix}$  changes with time as does the rotation angle  $\theta$ . We call eqn. (8.2) the *fixed basis* or *component* representation of the motion. It gives the components of the position in terms of base vectors that are fixed in space.

*Example:*

If in the reference position a particle on a rigid body is at  $\vec{r}_{P/0} = (1\hat{i} + 2\hat{j})$  m and the object displaces by  $\vec{r}_{0'/0} = (3\hat{i} + 4\hat{j})$  m and rotates by  $\theta\pi/3$  rad = 60 deg relative to that configuration, then its new position is:

$$\begin{aligned} [\vec{r}_{P/0}]_{xy} &= [\vec{r}_{0'/0}]_{xy} + [R(\theta)] [\vec{r}_{P/0'}]_{x'y'} \\ &= \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_{P/0}^* \\ y_{P/0}^* \end{bmatrix} \\ &= \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} \cos \pi/3 & \sin \pi/3 \\ -\sin \pi/3 & \cos \pi/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \text{m} \\ &= \begin{bmatrix} 3.5 + \sqrt{3} \\ 5 - \sqrt{3}/2 \end{bmatrix} \text{m} \\ \Rightarrow \vec{r}_{P/0} &= ((3.5 + \sqrt{3})\hat{i} + (5 - \sqrt{3}/2)\hat{j}) \text{ m} \end{aligned}$$

□

Finally, the *changing base* representation uses base vectors  $\hat{i}', \hat{j}'$  that are aligned with  $\hat{i}, \hat{j}$  in the reference configuration but which are glued to the rotating body. If we define  $x'$  and  $y'$  as the  $x$  and  $y$  components of  $P$  in the reference (\*) configuration we have that

$$[\vec{r}_{P/0}^*]_{xy} = [\vec{r}_{P/0'}^*]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{so} \quad \vec{r}_{P/0} = (x_{0'/0}\hat{i} + y_{0'/0}\hat{j}) + (x'\hat{i}' + y'\hat{j}').$$

Often the changing-base notation the clearest, the component or fixed base representation the best for computer calculations, and the base-independent or direct-vector notation the quickest and easiest.

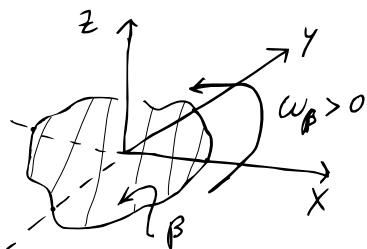


Figure 8.6: It is generally best to take positive  $\omega$  to be counterclockwise when viewed from the positive  $z$  axis.

(Filename:tfigure\_posomega)

## Angular velocity

Because all lines body  $\mathcal{B}$  rotate at the same rate (at a given instant)  $\mathcal{B}$ 's rotation rate is the single number we call  $\omega_{\mathcal{B}}$  ('omega b'). In order to make various formulas work out we define a vector angular velocity with magnitude  $\omega_{\mathcal{B}}$  which is perpendicular to the  $xy$  plane:

$$\vec{\omega}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}} \hat{k}$$

where  $\dot{\theta}$  is the rate of change of the angle of *any* line marked on body  $\mathcal{B}$ .

So long as you are careful to define angular velocity by the rotation of line segments and not by the motion of individual particles, the concept of angular velocity in general

motion is defined exactly as for a body rotating about a fixed axis. A legitimate way to think about planar motion of a rigid body is that any given point is moving in circles about any other given point (relative to that point). When a rigid body moves it always has an angular velocity (possibly zero). If we call the body  $\mathcal{B}$  (script B), we then call the body's angular velocity  $\vec{\omega}_{\mathcal{B}}$ . In general it is best to use the sign convention that when  $\omega_{\mathcal{B}} > 0$  the body is rotating counterclockwise when viewed looking in from the positive  $z$  axis (see Fig. 8.6).

The angular velocity vector  $\vec{\omega}_{\mathcal{B}}$  of a body  $\mathcal{B}$  describes its rate and direction of rotation. For planar motions  $\vec{\omega}_{\mathcal{B}} = \omega_{\mathcal{B}} \hat{k}$ .

## Relative velocity of two points on a rigid body

For any two points A and B glued to a rigid body  $\mathcal{B}$  the relative velocity of the points ('the velocity of B relative to A') is given by the cross product of the angular velocity of the body with the relative position of the two points:

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A = \vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}. \quad (8.3)$$

This formula says that the relative velocity of two points on a rigid body is the same as would be predicted for one of the points if the other were stationary. The derivation of this formula is the same as for planar circular motion.

Note that even though we are doing planar kinematics, it is convenient to use three dimensional cross products. Generally we will call the plane of motion the  $xy$  plane and  $\vec{\omega}$  will be in the  $z$  direction. Because  $\vec{\omega} \times \vec{r}$  must be perpendicular to  $\vec{\omega}$  it is perpendicular to the  $z$  axis. So this three dimensional cross product always gives a vector in the  $xy$  plane that is perpendicular to  $\vec{r}$ .

We can also represent the relative velocity in the changing base notation as

$$\begin{aligned} \vec{v}_{B/A} &= \frac{d}{dt} \left( x'_{B/A} \hat{i}' + x'_{B/A} \hat{j}' \right) \\ &= x'_{B/A} \frac{d}{dt} \hat{i}' + x'_{B/A} \frac{d}{dt} \hat{j}' \\ &= x'_{B/A} \vec{\omega}_{\mathcal{B}} \times \hat{i}' + x'_{B/A} \vec{\omega}_{\mathcal{B}} \times \hat{j}'. \end{aligned}$$

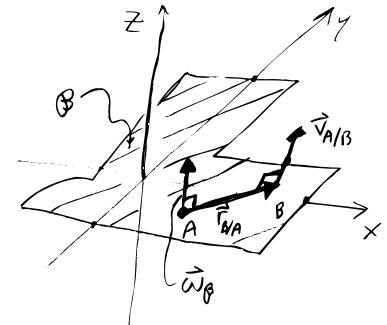


Figure 8.7: The relative velocity of points A and B is in the  $xy$  plane and perpendicular to the line segment AB.  
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Finally, we can use the fixed-base or component notation:

$$\begin{aligned} [\vec{v}_{B/A}]_{xy} &= \frac{d}{dt} \begin{bmatrix} x_{B/A} \\ y_{B/A} \end{bmatrix} \\ &= \frac{d}{dt} \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x^*_{B/A} \\ y^*_{B/A} \end{bmatrix} \right\} \\ &= \begin{bmatrix} -\dot{\theta} \sin \theta & \dot{\theta} \cos \theta \\ -\dot{\theta} \cos \theta & -\dot{\theta} \sin \theta \end{bmatrix} \begin{bmatrix} x^*_{B/A} \\ y^*_{B/A} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x^*_{B/A} \\ y^*_{B/A} \end{bmatrix} \end{aligned}$$

where  $x^*_{B/A}$  and  $y^*_{B/A}$  are the components of the position of B with respect to A in the reference configuration and hence do not change with time.

## Absolute velocity of a point on a rigid body

If one knows the velocity of one point on a rigid body and one also knows the angular velocity of the body, then one can find the velocity of any other point. How? By addition. Say we know the velocity of point A, the angular velocity of the body, and the relative position of A and B, then

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + (\vec{v}_B - \vec{v}_A) \\ &= \vec{v}_A + \vec{v}_{B/A} \\ &= \vec{v}_A + \vec{\omega}_{\mathcal{B}} \times \underbrace{\vec{r}_{B/A}}_{\vec{r}_B - \vec{r}_A}.\end{aligned}\quad (8.4)$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A. Because B and A are on the same rigid body, their relative velocity is given by formula 8.4 above. For ease of understanding one pretends one knows the quantities on the right and are trying to find the quantity on the left. But the equation is valid and useful no matter which quantities are known and which are not.

An alternative approach is to differentiate the coordinate expression eqn. (8.3) (see Box 8.1 on 442).

## Angular acceleration

We define the angular acceleration  $\vec{\alpha}$  ('alpha') of a rigid body as the rate of change of angular velocity,  $\vec{\alpha} = \dot{\vec{\omega}}$ . The angular acceleration of a body  $\mathcal{B}$  is  $\vec{\alpha}_{\mathcal{B}}$ . For two-dimensional bodies moving in the plane both the angular velocity and the angular acceleration are always perpendicular to the plane. That is  $\vec{\omega} = \omega \hat{k}$  and  $\vec{\alpha} = \alpha \hat{k} = \dot{\omega} \hat{k}$ . In 2-D the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e.,  $\alpha = \dot{\omega} = \ddot{\theta}$ .

### 8.1 THEORY

#### Using matrices to find velocity from position

An alternative derivation for the velocity eqn. (8.3) of a point on a rigid body comes from differentiating the matrix formula for the position (eqn. (8.3)). Denoting  $\vec{r}_{P/O}$  as the reference position of the particle and  $\vec{r}_{P'/O'}$  as the position relative to the reference point on the moved body at the time of interest, we have:

$$\begin{aligned}[\vec{v}_{P/0}]_{xy} &= \frac{d}{dt} [\vec{r}_{P/0}]_{xy} \\ &= \frac{d}{dt} \begin{bmatrix} x_{0/0} \\ y_{0/0} \end{bmatrix} \\ &\quad + \frac{d}{dt} \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_{P/0}^* \\ y_{P/0}^* \end{bmatrix} \right\} \\ &= \begin{bmatrix} \dot{x}_{0/0} \\ \dot{y}_{0/0} \end{bmatrix} + \begin{bmatrix} -\dot{\theta} \sin \theta & \dot{\theta} \cos \theta \\ -\dot{\theta} \cos \theta & -\dot{\theta} \sin \theta \end{bmatrix} \begin{bmatrix} x_{P/0}^* \\ y_{P/0}^* \end{bmatrix}.\end{aligned}$$

$$\begin{aligned}&= \begin{bmatrix} \dot{x}_{0/0} \\ \dot{y}_{0/0} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} -\cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_{P/0}^* \\ y_{P/0}^* \end{bmatrix} \\ &= \begin{bmatrix} \dot{x}_{0/0} \\ \dot{y}_{0/0} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} [\vec{r}_{P'/0'}]_{xy}.\end{aligned}$$

Thus, matrix product  $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} [\vec{r}_{P'/0'}]_{xy}$  is equivalent to the vector product  $\vec{\omega} \times \vec{r}_{P'/0'}$  and the matrix  $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$  is sometimes called the angular velocity matrix. It is an example of a so-called *skew symmetric* matrix because it is the negative of its own transpose.

## Relative acceleration of two points on a rigid body

For any two points A and B glued to a rigid body  $\mathcal{B}$ , the acceleration of B relative to A is

$$\begin{aligned}\vec{a}_{B/A} &= \frac{d}{dt} \vec{v}_{B/A} \\ &= \frac{d}{dt} \left\{ \vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A} \right\} \\ &= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} + \vec{\omega}_{\mathcal{B}} \times (\vec{v}_{B/A}), \\ &= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}), \\ &= \alpha_{\mathcal{B}} \hat{\mathbf{k}} \times \vec{r}_{B/A} + (-\omega_{\mathcal{B}}^2 \vec{r}_{B/A}),\end{aligned}\tag{8.5}$$

This is the base-independent or direct-vector expression for relative acceleration. If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles from chapter 7. On a rigid body in 2D all two points on rigid body can do relative to each other is to go in circles.

Equation (8.5) could also be derived, with some algebra, by taking two time derivatives of the relative position coordinate expression

$$\left[ \vec{r}_{B/A} \right]_{xy} = [R(\theta)] \left[ \vec{r}_{B/A}^* \right]_{x'y'}$$

or by taking two time derivatives of the changing base vector expression

$$\vec{r}_{B/A} = x'_{B/A} \hat{\mathbf{i}}' + y'_{B/A} \hat{\mathbf{j}}'.$$

## Absolute acceleration of a point on a rigid body

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How?

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + (\vec{a}_B - \vec{a}_A) = \vec{a}_A + \vec{a}_{B/A} \\ &= \vec{a}_A + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}) + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} \\ &= \vec{a}_A - \omega_{\mathcal{B}}^2 \vec{r}_{B/A} + \alpha_{\mathcal{B}} \hat{\mathbf{k}} \times \vec{r}_{B/A}\end{aligned}\tag{8.6}$$

This is the base-independent or direct-vector expression for acceleration. The fixed-base (component) and changing-base notations are somewhat more complex.

Equation 8.7 is often called the three term acceleration formula. The acceleration of a point B on a rigid body is the sum of three terms. The first,  $\vec{a}_A$ , is the acceleration of some point A on the body. The second term,  $\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A})$ , is the centripetal acceleration of B going in circles relative to A. It is directed from B towards A. The third term,  $\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A}$ , is due to the change in the magnitude of the angular velocity and is in the direction normal to the line from A to B.

**Example: Robot arm**

Given the configuration shown in Fig. 8.8 the acceleration of point B can be found by thinking of link AB as the body  $\mathcal{B}$  in eqn. (8.7) and using what you know about circular motion to find the acceleration of A:

$$\begin{aligned}\vec{a}_B &= \vec{a}_A - \omega_{\mathcal{B}}^2 \vec{r}_{B/A} + \alpha_{\mathcal{B}} \hat{\mathbf{k}} \times \vec{r}_{B/A} \\ &= (-\omega_{OA}^2 \ell \hat{\mathbf{j}} - \dot{\omega}_{OA} \ell \hat{\mathbf{i}}) - (\omega_{AB}^2 \ell \hat{\mathbf{i}}) + (\dot{\omega}_{AB} \hat{\mathbf{k}} \times (\ell \hat{\mathbf{i}})) \\ &= -(\dot{\omega}_{OA} \ell + \omega_{AB}^2 \ell) \hat{\mathbf{i}} + (-\omega_{OA}^2 \ell + \dot{\omega}_{AB} \ell) \hat{\mathbf{j}}\end{aligned}$$

[Note that  $\omega_{AB} \neq \dot{\theta}$  where  $\theta$  is the angle between the links. Rather  $\omega_{AB} = \omega_{OA} + \dot{\theta}$ .]  $\square$

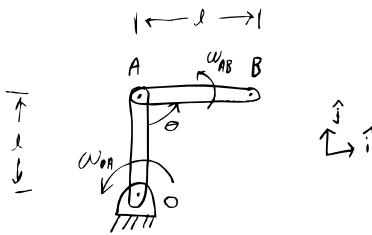


Figure 8.8: A two link robot arm.

(Filename:tfigure.robotarm)

**Computer graphics**

Given one point given by the  $xy$  pair  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  we can find out what happens to it by rotation  $[R]$  as

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

For example the point  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  gets changed by a 45 deg rotation to

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix}.$$

A translation is just a vector addition. For example the point  $\begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix}$  gets translated a distance 2 in the  $y$  direction by the addition of  $\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  like this

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{translated}} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 2.4 \end{bmatrix}.$$

Putting these together the point  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  gets rotated and translated by first multiplying by the rotation matrix and then adding the translation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.4 \\ 2.4 \end{bmatrix}.$$

A collection of points all rotated the same amount and then all translated the same amount keep their relative distances.

A picture is a set of points on a plane. If all the points are rotated and translated the same amount the picture is rotated and translated. Thus a picture of a rigid body described by points is rigidly rotated and translated. On a computer line drawings are often represented as a connect-the-dots picture. The picture is represented by the  $x$  and  $y$  coordinates of the reference dots at the corners. These can be stored in an array with the first row being the  $x$  coordinates and the second row the  $y$  coordinates as explained on page 381. Each column of this matrix represents one point of the

connect-the-dots picture. Thus a primitive picture of a house at the origin is given by the array

$$[P_0] \equiv [xy \text{ points originally}] = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{bmatrix}$$

with the lower left corner of the house at the origin.

To rotate this picture we rotate each of the columns of the matrix  $P_0$ . But this is exactly what is accomplished by the matrix multiplication  $[R][P_0]$ . To translate the points you add the translation vector to each of the columns of the resulting matrix. Thus the whole picture rotated by  $45^\circ$  and translated up by 1 is given by

$$[P_{\text{new}}] = [R][P_0] + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} [P_0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives a new array of points that, when connected give the picture shown. We have allowed the informal notation of adding a column matrix to a rectangular matrix, by which we mean adding to each column of the rectangular matrix.

To animate the motion of, say, a house flying in the Wizard of Oz you would first define the house as the set of points  $[P_0]$ . Then define, maybe by means of numerical solution of differential equations, a set of rotations and translations. Then for each rotation and translation the picture of the house should be drawn, one after the other. The sequence of such pictures is an animation of a flying and spinning house.

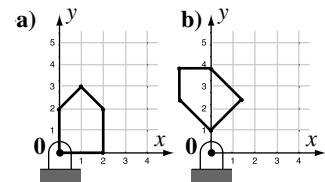


Figure 8.9: (a) A house drawn as 6 dots connected by line segments. The first and last point are the same. (b) The same house but rigidly rotated and translated.

(Filename:figure.rotatedhouse2)

## Summary of the kinematics of one rigid body in general 2D motion

You can use the position of one reference point and the rotation of the body as simple kinematic measures of the entire motion of the body. If you know the position, velocity, and acceleration of one point on a rigid body (represented by 6 scalars, say), and you know the rotation angle, angular rate and angular acceleration (3 scalars) then you can find the position, velocity and acceleration of any point on the body. In 2D, just 9 numbers tell you the position, velocity, and acceleration of any of the billions of points whose initial positions you know<sup>①</sup>.

<sup>①</sup> In 1D it takes just 3 numbers and in 3D just 18. The unusual pattern (3,9,18) comes from rotation being characterized by 0, 1, and 3 numbers in 1, 2, and 3 dimensions, respectively.

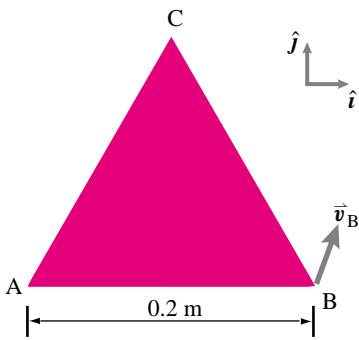


Figure 8.10: (Filename:sfig9.1.triang.1)

**SAMPLE 8.1** Velocity of a point on a rigid body in planar motion. An equilateral triangular plate ABC is in motion in the  $x$ - $y$  plane. At the instant shown in the figure, point B has velocity  $\vec{v}_B = 0.3 \text{ m/s} \hat{i} + 0.6 \text{ m/s} \hat{j}$  and the plate has angular velocity  $\vec{\omega} = 2 \text{ rad/s} \hat{k}$ . Find the velocity of point A.

**Solution** We are given  $\vec{v}_B$  and  $\vec{\omega}$ , and we need to find  $\vec{v}_A$ , the velocity of point A on the same rigid body. We know that,

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

Thus, to find  $\vec{v}_A$ , we need to find  $\vec{r}_{A/B}$ . Let us take an  $x$ - $y$  coordinate system whose origin coincides with point A of the plate at the instant of interest and the  $x$ -axis is along AB. Then,

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2 \text{ m} \hat{i}) = -0.2 \text{ m} \hat{i}$$

Thus,

$$\begin{aligned}\vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \\ &= (0.3 \hat{i} + 0.6 \hat{j}) \text{ m/s} + 2 \text{ rad/s} \hat{k} \times (-0.2 \hat{i}) \text{ m} \\ &= (0.3 \hat{i} + 0.6 \hat{j}) \text{ m/s} - 0.4 \hat{j} \text{ m/s} \\ &= (0.3 \hat{i} + 0.2 \hat{j}) \text{ m/s.}\end{aligned}$$

$$\boxed{\vec{v}_A = (0.3 \hat{i} + 0.2 \hat{j}) \text{ m/s}}$$

**SAMPLE 8.2** The instantaneous center of rotation. A rigid body is in planar motion. At some instant  $t$ , the angular velocity of the body is  $\vec{\omega} = 5 \text{ rad/s} \hat{k}$  and the linear velocity of a point C on the body is  $\vec{v}_C = 2 \text{ m/s} \hat{i} - 5 \text{ m/s} \hat{j}$ . Find a point on the body, assuming it exists, that has zero velocity. ①

**Solution** Let the point of zero velocity be O, with position vector  $\vec{r}_{O/C}$  with respect to point C. Since  $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$ , for  $\vec{v}_O$  to be zero,  $\vec{\omega} \times \vec{r}_{O/C}$  must be parallel to and in the opposite direction of  $\vec{v}_C$ . Since  $\vec{\omega}$  is out of plane,  $\vec{r}_{O/C}$  must be normal to  $\vec{v}_C$  for the cross product to be parallel to  $\vec{v}_C$ . Now, let  $\vec{v}_C = v_C \hat{\lambda}$ . Then,  $\vec{r}_{O/C} = r \hat{n}$  where  $\hat{n} \perp \hat{\lambda}$  and  $r = |\vec{r}_{O/C}|$ . Thus,

$$v_C \hat{\lambda} + \omega \hat{k} \times r \hat{n} = \vec{v}_O = \vec{0} \quad (8.7)$$

Dotting eqn. (8.7) with  $\hat{\lambda}$ , we get

$$v_C = \omega r \Rightarrow r = \frac{v_C}{\omega} = \frac{\sqrt{29} \text{ m/s}}{5 \text{ rad/s}} = 1.08 \text{ m.}$$

Since  $\hat{\lambda} = \vec{v}_C / |\vec{v}_C| = 0.37 \hat{i} - 0.93 \hat{j}$ ,  $\hat{n} = 0.93 \hat{i} + 0.37 \hat{j}$ . Thus

$$\vec{r}_{O/C} = r \hat{n} = 1.08 \text{ m}(0.93 \hat{i} + 0.37 \hat{j}) = 1 \text{ m} \hat{i} + 0.4 \text{ m} \hat{j}.$$

$$\boxed{\vec{r}_{O/C} = 1 \text{ m} \hat{i} + 0.4 \text{ m} \hat{j}}$$

**Note:** It is also possible to find  $\vec{r}_{O/C}$  purely by vector algebra. Assuming  $\vec{r}_{O/C} = (x \hat{i} + y \hat{j}) \text{ m}$  and plugging into  $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$  along with the given values, we get  $\vec{0} = (2 - 5y) \text{ m/s} \hat{i} + (-5 + 5x) \text{ m/s} \hat{j}$ . Dotting this equation with  $\hat{i}$  and  $\hat{j}$ , we get  $2 - 5y = 0$  and  $-5 + 5x = 0$ , which give  $x = 1$  and  $y = 0.4$ . Thus,  $\vec{r}_{O/C} = 1 \text{ m} \hat{i} + 0.4 \text{ m} \hat{j}$  as obtained above.

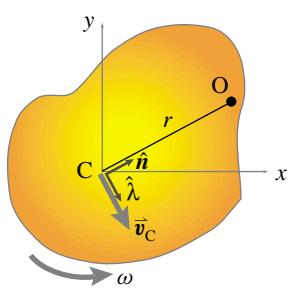


Figure 8.11: (Filename:sfig9.1.2.body)

**SAMPLE 8.3** A cheerleader throws her baton up in the air in the vertical  $xy$ -plane. At an instant when the baton axis is at  $\theta = 60^\circ$  from the horizontal, the velocity of end A of the baton is  $\vec{v}_A = 2 \text{ m/s} \hat{i} + \sqrt{3} \text{ m/s} \hat{j}$ . At the same instant, end B of the baton has velocity in the negative  $x$ -direction (but  $|\vec{v}_B|$  is not known). If the length of the baton is  $\ell = \frac{1}{2} \text{ m}$  and the center of mass is in the middle of the baton, find the velocity of the center of mass.

### Solution

$$\begin{aligned} \text{We are given: } \vec{v}_A &= (2\hat{i} + \sqrt{3}\hat{j}) \text{ m/s} \\ \text{and } \vec{v}_B &= -v_B \hat{i} \end{aligned}$$

where  $v_B = |\vec{v}_B|$  is unknown. We need to find  $\vec{v}_G$ . Using the relative velocity formula for two points on a rigid body, we can write:

$$\vec{v}_G = \vec{v}_A + \vec{\omega} \times \vec{r}_{G/A} \quad (8.8)$$

Here,  $\vec{v}_A$  and  $\vec{r}_{G/A}$  are known. Thus, to find  $\vec{v}_G$ , we need to find  $\vec{\omega}$ , the angular velocity of the baton. Since the motion is in the vertical  $xy$ -plane, let  $\vec{\omega} = \omega \hat{k}$ . Then,

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{A/B} = \vec{v}_A + \omega \hat{k} \times \underbrace{\ell(-\cos \theta \hat{i} + \sin \theta \hat{j})}_{\vec{r}_{A/B}} \\ \text{or } -v_B \hat{i} &= (2\hat{i} + \sqrt{3}\hat{j}) \text{ m/s} - \omega \ell(\cos \theta \hat{j} + \sin \theta \hat{i}) \\ &= (2\hat{i} + \sqrt{3}\hat{j}) \text{ m/s} - \omega \cdot \frac{1}{2} \text{ m} \cdot (\frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{i}) \end{aligned}$$

Dotting both sides of this equation with  $\hat{j}$  we get:

$$\begin{aligned} 0 &= \sqrt{3} \text{ m/s} - \frac{\omega}{2} \text{ m} \cdot \frac{1}{2} \\ \Rightarrow \omega &= \sqrt{3} \frac{\text{m}}{\text{s}} \cdot \frac{4}{1 \text{m}} = 4\sqrt{3} \text{ rad/s.} \end{aligned}$$

Now substituting the appropriate values in Eqn 8.8 we get:

$$\begin{aligned} \vec{v}_G &= \vec{v}_A + \omega \hat{k} \times \underbrace{\frac{\ell}{2}(\cos \theta \hat{i} - \sin \theta \hat{j})}_{\vec{r}_{G/A}} \\ &= \vec{v}_A + \frac{\omega \ell}{2}(\cos \theta \hat{j} + \sin \theta \hat{i}) \\ &= (2\hat{i} + \sqrt{3}\hat{j}) \text{ m/s} + \sqrt{3} \text{ m/s} \cdot (\frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{i}) \\ &= (2 + \frac{3}{2}) \text{ m/s} \hat{i} + (\sqrt{3} + \frac{\sqrt{3}}{2}) \text{ m/s} \hat{j} \\ &= 3.5 \text{ m/s} \hat{i} + 2.6 \text{ m/s} \hat{j} \end{aligned}$$

$\vec{v}_G = (3.5\hat{i} + 2.6\hat{j}) \text{ m/s}$

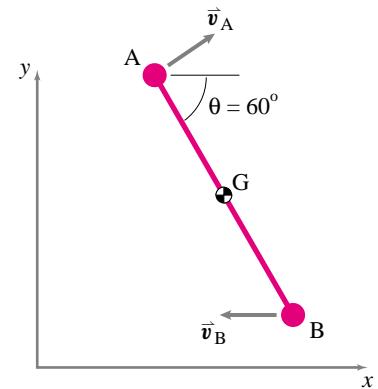


Figure 8.12: (Filename:sfig7.1.1)

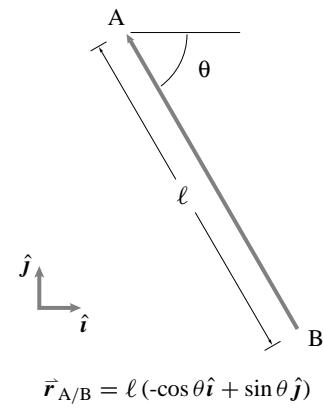


Figure 8.13: (Filename:sfig7.1.1a)

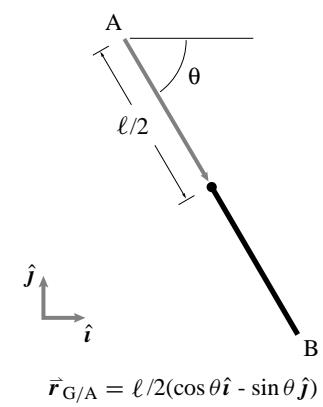


Figure 8.14: (Filename:sfig7.1.1b)

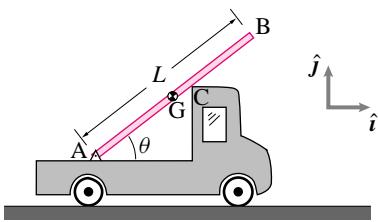


Figure 8.15: (Filename:sfig7.2.2a)

**SAMPLE 8.4** *A board in the back of an accelerating truck.* A 10 ft long board AB rests in the back of a flat-bed truck as shown in Fig. 8.15. End A of the board is hinged to the bed of the truck. The truck is going on a level road at 55 mph. In preparation for overtaking a vehicle in the front the trucker accelerates at a constant rate 3 mph/s. At the instant when the speed of the truck is 60 mph, the magnitude of the relative velocity and relative acceleration of end B with respect to the bed of the truck are 10 ft/s and 12 ft/s<sup>2</sup>, respectively. There is wind and at this instant, the board has lost contact with point C. If the angle θ between the board and the bed is 45° at the instant mentioned, find

- the angular velocity and angular acceleration of the board,
- the absolute velocity and absolute acceleration of point B, and
- the acceleration of the center of mass of the board.

**Solution** At the instant of interest

$$\begin{aligned}
 \vec{v}_A &= \text{velocity of the truck} = 60 \text{ mph } \hat{i} \\
 &= 88 \text{ ft/s } \hat{i} \\
 \vec{a}_A &= \text{acceleration of the truck} = 3 \text{ mph/s} \\
 &= 4.4 \text{ ft/s}^2 \hat{i} \\
 |\vec{v}_{B/A}| &= v_{B/A} = \text{magnitude of relative velocity of B} \\
 &= 10 \text{ ft/s} \\
 |\vec{a}_{B/A}| &= a_{B/A} = \text{magnitude of relative acceleration of B} \\
 &= 12 \text{ ft/s}^2.
 \end{aligned}$$

Let  $\vec{\omega} = \omega \hat{k}$  be the angular velocity and  $\vec{\dot{\omega}} = \dot{\omega} \hat{k}$  be the angular acceleration of the board.

- The relative velocity of end B of the board with respect to end A is

$$\begin{aligned}
 \vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\
 &= \omega \hat{k} \times L(\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= \omega L(\cos \theta \hat{j} - \sin \theta \hat{i}) \\
 \Rightarrow |\vec{v}_{B/A}| &= \omega L \\
 \Rightarrow \omega &= \frac{|\vec{v}_{B/A}|}{L} = \frac{v_{B/A}}{L} \\
 &= \frac{10 \text{ ft/s}}{10 \text{ ft}} = 1 \text{ rad/s}.
 \end{aligned}$$

Note that we have taken the positive value for  $\omega$  because the board is rotating counterclockwise at the instant of interest (it is given that the board has lost contact with point C).

Similarly, we can compute the angular acceleration:

$$\begin{aligned}
 \vec{a}_{B/A} &= \vec{\dot{\omega}} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \\
 &= \dot{\omega} \hat{k} \times L(\cos \theta \hat{i} + \sin \theta \hat{j}) - \omega^2 L(\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= \dot{\omega} L(\cos \theta \hat{j} - \sin \theta \hat{i}) - \omega^2 L(\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 \Rightarrow |\vec{a}_{B/A}| &= \sqrt{(\dot{\omega} L)^2 + (\omega^2 L)^2} = a_{B/A} \text{ (given)} \\
 \Rightarrow a_{B/A}^2 &= (\dot{\omega} L)^2 + (\omega^2 L)^2 \\
 \Rightarrow \dot{\omega} &= \sqrt{\frac{a_{B/A}^2}{L^2} - \omega^4}
 \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{\left(\frac{12 \text{ ft/s}^2}{10 \text{ ft}}\right)^2 - (1 \text{ rad/s})^4} \\
 &= \pm 0.663 \text{ rad/s}^2.
 \end{aligned}$$

Once again, we select the positive value for  $\dot{\omega}$  since we assume that the board accelerates counterclockwise.

$$\vec{\omega} = 1 \text{ rad/s} \hat{k}, \quad \dot{\vec{\omega}} = 0.663 \text{ rad/s}^2 \hat{k}$$

- (b) The absolute velocity and the absolute acceleration of the end point B can be found as follows.

$$\begin{aligned}
 \vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\
 &= v_A \hat{i} + v_{B/A} (\cos \theta \hat{j} - \sin \theta \hat{i}) \\
 &= 88 \text{ ft/s} \hat{i} + 10 \text{ ft/s} \left( \frac{1}{\sqrt{2}} \hat{j} - \frac{1}{\sqrt{2}} \hat{i} \right) \\
 &= 80.93 \text{ ft/s} \hat{i} + 7.07 \text{ ft/s} \hat{j}.
 \end{aligned}$$

$$\begin{aligned}
 \vec{a}_B &= \vec{a}_A + \vec{a}_{B/A} \\
 &= \vec{a}_A + \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \\
 &= a_A \hat{i} + \dot{\omega} \hat{k} \times L (\cos \theta \hat{i} + \sin \theta \hat{j}) - \omega^2 L (\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= (a_A - \dot{\omega} L \sin \theta - \omega^2 L \cos \theta) \hat{i} + (\dot{\omega} L \cos \theta - \omega^2 L \sin \theta) \hat{j} \\
 &= \left( 4.4 \text{ ft/s}^2 - \frac{0.66}{\text{s}^2} \cdot 10 \text{ ft} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2} \cdot 10 \text{ ft} \cdot \frac{1}{\sqrt{2}} \right) \hat{i} \\
 &\quad + \left( \frac{0.66}{\text{s}^2} \cdot 10 \text{ ft} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2} \cdot 10 \text{ ft} \cdot \frac{1}{\sqrt{2}} \right) \hat{j} \\
 &= -7.34 \text{ ft/s}^2 \hat{i} - 2.40 \text{ ft/s}^2 \hat{j}.
 \end{aligned}$$

$$\vec{v}_B = (80.93 \hat{i} + 7.07 \hat{j}) \text{ ft/s}, \quad \vec{a}_B = -(7.34 \hat{i} + 2.40 \hat{j}) \text{ ft/s}^2.$$

- (c) Now, we can easily calculate the acceleration of the center of mass as follows.

$$\begin{aligned}
 \vec{a}_G &= \vec{a}_A + \vec{a}_{G/A} \\
 &= a_A \hat{i} + \dot{\vec{\omega}} \times \vec{r}_{G/A} - \omega^2 \vec{r}_{G/A} \\
 &= a_A \hat{i} + \dot{\omega} \hat{k} \times \frac{L}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) - \omega^2 \frac{L}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= a_A \hat{i} + \dot{\omega} \frac{L}{2} (\cos \theta \hat{j} - \sin \theta \hat{i}) - \omega^2 \frac{L}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= 4.4 \text{ ft/s}^2 \hat{i} + 0.663 \text{ rad/s}^2 \cdot \frac{10 \text{ ft}}{2} \cdot \left( \frac{1}{\sqrt{2}} \hat{j} - \frac{1}{\sqrt{2}} \hat{i} \right) \\
 &\quad - (1 \text{ rad/s})^2 \cdot \frac{10 \text{ ft}}{2} \cdot \left( \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} \right) \\
 &= -1.48 \text{ ft/s}^2 \hat{i} - 1.19 \text{ ft/s}^2 \hat{j}.
 \end{aligned}$$

$$\vec{a}_G = -(1.48 \hat{i} + 1.19 \hat{j}) \text{ ft/s}^2$$

**Comments:** This problem is admittedly artificial. We, however, solve this problem to show kinematic calculations.

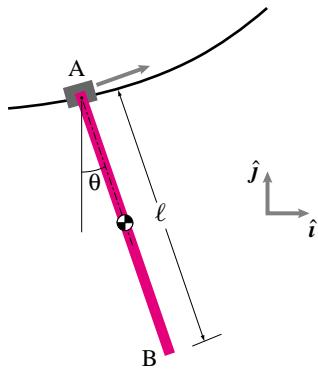


Figure 8.16: (Filename:sfig9.2.rodontrack)

**SAMPLE 8.5** *Tracking motion.* A cart moves along a suspended curved path. A rod AB of length  $\ell = 1$  m hangs from the cart. End A of the rod is attached to a motor on the cart. The other end B hangs freely. The motor rotates the rod such that  $\theta(t) = \theta_0 \sin(\lambda t)$  while the cart moves along the path such that  $\vec{r}_A = t\hat{i} + \frac{t^3}{18}\hat{j}$ , where all variables ( $r$ ,  $t$ , etc.) are nondimensional.

- Find the velocity and acceleration of point B as a function of nondimensional time  $t$ .
- Take  $\theta_0 = \pi/3$  and  $\lambda = 6$ . Find and plot the position of the bar at  $t = 0, 0.1, 0.3, 0.9, 1, 1.1, 1.2$ , and  $1.5$ . Find and draw  $\vec{v}_B$  and  $\vec{a}_B$  at the specified  $t$ .

### Solution

- The velocity and acceleration of point B are given by

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \\ \vec{a}_B &= \vec{a}_A + \vec{\dot{\omega}} \times \vec{r}_{B/A} - \vec{\omega}^2 \vec{r}_{B/A}.\end{aligned}$$

Thus, in order to find the velocity and acceleration of point B, we need to find the velocity and acceleration of point A and the angular velocity and angular acceleration of the bar. We are given the position of point A and the angular position of the bar as functions of  $t$ . We can, therefore, find  $\vec{v}_A$ ,  $\vec{a}_A$ ,  $\vec{\omega}$ , and  $\vec{\dot{\omega}}$  by differentiating the given functions with respect to  $t$ .

$$\vec{r}_A = t\hat{i} + \frac{t^3}{18}\hat{j} \quad (8.9)$$

$$\Rightarrow \vec{v}_A \equiv \dot{\vec{r}}_A = \hat{i} + (t^2/6)\hat{j} \quad (8.9)$$

$$\Rightarrow \vec{a}_A \equiv \ddot{\vec{r}}_A = (t/3)\hat{j} \quad (8.10)$$

and

$$\theta\hat{k} = \theta_0 \sin(\lambda t)\hat{k} \quad (8.11)$$

$$\Rightarrow \vec{\omega} \equiv \dot{\theta}\hat{k} = \theta_0 \lambda \cos(\lambda t)\hat{k} \quad (8.11)$$

$$\Rightarrow \vec{\dot{\omega}} \equiv \ddot{\theta}\hat{k} = -\theta_0 \lambda^2 \sin(\lambda t)\hat{k}. \quad (8.12)$$

So,

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega} \times \ell(\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= \hat{i} + (t^2/6)\hat{j} + \ell \dot{\theta} (\sin \theta \hat{j} + \cos \theta \hat{i}) \\ &= (1 + \ell \dot{\theta} \cos \theta)\hat{i} + (t^2/6 + \ell \dot{\theta} \sin \theta)\hat{j}\end{aligned} \quad (8.13)$$

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{\dot{\omega}} \times \ell(\sin \theta \hat{i} - \cos \theta \hat{j}) - \dot{\theta}^2 \ell(\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= (t/3)\hat{j} + \ell \ddot{\theta} \sin \theta \hat{j} + \ell \ddot{\theta} \cos \theta \hat{i} + \ell \dot{\theta}^2 \sin \theta \hat{i} + \ell \dot{\theta}^2 \cos \theta \hat{j} \\ &= \ell(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)\hat{i} + [t/3 + \ell(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)]\hat{j}\end{aligned} \quad (8.14)$$

where  $\theta = \theta_0 \sin(\lambda t)$ ,  $\dot{\theta} = \theta_0 \lambda \cos(\lambda t)$ , and  $\ddot{\theta} = -\theta_0 \lambda^2 \sin(\lambda t) = -\lambda^2 \theta$ . Thus  $\vec{v}_B$  and  $\vec{a}_B$  are functions of  $t$ .

- The position of the rod at any time  $t$  is specified by the position of the two end points A and B (or alternatively, the position of A and the angle of the rod  $\theta$ ). The position of point A is easily determined by substituting the value of  $t$  in the given expression for  $\vec{r}_A$ . The position of end B is given by

$$\begin{aligned}\vec{r}_B &= \vec{r}_A + \vec{r}_{B/A} = t\hat{i} + (t^3/18)\hat{j} + \ell(\sin \theta \hat{i} - \cos \theta \hat{j}) \\ &= (t + \ell \sin \theta)\hat{i} + (t^3/18 - \ell \cos \theta)\hat{j}.\end{aligned}$$

To compute the positions, velocities, and accelerations of end points A and B at the given instants, we first compute  $\theta$ ,  $\dot{\theta}$ , and  $\ddot{\theta}$ , and then substitute them in the expressions for  $\vec{r}_A$ ,  $\vec{r}_B$ ,  $\vec{v}_A$ ,  $\vec{v}_B$ ,  $\vec{a}_A$ , and  $\vec{a}_B$ . A pseudocode for computer calculation is given below.

```
t = [0 0.1 0.3 0.9 1.0 1.1 1.2 1.5]
theta0=pi/3, L=.5, lam=6
for each t, compute
    theta = theta0*sin(lam*t)
    w = lam*theta0*cos(lam*t)
    wdot = -lam^2*theta
        % Position of A and B
    xA=t, yA=t^3/18
    xB = xA + L*sin(theta)
    yB = yA - L*cos(theta)
        % Velocity of A and B
    uA = 1, vA = t^2/6
    uB = uA + L*w.*cos(theta)
    vB = vA + L*w.*sin(theta)
        % Acceleration of A and B
    axA = 0, ayA = t/3
    axB = L*wdot*cos(theta) - L*w^2*sin(theta)
    ayB = ayA + L*wdot*sin(theta) + L*w^2*cos(theta)
```

From the above calculation, we get the desired quantities at each  $t$ . For example, at  $t = 0$  we get,

$$\begin{aligned} xA &= 0, yA = 0, xB = 0, yB = -0.5 \\ uA &= 1, vA = 0, uB = 4.14, vB = 0, axB = 0, ayB = 19.74 \end{aligned}$$

which means,

$$\vec{r}_A = \vec{0}, \quad \vec{r}_B = -0.5\hat{j}, \quad \vec{v}_A = \hat{i}, \quad \vec{v}_B = 4.14\hat{i}, \quad \vec{a}_B = 19.74\hat{j}.$$

The position of the bar, the velocity vectors at points A and B, and the acceleration vector at B, thus obtained, are shown in Fig. 8.17 graphically. ①

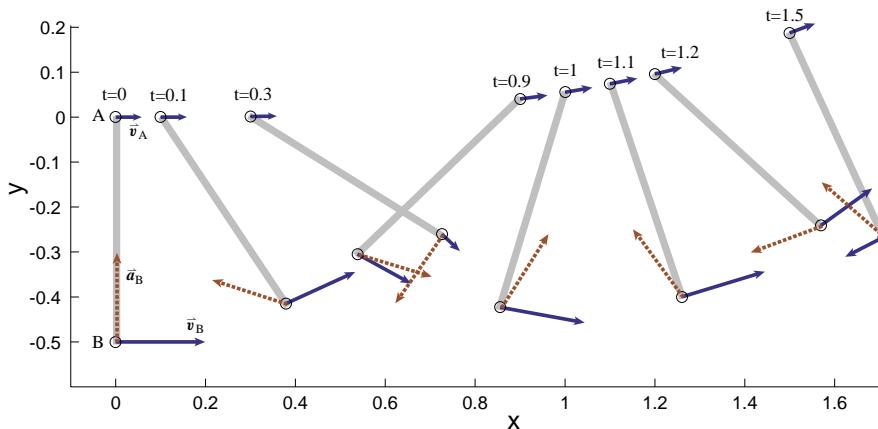


Figure 8.17: Position, velocity of the end points A and B, and acceleration of point B at various time instants.  
② (Filename:rodvelacc)

① We can take several values of  $t$ , say 400 equally spaced values between  $t = 0$  and  $t = 4$ , and draw the bar at each  $t$  to see its motion and the trajectory of its end points. Fig. 8.18 shows such a graph.

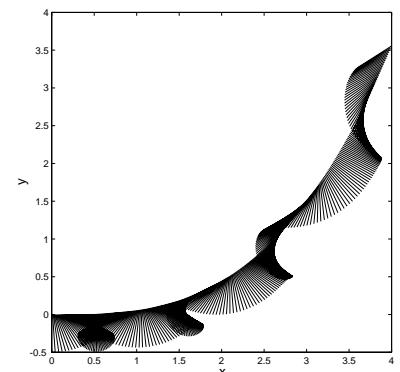


Figure 8.18: Graph of closely spaced configuration of the bar between  $t = 0$  to  $t = 4$ .  
③ (Filename:sfig9.1.rodconfig)

## 8.2 General planar mechanics of a rigid-body

**① Advanced aside:** What we call “simple measures” are examples of “generalized coordinates” in more advanced books. The idea sounds intimidating, but is simply this: If something can only move in a few ways, you should only keep track of the motion with that many variables. The kinematics of a rigid body (Sect. 8.1) allow us to “evaluate” the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change in terms of these “simple measures”. By “evaluate” we mean express the motion quantities in terms of these measures. The alternative is as sums over Avogadro’s number of particles (There are on the order of  $10^{23}$  atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we would still be stuck with integrals over complicated regions if we did not describe the motion with as few variables as possible. In the case of 2-D rigid body motion, the position of a reference point ( $x$  and  $y$ ) with the rotation  $\theta$  is called a set of *minimal coordinates*. These, and their time derivatives are the minimal information needed to describe all important mechanics motion quantities.

We now apply the kinematics ideas of the last section to the general mechanics principles in Table I in the inside cover. The goal is to understand the relation between forces and motion for a planar body in general 2-D motion. The simple measures<sup>①</sup> of motion will be the displacement, velocity and acceleration of one reference point  $0'$  on the body ( $\vec{r}_{0'}$ ,  $\vec{v}_{0'}$  and  $\vec{a}_{0'}$ ) and the rotation, angular velocity, and angular acceleration of the body ( $\theta$ ,  $\vec{\omega}$  and  $\vec{\alpha}$ ).

We will treat all bodies as if they are squished into the plane and moving in the plane. But the analysis is sensible for a body that is symmetric with respect to the plane containing the velocities (see Box 8.2 on page 458).

### The balance laws for a rigid body

As always, once you have defined the system and the forces acting on it by drawing a free body diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\begin{aligned} \text{Linear momentum balance: } & \sum \vec{F}_i = \dot{\vec{L}} \quad \text{and} \\ \text{Angular momentum balance: } & \sum \vec{M}_{i/0} = \dot{\vec{H}}_O. \end{aligned}$$

The same point  $O$ , any point, is used on both sides of the angular momentum balance equation.

We also have power balance which, for a system with no internal energy or dissipation, is

$$\text{Power balance: } P = \dot{E}_K.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one body or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power  $P$  only depends on the position of the forces and the motions of the material points at those positions. But the kinetic energy  $E_K$  and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the positions and accelerations are known (Chapter 5);
- Systems where all points have the same acceleration. That is, the system moves like a rigid body that does not rotate (Chapter 6); and
- Systems where all points move in circles about the same fixed axis. That is, the system moves like a rigid body that is rotating about a fixed skewer (Chapters 7 and 8).

Now we go on to consider the general 2-D motions of a planar rigid body. Its now a little harder to evaluate  $\vec{L}$ ,  $\dot{\vec{L}}$ ,  $\vec{H}_O$ ,  $\dot{\vec{H}}_O$ ,  $E_K$  and  $\dot{E}_K$ . But not much.

## Summary of the motion quantities

Table I in the back of the book describes the motion quantities for various special cases, including the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, is the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for any system with any motion, by the motion of the center of mass

$$\vec{L} = m_{\text{tot}} \vec{v}_{\text{cm}} \quad \text{and} \quad \dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}, \quad (8.15)$$

and the angular momentum has two contributions, one from the motion of the center of mass and one from rotation of the body about the center of mass,

Angular momentum  
due to motion of the  
center of mass

Angular momentum  
due to motion relative  
to the center of mass

$$\vec{H}_O = \overbrace{\vec{r}_{\text{cm}/O} \times (m_{\text{tot}} \vec{v}_{\text{cm}})} + \overbrace{I_{zz}^{\text{cm}} \vec{\omega}} \quad (8.16)$$

$$\text{and } \dot{\vec{H}}_O = \overbrace{\vec{r}_{\text{cm}/O} \times (m_{\text{tot}} \vec{a}_{\text{cm}})} + \overbrace{I_{zz}^{\text{cm}} \dot{\vec{\omega}}}. \quad (8.17)$$

An important special case for the angular momentum evaluation is when the reference point is coincident with the center of mass. Then the angular momentum and its rate of change simplify to

$$\vec{H}_{\text{cm}} = I_{zz}^{\text{cm}} \vec{\omega} \quad \text{and} \quad \dot{\vec{H}}_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\vec{\omega}}. \quad (8.18)$$

The kinetic energy and its rate of change are given by

kinetic energy from  
center of mass motion

kinetic energy rela-  
tive to the center of  
mass

$$E_K = \underbrace{\frac{1}{2} m_{\text{tot}} \underbrace{\vec{v}_{\text{cm}}^2}_{\vec{v}_{\text{cm}} \cdot \vec{v}_{\text{cm}}}} + \underbrace{\frac{1}{2} I_{zz}^{\text{cm}} \omega^2} \quad (8.19)$$

$$\text{and } \dot{E}_K = m_{\text{tot}} \underbrace{\vec{v}_{\text{cm}} \dot{\vec{v}}_{\text{cm}}}_{\vec{v}_{\text{cm}} \cdot \vec{a}_{\text{cm}}} + I_{zz}^{\text{cm}} \omega \dot{\vec{\omega}} \quad (8.20)$$

The relations above are easily derived from the general center of mass theorems at the end of chapter 5 (see box 8.2 on page 459 for some of these derivations).

## Equations of motion

Putting together the general balance equations and the expressions for the motion quantities we can now write linear momentum balance, angular momentum balance and power balance as:

$$\text{LMB : } \sum \vec{F}_i = m_{\text{tot}} \vec{a}_{\text{cm}}, \quad (\text{a})$$

$$\text{AMB : } \sum \vec{M}_O = \vec{r}_{\text{cm}/O} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I_{zz}^{\text{cm}} \dot{\vec{\omega}} \quad (\text{b})$$

$$\text{or } \sum \vec{M}_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\vec{\omega}},$$

$$\text{and Power : } \vec{F}_{\text{tot}} \cdot \vec{v}_{\text{cm}} + \vec{\omega} \cdot \vec{M}_{\text{cm}} = m_{\text{tot}} v \dot{v} + I_{zz}^{\text{cm}} \omega \dot{\omega}. \quad (\text{c})$$

(8.21)

## Independent equations?

Equations are only independent if no one of them can be derived from the others. When counting equations and unknowns one needs to make sure one is writing independent equations. How many independent equations are in the set eqns. (8.21)abc applied to one free body diagram? The short answer is 3.

The linear momentum balance equation eqn. (8.21)a yields two independent equations by dotting with any two non-parallel vectors (say,  $\hat{i}$  and  $\hat{j}$ ). Dotting with a third vector yields a dependent equation.

For any one reference point the angular momentum equation eqn. (8.21)a yields one scalar equation. It is a vector equation but always has zero components in the  $\hat{i}$  and  $\hat{j}$  directions. But angular momentum equation can yield up to three independent equations by being applied to any set of three non-collinear points.

The power balance equation is one scalar equation.

In total, however, the full set of equations above only makes up a set of three independent equations.

To avoid thinking about what is or is not an independent set of equations some people prefer to stick with one of the canonical sets of independent equations:

- The coordinate based set (“old standard”)
  - $\{\text{LMB}\} \cdot \hat{i}$  or, equivalently,  $\sum F_x = m_{\text{tot}} a_{\text{cm},x}$ ,
  - $\{\text{LMB}\} \cdot \hat{j}$  or, equivalently,  $\sum F_y = m_{\text{tot}} a_{\text{cm},y}$ , and
  - $\{\text{AMB}\} \cdot \hat{k}$  or, equivalently,  $\sum M_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\omega}$ .
- Moment only (good for eliminating unknown reaction forces)
  - $\{\text{AMB about pt A}\} \cdot \hat{k}$  (A is any point, on or off the body)
  - $\{\text{AMB about pt B}\} \cdot \hat{k}$  (B is any other point)
  - $\{\text{AMB about pt C}\} \cdot \hat{k}$  (C is a third point not on the line AB)
- Two moments and a force component
  - $\{\text{AMB about pt A}\} \cdot \hat{k}$  (A is any point, on or off the body)
  - $\{\text{AMB about pt B}\} \cdot \hat{k}$  (B is any other point)
  - $\{\text{LMB}\} \cdot \hat{\lambda}$  (where  $\hat{\lambda}$  is not perpendicular to the line AB)

- Two force components and a moment (also good for eliminating unknown forces)
  - $\{LMB\} \cdot \hat{\lambda}_1$  qquad (where  $\hat{\lambda}_1$  is any unit vector)
  - $\{LMB\} \cdot \hat{\lambda}_2$  qquad (where  $\hat{\lambda}_2$  is any other unit vector)
  - $\{AMB\}$  about pt A  $\cdot \hat{k}$  qquad (A is any point, on or off the body)

Any of these will always do the job. The power balance equation is often used as a consistency check rather than an independent equation.

From a theoretical point of view one might ask the related question of which of the equations of motion can be derived from the others. There are many answers. Here are some of them:

- Power balance follows from LMB and AMB,
- AMB about three non-collinear points implies LMB, and
- LMB and power balance yield AMB

Interestingly, there is no way to derive angular momentum balance from linear momentum balance without the questionable microscopic assumptions discussed in box 5.12 on page 323.

## Some simple examples

Here we consider some simple examples of unconstrained motion of a rigid body.

### *Example: The simplest case: no force and no moment.*

If the net force and moment applied to a body are zero we have:

$$\begin{aligned} LMB &\Rightarrow \vec{0} = m_{\text{tot}} \vec{a}_{\text{cm}} \quad \text{and} \\ AMB &\Rightarrow \vec{0} = I_{zz}^{\text{cm}} \dot{\omega} \hat{k} \end{aligned}$$

so  $\vec{a}_{\text{cm}} = \vec{0}$  and  $\dot{\omega} = 0$  and the object moves at constant speed in a constant direction with a constant rate of rotation, all determined by the initial conditions. Throw an object in space and its center of mass goes in a straight line and it spins at constant rate (subject to the 2-D restrictions of this chapter).  $\square$

### *Example: Constant force applied to the center of mass.*

In this case angular momentum balance about the center of mass again yields that the rotation rate is constant. Linear momentum balance is now the same as for a particle at the center of mass, *i.e.*, the center of mass has a parabolic trajectory.

Near-earth (constant) gravity provides a simple example. An ‘X’ marked at the center of mass of a clipboard tossed across a room follows a parabolic trajectory (see Fig. 8.19).  $\square$

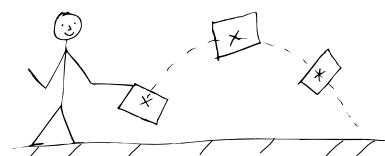


Figure 8.19: The X marked at the center of mass of a thrown spinning clipboard follows a parabolic trajectory.  
(Filename:tfigure.clipboard)

**Example: Constant force not at the center of mass.**

Assume the only force applied to an object is a constant force  $\vec{F} = F\hat{i}$  at A (see Fig. 8.20). Then linear momentum balance gives us that

$$\sum \vec{F}_i = \dot{\vec{L}} \Rightarrow F\hat{i} = m\vec{a}_G \Rightarrow \vec{a}_G = F/m\hat{i} = \text{constant.}$$

So if the object starts at rest, the point G will move in a straight line in the  $\hat{i}$  direction (The common intuition that point G will be pulled up is incorrect). Angular momentum about the center of mass gives

$$\begin{aligned} \sum \vec{M}_{cmi} = \dot{\vec{H}}_{cm} &\Rightarrow \left\{ \vec{r}_{A/G} \times F\hat{i} = I_{zz}^{cm}\ddot{\theta}\hat{k} \right\} \\ \{\} \cdot \hat{k} &\Rightarrow \ddot{\theta} + \frac{F\ell}{I_{zz}^{cm}} \sin \theta = 0, \end{aligned}$$

with  $\ell = |\vec{r}_{A/G}|$ , which is the pendulum equation. That is, the object can swing back and forth about  $\theta = 0$  just like a pendulum, approximately sinusoidally if the angle  $\theta$  starts small and with  $\dot{\theta}$  initially also small. [One might wonder how to do this experiment. One way would be with a jet on a space craft that keeps re-orienting itself to keep in a constant spatial direction as the object changes orientation. Another would be with a string tied to A and pulled from a great distance.]  $\square$

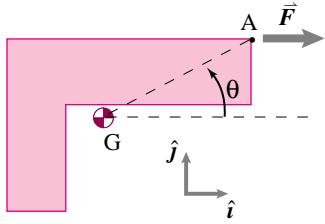


Figure 8.20: The only force applied to the object is the constant force  $\vec{F} = F\hat{i}$  applied at point A. The resulting motion is a constant acceleration of the center of mass  $\vec{a}_G = (F/m)\hat{i}$  and an oscillatory motion of  $\theta$  identical to that of a pendulum hinged at G.

(Filename:tfigure.constforce)

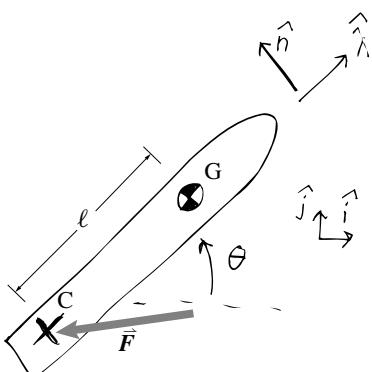


Figure 8.21: A rocket is pointed in the direction  $\lambda$  which makes an angle  $\theta$  with the positive  $x$  axis. The position and velocity of the center of mass at G are called  $\vec{r}$  and  $\vec{v}$ . The velocity of the tail is  $\vec{v}_C$ .

(Filename:tfigure.rocket)

**Example: The flight of an arrow or rocket.**

As a primitive model of an arrow or rocket assume that the only force is from drag on the fins at C and that this force opposes motion according to

$$\vec{F} = -c\vec{v}_C$$

where c is a drag coefficient (see Fig. 8.21). From linear momentum balance we have:

$$\begin{aligned} \sum \vec{F}_i = \dot{\vec{L}} &\Rightarrow \vec{F} = m\vec{a} \\ -c\vec{v}_C &= m\dot{\vec{v}} \\ m\dot{\vec{v}} &= -c(\vec{v} + \vec{\omega} \times \vec{r}_{C/G}) \\ &= -c(\vec{v} + \dot{\theta}\hat{k} \times (-\ell\hat{\lambda})) \\ (\hat{k} \times \hat{\lambda} = \hat{n}) &\Rightarrow \dot{\vec{v}} = \frac{c}{m}(\dot{\theta}\ell\hat{n} - \vec{v}). \end{aligned}$$

So if  $\vec{v}$ ,  $\theta$  and  $\dot{\theta}$  are known the acceleration  $\dot{\vec{v}}$  is calculated by the formula above.

Similarly angular momentum balance about G gives

$$\begin{aligned} \sum \vec{M}_G = \dot{\vec{H}}_G &\Rightarrow \{\vec{r}_{C/G} \times \vec{F}\} = I_{zz}^{cm}\dot{\theta}\hat{k} \\ \{\} \cdot \hat{k} &\Rightarrow I_{zz}^{cm}\dot{\theta} = \vec{r}_{C/G} \cdot \vec{F} \cdot \hat{k}. \end{aligned}$$

Then, making the same substitutions as before for  $\vec{r}_{C/G}$  and  $\vec{F}$  we get

$$\dot{\theta} = \frac{c\ell}{I_{zz}^{cm}}(\hat{\lambda} \times \vec{v} \cdot \hat{k} - \dot{\theta}\ell)$$

which determines the rate of change of  $\theta$  if the present values of  $\vec{v}$ ,  $\theta$  and  $\dot{\theta}$  are known.  $\square$

## Setting up differential equations for solution

If one knows the forces and torques on a body in terms of the bodies position, velocity, orientation and angular velocity one then has a ‘closed’ set of differential equations. That is, one has enough information to define the equations for a mathematician or a computer to solve them.

The full set of differential equations is gathered from linear and angular momentum balance and also from simple kinematics. Namely, one has the following set of 6 first order differential equations:

$$\begin{aligned}\dot{x} &= v_x, \\ \dot{v}_x &= F_x/m, \\ \dot{y} &= v_y, \\ \dot{v}_y &= F_y/m, \\ \dot{\theta} &= \omega, \text{ and} \\ \dot{\omega} &= M_{\text{cm}}/I_{zz}^{\text{cm}},\end{aligned}$$

where the positions and velocities are the positions and velocities of the center of mass. The expressions for  $F_x$ ,  $F_y$ , and  $M_{\text{cm}}$  may well be complicated, as in the rocket example above.

## 8.2 THEORY

### The utility of 2-D mechanics for understanding the 3-D world

The math for two-dimensional mechanics analysis is simpler than the math for three-dimensional analysis. And thus easier to learn first. Because we do actually live in a three-dimensional world you might wonder at the utility of learning something that is not right. There are three answers.

- (a) Two dimensional analysis can give partial information about the three-dimensional system that is exactly the same as the three-dimensional analysis would give by *projection*, no matter what the motion;
- (b) if the motion is planar the 2-D kinematics can be used; and
- (c) if the object is planar or symmetric about the motion plane, and any constraints that hold the object are also symmetric about the motion plane, the 2-D analysis is not only correct, but complete.

Of course no machine is exactly planar or exactly symmetric, but if the approximation seems reasonable most engineers will accept a small loss in accuracy for great gain in simplicity.

Similarly, the  $z$  component of the full angular momentum balance equation for a 3-D rigid body in planar motion is the same as the  $z$  component of eqn. (8.21)b.

$$\sum \vec{M}_O \cdot \hat{\mathbf{k}} = (\vec{r}_{cm/O} \times (m_{tot} \vec{\alpha}_{cm})) \cdot \hat{\mathbf{k}} + I_{zz}^{cm} \dot{\omega}$$

So for planar motion of 3-D rigid bodies one can do a correct 2-D analysis with the full ease of analyzing a planar body.

But this result is deceptively simple. The free body diagram in 3-D most likely shows forces in the  $z$  direction, pairs of forces in the  $x$  or  $y$  directions that are applied at points with the same  $x$  and  $y$  coordinates but different  $z$  values, or moments with components in the  $x$  or  $y$  directions. Full information about these force and moment components can't be found from 2-D analysis. That is,

the nature of the forces that it takes to *keep* a system in planar motion can't be found from a planar analysis.

## Projection

First lets relax our assumption of 2-D motion. Consider arbitrary 3-D motions of an arbitrarily complex system. If we take the dot product of the linear momentum equations with  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  and the angular momentum balance equation with  $\hat{\mathbf{k}}$  we get

$$\begin{aligned} \left\{ \sum \vec{F}_i = \sum m_i \vec{a}_i \right\} \cdot \hat{\mathbf{i}} &\Rightarrow \sum F_{ix} = \sum m_i a_{ix}, & (a) \\ \left\{ \sum \vec{F}_i = \sum m_i \vec{a}_i \right\} \cdot \hat{\mathbf{j}} &\Rightarrow \sum F_{iy} = \sum m_i a_{iy}, & \text{and} & (b) \\ \left\{ \sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times m_i \vec{a}_i \right\} \cdot \hat{\mathbf{k}} \\ &\Rightarrow \sum r_{ix} F_{iy} - r_{iy} F_{ix} = \sum m_i (r_{ix} a_{iy} - r_{iy} a_{ix}). & (c) \end{aligned} \quad (8.22)$$

These are exactly the equations of 2-D mechanics. That is, if we only consider the planar components of the forces, the planar components of the positions, and the planar components of the motions, we get a correct but partial set of the 3-D equations. In this sense 2-D analysis is correct but incomplete.

## Planar motion

If all the velocities of the parts of a 3-D system have no  $z$  component the motion is planar (in the  $xy$  plane). Thus the right-hand sides of eqns. (8.22) are not just projections, but the whole story. Further, in the case of rigid-body motion, the 2-D kinematics equation

$$\vec{v}_P = \vec{v}_G + \omega \hat{\mathbf{k}} \times \vec{r}_{P/G} = \vec{v}_G + \omega \hat{\mathbf{k}} \times (r_{P/G_x} \hat{\mathbf{i}} + r_{P/G_y} \hat{\mathbf{j}}) \quad (8.23)$$

also applies (the  $z$  component of the position drops out of the cross product) and the expression for, say, the  $z$  component of the angular momentum of a body about its center of mass is

$$H_{cmz} = I_{zz}^{cm} \omega.$$

Differentiating, or adding up the  $m_i \vec{a}_i$  terms we get,

$$H_{cmz} = I_{zz}^{cm} \dot{\omega}.$$

For example, a system rotating about the  $z$  axis which is statically balanced but is dynamically imbalanced (see section 11.5) has no *net*  $x$  or  $y$  reaction force, as a planar analysis would reveal, yet the bearing reaction forces are not zero.

Another example would be a plan view of a car in a turn (assuming a stiff suspension). A 2-D analysis could be accurate, but would no be complete enough to describe the tire reaction forces needed to keep the car flat.

## Symmetric bodies and planar bodies

If the rigid body has all its mass in the  $xy$  plane, or its mass is symmetrically distributed about the  $xy$  plane, and it is in planar motion in the  $xy$  plane then

$$\left\{ \sum \vec{r}_i \times m_i \vec{a}_i \right\} \cdot \hat{\mathbf{i}} = 0 \quad \text{and} \quad \left\{ \sum \vec{r}_i \times m_i \vec{a}_i \right\} \cdot \hat{\mathbf{j}} = 0$$

where  $\vec{r}$  is measured relative to any point in the plane. Thus, by linear and angular momentum balance,

$$\left\{ \sum \vec{r}_i \times \vec{F}_i \right\} \cdot \hat{\mathbf{i}} = 0 \quad \text{and} \quad \left\{ \sum \vec{r}_i \times \vec{F}_i \right\} \cdot \hat{\mathbf{j}} = 0$$

so

A planar object or a symmetric object in planar motion needs no force in the  $z$  direction and no moment in the  $x$  or  $y$  direction to keep it in the plane.

Systems that are symmetric or flat and moving in an approximately planar manner, are thus both accurately and completely modeled with a 2-D analysis. A slight generalization of the result is to any object or collection of objects whose center's of mass are on the plane and each of which is dynamically balanced for rotation about a  $\hat{\mathbf{k}}$  axis through its center of mass.

### 8.3 THEORY

#### The center of mass theorems for 2-D rigid bodies

That all the particles in a system are part of one planar body in planar motion (in that plane) allows highly useful simplification of the expressions for the motion quantities, namely Eqns. 8.15 to 8.19. We can derive these expressions from the center of mass theorems of chapter 5. For completeness, we repeat some of those derivations as the start of the derivations here. To save space, we only use the integral ( $\int$ ) forms for the general expressions; the derivations with sums ( $\sum$ ) are similar. In all cases position, velocity, and acceleration are relative to a fixed point in space (that is  $\vec{r}$ ,  $\vec{v}$ , and  $\vec{a}$  mean  $\vec{r}_{/0}$ ,  $\vec{v}_{/0}$ , and  $\vec{a}_{/0}$  respectively).

#### Linear momentum.

Here we show that to evaluate linear momentum and its rate of change you only need to know the motion of the center of mass.

$$\begin{aligned}\vec{L} &\equiv \int \vec{v} dm = \int \frac{d}{dt} \vec{r} dm = \frac{d}{dt} \int \vec{r} dm = \frac{d}{dt} (m_{\text{tot}} \vec{r}_{\text{cm}}) \\ &= m_{\text{tot}} \frac{d}{dt} \vec{r}_{\text{cm}} = m_{\text{tot}} \vec{v}_{\text{cm}}\end{aligned}$$

By identical reasoning, or by differentiating the expression above with respect to time,

$$\dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}$$

Thus for linear momentum balance one need not pay heed to rotation, only the center of mass motion matters.

#### Angular momentum.

Here we attempt a derivation like the one above but get slightly more complicated results. For simplicity we evaluate angular momentum and its rate of change relative to the origin, but a very similar derivation would hold relative to any fixed point C.

$$\begin{aligned}\vec{H}_O &\equiv \int \vec{r} \times \vec{v} dm \\ &= \int (\vec{r} - \vec{r}_{\text{cm}} + \vec{r}_{\text{cm}}) \times (\vec{v} - \vec{v}_{\text{cm}} + \vec{v}_{\text{cm}}) dm \\ &= \int (\vec{r}_{/\text{cm}} + \vec{r}_{\text{cm}}) \times (\vec{v}_{/\text{cm}} + \vec{v}_{\text{cm}}) dm \\ &= \int \vec{r}_{/\text{cm}} \times \vec{v}_{/\text{cm}} dm + \int \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} dm \\ &\quad + \int \vec{r}_{/\text{cm}} \times \vec{v}_{\text{cm}} dm + \int \vec{r}_{\text{cm}} \times \vec{v}_{/\text{cm}} dm \\ &= \underbrace{\int \vec{r}_{/\text{cm}} \times \vec{v}_{/\text{cm}} dm}_{\vec{0}} + \vec{r}_{\text{cm}} \times \underbrace{\int \vec{v}_{/\text{cm}} dm}_{\vec{0}} \\ &= \int \vec{r}_{/\text{cm}} \times \vec{v}_{/\text{cm}} dm + \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} m_{\text{tot}}.\end{aligned}$$

This much is true for any system in any motion. For a rigid body we know about the motions of the parts. Using the center of mass as a reference point we know that for all points on the body  $\vec{v}_{/\text{cm}} = \vec{\omega} \times \vec{r}_{/\text{cm}}$ . Thus we can continue the derivation above, following the same reasoning as was used in chapter 7 for circular motion of rigid bodies:

$$\vec{H}_O = \int \vec{r}_{/\text{cm}} \times (\vec{\omega} \times \vec{r}_{/\text{cm}}) dm + \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} m_{\text{tot}}.$$

Using the identity for the triple cross product (see box 11.1 on page 643) or using the geometry of the cross product directly with  $\vec{\omega} = \omega \hat{k}$  as in chapters 7 and 8 we get

$$\vec{H}_O = \omega \hat{k} \int r_{/\text{cm}}^2 dm + \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} m_{\text{tot}}.$$

Then defining  $I_{zz}^{\text{cm}} \equiv \int r_{/\text{cm}}^2 dm$  we get the desired result:

$$\vec{H}_O = \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} m_{\text{tot}} + I_{zz}^{\text{cm}} \omega \hat{k}.$$

A similar derivation, or differentiation of the result above (and using that  $(\frac{d}{dt} \vec{r}) \times \vec{v} = \vec{v} \times \vec{v} = \vec{0}$ ) gives

$$\dot{\vec{H}}_O = \vec{r}_{\text{cm}} \times \vec{a}_{\text{cm}} m_{\text{tot}} + I_{zz}^{\text{cm}} \dot{\omega} \hat{k}.$$

The results above hold for any reference point, not just the origin of the fixed coordinate system. Thus, relative to a point instantaneously coinciding with the center of mass

$$\vec{H}_{\text{cm}} = \underbrace{\vec{r}_{\text{cm}/\text{cm}} \times \vec{v}_{\text{cm}} m_{\text{tot}}}_{\vec{0}} + I_{zz}^{\text{cm}} \omega \hat{k} = I_{zz}^{\text{cm}} \omega \hat{k}.$$

and similarly

$$\dot{\vec{H}}_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\omega} \hat{k}.$$

#### Kinetic energy.

Unsurprisingly the expression for kinetic energy and it's rate of change are also simplified by derivations very similar to those above. Skipping the details (or leaving them as an exercise for the peppy reader):

$$\begin{aligned}E_K &\equiv \int \frac{1}{2} \vec{v} \cdot \vec{v} dm \\ &= \frac{1}{2} m_{\text{tot}} v_{\text{cm}}^2 + \frac{1}{2} I_{zz}^{\text{cm}} \omega^2\end{aligned}$$

and

$$\begin{aligned}\dot{E}_K &\equiv \frac{d}{dt} E_K \\ &= m_{\text{tot}} v \dot{v} + I_{zz}^{\text{cm}} \omega \dot{\omega}.\end{aligned}$$

## 8.4 THEORY

### The work of a moving force and of a couple

The work of a force acting on a body from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \underbrace{\vec{v} dt}_{d\vec{r}} \\ &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \end{aligned} \quad (8.24)$$

The validity of equation 8.24 depends on the force acting on the same material point of the moving body as it moves from position 1 to position 2; i.e., the force moves with the body. If the material point of force application changes with time, equation 8.24 is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = \int P dt \quad (8.25)$$

where  $\vec{v}$  is the velocity of the material point at the instantaneous location of the applied force.

### A subtlety in the concept of the work of a force

There is a subtle distinction between 8.24 and 8.25. As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train with the force

$$\vec{F}_{\text{hand on train}}.$$

It might seem that the work of the hand on the train is zero because your hand doesn't move; work is force times distance and the distance is zero and eqn. (8.24) superficially evaluates to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0.$$

But we have violated the condition for the validity of eqn. (8.24): the force be applied to a fixed material point as time progresses. Whereas on the train your hand smears a whole line of material points.

Clearly your hand does slow the train so it must do (negative) work. on it as Eqn. ?? correctly shows this because

$$P_{\text{force on train}} = \vec{F}_{\text{hand on train}} \cdot \vec{v}_{\text{train}} \neq 0.$$

The power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand. Thus, your hand does negative work on the train. Eqn. 8.25 applies to the train and 8.24 does not.

On the other hand (so to speak) if one looks at the power of the force on the hand we have:

$$\vec{F}_{\text{train on hand}} = -\vec{F}_{\text{hand on train}}$$

while the velocity of the hand is zero so

$$P_{\text{force on hand}} = \vec{F}_{\text{train on hand}} \cdot \vec{v}_{\text{hand}} = 0.$$

So the train does *no* work on your hand since while your hand does (negative) work on the train. The difference, of course, is mechanical energy lost to heat.

### Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid body the power of these forces turns out to be  $\vec{M} \cdot \vec{\omega}$ . At a given angular velocity a bigger torque applies more power. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces  $\vec{F}_i$  that add to zero acting at points with positions  $\vec{r}_i$  relative to a reference point on the body  $o'$ .

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= \sum \vec{F}_i \cdot (\vec{v}_{o'} + \vec{\omega} \times \vec{r}_{i/o'}) \\ &= \vec{v}_{o'} \cdot \underbrace{\sum \vec{F}_i}_{\vec{0}} + \sum \vec{F}_i \cdot (\vec{\omega} \times \vec{r}_{i/o'}) \\ &= \sum \vec{\omega} \cdot (\vec{r}_{i/o'} \times \vec{F}_i) \\ &= \vec{\omega} \cdot \sum \vec{r}_{i/o'} \times \vec{F}_i \\ &= \vec{\omega} \cdot \vec{M}_{o'} \end{aligned} \quad (8.26)$$

### Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{F}_{\text{tot}} \cdot \vec{v}_{o'} + \vec{\omega} \cdot \vec{M}_{o'}. \quad (8.27)$$

For a given force system applied to a given body in a given motion any point  $o'$  can be used. The terms in the formula above will depend on  $o'$ , but the sum does not. Besides the center of mass, another convenient locations for  $o'$  is a fixed hinge, in which case the applied force has no power.

**SAMPLE 8.6** Free planar motion. A rigid rod of length  $\ell = 1 \text{ m}$  and mass  $m_r = 1 \text{ kg}$ , and a rigid square plate of side  $1 \text{ m}$  and mass  $m_p = 10 \text{ kg}$  are launched in motion on a frictionless plane (e.g., an ice hockey rink) with exactly the same initial velocities  $\vec{v}_{cm}(0) = 10 \text{ m/s} \hat{i} + 1 \text{ m/s} \hat{j}$  and  $\vec{\omega}(0) = 1 \text{ rad/s} \hat{k}$ . Both the rod and the plate have their center of mass at the baseline at  $t = 0$ .

- Which of the two is farther from the base line in 3 seconds and which one has undergone more number of revolutions?
- Find and draw the position of the bar at  $t = 1 \text{ sec}$  and at  $t = 3 \text{ sec}$ .

### Solution

- The free body diagram of the rod is shown in Fig. 8.23. There are no forces acting on the rod in the  $xy$ -plane. Although there is force of gravity and the normal reaction of the surface acting on the rod, these forces are inconsequential since they act normal to the  $xy$ -plane. Therefore, we do not include these forces in our free body diagram. The linear momentum balance for the rod gives

$$\begin{aligned}\sum \vec{F} &= m_r \vec{a}_{cm} \\ \vec{0} &= m_r \vec{v}_{cm} \\ \Rightarrow \vec{v}_{cm} &= \int \vec{0} dt = \text{constant} = \vec{v}_{cm0} \\ \Rightarrow \vec{r}_{cm} &= \int \vec{v}_{cm0} dt = \vec{r}_{cm0} + \vec{v}_{cm0} t\end{aligned}\quad (8.28)$$

It is clear from the analysis above that in the absence of any applied forces, the position of the body depends only on the initial position and the initial velocity. Since both the plate and the rod start from the same base line with the same initial velocity, they travel the same distance from the base line during any given time period; mass or its geometric distribution play no role in the motion. Thus the center of mass of the rod and the plate will be exactly the same distance ( $|\vec{r}_{cm}(t) - \vec{r}_{cm0}| = |\vec{v}_{cm0}t|$ ) at time  $t$ . Similarly, the angular momentum balance about the center of mass of the rod gives

$$\begin{aligned}\sum \vec{M}_{cm} &= \dot{\vec{H}}_{cm} \\ \vec{0} &= I_{zz}^{cm} \vec{\omega} \\ \Rightarrow \vec{\omega} &= \int \vec{0} dt = \text{constant} = \vec{\omega}_0 = \dot{\theta}_0 \hat{k} \\ \Rightarrow \theta &= \int \dot{\theta}_0 dt = \theta_0 + \dot{\theta}_0 t\end{aligned}\quad (8.29)$$

Thus the angular position of the body is also, as expected, independent of the mass and mass distribution of the body, and depends entirely on the initial position and the initial angular velocity. Therefore, both the rod and the plate undergo exactly the same amount of rotation ( $\theta(t) - \theta_0 = \dot{\theta}_0 t$ ) during any given time.

- We can find the position of the rod at  $t = 1 \text{ s}$  and  $t = 3 \text{ s}$  by substituting these values of  $t$  in eqns. (8.28) and 8.29. For convenience, let us assume that  $\vec{r}_{cm0} = \vec{0}$ . From the initial configuration of the rod, we also know that  $\theta_0 = 0$ .
$$\begin{aligned}\vec{r}_{cm}(t = 1 \text{ s}) &= \vec{v}_{cm0} \cdot (1 \text{ s}) = (10 \text{ m/s} \hat{i} + 1 \text{ m/s} \hat{j}) \cdot (1 \text{ s}) 10 \hat{m} \hat{i} + 1 \hat{m} \hat{j} \\ \vec{r}_{cm}(t = 3 \text{ s}) &= \vec{v}_{cm0} \cdot (3 \text{ s}) = 30 \hat{m} \hat{i} + 3 \hat{m} \hat{j} \\ \theta(t = 1 \text{ s}) &= \dot{\theta}_0 \cdot (1 \text{ s}) = (1 \text{ rad/s}) \cdot (1 \text{ s}) = 1 \text{ rad} \\ \theta(t = 3 \text{ s}) &= \dot{\theta}_0 \cdot (3 \text{ s}) = 3 \text{ rad.}\end{aligned}$$

Accordingly, we show the position of the rod in Fig. 8.24.

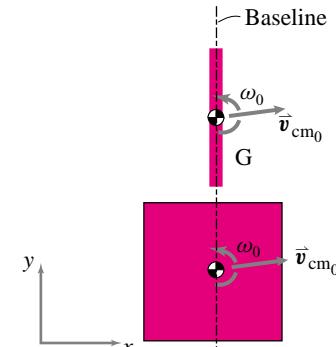


Figure 8.22: (Filename:sfig9.2.rodandplate)

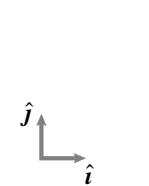


Figure 8.23: (Filename:sfig9.2.rodfdbd)

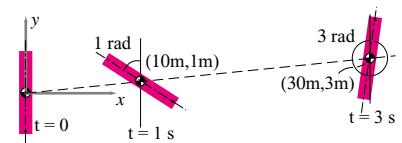


Figure 8.24: (Filename:sfig9.2.rodposition)

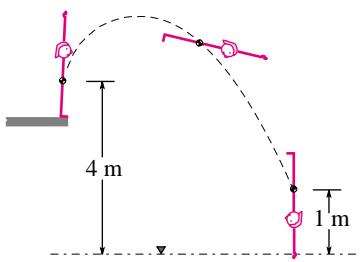


Figure 8.25: (Filename:sfig9.2.diver)

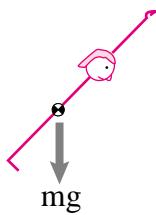


Figure 8.26: (Filename:sfig9.2.diver.a)

**SAMPLE 8.7** *A passive rigid diver.* An experimental model of a rigid diver is to be launched from a diving board that is 3 m above the water level. Say that the initial velocity of the center of mass and the initial angular velocity of the diver can be controlled at launch. The diver is launched into the dive in almost vertical position, and it is required to be as vertical as possible at the very end of the dive (which is marked by the position of the diver's center of mass at 1 m above the water level). If the initial vertical velocity of the diver's center of mass is 3 m/s, find the required initial angular velocity for the vertical entry of the diver into the water.

**Solution** Once the diver leaves the diving board, it is in free flight under gravity, i.e., the only force acting on it is the force due to gravity. The free body diagram of the diver is shown in Fig. 8.26. The linear momentum balance for the diver gives

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_{cm} \\ -mg\hat{j} &= m\ddot{y}\hat{j} \\ \Rightarrow \ddot{y} &= -g \\ \sum \vec{M}_{cm} &= \dot{\vec{H}}_{cm} \\ \vec{0} &= I_{zz}^{cm}\ddot{\theta}\hat{k} \\ \Rightarrow \ddot{\theta} &= 0.\end{aligned}$$

From these equations of motion, it is clear that the linear and the angular motions of the diver are uncoupled. We can easily solve the equations of motion to get

$$\begin{aligned}y(t) &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\ \theta(t) &= \theta_0 + \dot{\theta}_0 t.\end{aligned}$$

We need to find the initial angular speed  $\dot{\theta}_0$  such that  $\theta = \pi$  when  $y = 1$  m (the center of mass position at the water entry). From the expression for  $\theta(t)$ , we get,  $\dot{\theta}_0 = \pi/t$ . Thus we need to find the value of  $t$  at the instant of water entry. We can find  $t$  from the expression for  $y(t)$  since we know that  $y = 1$  m at that instant, and that  $y_0 = 3$  m and  $\dot{y}_0 = 3$  m/s. We have,

$$\begin{aligned}y &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\ \Rightarrow t &= \frac{\dot{y}_0 \pm \sqrt{\dot{y}_0^2 + 2g(y_0 - y)}}{g} \\ &= \frac{3 \text{ m/s} \pm \sqrt{(3 \text{ m/s})^2 + 2 \cdot 9.8 \text{ m/s}^2 \cdot (3 \text{ m} - 1 \text{ m})}}{9.8 \text{ m/s}^2} \\ &= 1.15 \text{ or } -0.53 \text{ s.}\end{aligned}$$

We reject the negative value of time as meaningless in this context. Thus it takes the diver 1.15 s to complete the dive. Since, the diver must rotate by  $\pi$  during this time, we have

$$\dot{\theta}_0 = \pi/t = \pi/(1.15 \text{ s}) = 2.73 \text{ rad/s.}$$

$\dot{\theta}_0 = 2.73 \text{ rad/s}$



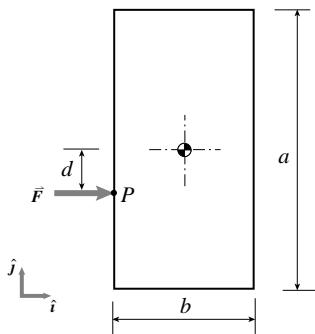


Figure 8.27: (Filename:sfig9.tumblingplate1)

**SAMPLE 8.8** *A plate tumbling in space.* A rectangular plate of mass  $m = 0.5 \text{ kg}$ ,  $I_{zz}^{\text{cm}} = 2.08 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ , and dimensions  $a = 200 \text{ mm}$  and  $b = 100 \text{ mm}$  is pushed by a force  $\vec{F} = 0.5 \text{ N}\hat{i}$ , acting at  $d = 30 \text{ mm}$  away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

- Find the initial acceleration of the mass-center.
- Find the initial angular acceleration of the plate.
- Write the equations of motion of the plate (for both linear and angular motion).

**Solution** The only force acting on the plate is the applied force  $\vec{F}$ . Thus, Fig. 8.27 is also the free body diagram of the plate at the start of motion.

- From the linear momentum balance we get,

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} = \frac{0.5 \text{ N}\hat{i}}{0.5 \text{ kg}} = 1 \text{ m/s}^2\hat{i}\end{aligned}$$

which is the initial acceleration of the mass-center.

$$\boxed{\vec{a}_{\text{cm}} = 1 \text{ m/s}^2\hat{i}}$$

- From the angular momentum balance about the mass-center, we get

$$\begin{aligned}\vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\ Fd\hat{k} &= I_{zz}^{\text{cm}}\dot{\vec{\omega}} \\ \Rightarrow \dot{\vec{\omega}} &= \frac{Fd}{I_{zz}^{\text{cm}}}\hat{k} = \frac{0.5 \text{ N} \cdot 0.03 \text{ m}}{2.08 \text{ kg} \cdot \text{m}^2} = 7.2 \text{ rad/s}^2\hat{k}\end{aligned}$$

which is the initial angular acceleration of the plate.

$$\boxed{\dot{\vec{\omega}} = 7.2 \text{ rad/s}^2\hat{k}}$$

- To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration,  $\vec{a}_{\text{cm}} = F/m\hat{i}$ , and the angular acceleration,  $\dot{\vec{\omega}} = Fd/I_{zz}^{\text{cm}}\hat{k}$ , qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

Let us first draw a free body diagram of the plate in a general configuration during its motion (see Fig. 8.28). Assume the center of mass to be displaced by  $x\hat{i}$  and  $y\hat{j}$ , and the longitudinal axis of the plate to be rotated by  $\theta\hat{k}$  with respect to the vertical (inertial  $y$ -axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} \\ \text{or} \quad \ddot{x}\hat{i} + \ddot{y}\hat{j} &= \frac{F}{m}\hat{i} \\ \Rightarrow \ddot{x} &= \frac{F}{m} \\ \ddot{y} &= 0\end{aligned}$$

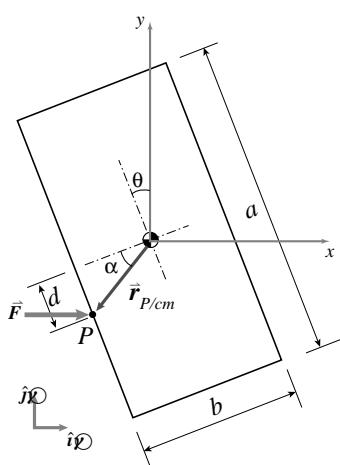


Figure 8.28: (Filename:sfig9.tumblingplate1.a)

Since  $F/m$  is constant, the equations of motion of the center of mass indicate that the acceleration is constant and that the mass-center moves in the  $x$ -direction.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned}\vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\ \vec{r}_{\text{P/cm}} \times \vec{F} &= I_{zz}^{\text{cm}} \ddot{\theta} \hat{k}\end{aligned}$$

Now,

$$\begin{aligned}\vec{r}_{\text{P/cm}} &= -r[\cos(\theta + \alpha)\hat{i} + \sin(\theta + \alpha)\hat{j}] \\ \vec{F} &= F\hat{i} \\ \Rightarrow \vec{r}_{\text{P/cm}} \times \vec{F} &= Fr \sin(\theta + \alpha) \hat{k}\end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fr}{I_{zz}^{\text{cm}}} \sin(\theta + \alpha)$$

where  $r = \sqrt{d^2 + (b/2)^2}$  and  $\alpha = \tan^{-1}(2d/b)$ .

Thus, we have got the equations of motion for both the linear and the angular motion.

$$\boxed{\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0, \quad \ddot{\theta} = \frac{Fr}{I_{zz}^{\text{cm}}} \sin(\theta + \alpha)}$$

- (d) The equations of linear motion of the plate are very simple and we can solve them at once to get

$$\begin{aligned}x(t) &= x_0 + \dot{x}_0 t + \frac{1}{2} \frac{F}{m} t^2 \\ y(t) &= y_0 + \dot{y}_0 t\end{aligned}$$

If the plate starts from rest ( $\dot{x}_0 = 0, \dot{y}_0 = 0$ ) with the center of mass at the origin ( $x_0 = 0, y_0 = 0$ ), then we have

$$x(t) = \frac{F}{2m} t^2, \quad \text{and} \quad y(t) = 0.$$

Thus the center of mass moves along the  $x$ -axis with acceleration  $F/m$ .

The equation of angular motion of the plate is, however, not so simple. In fact, it is a nonlinear ODE. It is very difficult to get an analytical solution of this equation. However, we can solve it numerically using, say, a Runge-Kutta ODE solver:

```
ODEs = {thetadot = w, wdot = (F*r/Icm)*sin(theta+a)}  
IC = {theta(0) = 0, w(0) = 0}  
Set F=.5, d=0.03; b=0.1; Icm=2.08e-03  
compute r = sqrt(d^2+0.25*b^2), a = atan(2*d/b)  
Solve ODEs with IC for t=0 to t=10  
Plot theta(t)
```

The plot obtained from this calculation is shown in Fig. 8.29.

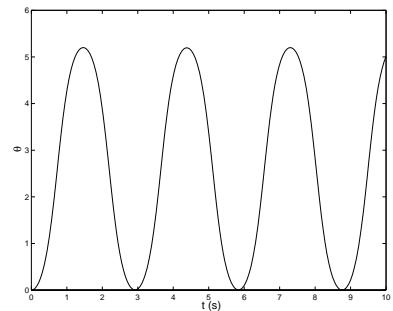


Figure 8.29: (Filename:fig9.2.odesoln)

**SAMPLE 8.9** *Impulse-momentum.* Consider the plate problem of Sample 8.8 (page 464) again. Assume that the plate is at rest at  $t = 0$  in the vertical upright position and that the force acts on the plate for 2 seconds.

- Find the velocity of the center of mass of the plate at the end of 2 seconds.
- Can you also find the angular velocity of the plate at the end of 2 seconds?

### Solution

- Since we are interested in finding the velocity at a particular instant  $t$ , given the velocity at another instant  $t = 0$ , we can use the impulse-momentum equations to find the desired velocity.

$$\begin{aligned}\vec{L}_2 - \vec{L}_1 &= \int_{t_1}^{t_2} \sum \vec{F} dt \\ m\vec{v}_{cm}(t) - m\vec{v}_{cm}(0) &= \int_0^t \vec{F} dt \\ \Rightarrow \vec{v}_{cm}(t) &= \vec{v}_{cm}(0) + \frac{1}{m} \int_0^t \vec{F} dt \\ &= \vec{0} + \frac{1}{0.5 \text{ kg}} \int_0^2 (0.5 \text{ N} \hat{i}) dt \\ &= 2 \text{ m/s} \hat{i}.\end{aligned}$$

$$\boxed{\vec{v}_{cm}(2 \text{ s}) = 2 \text{ m/s} \hat{i}}$$

- Now, let us try to find the angular velocity the same way, using angular impulse-momentum relation. We have,

$$\begin{aligned}(\vec{H}_{cm})_2 - (\vec{H}_{cm})_1 &= \int_{t_1}^{t_2} \sum \vec{M}_{cm} dt \\ I_{zz}^{cm} \vec{\omega}(t) - I_{zz}^{cm} \vec{\omega}(0) &= \int_0^t \sum \vec{M}_{cm} dt \\ \Rightarrow \vec{\omega}(t) &= \vec{\omega}(0) + \frac{1}{I_{zz}^{cm}} \int_0^t \sum \vec{M}_{cm} dt \\ &= \vec{\omega}(0) + \frac{1}{I_{zz}^{cm}} \int_0^t (\vec{r}_{P/cm} \times \vec{F}) dt \\ &= \vec{0} + \frac{1}{I_{zz}^{cm}} \int_0^t (Fr \sin(\theta + \alpha) \hat{k}) dt \\ &= \frac{Fr}{I_{zz}^{cm}} \left( \int_0^t \sin(\theta + \alpha) dt \right) \hat{k}\end{aligned}$$

Now, we are in trouble; how do we evaluate the integral? In the integrand, we have  $\theta$  which is an implicit function of  $t$ . Unless we know how  $\theta$  depends on  $t$  we cannot evaluate the integral. To find  $\theta(t)$  we have to solve the equation of angular motion we derived in the previous sample. However, we were not able to solve for  $\theta(t)$  analytically, we had to resort to numerical solution. Thus, it is not possible to evaluate the integral above and, therefore, we cannot find the angular velocity of the plate at the end of 2 seconds using impulse-momentum equations. We could, however, find the desired velocity easily from the numerical solution.

## 8.3 Kinematics of rolling and sliding

### Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, and rolling of round things on other round things like gears and cams.

### 2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in Fig. 8.30. For convenience, we pick a point  $D$  which was at  $x_D = 0$  at the start of rolling, when  $x_C = 0$ . The key to the kinematics is that:

*The arc length traversed on the wheel is the distance traveled by the wheel center.*

That is,

$$\begin{aligned} x_C &= s_D \\ &= R\phi \\ \Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\ \Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi} \end{aligned}$$

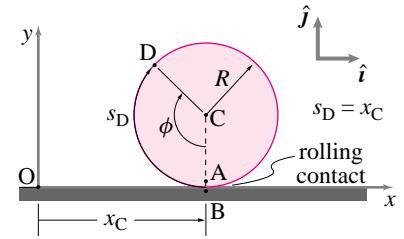


Figure 8.30: Pure rolling of a round wheel on a level support.  
(Filename:tfigure7.2D.pure.rolling)

So the rolling condition amounts to the following set of restrictions on the position of  $C$ ,  $\vec{r}_C$ , and the rotations of the wheel  $\phi$ :

$$\vec{r}_C = R\phi\hat{i} + R\hat{j}, \quad \vec{v}_C = R\dot{\phi}\hat{i}, \quad \vec{a}_C = R\ddot{\phi}\hat{i}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say  $D$ , we could do so by using the following parametric formula:

$$\begin{aligned} \vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\ &= R(\phi\hat{i} + \hat{j}) + R(-\sin\phi\hat{i} - \cos\phi\hat{j}) \\ &= R[(\phi - \sin\phi)\hat{i} + (1 - \cos\phi)\hat{j}] \\ \Rightarrow \vec{v}_D &= R[(\dot{\phi}(1 - \cos\phi)\hat{i} + \dot{\phi}\sin\phi\hat{j})] \quad (8.30) \\ \Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\hat{i} + \cos\phi\hat{j}). \end{aligned}$$

assuming  $\dot{\phi} = \text{constant}$

Note that if  $\phi = 0$  or  $2\pi$  or  $4\pi$ , etc., then the point  $D$  is on the ground and eqn. (8.30) correctly gives that

$$\vec{v}_D = R \left[ \underbrace{\dot{\phi}(1 - \cos(2n\pi))}_0 \right] \hat{i} + \underbrace{\dot{\phi}\sin(2n\pi)}_0 \hat{j} = \vec{0}.$$

## Instantaneous Kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

$$\boxed{\text{Velocity of point on the wheel touching the ground}} = \boxed{\text{Velocity of ground} = \vec{0}} \quad (8.31)$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C},$$

where, since A and C are on the same rigid body (Fig. 8.30), we have from eqn. (11.13) that

$$\vec{v}_{A/C} = \vec{\omega} \times \vec{r}_{A/C}.$$

Putting this equation together with eqn. (8.31), we get

$$\begin{aligned} \vec{v}_A &= \vec{v}_B \\ \Rightarrow \underbrace{\vec{v}_C}_{v_C \hat{i}} + \underbrace{\vec{\omega}}_{\omega \hat{k}} \times \underbrace{\vec{r}_{A/C}}_{-R \hat{j}} &= \vec{0} \\ \Rightarrow v_C \hat{i} + \omega R \hat{i} &= \vec{0} \\ \Rightarrow \boxed{v_C = -\omega R}. \end{aligned}$$

We use  $\vec{v}_C = v \hat{i}$  since the center of the wheel goes neither up nor down. Note that if you measure the angle by  $\phi$ , like we did before, then  $\vec{\omega} = -\dot{\phi} \hat{k}$  so that positive rotation rate is in the counter-clockwise direction. Thus,  $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$ .

Since there is always *some* point of the wheel touching the ground, we know that  $v_C = -\omega R$  for all time. Therefore,

$$\vec{a}_C = v_C \hat{i} = -\dot{\omega} R \hat{i}.$$

## Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. A common application is the so-called epicyclic, hypocyclic, or planetary gears (See Box 8.5 on planetary gears on page 470). Referring to Fig. 8.31, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

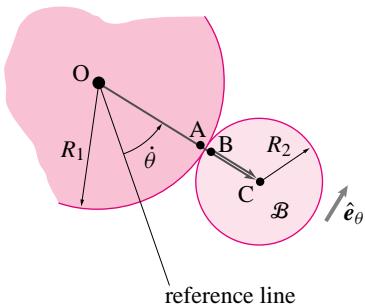


Figure 8.31: (Filename:tfigure7.rolling.on.another)

$$\begin{aligned} \vec{v}_C &= \vec{v}_B + \vec{v}_{C/B} \\ \vec{v}_C &= \underbrace{\vec{v}_A}_{\vec{0}} + \underbrace{\vec{v}_{B/A}}_{\vec{0}} + \vec{v}_{C/B}. \\ \dot{\theta}(R_1 + R_2)\hat{e}_\theta &= \omega_B R_2 \hat{e}_\theta \\ \Rightarrow \omega_B &= \frac{\dot{\theta}(R_1 + R_2)}{R_2} = \dot{\theta}\left(1 + \frac{R_1}{R_2}\right). \end{aligned}$$

**Example: Two quarters.**

The formula above can be tested in the case of  $R_1 = R_2$  by using two quarters or two dimes on a table. Roll one quarter, call it  $\mathcal{B}$ , around another quarter pressed fast to the table. You will see that as the rolling quarter  $\mathcal{B}$  travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of  $\mathcal{B}$  changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above,  $\omega_{\mathcal{B}} = 2\dot{\theta}$ .  $\square$

## Sliding

Although wheels and balls are known for rolling, they do sometimes slide such as when a car screeches at fast acceleration or sudden braking or when a bowling ball is released on the lane.

The *sliding velocity* is the velocity of the material point on the wheel (or ball) relative to its contacting substrate. In the case of pure rolling, the sliding velocity is zero. In the case of a ball or wheel moving against a stationary support surface, whether round or curved, the sliding velocity is

$$\vec{v}_{\text{sliding}} = \vec{v}_{\text{circle center}} + \vec{\omega} \times \vec{r}_{\text{contact/center}} \quad (8.33)$$

**Example: Bowling ball**

The velocity of the point on the bowling ball instantaneously in contact with the alley (ground) is  $\vec{v}_C = v_G \hat{i} + \omega \hat{k} \times \vec{r}_{C/G} = (v_G + \omega R) \hat{i}$ . So unless  $\omega = -v_G/R$  the ball is sliding.

Note that, if sliding, the friction force on the ball opposes the slip of the ball and tends to accelerate the ball's rotation towards rolling. That is, for example, if the ball is not rotating the sliding velocity is  $v_G \hat{i}$ , the friction force is in the  $-\hat{i}$  direction and angular momentum balance about the center of mass implies  $\dot{\omega} < 0$  and a counter-clockwise rotational acceleration. No matter what the initial velocity or rotational rate the ball will eventually roll.  $\square$

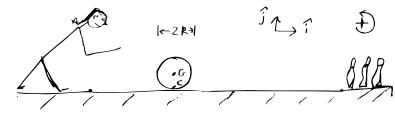
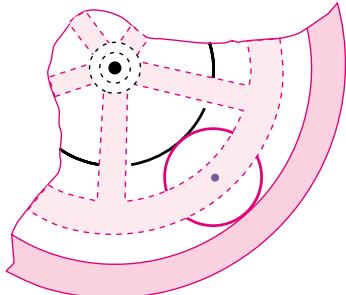


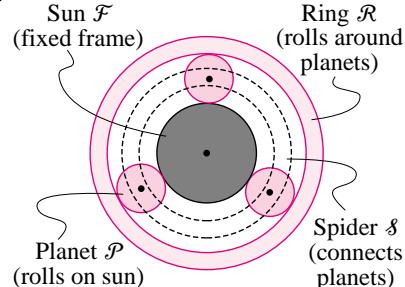
Figure 8.32: The bowling ball is sliding so long as  $v_G \neq -\omega R$   
(Filename:tfigure.bowlball)

## 8.5 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub three-speed bicycle transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. Now, a hundred years later, internal bicycle hubs (now made by Shimano and Sachs) are having something of a revival, particularly in Europe. These internal-hub transmissions utilize a system called *planetary gears*, gears which roll around other gears. See the figure below. In order to understand



this gear system, we need to understand its kinematics—the motion of its parts. Referring to figure above, the central ‘sun’ gear  $\mathcal{F}$  is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body  $\mathcal{F}$ . The ‘planet’ gears roll around the sun gear. Let’s call one of these planets  $\mathcal{P}$ . The spider  $\mathcal{S}$  connects the centers of the rolling planets. Finally, the ring gear  $\mathcal{R}$  rotates around the sun.



The gear transmission steps up the angular velocity when the spider  $\mathcal{S}$  is driven and ring  $\mathcal{R}$ , which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third ‘speed’ in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the ‘gear ratios’ in the planetary gear system? The ‘trick’ is to recognize that for rolling contact that the contacting points have the same velocity,  $\vec{v}_A = \vec{v}_B$  and  $\vec{v}_D = \vec{v}_E$ . Let’s define some terms.

$$\begin{aligned}\boldsymbol{\omega}_{\mathcal{S}} &= \omega_{\mathcal{S}} \hat{\mathbf{k}} && \text{angular velocity of the spider} \\ \boldsymbol{\omega}_{\mathcal{P}} &= \omega_{\mathcal{P}} \hat{\mathbf{k}} && \text{angular velocity of the planet} \\ \boldsymbol{\omega}_{\mathcal{R}} &= \omega_{\mathcal{R}} \hat{\mathbf{k}} && \text{angular velocity of the ring}\end{aligned}$$

Now, we can find the relation of these angular velocities as follows.

Look at the velocity of point  $C$  in two ways. First,

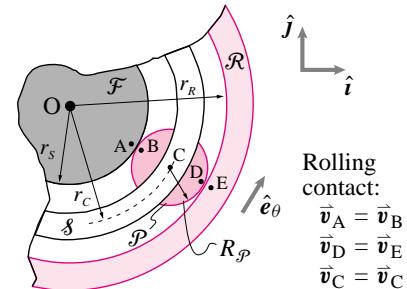
A point on the spider	A point on the planetary gear
-----------------------	-------------------------------

$$\begin{aligned}\vec{v}_C &= \underbrace{\vec{v}_C}_{\vec{0}} \\ \Rightarrow \boldsymbol{\omega}_{\mathcal{S}} \times \vec{r}_C &= \underbrace{\vec{v}_B}_{\vec{0}} + \underbrace{\boldsymbol{\omega}_{\mathcal{P}} \times \vec{r}_{C/B}}_{\vec{0}} \\ \Rightarrow \omega_{\mathcal{S}} r_C &= \omega_{\mathcal{P}} R_P \\ \Rightarrow \omega_{\mathcal{P}} &= \frac{r_C}{R_P} \omega_{\mathcal{S}}\end{aligned}\quad (8.32)$$

Next, let’s look at point  $D$  and  $E$ :

$$\begin{aligned}\vec{v}_D &= \vec{v}_E \\ \vec{v}_A + \vec{v}_{D/A} &= \boldsymbol{\omega}_{\mathcal{R}} \times \vec{r}_R \\ \vec{0} + \boldsymbol{\omega}_{\mathcal{P}} \times \vec{r}_{D/A} &= \omega_{\mathcal{R}} \hat{\mathbf{k}} \times \vec{r}_R \\ \omega_{\mathcal{P}} (2R_P) \hat{\mathbf{e}}_\theta &= \omega_{\mathcal{R}} r_R \hat{\mathbf{e}}_\theta \\ \Rightarrow \omega_{\mathcal{R}} &= \frac{2R_P}{r_R} \underbrace{\omega_{\mathcal{P}}}_{\omega_{\mathcal{P}} = \frac{r_C}{R_P} \omega_{\mathcal{S}}} \\ &= \frac{2R_P}{r_R} \frac{r_C}{R_P} \omega_{\mathcal{S}} \\ &= \frac{2(r_S + R_P)}{r_R} \omega_{\mathcal{S}} \\ &\downarrow \\ r_R &= r_S + 2R_P \\ \Rightarrow \frac{\omega_{\mathcal{R}}}{\omega_{\mathcal{S}}} &= \frac{1 + \frac{R_P}{r_S}}{1 + \frac{2R_P}{r_S}} = \text{angular velocity step-up.}\end{aligned}$$

Typically, the gears



have radius ratio of  $\frac{R_p}{r_s} = \frac{3}{2}$  which gives a gear ratio of  $\frac{5}{4}$ . Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is  $\frac{5}{4}/\frac{3}{2} = \frac{25}{16} = 1.5625$ . You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.

**SAMPLE 8.10 Falling ladder:** The ends of a ladder of length  $L = 3\text{ m}$  slip along the frictionless wall and floor shown in Figure 8.33. At the instant shown, when  $\theta = 60^\circ$ , the angular speed  $\dot{\theta} = 1.15\text{ rad/s}$  and the angular acceleration  $\ddot{\theta} = 2.5\text{ rad/s}^2$ . Find the absolute velocity and acceleration of end B of the ladder.

**Solution** Since the ladder is falling, it is rotating clockwise. From the given information:

$$\begin{aligned}\vec{\omega} &= \dot{\theta}\hat{k} = -1.15\text{ rad/s}\hat{k} \\ \vec{\alpha} &= \ddot{\theta}\hat{k} = -2.5\text{ rad/s}^2\hat{k}.\end{aligned}$$

We need to find  $\vec{v}_B$ , the absolute velocity of end B, and  $\vec{a}_B$ , the absolute acceleration of end B.

Since the end A slides along the wall and end B slides along the floor, we know the directions of  $\vec{v}_A$ ,  $\vec{v}_B$ ,  $\vec{a}_A$  and  $\vec{a}_B$ .

Let  $\vec{v}_A = v_A\hat{j}$ ,  $\vec{a}_A = a_A\hat{j}$ ,  $\vec{v}_B = v_B\hat{i}$  and  $\vec{a}_B = a_B\hat{i}$  where the scalar quantities  $v_A$ ,  $a_A$ ,  $v_B$  and  $a_B$  are unknown.

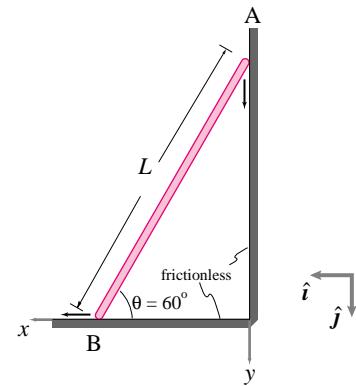


Figure 8.33: (Filename:fig7.2.2)

$$\begin{aligned}\text{Now, } \vec{v}_A &= \vec{v}_B + \vec{v}_{A/B} = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \\ \text{or } v_A\hat{j} &= v_B\hat{i} + \dot{\theta}\hat{k} \times \underbrace{L(-\cos\theta\hat{i} - \sin\theta\hat{j})}_{\vec{r}_{A/B}} \\ &= (v_B + \dot{\theta}L\sin\theta)\hat{i} - \dot{\theta}L\cos\theta\hat{j}.\end{aligned}$$

Dotting both sides of the equation with  $\hat{i}$ , we get:

$$\begin{aligned}v_A \underbrace{\hat{j} \cdot \hat{i}}_0 &= (v_B + \dot{\theta}L\sin\theta) \underbrace{\hat{i} \cdot \hat{i}}_1 + \dot{\theta}L\cos\theta \underbrace{\hat{j} \cdot \hat{i}}_0 \\ \Rightarrow 0 &= v_B + \dot{\theta}L\sin\theta \\ \Rightarrow v_B &= -\dot{\theta}L\sin\theta = -(-1.15\text{ rad/s}) \cdot 3\text{ m} \cdot \frac{\sqrt{3}}{2} \\ &= 2.99\text{ m/s}.\end{aligned}$$

$$\boxed{\vec{v}_B = 2.9\text{ m/s}\hat{i}}$$

Similarly,

$$\begin{aligned}\vec{a}_A &= \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} + \overbrace{\vec{\alpha} \times (\vec{\omega} \times \vec{r}_{A/B})}^{-\omega^2\vec{r}_{A/B}} \\ a_A\hat{j} &= a_B\hat{i} + \ddot{\theta}\hat{k} \times L(-\cos\theta\hat{i} - \sin\theta\hat{j}) - \dot{\theta}^2L(-\cos\theta\hat{i} - \sin\theta\hat{j}) \\ &= (a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta)\hat{i} + (-\ddot{\theta}L\cos\theta + \dot{\theta}^2L\sin\theta)\hat{j}.\end{aligned}$$

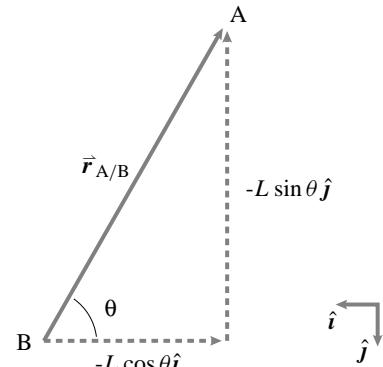


Figure 8.34: (Filename:fig7.2.2b)

Dotting both sides of this equation with  $\hat{i}$  (as we did for velocity) we get:

$$\begin{aligned}0 &= a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta \\ \Rightarrow a_B &= -\ddot{\theta}L\sin\theta - \dot{\theta}^2L\cos\theta \\ &= -(-2.5\text{ rad/s}^2 \cdot 3\text{ m} \cdot \frac{\sqrt{3}}{2}) - (-1.15\text{ rad/s})^2 \cdot 3\text{ m} \cdot \frac{1}{2} \\ &= 4.51\text{ m/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_B = 4.51\text{ m/s}^2}$$

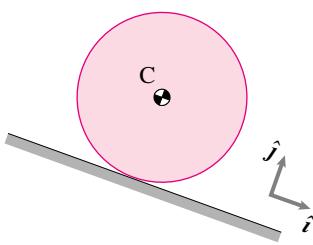


Figure 8.35: (Filename:sfig9.rolling.may00)

**SAMPLE 8.11** A cylinder of diameter 500 mm rolls down an inclined plane with uniform acceleration (of the center of mass)  $a = 0.1 \text{ m/s}^2$ . At an instant  $t_0$ , the mass-center has speed  $v_0 = 0.5 \text{ m/s}$ .

- Find the angular speed  $\omega$  and the angular acceleration  $\dot{\omega}$  at  $t_0$ .
- How many revolutions does the cylinder make in the next 2 seconds?
- What is the distance travelled by the center of mass in those 2 seconds?

**Solution** This problem is about simple kinematic calculations. We are given the velocity,  $\dot{x}$ , and the acceleration,  $\ddot{x}$ , of the center of mass. We are supposed to find angular velocity  $\omega$ , angular acceleration  $\dot{\omega}$ , angular displacement  $\theta$  in 2 seconds, and the corresponding linear distance  $x$  along the incline. The radius of the cylinder  $R = \text{diameter}/2 = 0.25 \text{ m}$ .

- From the kinematics of pure rolling,

$$\begin{aligned}\omega &= \frac{\dot{x}}{R} = \frac{0.5 \text{ m/s}}{0.25 \text{ m}} = 2 \text{ rad/s}, \\ \dot{\omega} &= \frac{\ddot{x}}{R} = \frac{0.1 \text{ m/s}^2}{0.25 \text{ m}} = 0.4 \text{ rad/s}^2.\end{aligned}$$

$$\boxed{\omega = 2 \text{ rad/s}, \quad \dot{\omega} = 0.4 \text{ rad/s}^2}$$

- We can find the number of revolutions the cylinder makes in 2 seconds by solving for the angular displacement  $\theta$  in this time period. Since,

$$\ddot{\theta} \equiv \dot{\omega} = \text{constant},$$

we integrate this equation twice and substitute the initial conditions,  $\dot{\theta}(t = 0) = \omega = 2 \text{ rad/s}$  and  $\theta(t = 0) = 0$ , to get

$$\begin{aligned}\theta(t) &= \omega t + \frac{1}{2}\dot{\omega}t^2 \\ \Rightarrow \theta(t = 2 \text{ s}) &= (2 \text{ rad/s}) \cdot (2 \text{ s}) + \frac{1}{2}(0.4 \text{ rad/s}) \cdot (4 \text{ s}^2) \\ &= 4.8 \text{ rad} = \frac{4.8}{2\pi} \text{ rev} = 0.76 \text{ rev}.\end{aligned}$$

$$\boxed{\theta = 0.76 \text{ rev}}$$

- Now that we know the angular displacement  $\theta$ , the distance travelled by the mass-center is the arc-length corresponding to  $\theta$ , *i.e.*,

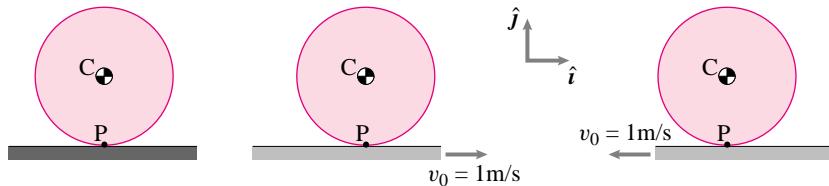
$$x = R\theta = (0.25 \text{ m}) \cdot (4.8) = 1.2 \text{ m}.$$

$$\boxed{x = 1.2 \text{ m}}$$

Note that we could have found the distance travelled by the mass-center by integrating the equation  $\ddot{x} = 0.1 \text{ m/s}^2$  twice

**SAMPLE 8.12 Condition of pure rolling.** A cylinder of radius  $R = 20\text{ cm}$  rolls on a flat surface with absolute angular speed  $\omega = 12\text{ rad/s}$  under the conditions shown in the figure (In cases (ii) and (iii), you may think of the ‘flat surface’ as a conveyor belt). In each case,

- Write the condition for pure rolling.
- Find the velocity of the center C of the cylinder.



(i) Fixed base    (ii) Base moves to the right    (iii) Base moves to the left

Figure 8.36: (Filename:sfig7.rolling1)

**Solution** At any instant during rolling, the cylinder makes a point-contact with the flat surface. Let the point of instantaneous contact on the cylinder be P, and let the corresponding point on the flat surface be Q. The condition of pure rolling, in each case, is  $\vec{v}_P = \vec{v}_Q$ , that is, there is no relative motion between the two contacting points (a relative motion will imply slip). Now, we analyze each case.

**Case(i)** In this case, the bottom surface is fixed. Therefore,

- The condition of pure rolling is:  $\vec{v}_P = \vec{v}_Q = \vec{0}$ .
- Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = \vec{0} + (-\omega \hat{k}) \times R \hat{j} \\ &= \omega R \hat{i} = (12 \text{ rad/s}) \cdot (0.2 \text{ m}) \hat{i} = 2.4 \text{ m/s} \hat{i}.\end{aligned}$$

**Case(ii)** In this case, the bottom surface moves with velocity  $\vec{v} = 1 \text{ m/s} \hat{i}$ . Therefore,  $\vec{v}_Q = 1 \text{ m/s} \hat{i}$ . Thus,

- The condition of pure rolling is:  $\vec{v}_P = \vec{v}_Q = 1 \text{ m/s} \hat{i}$ .
- Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = v_0 \hat{i} + \omega R \hat{i} \\ &= 1 \text{ m/s} \hat{i} + 2.4 \text{ m/s} \hat{i} = 3.4 \text{ m/s} \hat{i}.\end{aligned}$$

**Case(iii)** In this case, the bottom surface moves with velocity  $\vec{v} = -1 \text{ m/s} \hat{i}$ . Therefore,  $\vec{v}_Q = -1 \text{ m/s} \hat{i}$ . Thus,

- The condition of pure rolling is:  $\vec{v}_P = \vec{v}_Q = -1 \text{ m/s} \hat{i}$ .
- Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = -v_0 \hat{i} + \omega R \hat{i} \\ &= -1 \text{ m/s} \hat{i} + 2.4 \text{ m/s} \hat{i} = 1.4 \text{ m/s} \hat{i}.\end{aligned}$$

(a) :	(i) $\vec{v}_P = \vec{0}$ ,	(ii) $\vec{v}_P = 1 \text{ m/s} \hat{i}$ ,	(iii) $\vec{v}_P = -1 \text{ m/s} \hat{i}$ ,
(b) :	(i) $\vec{v}_C = 2.4 \text{ m/s} \hat{i}$ ,	(ii) $\vec{v}_C = 3.4 \text{ m/s} \hat{i}$ ,	(iii) $\vec{v}_C = 1.4 \text{ m/s} \hat{i}$

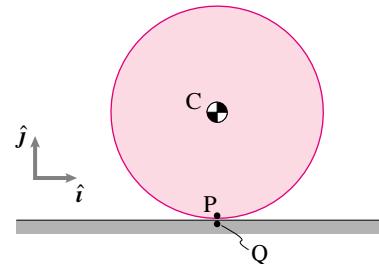


Figure 8.37: The cylinder rolls on the flat surface. Instantaneously, point P on the cylinder is in contact with point Q on the flat surface. For pure rolling, points P and Q must have the same velocity.

(Filename:sfig7.rolling1a)

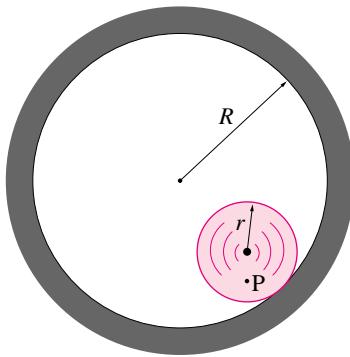


Figure 8.38: A uniform disk of radius  $r$  rolls without slipping inside a fixed cylinder.

(Filename: sfig6.5.3)

**SAMPLE 8.13 Motion of a point on a disk rolling inside a cylinder.** A uniform disk of radius  $r$  rolls without slipping with constant angular speed  $\omega$  inside a fixed cylinder of radius  $R$ . A point  $P$  is marked on the disk at a distance  $\ell$  ( $\ell < r$ ) from the center of the disk. at a general time  $t$  during rolling, find

- the position of point  $P$ ,
- the velocity of point  $P$ , and
- the acceleration of point  $P$

**Solution** Let the disk be vertically below the center of the cylinder at  $t = 0$  s such that point  $P$  is vertically above the center of the disk (Fig. 8.39). At this instant,  $Q$  is the point of contact between the disk and the cylinder. Let the disk roll for time  $t$  such that at instant  $t$  the line joining the two centers (line  $OC$ ) makes an angle  $\phi$  with its vertical position at  $t = 0$  s. Since the disk has rolled for time  $t$  at a constant angular speed  $\omega$ , point  $P$  has rotated counter-clockwise by an angle  $\theta = \omega t$  from its original vertical position  $P'$ .

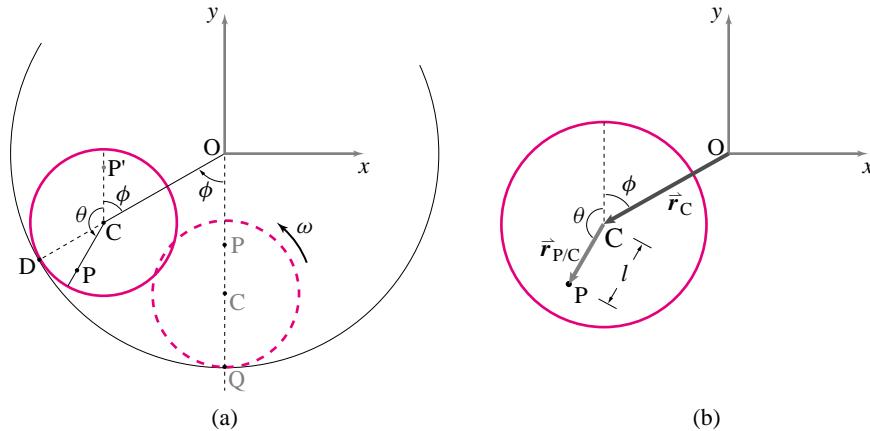


Figure 8.39: Geometry of motion: keeping track of point  $P$  while the disk rolls for time  $t$ , rotating by angle  $\theta = \omega t$  inside the cylinder.

(Filename: sfig6.5.3a)

**(a) Position of point  $P$ :** From Fig. 8.39(b) we can write

$$\vec{r}_P = \vec{r}_C + \vec{r}_{P/C} = (R - r)\hat{\lambda}_{OC} + \ell\hat{\lambda}_{CP}$$

where

$$\begin{aligned}\hat{\lambda}_{OC} &= \text{a unit vector along } OC = -\sin \phi \hat{i} - \cos \phi \hat{j}, \\ \hat{\lambda}_{CP} &= \text{a unit vector along } CP = -\sin \theta \hat{i} + \cos \theta \hat{j}.\end{aligned}$$

Thus,

$$\vec{r}_P = [-(R - r) \sin \phi - \ell \sin \theta] \hat{i} + [-(R - r) \cos \phi + \ell \cos \theta] \hat{j}.$$

We have thus obtained an expression for the position vector of point  $P$  as a function of  $\phi$  and  $\theta$ . Since we also want to find velocity and acceleration of point  $P$ , it will be nice to express  $\vec{r}_P$  as a function of  $t$ . As noted above,  $\theta = \omega t$ ; but how do we find  $\phi$  as a function of  $t$ ? Note that the center of the disk  $C$  is going around point  $O$  in circles with angular velocity  $-\dot{\phi} \hat{k}$ . The disk, however,

is rotating with angular velocity  $\vec{\omega} = \omega \hat{k}$  about the instantaneous center of rotation, point D. Therefore, we can calculate the velocity of point C in two ways:

$$\begin{aligned}\vec{v}_C &= \vec{v}_C \\ \text{or } \vec{\omega} \times \vec{r}_{C/D} &= -\dot{\phi} \hat{k} \times \vec{r}_{C/O} \\ \text{or } \omega \hat{k} \times r(-\hat{\lambda}_{OC}) &= -\dot{\phi} \hat{k} \times (R-r)\hat{\lambda}_{OC} \\ \text{or } -\omega r(\hat{k} \times \hat{\lambda}_{OC}) &= -\dot{\phi}(R-r)(\hat{k} \times \hat{\lambda}_{OC}) \\ \Rightarrow \frac{r}{R-r}\omega &= \dot{\phi}.\end{aligned}$$

Integrating the last expression with respect to time, we obtain

$$\phi = \frac{r}{R-r}\omega t.$$

Let

$$q = \frac{r}{R-r},$$

then, the position vector of point P may now be written as

$$\vec{r}_P = [-(R-r) \sin(q\omega t) - \ell \sin(\omega t)]\hat{i} + [-(R-r) \cos(q\omega t) + \ell \cos(\omega t)]\hat{j}. \quad (8.34)$$

- (b) **Velocity of point P:** Differentiating Eqn. (8.34) once with respect to time we get

$$\vec{v}_P = -\omega[(R-r)q \cos(q\omega t) + \ell \cos(\omega t)]\hat{i} + \omega[(R-r)q \sin(q\omega t) - \ell \sin(\omega t)]\hat{j}.$$

Substituting  $(R-r)q = r$  in  $\vec{v}_P$  we get

$$\vec{v}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r} \cos(\omega t)\}\hat{i} - \{\sin(q\omega t) - \frac{\ell}{r} \sin(\omega t)\}\hat{j}]. \quad (8.35)$$

- (c) **Acceleration of point P:** Differentiating Eqn. (8.35) once with respect to time we get

$$\vec{a}_P = -\omega^2 r[-\{q \sin(q\omega t) + \frac{\ell}{r} \sin(\omega t)\}\hat{i} - \{q \cos(q\omega t) - \frac{\ell}{r} \cos(\omega t)\}\hat{j}]. \quad (8.36)$$

$\vec{r}_P = [-(R-r) \sin(q\omega t) - \ell \sin(\omega t)]\hat{i} + [-(R-r) \cos(q\omega t) + \ell \cos(\omega t)]\hat{j}$
$\vec{v}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r} \cos(\omega t)\}\hat{i} - \{\sin(q\omega t) - \frac{\ell}{r} \sin(\omega t)\}\hat{j}]$
$\vec{a}_P = -\omega^2 r[-\{q \sin(q\omega t) + \frac{\ell}{r} \sin(\omega t)\}\hat{i} - \{q \cos(q\omega t) - \frac{\ell}{r} \cos(\omega t)\}\hat{j}]$

**SAMPLE 8.14** *The rolling disk: instantaneous kinematics.* For the rolling disk in Sample 8.13, let  $R = 4$  ft,  $r = 1$  ft and point P be on the rim of the disk. Assume that at  $t = 0$ , the center of the disk is vertically below the center of the cylinder and point P is on the vertical line joining the two centers. If the disk is rolling at a constant speed  $\omega = \pi$  rad/s, find

- (a) the position of point P and center C at  $t = 1$  s, 3 s, and 5.25 s,
- (b) the velocity of point P and center C at those instants, and
- (c) the acceleration of point P and center C at the same instants as above.

Draw the position of the disk at the three instants and show the velocities and accelerations found above.

**Solution** The general expressions for position, velocity, and acceleration of point P obtained in Sample 8.13 can be used to find the position, velocity, and acceleration of any point on the disk by substituting an appropriate value of  $\ell$  in equations (8.34), (8.35), and (8.36). Since  $R = 4r$ ,

$$q = \frac{r}{R-r} = \frac{1}{3}.$$

Now, point P is on the rim of the disk and point C is the center of the disk. Therefore,

$$\begin{aligned} \text{for point P: } \ell &= r, \\ \text{for point C: } \ell &= 0. \end{aligned}$$

Substituting these values for  $\ell$ , and  $q = 1/3$  in equations (8.34), (8.35), and (8.36) we get the following.

(a) **Position:**

$$\begin{aligned} \vec{r}_C &= -3r \left[ \sin\left(\frac{\omega t}{3}\right) \hat{i} + \cos\left(\frac{\omega t}{3}\right) \hat{j} \right], \\ \vec{r}_P &= \vec{r}_C + r \left[ -\sin(\omega t) \hat{i} + \cos(\omega t) \hat{j} \right]. \end{aligned}$$

(b) **Velocity:**

$$\begin{aligned} \vec{v}_C &= -\omega r \left[ \cos\left(\frac{\omega t}{3}\right) \hat{i} - \sin\left(\frac{\omega t}{3}\right) \hat{j} \right], \\ \vec{v}_P &= -\omega r \left[ \left\{ \cos\left(\frac{\omega t}{3}\right) + \cos(\omega t) \right\} \hat{i} - \left\{ \sin\left(\frac{\omega t}{3}\right) - \sin(\omega t) \right\} \hat{j} \right]. \end{aligned}$$

(c) **Acceleration:**

$$\begin{aligned} \vec{a}_C &= \frac{\omega^2 r}{3} \left[ \sin\left(\frac{\omega t}{3}\right) \hat{i} + \cos\left(\frac{\omega t}{3}\right) \hat{j} \right], \\ \vec{a}_P &= \omega^2 r \left[ \left\{ \frac{1}{3} \sin\left(\frac{\omega t}{3}\right) + \sin(\omega t) \right\} \hat{i} + \left\{ \frac{1}{3} \cos\left(\frac{\omega t}{3}\right) - \cos(\omega t) \right\} \hat{j} \right]. \end{aligned}$$

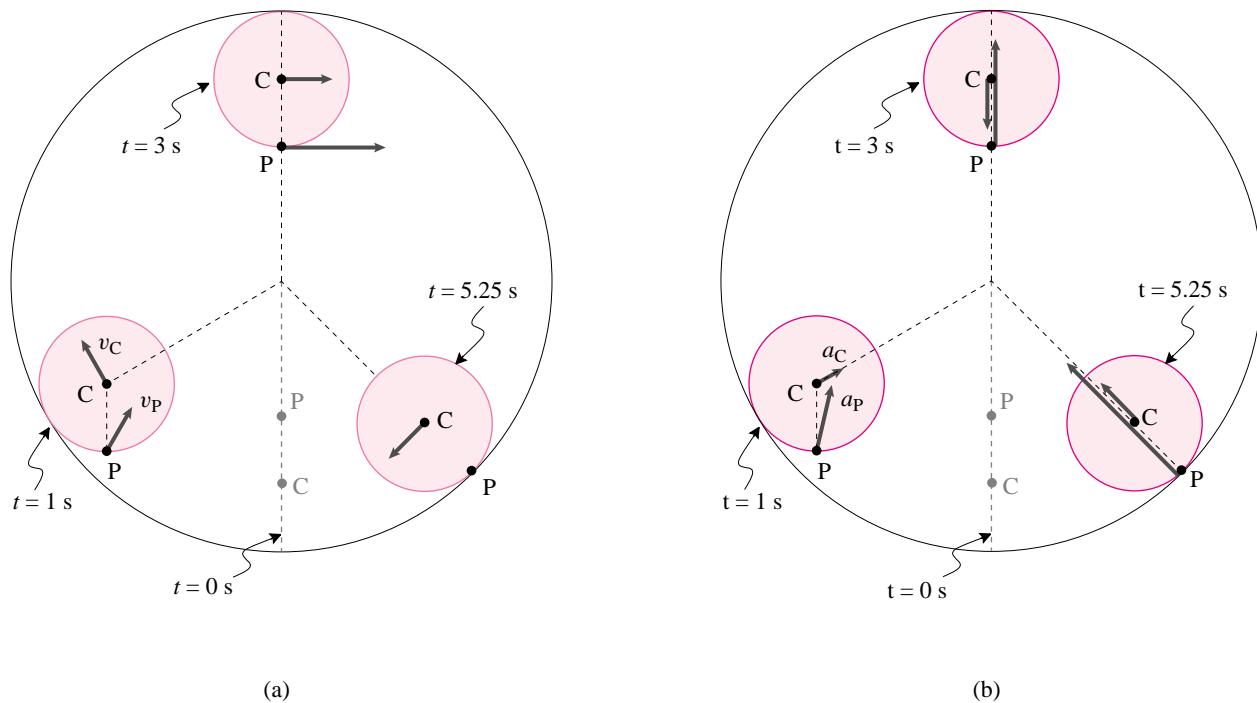
We can now use these expressions to find the position, velocity, and acceleration of the two points at the instants of interest by substituting  $r = 1$  ft,  $\omega = \pi$  rad/s, and appropriate values of  $t$ . These values are shown in Table 8.1.

The velocity and acceleration of the two points are shown in Figures 8.40(a) and (b) respectively.

It is worthwhile to check the directions of velocities and the accelerations by thinking about the velocity and acceleration of point P as a vector sum of the velocity (same for acceleration) of the center of the disk and the velocity (same for acceleration) of point P with respect to the center of the disk. Since the motions involved are circular motions at constant rate, a visual inspection of the velocities and the accelerations is not very difficult. Try it.

$t$	1 s	3 s	5.25 s
$\vec{r}_C$ (ft)	$3(-\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j})$	$3\hat{j}$	$3(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j})$
$\vec{r}_P$ (ft)	$\vec{r}_C - \hat{j}$	$\vec{r}_C - \hat{j}$	$4(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j})$
$\vec{v}_C$ (ft/s)	$\pi(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j})$	$\pi\hat{i}$	$\pi(-\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j})$
$\vec{v}_P$ (ft/s)	$\pi(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j})$	$2\pi\hat{i}$	$\vec{0}$
$\vec{a}_C$ (ft/s <sup>2</sup> )	$\frac{\pi^2}{3}(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j})$	$-\frac{\pi^2}{3}\hat{j}$	$\frac{\pi^2}{3}(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j})$
$\vec{a}_P$ (ft/s <sup>2</sup> )	$11.86(.24\hat{i} + .97\hat{j})$	$\frac{2\pi^2}{3}\hat{j}$	$13.16(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j})$

Table 8.1: Position, velocity, and acceleration of point P and point C

Figure 8.40: (a) Velocity and (b) Acceleration of points P and C at  $t = 1$  s, 3 s, and 5.25 s. (Filename:fig6.5.4a)

**SAMPLE 8.15** *The rolling disk: path of a point on the disk.* For the rolling disk in Sample 8.13, take  $\omega = \pi$  rad/s. Draw the path of a point on the rim of the disk for one complete revolution of the center of the disk around the cylinder for the following conditions:

- (a)  $R = 8r$ ,
- (b)  $R = 4r$ , and
- (c)  $R = 2r$ .

**Solution** In Sample 8.13, we obtained a general expression for the position of a point on the disk as a function of time. By computing the position of the point for various values of time  $t$  up to the time required to go around the cylinder for one complete cycle, we can draw the path of the point. For the various given conditions, the variable that changes in Eqn. (8.34) is  $q$ . We can write a computer program to generate the path of any point on the disk for a given set of  $R$  and  $r$ . Here is a pseudocode to generate the required path on a computer according to Eqn. (8.34).

**A pseudocode to plot the path of a point on the disk:**

```
(pseudo-code) program rollingdisk
%-----
% This code plots the path of any point on a disk of radius
% 'r' rolling with speed 'w' inside a cylinder of radius 'R'.
% The point of interest is distance 'l' away from the center of
% the disk. The coordinates x and y of the specified point P are
% calculated according to the relation mentioned above.
%-----

phi = pi/50*[1,2,3,...,100] % make a vector phi from 0 to 2*pi
x = R*cos(phi) % create points on the outer cylinder
y = R*sin(phi)
plot y vs x % plot the outer cylinder
hold this plot % hold to overlay plots of paths

q = r/(R-r) % calculate q.
T = 2*pi/(q*w) % calculate time T for going around-
% the cylinder once at speed 'w'.
t = T/100*[1,2,3, ..., 100] % make a time vector t from 0 to T-
% taking 101 points.

rcx = -(R-r)*(sin(q*w*t)) % find the x coordinates of pt. C.
rcy = -(R-r)*(cos(q*w*t)) % find the y coordinates of pt. C.
rpx = rcx-l*sin(w*t) % find the x coordinates of pt. P.
rpy = rcy + l*cos(q*t) % find the y coordinates of pt. P.

plot rpy vs rpx % plot the path of P and the path
plot rcy vs rcx % of C. For path of C
```

Once coded, we can use this program to plot the paths of both the center and the point P on the rim of the disk for the three given situations. Note that for any point on the rim of the disk  $l = r$  (see Fig 8.39).

- (a) Let  $R = 4$  units. Then  $r = 0.5$  for  $R = 8r$ . To plot the required path, we run our program `rollingdisk` with desired input,

```
R = 4
r = 0.5
w = pi
l = 0.5
execute rollingdisk
```

The plot generated is shown in Fig.8.41 with a few graphic elements added for illustrative purposes.

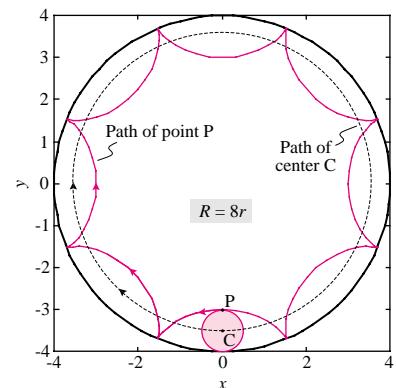


Figure 8.41: Path of point P and the center C of the disk for  $R = 8r$ .

(Filename:sfig6.5.5a)

- (b) Similarly, for  $R = 4r$  we type:

```
R = 4
r = 1
w = pi
l = 1
execute rollingdisk
```

to plot the desired paths. The plot generated in this case is shown in Fig.8.42

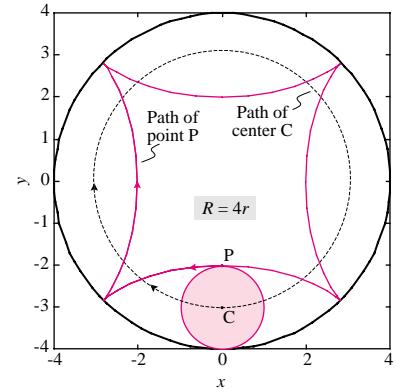


Figure 8.42: Path of point P and the center C of the disk for  $R = 4r$ .

(Filename:sfig6.5.5b)

- (c) The last one is the most interesting case. The plot obtained in this case by typing:

```
R = 4
r = 2
w = pi
l = 2
execute rollingdisk
```

is shown in Fig.8.43. Point P just travels on a straight line! In fact, every point on the rim of the disk goes back and forth on a straight line. Most people find this motion odd at first sight. You can roughly verify the result by cutting a whole twice the diameter of a coin (say a US quarter or dime) in a piece of cardboard and rolling the coin around inside while watching a marked point on the perimeter.

**A curiosity.** We just discovered something simple about the path of a point on the edge of a circle rolling in another circle that is twice as big. The edge point moves in a straight line. In contrast one might think about the motion of the center G of a *straight* line segment that *slides* against two *straight* walls as in sample 8.24. A problem couldn't be more different. Naturally the path of point G is a circle (as you can check physically by looking at the middle of a ruler as you hold it as you sliding against a wall-floor corner).

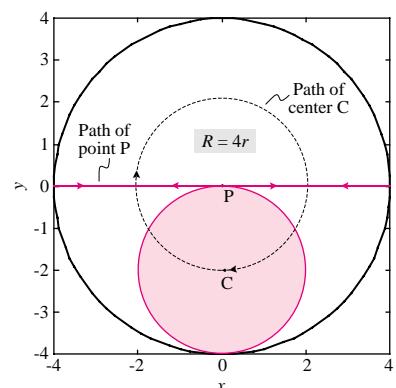


Figure 8.43: Path of point P and the center C of the disk for  $R = 2r$ .

(Filename:sfig6.5.5c)

## 8.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exception of gravity, most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free body diagrams have been contact forces: The force of the ground on an ideal wheel, of an axle on a bearing, etc.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid bodies in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book):  $\dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}$ ,  $\dot{\vec{H}}_C = \vec{r}_{\text{cm}/C} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I \dot{\omega} \hat{k}$  (where  $I = I_{zz}^{\text{cm}}$ ), and  $E_K = m_{\text{tot}} v_{\text{cm}}^2 / 2 + I \omega^2 / 2$ .

The key to success, as usual, is the drawing of appropriate free body diagrams (see Chapter 3 pages 88-91 and Chapter 6 pages 328-9). The two cases one needs to consider as possible are rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than  $\mu N$ , and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of  $\mu N$  but oppose the relative motion.

For friction forces in rolling refer to chapter 2 on free body diagrams. Note that in pure rolling contact, the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{diss} = F_{\text{friction}} \cdot v_{\text{slip}} \quad (8.37)$$

where  $v_{\text{slip}}$  is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation. Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

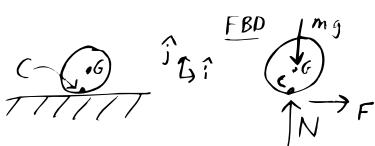


Figure 8.44: A ball rolls or slides on level ground.

(Filename: tfigure.levelrolling)

### Example: Pure rolling on level ground

A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see Fig. 8.44). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{aligned} \dot{\vec{M}}_C &= \dot{\vec{H}}_C & \Rightarrow \vec{r}_{G/C} \times -mg \hat{j} &= \vec{r}_{G/C} \times m \vec{a}_G + \dot{\omega} I_{zz}^{\text{cm}} \hat{k} \\ && \Rightarrow \vec{0} &= R \hat{j} \times (-m \dot{\omega} R \hat{i}) + \dot{\omega} I_{zz}^{\text{cm}} \hat{k} \\ \text{dotting with } \hat{k} &\Rightarrow \dot{\omega} = 0 & \Rightarrow \omega &= \text{constant.} \end{aligned}$$

Because for rolling  $v_G = -\omega R$  we thus have that  $v_G$  is a constant. [The result can also be obtained by combining angular momentum balance about the center of mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be  $\vec{F} = mg\hat{j}$ . So,

assuming point contact, there is no rolling resistance.

□

### Example: Bowling ball with initial sliding

A bowling ball is released with an initial speed of  $v_0$  and no rotation rate. What is its subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and  $F = -\mu N$  (see Fig. 8.44). Linear and angular momentum balance give:

$$\begin{aligned} \text{LMB:} \quad & \Rightarrow \quad \{-F\hat{i} + N\hat{j} - mg\hat{j} = ma\hat{i}\} \\ \{\} \cdot \hat{j} \quad & \Rightarrow \quad N = mg \\ \{\} \cdot \hat{i} \quad & \Rightarrow \quad a = -\mu g \\ \text{AMB}_G: \quad & \Rightarrow \quad -R\mu mg = I_{zz}^{\text{cm}}\dot{\omega} \\ \Rightarrow \quad v = v_0 - \mu gt \quad & \text{and} \quad \omega = -\mu Rmg t / I_{zz}^{\text{cm}} \end{aligned}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip,  $v_G > -\omega R$ . Just like for a sliding block, there is no impetus for reversal, and the block switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu gt = -(-\mu Rmg t / I_{zz}^{\text{cm}}) R \Rightarrow t = \frac{v_0}{\mu g \left( 1 + \frac{mR^2}{I_{zz}^{\text{cm}}} \right)}.$$

Note that the energy lost during sliding is less than  $\mu mg$  times the distance the center of the ball moves during slip. □

### Example: Ball rolling down hill.

Assuming rolling we can find the acceleration of a ball as it rolls downhill (see Fig. 8.45). We start out with the kinematic observations that  $\bar{a}_G = a_G\hat{\lambda}$ , that  $R\omega = -v_G$  and that  $R\dot{\omega} = -a_G$ . Angular momentum balance about the stationary point on the ground instantaneously coinciding with the contact point gives

$$\begin{aligned} \text{AMB}_C \quad & \Rightarrow \quad \vec{r}_{G/C} \times (-mg\hat{j}) = \vec{r}_{G/C} \times m\bar{a}_G + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \\ & \quad \left\{ -R \sin \phi mg\hat{k} = (R\hat{n}) \times (ma_G\hat{\lambda}) + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \right\} \\ \{\} \cdot \hat{k} \quad & \Rightarrow \quad -Rmg \sin \phi = -Rma_G - I_{zz}^{\text{cm}}a_G/R \\ & \Rightarrow \quad a_G = \frac{g \sin \phi}{1 + I_{zz}^{\text{cm}}/(mR^2)}. \end{aligned}$$

Which is less than the acceleration of a block sliding on a ramp without friction:  $a = g \sin \phi$  (unless the mass of the rolling ball is concentrated at the center with  $I_{zz}^{\text{cm}} = 0$ ). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant. □



Figure 8.45: A ball rolls or slides down a slope.  
(Filename:figure.sloperolling)

**Example: Ball rolling down hill: energy approach**

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$\begin{aligned} 0 &= \frac{d}{dt} E_T \Rightarrow \quad 0 = \dot{E}_K + \dot{E}_P \\ &= \frac{d}{dt} (mv^2/2 + I_{zz}^{cm}\omega^2/2) + \frac{d}{dt} (mgy) \\ &= mv\dot{v} + I_{zz}^{cm}\omega\dot{\omega} + mg\dot{y} \\ &= mv\dot{v} + I_{zz}^{cm}(v/R)\dot{v}/R - mg(\sin\phi)v \\ \text{assuming } v \neq 0 \Rightarrow \quad 0 &= (m + I_{zz}^{cm}/R^2)\dot{v} - mg \sin\phi \\ \Rightarrow \quad \dot{v} &= \frac{g \sin\phi}{1 + I_{zz}^{cm}/(mR^2)} \end{aligned}$$

as before.  $\square$

**Example: Does the ball slide?**

How big is the coefficient of friction  $\mu$  needed to prevent slip for a ball rolling down a hill? Use linear momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$\begin{aligned} \text{AMB } (\vec{F}_{\text{tot}} = m\vec{a}_G) \Rightarrow \quad &\left\{ N\hat{n} + F\hat{\lambda} - mg\hat{j} = ma_G\hat{\lambda} \right\} \\ \{\} \cdot \hat{n} \Rightarrow \quad &N = mg \cos\phi \\ \{\} \cdot \hat{\lambda} \Rightarrow \quad &F + mg \sin\phi = m \frac{g \sin\phi}{1 + I_{zz}^{cm}/(mR^2)} \\ F &= \frac{-mg \sin\phi}{1 + mR^2/I_{zz}^{cm}} \end{aligned}$$

Critical condition:  $\Rightarrow \mu = \frac{|F|}{N} = \frac{\tan\phi}{1 + mR^2/I_{zz}^{cm}}$

If  $I_{zz}^{cm}$  is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with  $\mu \approx 1$  and  $I_{zz}^{cm} \approx 2mR^2/5$ ) the steepest slope for rolling without slip is a steep  $\phi = \tan^{-1}(7/2) \approx 74^\circ$ . A metal hoop on the other hand (with  $\mu \approx .3$  and  $I_{zz}^{cm} \approx mR^2$ ) will only roll without slip for slopes less than about  $\phi = \tan^{-1}(.6) \approx 31^\circ$ .  $\square$

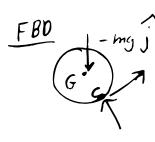
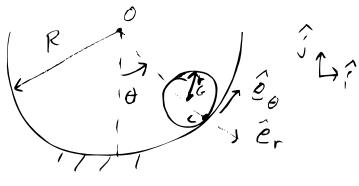


Figure 8.46: A ball rolls in a round cross-section bowl.

(Filename:tfigure.ballinbowl)

**Example: Oscillations of a ball in a bowl.**

A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see Fig. 8.46). Similarly, a cylindrical object can roll inside a pipe. What is the period of oscillation? Start with angular momentum balance about the contact point

$$\begin{aligned} \vec{r}_{G/C} \times (-mg\hat{j}) &= \vec{r}_{G/C} \times m\vec{a}_G + I_{zz}^{cm}\dot{\omega}\hat{k} \\ rmg \sin\theta\hat{k} &= -r\hat{e}_r \times (m((R-r)\ddot{\theta}\hat{e}_\theta - (R-r)\dot{\theta}^2\hat{e}_r)) \\ &\quad + I_{zz}^{cm}\dot{\omega}\hat{k}. \end{aligned}$$

Evaluating the cross products (using that  $\hat{e}_r \times \hat{e}_t = \hat{k}$ ) and using the kinematics from the previous section (that  $(R-r)\dot{\theta} = -r\omega$ ) and dotting the left and right sides with  $\hat{k}$  gives

$$(R-r)\ddot{\theta} = \frac{g \sin\theta}{1 + I_{zz}^{cm}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of  $-\theta$ . Using the small angle approximation that  $\sin \theta = \theta$  the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left( \frac{g}{(R-r)(1+I_{zz}^{\text{cm}}/mr^2)} \right) \theta,$$

If all the ball's mass were concentrated in its middle (so  $I_{zz}^{\text{cm}} = 0$ ) this is naturally the same as for a simple pendulum with length  $R-r$ . For any parameter values the period of small oscillation is

$$T = 2\pi \sqrt{\frac{(R-r)(1+I_{zz}^{\text{cm}}/mr^2)}{g}}.$$

For a marble, ball bearing, or AAA battery in a sideways glass (with  $R-r \approx 2 \text{ cm} = .04 \text{ m}$ ,  $I_{zz}^{\text{cm}}/mr^2 \approx 2/5$  and  $g \approx 10 \text{ m/s}^2$ ) this gives about one oscillation every half second. See page 608 for the energy approach to this problem.  $\square$

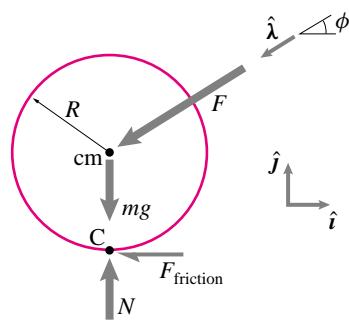


Figure 8.47: FBD of a wheel with mass  $m$ . Force  $F$  is applied by the axle.

(Filename:tfigure2.wheel.mass.lhs)

**SAMPLE 8.16** *A rolling wheel with mass.* Consider the wheel with mass  $m$  shown in figure 8.47. The free body diagram of the wheel is shown here again. Write the equation of motion of the wheel.

**Solution** We can write the equation of motion of the wheel in terms of either the center of mass position  $x$  or the angular displacement of the wheel  $\theta$ . Since in pure rolling, these two variables share a simple relationship ( $x = R\theta$ ), we can easily get the equation of motion in terms of  $x$  if we have the equation in terms of  $\theta$  and vice versa. Since all the forces are shown in the free body diagram, we can easily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force,  $-mg\hat{j}$ , the friction force  $-F_{friction}\hat{i}$ , and the normal reaction of the ground  $N\hat{j}$ , all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned} \sum \vec{M}_C &= \overbrace{\vec{r}_{cm/C} \times (F\hat{\lambda})}^{R\hat{j}} \\ &= R\hat{j} \times F(-\cos\phi\hat{i} - \sin\phi\hat{j}) \\ &= FR\cos\phi\hat{k} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_C &= \vec{r}_{cm/C} \times m\ddot{\vec{a}}_{cm} + I_{zz}^{cm}\dot{\vec{\omega}} \\ &= R\hat{j} \times m\underbrace{\ddot{x}}_{\dot{\omega}R}\hat{i} - I_{zz}^{cm}\dot{\omega}\hat{k} \\ &= -m\dot{\omega}R^2\hat{k} - I_{zz}^{cm}\dot{\omega}\hat{k} \\ &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}. \end{aligned}$$

Thus,

$$\begin{aligned} FR\cos\phi\hat{k} &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k} \\ \Rightarrow \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2} \end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute  $\ddot{\theta} = \ddot{x}/R$  in the equation of motion above to get the equation of motion in terms of the center of mass displacement  $x$  as

$$\ddot{x} = \frac{FR^2\cos\phi}{I_{zz}^{cm} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2}}$$

**Comments:** We could have, of course, used linear momentum balance with angular momentum balance about the center of mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the  $x$  and  $y$  directions involving all forces shown in the free body diagram. The angular momentum balance, on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

**SAMPLE 8.17** Energy and power of a rolling wheel. A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft·lbf. Assume the wheel to be a thin, uniform disk.

- Find the rate of rotation of the wheel.
- Find the average power required to bring the wheel to a complete stop in 5 s.

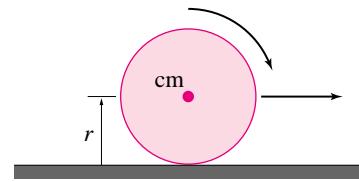


Figure 8.48: (Filename:sfig7.4.1a)

### Solution

- Let  $\omega$  be the rate of rotation of the wheel. Since the wheel rotates without slip, its center of mass moves with speed  $v_{\text{cm}} = \omega r$ . The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$\begin{aligned}
 E_K &= \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I^{\text{cm}}\omega^2 \\
 &= \frac{1}{2}m\omega^2r^2 + \frac{1}{2}I^{\text{cm}}\omega^2 \\
 &= \frac{1}{2}(mr^2 + \underbrace{I^{\text{cm}}}_{\frac{1}{2}mr^2})\omega^2 \\
 &= \frac{3}{4}mr^2\omega^2 \\
 \Rightarrow \omega^2 &= \frac{4E_K}{3mr^2} \\
 &= \frac{4 \times 1700 \text{ ft} \cdot \text{lbf}}{3 \times 20 \text{ lbm} \cdot 1 \text{ ft}^2} \\
 &= \frac{4 \times 1700 \times 32.2 \text{ lbm} \cdot \text{ft} / \text{s}^2}{3 \times 20 \text{ lbm} \cdot \text{ft}} \\
 &= 3649.33 \frac{1}{\text{s}^2} \\
 \Rightarrow \omega &= 60.4 \text{ rad/s.}
 \end{aligned}$$

$$\boxed{\omega = 60.4 \text{ rad/s}}$$

**Note:** This rotational speed, by the way, is extremely high. At this speed the center of mass moves at 60.4 ft/s!

- Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$\begin{aligned}
 P &= \frac{E_{K1} - E_{K2}}{\Delta t} \\
 &= \frac{1700 \text{ ft} \cdot \text{lbf} - 0}{5 \text{ s}} = 340 \text{ ft} \cdot \text{lbf/s} \\
 &= 340 \text{ ft} \cdot \text{lbf/s} \cdot \frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lbf/s}} \\
 &= 0.62 \text{ hp}
 \end{aligned}$$

$$\boxed{P = 0.62 \text{ hp}}$$

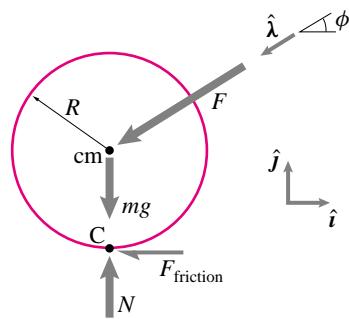


Figure 8.49: FBD of a rolling wheel.

(Filename:tfigure2.wheel.mass.lhs.en)

**SAMPLE 8.18** *Equation of motion of a rolling wheel from energy balance.* Consider the wheel with mass  $m$  from figure 8.49. The free body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance.

**Solution** From energy balance, we have

$$P = \dot{E}_K$$

where

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= F_{friction} \hat{i} \cdot \underbrace{\vec{v}_C}_{\mathbf{0}} + N \hat{j} \cdot \underbrace{\vec{v}_C}_{\mathbf{0}} - mg \hat{j} \cdot \underbrace{\vec{v}_{cm}}_{\hat{i}} + F \hat{\lambda} \cdot \underbrace{\vec{v}_{cm}}_{\hat{i}} \\ &= -mgv(\underbrace{\hat{i} \cdot \hat{j}}_0) + Fv(\underbrace{\hat{\lambda} \cdot \hat{i}}_{-\cos \phi}) \\ &= -Fv \cos \phi \end{aligned}$$

and

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt} \left( \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{zz}^{\text{cm}} \underbrace{\dot{\omega}^2}_{(\dot{x}/R)^2} \right) \\ &= \frac{1}{2} \frac{d}{dt} [(m + I_{zz}^{\text{cm}}/R^2) \dot{x}^2] \\ &= (m + I_{zz}^{\text{cm}}/R^2) \ddot{x}. \end{aligned}$$

Thus,

$$\begin{aligned} -Fv \cos \phi &= (m + I_{zz}^{\text{cm}}/R^2) \ddot{x} \\ \text{or } -F \ddot{x} \cos \phi &= (m + I_{zz}^{\text{cm}}/R^2) \ddot{x} \\ \Rightarrow \ddot{x} &= -\frac{F \cos \phi}{m + I_{zz}^{\text{cm}}/R^2}. \end{aligned}$$

We can also write the equation of motion in terms of  $\theta$  by replacing  $\ddot{x}$  with  $\ddot{\theta}R$  giving,

$$\ddot{\theta} = \frac{FR \cos \phi}{m + I_{zz}^{\text{cm}}/R^2}.$$

$$\boxed{\ddot{x} = -\frac{F \cos \phi}{m + I_{zz}^{\text{cm}}/R^2}}$$

**Comments:** In the equations above (for calculating  $P$ ), we have set  $\vec{v}_C = \mathbf{0}$  because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center of mass. So, the only power supplied to the wheel is due to the force  $F\hat{\lambda}$  acting at the center of mass.

**SAMPLE 8.19** *Equation of motion of a rolling disk on an incline.* A uniform circular disk of mass  $m = 1 \text{ kg}$  and radius  $R = 0.4 \text{ m}$  rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center of mass in 2 s.

**Solution** The free body diagram of the disk is shown in Fig. 8.51. In addition to the base unit vectors  $\hat{i}$  and  $\hat{j}$ , let us use unit vectors  $\hat{\lambda}$  and  $\hat{n}$  along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{M}_C = \vec{H}_C$$

where

$$\begin{aligned} \sum \vec{M}_C &= \vec{r}_{O/C} \times m\vec{g} = R\hat{n} \times (-mg\hat{j}) \\ &= -Rmg \sin \alpha \hat{k} \end{aligned}$$

and

$$\begin{aligned} \vec{H}_C &= -I_{zz}^{\text{cm}} \dot{\omega} \hat{k} + \vec{r}_{O/C} \times m \vec{a}_{\text{cm}} \\ &= -I_{zz}^{\text{cm}} \dot{\omega} \hat{k} + mR^2 \dot{\omega} (\hat{n} \times \hat{\lambda}) \\ &= -(I_{zz}^{\text{cm}} + mR^2) \dot{\omega} \hat{k}. \end{aligned}$$

Thus,

$$\begin{aligned} -Rmg \sin \alpha \hat{k} &= -(I_{zz}^{\text{cm}} + mR^2) \dot{\omega} \hat{k} \\ \Rightarrow \dot{\omega} &= \frac{g \sin \alpha}{R[1 + I_{zz}^{\text{cm}}/(mR^2)]} \end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g \sin \alpha}{R[1 + I_{zz}^{\text{cm}}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for  $\omega$  and  $\theta$  by simply integrating this equation and substituting the initial conditions  $\omega(t = 0) = 0$  and  $\theta(t = 0) = 0$ . Let us write the equation of motion as  $\dot{\omega} = \beta$  where  $\beta = g \sin \alpha / R(1 + I_{zz}^{\text{cm}} / mR^2)$ . Then,

$$\begin{aligned} \omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2} \beta t^2 + C_1 t + C_2. \end{aligned}$$

Substituting the given initial conditions  $\dot{\theta}(0) = 0$  and  $\theta(0) = 0$ , we get  $C_1 = 0$  and  $C_2 = 0$ , which implies that  $\theta = \frac{1}{2} \beta t^2$ . Now, in pure rolling,  $x = R\theta$ . Therefore,

$$\begin{aligned} x(t) &= R\theta(t) = \frac{1}{2} \beta t^2 = \frac{1}{2} \frac{g \sin \alpha}{R(1 + I_{zz}^{\text{cm}} / mR^2)} t^2 \\ &= \frac{1}{2} \frac{g \sin \alpha}{\frac{1}{2} mR^2} t^2 = \frac{1}{3} (g \sin \alpha) t^2 \\ x(2 \text{ s}) &= \frac{1}{3} \cdot 9.8 \text{ m/s}^2 \cdot \sin(30^\circ) \cdot (2 \text{ s})^2 = 6.53 \text{ m} \end{aligned}$$

$$\boxed{x(2 \text{ s}) = 6.53 \text{ m}}$$

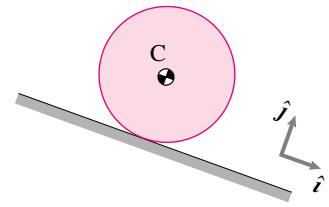


Figure 8.50: (Filename:sfig9.rolling.incline1)

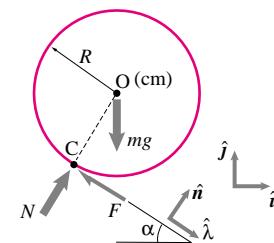


Figure 8.51: (Filename:sfig9.rolling.incline1a)

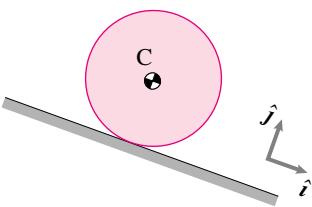


Figure 8.52: (filename:sfig9.rolling.incline2)

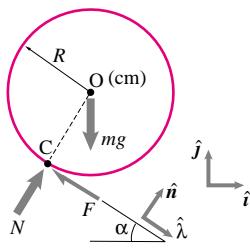


Figure 8.53: (filename:sfig9.rolling.incline2a)

**SAMPLE 8.20** *Using Work and energy in pure rolling.* Consider the disk of Sample 8.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center of mass when the disk is 2 m down the inclined plane.

**Solution** We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center of mass after it has travelled 2 m along the incline. We can, of course, solve this problem using equation of motion, by first solving for the time  $t$  the disk takes to travel the given distance and then evaluating the expression for speed  $\omega(t)$  or  $x(t)$  at that  $t$ . However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let  $\omega_1$  and  $\omega_2$  be the initial and final (after rolling down by  $d = 2$  m) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

$$E_K = \frac{1}{2}m \overbrace{v_{cm}^2} + \frac{1}{2}I_{zz}^{\text{cm}}\omega^2 = \frac{1}{2}(mR^2 + I_{zz}^{\text{cm}})\omega^2.$$

Therefore,

$$\Delta E_K = E_{K2} - E_{K1} = \frac{1}{2}(mR^2 + I_{zz}^{\text{cm}})(\omega_2^2 - \omega_1^2) \quad (8.38)$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of he mass-center by  $d$  along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{j}) \cdot (\hat{d}\hat{\lambda}) = -mgd(\hat{j} \cdot \hat{\lambda}) = mgd \sin \alpha \quad (8.39)$$

From work-energy principle (integral form of power balance,  $P = \dot{E}_K$ ), we know that  $W = \Delta E_K$ . Therefore, from eqn. (8.38) and eqn. (8.39), we get

$$\begin{aligned} mgd \sin \alpha &= \frac{1}{2}(mR^2 + I_{zz}^{\text{cm}})(\omega_2^2 - \omega_1^2) \\ \Rightarrow \omega_2^2 &= \omega_1^2 + \frac{2mgd \sin \alpha}{mR^2 + I_{zz}^{\text{cm}}} = \omega_1^2 + \frac{2gd \sin \alpha}{R^2 \left(1 + \frac{I_{zz}^{\text{cm}}}{mR^2}\right)} \\ &= \omega_1^2 + \frac{4gd \sin \alpha}{3R^2} \end{aligned}$$

Substituting the values of  $g$ ,  $d$ ,  $\alpha$ ,  $R$ , etc., and setting  $\omega_1 = 0$ , we get

$$\begin{aligned} \omega_2^2 &= \frac{4 \cdot (9.8 \text{ m/s}^2) \cdot (2 \text{ m}) \cdot (\sin(30^\circ))}{3 \cdot (0.4 \text{ m})^2} = 81.67/\text{s}^2 \\ \Rightarrow \omega_2 &= 9.04 \text{ rad/s.} \end{aligned}$$

The corresponding speed of the center of mass is

$$v_{cm} = \omega_2 R = 9.04 \text{ rad/s} \cdot 0.4 \text{ m} = 3.61 \text{ m/s.}$$

$v_{cm} = 3.61 \text{ m/s}$

**SAMPLE 8.21** *Impulse and momentum calculations in pure rolling.* Consider the disk of Sample 8.19 rolling down the incline again. Find an expression for the rolling speed ( $\omega$ ) of the disk after a finite time  $\Delta t$ , given the initial rolling speed  $\omega_1$ .

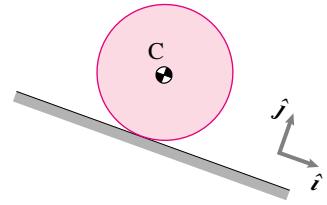


Figure 8.54: (Filename:sfig9.rolling.incline3)

**Solution** Once again, this problem can be solved by integrating the equation of motion (as done in Sample 8.19). However, we will solve this problem here using impulse-momentum relationship. Note that we need the speed of the disk  $\omega_2$ , after a finite time  $\Delta t$ , given the initial speed  $\omega_1$ . Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship,  $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$ , we have

$$(-F\hat{\lambda} + N\hat{i} - mg\hat{j})\Delta t = m(v_2 - v_1)\hat{\lambda} \quad (8.40)$$

Dotting eqn. (8.40) with  $\hat{\lambda}$  gives

$$\begin{aligned} (-F - mg(\underbrace{\hat{j} \cdot \hat{\lambda}}_{-\sin \alpha}))\Delta t &= m(v_2 - v_1) \\ (-F + mg \sin \alpha)\Delta t &= mR(\omega_2 - \omega_1) \end{aligned} \quad (8.41)$$

Similarly, the angular impulse-momentum relationship about the mass-center,  $\vec{M}_O \Delta t = (\vec{H}_O)_2 - (\vec{H}_O)_1$ , gives

$$\begin{aligned} (-FR\hat{k})\Delta t &= -I_{zz}^{\text{cm}}(\omega_2 - \omega_1)\hat{k} \\ \Rightarrow FR\Delta t &= I_{zz}^{\text{cm}}(\omega_2 - \omega_1) \end{aligned} \quad (8.42)$$

Note that the other forces ( $N$  and  $mg$ ) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force  $F$  from eqn. (8.41) and eqn. (8.42) by multiplying eqn. (8.41) with  $R$  and adding to eqn. (8.42):

$$\begin{aligned} mgR \sin \alpha \Delta t &= (I_{zz}^{\text{cm}} + mR^2)(\omega_2 - \omega_1) \\ \text{or} \quad g \sin \alpha \Delta t &= R \left( 1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right) (\omega_2 - \omega_1) \\ \Rightarrow \omega_2 &= \omega_1 + \frac{g \sin \alpha}{R \left( 1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right)} \Delta t \end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g \sin \alpha}{R \left( 1 + \frac{I_{zz}^{\text{cm}}}{mR^2} \right)} \Delta t}$$

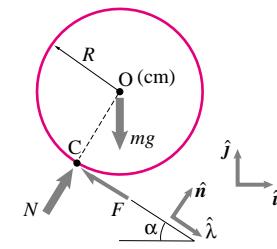


Figure 8.55: (Filename:sfig9.rolling.incline3a)

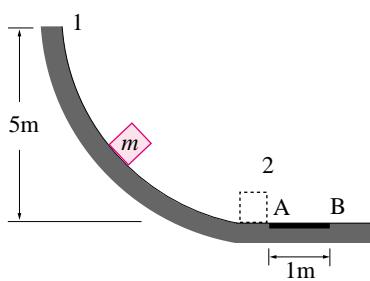


Figure 8.56: (Filename:sfig2.9.1a)

**SAMPLE 8.22 Work and energy calculations in sliding.** A block of mass  $m = 2.5 \text{ kg}$  slides down a frictionless incline from a 5 m height. The block encounters a frictional bed AB of length 1 m on the ground. If the speed of the block is 9 m/s at point B, find the coefficient of friction between the block and the frictional surface AB.

**Solution** We can divide the problem in two parts: We first find the speed of the block as it reaches point A using conservation of energy for its motion on the inclined surface, and then use work-energy principle to find the speed at B. Let the ground level be the datum for P.E. and let  $v$  be the speed at A. For the motion on the incline;

$$\begin{aligned} E_{K1} + E_{P1} &= E_{K2} + E_{P2} \\ 0 + mgh &= \frac{1}{2}mv^2 + 0 \\ \Rightarrow v &= \sqrt{2gh} \\ &= \sqrt{2 \cdot 9.81 \text{ m/s}^2 \cdot 5 \text{ m}} \\ &= 9.90 \text{ m/s}. \end{aligned}$$

Now, as the block slides on the surface AB, a force of friction  $= \mu N = \mu mg$  (since  $N = mg$ , from linear momentum balance in the vertical direction) acts in the opposite direction of motion (see Fig. 8.57). Work done by this force on the block is,

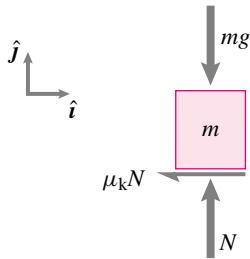
$$\begin{aligned} W &= \vec{F} \cdot \vec{r} \\ &= -\mu mg \hat{i} \cdot L \hat{i} \\ &= -\mu mg L \end{aligned}$$

From work energy relationship we have,

$$\begin{aligned} W &= \Delta E_K = E_{K2} - E_{K1} \\ \Rightarrow E_{K2} &= E_{K1} + W \\ \frac{1}{2}mv_B^2 &= \frac{1}{2}mv^2 - \mu mg L \\ -\mu mg L &= \frac{1}{2}m(v_B^2 - v^2) \\ \Rightarrow \mu &= \frac{1}{2gL}(v_B^2 - v^2) \\ &= \frac{(9.90 \text{ m/s})^2 - (9 \text{ m/s})^2}{2 \cdot 9.81 \text{ m/s}^2 \cdot 1 \text{ m}} \\ &= 0.87 \end{aligned}$$

$$\boxed{\mu = 0.87}$$

Figure 8.57: Free body diagram of the block when the block is on the rough surface.  
(Filename:sfig2.9.1b)





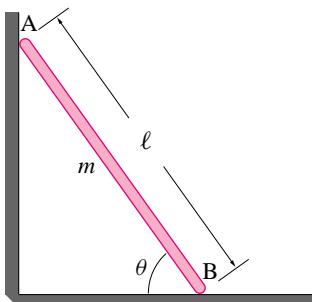


Figure 8.58: A ladder, modeled as a uniform rod AB, falls from a rest position at  $\theta = \theta_0$  ( $\theta_0 < \pi/2$ ) such that its ends slide along frictionless vertical and horizontal surfaces.

(Filename:sfig7.3.1)

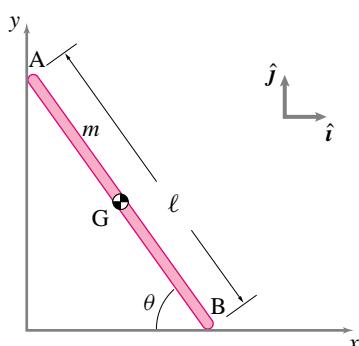


Figure 8.59: (Filename:sfig7.3.1a)

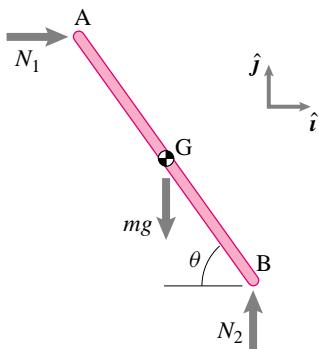


Figure 8.60: The free body diagram of the ladder.

(Filename:sfig7.3.1b)

**SAMPLE 8.23 Falling ladder.** A ladder AB, modeled as a uniform rigid rod of mass  $m$  and length  $\ell$ , rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at  $\theta = \theta_0$  ( $\theta_0 < \pi/2$ ). Assume the motion to be planar (in the vertical plane).

- As the ladder falls, what is the path of the center of mass of the ladder?
- Find the equation of motion (e.g., a differential equation in terms of  $\theta$  and its time derivatives) for the ladder.
- How does the angular speed  $\omega$  ( $= \dot{\theta}$ ) depend on  $\theta$ ?

**Solution** Since the ladder is modeled by a uniform rod AB, its center of mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle  $\theta$  (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

- Path of the center of mass:** Let the origin of our  $x$ - $y$  coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 8.59). The position vector of the center of mass G may be written as

$$\begin{aligned}\vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell \cos \theta \hat{i} + \frac{\ell}{2}(-\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= \frac{\ell}{2}(\cos \theta \hat{i} + \sin \theta \hat{j}).\end{aligned}\quad (8.43)$$

Thus the coordinates of the center of mass are

$$x_G = \frac{\ell}{2} \cos \theta \quad \text{and} \quad y_G = \frac{\ell}{2} \sin \theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius  $\frac{\ell}{2}$ . Therefore, the center of mass of the ladder follows a circular path of radius  $\frac{\ell}{2}$  centered at the origin. Of course, the center of mass traverses only that part of the circle which lies between its initial position at  $\theta = \theta_0$  and the final position at  $\theta = 0$ .

- Equation of motion:** The free body diagram of the ladder is shown in Fig. 8.60. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the ladder we get

$$N_1 \hat{i} + (N_2 - mg) \hat{j} = m \vec{a}_G = m \ddot{\vec{r}}_G.$$

Differentiating eqn. (8.43) twice we get  $\ddot{\vec{r}}_G$  as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta} \sin \theta - \dot{\theta}^2 \cos \theta) \hat{i} + (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \hat{j}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by  $\hat{i}$  and then by  $\hat{j}$  we get

$$\begin{aligned}N_1 &= -\frac{1}{2}m\ell(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + mg.\end{aligned}$$

Next, we write the angular momentum balance for the ladder about its center of mass,  $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ , where

$$\begin{aligned}\sum \vec{M}_{/G} &= \left( -N_1 \frac{\ell}{2} \sin \theta + N_2 \frac{\ell}{2} \cos \theta \right) \hat{k} \\ &= \frac{1}{2} m \ell (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \frac{\ell}{2} \sin \theta \hat{k} \\ &\quad + \left[ \frac{1}{2} m \ell (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + mg \right] \frac{\ell}{2} \cos \theta \hat{k} \\ &= \left( \frac{1}{4} m \ell^2 \ddot{\theta} + \frac{1}{2} m g \ell \cos \theta \right) \hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/G} = I_{zz/G} \dot{\vec{\omega}} = \frac{1}{12} m \ell^2 \dot{\theta} (-\hat{k}),$$

where  $\dot{\vec{\omega}} = \dot{\theta}(-\hat{k})$  because  $\theta$  is measured positive in the clockwise direction ( $-\hat{k}$ ). Now, equating the two quantities  $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$  and dotting both sides with  $\hat{k}$  we get

$$\begin{aligned}\frac{1}{4} m \ell^2 \ddot{\theta} + \frac{1}{2} m g \ell \cos \theta &= -\frac{1}{12} m \ell^2 \dot{\theta} \\ \text{or} \quad \left( \frac{1}{12} + \frac{1}{4} \right) \ell^2 \ddot{\theta} &= -\frac{1}{2} g \ell \cos \theta \\ \text{or} \quad \ddot{\theta} &= -\frac{3g}{2\ell} \cos \theta\end{aligned}\tag{8.44}$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for  $\theta(t)$ .

- (c) **Angular Speed of the ladder:** To solve for the angular speed  $\omega$  ( $= \dot{\theta}$ ) as a function of  $\theta$  we need to express eqn. (8.44) in terms of  $\omega$ ,  $\theta$ , and derivatives of  $\omega$  with respect to  $\theta$ . Now,

$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta}.$$

Substituting in eqn. (8.44) and integrating both sides from the initial rest position to an arbitrary position  $\theta$  we get

$$\begin{aligned}\int_0^\omega \omega d\omega &= - \int_{\theta_0}^\theta \frac{3g}{2\ell} \cos \theta d\theta \\ \Rightarrow \quad \frac{1}{2} \omega^2 &= - \frac{3g}{2\ell} (\sin \theta - \sin \theta_0) \\ \Rightarrow \quad \omega &= \pm \sqrt{\frac{3g}{\ell} (\sin \theta_0 - \sin \theta)}.\end{aligned}$$

Since end B is sliding to the right,  $\theta$  is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.*,

$$\vec{\omega} = \dot{\theta}(-\hat{k}) = -\sqrt{\frac{3g}{\ell} (\sin \theta_0 - \sin \theta)} \hat{k}.$$

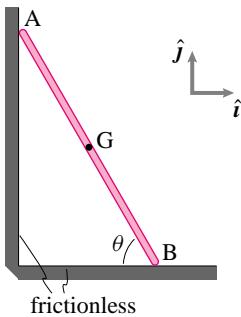


Figure 8.61: (Filename:sfig7.4.3)

**SAMPLE 8.24** *The falling ladder again.* Consider the falling ladder of Sample 8.10 again. The mass of the ladder is  $m$  and the length is  $\ell$ . The ladder is released from rest at  $\theta = 80^\circ$ .

- At the instant when  $\theta = 45^\circ$ , find the speed of the center of mass of the ladder using energy.
- Derive the equation of motion of the ladder using work-energy balance.

### Solution

- Since there is no friction, there is no loss of energy between the two states:  $\theta_0 = 80^\circ$  and  $\theta_f = 45^\circ$ . The only external forces on the ladder are  $N_1$ ,  $N_2$ , and  $mg$  as shown in the free body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls,  $N_1$  and  $N_2$ , respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

Let  $E_1$  and  $E_2$  be the total energy of the ladder at  $\theta_0$  and  $\theta_f$ , respectively. Then

$$E_1 = E_2 \quad (\text{conservation of energy})$$

$$\begin{aligned} \text{Now } E_1 &= \underbrace{E_{K1}}_{\text{K.E.}} + \underbrace{E_{P1}}_{\text{P.E.}} \\ &= 0 + mg h_1 \\ &= mg \frac{\ell}{2} \sin \theta_0 \end{aligned}$$

$$\text{and } E_2 = E_{K2} + E_{P2} = \underbrace{\frac{1}{2}mv_G^2}_{E_{K2}} + \underbrace{\frac{1}{2}I_{zz}^G\omega^2}_{mgh_2} + mgh_2$$

Figure 8.62: (Filename:sfig7.4.3a)

Equating  $E_1$  and  $E_2$  we get

$$\eta g \frac{\ell}{2} (\sin \theta_0 - \sin \theta_f) = \frac{1}{2} (\eta v_G^2 + \underbrace{\frac{1}{12} \eta \ell^2 \omega^2}_{I_{zz}^G})$$

$$\text{or } g \ell (\sin \theta_0 - \sin \theta_f) = v_G^2 + \frac{1}{12} \ell^2 \omega^2 \quad (8.45)$$

Clearly, we cannot find  $v_G$  from this equation alone because the equation contains another unknown,  $\omega$ . So we need to find another equation which relates  $v_G$  and  $\omega$ . To find this equation we turn to kinematics. Note that

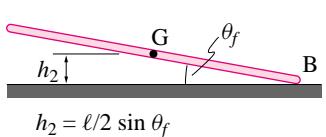
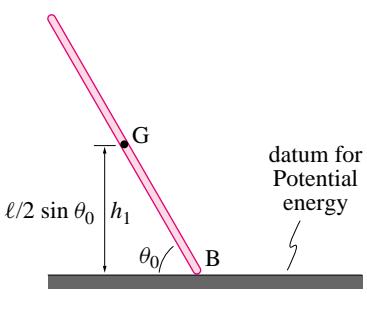


Figure 8.63: (Filename:sfig7.4.3b)

$$\begin{aligned} \vec{r}_G &= \frac{\ell}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ \Rightarrow \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2} (-\sin \theta \cdot \dot{\theta} \hat{i} + \cos \theta \cdot \dot{\theta} \hat{j}) \\ \Rightarrow v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4} (\cos^2 \theta + \sin^2 \theta) \dot{\theta}^2} \\ &= \frac{\ell}{2} \dot{\theta} = \frac{\ell}{2} \omega \\ \Rightarrow \omega &= \frac{2v_G}{\ell} \end{aligned}$$

Substituting the expression for  $\omega$  in Eqn 8.45 we get

$$\begin{aligned} g\ell(\sin \theta_0 - \sin \theta_f) &= v_G^2 + \frac{1}{12}\ell^2 \cdot \frac{4v_G^2}{\ell^2} \\ &= \frac{4}{3}v_G^2 \\ \Rightarrow v_G &= \sqrt{\frac{3g\ell}{4}(\sin \theta_0 - \sin \theta_f)} \\ &= 0.46\sqrt{g\ell} \end{aligned}$$

$$v_G = 0.46\sqrt{g\ell}$$

- (b) Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K$$

For the ladder, the only force that does work is  $mg$ . This force acts on the center of mass G. Therefore,

$$\begin{aligned} P &= \vec{F} \cdot \vec{v} = -mg\hat{j} \cdot \vec{v}_G \\ &= -mg\hat{j} \cdot \left[ \frac{\ell}{2}(-\sin \theta \hat{i} + \cos \theta \hat{j}) \dot{\theta} \right] \\ &= -mg\frac{\ell}{2}\dot{\theta} \cos \theta \end{aligned}$$

The rate of change of kinetic energy

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt} \left( \frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2 \right) \\ &= \frac{d}{dt} \left( \frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2 \right) \\ &= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\ &= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}) \end{aligned}$$

Now equating  $P$  and  $\dot{E}_K$  we get

$$\begin{aligned} \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} &= -mg\frac{\ell}{2}\dot{\theta} \cos \theta \\ \Rightarrow \ddot{\theta} &= -\frac{3g}{2\ell} \cos \theta \end{aligned}$$

which is the same expression as obtained in Sample 8.23 (b).

$$\ddot{\theta} = -\frac{3g}{2\ell} \cos \theta$$

**Note:** To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. One might wonder if this is a consistent assumption. Does this assumption correspond to the non-physical assumption that the wall is capable of pulling on the ladder? Or in other words, if a real ladder was sliding against a slippery wall and floor would it lose contact? The answer is yes. One way of finding when contact would be lost is to calculate the normal reaction  $N_1$  and finding out at what value of  $\theta$  it passes through zero. It turns out that  $N_1$  is zero at about  $\theta = 41^\circ$ .

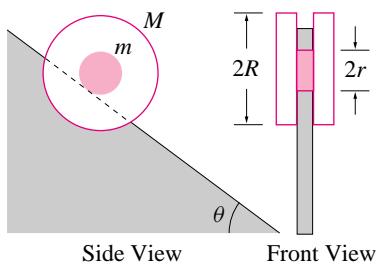


Figure 8.64: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

(Filename:sfig7.3.3)

**SAMPLE 8.25** *Rolling on an inclined plane.* A wheel is made up of three uniform disks—the center disk of mass  $m = 1 \text{ kg}$ , radius  $r = 10 \text{ cm}$  and two identical outer disks of mass  $M = 2 \text{ kg}$  each and radius  $R$ . The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is  $\theta = 30^\circ$ . The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed  $0.2g$ . Find the radius  $R$  of the bigger disks.

**Solution** Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius  $R$ .

The free body diagram of the wheel is shown in Fig. 8.65. In addition to the weight  $(m + 2M)g$  of the wheel and the normal reaction  $N$  of the wedge surface there is an unknown force of friction  $F_f$  acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that  $F_f \neq \mu N$  because there is neither slipping nor a condition of impending slipping. Thus the magnitude of  $F_f$  is not known yet.

Let the acceleration of the center of mass of the wheel be

$$\vec{a}_G = a_G \hat{\lambda}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\omega}} = -\dot{\omega} \hat{k}.$$

We assumed  $\dot{\vec{\omega}}$  to be in the *negative*  $\hat{k}$  direction. But, if this assumption is wrong, we will get a negative value for  $\dot{\omega}$ .

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned} \sum \vec{F} &= m_{\text{total}} \vec{a}_G \\ -(m + 2M)g \hat{j} + N \hat{n} - F_f \hat{\lambda} &= (m + 2M)a_G \hat{\lambda} \end{aligned}$$

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns:  $N$ ,  $F_f$ , and  $a_G$ . Thus we do not have enough equations to solve for the unknowns including the quantity of interest  $a_G$ . So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using  $\vec{r}_{G/C} = r \hat{n}$ ):

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned} \vec{M}_C &= \vec{r}_{G/C} \times (m + 2M)g(-\hat{j}) \\ &= r \hat{n} \times (m + 2M)g(-\hat{j}) \\ &= -(m + 2M)gr \sin \theta \hat{k} \quad (\text{see Fig. 8.66}) \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_C &= I_{zz}^G \dot{\vec{\omega}} + \vec{r}_{G/C} \times m_{\text{total}} \vec{a}_G \\ &= I_{zz}^G (-\dot{\omega} \hat{k}) - m_{\text{total}} \dot{\omega} r^2 \hat{k} \\ &= (I_{zz}^G + m_{\text{total}} r^2)(-\dot{\omega} \hat{k}) \\ &= [(\frac{1}{2}mr^2 + 2 \cdot \frac{1}{2}MR^2) + \overbrace{(m + 2M)}^{m_{\text{total}}} r^2](-\dot{\omega} \hat{k}) \\ &= -[\frac{3}{2}mr^2 + M(R^2 + 2r^2)]\dot{\omega} \hat{k}. \end{aligned}$$

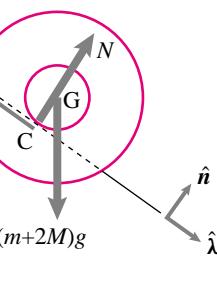


Figure 8.65: Free body diagram of the wheel.

(Filename:sfig7.3.3a)

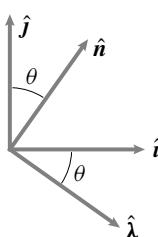


Figure 8.66: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example,  $\hat{n} \times \hat{j} = \sin \theta \hat{k}$ .

(Filename:sfig7.3.3b)

Thus,

$$\begin{aligned} -(m + 2M)gr \sin \theta \hat{k} &= -[\frac{3}{2}mr^2 + M(R^2 + 2r^2)]\dot{\omega}\hat{k} \\ \Rightarrow \dot{\omega} &= \frac{(m + 2M)gr \sin \theta}{\frac{3}{2}mr^2 + M(R^2 + 2r^2)}. \end{aligned} \quad (8.46)$$

Now we need to relate  $\dot{\omega}$  to  $a_G$ . From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (8.46) we get

$$a_G = \frac{(m + 2M)gr^2 \sin \theta}{\frac{3}{2}mr^2 + M(R^2 + 2r^2)}.$$

Now we can solve for  $R$  in terms of  $a_G$ :

$$\begin{aligned} \frac{3}{2}mr^2 + M(R^2 + 2r^2) &= \frac{(m + 2M)gr^2 \sin \theta}{a_G} \\ \Rightarrow M(R^2 + 2r^2) &= \frac{(m + 2M)g}{a_G} r^2 \sin \theta - \frac{3}{2}mr^2 \\ \Rightarrow R^2 &= \frac{(m + 2M)g}{Ma_G} r^2 \sin \theta - \frac{3m}{2M} r^2 - 2r^2. \end{aligned}$$

Since we require  $a_G \leq 0.2g$  we get

$$\begin{aligned} R^2 &\geq \left( \frac{(m + 2M)g}{M \cdot 0.2g} \sin \theta - \frac{3m}{2M} - 2 \right) r^2 \\ &\geq \left( \frac{5 \text{ kg}}{0.4 \text{ kg}} \cdot \frac{1}{2} - \frac{3 \text{ kg}}{4 \text{ kg}} - 2 \right) (0.1 \text{ m})^2 \\ &\geq 0.035 \text{ m}^2 \\ \Rightarrow R &\geq 0.187 \text{ m}. \end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$$R \geq 18.7 \text{ cm}$$

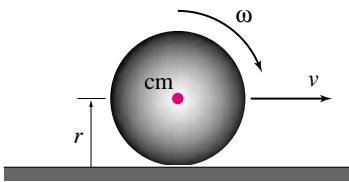
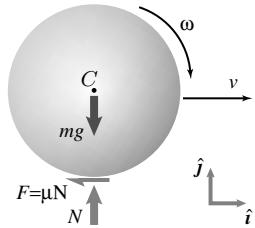


Figure 8.67: (Filename:sfig9.rollandslide.ball1)

**SAMPLE 8.26** Which one starts rolling first — a marble or a bowling ball? A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say  $v_0$ . The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

**Solution** Let us consider one ball, say the bowling ball, first. Let the radius of the ball be  $r$  and mass  $m$ . The ball starts with center of mass velocity  $\vec{v}_0 = v_0 \hat{i}$ . The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal  $\mu N$  (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition,  $v = \omega r$ . Let the instant of transition from the mixed motion to pure rolling be  $t^*$ . From linear momentum balance, we have

Figure 8.68: Free body diagram of the ball during sliding.  
(Filename:sfig9.rollandslide.ball1a)

$$m\dot{v}\hat{i} = -\mu N\hat{i} + (N - mg)\hat{j} \quad (8.47)$$

$$\begin{aligned} \text{eqn. (8.47)} \cdot \hat{j} &\Rightarrow N = mg \\ \text{eqn. (8.47)} \cdot \hat{i} &\Rightarrow m\dot{v} = -\mu N = -\mu mg \\ &\Rightarrow \dot{v} = -\mu g \\ &\Rightarrow v = v_0 - \mu gt \end{aligned} \quad (8.48)$$

Similarly, from angular momentum balance about the mass-center, we get

$$\begin{aligned} -I_{zz}^{cm}\dot{\omega}\hat{k} &= -\mu Nr\hat{k} = -\mu mgr\hat{k} \\ \Rightarrow \dot{\omega} &= \frac{\mu mgr}{I_{zz}^{cm}} \\ \Rightarrow \omega &= \underbrace{\omega_0}_{0} + \frac{\mu mgr}{I_{zz}^{cm}}t \end{aligned} \quad (8.49)$$

At the instant of transition from mixed rolling and sliding to pure rolling, i.e., at  $t = t^*$ ,  $v = \omega r$ . Therefore, from eqn. (8.48) and eqn. (8.49), we get

$$\begin{aligned} v_0 - \mu gt^* &= \frac{\mu mgr^2}{I_{zz}^{cm}}t^* \\ \Rightarrow v_0 &= \mu gt^*(1 + \frac{mr^2}{I_{zz}^{cm}}) \\ \Rightarrow t^* &= \frac{v_0}{\mu g(1 + \frac{mr^2}{I_{zz}^{cm}})} \end{aligned}$$

Now, for a sphere,  $I_{zz}^{cm} = \frac{2}{5}mr^2$ . Therefore,

$$t^* = \frac{v_0}{\mu g(1 + \frac{mr^2}{\frac{2}{5}mr^2})} = \frac{2v_0}{7\mu g}.$$

Note that the expression for  $t^*$  is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$t^* = \frac{2v_0}{7\mu g}$  for both.

**SAMPLE 8.27** Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum. Consider the problem in Sample 8.26 again: A ball of radius  $r = 10\text{ cm}$  and mass  $m = 1\text{ kg}$  is launched horizontally with initial velocity  $v_0 = 5\text{ m/s}$  on a surface with coefficient of friction  $\mu = 0.12$ . The ball starts sliding, rolls and slides simultaneously for a while, and then starts pure rolling. Find the time it takes to start pure rolling.

**Solution** Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by  $t^*$ . At  $t = 0$ , we know that  $v_{\text{cm}} = v_0 = 5\text{ m/s}$ , and  $\omega_0 = 0$ . We also know that at  $t = t^*$ ,  $v_{\text{cm}} = v_{t^*}\omega_{t^*}r$ , where  $r$  is the radius of the ball. We do not know  $t^*$  and  $v_{t^*}$ . However, we are considering a finite time event (during  $t^*$ ) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

$$\begin{aligned}\sum \vec{F} \cdot t^* &= mv_{t^*}\hat{i} - mv_0\hat{i} \\ (-\mu N\hat{i} + (N - mg)\hat{j})t^* &= m(v_{t^*} - v_0)\hat{i}\end{aligned}$$

Dotting the above equation with  $\hat{j}$  and  $\hat{i}$ , respectively, we get

$$\begin{aligned}N &= mg \\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\ \Rightarrow -\mu gt^* &= v_{t^*} - v_0\end{aligned}\quad (8.50)$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned}\sum \vec{M}_{\text{cm}} t^* &= (\vec{H}_{\text{cm}})_{t^*} - (\vec{H}_{\text{cm}})_0 \\ (-\mu Nr\hat{k})t^* &= (I_{zz}^{\text{cm}}\omega_{t^*} - I_{zz}^{\text{cm}}\underbrace{\omega_0}_0)(-\hat{k}) \\ \text{or} \quad -\mu mgrt^* &= -I_{zz}^{\text{cm}}\omega_{t^*} \\ \Rightarrow \omega_{t^*} &= \mu mgrt^*/I_{zz}^{\text{cm}} \\ \Rightarrow v_{t^*} &\equiv \omega_{t^*}r = \mu mgr^2t^*/I_{zz}^{\text{cm}}\end{aligned}$$

Substituting this expression for  $v_{t^*}$  in eqn. (8.50), we get

$$\begin{aligned}-\mu gt^* &= \mu mgr^2t^*/I_{zz}^{\text{cm}} - v_0 \\ \Rightarrow t^* &= \frac{v_0}{\mu g(1 + \frac{mr^2}{I_{zz}^{\text{cm}}})}\end{aligned}$$

which is, of course, the same expression we obtained for  $t^*$  in Sample 8.26. Again, noting that  $I_{zz}^{\text{cm}} = \frac{2}{5}mr^2$  for a sphere, we calculate the time of transition as

$$t^* = \frac{2v_0}{7\mu g} = \frac{2 \cdot (5\text{ m/s})}{7 \cdot (0.2) \cdot (9.8\text{ m/s}^2)} = 0.73\text{ s.}$$

$t^* = 0.73\text{ s}$

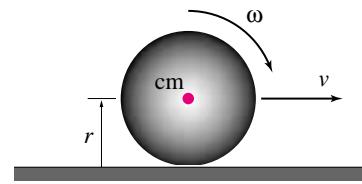


Figure 8.69: (Filename:sfig9.rollandslide.ball2)

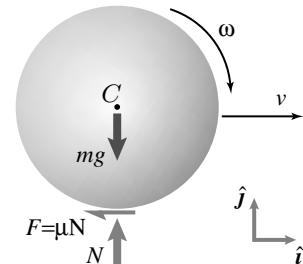


Figure 8.70: Free body diagram of the ball during sliding.  
(Filename:sfig9.rollandslide.ball2a)

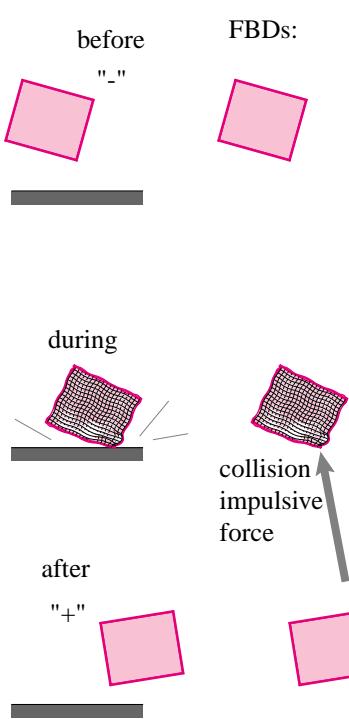


Figure 8.71: Just before a collision is called “-”, and just after is called “+”. The only forces that show on a collisional free body diagram are those that are large and part of the impact. Either a force or an impulse may be shown. This figure exaggerates the difference between the before (-) and after (+) states. In analysis we assume that there is no change in the body’s position or orientation from just before to just after the collision. The only net changes caused by the collision are the body’s velocity and rotation rate.

(Filename:figure.collisiongeneral)

① Nominally means “in name”. That is, what one calls “contacting points” are not points at all, but regions of complex interaction.

## 8.5 Collisions

Sometimes when things interact, they do so in a sudden manner. The extreme cases of collisions are dramatic, like car and plane crashes. More esoteric ‘sudden’ interactions include those between subatomic particles in an accelerator and near passes of satellites with planets. The possible collisions of consumer products with floors is of obvious interest to a design engineer. But many collisions are simply part of the way things work. The collisions of racquets, bats, clubs, sticks, hands and legs with balls, pucks and bodies is a key part of some sports. The clicking of the ratchet in the winding mechanism of an old mechanical clock involves a collision, as does the click of a camera shutter, and the flip of an electric light switch.

When things touch suddenly the contact forces can be much larger than during more smooth motions. But the collision time is short, so not much displacement occurs while these forces are applied. Thus the elementary analysis of rigid body collisions is based on these ideas:

- I. Collision forces are big, so non-collisional forces are neglected in collisional free body diagrams.
- II. Collision forces are of short duration, so the position and orientation of the colliding bodies do not change during the collision.

### What happens during a collision

During a collision between what would generally be called “rigid” bodies *all hell breaks loose*. There are huge contact forces and stresses in the regions near the nominally<sup>①</sup> contacting points, there could be plastic deformation, fracture, and frictional slip. Elastic waves may travel all over the body, reflect and scatter this way and that. Altogether the contact interaction during the collision is the result of very complex deformations in the contacting bodies (see Fig. 8.71).

It is the deformations (the lack of rigidity) that give rise to the forces between colliding bodies. So what is meant by “rigid body collisions”? The phrase is an oxymoron. Trying to understand the collision forces in detail, and how they are related to deformations, is way beyond this book. Actually, there is no unified theory of collisions so no book could write about it. Loosely one might imagine that during part of the collision material is being squeezed, this is called the *compression* phase and later on it expands back in a *restitution* phase. But the realities of collisions are not necessarily so simple; the forces and deformations can vary in complex ways.

Soon after the collision, however, the vibrations often die out, the body may have negligible permanent change in shape, and the body returns to motions that are well described by rigid-body kinematics. To find out the net effect of the collision forces we use this one key idea:

- III. The laws of mechanics apply *during* collisions even though rigid body kinematics does not.

While the motions during a collision may be wildly complex, the general linear and angular momentum balance laws are still applicable. Rather than applying these laws to understand the details *during* a collision, we use them to summarize the overall result of the collision.

That is, in rigid body collision analysis we do not pay attention to how the forces vary in time, or to the detailed trajectories, velocities or accelerations of any material points. Rather, we focus on the net change in the velocities of the colliding bodies that the collision forces cause. Thus, instead of using the differential equation form of the linear momentum balance, angular momentum balance and energy equations (Ia, IIa, and IIIa from the inside front cover) we use the time integrated forms (Ib, IIb, and IIIb). All that we note about a collisional force is its net impulse

$$\vec{P}_{\text{coll}} = \int_{\text{collision time}} \vec{F}_{\text{coll}} dt$$

in terms of which we have, for one body experiencing this impulse at point C

$$\vec{P}_{\text{coll}} = \Delta \vec{L}, \quad (8.51)$$

$$\vec{r}_{C/0} \times \vec{P}_{\text{coll}} = \Delta \vec{H}_O, \quad \text{and} \quad (8.52)$$

$$\text{Collisional dissipation} = \Delta E_K. \quad (8.53)$$

Most often the first two of these, the impulse-momentum equations are used to find the motion after collision. The energy equation is just a check to make sure that the collisional dissipation is positive (otherwise the collision would be an energy source).

## Extra assumptions are needed

The momentum balance equations, with the assumptions already discussed, are never enough in themselves to determine the outcome of a collision. The extra assumptions come in various forms. To minimize the algebra we discuss the issues first with one-dimensional collisions.

## One dimensional collisions

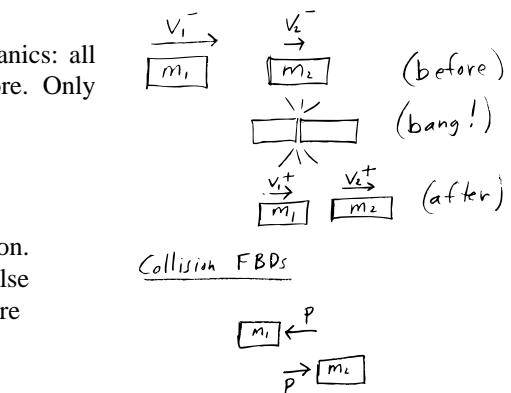
We start by considering collisions in the context of one-dimensional mechanics: all motion is constrained to one direction of motion by forces which we ignore. Only momentum and forces in, say, the  $\hat{i}$  direction are included.

### Example: 1-D collisions

Consider two masses which collide along their common line of motion. All velocities and momenta are positive if to the right and  $P$  is the impulse on mass 2 from mass 1. The relevant impulse-momentum relations are

$$\begin{aligned} \text{For mass 1} \quad -P &= m_1(v_1^+ - v_1^-), \\ \text{For mass 2} \quad P &= m_2(v_2^+ - v_2^-), \text{ and} \\ \text{For the system} \quad 0 &= (m_1v_1^+ + m_2v_2^+) - (m_1v_1^-) + m_2v_2^-. \end{aligned}$$

The third equation comes from a free body diagram of the system (ie, conservation of momentum) or by adding the first two equations. In any case, given the masses and initial velocities we have only two independent equations and we have three unknowns:  $v_1^+$ ,  $v_2^+$  and  $P$ . Momentum balance is not enough to determine the outcome of a collision.



To “close” (make solvable) the set of equations one needs to make extra assumptions.

Figure 8.72: Before the collisions the masses have velocities to the right of  $v_1^-$  and  $v_2^-$ . After the collision the velocities are  $v_1^+$  and  $v_2^+$ . During the collision  $P$  acts to the right on mass 2 and to the left on mass 1.  
(Filename:tfigure.1Dcollisions)

### Sticking collisions

The simplest assumption is that the masses *stick* together after the collision so

$$v_1^+ = v_2^+.$$

Such a collision is sometimes called a *perfectly plastic*, a *perfectly inelastic*, or a *dead* collision. Algebraic manipulations of the momentum equations and the “sticking” constitutive law give

$$\begin{aligned} v_1^+ v_2^+ &= = (m_1 v_1^- + m_2 v_2^-) / m_{\text{tot}} \quad (\text{where } m_{\text{tot}} = m_1 + m_2), \\ P &= (v_1^- - v_2^-) m_{\text{coll}}, \text{ and} \quad (\text{where } m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2}). \end{aligned}$$

The *collisional mass* or *contact mass*  $m_{\text{coll}}$  is not the mass of anything. It is just a quantity that shows up repeatedly in collision calculations and theory. It is the reciprocal of the sum of the reciprocals of the two masses. If one mass is much bigger than the other, the contact mass is  $m_{\text{coll}} \approx$  the smaller of the two masses.

### More general 1-D collisions

The momentum equations can be re-arranged to better get at the essence of the situation which is that

- In the collision the system’s center of mass velocity is unchanged, and
- The effect of the collision is to change the difference between the two mass velocities.

So we define the center of mass velocity  $v_{\text{cm}}$  and the velocity difference  $\Delta v$  as

$$v_{\text{cm}} \equiv (m_1 v_1 + m_2 v_2) / m_{\text{tot}} \quad \text{and} \quad \Delta v \equiv v_2 - v_1.$$

Note that before a collision the masses are approaching each other so  $v_1^- > v_2^-$  and  $\Delta v^- < 0$ . A little more algebra shows that for any  $P$ ,

$$\begin{aligned} v_2^+ &= v_{\text{cm}} + \frac{m_1}{m_1 + m_2} \Delta v^+, \\ v_1^+ &= v_{\text{cm}} - \frac{m_2}{m_1 + m_2} \Delta v^+, \quad \text{and} \\ P &= (\Delta v^+ - \Delta v^-) m_{\text{coll}} \end{aligned}$$

That is,  $P$  acts on  $\Delta v$  as if  $\Delta v$  were the velocity of an object with mass  $m_{\text{coll}}$ . If  $P = 0$  the equations above are a long winded way of saying that nothing happened,  $v_1^+ = v_1^-$  and  $v_2^+ = v_2^-$ , and the masses pass right through each other.

If  $P = -\Delta v^- m_{\text{coll}}$  there is a sticking collision.

### Elastic collisions

Application of the above formulas will show that if

$$P = -2\Delta v^- m_{\text{coll}}$$

then the kinetic energy of the system after the collision is the same as the kinetic energy before. That is

$$\frac{E_K^+}{2} = \frac{E_K^-}{2} \quad \frac{m_1 v_1^{+2} + m_2 v_2^{+2}}{2} = \frac{m_1 v_1^{-2} + m_2 v_2^{-2}}{2}.$$

Also,  $\Delta v^+ = -\Delta v^-$ , the relative velocity maintains its magnitude and reverses its sign.

### The coefficient of restitution

We have that as  $P$  ranges from  $-\Delta v^- m_{\text{coll}}$  to  $-2\Delta v^- m_{\text{coll}}$ , the collision ranges from sticking to an energy conserving reversal of relative velocities. The *coefficient of restitution*  $e$  is introduced as a way of interpolating between these cases. It can be defined by either of the following two equations

$$\begin{aligned}\Delta v^+ &= -e\Delta v^- \quad \text{or} \\ P &= -(1+e)\Delta v^- m_{\text{coll}}\end{aligned}$$

If  $e = 0$  we have a sticking collision. If  $e = 1$  we have an energy conserving elastic collision. If  $e$  is between 0 and 1 the collision is variously peppy or dead.

Somewhat of a miracle is that a given pair of objects seems to have a coefficient of restitution that is roughly independent of the velocities. This is the result of a conspiracy by all kinds of deformation mechanisms that we don't really understand. But that  $e$  is a constant of a pair of bodies is only an approximation that has roughly the same status as the friction coefficient. That is, much lower status than the momentum balance equations.

Note that a common mistake in many books is to take  $e$  as a material property. It is not. It generally depends on the shapes and sizes of the contacting objects also (see box 8.5 on 507).

## 2D collisions

For collisions between rigid bodies with more general motions before and after the collisions we depend on the three ideas from the start of this section, namely that

- I. Collision forces are big,
- II. Collisions are quick, and
- III. The laws of mechanics apply during the collision.

There are two extra assumptions that are needed in simple analysis:

- IV. Collision forces are few. For a given rigid body there is one, or at most two non-negligible collision forces. This is the real import of idea (I) above. Because collision forces are big most other forces can be neglected.
- V. The collision force(s) act at a well defined point which does not move during the collision.

Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form.

### Example: Two bodies in space

Two bodies collide at point C. The impulse acting on body 2 is  $\vec{P} = \int \vec{F}_{\text{coll}} dt$ . If the mass and inertia properties of both bodies are known, as are the velocities and rotation rates before the collision we have the following linear and angular momentum balance equations for the two bodies:

$$\begin{aligned}\vec{P} &= m_1 (\vec{v}_{G1}^+ - \vec{v}_{G1}^-) \\ -\vec{P} &= m_2 (\vec{v}_{G2}^+ - \vec{v}_{G2}^-) \\ \vec{r}_{C/G1} \times \vec{P} &= I_{zz}^{\text{cm}1} (\omega_1^+ - \omega_1^-) \hat{k} \\ \vec{r}_{C/G2} \times -\vec{P} &= I_{zz}^{\text{cm}2} (\omega_2^+ - \omega_2^-) \hat{k}\end{aligned}\quad (8.54)$$

These make up 6 scalar equations (2 for each momentum equation, 1 for each angular momentum equation). There are 8 scalar unknowns:

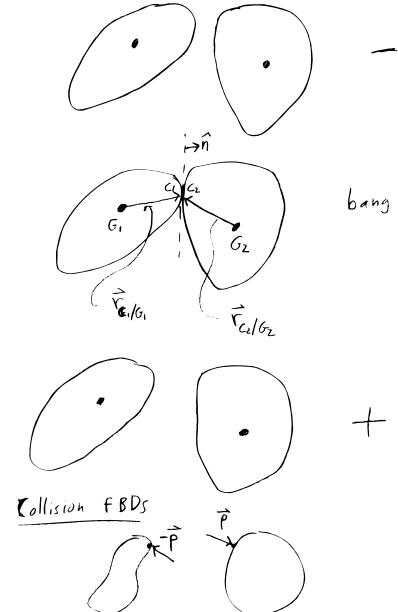


Figure 8.73: Two bodies collide at point C. The only non-negligible collision impulse is  $\vec{P}$  acting on body 2 (and  $-\vec{P}$  on body 1) at point C. The material points on the contacting bodies are  $C_1$  and  $C_2$ . The outward normal to body 1 at  $C_1$  is  $\hat{n}$ .

(Filename:tfigure.2Dcollision)

$\vec{v}_{G1}^+$  (2),  $\vec{v}_{G2}^+$  (2),  $\omega_1^+$  (1),  $\omega_2^+$  (1), and  $\vec{P}$  (2). Thus the motion after the collision cannot be determined.

[Note that we could write linear and angular momentum balance for the system, but this would only give equations which could be obtained by adding and subtracting combinations of the equations above. That is, the equations so obtained would not be independent.]  $\square$

So, as for 1-D collisions, momentum balance is not enough to determine the outcome of the collision. Eqns. 8.54 aren't enough. A thousand different models and assumptions could be added to make the system solvable. But there are only two cases that are non-controversial and also relatively simple: 1) sticking collisions, and 2) frictionless collisions.

## Sticking collisions

A ‘perfectly-plastic’ sticking collision is one where the relative velocities of the two contacting points are assumed to go suddenly to zero. That is

$$\vec{v}_{C1}^+ = \vec{v}_{C2}^+$$

Writing  $\vec{v}_{C1}^+ = \vec{v}_{G1}^+ (\omega_1^+ \hat{k}) \times \vec{r}_{C/G1}$  and similarly for  $\vec{v}_{C2}^+$  thus adds a vector equation (2 scalar equations) to the equation set 8.54. This gives 8 equations in 8 unknowns.

A little cleverness can reduce the problem to one of solving only 4 equations in 4 unknowns. Linear momentum balance for the system, angular momentum balance for the system and angular momentum balance for object 2 make up 4 scalar equations. None of these equations includes the impulse  $\vec{P}$ . Because the system moves as if hinged at  $C_1$  after the collision, the state of motion after the system is fully characterized by  $\vec{v}_{G1}^+$ ,  $\omega_1^+$ , and  $\omega_2^+$ . Thus we have 4 equations in 4 unknowns.

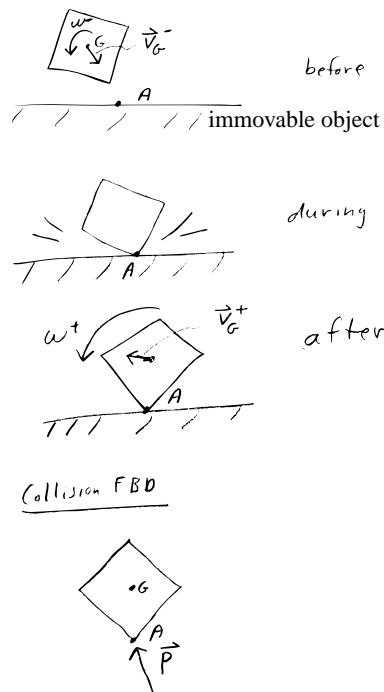


Figure 8.74: Sticking collision with an immovable object. The box sticks at A and then rotates about A. Angular momentum about point A is conserved in the collision.

(Filename:tfigure.collimmovable)

### Example: One body is hugely massive: collision with an immovable object

If body 2, say, is huge compared to body 1 then it can be taken to be immovable and collision problems can be solved by only considering body 1 (see Fig. 8.74). In the case of a sticking collision the full state of the system after the collision is determined by  $\omega_1^+$ . This can be found from the single scalar equation obtained from angular momentum balance about the collision point.

$$\begin{aligned} \vec{H}_A^- &= \vec{H}_A^+ \\ \vec{r}_{G/A} \times m\vec{v}_G^- + I_{zz}^{cm}\omega^- \hat{k} &= \vec{r}_{G/A} \times m\vec{v}_G^+ + I_{zz}^{cm}\omega^+ \hat{k} \end{aligned}$$

Because the state of the system before the collision is assumed known (the left “-” side of the equation, and because the post-collision (+) state is a rotation about A, this equation is one scalar equation in the one unknown  $\omega^+$ . Note that  $\vec{H}_A^+$  could also be evaluated as  $\vec{H}_A^+ = \omega^+ I_{zz}^A \hat{k}$ . So one way of expressing the post-collision state is as

$$\omega^+ = \frac{(\vec{r}_{G/A} \times m\vec{v}_G^- + I_{zz}^{cm}\omega^- \hat{k}) \cdot \hat{k}}{I_{zz}^A} \quad \text{and} \quad \vec{v}_G^+ = \omega^+ \hat{k} \times \vec{r}_{G/A}.$$

Note also that the same  $\vec{r}_{G/A}$  is used on the right and left sides of the equation because only the velocity and not the position is assumed to jump during the collision.

The collision impulse  $\vec{P}$  can then be found from linear momentum balance as

$$\vec{P} = m(\vec{v}_G^+ - \vec{v}_G^-).$$

□

Sticking collisions are used as models of projectiles hitting targets, of robot and animal limbs making contact with the ground, of monkeys and acrobats grabbing hand holds, and of some particularly dead and frictional collisions between solids (such as when a car trips on a curb).

## Frictionless collisions

The second special case is that of a frictionless collision. Here we add two assumptions:

- (a) There is no friction so  $\vec{P} = P\hat{n}$ . The number of unknowns is thus reduced from 8 to 7.
- (b) There is a coefficient of (normal) restitution  $e$ .

The normal restitution coefficient is taken as a property of the colliding bodies. It is a given number with  $0 < e < 1$  with this defining equation:

$$(\vec{v}_{C2}^+ - \vec{v}_{C1}^+) \cdot \hat{n} = -e(\vec{v}_{C2}^- - \vec{v}_{C1}^-) \cdot \hat{n}.$$

This says that the normal part of the relative velocity of the contacting points reverses sign and its magnitude is attenuated by  $e$ . This adds a scalar equation to the set Eqns. 8.54 thus giving 7 scalar equations (4 momentum, 2 angular momentum, 1 restitution) for 7 unknowns (4 velocity components, 2 angular velocities and the normal impulse).

The most popular application of the frictionless collision model is for billiard or pool balls, or carrom pucks. These things have relatively small coefficients of friction.

We state without proof that a frictionless collision with  $e = 1$  conserves energy.

### Example: Pool balls

Assume one ball approaches the other with initial velocity  $\vec{v}_{G1}^+ = v\hat{i}$  and has an elastic frictionless collision with the other ball at a collision angle of  $\theta$  as shown in Fig. 8.75. Defining  $\hat{n} \equiv \cos \theta \hat{i} - \sin \theta \hat{j}$  we have that  $\vec{P} = P\hat{n}$ . To determine the outcome of the equation we have the angular momentum balance equations which trivially tell us that

$$\omega_1^+ = \omega_2^+ = 0$$

because the balls start with no spin and the frictionless collision impulses  $\vec{P} = P\hat{n}$  and  $-\vec{P} = -P\hat{n}$  have no moment about the center of mass.

Linear momentum balance for each of the balls

$$\begin{aligned} -P\hat{n} &= m\vec{v}_{G1}^+ - mv\hat{i} \\ P\hat{n} &= m\vec{v}_{G2}^+ - \vec{0} \end{aligned}$$

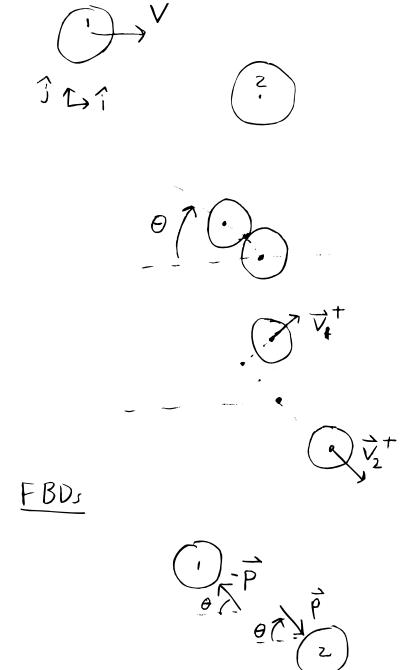


Figure 8.75: Frictionless collision between two identical round objects. Ball one is initially moving to the right, ball 2 is initially stationary. The impulse of ball 1 on ball 2 is  $\vec{P}$ .

(Filename: figure\_poolballs)

gives 4 scalar equations which are supplemented by the restitution equation (using  $e = 1$ )

$$\begin{aligned} (\Delta \vec{v}^+) \cdot \hat{n} &= -e (\Delta \vec{v}^-) \cdot \hat{n} \\ \Rightarrow -v \cos \theta &= \vec{v}_{G2}^+ \cdot \hat{n} - \vec{v}_{G1}^+ \cdot \hat{n} \end{aligned}$$

which together make 5 scalar equations in the 5 scalar unknowns  $\vec{v}_{G1}^+$ ,  $\vec{v}_{G2}^+$ , and  $P$  (each vector has 2 unknown components). These have the solution

$$\begin{aligned} \vec{v}_{G1}^+ &= v \sin \theta (\sin \theta \hat{i} + \cos \theta \hat{j}), \\ \vec{v}_{G1}^+ &= v \cos \theta (\cos \theta \hat{i} - \sin \theta \hat{j}), \quad \text{and} \\ P &= mv \cos \theta. \end{aligned}$$

The solution can be checked by plugging back into the momentum and restitution equations. Also, as promised, this  $e = 1$  solution conserves kinetic energy. The solution has the interesting property that the outgoing trajectories of the two balls are orthogonal for all  $\theta$  but  $\theta = 0$  in which case ball 1 comes to rest in the collision. [The solution can be found graphically by looking for two outgoing vectors which add to the original velocity of mass 1, where the sum of the squares of the outgoing speeds must add to the square of the incoming speed.]  $\square$

### *Frictional collisions*

If one wants to consider a collision with friction, but not so much friction that sticking is a good model, the modeling becomes complex and subtle. As of this writing there are no standard acceptable ways of dealing with such situations. Commercial simulation packages should be used for such with skeptical caution. They are generally defective in that either they can predict only a limited range of phenomena and/or they can create energy even with innocent input parameters. See the appendix on collisions for further discussion of these issues.

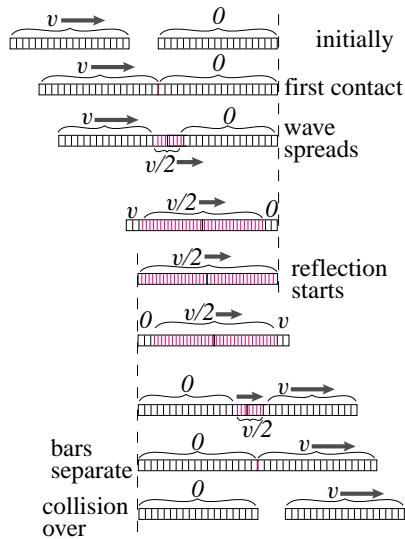
## 8.6 THEORY

### The axial collision of elastic rods

One approach to understanding collisions is to look at the stresses and deformations *during* the collision. This leads to the solution of partial differential equations. The material behavior needed to define those equations is usually not that well understood. So, hard as it is to solve such equations, even on a computer, the solution can be far from reality.

But to get a sense of things one can study an ideal system. The simple system we look at here was somewhat controversial amongst the great 19th century scientists Cauchy, Poisson and Saint-Venant (so said E.J. Routh in 1905).

**Two identical linear elastic rods.** Imagine two identical uniform linear elastic rods with length  $\ell$ . The right one is stationary and the left one approaches it with speed  $v$ .



No matter how the rods shake and vibrate, their elastic potential energy plus kinetic energy is constant.

Using reasoning beyond this book (see the paragraph for experts at the end of this box) one can explain this collision in detail, as illustrated in the sketches above. The pictures exaggerate the compression in the bar (For most materials the compression wouldn't be visible).

First the left rod moves like a rigid body towards the still rod at the right. Then contact is made and a compressional sound wave starts off spreading to the left and right. Behind the wave fronts is compressed material moving at speed  $v/2$  to the right. To the right of the right wave front the material is still. To the left of the left-moving wave front the material is still moves at  $v$ . When the wave fronts meet the ends of their respective bars, the bars are compressed and all material is going to the right at  $v/2$ . Then both wave-fronts reflect off the ends of the bars and head back towards the contact point. To the left of the right-moving wave front (on the left bar) the material is still and uncompressed. To the right of the left-moving wave front (on the right bar) the material is uncompressed but moving to the right at speed  $v$ . Finally, the waves meet in the center and the bars separate. The right bar is now uniformly moving to the right at speed  $v$  and the left bar is still.

The result of this collision is that all of the momentum of the left bar is transferred to the right bar. The separation velocity is equal in magnitude to the approach velocity. The coefficient of restitution  $e$  is 1, and the kinetic energy of the system is the same after the collision as it was before.

Note that the collision itself was quick. The wave-fronts move at the speed of sound, about 1000 m/s for metals. So for 1 me-

ter metal rods the collision takes a few thousandths of a second. But during that few thousandths of a second, the initial energy was partitioned into elastic strain energy and kinetic energy in different time-changing regions of the bar.

Despite all the complicated details, the elastic bars lead to the prediction of an 'elastic' collision. Maybe this is not surprising.

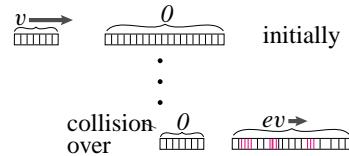
**An elastic rod hits a rigid wall** If you drop a 3 foot wooden dowel straight down on a thick concrete or stone floor it bounces quite well. Why? A wave analysis like that described above shows that a wave travelling from the first contact at the floor travels up the top and reflecting back to the bottom, leaving the rod moving uniformly up after the collisions just as fast as it was moving down before. Of course a wooden dowel is not perfectly described by the simple wave theory. And the ground is not perfectly rigid. So a real dowel's collision is not perfectly elastic.

But again we find that if we assume an elastic material that we predict an elastic collision. Again, no surprise. But

the previous two examples are completely misleading!

Actually these are maybe *the only* examples where a detailed elastic theory predicts an elastic collision. More commonly the details are more like the next example.

**Rods of different length** If the rods have length  $\ell_1$  and  $\ell_2 > \ell_1$  then the collision works out differently.



When the reflection from the left end of the left rod comes back to the contact point, the rods separate. The left rod is stationary but the right rod has waves moving up and back. The average speed of the right rod is  $(\ell_1/\ell_2)v$  so the effective coefficient of restitution is  $e = \ell_1/\ell_2 < 1$ . Later, after the vibrations have died out, the energy of the system will be less than initially. Or, even if the waves don't die out, the kinetic energy that can be accounted for in rigid-body mechanics is lost to remnant vibrations. Thus a totally elastic system leads to *inelastic* collisions. It is wrong to think that the restitution constant  $e$  depends on material; it also depends on the shapes and sizes of the objects. The amount of vibrational energy left after contact is lost depends on shape and size.

•   •   •

**For experts only: the wave equation** In one-dimensional linear elasticity the displacement  $u$  to the right, of a point at location  $x$  on one or the other rod follows this partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}.$$

That is, the collision mechanics in detail is the finding of  $u(x, t)$  that solves the wave equation above with the given initial conditions (one bar is moving the other isn't) and the boundary conditions (the ends of the bars have no stresses but when they are in contact where they can have equal compressive stresses). The solution is most easily found by constructing right and left going waves that add to meet the initial conditions and boundary conditions (Routh).

**SAMPLE 8.28** The vector equation  $m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}'_1 + m_2 \vec{v}'_2$  expresses the conservation of linear momentum of two masses. Suppose  $\vec{v}_1 = \vec{0}$ ,  $\vec{v}_2 = -v_0 \hat{j}$ ,  $\vec{v}_1^+ = v_1^+ \hat{i}$  and  $\vec{v}_2^+ = v_{2_t}^+ \hat{e}_t + v_{2_n}^+ \hat{e}_n$ , where  $\hat{e}_t = \cos \theta \hat{i} + \sin \theta \hat{j}$  and  $\hat{e}_n = -\sin \theta \hat{i} + \cos \theta \hat{j}$ .

- Obtain two independent scalar equations from the momentum equation corresponding to projections in the  $\hat{e}_n$  and  $\hat{e}_t$  directions.
- Assume that you are given another equation  $v'_{2_t} = -v_0 \sin \theta$ . Set up a matrix equation to solve for  $v_1^+$ ,  $v_{2_t}^+$ , and  $v_{2_n}^+$  from the three equations.

### Solution

- The given equation of conservation of linear momentum is

$$\begin{aligned} m_1 \underbrace{\vec{v}_1}_{\vec{0}} + m_2 \vec{v}_2 &= m_1 \vec{v}_1^+ + m_2 \vec{v}_2^+ \\ \text{or } -m_2 v_0 \hat{j} &= m_1 v_1^+ \hat{i} + m_2 (v_{2_t}^+ \hat{e}_t + v_{2_n}^+ \hat{e}_n) \end{aligned} \quad (8.55)$$

Dotting both sides of eqn. (8.55) with  $\hat{e}_n$  gives

$$\begin{aligned} -m_2 v_0 (\underbrace{\hat{e}_n \cdot \hat{j}}_{\cos \theta}) &= m_1 v_1^+ (\underbrace{\hat{e}_n \cdot \hat{i}}_{-\sin \theta}) + m_2 v_{2_t}^+ (\underbrace{\hat{e}_n \cdot \hat{e}_t}_{0}) + m_2 v_{2_n}^+ (\underbrace{\hat{e}_n \cdot \hat{e}_n}_{1}) \\ \text{or } -m_2 v_0 \cos \theta &= -m_1 v_1^+ \sin \theta + m_2 v_{2_n}^+. \end{aligned} \quad (8.56)$$

Dotting both sides of eqn. (8.55) with  $\hat{e}_t$  gives

$$\begin{aligned} -m_2 v_0 (\underbrace{\hat{e}_t \cdot \hat{j}}_{\sin \theta}) &= m_1 v_1^+ (\underbrace{\hat{e}_t \cdot \hat{i}}_{\cos \theta}) + m_2 v_{2_t}^+ (\underbrace{\hat{e}_t \cdot \hat{e}_t}_{1}) + m_2 v_{2_n}^+ (\underbrace{\hat{e}_t \cdot \hat{e}_n}_{0}) \\ \text{or } -m_2 v_0 \sin \theta &= m_1 v_1^+ \cos \theta + m_2 v_{2_t}^+. \end{aligned} \quad (8.57)$$

$$\boxed{-m_2 v_0 \cos \theta = -m_1 v_1^+ \sin \theta + m_2 v_{2_n}^+, \quad -m_2 v_0 \sin \theta = m_1 v_1^+ \cos \theta + m_2 v_{2_t}^+}$$

- Now, we rearrange eqn. (8.56) and 8.57 along with the third given equation,  $v'_{2_t} = -v_0 \sin \theta$ , so that all unknowns are on the left hand side and the known quantities are on the right hand side of the equal sign. These equations, in matrix form, are as follows.

$$\left[ \begin{array}{ccc} -m_1 \sin \theta & 0 & m_2 \\ -m_1 \cos \theta & m_2 & 0 \\ 0 & 1 & 0 \end{array} \right] \left\{ \begin{array}{c} v_1^+ \\ v_{2_t}^+ \\ v_{2_n}^+ \end{array} \right\} = \left\{ \begin{array}{c} -m_2 v_0 \cos \theta \\ -m_2 v_0 \sin \theta \\ -v_0 \sin \theta \end{array} \right\}$$

This equation can be easily solved on a computer for the unknowns.

**SAMPLE 8.29** *Cueing a billiard ball.* A billiard ball is cued by striking it horizontally at a distance  $d = 10 \text{ mm}$  above the center of the ball. The ball has mass  $m = 0.2 \text{ kg}$  and radius  $r = 30 \text{ mm}$ . Immediately after the strike, the center of mass of the ball moves with linear speed  $v = 1 \text{ m/s}$ . Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

**Solution** Let the force imparted during the strike be  $F$ . Since the ball is cued by giving a blow with the cue,  $F$  is an impulsive force. Impulsive forces, such as  $F$ , are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces ( $mg$ ,  $N$ ,  $f$ ) acting on the ball from its free body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{i} = \dot{\vec{L}} \quad \text{or} \quad (F\hat{i})dt = d\vec{L} \quad \Rightarrow \quad \int (F\hat{i})dt = \vec{L}_2 - \vec{L}_1$$

where  $\vec{L}_2 - \vec{L}_1 = \Delta\vec{L}$  is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike,  $\vec{L}_1 = m \underbrace{\vec{v}}_0 = \vec{0}$ . Immediately after the strike,  $\vec{v} = v\hat{i} = 1 \text{ m/s}$ .

$$\text{Thus } \vec{L}_2 = m\vec{v} = 0.2 \text{ kg} \cdot 1 \text{ m/s} \hat{i} = 0.2 \text{ N} \cdot \text{s} \hat{i}.$$

$$\text{Hence } \int (F\hat{i})dt = 0.2 \text{ N} \cdot \text{s} \hat{i} \quad \text{or} \quad \int Fdt = 0.2 \text{ N} \cdot \text{s}. \quad (8.58)$$

To find the angular speed we apply the angular momentum balance. Let  $\omega$  be the angular speed immediately after the strike and  $\vec{\omega} = \omega\hat{k}$ . Now,

$$\sum \vec{M}_{\text{cm}} = \dot{\vec{H}}_{\text{cm}} \quad \Rightarrow \quad \int \sum \vec{M}_{\text{cm}} dt = \int d\vec{H}_{\text{cm}} = (\vec{H}_{\text{cm}})_2 - (\vec{H}_{\text{cm}})_1.$$

Since  $\vec{H}_{\text{cm}} = I_{\text{cm}}^{zz}\vec{\omega}$  and just before the strike,  $\vec{\omega} = \vec{0}$ ,

$(\vec{H}_{\text{cm}})_1 \equiv \text{angular momentum just before the strike} = \vec{0}$

$(\vec{H}_{\text{cm}})_2 \equiv \text{angular momentum just after the strike} = I_{\text{cm}}^{zz}\omega\hat{k}$ ,

$$\int \sum \vec{M}_{\text{cm}} dt = I_{\text{cm}}^{zz}\omega\hat{k} = \frac{2}{5}mr^2\omega\hat{k} \quad (\text{since for a sphere, } I_{\text{cm}}^{zz} = \frac{2}{5}mr^2).$$

$$\text{But } \sum \vec{M}_{\text{cm}} = -Fd\hat{k},$$

$$\text{therefore } - \int (Fd)dt\hat{k} = \frac{2}{5}mr^2\omega\hat{k}$$

$$\text{or } - \underbrace{d}_{\text{constant}} \int Fdt = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2} \int Fdt.$$

Substituting the given values and  $\int Fdt = 0.2 \text{ N} \cdot \text{s}$  from equation 8.58 we get

$$\omega = -\frac{5(0.01 \text{ m})}{2 \cdot 0.2 \text{ kg} \cdot (0.03 \text{ m})^2} \cdot 0.2 \text{ N} \cdot \text{s} = -27.78 \text{ rad/s}$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that  $\omega$  was anticlockwise.

$$\boxed{\omega = -27.78 \text{ rad/s.}}$$

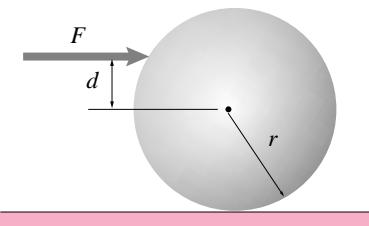


Figure 8.76: (filename:sfig7.3.DH1)

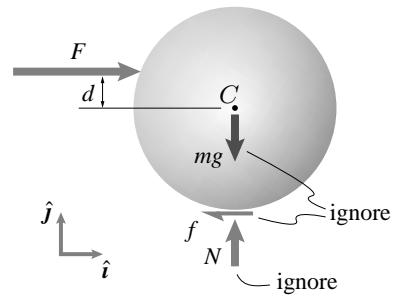


Figure 8.77: FBD of the ball during the strike. The nonimpulsive forces  $mg$ ,  $N$ , and  $f$  can be ignored in comparison to the strike force  $F$ .

(filename:sfig7.3.DH2)

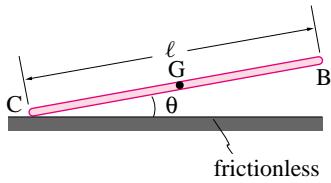


Figure 8.78: (Filename:sfig9.5.fallingbar)

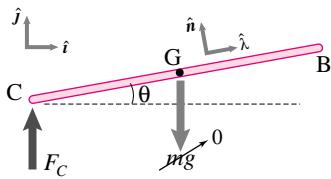


Figure 8.79: The free body diagram of the bar during collision. The impulsive force at the point of impact C is so large that the force of gravity can be completely ignored in comparison.

(Filename:sfig9.5.fallingbar.a)

Since C is a fixed point for the motion of the bar after impact, we could calculate  $\vec{H}_C^+$  as follows.

$$\vec{H}_C^+ = I_{zz}^C \vec{\omega} = \underbrace{\frac{1}{3} m \ell^2}_{I_{zz}^C} \omega(-\hat{k}).$$

**SAMPLE 8.30** *Falling stick.* A uniform bar of length  $\ell$  and mass  $m$  falls on the ground at an angle  $\theta$  as shown in the figure. Just before impact at point C, the entire bar has the same velocity  $v$  directed vertically downwards. Assume that the collision at C is plastic, *i.e.*, end C of the bar gets stuck to the ground upon impact.

- (a) Find the angular velocity of the bar just after impact.
- (b) Assuming  $\theta$  to be small, find the velocity of end B of the bar just after impact.

**Solution** We are given that the impact at point C is plastic. That is, end C of the bar has zero velocity after impact. Thus end C gets stuck to the ground. Then we expect the rod to rotate about point C as rest of the bar moves (perhaps faster) to touch the ground. The free body diagram of the bar is shown in Fig. 8.79 during the impact at point C. Note that we can ignore the force of gravity in comparison to the large impulsive force  $F_C$  due to impact at C.

- (a) Now, if we carry out angular momentum balance about point C, there will be no net moment acting on the bar, and therefore, angular momentum about the impact point C is conserved. Distinguishing the kinematic quantities before and after impact with superscripts ‘-’ and ‘+’, respectively, we get from the conservation of angular momentum about point C,

$$\begin{aligned} \vec{H}_C^- &= \vec{H}_C^+ \\ I_{zz}^{\text{cm}} \vec{\omega}^- + \vec{r}_{G/C} \times m \vec{v}_G^- &= I_{zz}^{\text{cm}} \vec{\omega}^+ + \vec{r}_{G/C} \times m \vec{v}_G^+ \end{aligned}$$

Now, we know that  $\vec{\omega}^- = \vec{0}$  since every point on the bar has the same vertical velocity  $\vec{v} = -v \hat{j}$ , and that just after impact,  $\vec{v}_G^+ = \vec{\omega}^+ \times \vec{r}_{G/C}$  where we can take  $\vec{\omega}^+ = \omega(-\hat{k})$ . Thus,

$$\begin{aligned} \vec{H}_C^- &= \vec{r}_{G/C} \times m \vec{v}_G^- = (\ell/2) \hat{\lambda} \times mv(-\hat{j}) \\ &= -\frac{mv\ell}{2} \cos \theta \hat{k} \quad (\text{since } \hat{\lambda} = \cos \theta \hat{i} + \sin \theta \hat{j}) \\ \vec{H}_C^+ &= I_{zz}^{\text{cm}} \vec{\omega}^+ + \vec{r}_{G/C} \times m(\vec{\omega}^+ \times \vec{r}_{G/C}) \\ &= -I_{zz}^{\text{cm}} \omega \hat{k} + (\ell/2) \hat{\lambda} \times m \underbrace{(-\omega \hat{k} \times \ell/2) \hat{\lambda}}_{\omega \ell/2(-\hat{n})} \\ &= -\frac{1}{12} m \ell^2 \omega \hat{k} - \frac{1}{4} m \ell^2 \omega \hat{k} = -\frac{1}{3} m \ell^2 \omega \hat{k}. \end{aligned}$$

Now, equating  $\vec{H}_C^-$  and  $\vec{H}_C^+$  we get

$$\omega = \frac{3v}{2\ell} \cos \theta, \quad \Rightarrow \quad \vec{\omega} = -\frac{3v}{2\ell} \cos \theta \hat{k}.$$

$$\boxed{\vec{\omega} = -\frac{3v}{2\ell} \cos \theta \hat{k}}$$

- (b) The velocity of the end B is now easily found using  $\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} = \vec{v}_{B/C}$  and  $\vec{v}_{B/C} = \vec{\omega} \times \vec{r}_{B/C}$ . Thus,

$$\begin{aligned} \vec{v}_{B/C} &= \vec{\omega} \times \vec{r}_{B/C} = -\omega \hat{k} \times \ell \hat{\lambda} \\ &= -\omega \ell \hat{n} = -\frac{3v}{2} \cos \theta (-\sin \theta \hat{i} + \cos \theta \hat{j}) \end{aligned}$$

but, for small  $\theta$ ,  $\cos \theta \approx 1$ , and  $\sin \theta \approx 0$ . Therefore,

$$\vec{v}_{B/C} = -\frac{3v}{2} \hat{j}.$$

Thus, end B of the bar speeds up by one and a half times its original speed due to the plastic impact at C.

$$\boxed{\vec{v}_{B/C} = -(3/2)v \hat{j}}$$

**SAMPLE 8.31** *The tipping box.* A box of mass  $m = 20 \text{ kg}$  and dimensions  $2a = 1 \text{ m}$  and  $2b = 0.4 \text{ m}$  moves along a horizontal surface with uniform speed  $v = 1 \text{ m/s}$ . Suddenly, it bumps into an obstacle at A. Assume that the impact is plastic and point A is at the lowest level of the box. Determine if the box can tip following the impact. If not, what is the maximum  $v$  the box can have so that it does not tip after the impact.

**Solution** Whether the box can tip or not depends on whether it gets sufficient initial angular speed just after collision to overcome the restoring moment due to gravity about the point of rotation A. So, first we need to find the angular velocity of the box immediately following the collision. The free body diagram of the box during collision is shown in Fig. 8.81. There is an impulse  $\vec{P}$  acting at the point of impact. If we carry out the angular momentum balance about point A, we see that the impulse at A produces no moment impulse about A, and therefore, the angular momentum about point A has to be conserved. That is,  $\vec{H}_A^+ = \vec{H}_A^-$ . Now,

$$\vec{H}_A^- = \vec{r}_{G/A} \times m\vec{v}_G^- = (-b\hat{i} + a\hat{j}) \times mv\hat{i} = -mav\hat{k}$$

Let the box have angular velocity  $\vec{\omega}^+ = \omega\hat{k}$  just after impact. Then,

$$\begin{aligned}\vec{H}_A^+ &= I_{zz}^{\text{cm}}\vec{\omega}^+ + \vec{r}_{G/A} \times m\vec{v}_G^+ = I_{zz}^{\text{cm}}\omega\hat{k} + r\hat{\lambda} \times m(\omega\hat{k} \times r\hat{\lambda}) \\ &= I_{zz}^{\text{cm}}\omega\hat{k} + mr^2\omega\hat{k} = \frac{1}{12}(4a^2 + 4b^2)m\omega\hat{k} + m(a^2 + b^2)\omega\hat{k} \\ &= \frac{4}{3}(a^2 + b^2)m\omega\hat{k}.\end{aligned}$$

Now equating the two momenta, we get

$$\omega = -\frac{3a}{4(a^2 + b^2)}v \Rightarrow \vec{\omega}^+ = -\frac{3a}{4(a^2 + b^2)}v\hat{k}.$$

Thus we know the angular velocity immediately after impact. Now let us find out if it is enough to get over the hill, so to speak. We need to find the equation of motion of the box for the motion that follows the impact. Once the impact is over (in a few milliseconds), the usual forces show up on the free body diagram (see Fig. 8.82).

We can find the equation of subsequent motion by carrying out angular momentum balance about point A (the box rotates about this point),  $\sum \vec{M}_A = \vec{H}_A$ .

$$\begin{aligned}\vec{r}_{G/A} \times mg(-\hat{j}) &= I_{zz}^A\dot{\omega}\hat{k} \\ \Rightarrow \dot{\omega} &= \frac{mgb}{I_{zz}^A} = \frac{3gb}{4(a^2 + b^2)}\end{aligned}$$

Thus the angular acceleration (due to the restoring moment of the weight of the box) is counterclockwise and constant. Therefore, we can use  $\omega^2 = \omega_0^2 + 2\dot{\omega}\Delta\theta$  to find if the box can make it to the tipping position (the center of mass on the vertical line through A). Let us take  $\theta$  to be positive in the clockwise direction (direction of tipping). Then  $\dot{\omega}$  is negative. Starting from the position of impact, the box must rotate by  $\Delta\theta = \tan^{-1}(b/a)$  in order to tip over. In this position, we must have  $\omega \geq 0$ .

$$\omega^2 = \omega_0^2 - 2\dot{\omega}\Delta\theta \geq 0 \Rightarrow \omega_0^2 \geq 2\dot{\omega}\Delta\theta \Rightarrow v^2 \geq \frac{24bg(a^2 + b^2)}{9a^2}\Delta\theta$$

Substituting the given numerical values for  $a$ ,  $b$ , and  $g = 9.8 \text{ m/s}^2$ , we get

$$v \geq 1.52 \text{ m/s}^2.$$

Thus the given initial speed of the box,  $v = 1 \text{ m/s}$ , is not enough for tipping over.

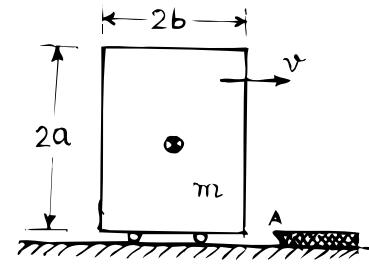


Figure 8.80: (Filename:sfig9.5.tipping)

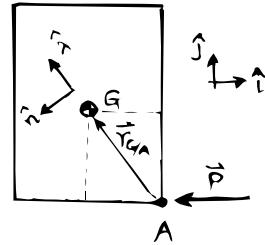


Figure 8.81: The free body diagram of the box during collision.

(Filename:sfig9.5.tipping.a)

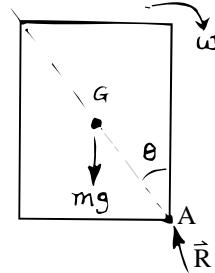


Figure 8.82: The free body diagram of the box just after the collision is over.

(Filename:sfig9.5.tipping.b)

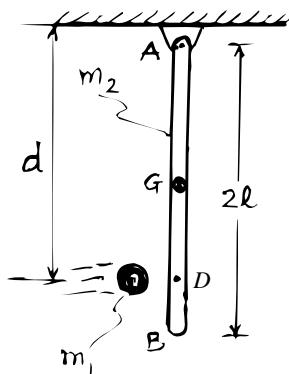


Figure 8.83: (Filename:sfig9.5.ballbar)

**SAMPLE 8.32 Ball hits the bat.** A uniform bar of mass  $m_2 = 1 \text{ kg}$  and length  $2\ell = 1 \text{ m}$  hangs vertically from a hinge at A. A ball of mass  $m_1 = 0.25 \text{ kg}$  comes and hits the bar horizontally at point D with speed  $v = 5 \text{ m/s}$ . The point of impact D is located at  $d = 0.75 \text{ m}$  from the hinge point A. Assume that the collision between the ball and the bar is plastic.

- Find the velocity of point D on the bar immediately after impact.
- Find the impulse on the bar at D due to the impact.
- Find and plot the impulsive reaction at the hinge point A as a function of  $d$ , the distance of the point of impact from the hinge point. What is the value of  $d$  which makes the impulse at A to be zero?

**Solution** The free body diagram of the ball and the bar as a single system is shown in Fig. 8.84 during impact. There is only one external impulsive force  $\vec{F}_A$  acting at the hinge point A. We take the ball and the bar together here so that the impulsive force acting between the ball and the bar becomes internal to the system and we are left with only one external force at A. Then, the angular momentum balance about point A yields  $\dot{\vec{H}}_A = \vec{0}$  since there is no net moment about A. Thus the angular momentum about A is conserved during the impact.

- Let us distinguish the kinematic quantities just before impact and immediately after impact with superscripts ‘-’ and ‘+’, respectively. Then, from the conservation of angular momentum about point A, we get  $\vec{H}_A^- = \vec{H}_A^+$ . Now,

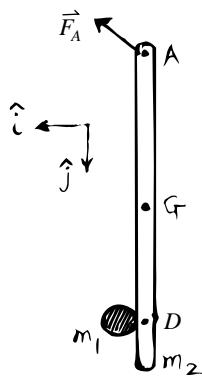


Figure 8.84: The free body diagram of the ball and the bar together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free body diagram.  
(Filename:sfig9.5.ballbar.a)

$$\begin{aligned}\vec{H}_A^- &= (\vec{H}_A^-)_{\text{ball}} + (\vec{H}_A^-)_{\text{bar}} \\ &= \vec{r}_{D/A} \times m_1 \vec{v}^- + I_{zz}^A \vec{\omega}^- \\ &= d \hat{i} \times m_1 v (-\hat{i}) + \vec{0} = m_1 d v \hat{k}.\end{aligned}$$

Similarly,

$$\vec{H}_A^+ = \vec{r}_{D/A} \times m_1 \vec{v}^+ + I_{zz}^A \vec{\omega}^+$$

but,  $\vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{D/A} = -\omega d \hat{i}$ , where  $\vec{\omega}^+ = \omega \hat{k}$  (let). Hence,

$$\begin{aligned}\vec{H}_A^+ &= d \hat{j} \times m_1 (-\omega d \hat{i}) + \frac{1}{3} m_2 (2\ell)^2 \omega \hat{k} \\ &= (m_1 d^2 + \frac{4}{3} m_2 \ell^2) \omega \hat{k}.\end{aligned}$$

Equating the two momenta, we get

$$\begin{aligned}\omega &= \frac{m_1 d v}{m_1 d^2 + (4/3)m_2 \ell^2} \\ &= \frac{v}{d \left( 1 + \frac{4}{3} \frac{m_2}{m_1} \left( \frac{\ell}{d} \right)^2 \right)} \\ \Rightarrow \vec{v}_D &= \vec{\omega}^+ \times \vec{r}_{D/A} = \omega d (-\hat{i}) \\ &= -\frac{v}{1 + \frac{4}{3} \frac{m_2}{m_1} \left( \frac{\ell}{d} \right)^2} \hat{i}.\end{aligned}$$

Now, substituting the given numerical values,  $v = 5 \text{ m/s}$ ,  $m_1 = 0.25 \text{ kg}$ ,  $m_2 = 1 \text{ kg}$ ,  $\ell = 0.5 \text{ m}$ , and  $d = 0.75 \text{ m}$ , we get  $\vec{v}_D = -2.08 \text{ m/s} \hat{i}$

$$\boxed{\vec{v}_D = -2.08 \text{ m/s} \hat{i}}$$

- To find the impulse at D due to the impact, we can consider either the ball or the bar separately, and find the impulse by evaluating the change in the linear

momentum of the body. Let us consider the ball since it has only one impulse acting on it. The free body diagram of the ball during impact is shown in Fig. 8.85. From the linear impulse-momentum relationship we get,

$$\begin{aligned}\vec{P}_D &= \int \vec{F}_D dt = \vec{L}^+ - \vec{L}^- = m_1(\vec{v}^+ - \vec{v}^-) \\ &= m_1 \left( -\frac{v}{1 + \frac{4}{3} \frac{m_2}{m_1} (\frac{\ell}{d})^2} \hat{i} + v \hat{i} \right) \\ &= m_1 v \left( 1 - \frac{1}{1 + \frac{4}{3} \frac{m_2}{m_1} (\frac{\ell}{d})^2} \right) \hat{i}\end{aligned}$$

Substituting the given numerical values, we get  $\vec{P}_D = 0.73 \text{ kg}\cdot\text{m}/\text{s}\hat{i}$ . The impulse on the bar is equal and opposite. Therefore, the impulse on the bar is  $-\vec{P}_D = -0.73 \text{ kg}\cdot\text{m}/\text{s}\hat{i}$ .

$$\boxed{\text{Impulse at D} = -0.73 \text{ kg}\cdot\text{m}/\text{s}\hat{i}}$$

- (c) Now that we know the impulse at D, we can easily find the impulse at A by applying impulse-momentum relationship to the bar. Since, the bar is stationary just before impact, its initial momentum is zero. Thus, for the bar,

$$\int (\vec{F}_A - \vec{F}_D) dt = \vec{L}^+ - \vec{L}^- = \vec{L}^+ = m_2 \vec{v}_{\text{cm}}^+$$

Denoting the impulse at A with  $\vec{P}_A$ , the mass ratio  $m_2/m_1$  by  $m_r$ , and the length ratio  $\ell/d$  by  $q$ , and noting that  $\vec{v}_{\text{cm}}^+ = \omega \hat{k} \times \ell \hat{j} = -\omega \ell \hat{i}$ , we get

$$\begin{aligned}\vec{P}_A &\equiv \int \vec{F}_A dt = \int \vec{F}_D dt + m_2(-\omega \ell \hat{i}) \\ &= m_1 v \left( 1 - \frac{1}{1 + \frac{4}{3} m_r q^2} \right) \hat{i} - m_2 \ell \frac{v}{d (1 + \frac{4}{3} m_r q^2)} \hat{i} \\ &= m_1 v \left( \frac{\frac{4}{3} m_r q^2}{1 + \frac{4}{3} m_r q^2} \right) \hat{i} - m_2 v \left( \frac{q}{1 + \frac{4}{3} m_r q^2} \right) \hat{i} \\ &= \frac{(4/3)m_2 q^2 - m_2 q}{1 + \frac{4}{3} m_r q^2} v \hat{i} = \frac{q(4q - 3)}{3(1 + \frac{4}{3} m_r q^2)} m_2 v \hat{i}\end{aligned}$$

Now, we are ready to graph the impulse at A as a function of  $q \equiv \ell/d$ . However, note that a better quantity to graph will be  $P_A/(m_1 v)$ , that is, the nondimensional impulse at A, normalized with respect to the initial linear momentum  $m_1 v$  of the ball. The plot is shown in Fig. ???. It is clear from the plot, as well as from the expression for  $\vec{P}_A$ , that the impulse at A is zero when  $q = 3/4$  or  $d = 4\ell/3 = 2/3(2\ell)$ , that is, when the ball strikes at two thirds the length of the bar. Note that this location of the impact point is independent of the mass ratio  $m_r$ .

$$\boxed{d = 2/3(2\ell) \text{ for } \vec{P}_A = \vec{0}}$$

**Comment:** This particular point of impact D (when  $d = 2/3(2\ell)$ ) which induces no impulse at the support point A is called the *center of percussion*. If you imagine the bar to be a bat or a racquet and point A to be the location of your grip, then hitting a ball at D gives you an impulse-free shot. In sports, point D is called a *sweet spot*.

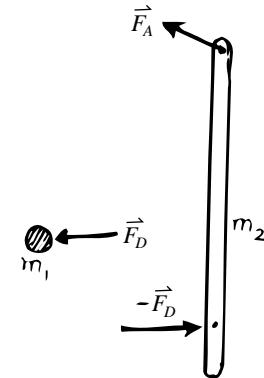


Figure 8.85: Separate free body diagrams of the ball and the bar during collision.

(Filename:sfig9.5.ballbar.b)

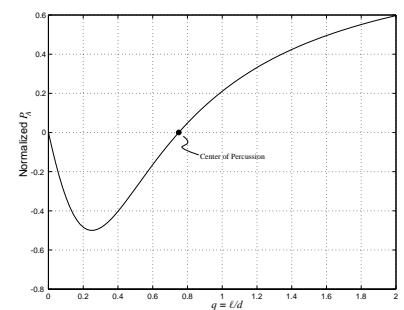


Figure 8.86: Plot of normalized impulse at A as a function of  $q = \ell/d$ .

(Filename:sweetspot)

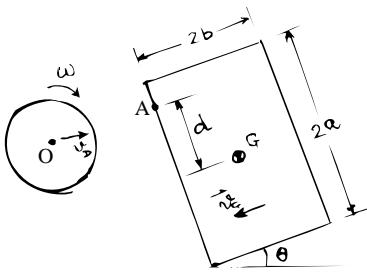


Figure 8.87: (filename:sfig9.5.diskplate)

**SAMPLE 8.33 Flying dish and the solar panel.** A uniform rectangular plate of dimensions  $2a = 2\text{ m}$  and  $2b = 1\text{ m}$  and mass  $m_P = 2\text{ kg}$  drifts in space at a uniform speed  $v_p = 10\text{ m/s}$  (in a local Newtonian reference frame) in the direction shown in the figure. Another circular disk of radius  $R = 0.25\text{ m}$  and mass  $m_D = 1\text{ kg}$  is heading towards the plate at a linear speed  $v_D = 1\text{ m/s}$  directed normal to the facing edge of the plate. In addition, the disk is spinning at  $\omega_D = 5\text{ rad/s}$  in the clockwise direction. The plate and the disk collide at point A of the plate, located at  $d = 0.8\text{ m}$  from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

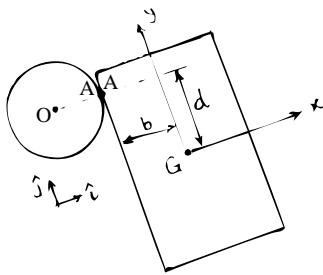


Figure 8.88: The free body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free body diagram .  
(filename:sfig9.5.diskplate.a)

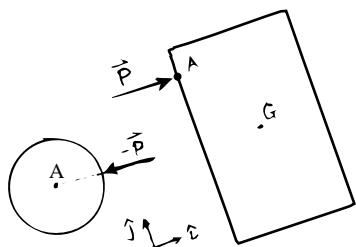


Figure 8.89: Separate free body diagrams of the disk and the plate during collision.  
(filename:sfig9.5.diskplate.b)

**Solution** To find the linear as well as the angular velocities of the disk and the plate, we will have to use linear and angular momentum-impulse relations. In total, we have 7 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components), 2 for the two angular velocities, and 1 for the collision impulse. Naturally, we need 7 independent equations. We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution  $e$  ( $=1$  for elastic collision). Thus we have enough equations. Let us set up all the required equations. We can then solve the equations using a computer.

The free body diagrams of the disk and the plate together and the two separately are shown in Fig. 8.88 and 8.89, respectively. Using an  $xy$  coordinate system oriented as shown in Fig. 8.88, we can write

$$\begin{aligned} \text{LMB for disk: } m_D(\vec{v}_D^+ - \vec{v}_D^-) &= -P\hat{i} \\ \text{LMB for plate: } m_P(\vec{v}_P^+ - \vec{v}_P^-) &= P\hat{i} \\ \text{AMB for disk: } I_D^{\text{cm}}(\vec{\omega}_D^+ - \vec{\omega}_D^-) &= \vec{0} \\ \text{AMB for plate: } I_P^{\text{cm}}(\vec{\omega}_P^+ - \vec{\omega}_P^-) &= \vec{r}_{A/G} \times P\hat{i} \\ \text{kinematics: } \hat{i} \cdot (\vec{v}_{A_D}^+ - \vec{v}_{A_P}^+) &= e(\vec{v}_{A_P}^- - \vec{v}_{A_D}^-) \end{aligned}$$

where, in the last equation  $\vec{v}_{A_D}$  and  $\vec{v}_{A_P}$  refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center of mass of the corresponding bodies. We are given that  $\vec{v}_D^- = v_D\hat{i}$ ,  $\vec{v}_P^- = -v_P\hat{i}$ ,  $\vec{\omega}_D^- = -\Omega_D\hat{k}$ , and  $\vec{\omega}_P^- = \vec{0}$ . Let us assume that  $\vec{\omega}_D^+ = \omega_D\hat{k}$ ,  $\vec{\omega}_P^+ = \omega_P\hat{k}$ ,  $v_{D_x}^+ = v_{D_x}^+\hat{i} + v_{D_y}^+\hat{j}$ , and similarly,  $v_{P_x}^+ = v_{P_x}^+\hat{i} + v_{P_y}^+\hat{j}$ . Then,

$$\begin{aligned} \vec{v}_{A_D}^- &= \vec{v}_D^- + \vec{\omega}_D^- \times \vec{r}_{A/O} = v_D\hat{i} - \omega_D R\hat{j} \\ \vec{v}_{A_D}^+ &= \vec{v}_D^+ + \vec{\omega}_D^+ \times \vec{r}_{A/O} = v_{D_x}^+\hat{i} + (v_{D_y}^+ + \omega_D R)\hat{j} \\ \vec{v}_{A_P}^- &= \vec{v}_P^- = -v_P\hat{i} \\ \vec{v}_{A_P}^+ &= \vec{v}_P^+ + \vec{\omega}_P^+ \times \vec{r}_{A/G} = (v_{P_x}^+ - \omega_P^+ d)\hat{i} + (v_{P_y}^+ - \omega_P^+ d)\hat{j} \end{aligned}$$

Substituting these quantities in the kinematics equation above and dotting with the normal direction at A,  $\hat{i}$ , we get

$$v_{D_x}^+ - v_{P_x}^+ + \omega_P^+ d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \quad (8.59)$$

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$m_D(v_{D_x}^+ - v_D) = -P \quad (8.60)$$

$$m_D v_{D_y}^+ = 0 \quad (8.61)$$

Dotting LMB for the plate with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$m_P(v_{P_x}^+ - v_P) = P \quad (8.62)$$

$$m_P v_{P_y}^+ = 0 \quad (8.63)$$

Dotting AMB for the disk and the plate with  $\hat{k}$ , we get

$$I_D^{\text{cm}}(\omega_D^+ - \omega_D) = 0 \quad (8.64)$$

$$I_P^{\text{cm}}\omega_P^+ = Pd \quad (8.65)$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (8.60)–(8.65), and then eqn. (8.59) as

$$\left[ \begin{array}{ccccccc} m_D & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_D^{\text{cm}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_P^{\text{cm}} & -d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{array} \right] \left\{ \begin{array}{l} v_{D_x}^+ \\ v_{D_y}^+ \\ v_{P_x}^+ \\ v_{P_y}^+ \\ \omega_D^+ \\ \omega_P^+ \\ P \end{array} \right\} = \left\{ \begin{array}{l} m_D v_D \\ 0 \\ m_P v_P \\ 0 \\ I_D^{\text{cm}} \omega_D \\ 0 \\ -v_P - v_D \end{array} \right\}$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia,  $I_D^{\text{cm}} = (1/2)m_D R^2$  and  $I_P^{\text{cm}} = (1/12)m_P(4a^2 + 4b^2)$ , and then solving the matrix equation on a computer, we get,

$$\begin{aligned} \vec{v}_D^+ &= 0.34 \text{ m/s}\hat{i}, \quad \vec{v}_P^+ = -9.67 \text{ m/s}\hat{i} \\ \vec{\omega}_D^+ &= -5 \text{ rad/s}\hat{k}, \quad \vec{\omega}_P^+ = -1.26 \text{ rad/s}\hat{k} \\ P &= -0.66 \text{ kg}\cdot\text{m/s} \end{aligned}$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

$$\boxed{\vec{v}_D^+ = 0.34 \text{ m/s}\hat{i}, \quad \vec{v}_P^+ = -9.67 \text{ m/s}\hat{i}, \quad \vec{\omega}_D^+ = -5 \text{ rad/s}\hat{k}, \quad \vec{\omega}_P^+ = -1.26 \text{ rad/s}\hat{k}}$$

**Comments:** In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (8.61), (8.63), and (8.64) are trivial to solve and immediately give,  $v_{D_y}^+ = 0$ ,  $v_{P_y}^+ = 0$ , and  $\omega_D^+ = \omega_D^D = 5 \text{ rad/s}$ . Rest of the equations can be solved by usual eliminations and substitutions, etc. However, it is important to learn how to set up these equations in matrix form so that no matter how complicated the equations are, they can be easily solved on a computer. What really counts is do you have 7 linear independent equations for the 7 unknowns. If you do, you are home.



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# 9

# Kinematics using time-varying base vectors

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Many parts of practical machines and structures move in ways that can be idealized as straight-line motion (Chapter 6) or circular motion (Chapters 7 and 8). But often an engineer must analyze parts with more general motions (as in chapter 9).

In principle one can study all motions of all things using one fixed coordinate system. If one knows a description of the  $x$ ,  $y$ , and  $z$  coordinates of all points at all times than one can evaluate the linear momentum, the angular momentum, and their rates of change. In this way one can do all of mechanics. But when a machine has various parts, each moving relative to the other, it turns out it is useful to make use of additional base vectors besides those fixed to a Newtonian (“fixed”) reference frame. That is, the formulas for velocity and acceleration are simplified (or clarified) by using moving base vectors, for example, base vectors that move with some of the parts.

You have seen time-varying base vectors, the polar coordinate base vectors  $\hat{e}_R$  and  $\hat{e}_\theta$  used to describe circular motion. These are the ideas on which we build here. Altogether we discuss 4 approaches that use time varying base vectors:

- I. In section 10.1 polar coordinates are extended for more general use than circular motion;
- II. Path coordinates and base vectors are also introduced in 10.1;
- III. Section 10.2 introduces general rotating base vectors and coordinate systems with an origin that moves; and
- IV. Section 10.3 shows formulas for differentiation in moving frames that don’t depend on any particular base vector choice.

Section 10.4 applies some of these ideas to the kinematics of mechanisms.

The basic idea is to try to use coordinate systems that most simply describe the motions of interest, even if these coordinate systems are somewhat confusing because they rotate and move. Most people find the ideas associated with time-varying base vectors difficult at first. Like many shortcuts, they have a cost in terms of the sophistication they demand. In the end, however, they aid intuition as well as calculation.

## 9.1 Polar coordinates and path coordinates

As you learned in Chapter 7, when a particle moves in a plane while going in circles around the origin its position velocity and acceleration can be described like this:

$$\begin{aligned}\vec{r} &= R\hat{e}_R \\ \vec{v} &= R\dot{\theta}\hat{e}_\theta = v_\theta\hat{e}_\theta \\ \vec{a} &= -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta = -\frac{v_\theta^2}{R}\hat{e}_R + \dot{v}_\theta\hat{e}_\theta\end{aligned}$$

Which we can describe in words like this. The position is the distance from the origin times a unit vector towards the point. The velocity is tangent to the circle of motion. And the acceleration has a centripetal component proportional to the speed squared, and a tangential component, tangent to the circle of motion with magnitude equal to the rate of change of speed.

We are now going to generalize these results two different ways. First we will use polar base vectors relaxing the restriction that  $R = \text{constant}$ . Next we will use path base vectors to show that, in some sense, the formulas above apply to any wild motion of a particle in 3D.

In principle these new methods are not needed. We could just use one fixed coordinate system with base vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  and write the velocity and acceleration of a point at position  $\vec{r} = x\hat{i} + y\hat{j}$  as

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \quad \text{and} \quad \vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

as was done in sections 5.7-10. But, as for circular motion, rotating base vectors are helpful for simplifying some kinematics and mechanics problems.

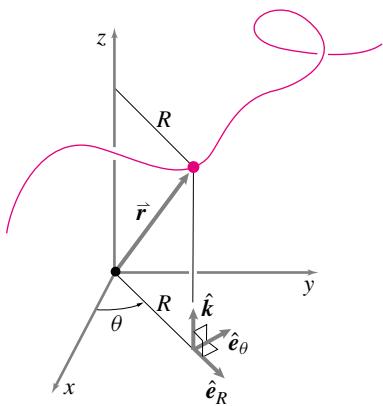


Figure 9.1: Polar coordinates.

(Filename:tfigure6.1)

### Polar coordinates

The extension of *polar coordinates* to 3 dimensions as *cylindrical coordinates* is shown in Fig. 9.1.

Rather than identifying the location of a point by its  $x$ ,  $y$  and  $z$  coordinates, a point is located by its cylindrical coordinates

$R$ , the distance to the point from the  $z$  axis,

$\theta$ , the angle that the most direct line from the  $z$  axis to the point makes with the positive  $x$  direction,

$z$ , the conventional  $z$  coordinate of the particle,

and base vectors:

$\hat{e}_R$ , a unit vector that most directly points from the  $z$  axis to the particle

(in 2-D  $\hat{e}_R = \vec{r}/r$ ,

in 3-D  $\hat{e}_R = (\vec{r} - (\vec{r} \cdot \hat{k})\hat{k})/|\vec{r} - (\vec{r} \cdot \hat{k})\hat{k}|$

= a unit vector in the direction of the shadow of  $\vec{r}$  in the  $xy$  plane)

$\hat{e}_\theta$ , a vector in the  $xy$  plane normal to  $\hat{e}_R$  (Formally  $\hat{e}_\theta = \hat{k} \times \hat{e}_R$ ),

$\hat{k}$ , the conventional  $\hat{k}$  base vector.

It is only the planar part of the position that is different with polar coordinates than with traditional Cartesian coordinates. The position vector of a particle is

$$\vec{r} = R\hat{e}_R + z\hat{k}.$$

or, if only two-dimensional problems are being considered

$$\vec{r} = R\hat{e}_R.$$

As the particle moves, the values of its coordinates  $R$ ,  $\theta$  and  $z$  change as do the base vectors  $\hat{e}_R$  and  $\hat{e}_\theta$ .

### Example: Oblong path

A particle that moves on the path  $R = A + B \cos(2\theta)$  with  $A > B$  moves on an oblong path something like that shown in Fig. 9.2 with

$$\vec{r} = R\hat{e}_R = (A + B \cos(2\theta))\hat{e}_R$$

Note that, unlike for circular motion,  $\hat{e}_\theta$  is *not* tangent to the particle's path in general. (In this example  $\hat{e}_\theta$  is only tangent to the path where the path is closest to or furthest from the origin.)  $\square$

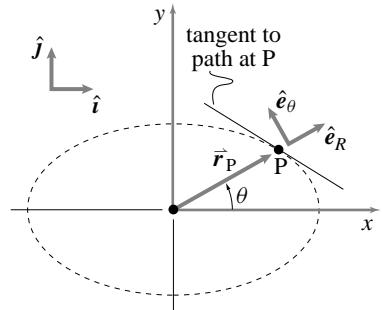


Figure 9.2: Particle  $P$  on an oblong path. Note that the velocity is not parallel to  $\hat{e}_\theta$  at most points on the path.

(Filename:tfigure6.polar.coord)

## Velocity in polar coordinates

The velocity and acceleration are found by differentiating the position  $\vec{r}$ , taking account that the base vectors  $\hat{e}_R$  and  $\hat{e}_\theta$  also change with time just as they did for circular motion:

$$\dot{\hat{e}}_R = \dot{\theta}\hat{e}_\theta \quad \text{and} \quad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R.$$

We find the velocity by taking the time derivative of the position, using the product rule of differentiation:

$$\begin{aligned} \vec{v} &= \frac{d}{dt}\vec{r} = \frac{d}{dt}[R\hat{e}_R + z\hat{k}] \\ &= \frac{d}{dt}(R\hat{e}_R) + \frac{d}{dt}(z\hat{k}) \\ &= (\dot{R}\hat{e}_R + R\underbrace{\dot{\hat{e}}_R}_{\dot{\theta}\hat{e}_\theta}) + (\dot{z}\hat{k}) \\ &= \underbrace{\dot{R}\hat{e}_R}_{v_R} + \underbrace{R\dot{\theta}\hat{e}_\theta}_{v_\theta} + \underbrace{\dot{z}\hat{k}}_{v_z}. \end{aligned} \quad (9.1)$$

This formula is intuitive: the velocity is the sum of three vectors: one due to moving towards or away from the  $z$  axis,  $\dot{R}\hat{e}_R$ , one having to do with the angle being swept,  $R\dot{\theta}\hat{e}_\theta$ , and in 3-D, one to motion perpendicular to the  $xy$  plane,  $\dot{z}\hat{k}$ . In 2-D this is shown in Fig. 9.3.

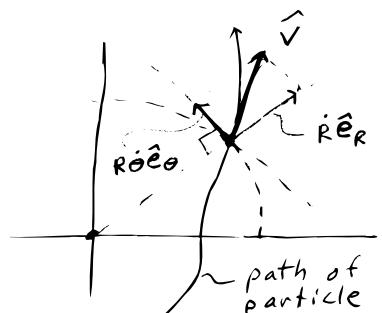


Figure 9.3: Velocity in polar coordinates for general planar motion. The velocity has a radial  $\hat{e}_R$  component  $\dot{R}$  and a circumferential  $\hat{e}_\theta$  component  $R\dot{\theta}$ .

(Filename:tfigure.polarvel)

But, as has been emphasized before, this isn't a new vector  $\vec{v}$  but just a new way of representing the same vector:

$$\begin{aligned}\vec{v} &= \vec{v} \\ v_x \hat{i} + v_y \hat{j} + v_z \hat{k} &= v_R \hat{e}_R + v_\theta \hat{e}_\theta + v_z \hat{k}.\end{aligned}$$

The vector  $\vec{v}$  can be represented in different base vector systems, in this case cartesian and polar.

Note that eqn. (9.1) adds two terms to the circular motion case from Chapter 7: one for variations in  $R$  and one for variations in  $z$ .

## Acceleration in polar coordinates

To find the acceleration, we differentiate once again. The resulting formula has new terms generated by the product rule of differentiation.

$$\begin{aligned}\vec{a} &= \frac{d}{dt} \vec{v} \\ &= \frac{d}{dt} (\dot{R} \hat{e}_R + R\dot{\theta} \hat{e}_\theta + \ddot{z} \hat{k}) \\ &= (\ddot{R} \hat{e}_R + \underbrace{\dot{R}}_{\dot{\theta} \hat{e}_\theta} \hat{e}_R) + (\dot{R}\dot{\theta} \hat{e}_\theta + R\ddot{\theta} \hat{e}_\theta + \underbrace{R\dot{\theta}}_{-\dot{\theta} \hat{e}_R} \hat{e}_\theta) + (\ddot{z} \hat{k}) \\ &= (\ddot{R} - \dot{\theta}^2 R) \hat{e}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta} \hat{e}_\theta + \underbrace{\ddot{z}}_{a_z} \hat{k}).\end{aligned}\quad (9.2)$$

The acceleration for an arbitrary planar path is shown in Fig. 9.4. Four of the five terms comprising the polar coordinate formula for acceleration are easy to understand.

$\ddot{R}$  is just the acceleration due to the distance from the origin changing with time.  
 $\dot{\theta}^2 R$  is the familiar *centripetal* acceleration.

$R\ddot{\theta}$  is the acceleration due to rotation proceeding at a faster and faster rate. And  
 $\ddot{z}$  is the same as for Cartesian coordinates.

**The Coriolis term.** The difficult term in the polar coordinate expression for the acceleration is the

$2\dot{\theta}\dot{R}$  term, called the *Coriolis* acceleration, after the civil engineer Gustave-Gaspard Coriolis who first wrote about it in 1835 (in a slightly more general context).

The presence of the '2' in this term is due to the two effects from which it derives: 1 from the change of the  $\dot{R}\hat{e}_R$  term in the velocity and 1 from the change of the  $R\dot{\theta}\hat{e}_\theta$  term in the velocity ( $1 + 1 = 2$ ).

The Coriolis acceleration occurs even if both  $\dot{\theta}$  and  $\dot{R}$  are constant, a situation which would be incorrectly characterized as 'constant velocity'. One way to understand the Coriolis term is to find a situation where the Coriolis acceleration is the only non-zero term in the general acceleration expression.

Most simply, a particle that moves on a straight line that is rotating does not have a straight-line path, and thus has some acceleration.

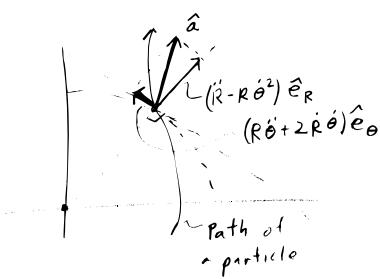


Figure 9.4: Acceleration in polar coordinates for general planar motion. The acceleration has a radial  $\hat{e}_R$  component  $\ddot{R} - R\dot{\theta}^2$  and a circumferential  $\hat{e}_\theta$  component  $R\ddot{\theta} + 2\dot{R}\dot{\theta}$ .

(Filename:tfigure.polaracc)

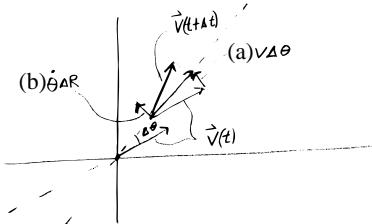


Figure 9.5: The simplest case for understanding of the *Coriolis* term in the acceleration. A particle moves at constant speed  $\dot{R}$  out on a line which is rotating at constant rate  $\dot{\theta}$ . The figure shows a velocity vector as an arrow when the particle is at the origin and a time  $\Delta t$  later. The change in the velocity vector comes from two effects: (a) the radial line has rotated, and (b) a circumferential component is added to the velocity when the particle is away from the origin. For small  $\Delta\theta$  these two changes to the velocity are both approximately perpendicular to  $\vec{v}$  and parallel to each other.

(Filename:tfigure.simpleCoriolis)

**Example: The simplest Coriolis example**

Imagine a particle moving at constant speed  $\dot{R}$  along a line which is itself rotating at constant  $\dot{\theta}$  about the origin. Let's look at the particle as it passes through the origin at time  $t$  and a small amount of time  $\Delta t$  later (see Fig. 9.5). At time  $t + \Delta t$  the direction of the scribed line has changed by an angle  $\Delta\theta = \dot{\theta}\Delta t$ . So that, even if at that later time  $\dot{\theta} = 0$  the direction of  $\vec{v}$  has changed so  $\vec{v}$  has changed by an amount  $v\Delta\theta$ . But the rotation of the line does continue, so the velocity includes a part in the  $\hat{e}_\theta$  direction with magnitude  $\dot{\theta}\Delta R$ . That is  $\vec{v}$  is changed by both  $\Delta\theta v$  and by  $\dot{\theta}\Delta R$  so

$$\begin{aligned}\Delta\vec{v} &\approx (v\Delta\theta + \dot{\theta}\Delta R)\hat{e}_\theta \\ &\approx (\dot{R}(\dot{\theta}\Delta t) + \dot{\theta}(\dot{R}\Delta t))\hat{e}_\theta \\ &\approx (\dot{R}\dot{\theta}) + \dot{\theta}\dot{R})\hat{e}_\theta\Delta t \\ \Rightarrow \frac{\Delta\vec{v}}{\Delta t} &\approx (\dot{R}\dot{\theta}) + \dot{\theta}\dot{R})\hat{e}_\theta \\ \Rightarrow \vec{a} &\approx (\dot{R}\dot{\theta}) + \dot{\theta}\dot{R})\hat{e}_\theta = 2\dot{R}\dot{\theta}\hat{e}_\theta\end{aligned}$$

as predicted by the general polar coordinate acceleration formula.  $\square$

But one need not get confounded by a desire to understand every term intuitively, eqn. (9.2) way of describing the same acceleration we have described with cartesian coordinates. Namely,

$$\begin{aligned}\vec{a} &= \vec{a} \quad \text{and} \\ \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} &= (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R}\hat{e}_R + \underbrace{(2\dot{R}\dot{\theta} + R\ddot{\theta})}_{a_\theta}\hat{e}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{k}.\end{aligned}$$

**Example:  $R(t)$  and  $\theta(t)$  are given functions.**

Say  $a$  and  $c$  are given constants and that

$$\theta = at \quad \text{and} \quad R = ct^2.$$

Then at any  $t$  the position, velocity, and acceleration are (see Fig. 9.6)

$$\begin{aligned}\vec{r} &= R\hat{e}_R = ct^2\hat{e}_R, \\ \vec{v} &= \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = 2ct\hat{e}_R + act^2\hat{e}_\theta, \quad \text{and} \\ \vec{a} &= (\ddot{R} - \dot{\theta}^2 R)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta \\ &= (2c - a^2ct^2)\hat{e}_R + (0 + 4act)\hat{e}_\theta\end{aligned}$$

with polar base vectors

$$\begin{aligned}\hat{e}_R &= \cos\theta\hat{i} + \sin\theta\hat{j} = \cos(at)\hat{i} + \sin(at)\hat{j} \quad \text{and} \\ \hat{e}_\theta &= -\sin\theta\hat{i} + \cos\theta\hat{j} = -\sin(at)\hat{i} + \cos(at)\hat{j}.\end{aligned}$$

If we substituted these expressions for the polar base vectors into the expressions for  $\vec{r}$ ,  $\vec{v}$ , and  $\vec{a}$  we would get the same cartesian representation that we would get from using  $x = R \cos\theta$  and  $y = R \sin\theta$  with  $\vec{r} = x\hat{i} + y\hat{j}$ ,  $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$ , and  $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$ . That is  $\vec{r} = \vec{r}$ ,  $\vec{v} = \vec{v}$ , and  $\vec{a} = \vec{a}$  even if the representation is different.  $\square$

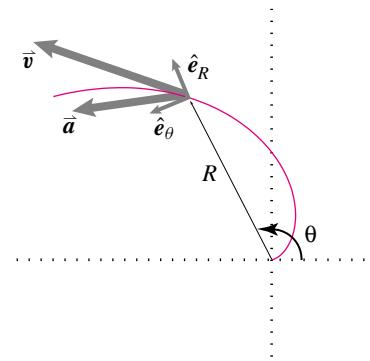


Figure 9.6: A particle moves with  $\theta = at$  and  $R = ct^2$ . In this drawing  $a = .5$  and  $c$  is anything since no scale is shown.

(Filename:tfigure.spiral)

## Path coordinates

Another way still to describe the velocity and acceleration is to use base vectors which are defined by the motion. In particular, the *path* base vectors used are:

- (a) the unit tangent to the path  $\hat{e}_t$ , and
- (b) the unit normal to the path  $\hat{e}_n$ .

Somewhat surprisingly at first glance, only two base vectors are needed to define the velocity and acceleration, even in three dimensions.

The base vectors can be described geometrically and analytically. Let's start out with a geometric description.

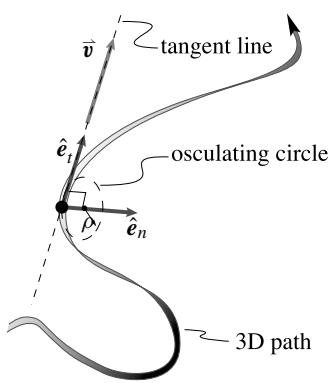


Figure 9.7: The base vectors for path coordinates are tangent and perpendicular to the path. The osculating circle is a circle that is tangent to the path and has the same curvature as the path. The circle is in the plane containing both the tangent to the path and the rate of change of that tangent. The osculating circle is not defined for a straight-line path.

(Filename:tfigure6.2)

## The geometry of the path basis vectors.

As a particle moves through space it traces a path  $\vec{r}(t)$ . At the moment of interest the path has a unique tangent line. The unit tangent  $\hat{e}_t$  is in the direction of this line, the direction of motion, as shown in figure 9.7.

Less clear is that the path has a unique ‘kissing’ plane. One line on this plane is the tangent line. The other line needed to define this plane is determined by the position of the particle just before and just after the time of interest. Just before and just after the time of interest the particle is a little off the tangent line (unless the motion happens to be a straight line and the tangent plane is not uniquely determined). Three points, the position of the particle just before, just at, and just after the moment of interest determine the tangent plane.

Another way to picture the tangent plane is to find the circle in space that is tangent to the path and which turns at the same rate and in the same direction as the path turns. This circle, which touches the path so intimately, is called the *osculating* or ‘kissing’ circle. The tangent plane is the plane of this circle. (See figure 9.7).

The unit normal  $\hat{e}_n$  is the unit vector which is perpendicular to the unit tangent and is in the tangent plane. It is pointed in the direction from the edge of the osculating circle towards the center of the circle as shown in figure 9.7. For 2-D motion in the  $xy$  plane the osculating plane is the  $xy$  plane and the osculating circle is in the  $xy$  plane. The path base vectors are unit vectors that vary along the path, always tangent and normal to the path (see Fig. 9.8).

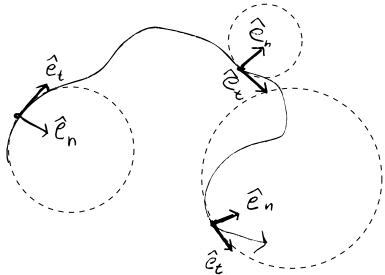


Figure 9.8: A path in the plane with the path base vectors and osculating circle marked at three points.

(Filename:tfigure.2Dpath)

## Formal definition of path basis vectors

The path of a particle is  $\vec{r}(t)$ . The path can also be parameterized by arc length  $s$  along the path, as explained in any introductory calculus text. So the path in space is  $\vec{r}(s)$ , where  $s$  is the path “coordinate”. The unit tangent is:

$$\hat{e}_t \equiv \frac{d\vec{r}(s)}{ds}.$$

Using the chain rule with  $\vec{r}(s(t))$  this is also

$$\hat{e}_t = \frac{d\vec{r}(t)}{dt} \frac{dt}{ds} = \frac{\vec{v}}{v}.$$

To define the unit normal let's first define the *curvature*  $\bar{\kappa}$  of the path as the rate of change of the tangent (rate in terms of arc length).

$$\bar{\kappa} \equiv \frac{d\hat{e}_t}{ds}$$

The unit normal  $\hat{e}_n$  is the unit vector in the direction of the curvature

$$\hat{e}_n = \frac{\vec{\kappa}}{|\vec{\kappa}|}.$$

Finally, the binormal  $\hat{e}_b$  is the unit vector perpendicular to  $\hat{e}_t$  and  $\hat{e}_n$ :

$$\hat{e}_b \equiv \hat{e}_t \times \hat{e}_n.$$

For 2-D motion the binormal  $\hat{e}_b$  is always in the  $\hat{k}$  direction. The radius of the osculating circle  $\rho$  is

$$\rho = \frac{1}{|\vec{\kappa}|}.$$

Note that, in general, the polar and path coordinate basis vectors are not parallel; i.e.,  $\hat{e}_n$  is *not* parallel to  $\hat{e}_R$  and  $\hat{e}_t$  is *not* parallel to  $\hat{e}_\theta$ . For example, consider a particle moving on an elliptical path in the plane shown in figure 9.9. In this case, the polar coordinate and path coordinate basis vectors are only parallel where the major and minor axes intersect the path. But for the special case of circular motion, as in Chapters 7 and 8, the polar and path coordinate vectors are everywhere parallel on the path.

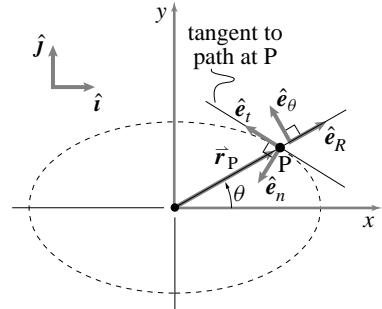


Figure 9.9: Particle  $P$  on an elliptical path.  
(Filename: tfigure6.path.coord)

## Velocity and acceleration in path coordinates.

Though it is not necessarily easy to compute the path basis vectors  $\hat{e}_t$  and  $\hat{e}_n$ , they lead to simple expressions for the velocity and acceleration:

$$\vec{v} = v\hat{e}_t = \frac{ds}{dt}\hat{e}_t, \quad \text{and} \quad (9.3)$$

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}(v\hat{e}_t) \quad (9.4)$$

$$= \dot{v}\hat{e}_t + v\dot{\hat{e}}_t \quad (9.5)$$

$$= \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{ds}\frac{ds}{dt} \quad (9.6)$$

$$= \dot{v}\hat{e}_t + v\vec{\kappa}v \quad (9.7)$$

$$= \underbrace{\dot{v}\hat{e}_t}_{\vec{a}_t} + \underbrace{\frac{v^2}{\rho}\hat{e}_n}. \quad (9.8)$$

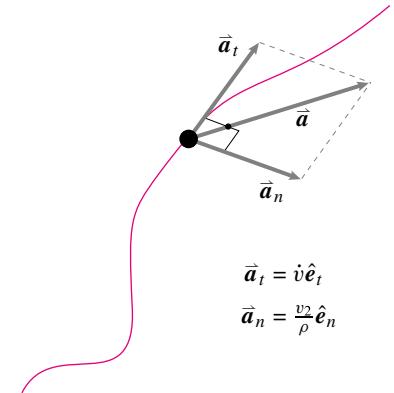


Figure 9.10: The acceleration of any particle can be broken into a part that is parallel to its path and a part perpendicular to its path.  
(Filename: tfigure6.3)

This formula for velocity is obvious: velocity is speed times a unit vector in the direction of motion. The formula for acceleration is more interesting. It says that the acceleration of any particle at any time is given by the same formula as the formula for acceleration of a particle going around in circles at non-constant rate. There is a term directed towards the center of the osculating circle  $v^2/\rho$  and a second term tangent to the path (also tangent to the osculating circle), as shown in figure 9.10.

The acceleration has two parts. A part associated with change of direction that is normal acceleration and does not vanish even if the speed is constant. This normal acceleration is perpendicular to the path. And a part associated with the change of speed, the tangential acceleration, which does not vanish even if the particle moves in a straight line. The tangential acceleration is, appropriately, tangent to the path.

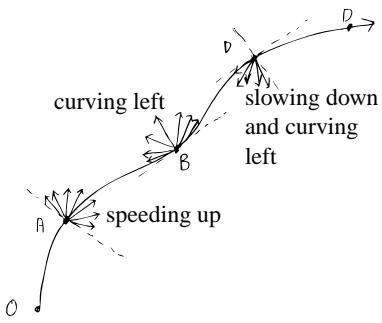


Figure 9.11: A particle moves from 0 to D on the path shown. At A it is speeding up so  $\dot{v} > 0$ . At B it is turning to the left so  $\hat{\mathbf{e}}_n$  points to the left. At C it is slowing down and turning to the right. So the acceleration must be a positive vector pointed in the quadrant shown.

(Filename:tfigure\_pathintuition)

### Example: Estimating the direction of acceleration

By looking at the path of a particle (e.g., see Fig. 9.11) and just knowing whether it is speeding up or slowing down one can estimate the direction of the acceleration.

If the particle is known to be speeding up at A then  $\dot{v} > 0$  so the tangential acceleration is in the direction of the velocity. Without thinking about the normal acceleration you know that the acceleration vector is pointed in the half plane of directions shown.

If at point B nothing is known about the rate of change of speed, you still know that the acceleration must be in the half plane shown because that is the direction of  $\vec{\kappa}$  and  $\hat{\mathbf{e}}_n$ .

If at C you know that the particle is slowing down then you know that  $\dot{v} < 0$ . But you also can see the curve is to the right so  $\hat{\mathbf{e}}_n$  is to the right. So the acceleration must be in the quadrant shown.

One can use the information about the curvature to further restrict the possible accelerations at point A also. At point B there is nothing more to know unless you know how the speed is changing with time.  $\square$

Earlier we found the curvature by assuming the particle's path was parameterized by arc length  $s$ . A second way of calculating the curvature  $\vec{\kappa}$  (and then the unit normal  $\hat{\mathbf{e}}_n$ ) is to calculate the normal part of the acceleration. First calculate the acceleration. Then subtract from the acceleration that part which is parallel to the velocity.

$$\vec{\mathbf{a}}_n = \vec{\mathbf{a}} - (\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_t) \hat{\mathbf{e}}_t = \vec{\mathbf{a}} - \frac{(\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}) \vec{\mathbf{v}}}{v^2}$$

The normal acceleration is  $\vec{\mathbf{a}}_n = v^2 \vec{\kappa}$  so

$$\vec{\kappa} = \frac{\vec{\mathbf{a}}_n}{v^2} = \frac{\vec{\mathbf{a}}}{v^2} - \frac{(\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}) \vec{\mathbf{v}}}{v^4}.$$

### Recipes for path coordinates.

Assume that you know the position as a function of time in either cartesian or polar coordinates. Then, say, at a particular time of interest when the particle is at  $\vec{r}$ , you can calculate the velocity of the particle using:

$$\vec{\mathbf{v}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \quad \text{or} \quad \vec{\mathbf{v}} = \dot{R}\hat{\mathbf{e}}_R + R\dot{\theta}\hat{\mathbf{e}}_\theta + \dot{z}\hat{\mathbf{k}}$$

and the acceleration using

$$\vec{\mathbf{a}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \quad \text{or} \quad \vec{\mathbf{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\mathbf{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\mathbf{e}}_\theta + \ddot{z}\hat{\mathbf{k}}.$$

From these expressions we can calculate all the quantities used in the path coordinate description. So we repeat what we have already said but in an algorithmic form. Here is one set of steps one can follow. This recipe is of little practical use, but does show that the motion explicitly determines the path base vectors as well as the osculating circle.

- Calculate  $\hat{\mathbf{e}}_t = \vec{\mathbf{v}} / |\vec{\mathbf{v}}|$ .
- Calculate  $a_t = \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} / v$ .
- Calculate  $\vec{\mathbf{a}}_n = \vec{\mathbf{a}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}) \vec{\mathbf{v}} / v^2$ .
- Calculate  $\hat{\mathbf{e}}_n = \frac{\vec{\mathbf{a}}_n}{|\vec{\mathbf{a}}_n|}$
- Calculate the radius of curvature as  $\rho = \frac{|\vec{\mathbf{v}}|^2}{|\vec{\mathbf{a}}_n|}$ .

(f) Write a parametric equation for the osculating circle as

$$\vec{r}_{\text{osculating}} = \underbrace{(\vec{r} + \rho \hat{e}_n)}_{\text{center}} + \underbrace{\rho(-\cos \phi \hat{e}_n + \sin \phi \hat{e}_t)}_{\text{circle}}$$

where  $\phi$  is the parameter used to parameterize the points on the circle  $\vec{r}_{\text{osculating}}$  of the point on the curve  $\vec{r}$ . As  $\phi$  ranges from 0 to  $2\pi$  the point  $\vec{r}_{\text{osculating}}$  goes from  $\vec{r}$  around the circle and back. The plane of the osculating circle is determined by  $\hat{e}_t$  and  $\hat{e}_n$ . For planar curves, the osculating circle is in the plane of the curve.

### Example: Particle on the rim of a tire

A particle on the rim of a tire whose center is moving at constant speed  $v$  has position  $r$  given by

$$\vec{r} = (vt - R \sin(vt/R)) \hat{i} + R(1 - \cos(vt/R)) \hat{j}$$

where the origin is at the ground contact time  $t = 0$ . When the particle is at its highest point  $vt = \pi R$  and

$$\vec{v} = 2v\hat{i} \quad \text{and} \quad \vec{a} = -(v^2/R)\hat{j}.$$

At that midpoint

$$\begin{aligned} \hat{e}_t &= \hat{i}, \quad \hat{e}_n = -\hat{j}, \quad \vec{\kappa} = -(1/(4R))\hat{j} \\ \text{and} \quad (2v)^2/\rho &= v^2/R \Rightarrow \rho = 4R \end{aligned}$$

as shown in Fig. 9.12. The osculating circle has 4 times the radius of the tire. Note the intimacy the osculating circle's kiss of the cycloidal path.

□

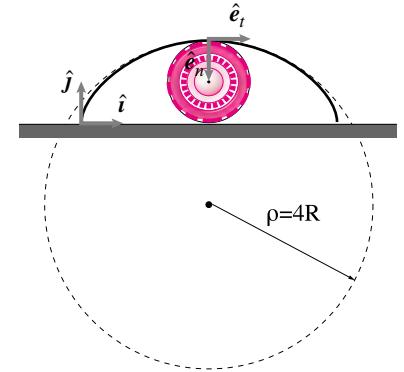


Figure 9.12: The path of a rock in a tire of radius  $R$  is shown as are the unit tangent, the unit normal and the osculating circle at the top position.

(Filename: tfigure.rockintire)

## Summary of polar(cylindrical) coordinates

See the inside back cover, table II, row 3 in for future reference:

$$\begin{aligned} \vec{r} &= R\hat{e}_R + z\hat{k} \\ \vec{v} &= v_R\hat{e}_R + v_\theta\hat{e}_\theta + v_z\hat{k} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \ddot{z}\hat{k} \\ \vec{a} &= a_R\hat{e}_R + a_\theta\hat{e}_\theta + a_z\hat{k} = (\ddot{R} - \dot{\theta}^2 R)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k} \\ \hat{e}_R &= (\vec{r} - z\hat{k})/|\vec{r} - z\hat{k}| \\ \hat{e}_\theta &= \hat{k} \times \hat{e}_R \end{aligned}$$

For 2-D problems just set  $z = 0$ ,  $\dot{z} = 0$ , and  $\ddot{z} = 0$  in these equations.

## Summary of path coordinates

See the inside back cover table II, row 4 and the text under the table for future reference:

$\vec{r}$  = no simple expression in terms of path base vectors

$$\vec{v} = \dot{s}\hat{e}_t = v\hat{e}_t$$

$$\vec{a} = \vec{a}_t + \vec{a}_n$$

$$\vec{a}_t = a_t \hat{e}_t = \dot{v}\hat{e}_t = \ddot{s}\hat{e}_t = (\vec{a} \cdot \hat{e}_t)\hat{e}_t$$

$$\vec{a}_n = a_n \hat{e}_n = (v^2/\rho)\hat{e}_n = \vec{a} - (\vec{a} \cdot \hat{e}_t)\hat{e}_t$$

$$\hat{e}_t = d\vec{r}/ds = \vec{v}/v$$

$$\hat{e}_n = \vec{\kappa}/|\vec{\kappa}| = \rho\vec{\kappa} = (\vec{a} - \vec{a} \cdot \hat{e}_t)/|\vec{a} - \vec{a} \cdot \hat{e}_t|$$

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n$$

$$\vec{\kappa} = d\hat{e}_t/ds = (\vec{a} - \vec{a} \cdot \hat{e}_t)/v^2 \quad \rho = 1/|\vec{\kappa}|$$

Both polar coordinates and path coordinates define base vectors in terms of the motion of a particle of interest relative to a fixed coordinate system.

**SAMPLE 9.1 Acceleration in polar coordinates.** A bug walks along the spiral section of a natural shell. The path of the bug is described by the equation  $R = R_0 e^{a\theta}$  where  $a = 0.182$  and  $R_0 = 5$  mm. The bug's radial distance from the center of the spiral is seen to be increasing at a constant rate of 2 mm/s. Find the  $x$  and  $y$  components of the acceleration of the bug at  $\theta = \pi$ .

**Solution** In polar coordinates, the acceleration of a particle in planar motion is

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

Since we know the position of the bug,

$$\begin{aligned} R &= R_0 e^{a\theta}, \\ \dot{R} &= R_0 a e^{a\theta} \dot{\theta} \Rightarrow \dot{\theta} = \frac{\dot{R}}{R_0 a e^{a\theta}}, \\ \ddot{R} &= R_0 a e^{a\theta} \ddot{\theta} + R_0 a^2 e^{a\theta} \dot{\theta}^2. \end{aligned}$$

Since the radial distance  $R$  of the bug is increasing at a constant rate  $\dot{R} = 2$  mm/s,  $\ddot{R} = 0$ , that is,

$$\begin{aligned} R_0 a e^{a\theta} (\ddot{\theta} + a\dot{\theta}^2) &= 0 \\ \Rightarrow \ddot{\theta} &= -a\dot{\theta}^2 \\ &= -\frac{\dot{R}^2}{R_0^2 a e^{2a\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= \left(0 - R_0 e^{a\theta} \cdot \frac{\dot{R}^2}{R_0^2 a^2 e^{2a\theta}}\right)\hat{e}_R + \left(\frac{2\dot{R}^2}{R_0 a e^{a\theta}} + R_0 e^{a\theta} \cdot \frac{-\dot{R}^2}{R_0^2 a e^{2a\theta}}\right)\hat{e}_\theta \\ &= \frac{\dot{R}^2}{R_0 a e^{a\theta}} \left[-\frac{1}{a}\hat{e}_R + (2 - 1)\hat{e}_\theta\right]. \end{aligned}$$

Now substituting  $R_0 = 5$  mm,  $a = 0.182$ ,  $\dot{R} = 2$  mm/s, and  $\theta = \pi$  in the above expression, we get

$$\vec{a} = (-13.63 \text{ mm/s}^2)\hat{e}_R + (2.48 \text{ mm/s}^2)\hat{e}_\theta.$$

But, at  $\theta = \pi$

$$\hat{e}_R = \cos \theta \hat{i} + \sin \theta \hat{j} = -\hat{i} \quad \text{and} \quad \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} = -\hat{j},$$

therefore,

$$\begin{aligned} \vec{a} &= (13.63 \text{ mm/s}^2)\hat{i} - (2.48 \text{ mm/s}^2)\hat{j}, \\ \Rightarrow a_x &= 13.63 \text{ mm/s}^2 \quad \text{and} \quad a_y = -2.48 \text{ mm/s}^2. \end{aligned}$$

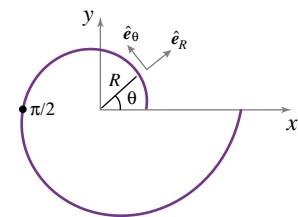


Figure 9.13: (Filename:fig10.1.polarshell)

**SAMPLE 9.2** *Going back and forth between  $(x, y)$  and  $(R, \theta)$ .* Given the position of a particle in polar coordinates  $(R, \theta)$  and its radial and angular velocity  $(\dot{R}, \dot{\theta})$  and radial and angular acceleration  $(\ddot{R}, \ddot{\theta})$ , find  $(\dot{x}, \dot{y})$  and  $(\ddot{x}, \ddot{y})$ . Also, find the inverse relationship.

### Solution

- **Polar to Cartesian:** In polar coordinates, we are given  $R, \theta, \dot{R}, \dot{\theta}, \ddot{R}$ , and  $\ddot{\theta}$ . We need to find  $\dot{x}, \dot{y}, \ddot{x}$ , and  $\ddot{y}$ . Let us consider the velocity first. The velocity of a point is  $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$  in cartesian coordinates and  $\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$  in polar coordinates. Thus,

$$\dot{x}\hat{i} + \dot{y}\hat{j} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta.$$

where  $\hat{e}_R = \cos\theta\hat{i} + \sin\theta\hat{j}$  and  $\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$ . Dotting this equation with  $\hat{i}$  and  $\hat{j}$ , respectively, we get

$$\begin{aligned}\dot{x} &= \underbrace{\dot{R}(\hat{e}_R \cdot \hat{i})}_{\sin\theta} + \underbrace{R\dot{\theta}(\hat{e}_\theta \cdot \hat{i})}_{-\sin\theta} \\ \dot{y} &= \underbrace{\dot{R}(\hat{e}_R \cdot \hat{j})}_{\cos\theta} + \underbrace{R\dot{\theta}(\hat{e}_\theta \cdot \hat{j})}_{\cos\theta}\end{aligned}$$

or,

$$\begin{aligned}\dot{x} &= \dot{R}\cos\theta + R\dot{\theta}(-\sin\theta) \\ \dot{y} &= \dot{R}\sin\theta + R\dot{\theta}\cos\theta\end{aligned}$$

or,

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{Bmatrix} \dot{R} \\ \dot{\theta} \end{Bmatrix}. \quad (9.9)$$

Thus given  $\dot{R}$  and  $\dot{\theta}$  at  $(R, \theta)$ , we can find  $\dot{x}$  and  $\dot{y}$ . Similarly, from the acceleration formula,  $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$ , we derive

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{Bmatrix} \ddot{R} \\ \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} -R\dot{\theta}^2 \\ 2\dot{R}\dot{\theta} \end{Bmatrix} \right). \quad (9.10)$$

It is not necessary to split the terms on the right hand side. We could have kept them together as  $(\ddot{R} - R\dot{\theta}^2)$  and  $(2\dot{R}\dot{\theta} + R\ddot{\theta})$  but we split them to keep the radial acceleration term  $\ddot{R}$  and angular acceleration  $\ddot{\theta}$  in evidence.

- **Cartesian to Polar:** Given  $\dot{x}, \dot{y}, \ddot{x}$ , and  $\ddot{y}$  at  $(x, y)$ , we can now find  $\dot{R}, \dot{\theta}, \ddot{R}$ , and  $\ddot{\theta}$  easily by inverting eqn. (9.9) and eqn. (9.10):

$$\begin{Bmatrix} \dot{R} \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}. \quad (9.11)$$

$$\begin{Bmatrix} \ddot{R} \\ \ddot{\theta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \left( \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} - \begin{Bmatrix} -R\dot{\theta}^2 \\ 2\dot{R}\dot{\theta} \end{Bmatrix} \right). \quad (9.12)$$

Note that in eqn. (9.12) we need  $\dot{R}$  and  $\dot{\theta}$  in order to compute  $\ddot{R}$  and  $\ddot{\theta}$ . This, however, is no problem since we have  $\dot{R}$  and  $\dot{\theta}$  from eqn. (9.11). Of course,  $R$  and  $\theta$  are required too, which are easily computed as  $R = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$ .

**SAMPLE 9.3 Velocity in path coordinates.** The path of a particle, stuck at the edge of a disk rolling on a level ground with constant speed, is called a cycloid. The parametric equations of a cycloid described by a particle is  $x = t - \sin t$ ,  $y = 1 - \cos t$  where  $t$  is a dimensionless time. Find the velocity of the particle at

- (a)  $t = \frac{\pi}{2}$ ,
- (b)  $t = \pi$ , and
- (c)  $t = 2\pi$

and express the velocity in terms of path basis vectors  $(\hat{e}_t, \hat{e}_n)$ .

**Solution** The position of the particle is given:

$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} \\ &= (t - \sin t)\hat{i} + (1 - \cos t)\hat{j} \\ \Rightarrow \quad \vec{v} &\equiv \frac{d\vec{r}}{dt} \\ &= (1 - \cos t)\hat{i} + \sin t\hat{j}, \end{aligned} \tag{9.13}$$

$$\text{and } v = |\vec{v}| \tag{9.14}$$

$$\begin{aligned} &= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}. \end{aligned} \tag{9.15}$$

In terms of path basis vectors, the velocity is given by

$$\vec{v} = v\hat{e}_t \quad \text{where } \hat{e}_t = \vec{v}/v.$$

Here,

$$\hat{e}_t = \frac{(1 - \cos t)\hat{i} + \sin t\hat{j}}{\sqrt{2 - 2\cos t}}. \tag{9.16}$$

Substituting the values of  $t$  in equations 9.15 and 9.16 we get

- (a) at  $t = \frac{\pi}{2}$ :

$$v = \sqrt{2}, \quad \hat{e}_t = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}), \quad \vec{v} = \sqrt{2}\hat{e}_t.$$

$$\boxed{\vec{v} = \sqrt{2}\hat{e}_t, \quad \hat{e}_t = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})}$$

- (b) at  $t = \pi$ :

$$v = 2, \quad \hat{e}_t = \hat{i}, \quad \vec{v} = 2\hat{e}_t.$$

$$\boxed{\vec{v} = 2\hat{e}_t, \quad \hat{e}_t = \hat{i}}$$

- (c) at  $t = 2\pi$ :

$$v = 0, \quad \hat{e}_t = \text{undefined}, \quad \vec{v} = \vec{0}.$$

$$\boxed{\vec{v} = \vec{0}}$$

**SAMPLE 9.4** Path coordinates in 2-D. A particle traverses a limacon  $R = (1 + 2 \cos \theta)$  ft, with constant angular speed  $\dot{\theta} = 3 \text{ rad/s}$ .

- Find the normal and tangential accelerations ( $a_t$  and  $a_n$ ) of the particle at  $\theta = \frac{\pi}{2}$ .
- Find the radius of the osculating circle and draw the circle at  $\theta = \frac{\pi}{2}$ .

### Solution

- The equation of the path is

$$R = (1 + 2 \cos \theta) \text{ ft}$$

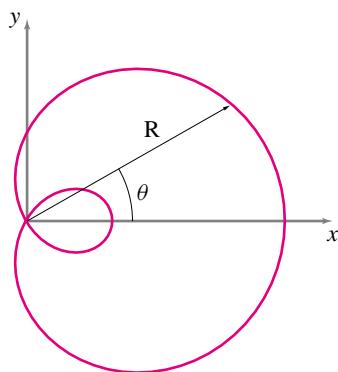


Figure 9.14: The limacon  $R = (1 + 2 \cos \theta)$  ft.

(Filename:fig6.3.1a)

The path is shown in Fig. 9.14. Since the equation of the path is given in polar coordinates, we can calculate the velocity and acceleration using the polar coordinate formulae:

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta \quad (9.17)$$

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta. \quad (9.18)$$

So, we need to find  $\dot{R}$ ,  $\ddot{R}$ ,  $\ddot{\theta}$  for computing  $\vec{v}$  and  $\vec{a}$ . From the given equation for  $R$

$$\begin{aligned} R &= (1 + 2 \cos \theta) \text{ ft} \\ \Rightarrow \dot{R} &= -(2 \text{ ft}) \sin \theta \dot{\theta} \\ \Rightarrow \ddot{R} &= -(2 \text{ ft}) \sin \theta \ddot{\theta} - (2 \text{ ft}) \cos \theta \dot{\theta}^2 \\ &= -(2 \text{ ft}) \dot{\theta}^2 \cos \theta \end{aligned}$$

where we set  $\ddot{\theta} = 0$  because  $\dot{\theta} = \text{constant}$ . Substituting these expressions in Eqn. (9.17) and (9.18), we get

$$\begin{aligned} \vec{v} &= -(2 \text{ ft})\dot{\theta} \sin \theta \hat{e}_R + \dot{\theta}(1 + 2 \cos \theta) \text{ ft} \hat{e}_\theta \\ \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= [-2\dot{\theta}^2 \cos \theta - (1 + 2 \cos \theta)\dot{\theta}^2] \text{ ft} \hat{e}_R + (-2\dot{\theta}^2 \sin \theta) \text{ ft} \hat{e}_\theta \\ &= -\dot{\theta}^2[(1 + 4 \cos \theta) \hat{e}_R + (2 \sin \theta) \hat{e}_\theta] \text{ ft} \end{aligned}$$

which give velocity and acceleration at any  $\theta$ . Now substituting  $\theta = \pi/2$  we get the velocity and acceleration at the desired point:

$$\begin{aligned} \vec{v}|_{\frac{\pi}{2}} &= \dot{\theta}[-2 \sin \frac{\pi}{2} \hat{e}_R + (1 + 2 \cos \frac{\pi}{2}) \hat{e}_\theta] \text{ ft} \\ &= 3 \text{ ft/s}(-2\hat{e}_R + \hat{e}_\theta) \\ \vec{a}|_{\frac{\pi}{2}} &= -9 \text{ ft/s}^2(\hat{e}_R + 2\hat{e}_\theta) \end{aligned}$$

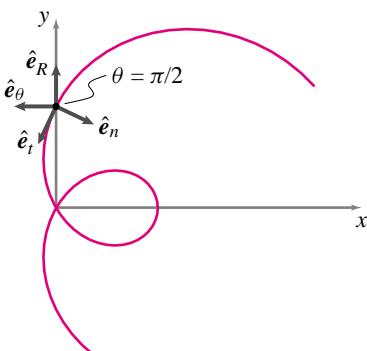
Thus we know the velocity and the acceleration of the particle in polar coordinates. Now we proceed to find the tangential and the normal components of acceleration (acceleration in path coordinates). In path coordinates

$$\begin{aligned} \vec{a} &= \vec{a}_t + \vec{a}_n \\ &\equiv a_t \hat{e}_t + a_n \hat{e}_n \end{aligned}$$

where  $\hat{e}_t$  and  $\hat{e}_n$  are unit vectors in the directions of the tangent and the principal normal of the path. We compute these unit vectors as follows.

Figure 9.15: The unit vectors  $\hat{e}_R$ ,  $\hat{e}_\theta$ , and  $\hat{e}_t$ ,  $\hat{e}_n$  at  $\theta = \pi/2$ .

(Filename:fig6.3.1b)



$$\begin{aligned}
 \hat{\mathbf{e}}_t &= \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \\
 &= \left( \frac{-6 \text{ ft/s} \hat{\mathbf{e}}_R + (3 \text{ ft/s}) \hat{\mathbf{e}}_\theta}{\sqrt{45} \text{ ft/s}} \right) \\
 &= -\frac{2}{\sqrt{5}} \hat{\mathbf{e}}_R + \frac{1}{\sqrt{5}} \hat{\mathbf{e}}_\theta.
 \end{aligned}$$

So,

$$\begin{aligned}
 \vec{\mathbf{a}}_t &= (\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_t) \hat{\mathbf{e}}_t \\
 &= 9 \text{ ft/s}^2 \left( -\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \right) \hat{\mathbf{e}}_t \\
 &= \vec{\mathbf{0}},
 \end{aligned}$$

and

$$\begin{aligned}
 \vec{\mathbf{a}}_n &= \vec{\mathbf{a}} - \vec{\mathbf{a}}_t \\
 &= -9 \text{ ft/s}^2 (\hat{\mathbf{e}}_R + 2\hat{\mathbf{e}}_\theta).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \hat{\mathbf{e}}_n &= \frac{\vec{\mathbf{a}}_n}{|\vec{\mathbf{a}}_n|} \\
 &= -\frac{1}{\sqrt{5}} (\hat{\mathbf{e}}_R + 2\hat{\mathbf{e}}_\theta).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \vec{\mathbf{a}} &= 9\sqrt{5} \text{ ft/s}^2 \hat{\mathbf{e}}_n \\
 \Rightarrow a_t &= 0 \quad \text{and} \quad a_n = 20.12 \text{ ft/s}^2.
 \end{aligned}$$

$$a_t = 0, a_n = 20.12 \text{ ft/s}^2$$

(b) In path coordinates the acceleration is also expressed as

$$\vec{\mathbf{a}} = \dot{v} \hat{\mathbf{e}}_t + \frac{v^2}{\rho} \hat{\mathbf{e}}_n$$

where  $\rho$  is the radius of the osculating circle. Since we already know the speed  $v$  and the normal component of acceleration  $a_n$  we can easily compute the radius of the osculating circle.

$$\begin{aligned}
 a_n &= \frac{v^2}{\rho} \\
 \Rightarrow \rho &= \frac{v^2}{a_n} = \frac{45(\text{ft/s})^2}{9\sqrt{5} \text{ ft/s}^2} = \sqrt{5} \text{ ft}.
 \end{aligned}$$

$$\rho = 2.24 \text{ ft}$$

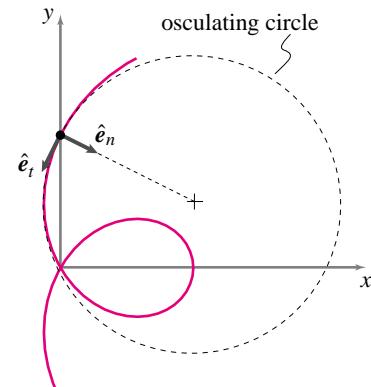


Figure 9.16: The osculating circle of radius  $\rho = 5^{1/2}$  ft at  $\theta = \pi/2$ . Note that  $\hat{\mathbf{e}}_n$  points to the center of the osculating circle.  
(Filename:sfig6.3.1c)

## 9.2 Rotating reference frames and their time-varying base vectors

In this section you will learn about rotating reference frames, how to take the derivative of a vector ‘in’ a rotating frame, and how to use that derivative to find the derivative in a Newtonian or fixed frame. We start by showing the alternative, just using one frame with one set of fixed base vectors.

### The fixed base vector method

To motivate the sections that follow we first show the “fixed base vector” method. Consider the task of determining the acceleration of a bug walking at constant speed as it walks on a straight line marked on the surface of a tire rolling at constant rate. Artificial as this problem seems, it is similar to the sort of calculation needed in the kinematics of mechanisms. For now, imagine you really care how strong the bugs legs need to be to hold on (unreasonably neglecting air friction). So knowing the bugs acceleration determines the net force on it by  $\vec{F} = m\vec{a}$ . Now we try to find  $\vec{a}$  by taking two time derivatives of position.

If we want to avoid using rotating base vectors we have to write an expression for the position of the bug in terms of  $x$  and  $y$  components. Choosing a suitable origin of the coordinate system we have

$$\begin{aligned}\vec{r}_{P/0} &= \vec{r}_{0'/0} + \vec{r}_{P/0'} \\ &= (R\hat{i} + R\hat{j}) + (s(\cos\theta\hat{i} - \sin\theta\hat{j}) + \ell(\sin\theta\hat{i} + \cos\theta\hat{j})) \\ &= (R\theta + s\cos\theta + \ell\sin\theta)\hat{i} + (R - s\sin\theta + \ell\cos\theta)\hat{j}. \quad (9.19)\end{aligned}$$

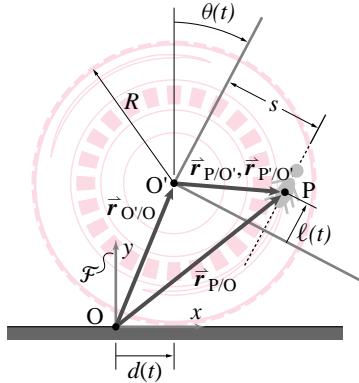


Figure 9.17: A bug walks at constant speed on a straight line marked on a tire.

(Filename:tfigure.fixedbase)

To find the velocity we take the time derivative, taking account that both  $\theta$  and  $\ell$  are functions of  $t$ . Thus, for example looking at the term  $\ell\cos\theta$  both the product rule and chain rule need be applied. Proceeding we get

$$\begin{aligned}\vec{v}_{P/0} &= (R\dot{\theta} - s\dot{\theta}\sin\theta + \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta)\hat{i} \\ &\quad + (-s\dot{\theta}\cos\theta + \dot{\ell}\cos\theta - \ell\dot{\theta}\sin\theta)\hat{j}. \quad (9.20)\end{aligned}$$

Now to get the acceleration of the bug we differentiate yet one more time. This time we use the product rule and chain rule again, but get to use the simplification for this problem that the rolling and bug walking are at constant rate so  $\ddot{\theta} = 0$  and  $\ddot{\ell} = 0$ . Proceeding, we get

$$\begin{aligned}\vec{a}_{P/0} &= (-s\dot{\theta}^2\cos\theta + \dot{\ell}\dot{\theta}\cos\theta + \dot{\ell}\dot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta)\hat{i} \\ &\quad + (s\dot{\theta}^2\sin\theta - \dot{\ell}\dot{\theta}\sin\theta - \dot{\ell}\dot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta)\hat{j} \\ &= (-\dot{\theta}^2(s\cos\theta + \ell\sin\theta) + 2\dot{\ell}\dot{\theta}\cos\theta)\hat{i} \\ &\quad + (\dot{\theta}^2(s\sin\theta - \ell\cos\theta) - 2\dot{\ell}\dot{\theta}\sin\theta)\hat{j} \quad (9.21)\end{aligned}$$

which is a bit of a mess. We could regroup the terms, but there would still be 6 of them.

The moving-reference-frame methods that follow don’t change this answer. But they give a somewhat simpler derivation. And they also group the terms in physically meaningful way. One would be hard pressed to say if all the terms in eqn. (9.21) made sense. With the time-varying base vector methods we can interpret the terms.

## Reference frames

A *reference frame* is a coordinate system<sup>①</sup>. It has an origin and a set of preferred mutually orthogonal directions represented by base vectors. You can think of a reference frame as a giant piece of graph paper, or in 3-D as a giant jungle gym, that permeates space. It has the look of a wire *frame*. Because we will use various frames, we name them. We always have one frame that we think of as *fixed* for the purposes of Newtonian mechanics. We call this frame  $\mathcal{F}$  (or sometimes  $\mathcal{N}$ ). Most often we choose a frame that is ‘glued’ to the ground with an origin at a convenient point and with at least one base vector lined up with something convenient (e.g., up, sideways, along a slope, along the edge of an important part, *etc.*).  $\mathcal{F}$  is a frame in which the mechanics laws we use are accurate. We define it by its origin and the direction of its coordinate axes, thus we would write

$$\mathcal{F} \text{ is } 0xyz \quad \text{or} \quad \mathcal{F} \text{ is } O\hat{i}\hat{j}\hat{k}.$$

where we would generally have a picture showing the position of the origin and the orientation of the coordinate axes (see Fig. 9.18).

When we write casually ‘position  $\vec{r}$ ’ of a point we mean  $\vec{r}_{P/0}$ . When we write ‘velocity  $\vec{v}$ ’ we mean  $\frac{d}{dt}\vec{r}$  as calculated in  $\mathcal{F}$ . That is, if  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  then we define the derivative of  $\vec{r}$  with respect to  $t$  in  $\mathcal{F}$  as

$$\frac{\mathcal{F} d\vec{r}}{dt} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}.$$

The script  $\mathcal{F}$  shows explicitly that when we take the time derivative of the vector we take the time derivative of its components, using the components associated with  $\mathcal{F}$  and holding constant the base vectors associated with  $\mathcal{F}$ . That is

$$\frac{\mathcal{F} d\vec{v}_{P/0}}{dt} = \dot{\vec{r}}_{P/0} \text{ are just fancy ways of writing what we have been calling } \vec{v}.$$

The elaborate notation just makes explicit how  $\vec{v}$  is defined. The only need for this elaborate notation is if there is ambiguity. There is only ambiguity if more than one reference frame is used in a given problem.

## Using more than one reference frame

Let’s add a second reference frame called  $\mathcal{B}$  glued to and oriented with the roof of the building. We will always use script capital letters ( $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$  or  $\mathcal{N}$ ) to name reference frames. We define  $\mathcal{B}$  by writing

$$\mathcal{B} \text{ is } 0'x'y'z' \quad \text{or} \quad \mathcal{B} \text{ is } O'\hat{i}'\hat{j}'\hat{k}'$$

and by drawing a picture (see Fig. 9.19). This new frame, as we have drawn it, is also a good Newtonian or fixed frame. So we could write all positions using the  $\mathcal{B}$  coordinates and base vectors and then proceed with all of our mechanics equations with the only confusion being that gravity doesn’t point in the  $-\hat{j}'$  direction, but in some crooked direction relative to  $\hat{i}'$  and  $\hat{j}'$  which we would have to work out from the angle of the roof. Although one hardly notices when using just a single fixed frame, we actually use frames for three somewhat distinct purposes:

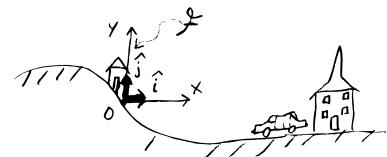


Figure 9.18: A fixed reference frame  $\mathcal{F}$  is defined by an origin  $O$  and coordinate axes  $xyz$  or base vectors  $\hat{i}\hat{j}\hat{k}$ . Once the  $xy$  (or  $\hat{i}\hat{j}$ ) directions are chosen the  $z$  (or  $\hat{k}$ ) direction is implicitly defined by the right hand rule.

(Filename:tfigure.firstrefframe)

<sup>①</sup> **A fine point for experts.** There is a semantic debate about the degree to which the phrases “coordinate system” and “reference frame” are synonymous. For simplicity we have taken the phrases to have the same meaning. An alternative definition distinguishes reference frames from coordinate systems. We note in the following text that coordinate systems which are rotated, but not rotating, with respect to each other both calculate the same time-derivative of a given vector. Because these coordinate systems are equivalent in this regard they are sometimes called the same reference frame. That is, some people consider one reference frame to be the set of all coordinate systems that are glued to each other, no matter what their position or orientation. In this way of thinking, a frame is made manifest by the use of one of its coordinate systems, but no particular coordinate system is unique to the frame.

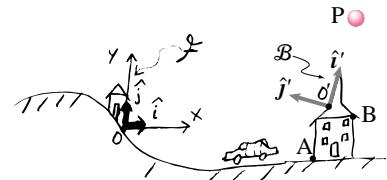


Figure 9.19: A second reference frame  $\mathcal{B}$  is defined by an origin  $O'$  and coordinate axes  $x'y'z'$  or base vectors  $\hat{i}'\hat{j}'\hat{k}'$ . Once the  $x'y'$  (or  $\hat{i}'\hat{j}'$ ) directions are chosen the  $z'$  (or  $\hat{k}'$ ) direction is implicitly defined by the right hand rule.

(Filename:tfigure.addrefframe)

- I. To define a vector.** For example if we were tracking the motion of a canon ball at P we could define its position vector  $\vec{r}$  as  $\vec{r}_{P/0}$ , using frame  $\mathcal{F}$  to define  $\vec{r}$ . Or we could define  $\vec{r}$  as  $\vec{r}_{P/O'}$  using frame  $\mathcal{B}$  to define  $\vec{r}$ .
- II. To assign coordinate values to a given vector.** For example, the vector  $\vec{r}_{B/A}$  could be written as

$$\begin{aligned}\vec{r}_{B/A} &= \vec{r}_{B/A}, \\ 4 m\hat{i} + 5 m\hat{j} &= 6 m\hat{i}' - 2.24 m\hat{j}'.\end{aligned}$$

Alternatively, if we just want to look at the components of a given vector we use  $[]_{\mathcal{F}}$  to indicate the components of the vector in []s using the base vectors of  $\mathcal{F}$ . Thus

$$[\vec{r}_{B/A}]_{\mathcal{F}} = [4 \text{ m}, \quad 5 \text{ m}]' \quad \text{and} \quad [\vec{r}_{B/A}]_{\mathcal{B}} = [6 \text{ m}, \quad -2.23 \text{ m}]'$$

where we have used  $[]'$  to put the components in their standard column form (though this is a picky detail). Note that although  $\vec{r}_{B/A} = \vec{r}_{B/A}$  that

$$\begin{aligned}[\vec{r}_{B/A}]_{\mathcal{F}} &\neq [\vec{r}_{B/A}]_{\mathcal{B}} \\ \text{because } \begin{bmatrix} 4 \text{ m} \\ 5 \text{ m} \end{bmatrix} &\neq \begin{bmatrix} 6 \text{ m} \\ 2.23 \text{ m} \end{bmatrix}\end{aligned}$$

- III. To find the rate of change of a given vector.** The position of P relative to A changes with time. We can calculate this rate of change two different ways. First using frame  $\mathcal{F}$

$$\begin{aligned}\frac{\mathcal{F}d\vec{v}_{P/A}}{dt} = \frac{\mathcal{F}\dot{\vec{r}}_{P/A}}{dt} &= \left( \frac{d}{dt}x_{P/A} \right) \hat{i} + \left( \frac{d}{dt}y_{P/A} \right) \hat{j} \\ \text{or more informally as } \frac{\mathcal{F}\dot{\vec{r}}}{dt} &= \dot{x}\hat{i} + \dot{y}\hat{j}\end{aligned}$$

if we are clear in our minds that  $x$  and  $y$  are the coordinates of P relative to A. But we can also calculate the rate of change of the same vector  $\vec{r}_{P/A}$  using frame  $\mathcal{B}$  as

$$\begin{aligned}\frac{\mathcal{B}d\vec{r}_{P/A}}{dt} = \frac{\mathcal{B}\dot{\vec{r}}_{P/A}}{dt} &= \left( \frac{d}{dt}x'_{P/A} \right) \hat{i}' + \left( \frac{d}{dt}y'_{P/A} \right) \hat{j}' \\ \text{or more informally as } \frac{\mathcal{B}\dot{\vec{r}}}{dt} &= \dot{x}'\hat{i}' + \dot{y}'\hat{j}'\end{aligned}$$

For the two frames  $\mathcal{F}$  and  $\mathcal{B}$   $\frac{\mathcal{B}\dot{\vec{r}}}{dt} = \frac{\mathcal{F}\dot{\vec{r}}}{dt}$  because the two frames are not rotating relative to each other. Specifically, for  $\mathcal{F}$  and  $\mathcal{B}$  the formula for finding  $x$  and  $y$  from  $x'$  and  $y'$  does not involve time. Similarly, the formulas for finding  $\hat{i}'$  and  $\hat{j}'$  from  $\hat{i}$  and  $\hat{j}$  do not involve time. For frames that are *rotated* with respect to each other but not *rotating*, the two time derivatives of a given vector are related the same way the vector itself is related to itself in the two frames. The vectors are the same but their coordinates are different. That is, for rotated but not relatively rotating frames

$$\begin{aligned}\vec{r}_{P/A} &= \vec{r}_{P/A} \quad \text{and} \quad \frac{\mathcal{F}d\vec{r}_{P/A}}{dt} = \frac{\mathcal{B}d\vec{r}_{P/A}}{dt} \\ \text{but } [\vec{r}_{P/A}]_{\mathcal{F}} &\neq [\vec{r}_{P/A}]_{\mathcal{B}} \quad \text{and} \quad [\frac{\mathcal{F}d\vec{r}_{P/A}}{dt}]_{\mathcal{F}} \neq [\frac{\mathcal{B}d\vec{r}_{P/A}}{dt}]_{\mathcal{B}}.\end{aligned}$$

Going back and forth between these three uses of frames with ease is one of the advanced skills of a person who can analyze the dynamics of complex systems (And being confused about the distinctions is an almost universal part of learning advanced dynamics).

**Example: Two fixed frames  $\mathcal{F}$  and  $\mathcal{B}$**

Consider  $\mathcal{F}$  and  $\mathcal{B}$  both to be fixed to the ground. Let's look at  $\vec{r}_{P/A}$  where P is moving at up at constant rate (see Fig. 9.20). First look at the position using both frames:

$$\begin{aligned}\vec{r}_{P/A} &= \vec{r}_{P/A} \quad \text{and} \\ ct\hat{j} &= (\sqrt{2}ct/2)\hat{i}' + (\sqrt{2}ct/2)\hat{j}' \quad \text{but} \\ [\vec{r}_{B/A}]_{\mathcal{F}} = [ct, 0]' &\neq [\vec{r}_{P/A}]_{\mathcal{B}} = [\sqrt{2}ct/2, \sqrt{2}ct/2]'.\end{aligned}$$

Now look at the rate of change of position using both frames. First  $\mathcal{F}$ :

$$\begin{aligned}\frac{\mathcal{F}d\vec{r}_{P/A}}{dt} = \frac{\mathcal{F}\vec{r}_{P/A}}{dt} &= \left(\frac{d}{dt}x_{P/A}\right)\hat{i} + \left(\frac{d}{dt}y_{P/A}\right)\hat{j} \\ &= c\hat{j}\end{aligned}$$

Then the rate of change of  $\vec{r}_{B/A}$  as calculated in  $\mathcal{B}$ :

$$\begin{aligned}\frac{\mathcal{B}d\vec{r}_{P/A}}{dt} = \frac{\mathcal{B}\vec{r}_{P/A}}{dt} &= \left(\frac{d}{dt}x'_{P/A}\right)\hat{i}' + \left(\frac{d}{dt}y'_{P/A}\right)\hat{j}' \\ &= (\sqrt{2}c/2)\hat{i}' + (\sqrt{2}c/2)\hat{j}' = c\hat{j}\end{aligned}$$

You can quickly verify that  $\frac{\mathcal{F}d\vec{r}_{P/A}}{dt} = \frac{\mathcal{B}d\vec{r}_{P/A}}{dt}$  by noting that  $\hat{i}' = \sqrt{2}(\hat{i} + \hat{j})/2$  and  $\hat{j}' = \sqrt{2}(-\hat{i} + \hat{j})/2$ .  $\square$

So long as  $\mathcal{B}$  is not rotating with respect to  $\mathcal{F}$  then the rate of change of a given vector is the same in both reference frames.

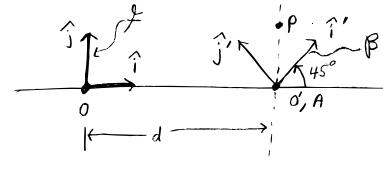


Figure 9.20: A particle P is moving up at speed  $c$ . Two fixed frames  $\mathcal{F}$  and  $\mathcal{B}$  are used to keep track of it.

(Filename: tfigure.gluedframes)

## Translating and rotating reference frames

Now look at a third reference frame  $\mathcal{C}$  that is glued to the roof of the car as it starts up hill (see Fig. 9.21). We define  $\mathcal{C}$  by the origin of its coordinate system  $0''$  and its time-varying base vectors  $\hat{i}''$  and  $\hat{j}''$ . The issues with **defining** a vector with  $\mathcal{C}$  and with **writing components** using  $\mathcal{C}$  are the same as for  $\mathcal{B}$ . However taking the **time derivative** of a given vector in  $\mathcal{C}$  is different than taking the time derivative in  $\mathcal{B}$  or  $\mathcal{F}$  because  $\mathcal{C}$  is rotating relative to them.

## Rate of change of a vector relative to a rotating frame: the $\dot{\vec{Q}}$ formula

Because dynamics involves the time derivatives of so many different vectors (e.g.  $\vec{r}$ ,  $\vec{v}$ ,  $\vec{L}$ ,  $\vec{H}_C$ , and  $\vec{\omega}$ ) it is easier to think about the derivative of some arbitrary or general vector, call it  $\vec{Q}$ , and then apply what we learn to these other vectors.

Recalling our three uses of frames:

- I. To define a vector.
- II. To express the coordinates of a given vector.
- III. To take the time derivative of a vector.

we see that items [III.] and [I.] can be combined. That is, once a vector  $\vec{Q}$  is defined clearly by some means then we can define a new vector as the derivative of that vector in, say, moving frame  $\mathcal{C}$ . Once this new vector  $\dot{\vec{Q}}$  is defined it can be expressed in terms of the coordinates of any convenient frame.

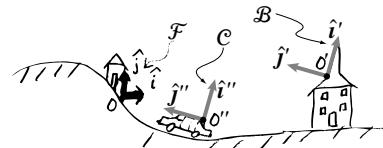


Figure 9.21: A third reference frame  $\mathcal{C}$  is defined by an origin  $0''$  and coordinate axes  $x''y''z''$  or base vectors  $\hat{i}''\hat{j}''\hat{k}''$ . In this case  $\mathcal{C}$  is moving and rotating with respect to  $\mathcal{F}$  and  $\mathcal{B}$ . Once the  $x''y''$  (or  $\hat{i}''\hat{j}''$ ) directions are chosen the  $z''$  (or  $\hat{k}''$ ) direction is implicitly defined by the right hand rule.

(Filename: tfigure.carframe)

**Example: Derivative in a moving frame of a constant vector**

Consider as  $\vec{Q}$  the relative position vector  $\vec{r}_{P/A}$  of the points A and P that do not move in the fixed frame  $\mathcal{F}$ . That is, the points A and B don't move in the ordinary sense of the words (see Fig. 9.22). Now also look at the frame  $\mathcal{B}$  that is rotating with respect to  $\mathcal{F}$  at the rate  $\dot{\theta}$ . We have

$$\begin{aligned}\vec{r}_{P/A} &= \vec{r}_{P/A} \\ \ell \hat{j} &= \ell \sin \theta \hat{i}' + \ell \cos \theta \hat{j}'\end{aligned}$$

So we can now calculate the derivative in each frame by holding the corresponding base vectors as constant. So

$$\begin{aligned}\frac{\mathcal{F} d\vec{r}_{P/A}}{dt} &= \dot{\ell} \hat{j} = \vec{0} \\ \text{and } \frac{\mathcal{B} d\vec{r}_{P/A}}{dt} &= \ell \dot{\theta} \cos \theta \hat{i}' - \ell \dot{\theta} \sin \theta \hat{j}' \\ &= \ell \dot{\theta} \underbrace{(\cos \theta \hat{i}' - \sin \theta \hat{j}')}_{\hat{i}} \\ &= \ell \dot{\theta} \hat{i}\end{aligned}$$

That is, the stationary vector  $\vec{r}_{P/A}$  and the rotating frame  $\mathcal{B}$  define a new vector, the derivative of  $\vec{r}_{P/A}$  in  $\mathcal{B}$ . This is also  $\vec{v}_{P/B} - \vec{v}_{A/B}$ , the difference between the velocity of P and the velocity of A in the frame  $\mathcal{B}$ . This new vector can be expressed in any coordinate system of choice for example the  $\hat{i} \hat{j}$  system. So we wrote above

$$\frac{\mathcal{B} d\vec{r}_{P/A}}{dt} = \ell \dot{\theta} \hat{i}$$

which looks mixed up but isn't. The frame  $\mathcal{B}$  is used to help define a vector which is then expressed in the coordinates of  $\mathcal{F}$ .  $\square$

**Using the moving-frame derivative to calculate the fixed-frame derivative**

Given a new vector  $\vec{Q}$ , the derivative of  $\vec{Q}$  as calculated in a rotating frame  $\mathcal{C}$ , one calculation of common use is the determination of the derivative  $\vec{Q}$  of the same vector in the fixed frame  $\mathcal{F}$ .

First think of a line segment that is marked between two points that are glued to a moving frame  $\mathcal{C}$ . We know (at least in 2-D and for fixed axis rotation) that

$$\vec{v}_{B/A} = \vec{\omega}_{\mathcal{C}} \times \vec{r}_{B/A}.$$

Likewise for any vector which is fixed in  $\mathcal{C}$ . It is especially useful to apply this formula to unit base vectors, so

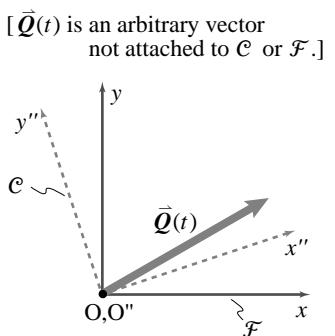


Figure 9.23: A fixed frame  $\mathcal{F}$  defined by  $Oxyz$  and a moving frame  $\mathcal{C}$  defined by  $O'x'y'z'$ . Also shown is an arbitrary vector  $\vec{Q}$  which changes relative to both frames.

(Filename:tfigure8.qdot1)

$$\begin{aligned}\dot{\hat{i}}'' &= \vec{\omega}_{\mathcal{C}} \times \hat{i}'', \\ \dot{\hat{j}}'' &= \vec{\omega}_{\mathcal{C}} \times \hat{j}'', \quad \text{and} \\ \dot{\hat{k}}'' &= \vec{\omega}_{\mathcal{C}} \times \hat{k}''.\end{aligned}\tag{9.22}$$

In some minds, Eqns. 9.22 are the core of rigid body kinematics. The two page box at the end of the section shows how these relations give ‘the Q dot’ formula: For any time dependent vector  $\vec{Q}$

$$\overset{\mathcal{F}}{\dot{\vec{Q}}} = \overset{\mathcal{C}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{C}/\mathcal{F}} \times \vec{Q}. \quad (9.23)$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}.$$

where  $\dot{\vec{Q}}_{rel}$  is the time derivative of  $\vec{Q}$  relative to the moving frame of interest (in this case  $\mathcal{C}$ ). This formula says that

*the derivative of a vector with respect to a Newtonian frame  $\mathcal{F}$  (or ‘absolute derivative’) can be calculated as the derivative  $\overset{\mathcal{C}}{\dot{\vec{Q}}}$  of the vector with respect to a moving frame  $\mathcal{C}$ , plus a term that corrects for the rotation of frame  $\mathcal{C}$  relative to frame  $\mathcal{F}$ ,  $\vec{\omega}_{\mathcal{C}} \times \vec{Q}$ .*

Note that if  $\vec{Q}$  is a constant in the frame  $\mathcal{C}$ , like the relative position vector of two points glued to  $\mathcal{C}$ , then  $\overset{\mathcal{C}}{\dot{\vec{Q}}} = \vec{Q}_{rel} = \vec{0}$  and the  $\dot{\vec{Q}}$  formula reduces to

$$\overset{\mathcal{F}}{\dot{\vec{Q}}} = \vec{\omega} \times \vec{Q}.$$

The  $\dot{\vec{Q}}$  formula 9.23 is useful for the derivation of a variety of formulas and is also useful in the solution of problems.

While we have shown how to use this formula to calculate the rate of change of a vector with respect to a Newtonian frame, the formula can be used to calculate its rate of change with respect to a non-Newtonian frame. Letting  $\mathcal{A}$  and  $\mathcal{B}$  be two possibly non-Newtonian frames, the  $\dot{\vec{Q}}$  formula for the rate of change of  $\vec{Q}$  with respect to frame  $\mathcal{A}$  is

$$\overset{\mathcal{A}}{\dot{\vec{Q}}} = \overset{\mathcal{B}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{Q}. \quad (9.24)$$

Both  $\mathcal{A}$  and  $\mathcal{B}$  could be non-Fixed (non-Newtonian).

## Summary of the $\dot{\vec{Q}}$ formula

For a vector  $\vec{Q}$  fixed in  $\mathcal{B}$ ,

$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}} \times \vec{Q} \quad \text{or} \quad \overset{\mathcal{F}}{\dot{\vec{Q}}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

For any time dependent vector  $\vec{Q}$ ,

$$\dot{\vec{Q}} = \overset{\mathcal{B}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{B}} \times \vec{Q} \quad \text{or} \quad \overset{\mathcal{F}}{\dot{\vec{Q}}} = \overset{\mathcal{B}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

Some examples of applying the  $\dot{\vec{Q}}$  formula are:

$$\begin{aligned}\dot{\vec{r}}_{P/O'} &= \dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} && \text{(absolute velocity of a point } P \text{ relative to } O') \\ \dot{\vec{i}}' &= \underbrace{\dot{\vec{i}}'_{rel}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{i}' && \text{(rate of change of a rotating unit vector in } \mathcal{B})\end{aligned}$$

## The varying base vectors method of computing velocity and acceleration

One way to calculate velocity, acceleration is to express the position of a particle in terms of a combination of based vectors, some of which change in time. Velocity and acceleration are then determined by directly differentiating the expression for position, taking account that the base vectors themselves are changing. This method is sometimes convenient for bodies connected in series, one body to the next, etc. The overall approach is as follows:

- 1) Glue a coordinate system to every moving body. If needed, also create moving frames that move independently of any particular body.
- 2) Call the basis vectors associated with these frames  $\hat{i}, \hat{j}, \hat{k}$  for the fixed frame  $\mathcal{F}$ ;  $\hat{i}', \hat{j}', \hat{k}'$  for the moving frame  $\mathcal{B}$ ; and  $\hat{i}'', \hat{j}'', \hat{k}''$  for the moving frame  $\mathcal{C}$ , etc.
- 3) Evaluate all of the relative angular velocities;  $\vec{\omega}_{\mathcal{B}/\mathcal{F}}, \vec{\omega}_{\mathcal{C}/\mathcal{B}}$ , etc. in terms of the scalar angular rates  $\dot{\theta}, \dot{\phi}$ , etc. and the base vectors glued to the frames.
- 4) Express all of the absolute angular velocities in terms of the relative angular velocities.
- 5) Differentiate to get the angular accelerations using, for example,

$$\begin{aligned}\dot{\vec{i}}' &= \vec{\omega}_{\mathcal{B}} \times \vec{i}' \text{ or} \\ \dot{\vec{j}}'' &= \vec{\omega}_{\mathcal{C}} \times \vec{j}''\end{aligned}$$

- 6) Write the position of all points of interest in terms of the various base vectors.
- 7) Differentiate the position to get the velocities (again using  $\dot{\vec{j}}'' = \vec{\omega}_{\mathcal{C}} \times \vec{j}''$ , etc.)
- 8) Differentiate again to get acceleration.

First, reconsider the bug crawling on the tire in figure 9.24.

*Example: Absolute velocity of a point moving relative to a moving frame: Bug crawling on a tire*

We write the position of the bug in terms of the various basis vectors as

$$\begin{aligned}\vec{r}_{P/O} &= \vec{r}_{O'/O} + \vec{r}_{P/O'} \\ &= \underbrace{\hat{d}\hat{i}}_{\vec{r}_{O'/O}} + \underbrace{R\hat{j}}_{\vec{r}_{P/O'}} + \underbrace{s\hat{i}'}_{\vec{r}_{P/O'}} + \underbrace{\ell\hat{j}'}_{\vec{r}_{P/O'}}.\end{aligned}$$

To get the absolute velocity  $\frac{d}{dt}(\vec{r}_{P/O})$  of the bug at the instant shown, we differentiate the position of the bug once, using the product rule and the rates of change of the rotating basis vectors with respect to the fixed frame, to get

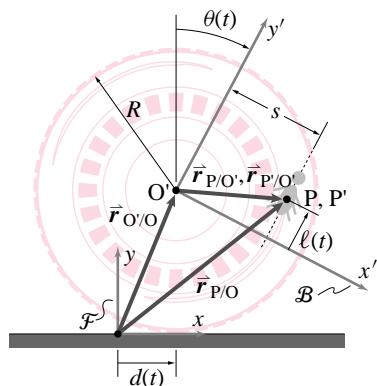


Figure 9.24: The  $xyz$  coordinate system is attached to the fixed frame with basis vectors  $\hat{i}, \hat{j}, \hat{k}$  and the  $x'y'z'$  coordinate system is attached to the rolling tire with basis vectors  $\hat{i}', \hat{j}', \hat{k}'$ . The absolute angular velocity of the tire is  $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}} = -\dot{\theta}\hat{k} = -\dot{\theta}\hat{k}'$ .

(Filename:tfigure8.alt.meth1)

$$\begin{aligned}
\dot{\vec{r}}_{P/O} = \dot{\vec{v}}_P &= \dot{d}\hat{i} + \underbrace{\dot{R}}_0 \hat{j} + \underbrace{\dot{s}}_0 \hat{i}' + s\dot{i}' + \dot{\ell}\hat{j}' + \ell\dot{\hat{j}'} \\
&= \dot{d}\hat{i} + s(\vec{\omega}_{\mathcal{B}} \times \hat{i}') + \dot{\ell}\hat{j}' + \ell(\vec{\omega}_{\mathcal{B}} \times \hat{j}') \\
&= \dot{d}\hat{i} + s(-\dot{\theta}\hat{k}' \times \hat{i}') + \dot{\ell}\hat{j}' + \ell(-\dot{\theta}\hat{k}' \times \hat{j}') \\
&= \dot{d}\hat{i} + \ell\dot{\theta}\hat{i}' + (\dot{\ell} - s\dot{\theta})\hat{j}'.
\end{aligned} \tag{9.25}$$

□

**Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

Differentiating equation 9.25 from the example above again, we get the absolute acceleration  $\frac{d^2}{dt^2}(\vec{r}_{P/O}) = \frac{d}{dt}(\vec{v}_P)$  of the bug at the instant shown,

$$\begin{aligned}
\ddot{\vec{r}}_{P/O} = \ddot{\vec{v}}_P = \vec{a}_P &= \ddot{d}\hat{i} + (\ell\ddot{\theta} + \dot{\ell}\dot{\theta})\hat{i}' + \ell\dot{\theta}(-\dot{\theta}\hat{k}' \times \hat{i}') \\
&\quad + (\ddot{\ell} - s\ddot{\theta} - \underbrace{\dot{s}\dot{\theta}}_0)\hat{j}' + (\dot{\ell} - s\dot{\theta})(-\dot{\theta}\hat{k}' \times \hat{j}') \\
&= \ddot{d}\hat{i} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\ell}\dot{\theta})\hat{i}' + (\ddot{\ell} - \ell\dot{\theta}^2 - s\ddot{\theta})\hat{j}'.
\end{aligned}$$

□

## Summary of the varying base-vector method

In the varying base vector method, we calculate the velocity of a point by looking at the position as the sum of two position vector, one of which is expressed in the moving base vectors. We then differentiate the position, taking account that the base vectors of the moving frame change with time. In general

$$\begin{aligned}
\vec{v}_P &= \frac{d}{dt} \vec{r}_P \\
&= \frac{d}{dt} [\vec{r}_{O'/O} + \vec{r}_{P/O'}] \\
&= \frac{d}{dt} [(x\hat{i} + y\hat{j} + z\hat{k}) + (x'\hat{i}' + y'\hat{j}' + z'\hat{k}')] \\
&= (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + (\dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}') + \\
&\quad [x'(\vec{\omega}_{\mathcal{B}} \times \hat{i}') + y'(\vec{\omega}_{\mathcal{B}} \times \hat{j}') + z'(\vec{\omega}_{\mathcal{B}} \times \hat{k}')].
\end{aligned}$$

We could calculate  $\vec{a}_P$  similarly using a combination of the product rule of differentiation and the facts that  $\dot{\hat{i}'} = \vec{\omega}_{\mathcal{B}} \times \hat{i}'$ ,  $\dot{\hat{j}'} = \vec{\omega}_{\mathcal{B}} \times \hat{j}'$ , and  $\dot{\hat{k}'} = \vec{\omega}_{\mathcal{B}} \times \hat{k}'$ , and would get a formula with 15 non-zero terms.

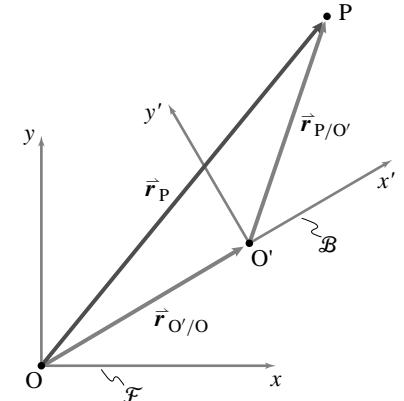


Figure 9.25: The position of a point P relative to the origin of a fixed frame  $\mathcal{F}$  is represented as the sum of two vectors: the position of the new origin relative to the old, and the position of P relative to the new origin. (Here we use “new” as an informal synonym for “moving frame”.)

(Filename: tfigure8.alt.app2)

## 9.1 The $\dot{\vec{Q}}$ formula

We think about some vector  $\vec{Q}$  as a quantity that could be represented by an arrow. We can write  $\vec{Q}$  using the coordinates of the fixed Newtonian frame with base vectors  $\hat{i}, \hat{j}, \hat{k}$ :  $\vec{Q} = Q_x \hat{i} + Q_y \hat{j} + Q_z \hat{k}$ . Similarly we could write  $\vec{Q}$  in terms of the coordinates of some moving and rotating frame  $\mathcal{B}$  with base vectors  $\hat{i}', \hat{j}', \hat{k}'$ :  $\vec{Q} = Q_{x'} \hat{i}' + Q_{y'} \hat{j}' + Q_{z'} \hat{k}'$ . Now of course

$$\vec{Q} = \vec{Q}$$

$$\text{so } Q_x \hat{i} + Q_y \hat{j} + Q_z \hat{k} = Q_{x'} \hat{i}' + Q_{y'} \hat{j}' + Q_{z'} \hat{k}'.$$

Similarly,  $\dot{\vec{Q}} = \vec{Q}$  so long as what we mean by  $\dot{\vec{Q}}$  is its derivative in a fixed frame. That is, we use  $\dot{\vec{Q}}$  as an informal notation for  $\dot{\vec{Q}} = \frac{\mathcal{F}_d \vec{Q}}{dt}$ . We can calculate  $\dot{\vec{Q}}$  the same way we have from the start of the book, namely,

$$\dot{\vec{Q}} = \dot{Q}_x \hat{i} + \dot{Q}_y \hat{j} + \dot{Q}_z \hat{k}.$$

We didn't have to use the product rule of differentiation because the unit vectors  $\hat{i}, \hat{j}$ , and  $\hat{k}$ , associated with a fixed frame, are constant in time.

What if we wanted to use the coordinate information that was given to us by a person who was moving and rotating with the moving frame  $\mathcal{B}$ ? Now we calculate  $\dot{\vec{Q}}$  taking account that the  $\mathcal{B}$  base vectors change in time.

$$\begin{aligned} \dot{\vec{Q}} &= \frac{d}{dt} [Q_{x'} \hat{i}' + Q_{y'} \hat{j}' + Q_{z'} \hat{k}'] \\ &= \underbrace{[\dot{Q}_{x'} \hat{i}' + \dot{Q}_{y'} \hat{j}' + \dot{Q}_{z'} \hat{k}']}_{\dot{\vec{Q}}} \\ &\quad + \underbrace{[Q_{x'} \dot{\hat{i}'} + Q_{y'} \dot{\hat{j}'} + Q_{z'} \dot{\hat{k}'}]}_? \end{aligned} \quad (9.26)$$

The first term in the product rule is just the derivative of  $\vec{Q}$  in the moving frame  $\dot{\vec{Q}}$ . That is,  $\dot{\vec{Q}}$  is calculated by differentiating the components in  $\mathcal{B}$  holding the base vectors in  $\mathcal{B}$  fixed. The second term depends on evaluating  $\dot{\hat{i}'}, \dot{\hat{j}'}$ , and  $\dot{\hat{k}'}$ . We know (at least for 2-D and for fixed-axis rotation in 3-D) that

$$\begin{aligned} \dot{\hat{i}'} &= \vec{\omega}_{\mathcal{B}} \times \hat{i}', \\ \dot{\hat{j}'} &= \vec{\omega}_{\mathcal{B}} \times \hat{j}', \quad \text{and} \\ \dot{\hat{k}'} &= \vec{\omega}_{\mathcal{B}} \times \hat{k}'. \end{aligned} \quad (9.27)$$

Eqns. 9.27 are the core of rigid body kinematics.

Now we can go back to the second group of terms in Eqn. 9.26.

$$\begin{aligned} ? &= [Q_{x'} \dot{\hat{i}'} + Q_{y'} \dot{\hat{j}'} + Q_{z'} \dot{\hat{k}'}] \\ &= [Q_{x'} \vec{\omega}_{\mathcal{B}} \times \hat{i}' + Q_{y'} \vec{\omega}_{\mathcal{B}} \times \hat{j}' + Q_{z'} \vec{\omega}_{\mathcal{B}} \times \hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times [Q_{x'} \hat{i}' + Q_{y'} \hat{j}' + Q_{z'} \hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times \vec{Q} \end{aligned}$$

Going back to Eqn. 9.26 we get the desired result:

$$\dot{\vec{Q}} = \dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}. \quad (9.28)$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}. \quad (9.29)$$

## Geometric ‘derivation’ of the $\dot{\vec{Q}}$ formula

Here is a geometrical ‘derivation’ of the  $\dot{\vec{Q}}$  formula in two dimensions. Referring to the figure at right, we look at a vector  $\vec{Q}$  at two successive times. We then look at how  $\vec{Q}$  seems to change in a frame that rotates slightly as  $\vec{Q}$  changes. The picture shows how to account for the difference between the change of  $\vec{Q}$  as perceived by the two different frames.

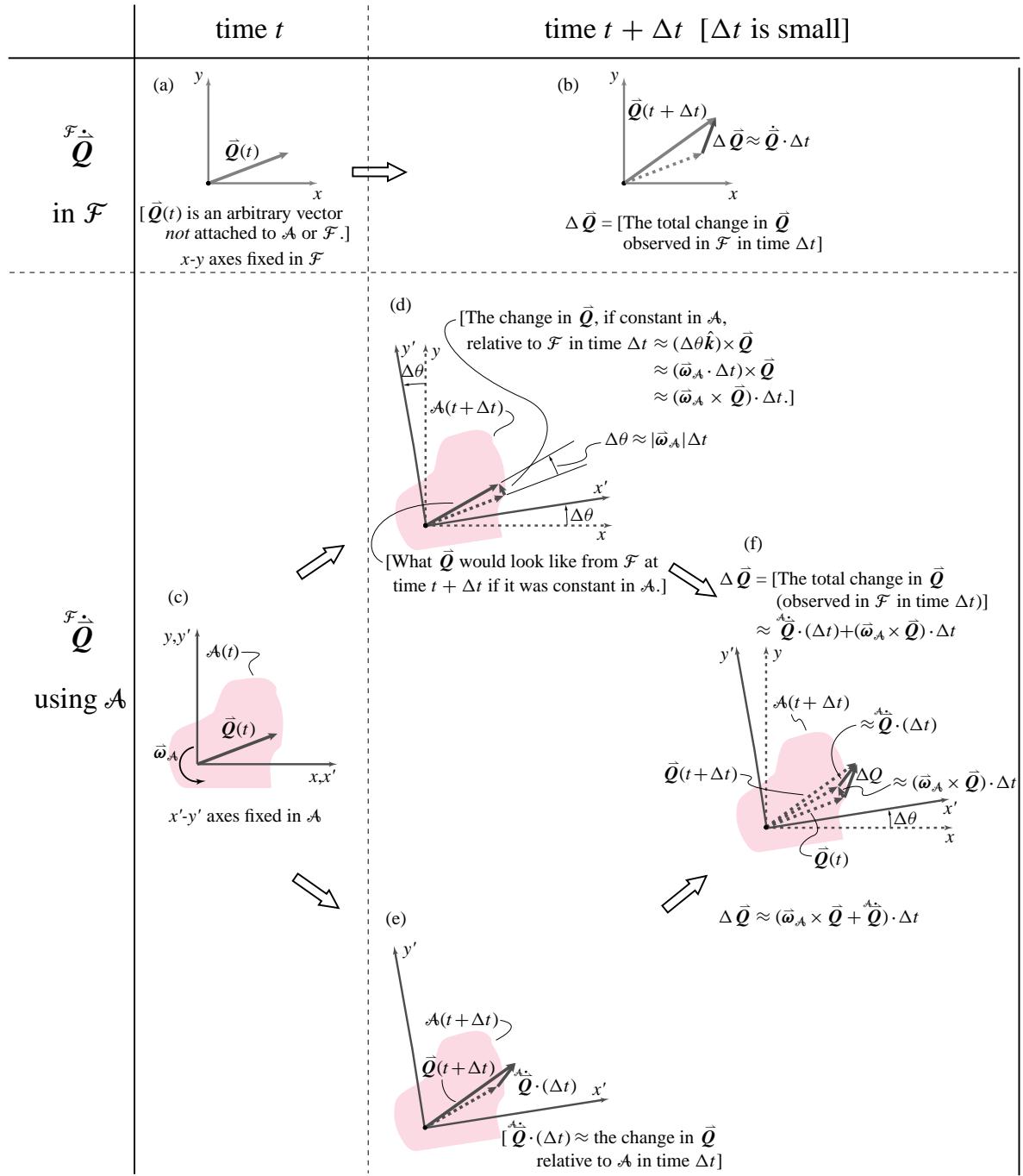
In detail the parts (a) to (e) of the picture show the following.

- Part (a) shows a vector  $\vec{Q}$  at time  $t$ .
- Part (b) shows  $\vec{Q}$  at time  $t + \Delta t$  and the change in  $\vec{Q}$ ,  $\Delta \vec{Q} \approx \dot{\vec{Q}} \cdot \Delta t$ .
- Part (c) is like (a) but shows a moving body or frame  $\mathcal{A}$ .
- Part (d) shows the change in  $\vec{Q}$ ,  $(\vec{\omega}_{\mathcal{A}} \times \vec{Q}) \cdot \Delta t$ , that would occur if  $\vec{Q}$  were fixed (constant) in  $\mathcal{A}$ .
- Part (e) shows the change in  $\vec{Q}$  that would be observed in the moving frame  $\mathcal{A}$ .
- Part (f) shows the net change in  $\vec{Q}$ ,  $\Delta \vec{Q}$ , that is the same as that in (b) above; here, it is shown as the sum of the two contributions from (d) and (e).

Thus, using  $\mathcal{A}, \Delta \vec{Q}$  for small  $\Delta t$  is composed of two parts: (1) the  $\Delta \vec{Q}$  observed in  $\mathcal{A}(t)$ , and (2) the change in  $\vec{Q}$  which would occur if  $\vec{Q}$  were constant in  $\mathcal{A}(t)$  and thus rotating with it. Dividing  $\Delta \vec{Q}$

by  $\Delta t$  gives the ‘ $\dot{\vec{Q}}$  formula’,  $\dot{\vec{Q}} = \dot{\vec{Q}} + \vec{\omega}_{\mathcal{A}} \times \vec{Q}$ .

## 2D Cartoon of the $\dot{\vec{Q}}$ Formula



Two different looks at the change in the vector  $\vec{Q}$ ,  $\Delta \vec{Q}$ , over a time interval  $\Delta t$ .

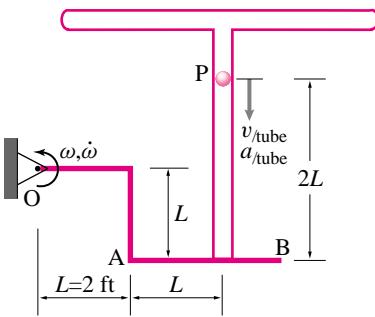


Figure 9.26: (Filename:sfig8.6.1)

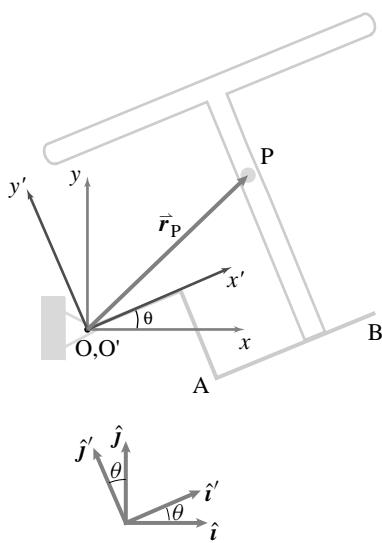


Figure 9.27: (Filename:sfig8.6.1a)

**SAMPLE 9.5 Acceleration of a point moving in a rotating frame.** Consider the rotating tube of Sample 9.9 again. It is given that the arm \$OAB\$ rotates with counterclockwise angular acceleration \$\dot{\omega} = 3 \text{ rad/s}^2\$ and at the instant shown the angular speed \$\omega = 5 \text{ rad/s}\$. Also, at the same instant, the particle \$P\$ is falling down with speed \$v\_{\text{tube}} = 4 \text{ ft/s}\$ and acceleration \$a\_{\text{tube}} = 2 \text{ ft/s}^2\$. Find the absolute acceleration of the particle at the given instant. Take \$L = 2 \text{ ft}\$ in the figure.

**Solution** Let us attach a body frame \$\mathcal{B}\$ to the rigid arm \$OAB\$. For calculations we fix a coordinate system \$x'y'z'\$ in this frame such that the origin \$O'\$ of the coordinate system coincides with \$O\$, and at the given instant, the axes are aligned with the inertial coordinate axes \$xyz\$. Since \$x'y'z'\$ is fixed in the frame \$\mathcal{B}\$ and \$\mathcal{B}\$ rotates with the rigid arm with \$\dot{\omega}\_{\mathcal{B}} = 3 \text{ rad/s}^2 \hat{k}\$ and \$\vec{\omega}\_{\mathcal{B}} = 5 \text{ rad/s} \hat{k}\$, the basis vectors \$\hat{i}'\$, \$\hat{j}'\$ and \$\hat{k}'\$ rotate with the same \$\dot{\omega}\_{\mathcal{B}}\$ and \$\vec{\omega}\_{\mathcal{B}}\$.

In the rotating (primed) coordinate system,

$$\begin{aligned}\vec{r}_P &= x'\hat{i}' + y'\hat{j}' \\ \vec{v}_P &= \frac{d}{dt}(\vec{r}_P) = \frac{d}{dt}(x'\hat{i}' + y'\hat{j}') \\ &= \dot{x}'\hat{i}' + x'\dot{i}' + \dot{y}'\hat{j}' + y'\dot{j}'\end{aligned}$$

Now, we use the \$\dot{\hat{Q}}\$ formula to evaluate \$\dot{i}'\$ and \$\dot{j}'\$, i.e.,

$$\begin{aligned}\dot{i}' &= \vec{\omega}_{\mathcal{B}} \times \hat{i}' = \omega \hat{k}' \times \hat{i}' = \omega \hat{j}' \\ \dot{j}' &= \vec{\omega}_{\mathcal{B}} \times \hat{j}' = \omega \hat{k}' \times \hat{j}' = -\omega \hat{i}'\end{aligned}$$

Also, note that \$x'\$ is constant since in frame \$\mathcal{B}\$, the motion of the particle is always along the tube, i.e., along the negative \$y'\$ axis (see Fig. 9.27). Thus, \$x' = 2L\$, \$\dot{x}' = 0\$, \$y' = L\$, and \$\dot{y}' = -v\_{\text{tube}}\$. Substituting these quantities in \$\vec{v}\_P\$, we get:

$$\begin{aligned}\vec{v}_P &= x'\omega \hat{j}' - v_{\text{tube}} \hat{j}' + y'(-\omega \hat{i}') \\ &= (x'\omega - v_{\text{tube}}) \hat{j}' - \omega y' \hat{i}'\end{aligned}\quad (9.30)$$

Now substituting \$x' = 2L = 4 \text{ ft}\$, \$\omega = 5 \text{ rad/s}\$, \$y' = L = 2 \text{ ft}\$, \$v\_{\text{tube}} = 4 \text{ ft/s}\$ and noting that \$\hat{i}' = \hat{i}\$, \$\hat{j}' = \hat{j}\$ at the given instant, we get:

$$\vec{v}_P = [(20 - 4)\hat{j} - 10\hat{i}] \text{ ft/s} = (-10\hat{i} + 16\hat{j}) \text{ ft/s}$$

We can find \$\vec{a}\_P\$ by differentiating Eq. (9.30) and noting again that \$\hat{i}' = \hat{i}\$, \$\hat{j}' = \hat{j}\$ at the given instant:

$$\begin{aligned}\vec{a}_P &= \frac{d}{dt}(\vec{v}_P) = \frac{d}{dt}[(x'\omega - v_{\text{tube}}) \hat{j}' - \omega y' \hat{i}'] \\ &= (\underbrace{x' \omega + x' \dot{\omega} - \dot{v}_{\text{tube}}}_{0} \hat{j}' + (x'\omega - v_{\text{tube}}) \dot{j}' - (\dot{\omega} y' + \omega \underbrace{\dot{y}'}_{-v_{\text{tube}}} \hat{j}') \hat{i}' - \omega y' \dot{i}') \\ &= (x'\dot{\omega} - a_{\text{tube}}) \hat{j}' + (x'\omega - v_{\text{tube}})(-\omega \hat{i}') - (\dot{\omega} y' - \omega v_{\text{tube}}) \hat{i}' - \omega y'(\omega \hat{j}') \\ &= a_{\text{tube}} \hat{j}' + 2\omega v_{\text{tube}} \hat{i}' - \omega^2(x'\hat{i}' + y'\hat{j}') + \dot{\omega}(x'\hat{j}' - y'\hat{i}') \\ &= -2 \text{ ft/s}^2 \hat{j} + 40 \text{ ft/s}^2 \hat{i} - 25(4\hat{i} + 2\hat{j}) \text{ ft/s}^2 + 3(4\hat{j} - 2\hat{i}) \text{ ft/s}^2 \\ &= -(66\hat{i} + 40\hat{j}) \text{ ft/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_P = -(66\hat{i} + 40\hat{j}) \text{ ft/s}^2}$$

① This problem is the same as Sample 9.9 and has the same solution. Here we use the \$\dot{\hat{Q}}\$ formula on various base vectors instead of using the relative velocity and acceleration formulae.

**SAMPLE 9.6** *Rate of change of unit vectors.* A circular disk  $\mathcal{D}$  is welded to a rigid rod AB. The rod rotates about point A with angular velocity  $\vec{\omega} = \omega \hat{k}$ . A frame  $\mathcal{B}$  is attached to the disk and therefore rotates with the same  $\vec{\omega}$ . Two coordinate systems,  $(\hat{i}', \hat{j}')$  and  $(\hat{e}_R, \hat{e}_\theta)$  are fixed in frame  $\mathcal{B}$  as shown in the figure.

- Find the rate of change of unit vectors  $\hat{e}_R$ ,  $\hat{e}_\theta$ ,  $\hat{i}'$  and  $\hat{j}'$  using the  $\dot{\vec{Q}}$  formula.
- Express the  $\hat{e}_R$  and  $\hat{e}_\theta$  vectors in terms of  $\hat{i}'$  and  $\hat{j}'$  and verify the results obtained above for  $\dot{\hat{e}}_R$  and  $\dot{\hat{e}}_\theta$  by direct differentiation.

**Solution** Since the disk is welded to the rod and frame  $\mathcal{B}$  is fixed in the disk, the frame rotates with  $\vec{\omega}_{\mathcal{B}} = \omega \hat{k}$ .

- To find the rate of change of the unit vectors using the  $\dot{\vec{Q}}$  formula, we substitute the desired unit vector in place of  $\vec{Q}$  in the formula (Eqn 8.12 of the Text). For example,

$$\dot{\hat{e}}_R = {}^{\mathcal{B}}\dot{\hat{e}}_R + \vec{\omega}_{\mathcal{B}} \times \hat{e}_R.$$

It should be clear that  ${}^{\mathcal{B}}\dot{\hat{e}}_R = 0$ , since  $\hat{e}_R$  does not change with respect to an observer sitting in frame  $\mathcal{B}$ . Therefore,

$$\dot{\hat{e}}_R = \vec{\omega}_{\mathcal{B}} \times \hat{e}_R = \omega \hat{k} \times \hat{e}_R = \omega \hat{e}_\theta.$$

Similarly,

$$\begin{aligned}\dot{\hat{e}}_\theta &= \underbrace{{}^{\mathcal{B}}\dot{\hat{e}}_\theta}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \hat{e}_\theta = \omega \hat{k} \times \hat{e}_\theta = -\omega \hat{e}_R. \\ \dot{\hat{i}'} &= \underbrace{{}^{\mathcal{B}}\dot{\hat{i}'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \hat{i}' = \omega \hat{k} \times \hat{i}' = \omega \hat{j}'. \\ \dot{\hat{j}'} &= \underbrace{{}^{\mathcal{B}}\dot{\hat{j}'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \hat{j}' = \omega \hat{k} \times \hat{j}' = -\omega \hat{i}'.\end{aligned}$$

- Since  $\hat{e}_R = \cos \theta \hat{i}' + \sin \theta \hat{j}'$  and  $\hat{e}_\theta = -\sin \theta \hat{i}' + \cos \theta \hat{j}'$ , we get their rates of change by direct differentiation as

$$\begin{aligned}\dot{\hat{e}}_R &= \cos \theta \dot{\hat{i}'} + \sin \theta \dot{\hat{j}'} \\ &= \cos \theta (\omega \hat{j}') + \sin \theta (-\omega \hat{i}') \\ &= \omega (-\sin \theta \hat{i}' + \cos \theta \hat{j}') = \omega \hat{e}_\theta, \\ \dot{\hat{e}}_\theta &= -\sin \theta \dot{\hat{i}'} + \cos \theta \dot{\hat{j}'} \\ &= -\sin \theta (\omega \hat{j}') + \cos \theta (-\omega \hat{i}') \\ &= -\omega (\cos \theta \hat{i}' + \sin \theta \hat{j}') = -\omega \hat{e}_R.\end{aligned}$$

Here we have used the fact that  $\theta$ , the angle between the unit vectors  $\hat{e}_R$  and  $\hat{i}'$ , remains constant during the motion. The results obtained are the same as in part (a).  $\triangleleft$

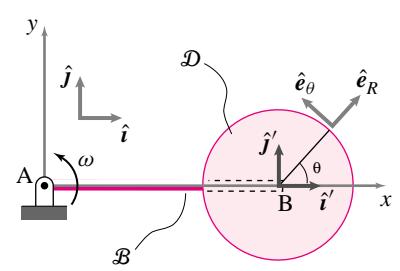


Figure 9.28: (Filename:sfig8.qdot.1)

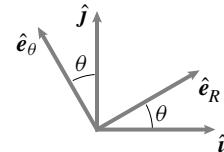


Figure 9.29: (Filename:sfig8.qdot.1a)

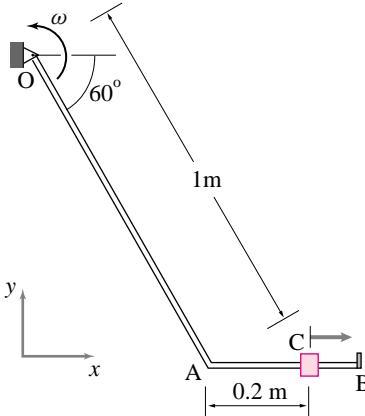


Figure 9.30: (Filename:sfig8.qdot.2)

**SAMPLE 9.7 Rate of change of a position vector.** A rigid rod OAB rotates counter-clockwise about point O with constant angular speed  $\omega = 5 \text{ rad/s}$ . A collar C slides out on the bent arm AB with constant speed  $v = 0.5 \text{ m/s}$  with respect to the arm. Find the velocity of the collar using the  $\vec{Q}$  formula.

**Solution** Let  $\vec{r}_C$  be the position vector of the collar. Then the velocity of the collar is  $\dot{\vec{r}}_C$ . Let the rod OAB be the rotating frame  $\mathcal{B}$ . Now we can find  $\dot{\vec{r}}_C$  using the  $\vec{Q}$  formula:

$$\dot{\vec{r}}_C = {}^{\mathcal{B}}\dot{\vec{r}}_C + \vec{\omega}_{\mathcal{B}} \times \vec{r}_C$$

To compute  $\dot{\vec{r}}_C$ , let us first find  ${}^{\mathcal{B}}\dot{\vec{r}}_C$ , the rate of change of  $\vec{r}_C$  as seen in frame  $\mathcal{B}$  (this term represents the velocity of the collar you see if you sit on the rod and watch the collar; also called  $\vec{v}_{rel}$ ).

$$\begin{aligned}\vec{r}_C &= \vec{r}_A + \vec{r}_{C/A} \\ {}^{\mathcal{B}}\dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_A + {}^{\mathcal{B}}\dot{\vec{r}}_{C/A}\end{aligned}$$

Note that the vector  $\vec{r}_A = \ell \hat{\lambda}$  does not change in frame  $\mathcal{B}$  since both its magnitude,  $\ell$ , and direction,  $\hat{\lambda}$ , remain fixed in  $\mathcal{B}$ . Therefore,

$${}^{\mathcal{B}}\dot{\vec{r}}_A = 0$$

$$\text{Now } \vec{r}_{C/A} = \vec{r}' \Rightarrow {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} = r \vec{i}' = 0.5 \text{ m/s} \vec{i}'$$

because  $\vec{i}'$  does not change in  $\mathcal{B}$  and  $r = \text{speed of the collar with respect to the arm}$ . (see Figure 9.31) Thus,

$${}^{\mathcal{B}}\dot{\vec{r}}_C = 0.5 \text{ m/s} \vec{i}'.$$

Hence,

$$\begin{aligned}\dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega \hat{k} \times (\ell \hat{\lambda} + r \vec{i}') \\ &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega L (\hat{k} \times \hat{\lambda}) + \omega r (\hat{k} \times \vec{i}') \\ &= \dot{r} \vec{i}' + \omega \ell (\sin \theta \hat{i} + \cos \theta \hat{j}) + \omega r \hat{j}' \\ &= 0.5 \text{ m/s} \vec{i}' + 5 \text{ m/s} \left( \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \right) + 1 \text{ m/s} \hat{j}' \\ &= 4.83 \text{ m/s} \hat{i} + 3.5 \text{ m/s} \hat{j}\end{aligned}$$

where we have used the fact that at the given instant,  $\vec{i}' = \hat{i}$  and  $\hat{j}' = \hat{j}$ .

$$\boxed{\vec{v}_C = 4.83 \text{ m/s} \hat{i} + 3.5 \text{ m/s} \hat{j}}$$

## 9.3 General expressions for velocity and acceleration

Now that we have some comfort with moving frames we can develop formulas that are not so strongly attached to base vectors. That is we take account that the base vectors rotate with the frame, but develop formulas that don't use the base vectors explicitly. Thus the formulas we develop here work equally for any frame that is glued to the rotating frame of choice, independent of its orientation.

### Absolute velocity of a point moving relative to a moving frame

Imagine that you know the absolute velocity of some point  $O'$  on a body  $\mathcal{B}$ , say the center of a car, tire and the angular velocity of the body,  $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$ . Finally, imagine you also know the relative velocity of point  $P$ ,  $\vec{v}_{P/\mathcal{B}}$ , say of a bug crawling on the tire.

If the frame  $\mathcal{B}$  is translating or rotating, the velocity of particle  $P$  relative to the frame  $\vec{v}_{P/\mathcal{B}}$  is not the absolute velocity (the velocity relative to a Newtonian frame). The absolute velocity in this case is  $\vec{v}_{P/\mathcal{F}}$ , or more simply  $\vec{v}_P$ , or more simply still, just  $\vec{v}$ . The relationship between the absolute velocity  $\vec{v}_{P/\mathcal{B}}$  and the relative velocity  $\vec{v}_{P/\mathcal{F}} \equiv \vec{v}$  is of interest.

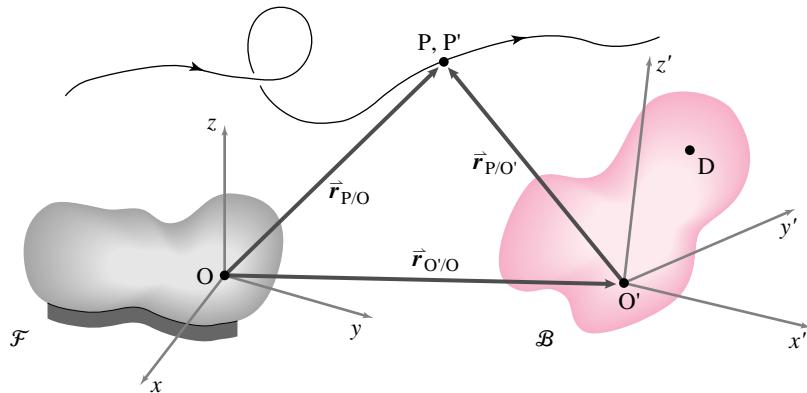


Figure 9.32: The position of point  $P$  relative to the origin  $O'$  of moving frame  $\mathcal{B}$  is  $\vec{r}_{P/O'}$ . The position of the origin of the frame  $\mathcal{B}$  relative to the origin  $O$  of the fixed frame  $\mathcal{F}$  is  $\vec{r}_{O'/O}$ . The position of point  $P$  relative to  $O$  is the sum of  $\vec{r}_{P/O'}$  and  $\vec{r}_{O'/O}$ . The motion of point  $P$  relative to the fixed frame may be complicated.

(Filename:tfigure8.v3)

Let's start by looking at the position. The position of a point  $P$  that is moving is:

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

where  $O'$  is the origin of a coordinate system which is glued to the rigid body, as shown in figure 9.32.

To find the absolute velocity of point  $P$  we will use the  $\dot{\vec{Q}}$  formula, equation 9.23, for computing the rate of change of a vector. The velocity of  $P$  is the rate of change of its position. Here, we use  $\dot{\vec{Q}} = \vec{r}_{P/O'}$

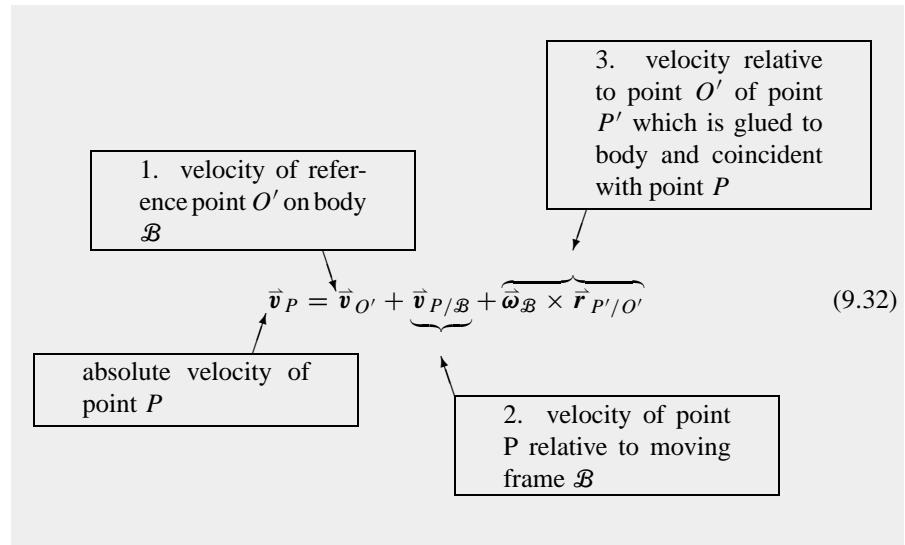
$$\begin{aligned}\vec{v}_{P/\mathcal{F}} &= \frac{\mathcal{F}_{\Delta}}{\mathcal{F}_{\Delta}} \vec{r}_{P/O} \\ &= \frac{\mathcal{F}_{\Delta}}{\mathcal{F}_{\Delta}} \vec{r}_{O'/O} + \frac{\mathcal{F}_{\Delta}}{\mathcal{F}_{\Delta}} \vec{r}_{P/O'}\end{aligned}$$

$$\begin{aligned} \overset{\mathcal{F}}{\dot{\vec{r}}}_{P/O'} &= \overset{\mathcal{F}}{\vec{v}}_{O'/\mathcal{F}} + \underbrace{\overset{\mathcal{B}}{\dot{\vec{r}}}_{P/O'}}_{\overset{\mathcal{B}}{\vec{v}}_{P/\mathcal{B}}} + \overset{\mathcal{B}}{\omega}_{\mathcal{B}/\mathcal{F}} \times \overset{\mathcal{F}}{\vec{r}}_{P'/O'} \end{aligned}$$

The  $\dot{\vec{Q}}$  formula 9.23 was used in the calculation to compute  $\overset{\mathcal{F}}{\dot{\vec{r}}}_{P/O'}$

$$\overset{\mathcal{F}}{\dot{\vec{r}}}_{P/O'} = \overset{\mathcal{B}}{\dot{\vec{r}}}_{P/O'} + \overset{\mathcal{B}}{\omega}_{\mathcal{B}/\mathcal{F}} \times \overset{\mathcal{F}}{\vec{r}}_{P'/O'}. \quad (9.31)$$

Thus, the ‘three term velocity formula’.



Another way to write the formula for absolute velocity is as

$$\overset{\mathcal{F}}{\vec{v}}_P = \overset{\mathcal{F}}{\vec{v}}_{P'} + \overset{\mathcal{B}}{\vec{v}}_{P/\mathcal{B}}$$

where  $P'$  is a point glued to  $\mathcal{B}$  which is instantaneously coincident with  $P$ , so the absolute velocity of  $P'$  is

$$\overset{\mathcal{F}}{\vec{v}}_{P'} = \overset{\mathcal{F}}{\vec{v}}_{O'} + \overset{\mathcal{B}}{\omega}_{\mathcal{B}} \times \overset{\mathcal{F}}{\vec{r}}_{P'/O'}. \quad (9.33)$$

Reconsider the bug crawling on the tire, body  $\mathcal{B}$ , in figure 9.33. To find the absolute velocity of the bug, we need be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

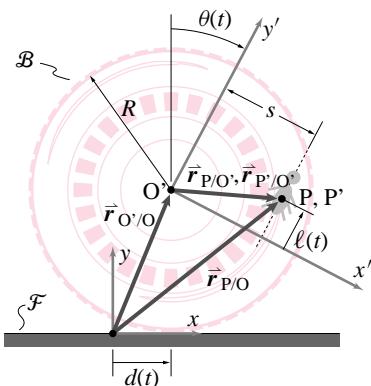


Figure 9.33: (Filename:tfigure8.velabsbug)

#### Example: Absolute velocity of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

Referring to equation 9.31 on page 546, the absolute velocity of the bug is

$$\begin{aligned} \overset{\mathcal{F}}{\vec{v}}_{P/\mathcal{F}} &= \overset{\mathcal{F}}{\vec{v}}_{O'/\mathcal{F}} + \underbrace{\overset{\mathcal{B}}{\dot{\vec{r}}}_{P/O'}}_{\overset{\mathcal{B}}{\vec{v}}_{P/\mathcal{B}}} + \overset{\mathcal{B}}{\omega}_{\mathcal{B}/\mathcal{F}} \times \overset{\mathcal{F}}{\vec{r}}_{P/O'} \\ &= R\dot{\theta}\hat{i} + \dot{\ell}\hat{j}' + (-\dot{\theta}\hat{k}') \times (s\hat{i}' + \ell\hat{j}') \\ &= R\dot{\theta}\hat{i} + \ell\dot{\theta}\hat{i}' + (\dot{\ell} - \dot{\theta})\hat{j}'. \end{aligned}$$

At the instant of interest, the direction of the bug’s absolute velocity depends upon the relative magnitudes of  $\dot{\ell}$  and  $\dot{\theta}$  as well as the orientation of  $\hat{i}'$  and  $\hat{j}'$ .  $\square$

As we noted earlier, another way to write the formula for absolute velocity is

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where, in the example above,  $\vec{v}_{P'} = R\dot{\theta}\hat{i} + \dot{\ell}(\ell\hat{i}' - s\hat{j}')$  and  $\vec{v}_{P/\mathcal{B}} = \dot{\ell}\hat{j}'$ . At the instant of concern, we can think of the absolute velocity of the bug as the velocity of the mark labeled  $P'$  under the bug *plus* the velocity of the bug relative to the tire.

## Acceleration

We would like to find acceleration of a point using information about its motion relative to a moving frame. The result, the ‘five term acceleration formula’ is the most complicated formula in this book. (For reference, it is in Table II, 5c).

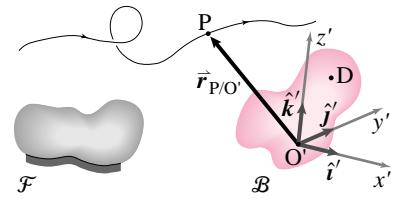


Figure 9.34: A body or reference frame  $\mathcal{B}$  and a point  $P$  move around. We are interested in the acceleration of point  $P$ .

(Filename:tfigure8.a1)

## Acceleration relative to a body or frame

The acceleration of a point relative to a body or frame is the acceleration you would calculate if you were looking at the particle while you translated and rotated with the frame and took no account of the outside world. That is, if the position of a particle  $P$  relative to the origin  $O'$  of a coordinate system in a moving frame  $\mathcal{B}$  is given by:

$$\vec{r}_{P/O'} = \underbrace{r_{Px'/O'}}_{x'} \hat{i}' + \underbrace{r_{Py'/O'}}_{y'} \hat{j}' + \underbrace{r_{Pz'/O'}}_{z'} \hat{k}',$$

then the acceleration of the particle  $P$  relative to the frame is:

$$\vec{a}_{P/\mathcal{B}} = \underbrace{\ddot{r}_{Px'/O'}}_{\ddot{x}'} \hat{i}' + \underbrace{\ddot{r}_{Py'/O'}}_{\ddot{y}'} \hat{j}' + \underbrace{\ddot{r}_{Pz'/O'}}_{\ddot{z}'} \hat{k}'.$$

That is, the acceleration relative to the frame takes no account of (a) the motion of the frame or of (b) the rotation of the base vectors with the frame to which they are fixed.

Reconsider the bug labeled point  $P$  crawling on the tire, body  $\mathcal{B}$ , in figure 9.35.

### Example: Acceleration relative to a frame (2-D): Bug crawling on a tire, again

If we are sitting on the tire, all that we see is the bug crawling in a straight line at non-constant rate relative to us. Thus, its acceleration relative to the tire is

$$\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{j}'.$$

So, at the instant of interest, the bug has an acceleration relative to the tire frame parallel to the  $y'$ -axis.  $\square$

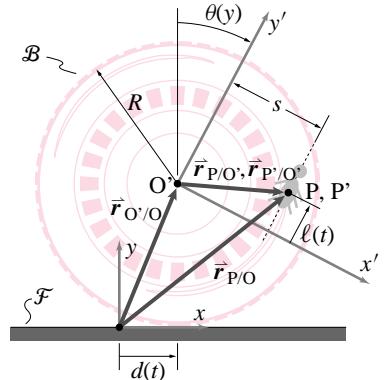


Figure 9.35: A bug crawling on a tire at point  $P$ . Point  $P'$  is on the tire and instantaneously corresponds to point  $P$ .

(Filename:tfigure8.accelrelbug)

## Absolute acceleration of a point $P'$ glued to a moving frame

Imagine that you know the absolute acceleration of some point  $O'$  at the center of a frame  $\mathcal{B}$ , say the center of a car tire. Imagine you also know the angular velocity of the tire,  $\vec{\omega}_{\mathcal{B}/F}$ , and the angular acceleration,  $\vec{\alpha}_{\mathcal{B}/F}$ . Then, you can find the absolute acceleration of a piece of gum labeled point  $D$  stuck to the sidewall (see Fig. 9.37). If we start with the equation 9.33 for the absolute velocity of a point glued to a moving frame on page 546 and differentiate with respect to time, we get the absolute acceleration of a point  $D$  fixed in a moving frame  $\mathcal{B}$  as follows:

$$\begin{aligned}\vec{a}_D &= \frac{d}{dt}[\vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'}] \\ &= \vec{a}_{O'/O} + [\vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'})] \\ &= \vec{a}_{O'/O} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{D/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'})\end{aligned}\quad (9.34)$$

*Example: Absolute acceleration of a point glued to a moving frame  
(2-D): Bug crawling on a tire, again*

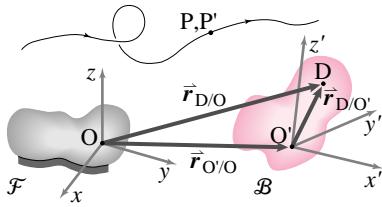


Figure 9.36: Tracking a moving particle from two different reference frames. (A copy of figure ?? on page ??.)

(Filename:tfigure8.a2)

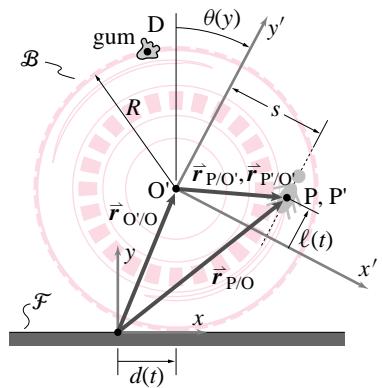


Figure 9.37: Gum is glued to a rolling tire at  $D$ . The point  $P'$  glued to the tire is just where the point  $P$  happens to be passing at the moment of interest.

(Filename:tfigure8.accelgluebug)

Here, the acceleration of point  $P'$  glued to the tire, relative to the tire is zero,  $\vec{a}_{P'/B} = 0$  (see Fig. 9.37). The angular velocity of the wheel with respect to the ground is  $\vec{\omega}_{\mathcal{B}/F} = -\dot{\theta}\hat{k} = -\dot{\theta}\hat{k}'$ . The angular speed is increasing at a rate  $\ddot{\theta}$ . Thus,  $\vec{\alpha}_{\mathcal{B}/F} = -\ddot{\theta}\hat{k} = -\ddot{\theta}\hat{k}'$ . The position of  $P'$  relative to  $O'$  is  $\vec{r}_{P'/O'} = s\hat{i}' + \ell\hat{j}'$ .

Using equation 9.34 on page 548, we get the absolute acceleration of point  $P'$  to be

$$\begin{aligned}\vec{a}_{P'/F} &= \vec{a}_{O'/F} + \vec{\omega}_{\mathcal{B}/F} \times (\vec{\omega}_{\mathcal{B}/F} \times \vec{r}_{P'/O'}) + \vec{\alpha}_{\mathcal{B}/F} \times \vec{r}_{P'/O'} \\ &\quad \boxed{\text{centripetal term}} \qquad \boxed{\text{tangential term}} \\ &= \underbrace{R\ddot{\theta}\hat{i}}_{\text{acceleration of origin of moving frame}} - \overbrace{\dot{\theta}^2(s\hat{i}' + \ell\hat{j}')} + \underbrace{R\ddot{\theta}(s\hat{j}' - \ell\hat{i}')}\end{aligned}$$

In this example, the absolute acceleration  $P'$  is due to:

1. the increase in the translational speed of the tire relative to the ground (acceleration of origin of moving frame),
2. its going in circles at non-constant rate about point  $O'$  relative to the ground ('tangential term'), and
3. 'centripetal term' towards the origin of the moving frame. (In three-dimensional problems, this term is directed towards an axis through  $\vec{\omega}$  that goes through  $O'$ ).

□

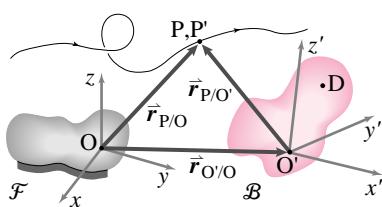


Figure 9.38: Keeping track of  $P$  which instantaneously coincides with point  $P'$  which is glued to frame  $\mathcal{B}$ .

(Filename:tfigure8.a3)

## Absolute acceleration of a point moving relative to a moving frame

If we start with the equation for absolute velocity 9.31 on page 546 and differentiate with respect to time we get the absolute acceleration of a point  $P$  using a moving frame

$\mathcal{B}$ . To do this calculation we need to use the product rule of differentiation. Refer to the  $\dot{\mathcal{Q}}$  formula, equation 9.28 on page 540 in section 9.2. Here is the calculation:

$$\begin{aligned}\vec{a}_P &= \frac{d}{dt} [\vec{v}_{O'/O} + \vec{v}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}] \\ &= \vec{a}_{O'/O} + (\vec{a}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}) \\ &\quad + [\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'})] \\ &= \underbrace{\vec{a}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'})}_{\vec{a}_{P'}} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}.\end{aligned}$$

The collection of terms  $\vec{a}_{P'}$  is the acceleration of a point  $P'$  which is glued to body  $\mathcal{B}$  and is instantaneously coincident with  $P$ . It is the same as  $\vec{a}_D$  using  $D = P'$  in equation 9.34. To repeat<sup>①</sup>, the result is

1. acceleration of reference point  $O'$  on body  $\mathcal{B}$

4. acceleration of  $P$  relative to moving frame  $\mathcal{B}$

$$\vec{a}_P = \overbrace{\vec{a}_{O'/O}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'})}_{\text{2. centripetal acceleration}} + \underbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'}}_{\text{3. tangential acceleration}} + \overbrace{\vec{a}_{P/\mathcal{B}}} + 2\overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}}_{\text{5. Coriolis accel.}}$$

2. centripetal acceleration: acceleration relative to  $O'$  of point glued to body, coincident with  $P$ , and going in circles around  $O'$  at constant rate

3. tangential acceleration: acceleration relative to  $O'$  of point glued to body, coincident with  $P$ , due to non-constant  $\vec{\omega}$

(9.35)

(9.36)

① True story. Young Professor X was excitedly teaching this material and got to eqn. (9.35) which he wrote in big letters on a fresh clean black board, with explanations in boxes. He then explained to the class that this was what they had been building up to, that this was the *climax* of the course (Young professors have different priorities than older professors, by the way.) There was still silence in the class. A student broke the quiet and gently asked “Was it as good for you as it was for us?”

eqn. (9.35) is the famous and infamous ‘five-term-acceleration’ formula. Famous because it is given a lot of emphasis by some instructors. Infamous because it takes some getting used to. Eqn. 9.36 is the three term acceleration formula. It combines the first three terms in the 5-term formula and interprets them as the acceleration of the point  $P'$  glued to the moving frame at the same point  $P$  now occupies.

The first three terms are acceleration of a point  $P'$  which is fixed relative to  $\mathcal{B}$ . One way to get used to this formula is to find situations where various of the terms drop out.

Reconsider the bug labeled point  $P$  crawling on the tire, body  $\mathcal{B}$ , in figure 9.39. To find the absolute acceleration of the bug we need to think about how the bug moves relative to the tire and how the tire moves relative to the ground.

**Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

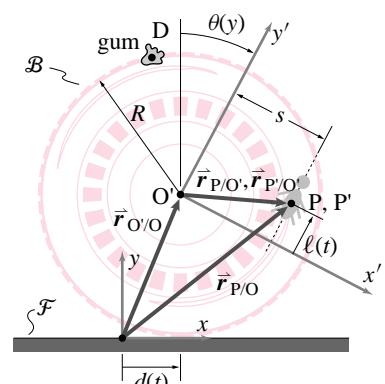


Figure 9.39: (Filename:tfigure8.accelabsbug)

From the previous bug examples on page 546 and page 547 we know that

$$\vec{v}_{P/\mathcal{B}} = \dot{\ell} \hat{j}', \text{ and } \vec{a}_{P/\mathcal{B}} = \ddot{\ell} \hat{j}'.$$

Referring to the five term acceleration formula, equation 9.35 on page 549, the absolute acceleration of the bug is

$$\begin{aligned}\vec{a}_{P/\mathcal{F}} &= \vec{a}_{O'/O} \\ &\quad + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}) \\ &\quad + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} \\ &\quad + \vec{a}_{P/\mathcal{B}} \\ &\quad + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\ &= \underbrace{R\ddot{\theta}\hat{i} - \dot{\theta}^2(s\hat{i}' + \ell\hat{j}') + \ddot{\theta}(\ell\hat{i}' - s\hat{j}')}_{\vec{a}_{P'/\mathcal{F}}} + \ddot{\ell}\hat{j}' + 2\dot{\theta}\dot{\ell}\hat{i}' \\ &= R\ddot{\theta}\hat{i} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\theta}\dot{\ell})\hat{i}' + (-s\ddot{\theta} + \ddot{\ell} - \ell\dot{\theta}^2)\hat{j}'.\end{aligned}$$

So, at the instant of interest, the bug's absolute acceleration is due to:

1. the translational acceleration of the tire,  $\vec{a}_{O'/O} = R\ddot{\theta}\hat{i}$
2. the centripetal acceleration of going in circles of radius  $\sqrt{s^2 + \ell^2}$  about the center of the tire as it rolls,  $-\dot{\theta}^2(s\hat{i}' + \ell\hat{j}')$ , pointing at the center of the tire,
3. the tangential acceleration of going in circles about the center of the tire as the tire rolls at non-constant rate,  $\ddot{\theta}(\ell\hat{i}' - s\hat{j}')$ ,
4. the acceleration of the bug relative to the tire as it crawls on the line,  $\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{j}'$ , and
5. the Coriolis acceleration caused, in part, by the change in direction, relative to the ground, of the velocity of the bug relative to the tire,  $2\dot{\theta}\dot{\ell}\hat{i}'$ .

Items 1, 2 and 3 sum to be the acceleration of point  $P'$  on the tire but instantaneously coinciding with moving point P.  $\square$

## Motion relative to a point versus motion relative to a frame

We can now give a different interpretation of the expressions we have been using  $\vec{v}_{B/A}$  and  $\vec{a}_{B/A}$ . Rather than thinking of  $\vec{v}_{B/A}$  as the difference between  $\vec{v}_B$  and  $\vec{v}_A$  we can think of  $\vec{v}_{B/A}$  as the  $\vec{v}_{B/\mathcal{A}}$  where  $\mathcal{A}$  is a frame with origin that moves with point A and which has no rotation rate relative to  $\mathcal{F}$ . That is

$$\vec{v}_{B/A} \text{ means } \vec{v}_{B/\mathcal{A}}$$

Similarly,

$$\vec{a}_{B/A} \text{ means } \vec{a}_{B/\mathcal{A}}.$$

## 9.2 THEORY

### Relation between moving frame formulae and polar coordinate formulae

formula, we get

A similarity exists between the polar coordinate velocity formula

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

and the second two terms in the ‘three-term’ velocity formula

$$\vec{v}_P = \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}}.$$

In fact, we have tried to build your understanding of moving frames by means of that connection.

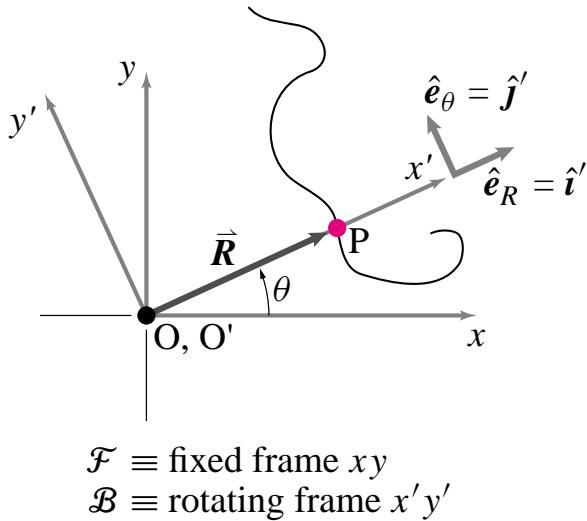
Similarly, the polar coordinate formula for acceleration

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$$

is somehow closely linked to the last four terms of the ‘5-term’ acceleration formula

$$\begin{aligned} \vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \vec{\omega} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}. \end{aligned}$$

Let’s make these connections explicit. Imagine a particle  $P$  moving around on the  $xy$ -plane.



Let’s create a moving frame  $\mathcal{B}$  with rotating coordinate system  $x'y'$  attached to it whose origin  $O'$  is coincident with origin  $O$  of a coordinate system  $xy$  attached to a fixed frame  $\mathcal{F}$ . Let this frame rotate in exactly such a way so that the particle is always on the  $x'$ -axis. So, in this frame,  $\vec{r}_{P/O'} = R\hat{i}'$ ,  $\vec{v}_{P/\mathcal{B}} = \dot{R}\hat{i}'$ , and  $\vec{a}_{P/\mathcal{B}} = \ddot{R}\hat{i}'$ . Also, the frame motion is characterized by  $\vec{v}_{O'/O} = \vec{0}$ ,  $\vec{a}_{O'/O} = \vec{0}$ ,  $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{k}$ , and  $\vec{\alpha}_{\mathcal{B}} = \ddot{\theta}\hat{k}$ . So, if we plug in the three-term velocity

$$\begin{aligned} \vec{v}_P &= \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + (\dot{\theta}\hat{k}) \times (R\hat{i}') + \dot{R}\hat{i}' \\ &= R\dot{\theta}\hat{j}' + \dot{R}\hat{i}' \\ &= R\dot{\theta}\hat{e}_\theta + \dot{R}\hat{e}_r \quad (\text{since } \hat{i}' \parallel \hat{e}_r \text{ and } \hat{j}' \parallel \hat{e}_\theta) \end{aligned}$$

which is the polar coordinate velocity formula.

Similarly, if we plug into the five-term acceleration formula, we get

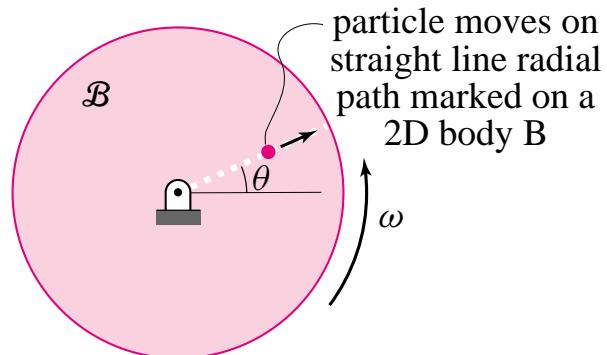
$$\begin{aligned} \vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \vec{\omega} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + \ddot{R}\hat{i}' + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times R\hat{i}') \\ &\quad + (\ddot{\theta}\hat{k}) \times (R\hat{i}') + 2(\dot{\theta}\hat{k}) \times \dot{R}\hat{i}' \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \end{aligned}$$

Again, we recover the appropriate polar coordinate formula.

We have just shown how the polar coordinate formulae are special cases of the relative motion formulae.

### Warning!

In problems where we want the rotating frame to be a rotating body on which a particle moves, the polar coordinate formulae only correspond term by term with the relative motion formulae if the particle path is a straight radial line fixed on a 2D body, as in the example of a bug walking on a straight line scribed on the surface of a rotating CD or a bead sliding in a tube rotating about an axis perpendicular to the tube.



Since we are sometimes interested in more general relative motions, the polar coordinate formulae do not always apply and we must make use of the more general relative motion formulae.

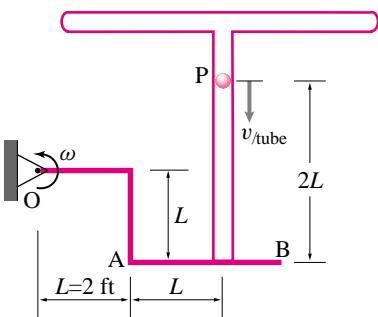


Figure 9.40: (Filename:sfig8.1.1a)

**SAMPLE 9.8** A ‘T’ shaped tube is welded to a massless rigid arm OAB which rotates about O at a constant rate  $\omega = 5 \text{ rad/s}$ . At the instant shown a particle P is falling down in the vertical section of the tube with speed  $v_{\text{tube}} = 4 \text{ ft/s}$ . Find the absolute velocity of the particle. Take  $L = 2 \text{ ft}$  in the figure.

**Solution** Let us attach a frame  $\mathcal{B}$  to arm OAB. Thus  $\mathcal{B}$  rotates with OAB with angular velocity  $\vec{\omega}_{\mathcal{B}} = \omega \hat{k}$  where  $\omega = 5 \text{ rad/s}$ . To do calculations in  $\mathcal{B}$  we attach a coordinate system  $x'y'z'$  to  $\mathcal{B}$  at point O. At the instant of interest the rotating coordinate system  $x'y'z'$  coincides with the fixed coordinate system  $xyz$ . (Since the entire motion is in the  $xy$ -plane, the  $z$ -axis is not shown in the figure). Let  $P'$  be a point coincident with  $P$  but fixed in  $\mathcal{B}$ . Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}$$

where

$$\begin{aligned}\vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} \\ &= \omega \hat{k} \times (2L\hat{i} + L\hat{j}) \\ &= 2\omega L\hat{j} - \omega L\hat{i},\end{aligned}$$

and

$$\begin{aligned}\vec{v}_{\text{rel}} &= \text{Velocity relative to the frame } \mathcal{B} \\ &= -v_{\text{tube}}\hat{j}.\end{aligned}$$

Thus,

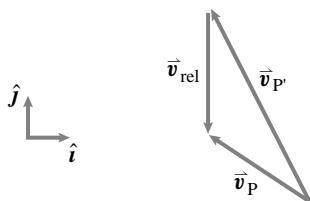


Figure 9.41: (Filename:sfig8.1.1b)

$$\begin{aligned}\vec{v}_P &= \vec{v}_{P'} + \vec{v}_{\text{rel}} \\ &= -\omega L\hat{i} + (2\omega L - v_{\text{tube}})\hat{j} \\ &= -5 \text{ rad/s} \cdot 2 \text{ ft} \hat{i} + (2 \cdot 5 \text{ rad/s} \cdot 2 \text{ ft} - 4 \text{ ft/s})\hat{j} \\ &= -10 \text{ ft/s} \hat{i} + 16 \text{ ft/s} \hat{j}.\end{aligned}$$

$$\boxed{\vec{v}_P = -10 \text{ ft/s} \hat{i} + 16 \text{ ft/s} \hat{j}}$$

**Comments:** The kinematics calculation is equivalent to the vector addition shown in Figure 9.41. The velocity of P is the sum of  $\vec{v}_{P'}$  and  $\vec{v}_{\text{rel}} = \vec{v}_{P/\mathcal{B}}$ .

**SAMPLE 9.9** Acceleration of a point in a rotating frame. Consider the rotating tube of Sample 9.8 again. The arm OAB rotates with counterclockwise angular acceleration  $\dot{\omega} = 3 \text{ rad/s}^2$  and, at the instant shown, its angular speed  $\omega = 5 \text{ rad/s}$ . Also, at the same instant, the particle P falls down with speed  $v_{\text{tube}} = 4 \text{ ft/s}$  and acceleration  $a_{\text{tube}} = 2 \text{ ft/s}^2$ . Find the absolute acceleration of the particle at the given instant. Take  $L = 2 \text{ ft}$  in the figure.

**Solution** We consider a frame  $\mathcal{B}$ , with coordinate axes  $x'y'z'$ , fixed to the arm OAB and thus rotating with  $\vec{\omega}_{\mathcal{B}} = \omega \hat{k} = 5 \text{ rad/s} \hat{k}$  and  $\vec{\alpha}_{\mathcal{B}} = \dot{\omega} \hat{k} = 3 \text{ rad/s}^2 \hat{k}$ . The acceleration of point P is given by

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}}$$

where

- $\vec{a}_{P'} =$  acceleration of a point  $P'$  that is fixed in  $\mathcal{B}$  and at the moment coincides with P,
- $\vec{a}_{\text{cor}} =$  Coriolis acceleration, and
- $\vec{a}_{\text{rel}} =$  acceleration of P relative to frame  $\mathcal{B}$ .

Now we calculate each of these terms separately. For calculating  $\vec{a}_{P'}$ , imagine a rigid rod from point O to point P, rotating with the frame  $\mathcal{B}$ . Mark the far end of the rod as  $P'$  (same as point P). The acceleration of this end of the rod is  $\vec{a}_{P'}$ . To find the relative terms  $\vec{v}_{\text{rel}}$  and  $\vec{a}_{\text{rel}}$ , freeze the motion of the frame  $\mathcal{B}$  at the given moment and watch the motion of point P. The non-intuitive term  $\vec{a}_{\text{cor}}$  has no such simple physical interpretation but has a simple formula. Thus,

$$\begin{aligned}\vec{a}_{P'} &= \overbrace{\vec{a}_{O'}}^{\vec{0}} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P'/O'} + \overbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'})}^{-\omega_{\mathcal{B}}^2 \vec{r}_{P'/O'}} \\ &= \dot{\omega} \hat{k} \times (2L\hat{i} + L\hat{j}) - \omega^2(2L\hat{i} + L\hat{j}) \\ &= 2\dot{\omega}L\hat{j} - \dot{\omega}L\hat{i} - 2\omega^2L\hat{i} - \omega^2L\hat{j} \\ &= -L(\dot{\omega} + 2\omega^2)\hat{i} + L(2\dot{\omega} - \omega^2)\hat{j} \\ &= -(106\hat{i} + 38\hat{j}) \text{ ft/s}^2,\end{aligned}$$

$$\begin{aligned}\vec{a}_{\text{cor}} &= 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\text{rel}} \\ &= 2\omega \hat{k} \times v_{\text{tube}}(-\hat{j}) \\ &= 2\omega v_{\text{tube}} \hat{i} = 2 \cdot 5 \text{ rad/s} \cdot 4 \text{ ft/s} \\ &= 40 \text{ ft/s}^2 \hat{i},\end{aligned}$$

$$\vec{a}_{\text{rel}} = a_{\text{tube}}(-\hat{j}) = -2 \text{ ft/s}^2 \hat{j}.$$

Adding the three terms together, we get

$$\begin{aligned}\vec{a}_P &= -106 \text{ ft/s}^2 \hat{i} - 38 \text{ ft/s}^2 \hat{j} + 40 \text{ ft/s}^2 \hat{i} - 2 \text{ ft/s}^2 \hat{j} \\ &= -66 \text{ ft/s}^2 \hat{i} - 40 \text{ ft/s}^2 \hat{j}.\end{aligned}$$

$$\boxed{\vec{a}_P = -(66\hat{i} + 40\hat{j}) \text{ ft/s}^2}$$

Note that the single term  $\vec{a}_{P'}$  encompasses three terms of the five term acceleration formula.

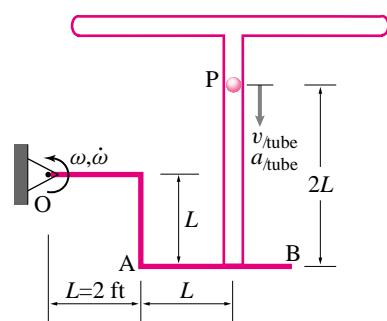


Figure 9.42: (Filename:sfig8.2.1)

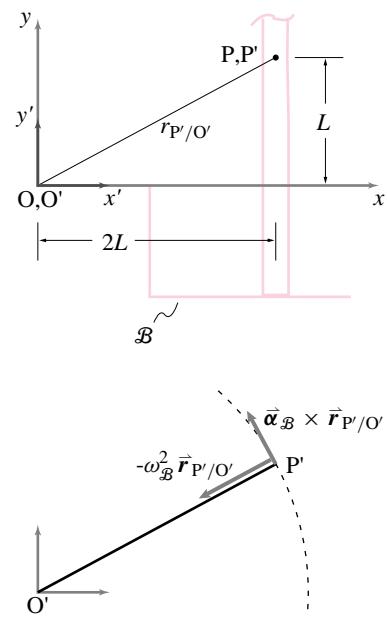


Figure 9.43: Acceleration of point  $P'$ .  $P'$  is fixed to the frame  $\mathcal{B}$  and at the moment coincides with point P. Therefore,  $\vec{a}_{P'} = \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P'/O'} - \omega_{\mathcal{B}}^2 \vec{r}_{P'/O'}$ . (Filename:sfig8.2.1a)

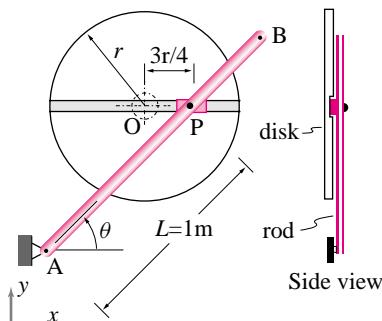


Figure 9.44: (Filename:fig8.1.2a)

**SAMPLE 9.10** A small collar P is pinned to a rigid rod AB at length  $L = 1$  m along the rod. The collar is free to slide in a straight track on a disk of radius  $r = 400$  mm. The disk rotates about its center O at a constant  $\omega = 2$  rad/s. At the instant shown, when  $\theta = 45^\circ$  and the collar is at a distance  $\frac{3}{4}r$  in the track from the center O, find

- the angular velocity of the rod AB and
- the velocity of point P relative to the disk.

**Solution** We will think of P in two ways: one as attached to the rod and the other as sliding in the slot. First, let us attach a frame  $\mathcal{B}$  to the disk. Thus  $\mathcal{B}$  rotates with the disk with angular velocity  $\vec{\omega}_{\mathcal{B}} = \omega \hat{k} = 2 \text{ rad/s} \hat{k}$ . We attach a coordinate system  $x'y'z'$  to  $\mathcal{B}$  at point O. At the instant of interest, the rotating coordinate system  $x'y'z'$  coincides with the fixed coordinate system  $xyz$ . Now let us consider point  $P'$  which is fixed on the disk (and hence in  $\mathcal{B}$ ) and coincides with point P at the moment of interest. We can write the velocity of P as:

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}$$

where

$$\begin{aligned}\vec{v}_{P'} &= \overset{\vec{0}}{\vec{v}_{O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} = \omega \hat{k} \times \left( \frac{3}{4}r \hat{i}' \right) = \frac{3}{4}\omega r \hat{j}' = \frac{3}{4}\omega r \hat{j}, \\ \vec{v}_{\text{rel}} &\equiv \vec{v}_{P/\mathcal{B}} = v_{\text{rel}} \hat{i}' = v_{\text{rel}} \hat{i}.\end{aligned}$$

In the last expression,  $\vec{v}_{\text{rel}} = v_{\text{rel}} \hat{i}$ , we do not know the magnitude of  $\vec{v}_{\text{rel}}$  and hence have left it as an unknown  $v_{\text{rel}}$ , but its direction is known because  $\vec{v}_{\text{rel}}$  has to be along the track and the track at the given instant is along the  $x$ -axis. Thus,

$$\vec{v}_P = \frac{3}{4}\omega r \hat{j} + v_{\text{rel}} \hat{i}. \quad (9.37)$$

Now let us consider the motion of rod AB. Let  $\vec{\Omega} = \Omega \hat{k}$  be the angular velocity of AB at the instant of interest where  $\Omega$  is unknown. Since P is pinned to the rod, it executes circular motion about A with radius  $AP = L$ . Therefore,

$$\vec{v}_P = \vec{\Omega} \times \vec{r}_{P/A} = \Omega \hat{k} \times L(\cos \theta \hat{i} + \sin \theta \hat{j}) = \Omega L(\cos \theta \hat{j} - \sin \theta \hat{i}) \quad (9.38)$$

But, and this trivial formula is the key,  $\vec{v}_P = \vec{v}_P$ . Therefore, from Eqn. (9.37) and (9.38),

$$\frac{3}{4}\omega r \hat{j} + v_{\text{rel}} \hat{i} = \Omega L(\cos \theta \hat{j} - \sin \theta \hat{i}). \quad (9.39)$$

Taking dot product of both sides of the above equation with  $\hat{j}$  we get

$$\begin{aligned}\frac{3}{4}\omega r &= \Omega L \cos \theta \\ \Rightarrow \Omega &= \frac{3\omega r}{4L \cos \theta} = \frac{3 \cdot 2 \text{ rad/s} \cdot 0.4 \text{ m}}{4 \cdot 1 \text{ m} \cdot \frac{1}{\sqrt{2}}} = 0.85 \text{ rad/s}.\end{aligned}$$

Again taking the dot product of both sides of Eqn. (9.39) with  $\hat{i}$  we get

$$v_{\text{rel}} = -\Omega L \sin \theta = -0.85 \text{ rad/s} \cdot 1 \text{ m} \cdot \frac{1}{\sqrt{2}} = -0.6 \text{ m/s}.$$

$(i) \vec{\Omega} = 0.85 \text{ rad/s} \hat{k}, \quad (ii) \vec{v}_{\text{rel}} = -0.6 \text{ m/s} \hat{i}$

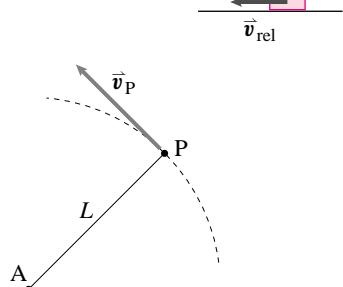


Figure 9.45: (Filename:fig8.1.2b)



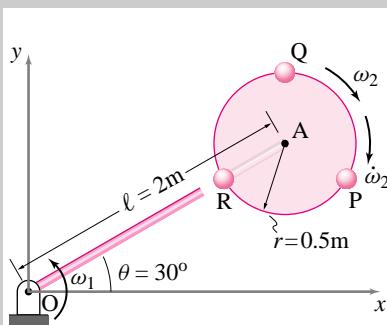
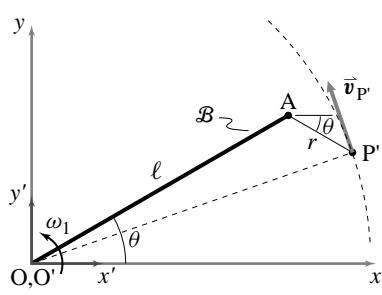


Figure 9.46: (Filename:sfig8.2.2)

Figure 9.47: P' goes in circles around point O with radius OP'. Therefore,  $\vec{v}_{P'}$  is tangential to the circular path at P'.  
(Filename:sfig8.2.2a)

**SAMPLE 9.11** Spinning wheel on a rotating rod in 2-D . A rigid body OA is attached to a wheel that is massless except for three point masses P, Q, and R, placed symmetrically on the wheel. Each of the three masses is  $m = 0.5 \text{ kg}$ . The rod OA rotates about point O in the counterclockwise direction at a constant rate  $\omega_1 = 3 \text{ rad/s}$ . The wheel rotates with respect to the arm about point A with angular acceleration  $\dot{\omega}_2 = 1 \text{ rad/s}^2$  and at the instant shown it has angular speed  $\omega_2 = 5 \text{ rad/s}$ . Note that both  $\omega_2$  and  $\dot{\omega}_2$  are given with respect to the arm.

Using a rotating frame  $\mathcal{B}$  attached to the rod and a coordinate system attached to the frame with origin at O, find

- (a) the velocity of the mass P and
- (b) the acceleration of the mass P.

**Solution** Frame  $\mathcal{B}$  is attached to the rod. We choose a coordinate system  $x'y'z'$  in frame  $\mathcal{B}$  with its origin at O and, at the instant, aligned with the fixed coordinate system  $xyz$ . We consider a point  $P'$  momentarily coincident with point P but fixed in frame  $\mathcal{B}$ . Since  $P'$  is fixed in  $\mathcal{B}$ , it rotates with  $\mathcal{B}$  with  $\vec{\omega}_{\mathcal{B}} = \omega_1 \hat{k}$ . To visualize the motion of  $P'$  imagine a rigid rod from O to  $P'$  (see Fig. 9.47). Now we can calculate the velocity and acceleration of point  $P'$  as follows.

- (a) **Velocity of point P:**

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}$$

Now we calculate the two terms separately:

$$\begin{aligned} \vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} \\ &= \omega_1 \hat{k} \times (\vec{r}_{A/O'} + \vec{r}_{P'/A}) \\ &= \omega_1 \hat{k} \times [\underbrace{\ell(\cos \theta \hat{i} + \sin \theta \hat{j})}_{\vec{r}_{A/O'}} + \underbrace{r(\cos \theta \hat{i} - \sin \theta \hat{j})}_{\vec{r}_{P'/A}}] \\ &= \omega_1 (\ell + r) \cos \theta \hat{j} - \omega_1 (\ell - r) \sin \theta \hat{i} \\ &= 3 \text{ rad/s} \cdot 2.5 \text{ m} \cdot \cos 30^\circ \hat{j} - 3 \text{ rad/s} \cdot 1.5 \text{ m} \cdot \sin 30^\circ \hat{i} \\ &= (6.50 \hat{j} - 2.25 \hat{i}) \text{ m/s}. \end{aligned}$$

Since the wheel rotates with angular speed  $\omega_2$  with respect to the rod, an observer sitting in frame  $\mathcal{B}$  would see a circular motion of point P about point A. Therefore,

$$\begin{aligned} \vec{v}_{\text{rel}} &= \vec{\omega}_{\text{wheel}/\mathcal{B}} \times \vec{r}_{P/A} \\ &= -\omega_2 \hat{k}' \times r(\cos \theta \hat{i}' - \sin \theta \hat{j}') \\ &= -\omega_2 r(\cos \theta \hat{j}' + \sin \theta \hat{i}') \\ &= -(2.16 \hat{j}' + 1.25 \hat{i}') \text{ m/s}. \end{aligned}$$

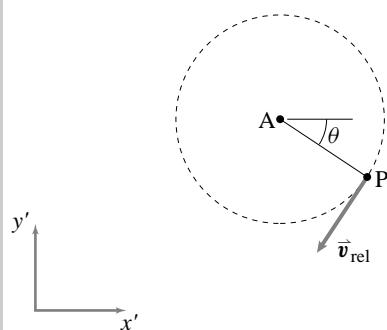
But at the instant of interest,  $\hat{i}' = \hat{i}$ ,  $\hat{j}' = \hat{j}$ , and  $\hat{k}' = \hat{k}$ . So,

$$\vec{v}_{\text{rel}} = -(2.16 \hat{j} + 1.25 \hat{i}) \text{ m/s}$$

Therefore,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = 4.33 \text{ m/s} \hat{j} - 3.50 \text{ m/s} \hat{i}$$

$$\boxed{\vec{v}_P = (-3.50 \hat{i} + 4.33 \hat{j}) \text{ m/s}}$$

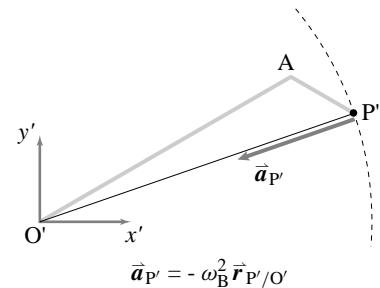
Figure 9.48: Velocity of point P with respect to the frame  $\mathcal{B}$ .  
(Filename:sfig8.2.2b)

(b) **Acceleration of point P:** We can similarly find the acceleration of point P:

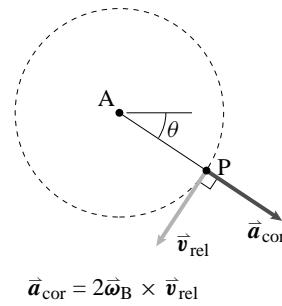
$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}}$$

where

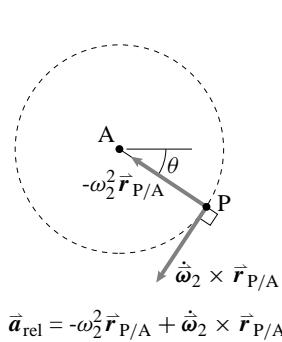
$$\begin{aligned}\vec{a}_{P'} &= \text{acceleration of point } P' \\ &= \underbrace{\vec{a}_{O'}}_{\vec{0}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\ &= -\omega_{\mathcal{B}}^2 \vec{r}_{P'/O'} \\ &= -\omega_1^2 [(\ell + r) \cos \theta \hat{i} + (\ell - r) \sin \theta \hat{j}] \\ &= -9 \text{ (rad/s)}^2 [2.5 \text{ m} \cdot \cos 30^\circ \hat{i} + 1.5 \text{ m} \cdot \sin 30^\circ \hat{j}] \\ &= -(19.48 \hat{i} + 6.75 \hat{j}) \text{ m/s}^2,\end{aligned}$$



$$\begin{aligned}\vec{a}_{\text{cor}} &= \text{Coriolis acceleration} \\ &= 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\text{rel}} \\ &= 2\omega_1 \hat{k} \times \vec{v}_{\text{rel}} \quad (\text{see part (a) above for } \vec{v}_{\text{rel}}) \\ &= (6 \text{ rad/s}) \hat{k} \times (-2.16 \hat{j} - 1.25 \hat{i}) \text{ m/s} \\ &= (12.99 \hat{i} - 7.50 \hat{j}) \text{ m/s}^2,\end{aligned}$$



$$\begin{aligned}\vec{a}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B} \\ &= \vec{a}_{P/\mathcal{B}} = \vec{\omega}_2 \times \vec{r}_{P/A} - \omega_2^2 \vec{r}_{P/A} \\ &= -\dot{\omega}_2 \hat{k}' \times r(\cos \theta \hat{i}' - \sin \theta \hat{j}') - \omega_2^2 r(\cos \theta \hat{i}' - \sin \theta \hat{j}') \\ &= -r[(\dot{\omega}_2 \sin \theta + \omega_2^2 \cos \theta) \hat{i}' + (\dot{\omega}_2 \cos \theta - \omega_2^2 \sin \theta) \hat{j}'] \\ &= -0.5 \text{ m} [(1 \text{ rad/s}^2 \cdot \sin 30^\circ + 25 \text{ (rad/s)}^2 \cdot \cos 30^\circ) \hat{i}' \\ &\quad + (1 \text{ rad/s}^2 \cdot \cos 30^\circ - 25 \text{ (rad/s)}^2 \cdot \sin 30^\circ) \hat{j}'] \\ &= (-11.08 \hat{i}' + 5.82 \hat{j}') \text{ m/s}^2 \\ &= (-11.08 \hat{i} + 5.82 \hat{j}) \text{ m/s}^2.\end{aligned}$$



The term  $\vec{a}_{P'}$  encompasses three terms of the five term acceleration formula. The last line in the calculation of  $\vec{a}_{\text{rel}}$  follows from the fact that at the instant of interest  $\hat{i}' = \hat{i}$  and  $\hat{j}' = \hat{j}$ .

Now adding the three parts of  $\vec{a}_P$  we get

$$\begin{aligned}\vec{a}_P &= \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}} \\ &= -17.57 \text{ m/s}^2 \hat{i} - 8.43 \text{ m/s}^2 \hat{j}.\end{aligned}$$

$$\boxed{\vec{a}_P = -(17.57 \hat{i} + 8.43 \hat{j}) \text{ m/s}^2}$$

Figure 9.49: (Filename:sfig8.2.2c)

## 9.4 Kinematics of 2-D mechanisms

An *ideal mechanism* or *linkage* is a collection of rigid bodies constrained to move relative to the ground and each other by hinges but which still has some possible motion(s). Generally people use the word mechanism or linkage more loosely to include any collection of machine parts connected by any means.

The analysis of the kinematics of mechanisms is an important part of machine design. Mechanisms *synthesis*, coming up with a mechanism design which has desired motions, is obviously key in creative design, and now days in computer aided design.

Finally, the determination of the dynamics of a mechanism, how it will move and with what forces, is completely dependent on understanding the kinematics of the mechanism. The whole subject of mechanism kinematic analysis, though in some sense a subset of dynamics, is actually a huge and infinitely complex subject in itself<sup>①</sup> and also a useful subject in itself. Often kinematics is the central interest in machine design , and mechanics (force and acceleration) analysis is only used if something breaks or moves too slowly. This section presents some of the basic ideas in kinematic analysis. The overall question in mechanism kinematic analysis is this:

Given a collection of parts and a description of how they are connected, in what ways can they move?

Without getting into the details of the motions yet, the first question to answer is How many ways can the mechanism move?

### Degrees of freedom (DOF)

The number of degrees of freedom (DOF)  $n_{\text{DOF}}$  of a mechanism is the number of different ways it can move. More precisely

The number of degrees of freedom  $n_{\text{DOF}}$  of a mechanism is the minimum number of configuration variables needed to describe all possible configurations of the mechanism.

The minimum number of configuration variables  $n_{\text{DOF}}$  is a property of the mechanism. The choice of what these variables are, however is not unique.

*Example: A particle in a plane has 2 degrees of freedom.*

The set of ‘configurations’ of a particle in a plane is the set of positions of the particle. This is fully described by its  $x$  and  $y$  coordinates. Thus  $n_{\text{DOF}} = 2$ . But the configuration is also determined by the particles polar coordinates  $R$  and  $\theta$ . And there are an infinite number of other pairs of numbers that could be used to describe the configurations (e.g., the  $x'$  and  $y'$  coordinates, the  $w$  and  $z$  coordinates with  $w = e^x$  and  $z = e^y$ , etc). The minimum number of configuration variables, 2, is unique, but the choice of variables is not.  $\square$

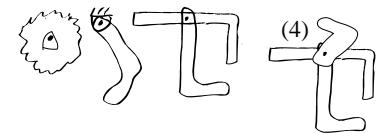
<sup>①</sup> **Very advanced aside.** The kinematics of planar linkages is as complicated as the classification of closed surfaces in any number of dimensions: spheres, donuts, spheres with two holes, etc. In math-speak: Any orientable manifold is a connected component of the configuration space of some planar linkage. This idea was proposed by Bill Thurston in the 1970’s and later proved by Kapovich and Millson.

For planar mechanisms one can often determine the number of degrees of freedom by the following formula

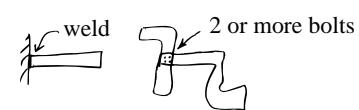
$$3 \cdot (\underbrace{\# \text{ of rigid bodies}}_{n_{\text{bod}}}) + 2 \cdot (\underbrace{\# \text{ of particles}}_{n_{\text{part}}}) - (\underbrace{\# \text{ of constraints}}_{n_{\text{con}}}) = n_{\text{DOF}} \quad (9.40)$$

The formula starts with the number of ways one rigid body can move (2 translations and a rotation makes 3) and one particle can move (just 2 translations) and then subtracts the restrictions to the motion. In eqn. (9.40) the number  $n_{\text{con}}$  of constraints is counted as the number of degrees of freedom restricted by the connections.

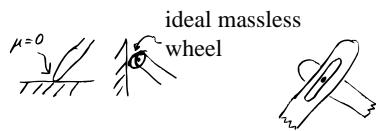
a) hinge (2 constraints)



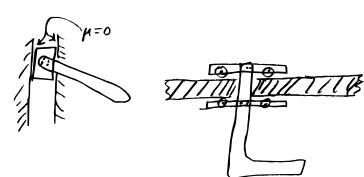
b) weld (3 constraints)



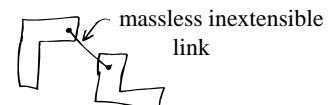
c) sliding contact (1 constraint)



d) keyed sliding (2 constraints)



e) inextensible link (1 constraint)



f) rolling contact (2 constraints)



Figure 9.50: The number of degrees of freedom that are eliminated by various constraints.  
(Filename: tfigure.countingconstraints)

## Examples of connections and their effect on $n_{\text{DOF}}$

See Fig. 9.50 for some standard idealized connections and their number of constraints.

- a) **2 for a pin joint:** a pin joint restricts relative motion in 2 directions but still allows relative rotation. If three bodies are connected at one pin then it counts as two pin joints and thus  $2 \times 2 = 4$  reductions in the number of degrees of freedom. There are 6 reductions for 4 bodies connected at one pin, etc.
- b) **3 for a welded connection:** a weld restricts relative translation in two directions as well as relative rotation ( $2 + 1 = 3$ ). So two parts that are welded together have  $2 \cdot 3 - 3 = 3$  degrees of freedom. That is, any collection of rigid bodies welded together is the same as one rigid body. The word ‘weld’ is meant to include any collection of bolts, glue, string, rivets or bailing wire that prevents any relative motion.
- c) **1 for a sliding contact:** the sliding contact restricts relative translation normal to the contact surfaces and allows translation tangent to the surfaces. Relative rotation is also allowed.
- d) **2 for a keyed sliding contact:** allows relative translation in one direction but disallows translation in one direction as well as rotation.
- e) **1 for a massless link hinged at its ends to two bodies:** this keeps the distance between two points fixed which is one restriction (alternatively the bar adds 3 degrees of freedom and each hinge subtracts 2 ( $+3 - 2 \times 2 = -1$  degree of freedom)).
- f) **2 for a rolling contact:** relative slip is not allowed nor is interpenetration.

Be warned that

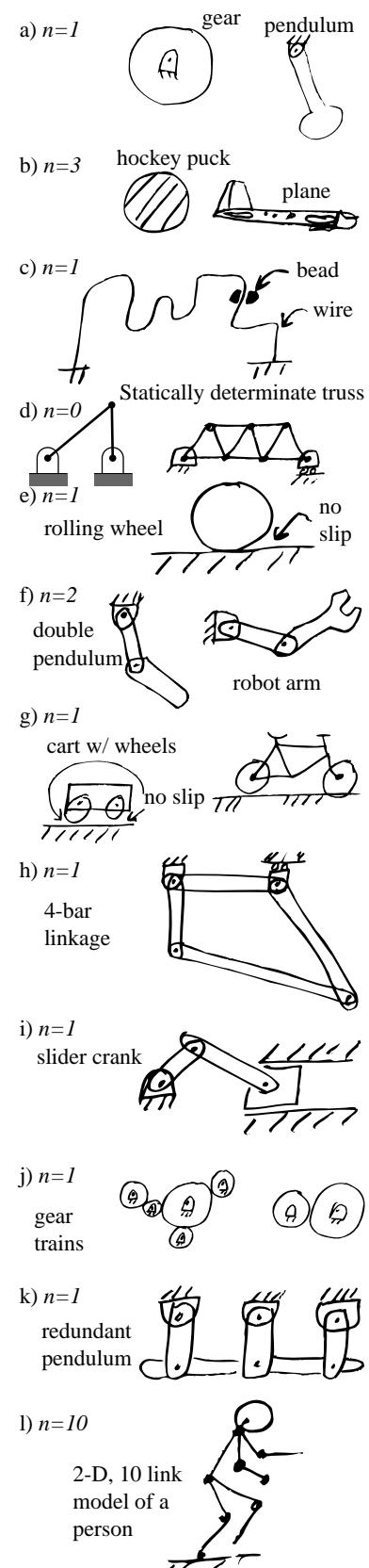
if some of the constraints are redundant then a system can have more degrees of freedom than eqn. (9.40) indicates.

## Some simple mechanisms

Fig. 9.51 shows some examples of simple mechanisms and the number of degrees of freedom. In each case we look at eqn. (9.40):  $3 \cdot n_{\text{bod}} + 2 \cdot n_{\text{part}} - n_{\text{con}} = n_{\text{DOF}}$ .

- a) A **body connected to ground by a hinge** has **1** degree of freedom; the set of all possible configurations can be described by the angle of the body:  $n_{\text{bod}} = 1, n_{\text{con}} = 2 \Rightarrow n_{\text{DOF}} = 1$ .
- b) An **unconstrained body** has **3** degrees of freedom; the set of all configurations can be described by the  $x$  and  $y$  coordinates of a reference point and by the rotation  $\theta$ :  $n_{\text{bod}} = 1, n_{\text{con}} = 0 \Rightarrow n_{\text{DOF}} = 3$ .
- c) A **bead on a wire** has **1** degree of freedom; its configuration is fully determined by the distance the bead has advanced along the wire relative to a reference mark:  $n_{\text{part}} = 1, n_{\text{con}} = 1 \Rightarrow n_{\text{DOF}} = 1$ .
- d) A **statically determinate truss** has **0** degrees of freedom; a statically determinate truss has no ways to move:  $n_{\text{bod}} = n_{\text{bars}}, n_{\text{part}} = 0, n_{\text{con}} = 2 \cdot n_{\text{pins}} + n_{\text{ground const}} \Rightarrow n_{\text{DOF}} = 0$ . Note that the number of pins is more than the number of joints when we studied trusses in statics. There we focussed on joints as restricted by bars. Here we look at bars as restricted by joints and a given joint counts 1,2, or 3 times depending on whether it connects 2,3, or 4 bars. Thus the 11 bar truss shown has  $3 \times 3 + 2 \times 2 + 2 = 15$  pin restrictions and three ground restrictions (or 16 pins and one sliding contact).
- e) A **rolling wheel** has **1** degree of freedom; its configuration is fully determined either by the net angle  $\theta$  it has rolled or by the  $x$  coordinate of its center:  $n_{\text{bod}} = 1, n_{\text{con}} = 2 \Rightarrow n_{\text{DOF}} = 1$ .
- f) A **double pendulum** or two-link robot arm has **2** degrees of freedom; its configuration is determined by the net rotations of its two links (or by the rotation of the first link and the relative rotation of the second link):  $n_{\text{bod}} = 2, n_{\text{con}} = 4 \Rightarrow n_{\text{DOF}} = 2$ .
- g) A **cart with two rolling wheels** or a planar rolling bicycle has **1** degree of freedom; both the rotation of the wheels and the position of the bicycle are determined by the 1 variable, say, the  $x$  coordinate of a reference point on the vehicle(*e.g.*, the bicycle seat).  $n_{\text{bod}} = 3, n_{\text{con}} = 4 \times 2 = 8$  (2 hinges and 2 rolling contacts, the hinge for bicycle steering isn't relevant for a planar analysis)  $\Rightarrow n_{\text{DOF}} = 1$ .
- h) A “**four**” **bar linkage** has **1** degree of freedom; the angle of any one of the bars determines the angles of the others:  $n_{\text{bod}} = 4, n_{\text{con}} = 2 \times 4 + 1 = 11$  (there are 4 pin joints between the bars, one pin joint to ground and one roller connection to the ground)  $\Rightarrow n_{\text{DOF}} = 1$ .
- i) A **slider crank** has **1** degree of freedom; the rotation of the crank determines the configuration of the system  $n_{\text{bod}} = 3, n_{\text{con}} = 3 \cdot 2 + 2$  (there are 3 pins and one keyed connection)  $\Rightarrow n_{\text{DOF}} = 1$ .
- j) An ideal **gear train** (with all gears pinned to ground) has **1** degree of freedom; the amount of rotation of any one gear determines the rotation of all of the gears: In this case the counting formula is *wrong*. Say there are 2 gears, then  $n_{\text{bod}} = 2, n_{\text{con}} = 3 \cdot 2 = 6$  (two pins and one rolling contact)  $\Rightarrow n_{\text{DOF}} = 0 \neq 1$ . The rolling constraint prevents interpenetration, but this was already prevented by the hinges at the center of the gears. The constraints are redundant and the system has more degrees of freedom than eqn. (9.40) indicates.
- k) A **redundant swing** with one horizontal bar suspended by 3 parallel struts has **1** degree of freedom; the angle of one upright links determines the full configuration of the mechanism. The counting formula is again wrong:  $n_{\text{bod}} = 4, n_{\text{con}} = 6 \cdot 2 = 12 \Rightarrow n_{\text{DOF}} = 0$  underestimates the number of degrees of freedom because the constraints are redundant.

- I) A **2-D 10-link model of a person** with one foot on the ground has **10** degrees of freedom; the angles of the 10 links determine the full configuration of the mechanism. There are no redundant constraints and the counting formula works:  $n_{\text{bod}} = 10$ ,  $n_{\text{con}} = 10 \cdot 2 = 20$  (counting a hinge at the ground contact and two hinges at both the hips and the shoulders)  $\Rightarrow n_{\text{DOF}} = 10$ .



## Configuration variables

Once we know the number of degrees of freedom  $n_{\text{DOF}}$  of a system it is often useful to settle on one set of  $n_{\text{DOF}}$  configuration variables. In this book  $n_{\text{DOF}}$  will be 1,2 or at most 3. Thus we pick 1,2 or 3 variables.

### Example: Straight line motion

Chapter 6 on straight line motion was mostly about one-degree-of-freedom systems ( $n_{\text{DOF}} = 1$ ). These systems could all be characterized by the single configuration variable  $x$ , the displacement along the line of a reference point on the body relative to a reference point on the ground. All the positions, velocities and accelerations of all points in the system could be found in terms of  $x$ ,  $\dot{x}$  and  $\ddot{x}$  (in fact all points had  $(\vec{v} = \dot{x}\hat{i}$  and  $\vec{a} = \ddot{x}\hat{i}$ .  $\square$

### Example: Circular motion about a fixed axis

In chapters 7 and 8 we were almost entirely focussed on systems with one degree of freedom well characterized by the one configuration variable, the rotation angle  $\theta$ . For such motions positions, velocities, and accelerations of all points were determined by the initial positions of the points  $\theta$ ,  $\dot{\theta}$  and  $\ddot{\theta}$  by equations which you know well by now.  $\square$

For more general motions we almost always take inspiration from the two examples above. We use the translation of a conspicuous point, or we use the rotation of a conspicuous body for a configuration variable. And more of the same if the system has more than 1 degree of freedom. The natural choice of configuration variables for some simple mechanisms is given in the text beside Fig. 9.51.

Often our main kinematic task is to express the full configuration of the system as well as all the velocities and accelerations of all its parts in terms of the positions of the parts, the configuration variables, and their first and second time derivatives. Fig. 9.50

## Adding relative angular velocities

One last simple kinematic fact is needed before we can plug and chug with the theory we have so far and apply it to general kinematic mechanisms. It concerns the addition of rotations and rotation rates. The following example basically tells the whole story

Figure 9.51: Some simple mechanisms and the number of degrees of freedom,  $n$  (called  $n_{\text{DOF}}$  in the text).

(Filename:figure.mechanismDOFs)

**Example: Double pendulum and the addition of rotation rates**

The commonly used configuration variables for the double pendulum shown in Fig. 9.52 are  $\theta_1$  and  $\theta_2$ . To actually know the configuration of the system obviously we need to know  $\phi$  which is given by

$$\phi = \theta_1 + \theta_2 \quad \text{so} \quad \dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 \quad \text{and so} \quad \ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2.$$

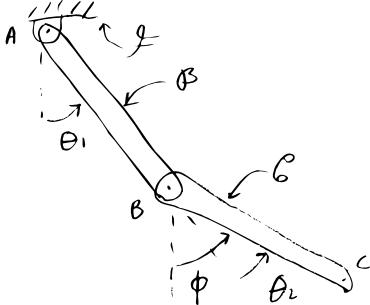


Figure 9.52: A double pendulum's configuration variables are most often taken to be  $\theta_1$  and  $\theta_2$ . But one then needs to know that  $\phi = \theta_1 + \theta_2$  and that  $\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2$  and  $\ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2$ .

(Filename: tfigure.doublependangles)

[Aside: One reason for choosing  $\theta_2$  instead of  $\phi$  as a configuration variable is that if one was measuring or controlling the second link, say as a robotic arm, the angle  $\theta_2$  can be measured more easily than  $\phi$ . Also, it turns out (in hindsight) that the differential equations of motion are slightly simpler using  $\theta_2$  instead of  $\phi$ .]  $\square$

Looking at the bars as being glued to reference frames ( $\mathcal{F}$  for the fixed frame,  $\mathcal{B}$  for bar AB, and  $\mathcal{C}$  for bar BC), the above example shows that

$$\vec{\omega}_{\mathcal{C}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} + \vec{\omega}_{\mathcal{C}/\mathcal{B}} \quad (9.41)$$

which is often written with the simple notation

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

Which can only be given strict meaning by the more elaborate eqn. (9.41) above it.

**Example: Double pendulum (see previous example)**

Take  $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \dot{\theta}_1 \hat{k}$ ,  $\vec{\omega}_{\mathcal{C}/\mathcal{B}} = \dot{\theta}_2 \hat{k}$  and  $\vec{\omega}_{\mathcal{C}/\mathcal{F}} = \dot{\phi} \hat{k}$  and eqn. (9.41) is self evident from the addition of angles.  $\square$

Similarly we have

$$\vec{\alpha}_{\mathcal{C}/\mathcal{F}} = \vec{\alpha}_{\mathcal{B}/\mathcal{F}} + \vec{\alpha}_{\mathcal{C}/\mathcal{B}} \quad (9.42)$$

which is often written with the simple notation

$$\vec{\alpha} = \vec{\alpha}_1 + \vec{\alpha}_2.$$

For those going on to study 3-D mechanics (Chapter 12), one should note that, unlike eqn. (9.41), eqn. (9.42) does *not* hold in 3-D.

## Kinematics of mechanisms

One approach to mechanisms is to do what one can with high-school geometry and trigonometry, the laws of sines (see page ??), and so on.

**Example: Rod on step using geometry and trigonometry**

One end of a rod slides on the ground. The other end slides on a corner at A (see Fig. 9.53). Given that  $\vec{v}_B = v_B \hat{i}$  we can find  $\dot{\phi}$  as follows:

$$\frac{h}{\ell_{DB}} = \tan \phi \Rightarrow \left\{ \frac{\ell_{DB}}{h} = \frac{\cos \phi}{\sin \phi} \right\}$$

$$\frac{d}{dt} \left\{ \right\} \Rightarrow \frac{\dot{\ell}_{DB}}{h} = \frac{-\dot{\phi}}{\sin^2 \phi} \Rightarrow \dot{\phi} = -\frac{v_B \sin^2 \phi}{h}$$

□

As the above calculation shows, this problem doesn't need the heavy machinery of our moving-frame vector methods. But it provides an instructive example.

**Example: Rod on step using moving-frame methods (see previous example)**

We look at point A and note that we can think of it as a fixed point in the fixed frame  $\mathcal{F}$  and also as a point that is moving relative to the translating and rotating frame  $\mathcal{B}$ . We evaluate its velocity both ways.

$$\begin{aligned} \vec{v}_A &= \vec{v}_A \\ \vec{0} &= \vec{v}_B + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{A/B} + \vec{v}_{A/\mathcal{B}} \quad (\text{eqn. (9.32)}) \\ \{\vec{0}\} &= v_B \hat{i} - \dot{\phi} \hat{k} \times (\ell_{BA} \hat{\lambda}_{BA}) + v_{A/\mathcal{B}} \hat{\lambda}_{BA} \} \\ \Rightarrow 0 &= v_B \underbrace{\hat{i} \cdot \hat{n}_{BA}}_{\sin \phi} + \dot{\phi} \ell_{BA} \quad (\{\} \cdot \hat{n}_{BA}) \\ \Rightarrow \dot{\phi} &= -\frac{v_B \sin \phi}{\ell_{BA}} = -\frac{v_B \sin^2 \phi}{h} \quad (\sin \phi = \frac{h}{\ell_{AB}}) \end{aligned}$$

as we had before. The key equation was the ‘three term velocity formula’ eqn. (9.32) on page 546 and the observation that relative to frame  $\mathcal{B}$  point A slides along the rod. Note that we never had to explicitly use the rotating coordinates associated with frame  $\mathcal{B}$  to do this calculation. □

You should understand the examples above, and the needed background material, before going on to the following examples.

**Example: Slider Crank using geometry and trigonometry**

The slider crank mechanism (Fig. 9.54) was briefly introduced in the context of statics where its forces could be analyzed assuming inertial terms were negligible (see 186). But it is a commonly used mechanism (*e.g.*, in every car) and its motions are of central interest. The angle  $\theta$  is the most natural configuration variable for this  $n_{DOF} = 1$  system. One would like to know the position, velocity and acceleration of the slider ( $x_C$ ,  $\dot{x}_C$  and  $\ddot{x}_C$ ) in terms of  $\theta$ ,  $\dot{\theta}$  and  $\ddot{\theta}$ .

$$\begin{aligned} x_C &= x_C + \ell_{DC} \\ &= d \cos \theta + \sqrt{\ell^2 - h^2} \\ &= d \cos \theta + \sqrt{\ell^2 - (d \sin \theta)^2} \\ &= d \left( \cos \theta + \sqrt{(\ell/d)^2 - \sin^2 \theta} \right) \\ &= d \left( \cos \theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2} \right) \quad (9.43) \end{aligned}$$

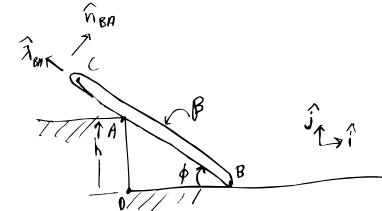


Figure 9.53: A rigid rod slides so that it always touches the ground and the corner at A. One would like to know the relation between  $\dot{\phi}$  and  $\vec{v}_B$ .

(Filename:tfigure.rodoncorner)

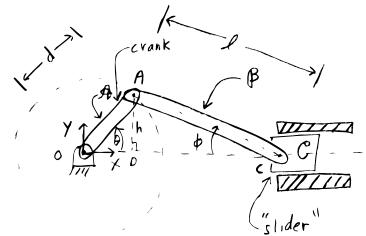


Figure 9.54: The slider crank mechanism.

(Filename:tfigure.slidercrankkin)

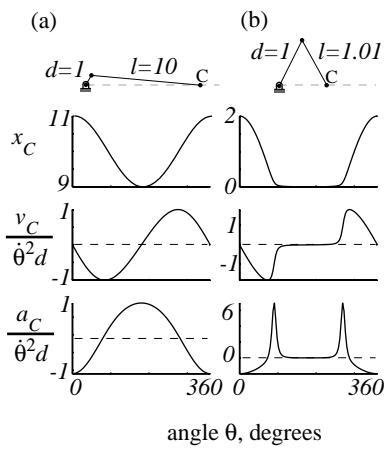


Figure 9.55: Position  $x_C$  of the slider, its velocity  $v_C$  and its acceleration  $a_C$  (Strictly, e.g.,  $\ddot{\mathbf{a}}_C = a_C \hat{\mathbf{i}}$ ). Two sets of curves are shown. (a) a long connecting rod, and (b) a connecting rod a hair longer than the crank.

In case (a) the connecting rod is nearly horizontal at all times and the displacement of point C is almost entirely due to the horizontal displacement of the end of the crank arm. Thus point C moves with almost exactly a cosine wave with amplitude equal to the length of the crank. The velocity and acceleration curves are thus also sine and cosine waves.

In case (b) the motion is close to a cosine curve approximately when  $-60^\circ < \theta < 60^\circ$ . That is, the displacement of C is about twice the horizontal displacement of the end of the crank arm when the end is to the right of its base bearing. When the end of the crank arm is to the left of the bearing point C is nearly stationary and is just to the right of the crank base bearing. The transition between these two cases involves a sudden large acceleration.

(Filename: tfigure.crankcurves)

The positive  $\sqrt{\cdot}$  corresponds to C being to the right of 0. The negative  $\sqrt{\cdot}$  corresponds to point C being to the left of 0. The mechanism just doesn't work for a full revolution of the link OA if  $\ell < d$  as you can see from the picture or from that the  $\sqrt{\cdot}$  above giving imaginary values for  $\cos 2\theta$  near -1,  $\sin \theta$  near 1, and  $\theta$  near  $\pi/2$ .

To get the velocity of point C we just take the derivative of eqn. (9.43) above.

$$\begin{aligned} v_C &= \dot{x}_C & (9.44) \\ &= \frac{d}{dt} \left\{ d \left( \cos \theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2} \right) \right\} \\ &= d\dot{\theta} \left\{ -\sin(\theta) - \sin(2\theta) \frac{1}{\sqrt{4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2}} \right\} \end{aligned}$$

To get the acceleration we differentiate once again. For simplicity lets assume the crank rotates at constant rate, so  $\dot{\theta}$  is a constant and  $\ddot{\theta} = 0$ . Cranking out the derivative of eqn. (9.44), so to speak, we get

$$\begin{aligned} a_c = \dot{v}_C &= \frac{d}{dt} \left\{ \text{the mess on the right of eqn. (9.44)} \right\} & (9.45) \\ &= -d\dot{\theta}^2 \left\{ \cos(\theta) \right. \\ &\quad \left. + 2 \sin^2(2\theta) \left( 4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2 \right)^{-3/2} \right. \\ &\quad \left. + 2 \cos(2\theta) \frac{1}{\sqrt{4 \frac{l^2}{d^2} + 2 \cos(2\theta) - 2}} \right\} \end{aligned}$$

So we now know the position, velocity and acceleration of point C in terms of  $\theta$ ,  $\dot{\theta}$  and  $\ddot{\theta}$ . You should commit the solution eqn. (9.45) to memory. Just kidding.

Plots of  $x_C$ ,  $v_C$ , and  $a_C$  from these equations are shown in Fig. 9.55ab for two different extremes of slider crank design: one with a very long connecting rod that gives sinusoidal motion, and one with a connecting rod just barely long enough to prevent locking that gives intermittent motion.  $\square$

Unlike some more complex mechanisms, the slider crank is solvable in that one can write a formula for the position of any point of interest in terms of the single configuration variable  $\theta$ . For more complex mechanisms this may not be possible. Further, even if possible the above example shows that the differentiation required to find velocity and acceleration can lead to a bit of a mess.

A different approach is to assume that at some value of the configuration variable ( $\theta$  for the slider crank) that the full configuration of the system is known. That is, that the locations of all points are known. Then we can use our vector methods to find velocities and accelerations of all points of interest.

#### Example: Slider crank using vector methods (see previous example)

Take the slider crank of Fig. 9.54 to be in some known configuration. We now try to find the velocity and acceleration of point C in terms of the positions of the points 0, B, and C as well as  $\theta$  and  $\dot{\theta}$ .

The basic approach is to write true things, and then solve for unknowns. First work on velocities. The basic idea is to look at the *closure* condition. That is, the velocity of point C as calculated by working down the linkage from 0 to A to C has to be consistent with the velocity of C as calculated in the fixed frame.

$$\begin{aligned}\vec{v}_C &= \vec{v}_C \\ v_C \hat{i} &= \vec{v}_{A/0} + \vec{v}_{C/A} \\ v_C \hat{i} &= (\dot{\theta} \hat{k}) \times \vec{r}_{A/0} + (-\dot{\phi} \hat{k}) \times \vec{r}_{C/A}\end{aligned}\quad (9.46)$$

eqn. (9.46) is a 2-D vector equation in the 2 unknown scalars  $v_C$  and  $\dot{\phi}$ . It could be solved as a pair of equations, or solved directly by first dotting both sides with  $\hat{j}$  to find  $\dot{\phi}$  and dotting both sides with  $\vec{r}_{C/A}$  to find  $v_C$ . These yield

$$\begin{aligned}\dot{\phi} &= \frac{(\dot{\theta} \hat{k}) \times \vec{r}_{A/0} \cdot \hat{j}}{(\hat{k} \times \vec{r}_{C/A}) \cdot \hat{j}} \quad \text{and} \\ v_C &= \frac{\dot{\theta} (\hat{k} \times \vec{r}_{A/0}) \cdot \vec{r}_{C/A}}{\hat{i} \cdot \vec{r}_{C/A}}\end{aligned}$$

where everything on the right of these equations is assumed known. Without grinding out the vector products in terms, say, of components, we can just know that we can at this point know  $\dot{\phi}$  and  $v_C$ .

We proceed to find the accelerations by similar means, assuming  $\dot{\phi}$  is a constant so  $\vec{\alpha}_A = \vec{0}$ :

$$\begin{aligned}\vec{a}_C &= \vec{a}_C \\ a_C \hat{i} &= \vec{a}_{A/0} + \vec{a}_{C/A} \\ a_C \hat{i} &= -\ddot{\theta}^2 \vec{r}_{A/0} + (-\ddot{\phi} \hat{k}) \times \vec{r}_{C/A} - -\dot{\phi}^2 \vec{r}_{C/A}\end{aligned}\quad (9.47)$$

eqn. (9.47) is a 2-D vector equation in the two scalar unknowns  $a_C$  and  $\ddot{\phi}$ . We can set this up as two equations in two unknowns. Or we can solve for  $\ddot{\phi}$  directly by dotting both sides with  $\hat{j}$  and we can solve for  $a_C$  directly by crossing both sides with  $\vec{r}_{C/A}$  or by dotting with a vector perpendicular to  $\vec{r}_{C/A}$ .

Although we have presented an algorithm rather than a formula, we have found the velocity and acceleration of C without writing any large equations of the type needed in the previous example. The shortcoming is that this method depends on knowing the full configuration at the time of interest.  $\square$

#### Example: Four bar linkage using geometry and trigonometry

Fig. 9.56 shows a “four-bar linkage”. Please see 185 for an introduction to 4-bar linkages in the context of statics. Four bar linkages are solvable in the sense that one can write equations for the positions of any point of interest in terms of the single configuration variable  $\theta$  marked in Fig. 9.56. But the formulas are really a mess. And the first and second time derivatives are an unbelievable mess.

The four-bar linkage is about as complex a system as can be solved in this sense, and it is probably too-complex for this solution to be useful in the kinematic analysis of accelerations.  $\square$

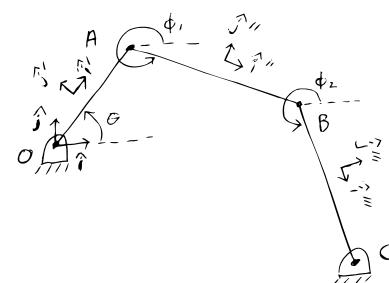


Figure 9.56: A four bar linkage.

(Filename: tfigure.4barkin)

For complex mechanisms one is often stuck using vector methods, like we are for practical purposes stuck with the 4-bar linkage. But the vector methods based on the current configuration are not crippled by complexity.

**Example: Four-bar linkage using relative velocities and accelerations**

Assuming the configuration is known (*i.e.*, that  $\theta$ ,  $\phi_1$ , and  $\phi_2$  are known), we can proceed with the 4-bar linkage just as we did for the slider crank. We enforce closure by picking a point and thinking about its velocity two different ways. We could pick any point, say C. From the fixed frame we know that the velocity of C is zero. Working around the linkage, link by link, we know it is the sum of relative velocities as

$$\begin{aligned}\vec{v}_C &= \vec{v}_C \\ \vec{0} &= \vec{v}_{A/0} + \vec{v}_{B/A} + \vec{v}_{C/B} \\ &= (\dot{\theta}\hat{k}) \times \vec{r}_{A/0} + (\dot{\phi}_1\hat{k}) \times \vec{r}_{B/A} + (\dot{\phi}_2\hat{k}) \times \vec{r}_{C/B}\end{aligned}$$

which is equivalent to two scalar equations in the two unknowns  $\dot{\phi}_1$  and  $\dot{\phi}_2$ . This equation can be solved directly for  $\dot{\phi}_2$  by taking the dot product of both sides with a vector perpendicular to  $\vec{r}_{B/A}$  (such as  $\hat{j}''$  or  $\hat{k} \times \vec{r}_{B/A}$ ) and for  $\dot{\phi}_1$  by taking the dot product of both sides with for a vector perpendicular to  $\vec{r}_{C/B}$  (such as  $\hat{j}'''$  or  $\hat{k} \times \vec{r}_{C/B}$ ) to get

$$\begin{aligned}\dot{\phi}_1 &= -\dot{\theta} \frac{(\hat{k} \times \vec{r}_{A/0}) \cdot \hat{j}'''}{(\hat{k} \times \vec{r}_{B/A}) \cdot \hat{j}''} \quad \text{and} \\ \dot{\phi}_2 &= -\dot{\theta} \frac{(\hat{k} \times \vec{r}_{A/0}) \cdot \hat{j}''}{(\hat{k} \times \vec{r}_{C/B}) \cdot \hat{j}''}.\end{aligned}$$

The dot product with  $\hat{k}$  is used to get a scalar on the top and bottom of the fraction, both vectors are already only in the  $\hat{k}$  direction. Now that  $\dot{\phi}_1$  and  $\dot{\phi}_2$  are known the velocity of any point on the mechanism is known. For example

$$\vec{v}_B = (\dot{\phi}_2\hat{k}) \times \vec{r}_{B/C}.$$

The angular accelerations of the two links are found by the same method. For simplicity lets assume that the driving crank OA spins at constant rate so  $\ddot{\theta} = 0$ . Looking at the acceleration of point C two ways we have

$$\begin{aligned}\vec{a}_C &= \vec{a}_C \\ \vec{0} &= \vec{a}_{A/0} + \vec{a}_{B/A} + \vec{a}_{C/B} \\ &= \underbrace{(\ddot{\theta}\hat{k}) \times \vec{r}_{A/0}}_0 - \dot{\theta}^2 \vec{r}_{A/0} \\ &\quad + (\ddot{\phi}_1\hat{k}) \times \vec{r}_{B/A} - \dot{\phi}_1^2 \vec{r}_{B/A} \\ &\quad + (\ddot{\phi}_2\hat{k}) \times \vec{r}_{C/B} - \dot{\phi}_2^2 \vec{r}_{C/B}\end{aligned}$$

Because  $\dot{\phi}_1$  and  $\dot{\phi}_2$  are already known, this is one equation in the two unknowns  $\ddot{\phi}_1$  and  $\ddot{\phi}_2$ . They can be solved for  $\ddot{\phi}_1$  by taking the dot product of both sides with  $\vec{j}'''$  and for  $\ddot{\phi}_2$  by taking the dot product of both sides with  $\vec{j}''$ .

At this point you know  $\theta$ ,  $\dot{\theta}$ ,  $\phi_1$ ,  $\dot{\phi}_1$ ,  $\ddot{\phi}_1$ ,  $\phi_2$ ,  $\dot{\phi}_2$ , and  $\ddot{\phi}_2$  and can thus calculate the position, velocity and acceleration of any point in the mechanism.  $\square$

In the examples above we used the absolute rotations of the links, however we could also have done the calculations using the relative rotations and then used the formulas for velocity and acceleration relative to a rotating frame. For the examples above this would make the calculations slightly longer.

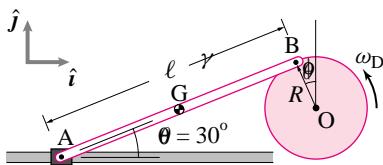


Figure 9.57: (Filename:sfig7.1.2a)

**SAMPLE 9.12 Velocity of points in a one-DOF mechanism.** In machines we often encounter mechanisms and links in which the ends of a link or a rod are constrained to move on a specified geometric path. A simplified typical link AB is shown in Fig. 9.57.

Link AB is a uniform rigid rod of length  $\ell = 2 \text{ m}$ . End A of the rod is attached to a collar which slides on a horizontal track. End B of the rod is attached to a uniform disk of radius  $R = 0.5 \text{ m}$  which rotates about its center O. At the instant shown, when  $\theta = 30^\circ$ , end A is observed to move at  $2 \text{ m/s}$  to the left.

- Find the angular velocity of the rod.
- Find the angular velocity of the disk.
- Find the velocity of the center of mass of the rod.

**Solution** Let the angular velocities of the rod and the disk be  $\vec{\omega}_{AB} = \dot{\theta}\hat{k}$  and  $\vec{\omega}_D = \dot{\phi}\hat{k}$  respectively, where  $\dot{\theta}$  and  $\dot{\phi}$  are unknowns. We are given  $\vec{v}_A = -v_A\hat{i}$  where  $v_A = 2 \text{ m/s}$ .

- Point B is on the rod as well as the disk. Hence, the velocity of point B can be found by considering either the motion of the rod or the disk. Considering the motion of the rod we write,

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\ &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\ &= -v_A\hat{i} + \dot{\theta}\hat{k} \times \ell(\cos\theta\hat{i} + \sin\theta\hat{j}) \\ &= -v_A\hat{i} + \dot{\theta}\ell \cos\theta\hat{j} - \dot{\theta}\ell \sin\theta\hat{i} \\ &= -(v_A + \dot{\theta}\ell \sin\theta)\hat{i} + \dot{\theta}\ell \cos\theta\hat{j}. \end{aligned} \quad (9.48)$$

Now considering the motion of the disk we write,

$$\begin{aligned}\vec{v}_B &= \vec{\omega}_D \times \vec{r}_{B/O} \\ &= \dot{\phi}\hat{k} \times R(-\sin\theta\hat{i} + \cos\theta\hat{j}) \\ &= -\dot{\phi}R \sin\theta\hat{j} - \dot{\phi}R \cos\theta\hat{i}. \end{aligned} \quad (9.49)$$

But  $\vec{v}_B = \vec{v}_B$ , therefore, from equations (9.48) and (9.49) we get

$$-(v_A + \dot{\theta}\ell \sin\theta)\hat{i} + \dot{\theta}\ell \cos\theta\hat{j} = -\dot{\phi}R \sin\theta\hat{j} - \dot{\phi}R \cos\theta\hat{i}$$

By equating the  $\hat{i}$  and  $\hat{j}$  components of the above equation we get

$$-(v_A + \dot{\theta}\ell \sin\theta) = -\dot{\phi}R \cos\theta, \quad (9.50)$$

$$\text{and } \dot{\theta}\ell \cos\theta = -\dot{\phi}R \sin\theta$$

$$\Rightarrow \dot{\theta} = -\dot{\phi} \frac{R}{\ell} \tan\theta. \quad (9.51)$$

Dividing Eqn. (9.50) by (9.51) we get

$$\begin{aligned}-\left(\frac{v_A}{\dot{\theta}} + \ell \sin\theta\right) &= \frac{\ell \cos\theta}{\tan\theta} = \ell \frac{\cos^2\theta}{\sin\theta} \\ \Rightarrow -\frac{v_A}{\dot{\theta}} &= \ell \left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right) \\ &= \ell \left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\right) \\ &= \frac{\ell}{\sin\theta}\end{aligned}$$

$$\Rightarrow \dot{\theta} = -\frac{v_A}{\ell} \sin \theta = -\frac{v_A}{\ell} \sin 30^\circ \\ = -\frac{2 \text{ m/s}}{2 \text{ m}} \cdot \frac{1}{2} = -0.5 \text{ rad/s.}$$

Thus  $\vec{\omega}_{AB} = \dot{\theta} \hat{k} = -0.5 \text{ rad/s} \hat{k}$ .

$$\boxed{\vec{\omega}_{AB} = -0.5 \text{ rad/s} \hat{k}}$$

(b) From Eqn. (9.51)

$$\begin{aligned} \dot{\phi} &= -\frac{\dot{\theta} \ell}{R \tan \theta} = -\frac{\overbrace{-\frac{v_A}{\ell} \sin \theta \ell}^{\dot{\theta}}}{R \frac{\sin \theta}{\cos \theta}} = \frac{v_A}{R} \cos \theta \\ &= -\frac{2 \text{ m/s}}{0.5 \text{ m}} \frac{\sqrt{3}}{2} = 2\sqrt{3} \text{ rad/s.} \end{aligned}$$

Thus  $\vec{\omega}_D = \dot{\phi} \hat{k} = 3.46 \text{ rad/s} \hat{k}$ .

$$\boxed{\vec{\omega}_D = 3.46 \text{ rad/s} \hat{k}}$$

[At this point, it is a good idea to check our algebra by substituting the values of  $\dot{\theta}$  and  $\dot{\phi}$  in equations (9.48) and (9.49) to calculate  $\vec{v}_B$ .] ①

(c) Now we can calculate the velocity of the center of mass of the rod by considering either point A or point B as a reference:

$$\begin{aligned} \vec{v}_G &= \vec{v}_A + \vec{v}_{G/A} \\ &= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{G/A} \\ &= -v_A \hat{i} + \dot{\theta} \hat{k} \times \frac{\ell}{2} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= -(v_A + \dot{\theta} \frac{\ell}{2} \sin \theta) \hat{i} + \dot{\theta} \frac{\ell}{2} \cos \theta \hat{j} \\ &= -(2 \text{ m/s} - 0.5 \text{ rad/s} \cdot \frac{2 \text{ m}}{2} \cdot \frac{1}{2}) \hat{i} + (-0.5 \text{ rad/s} \cdot \frac{2 \text{ m}}{2} \cdot \frac{\sqrt{3}}{2}) \hat{j} \\ &= -\frac{7}{4} \text{ m/s} \hat{i} - \frac{\sqrt{3}}{4} \text{ m/s} \hat{j}. \end{aligned}$$

$$\boxed{\vec{v}_G = -(1.75 \hat{i} + 0.43 \hat{j}) \text{ m/s}}$$

We could easily check our calculation by taking point B as a reference and writing

$$\begin{aligned} \vec{v}_G &= \vec{v}_B + \vec{v}_{G/B} \\ &= \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{G/B} \end{aligned}$$

By plugging in appropriate values we get, of course, the same value as above.

**Comment:** We used the standard basis vectors  $\hat{i}$  and  $\hat{j}$  for all our vector calculations in this sample. We can shorten these calculations by choosing other appropriate basis vectors as we show in the following samples.

① Substituting  $\dot{\theta} = -0.5 \text{ rad/s}$  in Eqn. (9.48) and plugging in the given values of other variables we get

$$\vec{v}_B = -\left(\frac{3}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j}\right) \text{ m/s.}$$

Similarly, substituting  $\dot{\phi} = 2\sqrt{3} \text{ rad/s}$  in Eqn. (9.48) and plugging in the other given values we get

$$\vec{v}_B = -\left(\frac{3}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j}\right) \text{ m/s,}$$

which checks with the  $\vec{v}_B$  found above.

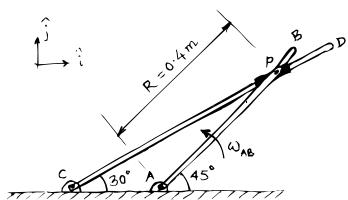


Figure 9.58: (Filename:sfig10.4.rods1dof)

**SAMPLE 9.13 Another one-DOF mechanism.** A mechanism consists of two rods AB and CD connected together at P with a collar pinned to AB but free to slide on CD. Rod AB is driven with  $\vec{\omega}_{AB} = 10 \text{ rad/s} \hat{k}$  and  $\vec{\alpha}_{AB} = 4 \text{ rad/s}^2 \hat{k}$ . At the instant shown,  $\theta = 45^\circ$  and  $\beta = 30^\circ$ . The length of rod AB is  $R = 0.4 \text{ m}$ . At the instant shown,

- Find the angular velocity and angular acceleration of rod CD.
- Find the velocity and acceleration of the collar with respect to rod CD.

**Solution** Here, we are interested in instantaneous kinematics of this mechanism. Since point P is on rod AB as well as on rod CD, its velocity and acceleration can be expressed in terms of the angular motion of rod AB or that of rod CD. Let us consider rod AB first. Let  $\hat{e}_R$  and  $\hat{e}_\theta$  be basis vectors attached to rod AB that rotate with the rod. Since P is fixed on rod AB, it executes simple circular motion about A with  $\vec{\omega}_{AB} = \dot{\theta} \hat{k}$  and  $\vec{\alpha}_{AB} = \ddot{\theta} \hat{k}$  where  $\dot{\theta} = 10 \text{ rad/s}$  and  $\ddot{\theta} = 4 \text{ rad/s}^2$ , respectively. Then

$$\vec{v}_B = R\dot{\theta}\hat{e}_\theta \quad (9.52)$$

$$\vec{a}_B = -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta. \quad (9.53)$$

Now let us consider rod CD and express the velocity and acceleration of point P in terms of motion of rod CD. Let the angular velocity and angular acceleration of rod CD be  $\vec{\omega}_{CD} = \dot{\beta} \hat{k}$  and  $\vec{\alpha}_{CD} = \ddot{\beta} \hat{k}$ , respectively. Let  $\hat{\lambda}$  and  $\hat{n}$  be basis vectors attached to rod CD. Let the instantaneous position of point be on CD be  $\vec{r}_P = r\hat{\lambda}$ . Since the collar can slide along CD, we can write the velocity and acceleration of point P as

$$\vec{v}_B = \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} \quad (9.54)$$

$$\vec{a}_B = (\ddot{r} - r\dot{\beta}^2)\hat{\lambda} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{n}. \quad (9.55)$$

From eqn. (9.52) and 9.54 we get,

$$\begin{aligned} \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} &= R\dot{\theta}\hat{e}_\theta \\ \Rightarrow \dot{r} &= R\dot{\theta}(\hat{e}_\theta \cdot \hat{\lambda}) \\ \dot{\beta} &= (1/r)R\dot{\theta}(\hat{e}_\theta \cdot \hat{n}) \end{aligned}$$

Similarly, from eqn. (9.53) and 9.55 we get,

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= -R\dot{\theta}^2(\hat{e}_R \cdot \hat{\lambda}) + R\ddot{\theta}(\hat{e}_\theta \cdot \hat{\lambda}) \\ r\ddot{\beta} + 2\dot{r}\dot{\beta} &= -R\dot{\theta}^2(\hat{e}_R \cdot \hat{n}) + R\ddot{\theta}(\hat{e}_\theta \cdot \hat{n}). \end{aligned}$$

Thus, to find all kinematic quantities of interest, all we need now is to figure out a few dot products between the two sets of basis vectors. This is easily done by writing out  $\hat{e}_R$ ,  $\hat{e}_\theta$ ,  $\hat{\lambda}$ , and  $\hat{n}$ . ① Substituting the dot products in the expressions for  $\dot{r}$ ,  $\dot{\beta}$ ,  $\ddot{r}$ , and  $\ddot{\beta}$  we get

$$\begin{aligned} \dot{r} &= R\dot{\theta} \sin(\beta - \theta), \\ \dot{\beta} &= R\dot{\theta} \cos(\beta - \theta) \\ \ddot{r} &= r\dot{\beta}^2 - R\dot{\theta}^2 \cos(\beta - \theta) + R\ddot{\theta} \sin(\beta - \theta), \\ \ddot{\beta} &= r^{-1}[-R\dot{\theta}^2 \sin(\theta - \beta) + R\ddot{\theta} \cos(\theta - \beta) - 2\dot{r}\dot{\beta}]. \end{aligned}$$

Substituting the given values of  $R$ ,  $\dot{\theta}$ ,  $\ddot{\theta}$ ,  $r$ ,  $\theta$ ,  $\beta$ , and  $r = R \sin \theta / \sin \beta$ , we get

$$\dot{r} = -1.04 \text{ m/s}, \quad \dot{\beta} = 6.83 \text{ rad/s}, \quad \ddot{r} = -12.66 \text{ m/s}^2, \quad \ddot{\beta} = 9.43 \text{ rad/s}^2.$$

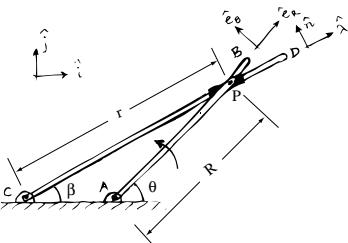
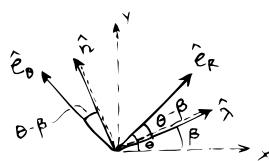


Figure 9.59: (Filename:sfig10.4.rods1dof.a)

Figure 9.60: Geometry of the unit vectors.  
(Filename:sfig10.4.rods1dof.b)

- ① We have,  
 $\hat{e}_R = \cos \theta \hat{i} + \sin \theta \hat{j}$ ,  
 $\hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$ ,  
 $\hat{\lambda} = \cos \beta \hat{i} + \sin \beta \hat{j}$ ,  
 $\hat{n} = -\sin \beta \hat{i} + \cos \beta \hat{j}$ .  
Therefore,  
 $\hat{e}_R \cdot \hat{\lambda} = \cos(\beta - \theta)$ ,  
 $\hat{e}_R \cdot \hat{n} = \sin(\beta - \theta)$ ,  
 $\hat{e}_\theta \cdot \hat{\lambda} = \sin(\theta - \beta)$ ,  
 $\hat{e}_\theta \cdot \hat{n} = \cos(\theta - \beta)$ .

(a) $\vec{\omega}_{CD} = 6.83 \text{ rad/s} \hat{k}$ ,	$\vec{\alpha}_{CD} = 9.43 \text{ rad/s}^2 \hat{k}$
(b) $\vec{v}_{/CD} = -1.04 \text{ m/s} \hat{\lambda}$ ,	$\vec{a}_{/CD} = -12.66 \text{ m/s}^2 \hat{\lambda}$

**SAMPLE 9.14 A two-DOF mechanism.** A two degree-of-freedom mechanism made of three rods and two sliders is shown in the figure. At the instant shown, the crank AB is rotating with angular velocity  $\vec{\omega}_{AB} = 12 \text{ rad/s} \hat{k}$  and angular acceleration  $\dot{\omega}_{AB} = 10 \text{ rad/s}^2 \hat{k}$ . At the same instant, the collar at end C of the link rod CD is sliding on the vertical rod with velocity  $\vec{v}_C = 0.5 \text{ m/s} \hat{j}$  and acceleration  $\vec{a}_C = 10 \text{ m/s}^2 \hat{j}$ . Find the angular velocity and angular acceleration of the link rod CD.

**Solution** Once again, we are interested in instantaneous kinematics — we wish to find the angular velocity and acceleration of rod CD at the given instant. This problem is just like the previous sample problem except that end C of the link rod CD is not fixed but free to slide on the vertical bar. But the velocity and acceleration of point C is given; so it is exactly like the previous sample (there, the velocity and acceleration of point C was identically zero). So, we adopt the same line of attack. We figure out the velocity and acceleration of point B using the kinematics of rod AB. We then write the velocity and acceleration of the same point using the kinematics of rod CD (this will involve the unknown angular velocity and acceleration of CD that we are interested in). Equate the two and solve for the unknowns we are interested in.

Let the angular velocity and acceleration of rod CD be  $\vec{\omega}_{CD} = \dot{\beta} \hat{k}$  and  $\dot{\omega}_{CD} = \ddot{\beta} \hat{k}$ , respectively. Let  $\hat{e}_R$  and  $\hat{e}_\theta$  be base vectors rotating with rod AB, and  $\hat{\lambda}$  and  $\hat{n}$  be the base vectors rotating with rod CD (see Fig. 9.62). Considering rod AB, we have

$$\vec{v}_B = R\dot{\theta}\hat{e}_\theta \quad (9.56)$$

$$\vec{a}_B = -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta. \quad (9.57)$$

Considering rod CD, we have

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} = v_C \hat{j} + \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} \quad (9.58)$$

$$\vec{a}_B = \vec{a}_C + \vec{a}_{B/C} = a_C \hat{j} + (\ddot{r} - r\dot{\beta}^2)\hat{\lambda} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{n}. \quad (9.59)$$

Now equating eqn. (9.56) and (9.58), and dotting both sides with  $\hat{\lambda}$  and  $\hat{n}$ , we get

$$\dot{r} = R\dot{\theta} \underbrace{(\hat{e}_\theta \cdot \hat{\lambda})}_{-\sin(\theta-\beta)} - v_C \underbrace{(\hat{j} \cdot \hat{\lambda})}_{\sin\beta} = -R\dot{\theta} \sin(\theta - \beta) - v_C \sin\beta \quad (9.60)$$

$$r\dot{\beta} = R\dot{\theta} \underbrace{(\hat{e}_\theta \cdot \hat{n})}_{\cos(\theta-\beta)} - v_C \underbrace{(\hat{j} \cdot \hat{n})}_{\cos\beta} = R\dot{\theta} \cos(\theta - \beta) - v_C \cos\beta \quad (9.61)$$

where the dot products among the basis vectors are easily found from either their geometry (see Fig. 9.63) or from their component representation (see previous sample). Following exactly the same procedure, we get, from eqn. (9.57) and 9.59,

$$\ddot{r} = -R\dot{\theta}^2 \cos(\theta - \beta) + R\ddot{\theta}(-\sin(\theta - \beta)) - a_C \sin\beta + r\dot{\beta}^2 \quad (9.62)$$

$$r\ddot{\beta} = -R\dot{\theta}^2 \sin(\theta - \beta) + R\ddot{\theta} \cos(\theta - \beta) - a_C \cos\beta - 2\dot{r}\dot{\beta}. \quad (9.63)$$

Now, note that although we are only interested in finding  $\dot{\beta}$  and  $\ddot{\beta}$ . So, we only need eqn. (9.61) and eqn. (9.63). But, eqn. (9.63) requires  $\dot{r}$  on the right hand side and, therefore, we do need eqn. (9.60). We can, however, happily ignore eqn. (9.62).

Now, to find the numerical values of  $\dot{\beta}$  and  $\ddot{\beta}$ , we need to find  $r$  and  $\theta$  in addition to all other given values. Consider triangle ABC in Fig. 9.62. Using the law of sines ( $\frac{R}{\sin\beta} = \frac{r}{\sin\theta} = \frac{d}{\sin(\theta-\beta)}$ ), we get  $r = 0.7 \text{ m}$  and  $\theta = 79.45^\circ$ . Now, substituting all known numerical values in eqns. (9.60), (9.61), and (9.63), we get

$$\dot{r} = -3.75 \text{ m/s}, \quad \dot{\beta} = 6.61 \text{ rad/s}, \quad \ddot{\beta} = 8.43 \text{ rad/s}^2.$$

$$\vec{\omega}_{CD} = (6.61 \text{ rad/s})\hat{k}, \quad \dot{\omega}_{CD} = (8.43 \text{ rad/s}^2)\hat{k}$$

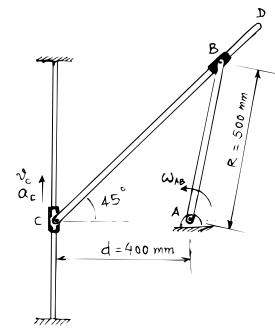


Figure 9.61: (Filename:sfig10.4.rods2dof)

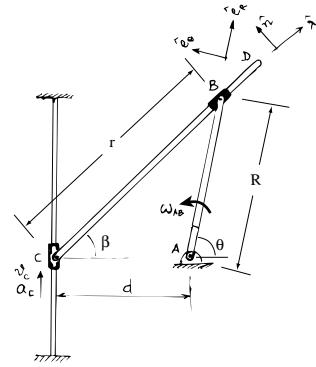


Figure 9.62: (Filename:sfig10.4.rods2dof.a)

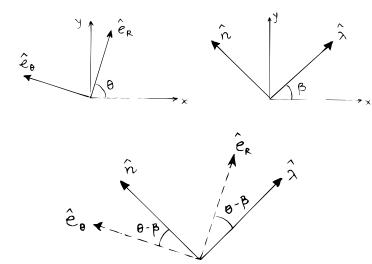


Figure 9.63: (Filename:sfig10.4.rods2dof.b)

## 9.5 Advance kinematics of planar motion

?? In this section we consider three types of problems where the kinematics involves solution of differential equations. In most cases this means computer solution is involved for this type of problem. Here are the three problem types:

- **I. Closed kinematic chains.** The main simple example is a 4-bar linkage with one bar grounded. This system has one degree of freedom, but it is difficult to directly calculate the positions, velocities and accelerations of all points in terms of one variable. Instead, the constraint that the linkage is closed is expressed as a differential equation.
- **II. Rolling contact with not-round objects.** For non-round rollers and cams solving for configuration, velocity and acceleration can sometimes be best done with integration. A side benefit from studying this topic is the observation that all non-translational motions are equivalent to rolling of some kind.
- **III. Contact with ideal wheels and skates, looking down.** Cars, tricycles, trailers, grocery carts and sleighs have wheels and have dynamics that is sometimes well characterized by planar analysis, where the plane is the horizontal plane. In this view the simple model of a wheel is as something that prevents sideways motion but allows motion in the direction of travel (like for some of the trike and car problems in 1-D constrained motion). Such problems are called *non-holonomic* (see box 9.3 on page 576).

### Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than do they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

#### *Example: Four bar linkage: configuration variables*

It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of  $\theta$ ,  $\dot{\theta}$  and  $\ddot{\theta}$  of any one of its joints.

□

However, given a configuration, the constraint on the rates and accelerations is relatively easy to express, always yielding linear equations.

**Example: Four bar linkage: configuration rates**

If you write the relative velocities of the ends of the bars in terms of configuration rates  $\dot{\theta}_1$ ,  $\dot{\theta}_2$ , and  $\dot{\theta}_3$  and then write the chain closure equation you get a linear equation in the rates. Likewise if you write the closure condition in terms of acceleration. The coefficients in these equations are likely to be complex functions of the configuration, so integrating these equations requires numerics. But the constraint is linear.  $\square$

Thus, as shown in the last sample of Sect. 10.4, one way to calculate the evolving configurations of a closed chain is to integrate the velocity relations numerically.

## Rolling of not round objects

When two objects roll on each other they maintain contact and do not slip relative to each other. That is to say rolling of one rigid curve  $\mathcal{B}$  on another  $\mathcal{A}$  means:

- The instantaneous relative motion of  $\mathcal{B}$  with respect to  $\mathcal{A}$  is a rotation about the contact point at the common tangent C, and
- The sequence of points C moves the same distance on both curves.

For simplicity let's take  $\mathcal{A}$  to be a curve fixed in space on which rigid curve  $\mathcal{B}$  rolls. Take a reference point of interest fixed on body  $\mathcal{B}$  to be O'. So,

$$\vec{v}_{O'} = \vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/C} \quad \text{where } \vec{\omega} = \dot{\theta}_{\mathcal{B}} \hat{k} \quad (9.64)$$

and  $\theta_{\mathcal{B}}$  is, say, the rotation of a  $\hat{t}'$  axis fixed in  $\mathcal{B}$  relative to a  $\hat{t}$  axis fixed in  $\mathcal{A}$ . If we use the rotation  $\theta_{\mathcal{B}}$  of body  $\mathcal{B}$  as our configuration variable, we now know how to find the velocity of all points in terms of their positions and the rotation rate. Thus we can find the rate of change of the configuration. To proceed as time progresses we also need to know how the position of point C evolves. Not the material point C on either body, but the location of mutual contact.

If we assume that both curves are parameterized by arc-length going counter-clock wise, if we take curvature as positive if directed towards the interior of each curve's body, then the condition of maintaining contact requires that

$$\vec{v}_C = \dot{s} \hat{e}_t \quad \text{where } \hat{e}_t \text{ is the tangent to fixed curve } \mathcal{A}.$$

and  $s$  is the advance along curve  $\mathcal{A}$ . To maintain tangency, the angles must be maintained so

$$\dot{s} = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}} \dot{\theta}.$$

Altogether this gives

$$\vec{v}_C = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}} \dot{\theta} \hat{e}_t \quad (\text{not the velocity of any material point}).$$

To find the acceleration of material point O' on  $\mathcal{B}$  we differentiate eqn. (9.64) with respect to time:

$$\begin{aligned}
 \vec{a}_{O'} &= \frac{d}{dt} \vec{v}_{O'} \\
 &= \frac{d}{dt} (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/C}) \\
 &= \frac{d}{dt} (\vec{\omega}_{\mathcal{B}} \times (\vec{r}_{O'} - \vec{r}_C)) \\
 &= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{O'} - \vec{\omega} \times \vec{v}_C \\
 &= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} - \omega_{\mathcal{B}}^2 \vec{r}_{O'/C} - \vec{\omega} \times \left( \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}} \dot{\theta} \hat{e}_t \right) \\
 &= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} - \omega_{\mathcal{B}}^2 \vec{r}_{O'/C} + \left( \frac{\dot{\theta}_{\mathcal{B}}^2}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}} \hat{e}_n \right)
 \end{aligned}$$

where  $\hat{e}_n$  is normal to the curves and directed towards the interior of  $\mathcal{B}$ . Thus the acceleration of all points on  $\mathcal{B}$  is the same as if the body were pinned at C *plus* an acceleration due to rolling. This rolling acceleration is small if either of the bodies is sharp (has very large  $\kappa$ ) and large if the bodies are nearly conformal.

#### *Body curve and space curve*

As one rigid body moves arbitrarily on a plane with some non-zero rotation rate we can find a point at relative position  $\vec{r}_{C/O'}$  where  $\vec{r}_C = \vec{0}$ . That is a place where the velocity due to rotation about O' exactly cancels the velocity of O'.

$$\vec{0} = \vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{C/O'}$$

Crossing both sides with  $\hat{k}$  and using that  $\vec{\omega} = \omega \hat{k}$  we get

$$\vec{r}_{C/O'} = \frac{\hat{k} \times \vec{v}_{O'}}{\omega_{\mathcal{B}}}$$

as the point “on” the body that has no velocity. This point does not literally have to be on the body, rather it is fixed to the reference frame defined by the body.

As motion progresses a sequence of such points C is traced on the ground. Similarly a sequence of points is traced on the body. These two sequences are called the space curve and the body curve (or “polohodie” and “herpolhodie” in older books). The motion of body  $\mathcal{B}$  is thus a rolling of the body curve on the space curve.

As a machine designer this means you can generate any desired motion by rolling of appropriate shapes. Move the object in the desired manner, draw the space curve and body curve, make parts with those shapes, and the desired motion occurs by a rolling of those shapes.

#### **Ideal wheels and skates, looking down**

If we look down on an ideal skate or wheel at point C on a rigid body and assume that the skate is oriented with the positive  $\vec{i}'$  axis at point C on the body then we know that

$$\vec{v}_C = v_C \vec{i}'$$

and hence the velocity of any point G on the body of interest is

$$\begin{aligned}\vec{v}_G &= \vec{v}_C + \vec{v}_{G/C} \\ &= v_c \hat{i}' + \dot{\theta} \hat{k} \times \vec{r}_{G/C}.\end{aligned}$$

The acceleration is found by differentiating this expression as

$$\begin{aligned}\vec{a}_G = \frac{d}{dt} \vec{v}_G &= \vec{v}_C + \vec{v}_{G/C} \\ &= \frac{d}{dt} (v_c \hat{i}' + \dot{\theta} \hat{k} \times \vec{r}_{G/C}) \\ &= \dot{v}_c \hat{i}' + v_c \dot{\hat{i}'} + \ddot{\theta} \hat{k} \times \vec{r}_{G/C} - \dot{\theta}^2 \vec{r}_{G/C} \\ &= \underbrace{\dot{v}_c \hat{i}' + v_c \dot{\theta} \hat{j}'}_{\vec{a}_C} + \underbrace{+ \ddot{\theta} \hat{k} \times \vec{r}_{G/C} - \dot{\theta}^2 \vec{r}_{G/C}}_{\vec{a}_{G/C}}.\end{aligned}$$

It is interesting to note that the Coriolis-like term  $v_C \dot{\theta} \hat{j}'$  does not have the usual factor of 2 one encounters in holonomic problems. To find the trajectory of the point C, say, one needs to integrate the velocity like this:

$$\begin{aligned}\dot{x} &= \vec{v}_C \cdot \hat{i} \\ &= v_C \hat{i}' \cdot \hat{i} \\ &= v_C \cos \theta \\ \dot{y} &= \vec{v}_C \cdot \hat{j} \\ &= v_C \hat{i}' \cdot \hat{j} \\ &= v_C \sin \theta.\end{aligned}$$

### 9.3 THEORY

#### Skates, wheels and non-holonomic constraints

Of the words in this book “non-holonomic” is probably the most obscure. This is because the subject of mechanics was mostly stolen from engineers by physicists about 100 years ago. And physicists, the authors of most introductions to mechanics, had no use for non-holonomic mechanics as it wasn’t useful for the development of quantum mechanics. So many people are unaware of the word, the subject or its utility.

In two dimensions the word non-holonomic in effect means the mechanics of objects constrained by ideal skates or massless ideal wheels. Often these decades non-holonomic constraints are described as “non integrable”. Literally, the word non-holonomic means “not whole”. But in what sense is a rolling ideal wheel “non-integrable” or less “whole” than anything else?

**A constrained rigid body.** Consider a rigid body that is free to slide on a plane. It has three degrees of freedom described by  $x_{O'}$ ,  $y_{O'}$  and  $\theta$ , all measured relative to a fixed reference frame  $O\hat{i}\hat{j}$ . Point C on the body has relative position  $\vec{r}_{C/O'} = x'_C \hat{i}' + y'_C \hat{j}'$  where  $x'_C$  and  $y'_C$  are constants. Now lets constrain the body at point C one of these two different ways:

- a) Pin the body to the ground with an ideal hinge at point C. This keeps point C from moving but allows the body to rotate (holonomic).
- b) Put an ideal wheel or skate under the body at C that prevents sliding sideways to the skate but allows point C to move parallel to the skate and also allows rotation about the skate (non-holonomic).

**Pin Constraint.** In the first case, for the pin, we could describe the constraint with the phrase ‘point C on the body can have no velocity’ and the write and calculate:

$$\begin{aligned} \vec{0} &= \vec{v}_C = \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{C/O'} & (9.65) \\ \vec{0} &= \dot{x}_{O'} \hat{i} + \dot{y}_{O'} \hat{j} + \dot{\theta} \hat{k} \times (x'_C \hat{i}' + y'_C \hat{j}') \\ \{ \vec{0} &= \dot{x}_{O'} \hat{i} + \dot{y}_{O'} \hat{j} + (\dot{\theta} x'_C \hat{j}' - \dot{\theta} y'_C \hat{i}') \} \\ \{ \cdot \hat{i} &\Rightarrow \dot{x}_{O'} - \dot{\theta} \sin \theta \dot{x}'_C - \dot{\theta} \cos \theta \dot{y}'_C = 0 \\ \{ \cdot \hat{j} &\Rightarrow \dot{y}_{O'} + \dot{\theta} \cos \theta \dot{x}'_C - \dot{\theta} \sin \theta \dot{y}'_C = 0. \\ &\Rightarrow \frac{d}{dt} (x_{O'} + \cos \theta \dot{x}'_C - \sin \theta \dot{y}'_C) = 0 \\ &\Rightarrow \frac{d}{dt} (y_{O'} + \sin \theta \dot{x}'_C + \cos \theta \dot{y}'_C) = 0 \end{aligned}$$

The last two equations are two differential equations in the three variables  $x_{O'}$ ,  $y_{O'}$  and  $\theta$ . They are “integrable” in the sense that they are equivalent to

$$\begin{aligned} x_{O'} + \cos \theta \dot{x}'_C - \sin \theta \dot{y}'_C &= C_1 \\ \text{and } y_{O'} + \sin \theta \dot{x}'_C + \cos \theta \dot{y}'_C &= C_2 & (9.66) \end{aligned}$$

where  $C_1$  and  $C_2$  are integration constants that need to be set by the starting configuration. Solving for  $x_{O'}$  and  $y_{O'}$  in terms of  $\theta$ :

$$\begin{aligned} x_{O'} &= C_1 - \cos \theta \dot{x}'_C + \sin \theta \dot{y}'_C \\ y_{O'} &= C_2 - \sin \theta \dot{x}'_C - \cos \theta \dot{y}'_C. \end{aligned}$$

Here we have derived the obvious, that a pinned body has one independent configuration variable  $\theta$ , but we did so starting with a vector expression of constraint in terms of velocities (eqn. (9.65)). Then we wrote the constraint as two scalar constraints on the derivatives of configuration variables and then “integrated” them to write constraints on the configuration variables, finally eliminating two of the configuration variables.

**Skate constraint.** Now consider the same body constrained by a skate or ideal wheel at C instead of a pin. The skate is aligned with the  $\hat{i}'$  so point C can only move in the  $\hat{i}'$  direction. The body is still free to rotate about the point C (to steer). Thus,

$$\begin{aligned} 0 &= \vec{v}_C \cdot \hat{j}' = \left( \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{C/O'} \right) \cdot \hat{j}' & (9.67) \\ 0 &= \left( \dot{x}_{O'} \hat{i} + \dot{y}_{O'} \hat{j} + \dot{\theta} \hat{k} \times (x'_C \hat{i}' + y'_C \hat{j}') \right) \cdot \hat{j}' \\ 0 &= -\dot{x}_{O'} \sin \theta + \dot{y}_{O'} \cos \theta + \dot{\theta} x'_C \\ \Rightarrow 0 &= \frac{d}{dt} F(x_{O'}, y_{O'}, \theta) ? \end{aligned}$$

As for the hinge where we found 2 constant functions, we might want to find the function  $F(x_{O'}, y_{O'}, \theta)$  that satisfies the differential equation above, namely

$$\frac{d}{dt} F(x_{O'}, y_{O'}, \theta) = -\dot{x}_{O'} \sin \theta + \dot{y}_{O'} \cos \theta + \dot{\theta} x'_C. & (9.68)$$

Another math nightmare. How do we find this  $F$ ? You can’t find one. This is the crux of the matter. Neither your calculus professor nor Ramanujan could find one either. No computer can find one, or even a numerical approximation of a solution. There is no function  $F(x_{O'}, y_{O'}, \theta)$  that solves eqn. (9.68). The solution fundamentally does not exist. That is why we say the skate/wheel constraint eqn. (9.67) is “non-integrable”.

**Parallel parking** We can use physical reasoning to show that no function  $F$  can solve eqn. (9.68). If such an  $F$  did exist it would mean that only the set of configurations with position  $x_{O'}$ ,  $y_{O'}$  and angles  $\theta$  consistent with  $F = \text{constant}$  would be allowed by the skate constraint (assuming  $F$  depends nontrivially on at least one of the variables). This means there would be some angles and positions that the body couldn’t get to. Remember, we are not doing mechanics, just kinematics. So we can see what configurations are geometrically allowed while still respecting the constraint. The simple observation that motivates the answer is this:

Even though the skate constrains  $\vec{v}_C$  to not have a sideways component, point C can get to a point that is straight sideways.

How? Like a car can move sideways into a parking space without skidding sideways; by parallel parking. More generally, the body can get to any position and any orientation by the following moves. First rotate the body so the skate aims to its new goal. Then slide the skate to its new goal. And finally rotate the body to its new desired orientation.

Thus, the skate constraint does not disallow *any* configurations! Yet the constraint does disallow some velocities (the skate can’t go sideways). In this way, the skate constraint is not “whole”. It constrains velocities without constraining configurations.

**Counting degrees of freedom.** How many degrees of freedom does a body with a skate constraint have? There are two different answers. By counting possible configurations there are three degrees of freedom (it takes three variables to describe all possible configurations). But at any configuration the velocity can be described by 2 numbers ( $\dot{\theta}$  and  $v_{\hat{i}'}^*$ ). Whenever the number of configuration degrees of freedom is greater than the number of velocity degrees of freedom (for example, 3>2) there are non-holonomic constraints.

One might like more examples. But besides artificial mathematical ones, there are none. The only smooth non-holonomic constraint in 2D mechanics is the ideal skate or wheel.



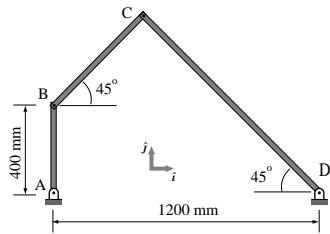


Figure 9.64: (filename:sfig10.4.fourbar)

**SAMPLE 9.15 Kinematics of a four bar linkage.** A four bar linkage ABCD is shown in the figure (fourth bar is the ground AD) at some instant  $t_0$ . The driving bar AB rotates with angular velocity  $\vec{\omega}_{AB} = \dot{\theta}(t)\hat{k}$ . Find the angular velocities of rods BC and CD as a function of  $\dot{\theta}$ . How can you solve for the positions of the bars at any  $t$  if the initial configuration is as shown in the figure?

**Solution** Let the angles that rods AB, BC, and CD make with the horizontal ( $x$ -axis) be  $\theta$ ,  $\beta$ , and  $\phi$ , respectively. Then, we can write  $\vec{\omega}_{BC} = \dot{\beta}\hat{k}$  and  $\vec{\omega}_{CD} = \dot{\phi}\hat{k}$ . We have to find  $\dot{\beta}$  and  $\dot{\phi}$ .

Note that the motion of point C is a simple circular motion about point A with given angular velocity  $\vec{\omega}_{AB}$ . Thus, the velocity of point B is known. Now, we can find the velocity of point C two ways: (i) by considering rod CD:  $\vec{v}_C = \vec{v}_B + \vec{v}_{C/B} = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$ , and (ii) by considering rod CD:  $\vec{v}_C = \vec{\omega}_{CD} \times \vec{r}_{C/D}$ . Either way the velocity must be the same. Thus, we have a 2-D vector equation with two unknowns  $\dot{\beta}$  and  $\dot{\phi}$ . We can get two independent scalar equations from the vector equation and thus we can solve for the desired unknowns.

Let us use the rotating base vectors  $(\hat{\lambda}_1, \hat{n}_1)$ ,  $(\hat{\lambda}_2, \hat{n}_2)$ , and  $(\hat{\lambda}_3, \hat{n}_3)$  with rods AB, BC, and CD, respectively. Note that these base vectors are basically the  $(\hat{e}_R, \hat{e}_\theta)$  pairs; we use  $(\hat{\lambda}, \hat{n})$  just for the sake of easy subscripting. Now,

$$\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_{B/A} = \ell_1 \dot{\theta} \hat{n}_1 \quad (9.69)$$

$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B} = \ell_1 \dot{\theta} \hat{n}_1 + \ell_2 \dot{\beta} \hat{n}_2 \quad (9.69)$$

$$\text{also, } \vec{v}_C = \vec{\omega}_{CD} \times \vec{r}_{C/D} = \ell_3 \dot{\phi} \hat{n}_3 \quad (9.70)$$

Thus, from eqn. (9.69) and (9.70), we have,

$$\ell_1 \dot{\theta} \hat{n}_1 + \ell_2 \dot{\beta} \hat{n}_2 = \ell_3 \dot{\phi} \hat{n}_3 \quad (9.71)$$

Dotting eqn. (9.71) with  $\hat{\lambda}_2$  (to eliminate  $\dot{\beta}$  term), we get

$$\begin{aligned} \ell_1 \dot{\theta} (\hat{n}_1 \cdot \hat{\lambda}_2) &= \ell_3 \dot{\phi} (\hat{n}_3 \cdot \hat{\lambda}_2) \\ \Rightarrow \dot{\phi} &= \frac{\ell_1 (\hat{n}_1 \cdot \hat{\lambda}_2)}{\ell_3 (\hat{n}_3 \cdot \hat{\lambda}_2)} \dot{\theta}. \end{aligned} \quad (9.72)$$

Similarly, dotting eqn. (9.71) with  $\hat{\lambda}_3$  (to eliminate  $\dot{\phi}$  term), we get

$$\dot{\beta} = -\frac{\ell_1 (\hat{n}_1 \cdot \hat{\lambda}_3)}{\ell_2 (\hat{n}_2 \cdot \hat{\lambda}_3)} \dot{\theta}. \quad (9.73)$$

We are practically done at this point with the kinematics — we have found  $\dot{\beta}$  and  $\dot{\phi}$  as functions of  $\dot{\theta}$ . The various dot products are just geometry and vector algebra. To write them explicitly, we note that  $\hat{\lambda}_1 = \cos \theta \hat{i} + \sin \theta \hat{j}$ ,  $\hat{n}_1 = -\sin \theta \hat{i} + \cos \theta \hat{j}$ ,  $\hat{\lambda}_2 = \cos \beta \hat{i} + \sin \beta \hat{j}$ , etc. Thus,

$$\hat{n}_1 \cdot \hat{\lambda}_2 = -\sin \theta \cos \beta + \cos \theta \sin \beta = \sin(\beta - \theta)$$

$$\hat{n}_3 \cdot \hat{\lambda}_2 = \sin(\beta - \phi)$$

$$\hat{n}_1 \cdot \hat{\lambda}_3 = \sin(\phi - \theta), \quad \hat{n}_2 \cdot \hat{\lambda}_3 = \sin(\phi - \beta).$$

Substituting the appropriate expressions in eqn. (9.73) and (9.72), we get

$$\dot{\beta} = -\frac{\ell_1 \sin(\phi - \theta)}{\ell_2 \sin(\phi - \beta)} \dot{\theta}, \quad \dot{\phi} = \frac{\ell_1 \sin(\beta - \theta)}{\ell_3 \sin(\beta - \phi)} \dot{\theta}.$$

$\vec{\omega}_{BC} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi - \theta)}{\sin(\phi - \beta)} \dot{\theta} \hat{k},$	$\vec{\omega}_{CD} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta - \theta)}{\sin(\beta - \phi)} \dot{\theta} \hat{k}$
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Note that the expressions for  $\dot{\beta}$  and  $\dot{\phi}$  are coupled, nonlinear, first order ordinary differential equations. To be able to find  $\dot{\theta}$ ,  $\beta(t)$  and  $\phi(t)$ , we need to integrate  $\dot{\theta}$ ,  $\dot{\beta}$ , and  $\dot{\phi}$ . Here, we set up these differential equations for numerical integration. Although, we can use any given  $\dot{\theta}(t)$  (e.g.,  $\alpha t$  or  $\dot{\theta}_0 \sin(\Omega t)$  or whatever), for definiteness in our numerical integration, let us take a constant  $\dot{\theta}$ , that is, let  $\dot{\theta} = C = 10 \text{ rad/s}$  (say). So, our equations are,

$$\dot{\theta} = C, \quad \dot{\beta} = \frac{\ell_1 \sin(\phi - \theta)}{\ell_2 \sin(\beta - \phi)} C, \quad \dot{\phi} = \frac{\ell_1 \sin(\beta - \theta)}{\ell_3 \sin(\beta - \phi)} C,$$

and the initial conditions are  $\theta(0) = \pi/2$ ,  $\beta(0) = \pi/4$ ,  $\phi(0) = 3\pi/4$ .

Here is a pseudocode that we use to integrate these equations numerically for a period of  $2\pi/C = \pi/5$  seconds (one complete revolution of AB).

```

ODEs = {thetadot = C,
        betadot = (11/12)*sin(phi-theta)/sin(beta-phi)*C,
        phidot = (11/13)*sin(beta-theta)/sin(beta-phi)*C}
IC = {theta(0) = pi/2, beta(0) = pi/4, phi(0) = 3*pi/4}
Set C=10, 11=.4, 12=.4*sqrt(2), 13=.8*sqrt(2)
Solve ODEs with IC for t=0 to t=pi/5

```

After we get the angles, we can compute the  $xy$  coordinates of points B and C at each instant and plot the mechanism at those instants. Plots thus obtained from our numerical solution are shown in Fig. 9.66 where the configuration of the mechanism is shown at 9 equally spaced times between  $t = 0$  to  $t = \pi/5$ .

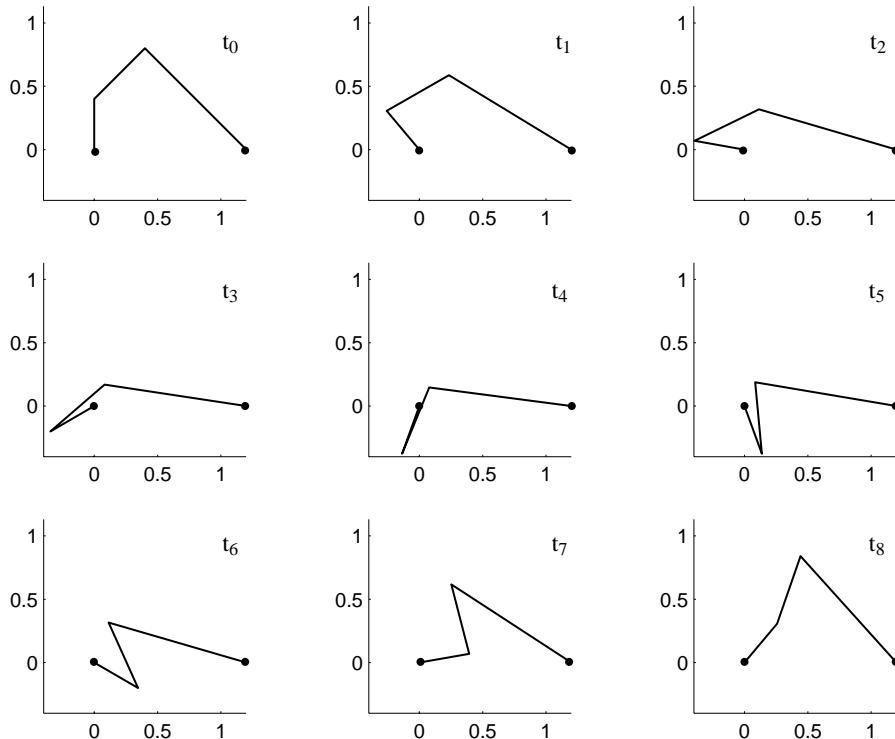


Figure 9.66: Several configurations of the mechanism at equal intervals of time during one complete revolution of the driving link. After  $t_8$  the mechanism returns to the initial configuration.

(Filename:sfig10.4.fourbar.b)



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# 10

# Mechanics of constrained particles and rigid bodies

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We have studied the mechanics of particles and rigid bodies with constraints that require progressively more involved kinematics. We now proceed to study the mechanics of more complex systems: particles with constrained paths, particles moving relative to moving frames, and mechanisms with several parts.

The basic strategy throughout is to use, in combination, the following skills which you have been developing:

1. Basic modeling. Describe a system in an appropriate way using the language of particle and rigid body mechanics. As described in Chapter 2, the force modeling and kinematic modeling are coupled. Where relative motion is freely allowed there is no force. And where motion is caused or prevented there is a force. Here is where you decide the constitutive (force) laws you are using for springs, contact, gravity, etc.
2. Draw free body diagrams of the system of interest and of its parts. These diagrams should show what you do and do not know about the constraint forces (*e.g.*, at a pin connection cut free in a free body diagram the FBD should show an arbitrary force and no moment). These are exactly the same free body diagrams that one would draw for statics.
3. Kinematics calculations. Pick appropriate configuration variables, as many as there are degrees of freedom. Then write the velocities, accelerations, angular

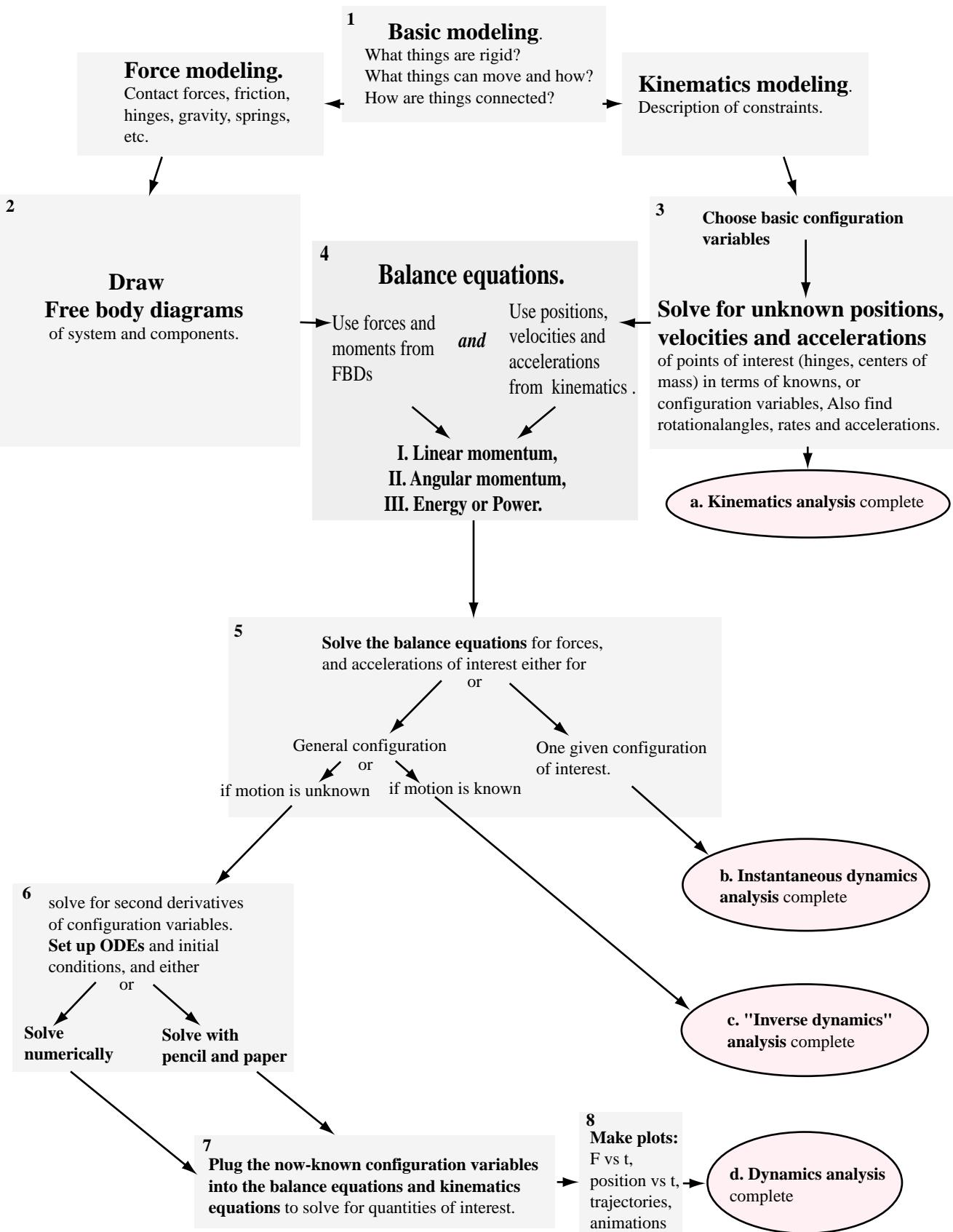


Figure 10.1: Basic flow chart for solving the various types of dynamics problems. (Filename: tfigure.conceptmap)

velocities and angular accelerations of interest in terms of the configuration variables and their first and second time derivatives, possibly using methods from Chapter 10. Often this is the hardest part of the analysis.

4. Use appropriate balance equations: linear momentum, angular momentum, or power balance equations.
5. Solve the balance equations for unknown forces or accelerations of interest. Sometimes this can be done by hand by writing out components and solving simultaneous equations or by using appropriate dot products. And sometimes it is best done on by setting up a matrix equation and solving on the computer.
6. Solve the differential equations to find how the basic configuration variables change with time. For some special problems this can be done by hand, but most often involves computer solution.
7. Plug the ODE solution from (6) above into the equations from kinematics (3 above) and the balance laws (4 above). This is not a different skill from (3) or (4), it is just applied at a different time in the work.
8. Make plots of how forces, positions and velocities change with time, or of trajectories. Animations are also often nice.

These skills are used to solve dynamics problems which often fall into one of these 4 categories.

- a. Kinematics. These are problems where *only* geometry is used, where the kinematics constraints determine what you are interested in, independent of the forces or time history. A classic example is determining the path of a point on a given four-bar linkage. More basic examples include finding position or acceleration from a given velocity history.
- b. Instantaneous dynamics. These are problems where the positions and velocities of all points are given and you need to find forces or accelerations. Often these are “first-motion” problems: what are accelerations and forces immediately after something is released from rest?
- c. “Inverse dynamics.” These problems are called “inverse” because they are backwards of the original hard dynamics problems ((d) below). In these problems the motion is given as a function of time, and you have to calculate the forces. These problems are easier than non “inverse” problems because the differential equations from the balance laws don’t need to be solved. A classic example is a slider-crank where the motion of the crank is known *a priori* to be at constant rate and you need to find the torque required to keep that motion. Usually in science “inverse” problems are harder. In dynamics this kind of “inverse” problem is easier than the non “inverse” problems.
- d. Dynamics analysis. You are given some information about forces and constraints and you have to find the motion and more about the forces. These are the capstone problems that require use of all the skills.

A flow chart showing how these problem types are solved using the basic skill components ideas is shown in Fig. 10.1. As you solve a problem, at any instant in time you should be able to place your work on this chart.

In terms of putting all the ideas together, this chapter completes the book. But we only consider two dimensional models and motions in this chapter. Three dimensional models and motions involve more difficult kinematics and are postponed until Chapter 12.

## 10.1 Mechanics of a constrained particle and of a particle relative to a moving frame

With the kinematics tools of Chapter 10 we can deal with the mechanics of a new set of interesting particle motion problems. For one point mass it is easy to write balance of linear momentum. It is:

$$\vec{F} = m\vec{a}.$$

The mass of the particle  $m$  times its vector acceleration  $\vec{a}$  is equal to the total force on the particle  $\vec{F}$ . No problem.

Now, however, we can write this equation in five somewhat distinct ways.

- (a) In general abstract vector form:  $\vec{F} = m\vec{a}$ .
- (b) In cartesian coordinates:  $F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = m[\ddot{x} \hat{i} + \ddot{y} \hat{j} + \ddot{z} \hat{k}]$ .
- (c) In polar coordinates:  

$$F_R \hat{e}_R + F_\theta \hat{e}_\theta + F_z \hat{k} = m[(\ddot{R} - R\dot{\theta}^2) \hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta}) \hat{e}_\theta + \ddot{z} \hat{k}]$$
- (d) In path coordinates:  $F_t \hat{e}_t + F_n \hat{e}_n = m[\ddot{v} \hat{e}_t + (v^2/\rho) \hat{e}_n]$

All of these equations are always right. Additionally, for a given particle moving under the action of a given force there are many more correct equations that can be found by shifting the origin and orientation of the coordinate systems. For example for a moving frame  $\mathcal{B}$  with origin  $0'$ , rotation rate  $\vec{\omega}_{\mathcal{B}}$  and angular acceleration  $\vec{\alpha}_{\mathcal{B}}$ :

$$(e) \vec{F} = m \left\{ \vec{a}_{0'} - \omega_{\mathcal{B}}^2 \vec{r} + \vec{\alpha}_{\mathcal{B}} \times \vec{r} + \vec{a}_{/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{/\mathcal{B}} \right\}$$

where, to simplify the notation, all motions are relative to  $\mathcal{F}$  and positions relative to  $0$  unless explicitly indicated by a  $/\mathcal{B}$  or  $/0'$ . This is quite a collection of kinematic tools. In general we want to choose the best tools for the job. But to get a sense lets first look at a simple problem using each of these kinematic approaches, some of which are rather inappropriate.

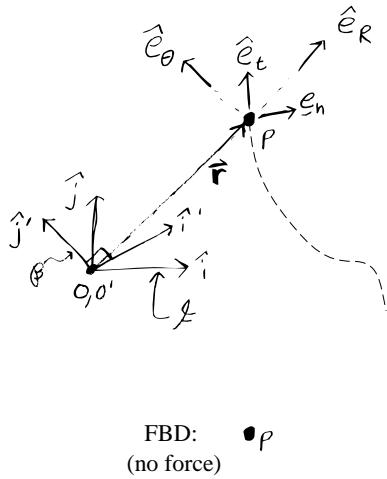


Figure 10.2: A particle P moves. One can track its motion using the general vector form  $\vec{r}$ , Cartesian coordinates in the fixed frame  $\mathcal{F} = 0\hat{i}$ , Polar coordinates using  $\hat{e}_R$  &  $\hat{e}_\theta$ , path coordinates  $\hat{e}_t$  &  $\hat{e}_n$  and cartesian coordinates in a rotating frame  $\mathcal{B} = 0'\hat{i}'\hat{j}'$ .

### A particle that moves with no net force

In the special case that a particle has no force on it we know intuitively, or from the verbal statement of Newton's First Law, that the particle travels in a straight line at constant speed. As a first example, let's try to find this result using the vector equations of motion five different ways: in the general abstract form, in cartesian coordinates, in polar coordinates, in path coordinates, and relative to a moving frame (see Fig. 10.2).

**General abstract form.** The equation of linear momentum balance is  $\vec{F} = m\vec{a}$  or, if there is no force,  $\vec{a} = \vec{0}$ , which means that  $d\vec{v}/dt = \vec{0}$ . So  $\vec{v}$  is a constant. We can call this constant  $\vec{v}_0$ . So after some time the particle is where it was at  $t = 0$ , say,  $\vec{r}_0$ , plus its velocity  $\vec{v}_0$  times time. That is:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t. \quad (10.1)$$

This vector relation is a parametric equation for a straight line. The particle moves in a straight line, as expected.

**Cartesian coordinates.** If instead we break the linear momentum balance equation into cartesian coordinates we get

$$F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = m(\ddot{x} \hat{i} + \ddot{y} \hat{j} + \ddot{z} \hat{k}).$$

Because the net force is zero and the net mass is not negligible,

$$\ddot{x} = 0, \quad \ddot{y} = 0, \text{ and } \ddot{z} = 0.$$

These equations imply that  $\dot{x}$ ,  $\dot{y}$ , and  $\dot{z}$  are all constants, lets call them  $v_{x0}$ ,  $v_{y0}$ ,  $v_{z0}$ . So  $x$ ,  $y$ , and  $z$  are given by

$$x = x_0 + v_{x0}t, \quad y = y_0 + v_{y0}t, \quad \& z = z_0 + v_{z0}t.$$

We can put these components into their place in vector form to get:

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k} = (x_0 + v_{x0}t) \hat{i} + (y_0 + v_{y0}t) \hat{j} + (z_0 + v_{z0}t) \hat{k}. \quad (10.2)$$

Note that there are six free constants in this equation representing the initial position and velocity. Equation 10.2 is a cartesian representation of equation 10.1; it describes a straight line being traversed at constant rate.

**Polar/cylindrical coordinates.** When there is no force, in polar coordinates we have:

$$\underbrace{F_R}_{0} \hat{e}_R + \underbrace{F_\theta}_{0} \hat{e}_\theta + \underbrace{F_z}_{0} \hat{k} = m[(\ddot{R} - R\dot{\theta}^2) \hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta}) \hat{e}_\theta + \ddot{z} \hat{k}].$$

This vector equation leads to the following three scalar differential equations, the first two of which are coupled non-linear equations (neither can be solved without the other).

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

A tedious calculation will show that these equations are solved by the following functions of time:

$$\begin{aligned} R &= \sqrt{d^2 + [v_0(t - t_0)]^2} \\ \theta &= \theta_0 + \tan^{-1}[v_0(t - t_0)/d] \\ z &= z_0 + v_{z0}t, \end{aligned} \quad (10.3)$$

where  $\theta_0$ ,  $d$ ,  $t_0$ ,  $v_0$ ,  $z_0$ , and  $v_{z0}$  are constants. Note that, though eqn. (10.4) looks different than eqn. (10.2), there are still 6 free constants. From the physical interpretation you know that eqn. (10.4) must be the parametric equation of a straight line. And, indeed, you can verify that picking arbitrary constants and using a computer to make a polar plot of eqn. (10.4) does in fact show a straight line. From eqn. (10.4) it seems that polar coordinates' main function is to obfuscate rather than clarify. For the simple case that a particle moves with no force at all, we have to solve non-linear differential equations whereas using cartesian coordinates we get linear equations which are easy to solve and where the solution is easy to interpret.

But, if we add a central force, a force like earth's gravity acting on an orbiting satellite (the force on the satellite is directed towards the center of the earth), the equations become almost intolerable in cartesian coordinates. But, in polar coordinates, the solution is almost as easy (which is not all that easy for most of us) as the solution 10.4. So the classic analytic solutions of celestial mechanics are usually expressed in terms of polar coordinates.

**Path coordinates.** When there is no force,  $\vec{F} = m\vec{a}$  is expressed in path coordinates as

$$\underbrace{F_t}_{0} \hat{\mathbf{e}}_t + \underbrace{F_n}_{0} \hat{\mathbf{e}}_n = m(\dot{v}\hat{\mathbf{e}}_t + \underbrace{(v^2/\rho)\hat{\mathbf{e}}_n}_{v^2\hat{\mathbf{k}}}).$$

That is,

$$\dot{v} = 0 \quad \text{and} \quad v^2/\rho = 0.$$

So the speed  $v$  must be constant and the radius of curvature  $\rho$  of the path infinite. That is, the particle moves at constant speed in a straight line.

**Relative to a rotating reference frame** Let's look at the equations using a frame  $\mathcal{B}$  that shares an origin with  $\mathcal{F}$  but is rotating at a constant rate  $\vec{\omega}_{\mathcal{B}} = \omega\hat{\mathbf{k}}$  relative to  $\mathcal{F}$ . Thus

$$\vec{\alpha}_{\mathcal{B}} = \vec{0} \quad \text{and} \quad \vec{a}_{0'/0} = \vec{0}$$

and we have that  $\vec{F} = m\vec{a}$  is written as

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \vec{0} &= m \left\{ \underbrace{\vec{a}_{0'}}_{\vec{0}} - \omega_{\mathcal{B}}^2 \vec{r} + \underbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}}_{\vec{0}} + \vec{a}_{/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{/\mathcal{B}} \right\} \\ &= -\omega^2 \vec{r} + \vec{a}_{/\mathcal{B}} + 2\omega\hat{\mathbf{k}} \times \vec{v}_{/\mathcal{B}}. \end{aligned} \quad (10.4)$$

Now, using the rotating base vectors  $\hat{\mathbf{i}}'$  and  $\hat{\mathbf{j}}'$ , we have that  $F_{x'}\hat{\mathbf{i}}' + F_{y'}\hat{\mathbf{j}}' = 0\hat{\mathbf{i}}' + 0\hat{\mathbf{j}}'$  and

$$\vec{r} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}', \quad \vec{v}_{/\mathcal{B}} = \dot{x}'\hat{\mathbf{i}}' + \dot{y}'\hat{\mathbf{j}}', \quad \text{and} \quad \vec{a}_{/\mathcal{B}} = \ddot{x}'\hat{\mathbf{i}}' + \ddot{y}'\hat{\mathbf{j}}'$$

so eqn. (10.4) can be rewritten as

$$\vec{0} = -\omega^2(x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}') + (\ddot{x}'\hat{\mathbf{i}}' + \ddot{y}'\hat{\mathbf{j}}') + 2\omega\hat{\mathbf{k}} \times (x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}')$$

which in turn can be broken into components and written as:

$$\begin{aligned} \ddot{x}' &= \omega^2 x' + 2\omega\dot{y}' \\ \ddot{y}' &= \omega^2 y' - 2\omega\dot{x}' \end{aligned} \quad (10.5)$$

which makes up a pair of second order linear differential equations. With some work, someone good at ODEs can solve this with pencil and paper. But most of us would use a computer for such a system. If, in some consistent units, we had

$$y'(0) = 0, \dot{y}'(0) = 0, x'(0) = 0, \dot{x}'(0) = 1, \omega = 1$$

then the solution turns out to be

$$\begin{aligned} x' &= t \cos(\omega t) \\ y' &= -t \sin(\omega t) \end{aligned}$$

as you can check by substituting into eqn. (10.6). That is, a particle which we goes in a straight line away from the origin goes, as seen in the rotating frame in spirals<sup>①</sup>.

## Constrained motion

A particle in a plane has 2 degrees of freedom. There is basically only one kind of constraint — to a path. When constrained to a path the particle has one remaining degree of freedom so its configuration can be described with one variable. For a given problem you must think about

① This is why we are lucky the earth is not spinning fast. And it could have been more complicated yet with variable rate rotation and acceleration. If any of these were the case, we would see particles spiraling around all over the place, as would have Isaac Newton. He would have written “A particle in motion tends to go in crazy spirals, and so does a particle that is initially stationary” and the equations we have used through out this book would have been much harder, if not impossible, to discover.

- What force(s) constrain the motion to the path?
- What do you want to use for a configuration variable?

You use these ideas to

- draw an appropriate free body diagram, and then
- calculate the velocity and acceleration in terms of the configuration variable and its derivatives.

After these key first steps you plug into equations of motion and solve for what you are interested in. Of course the needed math could be difficult or impossible, but the work is somewhat routine from a mechanics point of view.

### Example: Bead on frictionless wire

A bead slides on a frictionless wire with a crazy but smooth shape. No forces are applied to the bead besides the constraint force (see Fig. 10.3).

Fig. 10.3b shows a free body diagram where  $\hat{n}$  is the normal to the wire at the point of interest. It doesn't matter if you use for  $\hat{n} = \hat{e}_n$ , or  $\hat{n} =$  a vector always, say, to the left, just so long as you know what you mean by  $\hat{n}$ . The free body diagram shows that you know that the constraint force is normal to the path (the frictionless wire) but that you don't know how big it is ( $F$  is an unknown scalar).

For some purposes, especially general problems like this where no specific path is given, the most appropriate configuration variable is  $s$ , the arc length along the path. If the path is given we assume we know the position at any given arc length by the functions

$$x(s) \quad \text{and} \quad y(s).$$

So

$$\vec{r} = \vec{r}(s), \vec{v} = \dot{s}\hat{e}_t, \quad \text{and} \quad \vec{a} = (\dot{s}^2/\rho)\hat{e}_n + \ddot{s}\hat{e}_t.$$

Now we can write linear momentum balance

$$\begin{aligned} \vec{F} &= m\vec{a} & (10.6) \\ \{ F\hat{n} \} &= m(\dot{s}^2/\rho)\hat{e}_n + \ddot{s}\hat{e}_t \} \\ \{ \} \cdot \hat{e}_t &\Rightarrow \dot{v} = 0 \quad \text{and} \\ \{ \} \cdot \hat{e}_n &\Rightarrow F = mv^2/\rho. \end{aligned}$$

Eqn. 10.6 tells us that the bead moves at constant speed, no matter what the shape of the wire. It also tells us that the more curved the wire, the bigger the constraint force needed to keep the bead on the wire.

Because this is a 1 DOF system, any one equation of motion should give us the result. Instead of linear momentum balance we could have used power balance to get the same result like this:

$$\begin{aligned} P &= E_K \\ \Rightarrow \vec{F}_{\text{tot}} \cdot \vec{v} &= \frac{d}{dt} \left\{ \frac{1}{2}mv^2 \right\} \\ \Rightarrow F\hat{e}_n \cdot v\hat{e}_t &= mv\dot{v} \\ \Rightarrow 0 &= \dot{v}. \end{aligned}$$

This is natural enough. The only force on the particle is perpendicular to its motion, so does no work. So the particle must have constant kinetic energy and its speed must be constant.  $\square$

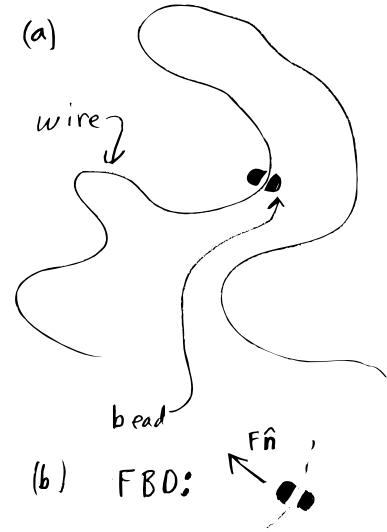


Figure 10.3: A point-mass bead slides on a rigid immobile frictionless wire. The free body diagram shows that the only force on the bead is in the direction normal to the wire.

(Filename: tfigure.beandonwire)

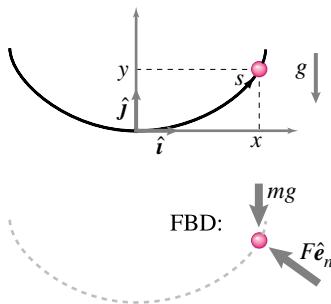


Figure 10.4: A bead slides on a frictionless wire on the curve implicitly defined by  $y = cs^2/2$ , where  $s$  is arc-length along the curve measured from the origin.

(Filename:tfigure.brachistochrones)

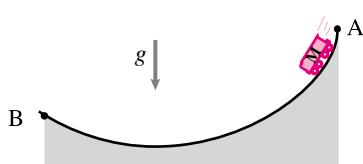


Figure 10.5: The roller coaster that gets from A to B the fastest is the one with a track in the shape of the brachistochrone  $y = cs^2/2$ .

(Filename:tfigure.brachsln)

① The original brachistochrone (least time) puzzle:

"I, Johann Bernoulli, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose possible solution will bestow fame and remain as lasting monument. Following the example set by Pascal, Fermat, etc., I hope to earn the gratitude of the entire scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall then publicly declare him worthy of praise."

"Let two points A and B be given in a vertical plane. Find the curve that a point M, moving on a path AMB must follow such that, starting from A, reaches B in the shortest time under its own gravity."

**Newton.** Besides Bernoulli's son Dan, one of the people to solve the puzzle was 55 year old Isaac Newton. He attempted to keep his solution, said to have been worked out in one evening while he also invented the calculus of variations, anonymous. But Bernoulli supposedly read through this deception, commenting "I recognize the lion by his paw print", which was presumably not a comment about Newton's handwriting.

The example above doesn't seem that applicable; how often does one see beads on frictionless wires? It is a bit more useful than it appears. For example, if a car is coasting on an open road and tire resistance and sideslip can be ignored, the reasoning above shows that the car is neither speeded nor slowed by turning. Similarly, an idealization of an airplane wing is as something which only causes force perpendicular to motion. So in quick maneuvers where the gravity force is relatively small, the plane maintains its speed.

#### Example: The hard brachistochrone problem.

Here is a puzzle proposed by Johann Bernoulli on January 1, 1669 ①: given that a roller coaster has to coast from rest at one place to another place that is no higher, what shape should the track be to make the trip as quick as possible.

**The solution.** Finding the solution, or even verifying it, is a problem in the calculus of variations, *i.e.*, too advanced for this book. The solution turns out to be the brachistochrone curve that obeys the following relationship between arc-length  $s$  from the origin and vertical position  $y$  (drawn accurately in Fig. 10.4):

$$y = \frac{1}{2}cs^2 \quad (\text{for } |s| \leq 1/c). \quad (10.7)$$

Starting at the origin this curve is close to  $y = cx^2/2$  but gets a bit higher (bigger  $y$ ) because for a given value of  $x$ ,  $s$  is greater than  $x$ . The curve terminates at vertical tangents at  $s = \pm 1/c$  where  $y = 1/2c$ . (see Fig. 10.4). To solve the puzzle this curve is scaled (by choosing a value of  $c$ ) and displaced so that it has a vertical tangent at A and also so the curve goes through B. The idea that the hard math seems to be expressing, is that the particle should first build up as much speed as it can (by going straight down) and then head off in the right direction (see Fig. 10.5). □

On the other hand, here is an easier problem that is a virtual setup for the techniques now at hand.

#### Example: The easy brachistochrone problem

How long does it take for a particle to slide back and forth on a frictionless wire with  $y = cs^2/2$  as driven by gravity? (see Fig. 10.4)

Let's use  $s$  as our configuration variable. The power balance equation is:

$$\begin{aligned} P &= \dot{E}_K \\ (F\hat{e}_n) \cdot (v\hat{e}_t) = 0 &\Rightarrow -mg\hat{j} \cdot (\dot{x}\hat{i} + \dot{y}\hat{j}) = \frac{d}{dt} \left\{ \frac{1}{2}mv^2 \right\} \\ &\Rightarrow -mg\dot{y} = m\frac{d}{dt} \frac{\dot{s}^2}{2} \\ &\Rightarrow -mgc\dot{s} = m\ddot{s}\dot{s} \\ \text{Assuming } \dot{s} \neq 0 &\Rightarrow \ddot{s} + gcs = 0. \end{aligned}$$

This, remarkably, is the simple harmonic oscillator equation with general solution

$$s = A \cos(\sqrt{gc} t) + B \sin(\sqrt{gc} t).$$

Thus the period of oscillation ( $T$  such that  $\sqrt{gc}T = 2\pi$ ) is

$$T = \frac{2\pi}{\sqrt{gc}}$$

which is independent of the amplitude of oscillation <sup>①</sup>.

The key to this quick solution was using a configuration variable that made the expression for the velocity simple, and using an equation of motion that didn't involve the unknown reaction force  $F$  which we also didn't care about. We could have got the same equation of motion by writing  $\vec{F} = m\vec{a}$  and eliminated  $F\hat{e}_n$  by dotting both sides with a convenient vector orthogonal to  $F\hat{e}_n$ , say  $\vec{v}$ .

The brachistochrone is a famous curve that has various interesting properties (e.g., Box 10.1 on page 592).  $\square$

### Example: A collar on two rotating rods

Consider a pair of collars hinged together as a point mass  $m$  at P. Each slides frictionlessly on a rod about whose rotation everything is known (see Fig. 10.6. What is the force of rod 1 on the mass? For this 2 degree of freedom system lets use configuration variables  $\theta_1$  and  $\theta_2$ , and two sets of rotating base vectors:  $\hat{\lambda}_1\hat{n}_1$  and  $\hat{\lambda}_2\hat{n}_2$ . These rotating base vectors can be written in terms of the  $\theta$ s,  $\hat{i}$  and  $\hat{j}$  in the standard manner. Assume we know  $\ell_1$  and  $\ell_2$  in this configuration. First find  $\dot{\ell}_1$  and  $\dot{\ell}_2$  by thinking of the velocity of the collar two different ways:

$$\begin{aligned} \vec{v} &= \vec{v} \\ \{ \dot{\ell}_1\hat{\lambda}_1 + \dot{\theta}_1\ell_1\hat{n}_1 &= \dot{\ell}_2\hat{\lambda}_2 + \dot{\theta}_2\ell_2\hat{n}_2 \} \\ \{ \cdot \hat{n}_2 &\Rightarrow \dot{\ell}_1 = \frac{\dot{\theta}_2\ell_2 - \dot{\theta}_1\ell_1\hat{n}_1 \cdot \hat{n}_2}{\hat{\lambda}_1 \cdot \hat{n}_2} \\ \{ \cdot \hat{n}_1 &\Rightarrow \dot{\ell}_2 = \frac{\dot{\theta}_1\ell_1 - \dot{\theta}_2\ell_2\hat{n}_2 \cdot \hat{n}_1}{\hat{\lambda}_2 \cdot \hat{n}_1}. \end{aligned} \quad (10.8)$$

Having found  $\dot{\ell}_1$  and  $\dot{\ell}_2$  we can find the velocity  $\vec{v}$  by evaluating either side of eqn. (10.8). Now we apply identical reasoning with the acceleration. The result looks messy, but the approach is straightforward:

$$\begin{aligned} \vec{a} &= \vec{a} \\ \{ \ddot{\ell}_1\hat{\lambda}_1 + \ddot{\theta}_1\ell_1\hat{n}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{n}_1 &= \ddot{\ell}_2\hat{\lambda}_2 + \ddot{\theta}_2\ell_2\hat{n}_2 + 2\dot{\ell}_2\dot{\theta}_2\hat{n}_2 \} \\ \{ \cdot \hat{n}_2 &\Rightarrow \ddot{\ell}_1 = \frac{\ddot{\theta}_2\ell_2 + 2\dot{\ell}_2\dot{\theta}_2 - \ddot{\theta}_1\ell_1\hat{n}_1 \cdot \hat{n}_2 - 2\dot{\ell}_1\dot{\theta}_1\hat{n}_1 \cdot \hat{n}_2}{\hat{\lambda}_1 \cdot \hat{n}_2} \\ \{ \cdot \hat{n}_1 &\Rightarrow \ddot{\ell}_2 = \frac{\ddot{\theta}_1\ell_1 + 1\dot{\ell}_1\dot{\theta}_1 - \ddot{\theta}_2\ell_2\hat{n}_2 \cdot \hat{n}_1 - 2\dot{\ell}_2\dot{\theta}_2\hat{n}_2 \cdot \hat{n}_1}{\hat{\lambda}_2 \cdot \hat{n}_1} \end{aligned} \quad (10.9)$$

We use the results from eqn. (10.8) for  $\dot{\ell}_1$  and  $\dot{\ell}_2$  to evaluate the right hand sides of the expressions for  $\ddot{\ell}_2$  and  $\ddot{\ell}_3$  in eqn. (10.9). So now either the left hand side or the right hand side of the second of Eqns. 10.9 can be used to evaluate the acceleration  $\vec{a}$ , all the terms in both expressions have been found.

To find the forces we use linear momentum balance and the free body diagram

$$\vec{F}_{\text{tot}} = m\vec{a} \quad (10.10)$$

<sup>①</sup> Actually, the amplitude can't be arbitrarily large. The solution to the defining eqn. (10.7) only makes sense for  $|s| < 1/c$ . For  $|s| > 1/c$  there is no curve satisfying eqn. (10.7).

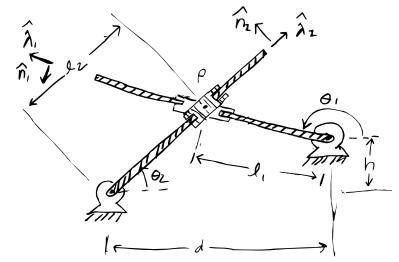


Figure 10.6: A point-mass collar slides simultaneously on 2 rods.  
(Filename:tfigure.collaron2rods)

$$\begin{aligned} \{ F_1 \hat{\mathbf{n}}_1 + F_2 \hat{\mathbf{n}}_2 &= m\{\ddot{\ell}_1 \hat{\lambda}_1 + \ddot{\theta}_1 \ell_1 \hat{\mathbf{n}}_1 + 2\dot{\ell}_1 \dot{\theta}_1 \hat{\mathbf{n}}_1\} \} \\ \{ \} \cdot \hat{\lambda}_2 &\Rightarrow F_2 = \frac{\{\ddot{\ell}_1 \hat{\lambda}_1 + \ddot{\theta}_1 \ell_1 \hat{\mathbf{n}}_1 + 2\dot{\ell}_1 \dot{\theta}_1 \hat{\mathbf{n}}_1\} \cdot \hat{\lambda}_2}{\hat{\mathbf{n}}_1 \cdot \hat{\lambda}_2} \\ \{ \} \cdot \hat{\lambda}_1 &\Rightarrow F_2 = \frac{\{\ddot{\ell}_1 \hat{\lambda}_1 + \ddot{\theta}_1 \ell_1 \hat{\mathbf{n}}_1 + 2\dot{\ell}_1 \dot{\theta}_1 \hat{\mathbf{n}}_1\} \cdot \hat{\lambda}_1}{\hat{\mathbf{n}}_2 \cdot \hat{\lambda}_1} \end{aligned}$$

When actually evaluating the expressions above one can write the base vectors in terms of  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  or use geometry.

Often when working out a problem it is best to not substitute numbers until the end of a problem. This example shows the opposite. If we left the expressions for  $\dot{\ell}_1$  and  $\dot{\ell}_2$  with letters and substituted that into the expressions for  $\ddot{\ell}_1$  and  $\ddot{\ell}_2$  and left those expressions intact while substituting for the acceleration  $\vec{a}$  we would have large expressions for the force components  $F_1$  and  $F_2$ . On the other hand, by using numbers as the calculation progresses the formulas do not grow so much in complexity.

As is the case with most mechanism-mechanics problems, the hard work in getting the dynamics equations is in the kinematics. Generally there are no great short-cuts. There are alternative methods. In this case the location of the base points and the two angles determine the base and two angles of a triangle. This triangle can be solved for the location of the point P. Once that position is known in terms of  $\theta_1$  and  $\theta_2$  the velocity and acceleration can be found by differentiation.

As a robot manipulator, this design has the advantage that no motors need to be displaced. It has the disadvantage of requiring good sliding joints.

An alternative solution of the kinematics of this problem would be to use trigonometry to find the position of point P in terms of the angles  $\theta_1$  and  $\theta_2$ . Then the acceleration of point P is found by taking two time derivatives. The result is approximately equal in the complexity of its appearance to the results used above. That method requires more cleverness at the start (solving an angle-side-angle triangle) and then just brute force differentiation using the chain rule and the product rule.  $\square$

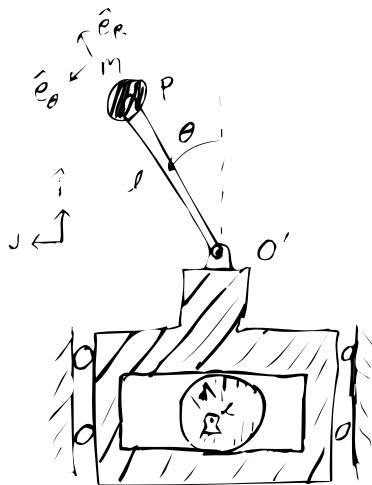


Figure 10.7: The base of a pendulum is vertically vibrated.

(Filename: tfigure.vibratingpend)

#### Example: Inverted pendulum with a vibrating base

Assume that the base  $O'$  of an inverted point-mass pendulum of length  $\ell$  is vibrated according to (see Fig. 10.7)

$$\vec{r}_{O'} = d \sin \omega t \hat{\mathbf{i}}.$$

The point P thus has acceleration

$$\begin{aligned} \vec{a}_P &= \vec{a}_{O'} + \vec{a}_{P/O'} \\ &= -d\omega^2 \sin \omega t \hat{\mathbf{i}} + \ddot{\theta} \ell \hat{\mathbf{e}}_\theta - \dot{\theta}^2 \ell \hat{\mathbf{e}}_R \end{aligned}$$

Now apply linear momentum balance as

$$\begin{aligned} \vec{F}_{\text{tot}} &= m \vec{a}_P \\ \{ -mg \hat{\mathbf{i}} + -T \hat{\mathbf{e}}_R &= m\{-d\omega^2 \sin \omega t \hat{\mathbf{i}} + \ddot{\theta} \ell \hat{\mathbf{e}}_\theta - \dot{\theta}^2 \ell \hat{\mathbf{e}}_R\} \} \\ \{ \} \cdot \hat{\mathbf{e}}_\theta &\Rightarrow -g \hat{\mathbf{i}} \cdot \hat{\mathbf{e}}_\theta = -d\omega^2 \sin \omega t \hat{\mathbf{i}} \cdot \hat{\mathbf{e}}_\theta + \ddot{\theta} \ell \\ &\Rightarrow g \sin \theta = d\omega^2 \sin \omega t \sin \theta + \ddot{\theta} \ell \end{aligned}$$

which you write as

$$\ddot{\theta} + (d\omega^2 \sin \omega t - g) \sin \theta / \ell \quad \text{or} \quad \ddot{\theta} = (g - d\omega^2 \sin \omega t) \sin \theta / \ell$$

depending on whether you are analytically or numerically inclined. This is a second order non-linear ordinary differential equation. If  $\omega = 0$  or  $d = 0$  then this is the classic inverted pendulum equation and has solutions that show that the pendulum doesn't stay near upright. But, you can find by analytic cleverness or numerical integration that for some values of  $d$  and  $\omega$  that the pendulum does not fall down! Just shaking the base keeps the pendulum up ( $\omega^2 d > g$  for all cases where this is possible). This isn't just academic nonsense, the device can be built and the balancing demonstrated.

One alternative to using linear momentum balance in the equations above would be to use angular momentum balance about the point  $0'$ . The resulting vector equation

$$\ell \hat{\mathbf{e}}_R \times (-mg\hat{\mathbf{i}}) = \vec{r}_{P/0'} \times (m\vec{a})$$

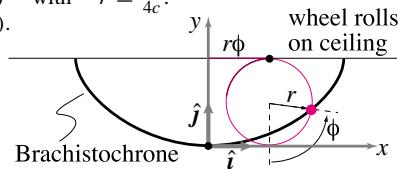
yields the same second order scalar ODE.

The vibrating mechanism shown is a “Scotch yoke”. An eccentric disk is mounted to the shaft of a constant angular velocity motor. The rectangular slot moves up and down sinusoidally as the disk wobbles.  $\square$

## 10.1 Some brachistochrone curiosities

**The brachistochrone is a cycloid.** There is no straightforward way to draw the curve  $y = cs^2/2$  because the formula doesn't tell you the  $x$  coordinates of the points. You could find them by integrating  $dx = \sqrt{ds^2 - dy^2}$  numerically or with calculus tricks. But it turns out (see below) that the curve with  $y = cs^2/2$  is described by the parametric equations

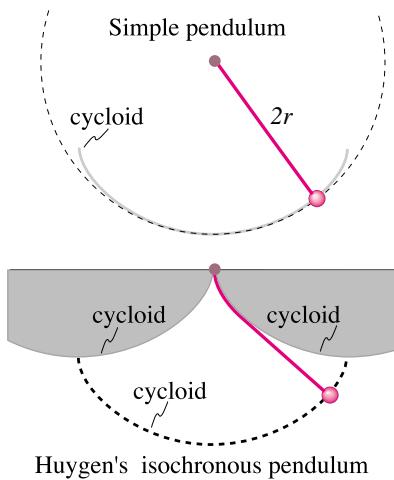
$$x = r(\phi + \sin \phi) \quad \text{with} \quad r = \frac{1}{4c}, \\ y = r(1 - \cos \phi).$$



This is the path of a particle on the perimeter of a wheel that rolls against a horizontal ceiling a distance  $2r$  above the origin, as you can verify by adding up distances in the picture above (see page 525). We will show below that the upside down cycloid and the curve  $y = cs^2/2$  are one and the same.

Note that the osculating circle of this cycloid at its lowest point has radius  $4r = 1/c$ , just the length of a simple pendulum that, for small oscillations, has the same frequency of oscillation as the bead on the brachistochrone. A point mass swinging on a string is like a bead on a frictionless circular wire and this, in turn, is close to the motion of a bead on a brachistochrone wire for small oscillations.

**Galileo** (1564-1642). Well before Bernoulli's challenge, Galileo was interested in things rolling and sliding on ramps. He knew that the shortest distance between two points is a straight line, and had noted that a ball rolling down an appropriately curved ramp gets to its destination faster than a ball traveling the shortest route. A ball going on a straight ramp just doesn't pick up much speed, and when it finally has its greatest speed the trip is over. Imagine sliding straight sideways; it takes forever on a straight-line route. Better, he must have reasoned, to get the ball rolling fast at the start and then go fast for most of its journey, possibly slowing at the end. Galileo thought the best shape was the bottom of a circle (or fraction thereof), which isn't far off either in shape or concept, but isn't quite right. Galileo was apparently obsessed with cycloids for other reasons but didn't see their connection to this problem.



**A constant period pendulum.** For clock time keeping, a pendulum is better than a bead on a wire because the friction of sliding is avoided. Unfortunately, a simple pendulum has a period which is longer if the amplitude is bigger. Not much longer, 18% if

the swinging is  $\pm 90^\circ$  and only 1.7% longer if the amplitude is  $\pm 30^\circ$ , but enough to annoy clock designers. A bead sliding frictionlessly on the path  $y = s^2/2$  has the nice property that the period does not depend on the amplitude. But any real bead sliding on any real wire has substantial friction. So, at first blush the brachistochrone curve, despite its nice constant-period property, cannot be used to keep time.

But Huygens, one of the smart old timers, looked for a curve that, when a string wraps around it, makes the end follow the brachistochrone curve. To this day you can see fancy old clocks with this wrapping device, a solid piece with the cuspidal shape of neighboring cycloids, near the hinge of the swinging ("isochronous" or "tautochrone") pendulum which wraps around it.

**Geometry.** The two key features discussed above, that the curve  $y = cs^2/2$  is a cycloid, and that a cycloid can be generated by wrapping a string around another cycloid, can be found from the geometric construction below. Two cycloids are shown, one from wheel 1 rolling under line  $L_1$  and another from wheel 2 rolling under line  $L_2$  a distance  $2r$  below. Both wheels have radius  $r$ . Imagine that the cycloids  $A_1M_1B$  and  $A_2M_2B$  are drawn by wheels always arranged with vertically aligned rolling contact points  $C_1$  and  $C_2$  and with points  $M_1$  and  $M_2$  initially aligned vertically a distance  $4r$  apart at  $A_1$  and  $A_2$ .

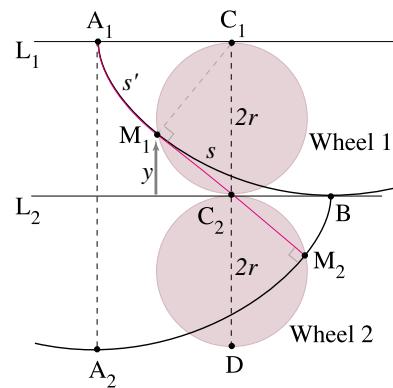
The two cycloids are thus the same shape but are displaced with one being  $2r$  below and  $\pi r$  sideways from the other.

Because both wheels have rolled the same distance ( $A_1C_1$ ) they have rotated the same amount and  $M_2$  is as far forward of  $C_1C_2D$  as  $M_1$  is behind. Similarly  $M_1$  is as far above  $L_2$  as  $M_2$  is below. So the line  $M_1M_2$  is bisected by the point  $C_2$ .

Because  $C_1C_2$  is the diameter of a circle with  $M_1$  on the perimeter, angle  $C_1M_1C_2$  is a right angle. Because material point  $C_1$  on the wheel has zero velocity the velocity of  $M_1$  (and thus the tangent to the curve) is orthogonal to  $C_1M_1$ . Thus the line  $M_1M_2$  is tangent to the upper cycloid.

The rolling of wheel 2 instantaneously rotating about  $C_2$  makes the tangent to the lower cycloid orthogonal to  $M_1M_2$ , the condition for the motion of  $M_2$  to be from the wrapping of an inextensible line around the curve  $A_1M_1B$ . This shows that cycloid  $A_2M_2B$  is generated by the wrapping of a line anchored at  $A_1$  about the upper cycloid. And this is Huygen's wrapping mechanism for making a pendulum bob follow a cycloid. Because of this wrapping generation, the arc-length  $s' + s$  of  $A_1M_1B$  must be  $4r$  and the arc length  $s$  of  $M_1B$  is  $M_1M_2$  so the length  $M_1C_2$  is  $s/2$ . By the similarity of the two right triangles that share the length  $s/2$  of  $M_1C_2$ :

$$\frac{y}{s/2} = \frac{s/2}{2r} \Rightarrow y = \frac{1}{4r} \frac{s^2}{2} = c \frac{s^2}{2}$$



which shows that the upper cycloid is the curve  $y = cs^2/2$  if  $c = 1/4r$ , where  $s$  is measured from  $B$ . This was the equation used to show the constant period nature of the sliding motion of a bead on a frictionless cycloidal curve using power balance.

**SAMPLE 10.1** A bead on a straight wire. A straight wire is hung between points A and B in the  $xy$  plane as shown in the figure. A bead slides down the wire from point A. Write the geometric constraint equation for the bead's motion and derive the conditions on velocity and acceleration components of the bead due to the constraint.

**Solution** The constraint on the bead's motion is that its path must be along the wire, i.e., a straight line between points A and B. Thus the geometric constraint on the motion is expressed by the equation of the path which is

$$y = h - \frac{h}{\ell}x.$$

Since the bead is constrained to move on this path, its velocity and acceleration vectors are also constrained to be directed along AB. This imposes conditions on their  $x$  and  $y$  components that are easily derived by differentiating the geometric constraint equation with respect to  $t$ . Thus,

$$\begin{aligned}\dot{y} &= -\frac{h}{\ell}\dot{x}, \\ \ddot{y} &= -\frac{h}{\ell}\ddot{x}.\end{aligned}$$

$$y = h - \frac{h}{\ell}x, \quad \dot{y} = -\frac{h}{\ell}\dot{x}, \quad \ddot{y} = -\frac{h}{\ell}\ddot{x}$$

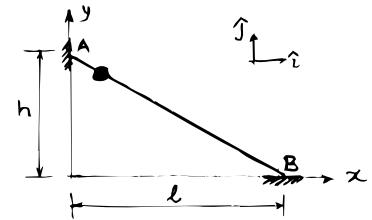


Figure 10.8: (Filename:sfig11.1.beadonline.kin)

**SAMPLE 10.2** A particle sliding on a parabolic path. A particle slides on a parabolic trough given by  $y = ax^2$  where  $a$  is a constant. Write the geometric constraints of motion (on the path, velocity, and acceleration) of the particle. Write the velocity and acceleration of the particle at a generic location  $(x, y)$  on its path.

**Solution** The geometric constraint on the path of the particle is already given,  $y = ax^2$ . Differentiating the path constraint with respect to time, we get the constraint on velocity and acceleration components.

$$\begin{aligned}\dot{y} &= 2ax\dot{x}, \\ \ddot{y} &= 2a\ddot{x} + 2a\dot{x}^2.\end{aligned}$$

Now, at a point  $(x, y)$ , we can write the velocity and acceleration of the particle as

$$\begin{aligned}\vec{v} &= \dot{x}\hat{i} + \dot{y}\hat{j} = \dot{x}\hat{i} + 2a\dot{x}\hat{j}, \\ \vec{a} &= \ddot{x}\hat{i} + \ddot{y}\hat{j} = \ddot{x}\hat{i} + (2a\ddot{x} + 2a\dot{x}^2)\hat{j}.\end{aligned}$$

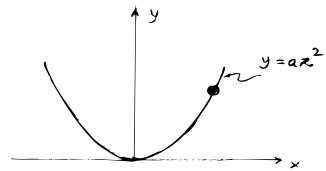


Figure 10.9: (Filename:sfig11.1.parabola.kin)

$$\vec{v} = \dot{x}\hat{i} + 2a\dot{x}\hat{j}, \quad \vec{a} = \ddot{x}\hat{i} + (2a\ddot{x} + 2a\dot{x}^2)\hat{j}$$

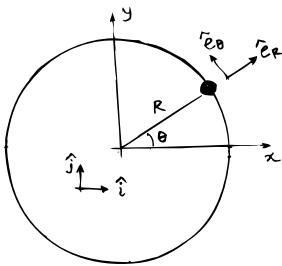


Figure 10.10: (Filename:sfig11.1.circle)

**SAMPLE 10.3 Circular motion of a particle.** A particle is constrained to move on a frictionless circular path of radius  $R_0$  with constant angular speed  $\dot{\theta}$ . There is no gravity. Find the equation of motion of the particle in the  $x$ -direction and show that this motion is simple harmonic.

**Solution** This is simple problem that you have solved before, probably a few times. Here, we do this problem again just to show how it works out with the constraint machinery in evidence. The geometric constraint on the path of the particle is  $R = R_0$  (in polar coordinates). This constraint gives us  $\dot{R} = 0$  and  $\ddot{R} = 0$ . Then the acceleration of the particle (in polar coordinates),  $\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$  reduces to  $\vec{a} = -R\dot{\theta}^2\hat{e}_R$ , (of course).

The free body diagram of the particle shows that there is only one force acting on the particle, the normal reaction  $N$  of the path acting in the  $\hat{e}_R$  direction. Therefore, the linear momentum balance gives,

$$N\hat{e}_R = m\vec{a} = m(-R\dot{\theta}^2\hat{e}_R) \Rightarrow N = -mR\dot{\theta}^2.$$

But, to write the equation of motion in the  $x$ -direction, we need to write the linear momentum balance in the  $x$ -direction. We can write  $\sum \vec{F} = m\vec{a}$  using mixed basis vectors as  $N\hat{e}_R = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})$ . Dotting this equation with  $\hat{i}$ , we get

$$\ddot{x} = \frac{N}{m}(\hat{e}_R \cdot \hat{i}) = \frac{-mR\dot{\theta}^2}{m} \cos\theta = -\dot{\theta}^2(R \cos\theta) = -\dot{\theta}^2x$$

or,  $\ddot{x} + \dot{\theta}^2x = 0$ , which is the equation of simple harmonic motion in  $x$ . You can easily show that the motion in the  $y$ -direction is also simple harmonic ( $\ddot{y} + \dot{\theta}^2y = 0$ ).  $\square$

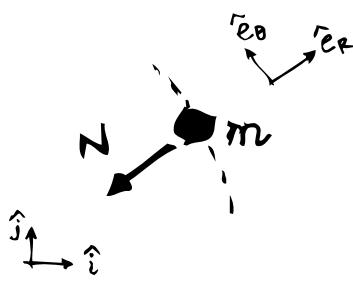


Figure 10.11: (Filename:sfig11.1.circle.a)

**SAMPLE 10.4 A bead slides on a straight wire.** Consider the problem of the bead sliding on a straight, inclined, frictionless wire of Sample 10.1 again. Find the position of the bead  $x(t)$  and  $y(t)$  assuming it slides under gravity starting from rest at A.

**Solution** To find the position of the bead, we need to write the equation of motion and solve it. This is single DOF system and, therefore, one scalar equation of motion should suffice.

The free body diagram of the bead is shown in Fig. 10.12. Using basis vectors  $(\hat{\lambda}, \hat{n})$  and  $(\hat{i}, \hat{j})$  we write the LMB for the bead as

$$-mg\hat{j} + N\hat{n} = m\vec{a} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}).$$

We can easily eliminate the constraint force  $N$  from this equation by dotting this equation with  $\hat{\lambda}$ , which gives

$$\begin{aligned} -mg(\hat{j} \cdot \hat{\lambda}) &= m[\ddot{x}(\hat{i} \cdot \hat{\lambda}) + \ddot{y}(\hat{j} \cdot \hat{\lambda})] \\ \Rightarrow g \sin\theta &= \ddot{x} \cos\theta - \ddot{y} \sin\theta. \end{aligned}$$

But, from the geometric constraint  $y = h - (h/\ell)x$ , we have  $\ddot{y} = -(h/\ell)\ddot{x} = -(\tan\theta)\ddot{x}$ . Therefore,

$$g \sin\theta = \ddot{x} \cos\theta + \ddot{x} \tan\theta \sin\theta \Rightarrow \ddot{x} = g \sin\theta \cos\theta.$$

Since  $g \sin\theta \cos\theta$  is constant, we integrate the equation of motion easily to find  $x(t) = \frac{1}{2}g \sin\theta \cos\theta t^2$  since  $x(0) = 0$ ,  $\dot{x}(0) = 0$ . And, since  $y = h - x \tan\theta$ , we have  $y(t) = h - \frac{1}{2}g \sin^2\theta t^2$ .

$$x(t) = \frac{1}{2}ght^2 \sin\theta \cos\theta, \quad y(t) = h - \frac{1}{2}gt^2 \sin^2\theta$$

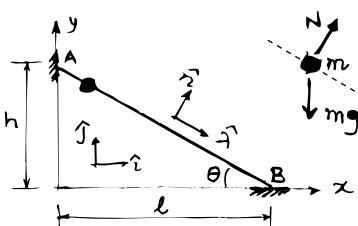


Figure 10.12: (Filename:sfig11.1.beadonline.a)

**SAMPLE 10.5** A bead sliding down a parabolic trough. Consider the problem of Sample 10.2 again. Find the equation of motion of the bead.

**Solution** This is, again, a one DOF system. Therefore, we will get a single scalar equation of motion. The free body diagram shown in Fig. 10.14 shows two forces acting on the bead. The constraint force  $N$  acts normal to the path. Let  $\hat{e}_t$  and  $\hat{e}_n$  be unit vectors tangential and normal to the path, respectively. Then the linear momentum balance gives

$$-mg\hat{j} + N\hat{e}_n = m\vec{a} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}).$$

To eliminate the unknown constraint force  $N$  from this equation, we can take a dot product of this equation with  $\hat{e}_t$ . However, we must first find  $\hat{e}_t$ . Now  $\hat{e}_t$  is the unit tangent vector. So, we can find it by finding a tangent vector to the path (remember gradient of a function  $\nabla f$ ?) and then dividing it by the length of the vector. That is doable but a little complicated. All we need here is the dot product with a vector *normal* to  $\hat{e}_n$ . Why not use the velocity vector  $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$ ? The velocity vector is always tangential to the path. Furthermore, we know that from the geometric constraint ( $y = ax^2$ ),  $\dot{y} = 2ax\dot{x}$  and  $\ddot{y} = 2a\ddot{x}\dot{x} + 2a\dot{x}^2$ . Therefore,  $\vec{v} = \dot{x}\hat{i} + 2a\dot{x}\dot{x}\hat{j}$ . Now, dotting the LMB equation with  $\vec{v}$  we get,

$$\begin{aligned} -\eta\cancel{g}(\hat{j} \cdot \vec{v}) &= \cancel{\eta}[\dot{x}(\hat{i} \cdot \vec{v}) + \dot{y}(\hat{j} \cdot \vec{v})] \\ -2gax\ddot{x} &= \dot{x}\ddot{x} + 2a\dot{y}\ddot{x} \\ &= \dot{x} + 2a\underbrace{(2a\dot{x}\dot{x} + 2a\dot{x}^2)}_{\ddot{y}} \\ &= \dot{x}(1 + 4a^2x) + 4a^2\dot{x}^2. \end{aligned}$$

Rearranging the terms above, we get the required equation of motion:

$$\ddot{x} + \frac{4a^2}{1+4a^2x}\dot{x}^2 + \frac{2gax}{1+4a^2x} = 0.$$

As you can see, this is a nonlinear ODE. Analytical solution of this equation is rather difficult. We can, however, always solve it numerically. Note that a solution of this equation only gives you  $x(t)$ , i.e., the  $x$  coordinate of the position of the bead. But, you can always find the  $y$  coordinate since  $y = ax^2$ .

$$\boxed{\ddot{x} + \frac{4a^2}{1+4a^2x}\dot{x}^2 + \frac{2gax}{1+4a^2x} = 0}$$

**Comment:** Note that if we consider  $x$  and  $\dot{x}$  to be very small so that we can ignore the  $\dot{x}^2$  term completely and take  $1 + 4a^2x \approx 1$ , then the equation of motion becomes

$$\ddot{x} + (2ga)x = 0$$

which is the equation of simple harmonic motion with frequency  $\sqrt{2ga}$ . Thus, if we consider a shallow parabola, and release the bead close to the origin, it executes simple harmonic motion, much like a simple pendulum. This is an intuitively realizable motion.

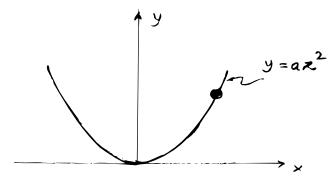


Figure 10.13: (Filename:sfig11.1.parabola)

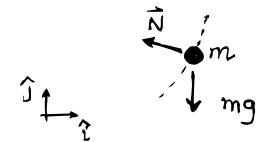


Figure 10.14: (Filename:sfig11.1.parabola.a)

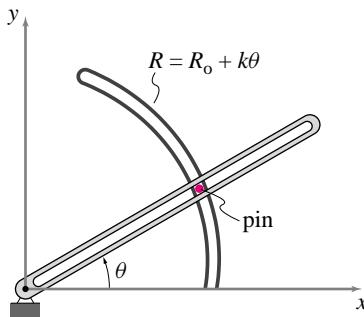


Figure 10.15: A pin is constrained to move in a groove and a slotted arm.

(Pilename:sfig6.2.2)

① Note that  $R$  is a function of  $\theta$  and  $\theta$  is a function of time, therefore  $R$  is a function of time. Although we are interested in finding  $\dot{R}$  and  $\ddot{R}$  at  $\theta = 60^\circ$ , we cannot first substitute  $\theta = 60^\circ$  in the expression for  $R$  and then take its time derivatives (which will be zero).

**SAMPLE 10.6** *Constrained motion of a pin.* During a small interval of its motion, a pin of 100 grams is constrained to move in a groove described by the equation  $R = R_0 + k\theta$  where  $R_0 = 0.3 \text{ m}$  and  $k = 0.05 \text{ m}$ . The pin is driven by a slotted arm AB and is free to slide along the arm in the slot. The arm rotates at a constant speed  $\omega = 6 \text{ rad/s}$ . Find the magnitude of the force on the pin at  $\theta = 60^\circ$ .

**Solution** Let  $\vec{F}$  denote the net force on the pin. Then from the linear momentum balance

$$\vec{F} = m\vec{a}$$

where  $\vec{a}$  is the acceleration of the pin. Therefore, to find the force at  $\theta = 60^\circ$  we need to find the acceleration at that position.

From the given figure, we assume that the pin is in the groove at  $\theta = 60^\circ$ . Since the equation of the groove (and hence the path of the pin) is given in polar coordinates, it seems natural to use polar coordinate formula for the acceleration. For planar motion, the acceleration is

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

We are given that  $\dot{\theta} \equiv \omega = 6 \text{ rad/s}$  and the radial position of the pin  $R = R_0 + k\theta$ . Therefore, ①

$$\begin{aligned}\ddot{\theta} &= \frac{d\dot{\theta}}{dt} = 0 \quad (\text{since } \dot{\theta} = \text{constant}) \\ \dot{R} &= \frac{d}{dt}(R_0 + k\theta) = k\dot{\theta} \quad \text{and} \\ \ddot{R} &= k\ddot{\theta} = 0.\end{aligned}$$

Substituting these expressions in the acceleration formula and then substituting the numerical values at  $\theta = 60^\circ$ , (remember,  $\theta$  must be in radians!), we get

$$\begin{aligned}\vec{a} &= -\overbrace{(R_0 + k\theta)\dot{\theta}^2}^{R\dot{\theta}^2} \hat{e}_R + \overbrace{2k\dot{\theta}^2}^{2\dot{R}\dot{\theta}} \hat{e}_\theta \\ &= -(0.3 \text{ m} + 0.05 \text{ m} \cdot \frac{\pi}{3}) \cdot (6 \text{ rad/s})^2 \hat{e}_R + 2 \cdot 0.3 \text{ m} \cdot (6 \text{ rad/s})^2 \hat{e}_\theta \\ &= -13.63 \text{ m/s}^2 \hat{e}_R + 21.60 \text{ m/s}^2 \hat{e}_\theta.\end{aligned}$$

Therefore the net force on the pin is

$$\begin{aligned}\vec{F} &= m\vec{a} \\ &= 0.1 \text{ kg} \cdot (-13.63 \hat{e}_R + 21.60 \hat{e}_\theta) \text{ m/s}^2 \\ &= (-1.36 \hat{e}_R + 2.16 \hat{e}_\theta) \text{ N}\end{aligned}$$

and the magnitude of the net force is

$$F = |\vec{F}| = \sqrt{(1.36 \text{ N})^2 + (2.16 \text{ N})^2} = 2.55 \text{ N}.$$

$F = 2.55 \text{ N}$



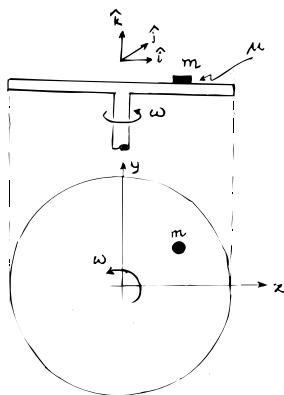
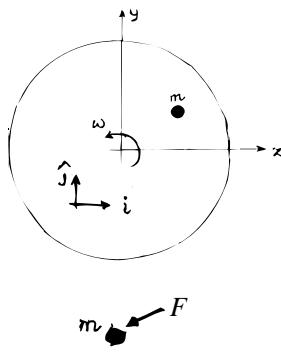


Figure 10.16: (Filename:sfig11.1.puckontable)

Figure 10.17: A partial free body diagram of the puck. For linear momentum balance we need to consider only the forces acting in the plane of motion.  
(Filename:sfig11.1.puckontable.a)

**SAMPLE 10.7** *A puck sliding on a rough rotating table.* A horizontal turntable rotates with constant angular speed  $\omega = 100 \text{ rpm}$ . A puck of mass  $m = 0.1 \text{ kg}$  gently placed on the rotating turntable. The puck begins to slide. The coefficient of friction between the puck and the turntable is 0.25. Find the equation of motion of the puck using

- (a) A fixed reference frame with cartesian coordinates
- (b) A rotating reference frame with cartesian coordinates

### Solution

- (a) **Equation of motion using a fixed reference frame:** The puck has two DOF on the turntable. So, we will need two configuration variables, say  $x$  and  $y$ , and we will have to find equation of motion for each variable.

Let us use a fixed cartesian coordinate system with the origin at the center of the turntable. Let  $\vec{r}_P = x\hat{i} + y\hat{j}$  be the position of the puck at some instant  $t$ , so that its velocity is  $\vec{v}_P = \dot{x}\hat{i} + \dot{y}\hat{j}$  and acceleration is  $\vec{a}_P = \ddot{x}\hat{i} + \ddot{y}\hat{j}$ .

The free body diagram of the puck should show three forces — the force of gravity (in  $-\hat{k}$  direction), the normal reaction of the turntable (in  $\hat{k}$  direction) and the friction force  $\vec{F}$ . Since, there is no motion in the vertical direction, we know that  $N = mg$  and that  $F = |\vec{F}| = \mu N = \mu mg$ . But, what is the direction of the friction force? Well, we know that it acts in the opposite direction of the *relative slip*, so that

$$\vec{F} = -\mu N \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|}.$$

So, we need to find  $\vec{v}_{\text{rel}}$ . Now  $\vec{v}_{\text{rel}}$  is the velocity of the puck relative to the turntable, or more precisely, relative to the point on the turntable just underneath the puck. Let us denote that point by  $P'$ . Clearly,  $P'$  goes in circles with constant speed, so that its velocity is

$$\vec{v}_{P'} = \vec{\omega} \times \vec{r}_{P'} = \dot{\theta}\hat{k} \times (x\hat{i} + y\hat{j}) = \dot{\theta}x\hat{j} - \dot{\theta}y\hat{i}.$$

Therefore, the relative velocity,  $\vec{v}_{\text{rel}}$  is

$$\vec{v}_{\text{rel}} = \vec{v}_P - \vec{v}_{P'} = (\dot{x} + \dot{\theta}y)\hat{i} + (\dot{y} - \dot{\theta}x)\hat{j}$$

Now the linear momentum balance for the puck in the  $xy$  plane gives

$$\begin{aligned} -\mu \eta g \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} &= \eta(\ddot{x}\hat{i} + \ddot{y}\hat{j}) \\ \Rightarrow \ddot{x} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|} (\vec{v}_{\text{rel}} \cdot \hat{i}) = -\frac{\mu g(\dot{x} + \dot{\theta}y)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}} \\ \text{and } \ddot{y} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|} (\vec{v}_{\text{rel}} \cdot \hat{j}) = -\frac{\mu g(\dot{y} - \dot{\theta}x)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}. \end{aligned}$$

These are coupled nonlinear ODEs that represent the equations of motion of the puck.

$$\ddot{x} = -\frac{\mu g(\dot{x} + \dot{\theta}y)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}, \quad \ddot{y} = -\frac{\mu g(\dot{y} - \dot{\theta}x)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}$$

Note that these equations are valid only as long as there is relative slip between the puck and the turntable. If the puck stops sliding due to friction, it simply goes in circles with the turntable and, therefore, its equations of motion then are  $\ddot{x} = -\dot{\theta}^2 x$ ,  $\ddot{y} = -\dot{\theta}^2 y$ .

- (b) **Equation of motion using a fixed reference frame:** Now we derive the equations of motion using a rotating reference frame,  $\mathcal{B}$ , with  $(x', y')$  coordinate axes, fixed to the rotating turntable. Let the position of the puck in the rotating frame be  $\vec{r}_{P/O'} = x'\hat{i}' + y'\hat{j}'$ . Note that the velocity of the puck in the rotating reference frame is  $\vec{v}_{P/\mathcal{B}} = \dot{x}'\hat{i}' + \dot{y}'\hat{j}'$ , and acceleration is  $\vec{a}_{P/\mathcal{B}} = \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}'$ . Now, from the linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the puck, we get

$$-\mu mg \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} = m(\vec{a}_P + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}})$$

where we have used the three term acceleration formula for  $\vec{a}_P$ . Here,

$$\begin{aligned}\vec{a}_P &= -\dot{\theta}^2(x'\hat{i}' + y'\hat{j}') \\ \vec{a}_{P/\mathcal{B}} &= \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' \\ 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} &= -2\dot{\theta}\dot{y}'\hat{i}' + 2\dot{\theta}\dot{x}'\hat{j}'\end{aligned}$$

Note that the point  $P'$ , coincident with  $P$  and fixed on the turntable, is stationary with respect to the rotating frame. Therefore, the relative velocity of  $P$  as observed in the rotating frame is  $\vec{v}_{\text{rel}} = \vec{v}_{P/\mathcal{B}} = \dot{x}'\hat{i}' + \dot{y}'\hat{j}'$ . Substituting these terms in the LMB equation above, we have

$$-\mu g \frac{\dot{x}'\hat{i}' + \dot{y}'\hat{j}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} = (-\dot{\theta}^2 x' + \ddot{x}' - 2\dot{\theta}\dot{y}')\hat{i}' + (-\dot{\theta}^2 y' + \ddot{y}' + 2\dot{\theta}\dot{x}')\hat{j}'$$

Dotting this equation with  $\hat{i}'$  and  $\hat{j}'$ , respectively, we get

$$\begin{aligned}\ddot{x}' &= \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\dot{y}' \\ \ddot{y}' &= \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\dot{x}'\end{aligned}$$

These are the required equations of motion for the puck in the rotating frame.

$$\boxed{\ddot{x}' = \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\dot{y}', \quad \ddot{y}' = \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\dot{x}'}$$

Once we find a solution  $x'(t)$  and  $y'(t)$  of these equations, we can find the solution in the fixed frame by transforming  $(x', y')$  to  $(x, y)$  through

$$\begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) & -\sin(\dot{\theta}t) \\ \sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{bmatrix} \begin{Bmatrix} x'(t) \\ y'(t) \end{Bmatrix}$$

Also, note that when the solution of the equations of motion in the rotating reference frame brings the puck to halt, the puck stops with respect to the rotating turntable. To an observer in the fixed frame, the puck will be going in circles with a constant  $\dot{\theta}$ .

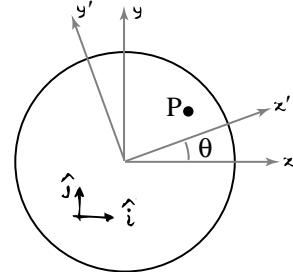


Figure 10.18: Axes  $(x', y')$  represent the rotating frame  $\mathcal{B}$  fixed to the rotating turntable; ie  $(x', y')$  rotate with angular velocity  $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{k}$ .

(Filename:sfig11.1.puckontable.b)

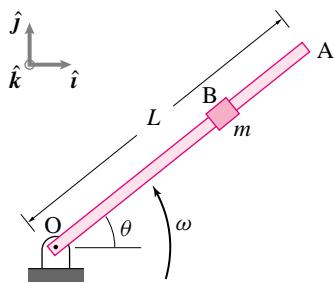


Figure 10.19: A collar slides on a rough bar and finally shoots off the end of the bar as the bar rotates with constant angular speed.

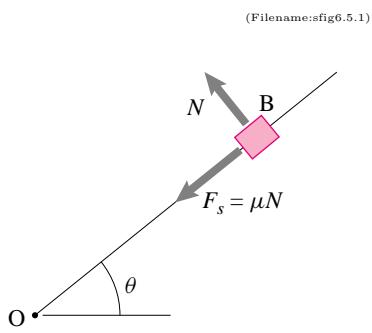


Figure 10.20: Free Body Diagram of the collar. The only forces on the collar are the interaction forces of the bar, which are the normal force  $N$  and the friction force  $F_s = \mu N$ .

(Filename:fig6.5.1a)

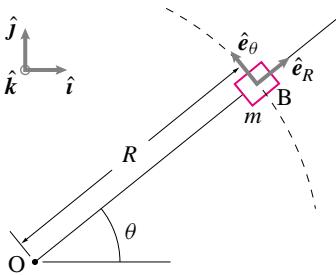


Figure 10.21: Geometry of the collar position at an arbitrary time during its slide on the rod.

**SAMPLE 10.8** *A collar sliding on a rough rod.* A collar of mass  $m = 0.5$  lb slides on a massless rigid rod OA of length  $L = 8$  ft. The rod rotates counterclockwise with a constant angular speed  $\dot{\theta} = 5$  rad/s. The coefficient of friction between the rod and the collar is  $\mu = 0.3$ . At time  $t = 0$  s, the bar is horizontal and the collar is at rest at 1 ft from the center of rotation O. Ignore gravity.

- How does the position of the collar change with time (*i.e.*, what is the equation of motion of the rod)?
- Plot the path of the collar starting from  $t = 0$  s till the collar shoots off the end of the bar.
- How long does it take for the collar to leave the bar?

### Solution

- First, we draw a Free Body Diagram of the the collar at a general position  $(R, \theta)$ . The FBD is shown in Fig. 10.20 and the geometry of the position vector and basis vectors is shown in Fig. 10.21. In the Free Body Diagram there are only two forces acting on the collar (forces exerted by the bar) — the normal force  $\vec{N} = N\hat{e}_\theta$  acting normal to the rod and the force of friction  $\vec{F}_s = -\mu N\hat{e}_R$  acting along the rod. Now, we can write the linear momentum balance for the collar:

$$\begin{aligned} \sum \vec{F} &= m\vec{a} && \text{or} \\ -\mu N\hat{e}_R + N\hat{e}_\theta &= m[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_0)\hat{e}_\theta] \end{aligned} \quad (10.11)$$

Note that  $\ddot{\theta} = 0$  because the rod is rotating at a constant rate. Now dotting both sides of Eqn. (10.11) with  $\hat{e}_R$  and  $\hat{e}_\theta$  we get

$$\begin{aligned} [\text{Eqn. (10.11)}] \cdot \hat{e}_R &\Rightarrow -\mu N = m(\ddot{R} - R\dot{\theta}^2) \\ \text{or } \ddot{R} - R\dot{\theta}^2 &= -\frac{\mu N}{m} \\ [\text{Eqn. (10.11)}] \cdot \hat{e}_\theta &\Rightarrow N = 2m\dot{R}\dot{\theta}. \end{aligned}$$

Eliminating  $N$  from the last two equations we get

$$\ddot{R} + 2\mu\dot{R}\dot{\theta} - \dot{\theta}^2 R = 0.$$

Since  $\dot{\theta} = \omega$  is constant, the above equation is of the form

$$\ddot{R} + C\dot{R} - \omega^2 R = 0 \quad (10.12)$$

where  $C = 2\mu\omega$  and  $\omega = \dot{\theta}$ .

**Solution of equation (10.12):** The characteristic equation associated with Eqn. (10.12) (time to pull out your math books and see the solution of ODEs) is

$$\begin{aligned} \lambda^2 + C\lambda - \omega^2 &= 0 \\ \Rightarrow \lambda &= \frac{-C \pm \sqrt{C^2 + 4\omega^2}}{2} \\ &= \omega(-\mu \pm \sqrt{\mu^2 + 1}). \end{aligned}$$

Therefore, the solution of Eqn. (10.12) is

$$\begin{aligned} R(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ &= Ae^{(-\mu+\sqrt{\mu^2+1})\omega t} + Be^{(-\mu-\sqrt{\mu^2+1})\omega t}. \end{aligned}$$

Substituting the given initial conditions:  $R(0) = 1$  ft and  $\dot{R}(0) = 0$  we get

$$R(t) = \frac{1 \text{ ft}}{2} \left[ e^{(-\mu + \sqrt{\mu^2 + 1})\omega t} + e^{(-\mu - \sqrt{\mu^2 + 1})\omega t} \right]. \quad (10.13)$$

$$R(t) = \frac{1 \text{ ft}}{2} \left[ e^{(-\mu + \sqrt{\mu^2 + 1})\omega t} + e^{(-\mu - \sqrt{\mu^2 + 1})\omega t} \right].$$

- (b) To draw the path of the collar we need both  $R$  and  $\theta$ . Since  $\dot{\theta} = 5 \text{ rad/s}$  = constant,

$$\theta = \dot{\theta} t = (5 \text{ rad/s}) t.$$

Now we can take various values of  $t$  from 0 s to, say, 1 s, and calculate values of  $\theta$  and  $R$ . Plotting all these values of  $R$  and  $\theta$ , however, does not give us an entirely correct path of the collar, since the equation for  $R(t)$  is valid only till  $R = \text{length of the bar} = 8$  ft. We, therefore, need to find the final time  $t_f$  such that  $R(t_f) = 8$  ft. Equation (10.13) is a nonlinear algebraic equation which is hard to solve for  $t$ . We can, however, solve the equation iteratively on a computer, or with some patience, even on a calculator using trial and error. One way to find  $t_f$  would be to simply plot  $R(t)$  and find the intersection with  $R = 8$  (see Fig. 10.22) and read the corresponding value of  $t$ . Either by refining the time interval around the intersection or by interpolation, we can find  $t_f$ . Following this method, we find that  $t_f = 0.74518$  s here. Now, we can plot the path of the collar by computing  $R$  and  $\theta$  from  $t = 0$  to  $t = t_f$  and making a polar plot on a computer as follows (pseudocode).

```

tf = 0.74518 % final value of t
t = 0:tf/100:tf; % take 101 points in [0 tf]
R0 = 1; w = 5; mu = .3; % initialize variables
f1 = -mu + sqrt(mu^2 + 1); % first partial exponent
f2 = -mu - sqrt(mu^2 + 1); % second partial exponent
R = 0.5*R0*(exp(f1*w*t) + exp(f2*w*t)); % calculate R
theta = w*t; % calculate theta
polarplot(theta,r)

```

The plot produced thus is shown in Fig. 10.23.

- (c) The time  $t_f$  computed above was

$$t_f = 0.7452 \text{ s.}$$

By plugging this value in the expression for  $R(t)$  (Eqn. (10.13)) we get, indeed,

$$R = 8 \text{ ft.}$$

$$t_f = 0.7452 \text{ s}$$

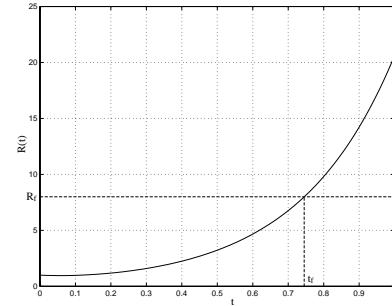


Figure 10.22: N (Filename:sfig11.1.newgunRoft)

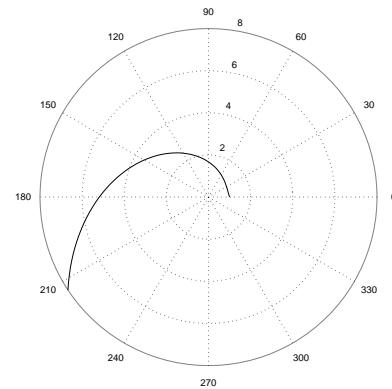


Figure 10.23: Plot of the path of the collar till it leaves the rod.  
(Filename:sfig11.1.newgunpath)

## 10.2 Mechanics of one-degree-of-freedom 2-D mechanisms

① No physical system can move in only one way. The idea that a machine has one degree of freedom is an idealization that takes literally the rigid body description of the parts and the ideal nature of the connections. In fact all parts can deform and no connections are so strict as to be exact geometric constraints. It may be reasonable to respect the standard rigid body idealizations and consider a machine as one-degree-of-freedom for basic analysis. But it may also be reasonable to relax some of those assumptions and consider more degrees of freedom when trying to figure out why that same machine vibrates in an undesirable way.

A one-degree-of-freedom mechanism is a collection of parts linked together so that they can move in only one way<sup>①</sup>. The word “freedom” has to be taken lightly here because in practice even the one freedom is often controlled or restricted. Frankly, most machine designers don’t trust the laws of mechanics to enforce motions that they want. Instead they choose kinematic restrictions that enforce the desired motions and then use a motor, a computer controlled actuator or big flywheel to keep that motion moving at a prescribed rate.

We consider here machines that can move in just one way, whether or not that one motion is free. So in this sense, “one”-degree-of-freedom machines include machines with no freedom at all, just so long as they move in only one way.

Some familiar examples of one-degree-of-freedom mechanisms are a 1-D spring and mass, a pendulum, a slider-crank, a grounded 4-bar linkage, and a gear train.

### Most ideal constraints are workless constraints

A fruitful equation for studying one-degree-of-freedom mechanisms is power balance, or for conservative systems, energy balance. The reason these equations are so useful is because most ideal connections are workless. That is:

the net work of the interaction forces (and moments) of a pair of parts that are connected with the standard ideal connections is zero. This includes welds, frictionless hinges, frictionless sliding contact, rolling contact, or parts connected by a massless inextensible link.

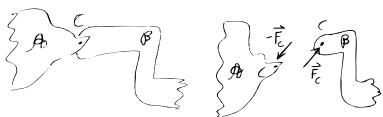


Figure 10.24: A frictionless hinge is a workless constraint. The *net* work of the interaction force on the two contacting bodies is zero.

(Filename:tfigure.hingeisworkless)

#### *Example: A frictionless hinge is a workless constraint*

Body  $\mathcal{A}$  is connected to body  $\mathcal{B}$  by a frictionless hinge at  $C$  (see Fig. ??). The force on body  $\mathcal{B}$  at  $C$  from  $\mathcal{A}$  is  $\vec{F}_C$  and the force on  $\mathcal{A}$  from body  $\mathcal{B}$  at  $C$  is  $-\vec{F}_C$ . The power of the interaction force on body  $\mathcal{B}$  is  $P_{\mathcal{B}\text{on}\mathcal{A}} = \vec{F}_C \cdot \vec{v}_C$ . This power contributes to the increase in the kinetic energy of  $\mathcal{B}$ . The power of the interaction force on  $\mathcal{A}$  is  $P_{\mathcal{A}\text{on}\mathcal{B}} = -\vec{F}_C \cdot \vec{v}_C = -P_{\mathcal{B}}$ . So the contribution to the increase in the kinetic energy of body  $\mathcal{A}$  is minus the contribution to body  $\mathcal{B}$  and the net power on the system of two bodies is zero. Writing this out,

The net power of the pair of interaction forces on the pair of bodies

$$\begin{aligned} \overbrace{P_{\text{total}}} &= P_{\mathcal{B}\text{on}\mathcal{A}} + P_{\mathcal{A}\text{on}\mathcal{B}} \\ &= \vec{F}_C \cdot \vec{v}_C + (-\vec{F}_C) \cdot \vec{v}_C \\ &= (\vec{F}_C - \vec{F}_C) \cdot \vec{v}_C \\ &= 0. \end{aligned}$$

□

Basically the same situation holds for all the standard ideal connections as explained in the box on page 603.

If one of two interacting bodies is known to be stationary, like the ground, then the work of the constraint forces is zero on both of the bodies. Thus the work of the hinge force on a pendulum, and the ground reaction forces on a frictionlessly sliding body or the ground force on a perfectly rolling body is zero. But be careful with the words “workless constraint forces”, however.

The workless constraint connecting moving bodies  $\mathcal{A}$  and  $\mathcal{B}$  is likely to do positive work on one of the bodies and negative work on the other.

It is just the net work on the two bodies which is zero.

## Energy method: single degree of freedom systems

Although linear and angular momentum balance apply to a single degree of freedom system and all of its parts, often one finds what one wants with a single scalar equation, namely energy or power balance.

Imagine a complex machine that only has one degree of freedom, meaning the position of the whole machine is determined by a single configuration variable, call it  $q$ . Further assume that the machine has no motion when  $\dot{q} = 0$ . The variable  $q$  could be, for example, the angle of one of the linked-together machine parts. Also, assume that the machine has no dissipative parts: no friction, no collisions, no inelastic deformation. Because  $q$  characterizes the position of all of the parts of the system we can, in principle, calculate the potential energy of the system as a function of  $q$ ,

$$E_P = E_P(q).$$

We find this function by adding up the potential energies of all the springs in the machine and the gravitational potential energies of the parts. Similarly we can write the system's kinetic energy in terms of  $q$  and its rate of change  $\dot{q}$ . Because at any configuration the velocity of every point in the system is proportional to  $\dot{q}$  we can write the kinetic energy as:

$$E_K = M(q)q^2/2$$

## 10.2 THEORY

### *Ideal constraints and workless constraints*

All of the ideal constraints we consider are interactions between two bodies  $\mathcal{A}$  and  $\mathcal{B}$ . One of these could be the ground. Let's take the interaction force  $\vec{F}$  and moment  $\vec{M}$  to be the force and moment of  $\mathcal{A}$  on  $\mathcal{B}$ . The point of interaction is A on  $\mathcal{A}$  and B on  $\mathcal{B}$ . By the principle of action and reaction, the net power of the interaction force on the two bodies is

$$\begin{aligned} P &= \vec{F} \cdot \vec{v}_B + \vec{M} \cdot \vec{\omega}_{\mathcal{B}} + (-\vec{F}) \cdot \vec{v}_A + (-\vec{M}) \cdot \vec{\omega}_{\mathcal{A}} \\ &= \vec{F} \cdot \vec{v}_{B/A} + \vec{M} \cdot \vec{\omega}_{\mathcal{B}/\mathcal{A}}. \end{aligned}$$

All of our ideal constraints are designed to exactly make these dot products zero. The ideal hinge is considered in the text. Another

example is perfect rolling. In that case the interaction moment is assumed to be zero. The no-slip condition means that  $\vec{v}_{B/A}$ . On the other hand for frictionless sliding there  $\vec{v}_{B/A}$  can have a component tangent to the surfaces. But that is exactly the direction where the friction force is assumed to be zero.

And so it is for all of the ideal “workless” constraints.

Examples of non-workless constraints, that is, interactions that contribute to the energy equations are: sliding with non-zero friction, joints with non-zero friction torques, joints with motors, or interactions mediated by springs, dampers or actuators.

where  $M(q)$  is a function that one can determine by calculating the machine's total kinetic energy in terms of  $q$  and  $\dot{q}$  and then factoring  $\dot{q}^2$  out of the resulting expression.

Now, if we accept the equation of mechanical energy conservation we have

$$\begin{aligned} \text{constant} &= E_T && \text{by conservation of energy,} \\ \Rightarrow 0 &= \frac{d}{dt} E_T && \text{taking one time derivative,} \\ &= \frac{d}{dt} [E_P + E_K] && \text{breaking energy into total potential, plus kinetic} \\ &= \frac{d}{dt} [E_P(q) + \frac{1}{2} M(q) \dot{q}^2] && \text{substituting from paragraphs above} \end{aligned}$$

so,

$$\begin{aligned} 0 &= \frac{d}{dq} [E_P(q)] \dot{q} + \frac{1}{2} \frac{d}{dq} [M(q)] \dot{q} \dot{q}^2 + M(q) \ddot{q} \\ 0 &= \frac{d}{dq} [E_P(q)] + \left( \frac{1}{2} \frac{d}{dq} [M(q)] \right) \dot{q}^2 + M(q) \ddot{q} \quad \text{cancelling } \dot{q} \\ 0 &= f_1(q) + f_2(q) \dot{q}^2 + f_3(q) \ddot{q} \quad (10.14) \end{aligned}$$

with  $f_1(q) \equiv \frac{d}{dq} [E_P(q)]$ ,  
 $f_2(q) \equiv \frac{1}{2} \frac{d}{dq} [M(q)]$ , and  
 $f_3(q) \equiv M(q)$ .

① The cancellation of the factor  $\dot{q}$  from equation 10.15 depends on  $\dot{q}$  being other than zero. While moving  $\dot{q}$  is not zero. Strictly we cannot cancel the  $\dot{q}$  term from the equation at the instants when  $\dot{q} = 0$ . However, to say that a differential equation is true except for certain instants in time is, in practice to say that it is always true, at least if we make reasonable assumptions about the smoothness of the motions.

The cancellation of  $\dot{q}$  above lacks a bit of mathematical rigor, but doesn't cause problems<sup>①</sup>. The equation of motion is complicated because when we take the time derivative of a function of  $M(q)$  and  $E_P(q)$  we have to use the chain rule. Also, because we have products of terms, we had to use the product rule. Eqn. 10.15 is the general equation of motion of a conservative one-degree-of-freedom system. It is really just a special case of the equation of motion for one-degree-of-freedom systems found from power balance. Rather than memorizing eqn. (10.15) it is probably best to look at its derivation as an algorithm to be reproduced on a problem by problem basis.

### Example: Spring and mass

Although the motion of a spring and mass system can be found easily enough from linear momentum balance, it is also a good example for energy balance (see Fig. 10.25). Using conservation of energy for the spring and mass system:



Figure 10.25: The familiar one degree of freedom spring and mass system.

(Filename:tfigure.springmassas1DOF)

$$\begin{aligned} E_T &= \text{constant} \\ 0 &= \frac{d}{dt} E_T \\ &= \dot{E}_K + \dot{E}_P \\ &= \frac{d}{dt} (mv^2/2) + \frac{d}{dt} (kx^2/2) \\ &= mv\dot{v} + kx\dot{x} \\ v = \dot{x} &\Rightarrow 0 = m\ddot{x} + kx. \end{aligned}$$

Similarly power balance could have been used to get the same result, looking at just the mass

$$P = \frac{d}{dt} E_K$$

$$\begin{aligned} (-kx)(\dot{x}) &= \frac{d}{dt}(mv^2/2) \\ &= mv\dot{v} \\ \Rightarrow 0 &= kx + m\ddot{x} \end{aligned}$$

as before.  $\square$

### Example: Pendulum

Consider a rigid body with mass  $m$  and moment of inertia  $I^o$  about a hinge which is a distance  $\ell$  from the center of mass (see Fig. 10.26). The familiar simple pendulum is another single degree of freedom system for which the equation of motion can be found from conservation of energy.

$$\begin{aligned} E_T &= \text{constant} \\ \rightarrow 0 &= \frac{d}{dt}E_T \\ &= \dot{E}_K + \dot{E}_P \\ &= \frac{d}{dt}(I^o\omega^2/2) + \frac{d}{dt}(-gml\cos\theta) \\ &= I^o\omega\dot{\omega} + -g\ell(-\sin\theta)\dot{\theta} \\ \omega = \dot{\theta} &\Rightarrow 0 = \ddot{\theta} + \frac{mg\ell}{I^o}\sin\theta. \end{aligned}$$

the pendulum equation that we have derived before by this and other means (angular momentum balance about point o).  $\square$

The above examples are old friends which are handled easily with other techniques. Here is a problem which is much more difficult without the energy method.

### Example: Three bars act like a simple pendulum

Assume all three bars in the structure shown in Fig. 10.27 are of equal length  $\ell$  and have mass  $m$  uniformly distributed along their length. It is intuitively obvious that this device swings back and forth something like a simple pendulum. But how can we get the laws of mechanics to tell us this? One approach, which will work in the end, is to draw free body diagrams of all the parts, write linear and angular momentum balance for each, and then add and subtract equations to eliminate the unknown constraint forces at the various hinges.

The more direct approach is to write the energy equation, adding up the potential and kinetic energies of the parts, all evaluated in terms of the single configuration variable  $\theta$ . Taking the potential energy to be zero at  $\theta = \pi/2$  (when all centers of mass are at hinge height) we have

$$\begin{aligned} E_T &= \text{constant} \\ 0 &= \frac{d}{dt}E_T \\ &= \dot{E}_P + \dot{E}_K \\ &= \frac{d}{dt} \left( (I^o\omega^2/2) + (I^o\omega^2/2) + (m(\ell\omega)^2/2) \right) \\ &\quad + \frac{d}{dt} \left( (-gm(\ell/2)\cos\theta - gm(\ell/2)\cos\theta) - gml\cos\theta \right) \\ I^o = m\ell^2/3 &\Rightarrow 0 = \frac{d}{dt} \left( 5m\ell^2\omega^2/6 \right) + \frac{d}{dt} \left( -2gml\cos\theta \right) \end{aligned}$$

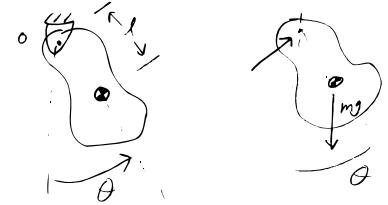


Figure 10.26: A rigid body suspended from a frictionless hinge is an energy conserving one-degree-of-freedom mechanism.

(Filename:tfigure.pendulumas1DOF)

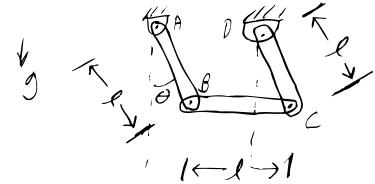


Figure 10.27: Three identical uniform bars are pinned with frictionless hinges and swing, obviously, something like a simple pendulum.

(Filename:tfigure.threelinkpend)

$$\begin{aligned}
 &= (5m\ell^2\omega\dot{\omega}/3 + 2gm\ell\omega \sin \theta) \\
 \Rightarrow 0 &= 5\ell\dot{\omega}/3 + 2g \sin \theta \\
 \Rightarrow 0 &= \ddot{\theta} + \frac{6g}{5\ell} \sin \theta
 \end{aligned}$$

which is the same governing equation as for a point-mass pendulum with length  $5\ell/6$ . This is just half way between the following two cases. If the side links had no mass the equation would have been the same as for a point mass pendulum with length  $\ell$

$$0 = \ddot{\theta} + \frac{g}{\ell} \sin \theta$$

and if the bottom link had no mass the equation would be the same as a stick hanging from one end which goes back and forth like a point mass pendulum with length  $2\ell/3$  according to

$$0 = \ddot{\theta} + \frac{3g}{2\ell} \sin \theta.$$

□

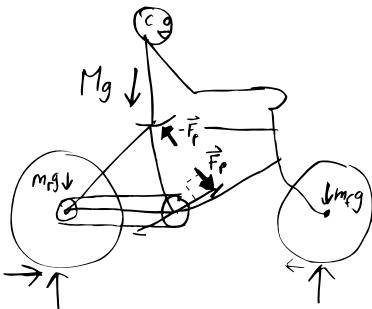


Figure 10.28: A person rides a bike. The pedaling leg is idealized as a pair of equal and opposite forces acting on the seat and pedal.

(Filename:tfigure.bikeandpedal)

### Example: One on the rim is like two on the frame.

A bicycle transmission is such that the speed of the bike relative to the ground is  $n$  times the speed pedal relative to the frame :

$$v_{\text{bike}} = n v_{\text{pedal/bike}}$$

Assume the kinetic energy of the relative motion of a rider's legs can be neglected, as can be the weight of the rider's leg. At the moment in question the velocity of the pedal is parallel to the direction from the seat to the leg. Thus the free body diagram of the bike/person system, leaving out the pedaling leg is as shown in Fig. 10.28. Let's assume the bike and rider have mass  $M$  and that the wheels have mass  $m_r$  and  $m_f$  concentrated on the rim (the hubs are considered part of the frame and the spokes are neglected). Neglecting air resistance etc. the power balance equation is:

$$P = \dot{E}_K \quad (10.15)$$

Let's do some side calculations for evaluating the terms in eqn. (10.15). First, the only forces that do work on the system as drawn are the force on the pedal and the force on the seat.

$$\begin{aligned}
 P &= -\vec{F}_P \cdot \vec{v}_{\text{seat}} + \vec{F}_P \cdot \vec{v}_{\text{pedal}} \\
 &= -\vec{F}_P \cdot \vec{v}_{\text{seat}} + \vec{F}_P \cdot (\vec{v}_{\text{bike}} + \vec{v}_{\text{pedal/bike}}) \\
 &= \underbrace{-\vec{F}_P \cdot \vec{v}_{\text{seat}}}_{0} + \underbrace{\vec{F}_P \cdot \vec{v}_{\text{bike}}}_{F_P v_{\text{pedal/bike}}} + \underbrace{\vec{F}_P \cdot \vec{v}_{\text{pedal/bike}}}_{F_P v_{\text{pedal/bike}}} \\
 &= F_P v_{\text{bike}}/n
 \end{aligned}$$

The net power of the leg is expressed by the compression it carries times its extension rate. The kinetic energy of the wheel comes from both its

rotation and its translation. The moment of inertia of a hoop about its center is  $I = mr^2$ . For rolling contact  $|\omega R| = v$  so, for *one* wheel:

$$\begin{aligned} E_{\text{Kwheel}} &= mv_{\text{bike}}^2/2 + I\omega^2/2 \\ &= mv_{\text{bike}}^2/2 + (mR^2)(v/R)^2/2 \\ &= mv_{\text{bike}}^2. \end{aligned}$$

The kinetic energy of a rolling hoop is twice that of a point mass moving at the same speed. Putting these results back in to eqn. (10.15) we have

$$P = \dot{E}_{\text{K}} \quad (10.16)$$

$$F_P v_{\text{bike}}/n = \frac{d}{dt} (Mv_{\text{bike}}^2/2 + (m_r + m_f)v_{\text{bike}}^2) \quad (10.17)$$

$$= (M\dot{v}_{\text{bike}} + 2(m_r + m_f)\dot{v}_{\text{bike}}) v_{\text{bike}} \quad (10.18)$$

$$\Rightarrow F_P/n = (M + 2(m_r + m_f))\dot{v}_{\text{bike}} \quad (10.19)$$

$$\Rightarrow \dot{v}_{\text{bike}} = \frac{F_P}{n(M + 2(m_r + m_f))}. \quad (10.20)$$

$$(10.21)$$

The bigger the pedal force, the bigger the acceleration, obviously. The higher the gear ratio, the less the acceleration; the faster gears let you pedal slower for a given bike speed, but demand more pedal force for a given acceleration. A heavier bike accelerates less. But the contribution to slowing a bike is twice as much for mass added to the rim as for mass added to the frame or body.

**Some comments.**  $n$  typically ranges from about 1.7 to 8 for a new 21 speed bike and is about 5 for an adult European, Indian or Chinese 1-speed. For a given speed of bicycle riding your feet go  $n$  times slower relative to your body than for walking or running at that speed. This calculation is for accelerating a bike on level ground with no wind and rolling resistance. The net speed of a bike in a bike race is not so dependent on weight, because the main enemy is wind resistance. To the extent that weight is a problem it is for steady uphill travel. In this case the mass on the rim makes the same contribution as mass on the frame.

□

## Vibrations

The preponderance of systems where vibrations occur is not due to the fact that so many systems look like a spring connected to a mass, a simple pendulum, or a torsional oscillator. Instead there is a general class of systems which can be expected to vibrate sinusoidally near some equilibrium position. These systems are one-degree-of-freedom (one DOF) near an energy minimum.

In detail why this works out is explained in Box ??.

## Examples of 1 DOF harmonic oscillators

In the previous section, we have shown that any non-dissipative one-degree-of-freedom system that is near a potential energy minimum can be expected to have simple harmonic motion. Besides the three examples we have given so far, namely,

- a spring and mass,
- a simple pendulum, and
- a rigid body and a torsional spring,

there are examples that are somewhat more complex, such as

- a cylinder rolling near the bottom of a valley,
- a cart rolling near the bottom of a valley, and a
- a four bar linkage swinging freely near its energy minimum.

The restriction of this theory to systems with only one-degree-of-freedom is not so bad as it seems at first sight. First of all, it turns out that simple harmonic motion is important for systems with multiple-degrees-of-freedom. We will discuss this generalization in more detail later with regard to normal modes. Secondly, one can also get a good understanding of a vibrating system with multiple-degrees-of-freedom by modeling it as if it has only one-degree-of-freedom.

### *Example: Cylinder rolling in a valley*

Consider the uniform cylinder with radius  $r$  rolling without slip in an cylindrical ‘ideal’ valley of radius  $R$ .

## 10.3 THEORY

### *One degree of freedom systems near a potential energy minimum are harmonic oscillators*

In order to specialize to the case of oscillations, we want to look at a one degree of freedom system near a stable equilibrium point, a potential energy minimum.

At a potential energy minimum we have, as you will recall from ‘max-min’ problems in calculus, that  $dE_P(q)/dq = 0$ . To keep our notation simple, let’s assume that we have defined  $q$  so that  $q = 0$  at this minimum. Physically this means that  $q$  measures how far the system is from its equilibrium position. That means that if we take a Taylor series approximation of the potential energy the expression for potential energy can be expressed as follows:

$$\begin{aligned} E_P &\approx \text{const} + \underbrace{\frac{dE_P}{dq}}_0 \cdot q + \frac{1}{2} \underbrace{\frac{d^2E_P}{dq^2}}_{K_{\text{equiv}}} \cdot q^2 \quad (10.22) \\ \Rightarrow \quad \frac{dE_P}{dq} &\approx K_{\text{equiv}} \cdot q \quad (10.23) \end{aligned}$$

Applying this result to equation( 10.15) we get:

$$0 = K_{\text{equiv}}q + \frac{1}{2} \frac{d}{dq}[M(q)]\dot{q}^2 + M(q)\ddot{q}. \quad (10.24)$$

We now write  $M(q)$  in terms of its Taylor series. We have

$$M(q) = M(0) + dM/dq|_0 \cdot q + \dots \quad (10.25)$$

and substitute this result into equation 10.24. We have not finished using our assumption that we are only going to look at motions that are close to the equilibrium position  $q = 0$  where  $q$  is small. The

nature of motion close to an equilibrium is that when the deflections are small, the rates and accelerations are also small. Thus, to be consistent in our approximation we should neglect any terms that involve products of  $q$ ,  $\dot{q}$ , or  $\ddot{q}$ . Thus the middle term involving  $\dot{q}^2$  is negligibly smaller than other terms. Similarly, using the Taylor series for  $M(q)$ , the last term is well approximated by  $M(0)\ddot{q}$ , where  $M(0)$  is a constant which we will call  $M_{\text{equiv}}$ . Now we have for the equation of motion:

$$0 = \frac{d}{dt}E_T \quad \Rightarrow \quad 0 = K_{\text{equiv}}q + M_{\text{equiv}}\ddot{q}, \quad (10.26)$$

which you should recognize as the harmonic oscillator equation. So we have found that for any energy conserving one degree of freedom system near a position of stable equilibrium, the equation governing small motions is the harmonic oscillator equation. The effective stiffness is found from the potential energy by  $K_{\text{equiv}} = d^2E_P/dq^2$  and the effective mass is the coefficient of  $\dot{q}^2/2$  in the expansion for the kinetic energy  $E_K$ . The displacement of any part of the system from equilibrium will thus be given by

$$A \sin(\lambda t) + B \cos(\lambda t) \quad (10.27)$$

with  $\lambda^2 = K_{\text{equiv}}/M_{\text{equiv}}$ , and  $A$  and  $B$  determined by the initial conditions. So we have found that *all stable non-dissipative one-degree-of-freedom systems oscillate when disturbed slightly from equilibrium* and we have found how to calculate the frequency of vibration.

For this problem we can calculate  $E_K$  and  $E_P$  in terms of  $\theta$ . Briefly,

$$\begin{aligned} E_P &= -mg(R-r)\cos\theta \\ E_K &= \frac{1}{2}\left(\frac{3}{2}mr^2\right)\left(\frac{\dot{\theta}(R-r)}{r}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot{\theta}^2 \end{aligned}$$

So we can derive the equation of motion using the fact of constant total energy.

$$\begin{aligned} 0 &= \frac{d}{dt}(E_T) \\ &= \frac{d}{dt}(E_K + E_P) \\ &= \frac{d}{dt}\left(\underbrace{-mg(R-r)\cos\theta}_{E_P} + \underbrace{\frac{3}{4}m(R-r)^2\dot{\theta}^2}_{E_K}\right) \\ &= (mg(R-r)\sin\theta)\dot{\theta} + \frac{3}{2}(R-r)^2\dot{\theta}\ddot{\theta} \\ \Rightarrow 0 &= mg(R-r)\sin\theta + \frac{3}{2}(R-r)^2m\ddot{\theta} \end{aligned}$$

Now, assuming small angles, so  $\theta \approx \sin\theta$ , we get

$$g(R-r)\theta + \frac{3}{2}(R-r)^2\ddot{\theta} = 0 \quad (10.28)$$

$$\theta + \underbrace{\left(\frac{2}{3}\frac{g}{(R-r)}\right)\ddot{\theta}}_{\lambda^2} = 0 \quad (10.29)$$

This equation is our old friend the harmonic oscillator equation, as expected. The period is a funny combination of terms. If  $r \ll R$  it looks like a point mass pendulum with length  $3R/2$ , more than  $R$ . That is, the rolling effect doesn't go away and make the roller act like a point mass even when the radius goes to zero. See page 482 for the angular momentum approach to this problem.  $\square$

Although, in some abstract way the energy approach always works, practically speaking it has limitations for systems where the configuration is not easily found from one configuration variable.

#### **Example: A four bar linkage does not easy with energy methods**

Take a four-bar linkage with one bar grounded. Assume the bars all have different lengths. This is a one-DOF system which can use the angle  $\theta$  of one of the links as a configuration variable. But finding the potential energy as a single formula in terms of all of the links in terms of  $\theta$  is more trigonometry than most of us like. And then finding the kinetic energy in terms of  $\theta$  and  $\dot{\theta}$  is close enough to impossible that people don't do it.

So, though it is true that there are functions  $E_P(\theta)$  and  $E_K(\theta, \dot{\theta})$  and that the equations of motion could be written in terms of them, it is really not practical to do so.

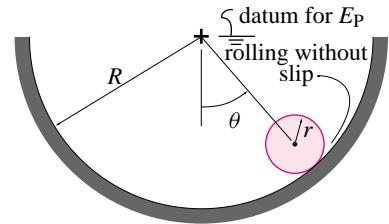


Figure 10.29: Cylinder rolling without slip in a cylinder.  
(Filename:tfigure12.bigcyl.smallcyl)

How do you find the motions of a 4-bar linkage in practice? It's more tricky. One approach is to solve the kinematics by integrating kinematic differential equations, as in Sample 9.5 on page 577. Then you set up and solve the balance equations of the separate parts as in Sample 10.3 on page 632  $\square$

**SAMPLE 10.9** A plate pendulum. A  $2a \times 2b$  rectangular plate of mass  $m$  hangs from two parallel, massless links EA and FD of length  $\ell$  each. The links are hinged at both ends so that when the plate swings, its edges AD and BC remain horizontal at all times. The only driving force present is gravity. Find the equation of motion of the plate.

**Solution** The given system is a single DOF system. So, we need just one configuration variable and the equation of motion be just one scalar equation in this variable. Let us take angle  $\theta$  (Fig. 10.31) as our configuration variable.

The free body diagram of the plate is shown in Fig. 10.32. Note that the link forces  $\vec{F}_1$  and  $\vec{F}_2$  act along the links because massless links are two force bodies. Let  $(x, y)$  be the coordinates of the center of mass. Then the linear momentum balance for the plate gives

$$\vec{F}_1 + \vec{F}_2 - mg\hat{j} = m\vec{a} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}).$$

Now, we can eliminate both the unknown link forces from this equation by dotting the equation with  $\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j}$ , a unit vector normal to the links. Then, we have

$$\begin{aligned} -mg(\hat{j} \cdot \hat{n}) &= m[\ddot{x}(\hat{i} \cdot \hat{n}) + \ddot{y}(\hat{j} \cdot \hat{n})] \\ -g \sin \theta &= \ddot{x} \cos \theta + \ddot{y} \sin \theta. \end{aligned} \quad (10.30)$$

Now we need to find a relationship between  $x$  and  $\theta$ , and  $y$  and  $\theta$ , so that we can write  $\ddot{x}$  and  $\ddot{y}$  in terms of our configuration variable  $\theta$  and its derivatives. From Fig. 10.31, we have

$$\begin{aligned} \vec{r}_G &= (\overbrace{\ell \sin \theta + a}^x)\hat{i} + (\overbrace{-\ell \cos \theta - b}^y)\hat{j} \\ \Rightarrow \ddot{x} &= \ell(\cos \theta \cdot \dot{\theta} - \sin \theta \cdot \dot{\theta}^2) \\ \ddot{y} &= \ell(\sin \theta \cdot \dot{\theta} + \cos \theta \cdot \dot{\theta}^2) \end{aligned}$$

Substituting these expressions for  $\ddot{x}$  and  $\ddot{y}$  in eqn. (10.30), we get

$$\begin{aligned} -g \sin \theta &= \ell \ddot{\theta} \\ \Rightarrow \ddot{\theta} + \frac{g}{\ell} \sin \theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0}$$

This is the equation of a simple pendulum! Well, the plate does behave just like a simple pendulum in the given mechanism. From the expressions for the  $x$  and  $y$  coordinates of the center of mass, we have

$$\begin{aligned} x - a &= \ell \sin \theta \\ y + b &= -\ell \cos \theta \\ \Rightarrow (x - a)^2 + (y + b)^2 &= \ell^2 \end{aligned}$$

that is, the center of mass follows a circle of radius  $\ell$  centered at  $(-a, b)$ . Since the orientation of the plate never changes (AD and BC always remain horizontal), the plate has no angular velocity. Thus the motion of the plate is equivalent to the motion of a particle of mass  $m$  going in a circle centered at  $(-a, b)$  and driven by gravity. That is the simple pendulum.

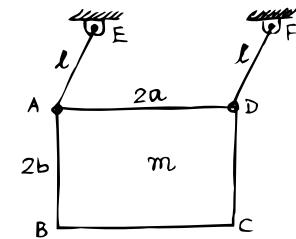


Figure 10.30: (Filename:sfig11.2.platepend)

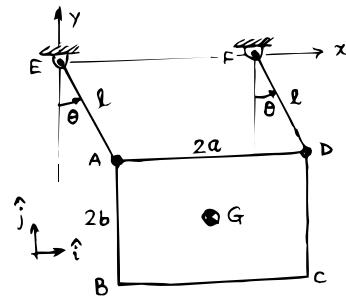


Figure 10.31: (Filename:sfig11.2.platepend.a)

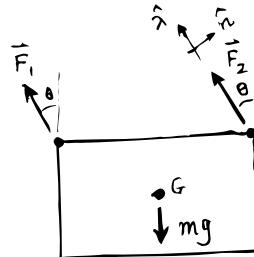


Figure 10.32: (Filename:sfig11.2.platepend.b)

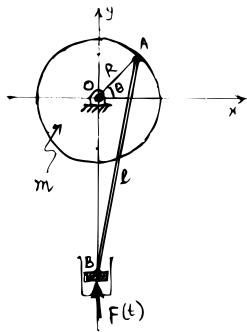


Figure 10.33: (Filename:sfig11.2.sliderenergy)

**SAMPLE 10.10** *Equation of motion from power balance.* A slider crank mechanism is shown in Fig. 10.33 where the crank is a uniform wheel of mass  $m$  and radius  $R$  anchored at the center and the connecting rod AB is a massless rod of length  $\ell$ . The rod is driven by a piston at B with a known force  $F(t) = F_0 \cos \Omega t$ . There is no gravity. Find the equation of motion of the wheel.

**Solution** The given mechanism is a one DOF system. So, let us choose a single configuration variable,  $\theta$  for specifying the configuration of the system and derive an equation that determines  $\theta$ . Since the applied force is given and the point of application of the force has a simple motion (vertical), it will be easy to calculate power of this force. Also, the connecting rod is massless, so it does not enter into dynamic calculations. The wheel rotates about its center and, therefore, it is easy to calculate its kinetic energy. So, we use the power balance,  $\dot{E}_K = P$ , here to find the equation of motion of the wheel. Since  $E_K = \frac{1}{2}I_{zz}^{\text{cm}}\dot{\theta}^2$  and  $P = \vec{F} \cdot \vec{v}_B$ , we have

$$I_{zz}^{\text{cm}}\ddot{\theta}\dot{\theta} = F(t)\hat{j} \cdot v_B \hat{j} = F(t)v_B \quad (10.31)$$

Now, we need to find  $v_B$  and express it using the configuration variable  $\theta$  and its derivatives. There are several ways we could find  $v_B$ . Vectorially, we could write,  $\vec{v}_B \equiv v_B \hat{j} = \vec{v}_A \hat{j} + \vec{\omega}_{AB} \times \vec{r}_{B/A}$  where  $\vec{\omega}_{AB} = \dot{\phi} \hat{k}$ . Dotting both sides of this equation with  $\hat{i}$  and  $\hat{j}$  we can find  $\phi$  in terms of  $\theta$  and  $v_B$  in terms of  $\theta$  and  $\dot{\theta}$ . But, for a change, let us use geometry here.

See Fig. 10.34. From triangle ABO, we have

$$\begin{aligned} \frac{R}{\sin(90^\circ - \phi)} &= \frac{\ell}{\sin(90^\circ + \theta)} \\ \Rightarrow \ell \cos \phi &= R \cos \theta \\ \Rightarrow -\ell \sin \phi \cdot \dot{\phi} &= -R \sin \theta \cdot \dot{\theta} \\ \dot{\phi} &= \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta} \end{aligned}$$

Now,

$$\begin{aligned} y_B &= R \sin \theta - \ell \sin \phi \\ \Rightarrow v_B &\equiv \dot{y}_B = R \cos \theta \cdot \dot{\theta} - \ell \cos \phi \cdot \dot{\phi} \\ &= R \dot{\theta} \cos \theta - R \cos \theta \cdot \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta} \\ &= R \dot{\theta} \left( \cos \theta - \frac{R \sin \theta \cos \theta}{\sqrt{\ell^2 - R^2 \cos^2 \theta}} \right) \end{aligned}$$

Substituting this expression for  $v_B$  in power balance eqn. (10.31), we get

$$\begin{aligned} I_{zz}^{\text{cm}}\ddot{\theta}\dot{\theta} &= F(t) \cdot R \dot{\theta} \left( \cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right) \\ \ddot{\theta} &= \frac{RF_0 \cos \Omega t}{\frac{1}{2}mR^2} \left( \cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right). \end{aligned}$$

This is the required equation of motion. As is evident, it is a nonlinear ODE which requires numerical solution on a computer if we would like to plot  $\theta(t)$ .

$$\boxed{\ddot{\theta} = \frac{2F_0 \cos \Omega t}{mR} \left( \cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right)}$$



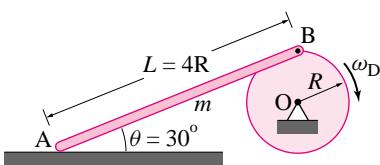


Figure 10.35: End A of bar AB is free to slide on the frictionless horizontal surface while end B is going in circles with a disk rotating at a constant rate.

(Filename:sfig7.3.2)

**SAMPLE 10.11** Instantaneous dynamics of slider crank. A uniform rigid rod AB of mass  $m$  and length  $L = 4R$  has one of its ends pinned to the rim of a disk of radius  $R$ . The other end of the bar is free to slide on a frictionless horizontal surface. A motor, connected to the center of the disk at O, keeps the disk rotating at a constant angular speed  $\omega_D$ . At the instant shown, end B of the rod is directly above the center of the disk making  $\theta$  to be  $30^\circ$ .

- Find all the forces acting on the rod.
- Is there a value of  $\omega_D$  which makes end A of the rod lift off the horizontal surface when  $\theta = 30^\circ$ ?

**Solution** The disk is rotating at constant speed. Since end B of the rod is pinned to the disk, end B is going in circles at constant rate. The motion of end B of the rod is completely prescribed. Since end A can only move horizontally (assuming it has not lifted off yet), the orientation (and hence the position of each point) of the rod is completely determined at any instant during the motion. Therefore, the rod represents a zero degree of freedom system.

- Forces on the rod:** The free body diagram of the rod is shown in Fig. 10.36. The pin at B exerts two forces  $B_x$  and  $B_y$  while the surface in contact at A exerts only a normal force  $N$  because there is no friction. Now, we can write the momentum balance equations for the rod. The linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the rod gives

$$B_x \hat{i} + (B_y + N - mg) \hat{j} = m \vec{a}_G. \quad (10.32)$$

The angular momentum balance about the center of mass G ( $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ ) of the rod gives

$$\vec{r}_{A/G} \times N \hat{j} + \vec{r}_{B/G} \times (B_x \hat{i} + B_y \hat{j}) = I_{zz/G} \alpha_{rod} \hat{k}. \quad (10.33)$$

From these two vector equations we can get three scalar equations (the Angular Momentum Balance gives only one scalar equation in 2-D since the quantities on both sides of the equation are only in the  $\hat{k}$  direction), but we have six unknowns —  $B_x$ ,  $B_y$ ,  $N$ ,  $\vec{a}_G$  (counts as two unknowns), and  $\alpha_{rod}$ . Therefore, we need more equations. We have already used the momentum balance equations, hence, the extra equations have to come from kinematics.

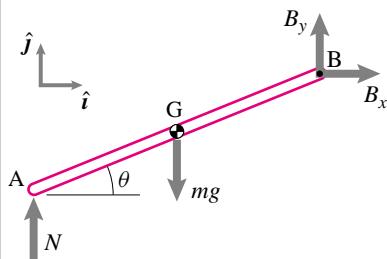


Figure 10.36: Free body diagram of the bar.

(Filename:sfig7.3.2a)

$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \overbrace{\vec{\omega}_{rod} \times \vec{r}_{A/B}}^{\vec{v}_{A/B}} \\ \text{or } v_A \hat{i} &= \omega_D R \hat{i} + \omega_{rod} \hat{k} \times L(-\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= (\omega_D R + \omega_{rod} L \sin \theta) \hat{i} - \omega_{rod} L \cos \theta \hat{j} \end{aligned}$$

Dotting both sides of the equation with  $\hat{j}$  we get

$$0 = \omega_{rod} L \cos \theta \quad \Rightarrow \quad \omega_{rod} = 0.$$

Also,

$$\begin{aligned} \vec{a}_A &= \vec{a}_B + \overbrace{\vec{\omega} \times \vec{r}_{A/B} + \underbrace{\vec{\omega}_{rod} \times (\vec{\omega}_{rod} \times \vec{r}_{A/B})}_{\vec{0}}}^{\vec{a}_{A/B}} \\ \text{or } a_A \hat{i} &= -\omega_D^2 R \hat{j} + \dot{\omega}_{rod} \hat{k} \times L(-\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= -(\omega_D^2 R + \dot{\omega}_{rod} L \cos \theta) \hat{j} + \dot{\omega}_{rod} L \sin \theta \hat{i}. \end{aligned}$$

Dotting both sides of this equation by  $\hat{\mathbf{j}}$  we get

$$\dot{\omega}_{rod} = -\frac{\omega_D^2 R}{L \cos \theta}. \quad (10.34)$$

Now, we can find the acceleration of the center of mass:

$$\begin{aligned}\vec{a}_G &= \vec{a}_B + \overbrace{\hat{\boldsymbol{\omega}} \times \vec{r}_{G/B} + \underbrace{\vec{\omega}_{rod} \times (\vec{\omega}_{rod} \times \vec{r}_{G/B})}_{\vec{0}}} \\ &= -\omega_D^2 R \hat{\mathbf{j}} + \dot{\omega}_{rod} \hat{\mathbf{k}} \times \frac{1}{2} L (-\cos \theta \hat{\mathbf{i}} - \sin \theta \hat{\mathbf{j}}) \\ &= -(\omega_D^2 R + \frac{1}{2} \dot{\omega}_{rod} L \cos \theta) \hat{\mathbf{j}} + \frac{1}{2} \dot{\omega}_{rod} L \sin \theta \hat{\mathbf{i}}.\end{aligned}$$

Substituting for  $\dot{\omega}_{rod}$  from eqn. (10.34) and  $30^\circ$  for  $\theta$  above, we obtain

$$\vec{a}_G = -\frac{1}{2} \omega_D^2 R \left( \frac{1}{\sqrt{3}} \hat{\mathbf{i}} + \hat{\mathbf{j}} \right).$$

Substituting this expression for  $\vec{a}_G$  in eqn. (10.32) and dotting both sides by  $\hat{\mathbf{i}}$  and then by  $\hat{\mathbf{j}}$  we get

$$\begin{aligned}B_x &= -\frac{1}{2\sqrt{3}} m \omega_D^2 R, \\ B_y + N &= -\frac{1}{2} m \omega_D^2 R + mg \quad (10.35)\end{aligned}$$

From eqn. (10.33)

$$\begin{aligned}\frac{1}{2} L [(B_y - N) \cos \theta - B_x \sin \theta] \hat{\mathbf{k}} &= \frac{1}{12} m L^2 \left( -\frac{\omega_D^2 R}{L \cos \theta} \right) \hat{\mathbf{k}} \\ \text{or} \quad B_y - N &= -\frac{1}{6} \frac{m \omega_D^2 R}{\cos^2 \theta} + B_x \tan \theta \\ &= -\frac{2}{9} m \omega_D^2 R - \frac{1}{6} m \omega_D^2 R \\ &= -\frac{7}{18} m \omega_D^2 R \quad (10.36)\end{aligned}$$

From eqns. (10.35) and (10.36)

$$B_y = \frac{1}{2} \left( mg - \frac{8}{9} m \omega_D^2 R \right)$$

and

$$N = \frac{1}{2} \left( mg - \frac{1}{9} m \omega_D^2 R \right).$$

- (b) **Lift off of end A:** End A of the rod loses contact with the ground when normal force  $N$  becomes zero. From the expression for  $N$  from above, this condition is satisfied when

$$\begin{aligned}\frac{2}{9} m \omega_D^2 R &= mg \\ \Rightarrow \quad \omega_D &= 3 \sqrt{\frac{g}{R}}.\end{aligned}$$

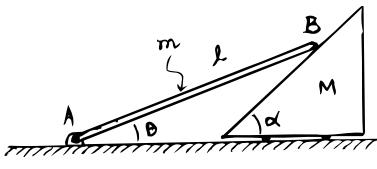


Figure 10.37: (Filename:sfig11.2.slidingbar)

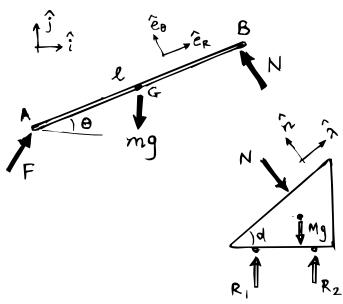


Figure 10.38: (Filename:sfig11.2.slidingbar.a)

**SAMPLE 10.12 A bar sliding on a sliding wedge.** A bar AB of mass  $m$  and length  $\ell$  is hinged at end A rests on a wedge of mass  $M$  at the other end B. The contact at B is frictionless. The wedge is free to slide horizontally without any friction. The motion of the system is driven only by gravity. Find the equation of motion of the bar using

- (a) momentum balance
- (b) energy (or power) balance.

**Solution (a) Momentum Balance:** The bar and the wedge make up a single DOF system. To derive the equations of motion of the bar, let us choose  $\theta$  as the configuration variable. The free body diagram of the bar and the wedge are shown in Fig. 10.38. Note that the normal reaction at B is normal to the wedge surface, *i.e.*,  $\vec{N} = N\hat{n}$ . Now, the angular momentum balance for the bar about point A gives,

$$\begin{aligned}\dot{\vec{H}}_A &= \sum \vec{M}_A \\ I_{zz}^A \ddot{\theta} \hat{k} &= \frac{\ell}{2} \hat{e}_R \times (-mg \hat{j}) + \ell \hat{e}_R \times N \hat{n} \\ &= -\frac{1}{2} mg \ell \cos \theta \hat{k} + N \ell \cos(\alpha - \theta) \hat{k}\end{aligned}\quad (10.37)$$

where the last line follows from the fact that  $\hat{e}_R = \cos \theta \hat{i} + \sin \theta \hat{j}$ , the unit normal  $\hat{n} = -\sin \alpha \hat{i} + \cos \alpha \hat{j}$ , and so,  $\hat{e}_R \times \hat{j} = \cos \theta \hat{k}$  and  $\hat{e}_R \times \hat{n} = \cos(\alpha - \theta) \hat{k}$ . We now need to eliminate the unknown normal reaction  $N$  from the above equation. Since the wedge is constrained to move only horizontally, we can write the linear momentum balance for the wedge as

$$N \hat{n} \cdot \hat{i} = M \ddot{x} \Rightarrow N = \frac{M \ddot{x}}{\hat{n} \cdot \hat{i}} = \frac{M \ddot{x}}{\sin \alpha} \quad (10.38)$$

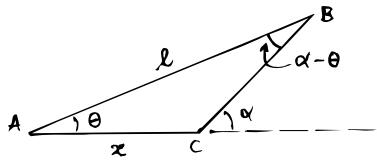


Figure 10.39: (Filename:sfig11.2.slidingbar.b)

Thus, we have found  $N$  in terms of  $\ddot{x}$  that we can use in eqn. (10.37) to get rid of  $N$ . But, we now need to express  $\ddot{x}$  in terms of our configuration variable  $\theta$  and its derivatives. Consider the triangle ABC formed by the bar and the slanted edge of the wedge. Let  $x = AC$  denote the horizontal position of the wedge. Then, from the law of sines, we have  $\frac{x}{\sin(\alpha-\theta)} = \frac{\ell}{\sin \alpha}$ , so that

$$\begin{aligned}x &= \frac{\ell}{\sin \alpha} \sin(\alpha - \theta) \\ \Rightarrow \dot{x} &= \frac{\ell}{\sin \alpha} \cos(\alpha - \theta) \cdot (-\dot{\theta})\end{aligned}\quad (10.39)$$

$$\Rightarrow \ddot{x} = \frac{\ell}{\sin \alpha} [\ddot{\theta} \cos(\alpha - \theta) + \dot{\theta}^2 \sin(\alpha - \theta)]. \quad (10.40)$$

Now, substituting for  $N$  in eqn. (10.37) from eqn. (10.38), using the expression for  $\ddot{x}$  from above, and dotting the resulting equation with  $\hat{k}$ , we get

$$\begin{aligned}\frac{1}{3} m \ell^2 \ddot{\theta} &= -\frac{mg \ell}{2} \cos \theta + \frac{M \ell^2 \cos(\alpha - \theta)}{\sin^2 \alpha} [\ddot{\theta} \cos(\alpha - \theta) + \dot{\theta}^2 \sin(\alpha - \theta)] \\ \Rightarrow \ddot{\theta} &= -\frac{3mg \sin^2 \alpha \cos \theta + 3M \ell \dot{\theta}^2 \sin 2(\alpha - \theta)}{2m \ell \sin^2 \alpha + 6M \ell \cos^2(\alpha - \theta)} \\ &= -\frac{3[(g/\ell) \sin^2 \alpha \cos \theta + (M/m) \dot{\theta}^2 \sin 2(\alpha - \theta)]}{2[\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)]}\end{aligned}\quad (10.41)$$

$$\boxed{\ddot{\theta} = -\frac{3[(g/\ell) \sin^2 \alpha \cos \theta + (M/m) \dot{\theta}^2 \sin 2(\alpha - \theta)]}{2[\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)]}}$$

**(b) Power balance:** Now we derive the equation of motion for the bar using power balance  $E_K = P$ . For power balance, we have to consider both the bar and the wedge. The bar rotates about the fixed point A, therefore, its kinetic energy is  $(1/2)I_z^A z\dot{\theta}^2$ . The wedge moves with horizontal speed  $\dot{x}$ , therefore, its kinetic energy is  $(1/2)M\dot{x}^2$ . Thus,  $E_K = 1/2I_z^A z\dot{\theta}^2 + 1/2M\dot{x}^2$ . The only force that contributes to power is the force of gravity on the rod because  $\vec{v}_G \cdot (-mg\hat{j})$  is non-zero. The sliding contact at B is frictionless and hence the net power due to the contact force there is zero. Now,  $\vec{v}_G = \frac{1}{2}\ell\dot{\theta}\hat{e}_\theta$ . So,  $P = -\frac{1}{2}mg\ell\dot{\theta}(\hat{e}_\theta \cdot \hat{j}) = -\frac{1}{2}mg\ell\dot{\theta} \cos\theta$ . Thus the power balance for the system gives

$$I_{zz}^A \ddot{\theta} + M\ddot{x}\dot{x} = -\frac{1}{2}mg\ell\dot{\theta} \cos\theta. \quad (10.42)$$

We can simplify this equation further. Note that,  $\ddot{\theta} = \frac{d}{dt}(\frac{1}{2}\dot{\theta}^2)$  and  $\ddot{x}\dot{x} = \frac{d}{dt}(\frac{1}{2}\dot{x}^2)$ . So that we can write the above equation as

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} I_{zz}^A \dot{\theta}^2 \right) + \frac{d}{dt} \left( \frac{1}{2} M \dot{x}^2 \right) &= -\frac{1}{2} mg\ell\dot{\theta} \cos\theta \\ \text{or} \quad \int d(I_{zz}^A \dot{\theta}^2 + M \dot{x}^2) &= -mg\ell \int \cos\theta \, d\theta \\ \Rightarrow I_{zz}^A \dot{\theta}^2 + M \dot{x}^2 &= C - mg\ell \sin\theta \end{aligned}$$

where  $C$  is a constant of integration to be determined from initial conditions. For example, when the bar begins to slide from rest, we have, at  $t = 0$ ,  $\dot{x} = 0$  and  $\dot{\theta} = 0$ . At that instant, if  $\theta(0) = \theta_0$ , then  $C = mg\ell \sin\theta_0$ . So, we can write

$$I_{zz}^A \dot{\theta}^2 + M \dot{x}^2 = mg\ell(\sin\theta_0 - \sin\theta).$$

Now, replacing  $\dot{x}$  in this equation with the expression we obtained in eqn. (10.39), we have

$$\begin{aligned} I_{zz}^A \dot{\theta}^2 + M\ell^2 \dot{\theta}^2 \frac{\cos^2(\alpha - \theta)}{\sin^2 \alpha} &= mg\ell(\sin\theta_0 - \sin\theta) \\ \Rightarrow \dot{\theta}^2 &= \frac{mg\ell(\sin\theta_0 - \sin\theta)}{I_{zz}^A + M\ell^2 \frac{\cos^2(\alpha - \theta)}{\sin^2 \alpha}} \\ &= \frac{mg\ell \sin^2 \alpha (\sin\theta_0 - \sin\theta)}{(1/3)m\ell^2 \sin^2 \alpha + M\ell^2 \cos^2(\alpha - \theta)} \\ \Rightarrow \dot{\theta} &= \sqrt{\frac{3g}{\ell} \sin \alpha} \sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)}}. \end{aligned}$$

This is a first order, nonlinear, ODE compared to the second order equation we got in eqn. (10.41). However, again, we need to resort to numerical solution if we wish to solve for  $\theta(t)$ , in which case, this reduction to the first order equation does not save much work.

$$\dot{\theta} = \sqrt{\frac{3g}{\ell} \sin \alpha} \sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2 \alpha + 3(M/m) \cos^2(\alpha - \theta)}}$$

## 10.3 Dynamics of rigid bodies in multi-degree-of-freedom 2-D mechanisms

To solve problems with multiple degrees of freedom the basic strategy is as described at the start of the chapter

- draw FBDs of each body,
- pick configuration variables,
- write linear and angular momentum balance equations
- solve the equations for variables of interest (forces, second derivatives of the configuration variables).
- set up and solve the resulting differential equations if you are trying to find the motion.

There are two basic approaches to these multi-body problems which, for lack of better language we call “brute-force” and “clever”:

- A. In the brute force approach you write three times as many scalar balance equations as you have bodies. That is, for example, for each free body diagram you write linear momentum balance and angular momentum balance about the center of mass. Then you take this set of  $3n$  equations and add and subtract them to solve for variables of interest.

But is quite suitable for computers and most commercial general purpose dynamic simulators use a variant of this approach. For individual use the brute-force approach is generally more reliable and more time consuming.

- B. In the clever approach you write as many scalar momentum balance equations as you have unknowns. For example, if you have 2 degrees of freedom and you are concerned with motions and not reaction forces, you write 2 equations. You do this by finding momentum balance equations that do not include the variables you are not interested in. Usually this involves using angular momentum balance about hinge points, or linear momentum balance orthogonal to sliding contacts.

The clever approach does not always work; the four-bar linkage is the classic problem case. However the desire to find minimal sets of equations of motion it is historically important<sup>①</sup>.

At this point in the subject all problems are involved if taken from start all the way to plotting solutions to the differential equations. The examples that follow emphasize getting to the equations of motion. The skills for numerically solving the differential equations and plotting the solutions are the same as from the start of dynamics. The sample problems then show all the work from beginning to end for related problems.

### *Example: Block sliding on sliding block: clever approach*

Block 1 with mass  $m_1$  rolls without friction on ideal massless wheels at A and B (see Fig. 10.40). Block 2 with mass  $m_2$  rolls down the tipped top of block 1 on ideal massless rollers at C and D. The locations of  $G_1$  relative to A and B, and of  $G_2$  relative to C and D are known. How do blocks 1 and 2 move. First look at the free body diagram of the system

<sup>①</sup> Attempts to automate the clever approach, to quickly find minimal equations of motion, led to Lagrange equations which led to Hamilton’s equations which led to quantum mechanics (but we won’t be that clever here).

and note that there are no unknown forces in the  $\hat{i}$  direction. So, for the system

$$\begin{aligned} \left\{ \sum \vec{F}_i = \dot{\vec{L}} \right\} \cdot \hat{i} \\ \Rightarrow 0 = m_1 \ddot{x} + m_2 (\ddot{x} \hat{i} + \ddot{y}' \hat{j}') \cdot \hat{i} \\ = (m_1 + m_2) \ddot{x} - \ddot{y}' m_2 \cos \theta. \end{aligned} \quad (10.43)$$

Looking at the free body diagram of mass 2 note that there are no unknown forces in the  $\hat{j}'$  direction, so

$$\begin{aligned} \left\{ \sum \vec{F}_i = \dot{\vec{L}} \right\} \cdot \hat{j}' \\ \Rightarrow m_2 g \sin \theta = m_2 (\ddot{x} \hat{i} + \ddot{y}' \hat{j}') \cdot \hat{j}' \\ m_2 g \sin \theta = m_2 (-\cos \theta \ddot{x} + \ddot{y}'). \end{aligned} \quad (10.44)$$

Eqns. 10.43 and 10.44 are a system of two equations in the two unknowns  $\ddot{x}$  and  $\ddot{y}'$

$$\begin{aligned} (m_1 + m_2) \ddot{x} - m_2 \cos \theta \ddot{y}' &= 0 \\ -m_2 \cos \theta \ddot{x} + m_2 \ddot{y}' &= m_2 g \sin \theta \end{aligned}$$

which can be solved for  $\ddot{x}$  and  $\ddot{y}'$  by hand or on the computer. Finding  $x(t)$  and  $y'(t)$  is then easy because both  $\ddot{x}$  and  $\ddot{y}'$  are constants.  $\square$

Now we look at the same example, but proceed in a more naive manner.

#### Example: Block sliding on sliding block: brute force approach

Now we look at the free body diagrams of the two separate blocks. We will use 3 balance equations from each free body diagram, taking account of the kinematic constraints.

For the lower block we have

$$\text{AMB}_{G_1} \Rightarrow \sum \vec{M}_{/G_1} = \dot{\vec{H}}_{/G_1} \quad (10.45)$$

where  $\sum \vec{M}_{/G_1} = \vec{r}_{D/G_1} \times (-F_D \hat{i}') + \vec{r}_{C/G_1} \times (-F_C \hat{i}') + \vec{r}_{B/G_1} \times F_B \hat{j} + \vec{r}_{A/G_1} \times F_A \hat{j}$

$$\text{and } \dot{\vec{H}}_{/G_1} = \vec{0} \quad (\vec{\omega}_1 = \vec{0})$$

$$\text{and LMB} \Rightarrow \sum \vec{F}_i = \dot{\vec{L}} \quad (10.46)$$

where  $\sum \vec{F}_i = -F_D \hat{i}' - F_C \hat{i}' + F_B \hat{j} + F_A \hat{j} - m_1 g \hat{j}$

and  $\dot{\vec{L}} = m_1 \ddot{x} \hat{i}$

Similarly for the upper block:

$$\text{AMB}_{G_2} \Rightarrow \sum \vec{M}_{/G_2} = \dot{\vec{H}}_{/G_2} \quad (10.47)$$

where  $\sum \vec{M}_{/G_2} = \vec{r}_{D/G_2} \times F_D \hat{i}' + \vec{r}_{C/G_2} \times F_C \hat{i}'$

and  $\dot{\vec{H}}_{/G_2} = \vec{0} \quad (\vec{\omega}_1 = \vec{0})$

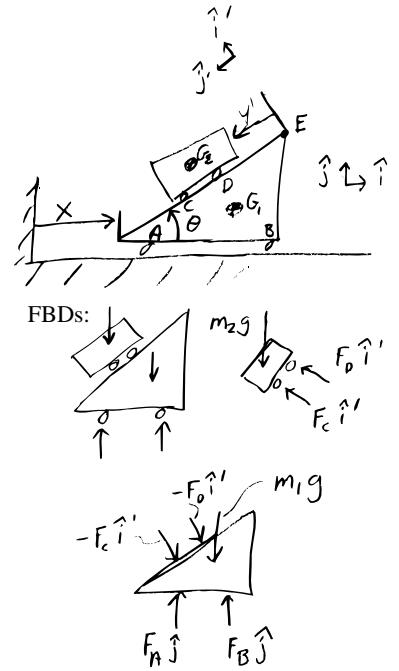


Figure 10.40: Block 2 rolls on block 1 which rolls on the ground. All rollers are ideal (frictionless and massless). The moving  $\hat{i}' \hat{j}'$  frame moves with the lower block and is oriented with the slope.

(Filename:tfigure.blockslidesonblock)

$$\text{and LMB} \Rightarrow \sum \vec{F}_i = \dot{\vec{L}} \quad (10.48)$$

where  $\sum \vec{F}_i = F_D \hat{i}' + F_C \hat{j}' - m_2 g \hat{j}'$   
and  $\dot{\vec{L}} = m_2 (\ddot{x} \hat{i} + \ddot{y}' \hat{j}')$

Eqns. 10.45-10.48 can be written as scalar equations by dotting the LMB equations with  $\hat{i}$  and  $\hat{j}$  and the AMB equations with  $\hat{k}$ . All is known in these 6 equations but the six scalars:  $F_A, F_B, F_C, F_D, \ddot{x}$ , and  $\ddot{y}'$ . These could be set up as a matrix equation and solved on the computer, or you could try to find your way through by adding and subtracting equations. In any case you could solve for  $\ddot{x}$  and  $\ddot{y}'$  and thus have differential equations to solve to find the motions.

One quick inference one can make is from looking at the equations. No term in the coefficients of the unknowns depends on  $x, \dot{x}, y$ , or  $\dot{y}'$ . So all of the reactions  $F_A, F_B, F_C$ , and  $F_D$  as well as the accelerations  $\ddot{x}$  and  $\ddot{y}'$  are constants in time (until the upper mass hits the ground).  $\square$

#### Example: Block sliding on sliding block: even more brute force

This “multi-body” problem can be solved in an even more naive and more brute force manner. The method is the same as shown in Section 6.1 on page 331 for a one dimensional problem.

We would use 6 configuration variables: the  $x$  and  $y$  coordinates of  $G_1$  and  $G_2$  and the rotations of the two bodies:

$$x_1, y_1, \theta_1, x_2, y_2, \text{ and } \theta_2$$

That the two bodies don’t rotate would be expressed indirectly by noting that the velocities of points A and B on the lower mass must have acceleration in the  $\hat{i}$  direction. These two equations would be added to the 6 linear and angular momentum balance equations. Similar constraint equations would be written for the interactions at C and D. Altogether there are now 6 configuration variables and 4 constraint forces. But there are 6 differential equations of motion and 4 constraint equations. Thus at one instant in time a set of 10 simultaneous equations needs to be solved. Then these are used to evaluate the right hand sides in the differential equations.

All this is an impractical mess for solving one problem. But it lends itself to easy automation and is closest to the approach used by general purpose dynamics simulators.  $\square$

The example above is particularly simple because block 1 moves in a straight line without rotating and block 2 moves in a straight line without rotating relative to block 1. Even for a system of just two bodies the situation could be much more complex if the first body had a complex motion and the second a complex motion relative to the first. But because of the preponderance of hinges in the world, circular motion, and motion relative to circular motion, is the most complex motion that need be considered by many engineers. Here is a version of the most common example of that class.

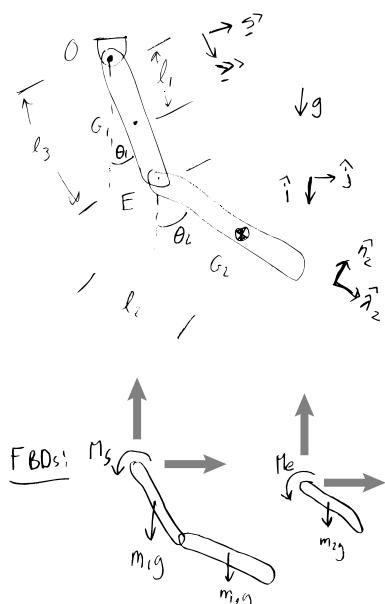


Figure 10.41: A two link robot arm. Free body diagrams are shown of the whole system (including the motor torque at the shoulder) and of the fore-arm (including the motor torque at the elbow).

(Filename:tfigure.robotarm2)

#### Example: A two link robot arm: finesse the finding of reactions

A robot arm has two links. There are motors that apply known torque  $M_s$  at the shoulder (reacted by the base and a torque  $M_e$  at the elbow (reacted by the upper arm). Dimensions are as marked. This system

has 2 degrees of freedom. So we need 2 configuration variables and 2 independent balance equations to find the motion.

The natural configuration variables are the angles of the upper arm relative to a fixed reference and the angle of the lower arm relative to a fixed reference. It would also be natural to instead use the angle of the lower arm relative to the upper arm. This leads to simpler equations in the end, but more work in set up.

Angular momentum balance for the system about the shoulder contains no unknown reaction forces, nor does angular momentum balance of the fore-arm about the hinge. So we base our work on these two equations:

$$\text{System AMB}_{/O} \Rightarrow \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O} \quad (10.49)$$

$$\text{Forearm AMB}_{/E} \Rightarrow \sum \vec{M}_{/E} = \dot{\vec{H}}_{/E}. \quad (10.50)$$

The goal, equations of motion, is reached by evaluating the left and right sides of these equations in terms of known geometric and mass quantities as well as the configuration variables. When we write  $\sum \vec{M}_{/O} \dot{\vec{H}}_{/O}$  we implicitly mean for the whole system. Likewise  $\sum \vec{M}_{/E}$  and  $\dot{\vec{H}}_{/E}$  apply to the forearm.

At each step in the calculations below imagine the results can be substituted into the later steps. We don't do that here because the expressions grow in size. Further, if the angles and their rates of change are known, as they are when doing most dynamics problems, the intermediate calculations will result in numbers, rather than algebraic expressions which grow in size.

$$\begin{aligned} \hat{\lambda}_1 &= \cos \theta_1 \hat{i} + \sin \theta_1 \hat{j} & \text{and} & \hat{\lambda}_2 = \cos \theta_2 \hat{i} + \sin \theta_2 \hat{j} \\ \vec{r}_{G_1/O} &= \ell_1 \hat{\lambda}_1, \quad \vec{r}_{E/O} = \ell_3 \hat{\lambda}_1 & \text{and} & \vec{r}_{G_2/E} = \ell_2 \hat{\lambda}_2 \\ && \text{and} & \text{then} \quad \vec{r}_{G_2/O} = \vec{r}_{E/O} + \vec{r}_{G_2/E}. \end{aligned}$$

$$\begin{aligned} \vec{a}_{G_1/O} &= -\ddot{\theta}_1^2 \vec{r}_{G_1/O} + \ddot{\theta}_1 \hat{k} \times \vec{r}_{G_1/O} & , \\ \vec{a}_{E/O} &= -\ddot{\theta}_1^2 \vec{r}_{E/O} + \ddot{\theta}_1 \hat{k} \times \vec{r}_{E/O} & , \\ \vec{a}_{G_2/E} &= -\ddot{\theta}_2^2 \vec{r}_{G_2/E} + \ddot{\theta}_2 \hat{k} \times \vec{r}_{G_2/E} & \text{and} & \vec{a}_{G_2/O} = \vec{a}_{E/O} + \vec{a}_{G_2/E}. \end{aligned}$$

These terms are all we need to evaluate the 4 terms in Eqns. 10.49 and 10.50.

$$\begin{aligned} \sum \vec{M}_{/O} &= \vec{r}_{G_1/O} \times (-m_1 g \hat{j}) + \vec{r}_{G_2/O} \times (-m_2 g \hat{j}) + M_s \hat{k}, \\ \sum \vec{M}_{/E} &= \vec{r}_{G_2/E} \times (-m_2 g \hat{j}) + M_e \hat{k}, \end{aligned}$$

$$\begin{aligned} \dot{\vec{H}}_{/O} &= m_1 \vec{r}_{G_1/O} \times \vec{a}_{G_1/O} + \ddot{\theta}_1 I_1 \hat{k} + m_2 \vec{r}_{G_2/O} \times \vec{a}_{G_2/O} + \ddot{\theta}_2 I_2 \hat{k}, \\ \text{and} \quad \dot{\vec{H}}_{/E} &= m_2 \vec{r}_{G_2/E} \times \vec{a}_{G_2/E} + \ddot{\theta}_2 I_2 \hat{k}. \end{aligned}$$

Once these are substituted into Eqns. 10.49 and 10.50 one has 2 vector equations with only  $\hat{k}$  components. In other words we have two scalar equations in the two unknowns  $\ddot{\theta}_1$  and  $\ddot{\theta}_2$ . One can go through the algebra and solve for them explicitly, but the expressions are quite complex, even when simplified. At given values of  $\theta_1$ ,  $\dot{\theta}_1$ ,  $\theta_2$ , and  $\dot{\theta}_2$  however these are just two linear equations in two unknowns.  $\square$

## Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than do they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

### *Example: Four bar linkage.*

It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of  $\theta$ ,  $\dot{\theta}$  and  $\ddot{\theta}$  of any one of its joints.

□

**SAMPLE 10.13** *Dynamics of sliding wedges.* A wedge shaped body of mass  $m_2$  sits on a frictionless ground. Another wedge shaped body of mass  $m_1$  is gently placed on the inclined face of the stationary wedge. The top wedge starts to slide down. The coefficient of friction between the two wedges is  $\mu$ . Find the sliding acceleration of the top wedge along the incline (*i.e.*, the relative acceleration of  $m_1$  with respect to  $m_2$ ).

**Solution** The free body diagrams of the two wedges are shown in Fig. 10.43. Note that the friction force is  $\mu N$  since the wedges are sliding with respect to each other (if they were not sliding already then the friction force is an unknown force  $F \leq \mu N$ ). Let the absolute acceleration of  $m_2$  be  $\vec{a}_2 = a_2 \hat{i}$ . Then, the absolute acceleration of  $m_1$  is  $\vec{a}_1 = \vec{a}_2 + \vec{a}_{1/2} = a_2 \hat{i} + a_{\text{rel}} \hat{\lambda}$ . Now, we can write the linear momentum balance for  $m_1$  and  $m_2$  as follows.

$$N \hat{n} - m_1 g \hat{j} - \mu N \hat{\lambda} = m_1 (a_2 \hat{i} + a_{\text{rel}} \hat{\lambda}) \quad (10.51)$$

$$(R - m_2 g) \hat{j} - N \hat{n} + \mu N \hat{\lambda} = m_2 a_2 \hat{i} \quad (10.52)$$

where  $\hat{\lambda} = \cos \alpha \hat{i} - \sin \alpha \hat{j}$  and  $\hat{n} = \sin \alpha \hat{i} + \cos \alpha \hat{j}$ . Here, we have 4 independent scalar equations (from the two 2-D vector equations) in four unknowns  $N$ ,  $R$ ,  $a_2$ , and  $a_{\text{rel}}$ . Thus, we can certainly solve for them. We are, however, only interested in  $a_{\text{rel}}$ . So, we should try to find the answer with fewer calculations. Dotting eqn. (10.51) with  $\hat{\lambda}$ , we have

$$m_1 a_{\text{rel}} = -m_1 a_2 \cos \alpha - \mu N + m_1 g \sin \alpha \quad (10.53)$$

So, to find  $a_{\text{rel}}$ , we need  $a_2$  and  $N$ . Dotting eqn. (10.51) with  $\hat{n}$ , we have

$$m_1 a_2 \sin \alpha = N - m_1 g \cos \alpha, \quad (10.54)$$

and dotting eqn. (10.52) with  $\hat{i}$ , we have

$$m_2 a_2 = -N \sin \alpha + \mu N \cos \alpha. \quad (10.55)$$

Solving eqn. (10.54) and (10.55) simultaneously, and using new variables (for convenience)  $M = m_1/m_2$ ,  $C = \cos \alpha$ , and  $S = \sin \alpha$ , we get

$$a_2 = \frac{MC(\mu C - S)}{1 - MS(\mu C - S)} g, \quad N = m_1 g \frac{C}{1 - MS(\mu C - S)}$$

Substituting these expression in eqn. (10.53), we get

$$a_{\text{rel}} = gS - \frac{gC[M(\mu C - S) - \mu]}{1 - MS(\mu C - S)}$$

$$a_{\text{rel}} = g \sin \alpha - \frac{g \cos \alpha \left[ \frac{m_1}{m_2} \cos \alpha (\mu \cos \alpha - \sin \alpha) - \mu \right]}{1 - \frac{m_1}{m_2} \sin \alpha (\mu \cos \alpha - \sin \alpha)}$$

Note that when there is no friction ( $\mu = 0$ ), the expression for  $a_{\text{rel}}$  reduces to

$$a_{\text{rel}} = gS + \frac{gMC^2S}{1 + MS^2}$$

and if we let  $m_2 \rightarrow \infty$  (*i.e.*,  $m_2$  represents fixed ramp) so that  $M \rightarrow 0$ , then  $a_{\text{rel}} = g \sin \alpha$  which is the acceleration of a point mass down a frictionless ramp of slope  $\tan \alpha$ .

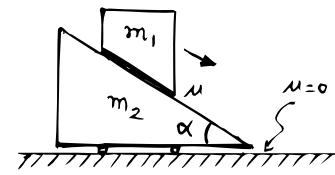


Figure 10.42: (Filename:sfig11.3.wedges)

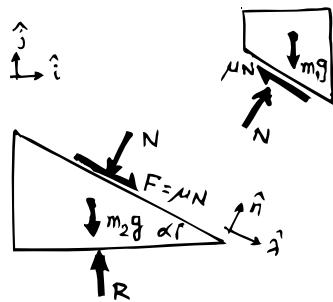


Figure 10.43: (Filename:sfig11.3.wedges.a)

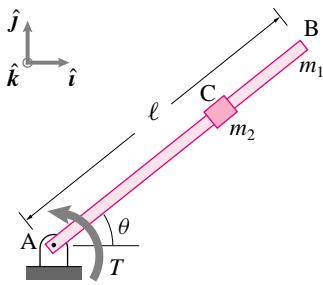


Figure 10.44: (Filename:sfig11.3.newgun)

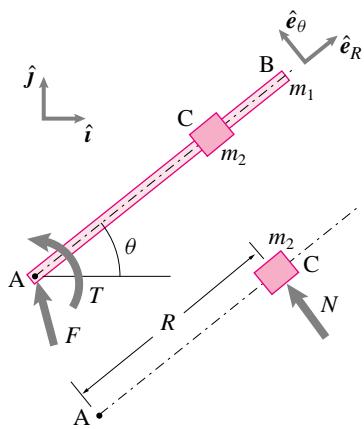


Figure 10.45: (Filename:sfig11.3.newgun.a)

**SAMPLE 10.14** *Dynamics of a new gun.* A new gun consists of a uniform rod AB of mass  $m_1$  and a small collar C of mass  $m_2$  that slides freely on the rod. A motor at A rotates the rod with constant torque  $T$ .

- Find the equations of motion of the collar.
- Show that if  $T = 0$  then the equations of motion imply conservation of angular momentum about point A.

### Solution

(a) Let us denote the configuration of the collar with  $R$ , the radial distance from the fixed point A along the rod, and  $\theta$ , the angular displacement of the rod. We need to find differential equations that determine  $R$  and  $\theta$  as functions of time. The free body diagram of the whole system (rod and collar together) and that of the collar is shown in Fig. 10.45. We can write angular momentum balance for the whole system about point A so that the unknown reaction force  $F$  at A does not enter the equations. Noting that the acceleration of the collar is  $\vec{a}_C = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$ , and letting  $I_1 \equiv I_z^A$  be the moment of inertia of the rod about A, we have,

$$\begin{aligned}\sum \vec{M}_A &= \dot{H}_A \\ T\hat{k} &= I_1\ddot{\theta}\hat{k} + R\hat{e}_R \times m_2[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta] \\ &= I_1\ddot{\theta}\hat{k} + m_2R^2\ddot{\theta}\hat{k} + 2m_2R\dot{R}\dot{\theta}\hat{k}\end{aligned}$$

Dotting this equation with  $\hat{k}$ , we have

$$\ddot{\theta} = \frac{T}{I_1 + m_2R^2} - \frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \quad (10.56)$$

Thus we have obtained the equation of motion for  $\theta$ . Now we consider the free body diagram of the collar alone and write the linear momentum balance for it in the  $\hat{e}_R$  direction, i.e.,  $\hat{e}_R \cdot (\sum \vec{F} = m\vec{a})$ , so that we do not have to care about the unknown normal reaction  $\vec{N}$ . So, we have,

$$\begin{aligned}0 &= \hat{e}_R \cdot m_2[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta] \\ &= m_2(\ddot{R} - R\dot{\theta}^2) \\ \Rightarrow \ddot{R} &= R\dot{\theta}^2. \quad (10.57)\end{aligned}$$

Thus we have the required equations of motion. Note that eqn. (10.56) and (10.57) are coupled nonlinear differential equations. So, to find  $\theta(t)$  and  $R(t)$  we need to solve them numerically.

$$\boxed{\ddot{\theta} = \frac{T}{I_1 + m_2R^2} - \frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}, \quad \ddot{R} = R\dot{\theta}^2}$$

- Now we set  $T = 0$  in our equations of motion. Note that the equation for  $R$  is independent of  $T$ . The equation for  $\theta$  becomes

$$\ddot{\theta} = -\frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \Rightarrow (I_1 + m_2R^2)\ddot{\theta} + 2m_2R\dot{R}\dot{\theta} = 0$$

But the last expression is simply  $\dot{H}_A$  for the system. Thus we have  $\dot{H}_A = 0$  which implies that  $H_A = \text{constant}$ . That is conservation of angular momentum about point A.  $\triangleleft$



**SAMPLE 10.15** Numerical solutions of new gun equations. Consider Sample 10.14 again. Set up the equations of motion for numerical solution. Take  $T = 1 \text{ N}\cdot\text{m}$ ,  $\ell = 1 \text{ m}$ ,  $m_2 = 1 \text{ kg}$ , and  $m_1 = m_2/3$ . Carry out numerical solutions for the following cases.

- Let the system start from rest at  $\theta = 0$  and  $R = 0.1 \text{ m}$ . Find the solution from  $t = 0$  to  $t = 1 \text{ s}$ . Plot  $R(t)$ ,  $\theta(t)$  and  $R(\theta)$  (in polar coordinates).
- Find the solution till the collar leaves the rod. What is the speed of the collar at this instant?
- Compute and plot the total energy of the system as a function of time. Also, plot the work done by the torque as a function of time and show that the work done is equal to the total energy of the system at each instant.
- Vary torque  $T$  and carry out solutions for several values of  $T$ . Find the terminal value of  $\theta_f$  (when the collar leaves the rod) for each  $T$ . Justify your observation about  $\theta_f$  by plotting  $\dot{R}/\dot{\theta}$  as a function of  $T$ .

**Solution** We first need to write the equations of motion, eqn. (10.56) and (10.57), as a set of first order ODEs. We can easily do so by introducing new variables  $\omega \equiv \dot{\theta}$  and  $v_R \equiv \dot{R}$ , so that we have,

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{R} \\ \dot{v}_R \end{pmatrix} = \begin{pmatrix} \frac{T}{I_1+m_2R^2} - \frac{\omega}{I_1+m_2R^2} v_R \omega \\ \frac{2m_2R}{I_1+m_2R^2} v_R \\ v_R \\ R\omega^2 \end{pmatrix}$$

Given the values of all constants, we only need to specify the initial conditions for  $\theta$ ,  $\omega$ ,  $R$ , and  $v_R$  for solving these equations numerically.

- We use the following pseudocode to carry out the numerical solution.

```
Set T = 1, L = 1, m2 = 1, m1 = m2/3
Let I1 = m1*L^2/3, I2 = m2*R^2,
ODEs = {thetadot = w,
         wdot = (T-2*m2*R*vR*w)/(I1+I2),
         Rdot = vR,
         vRdot = R*w^2}
IC = {theta = 0, w = 0, R = 0.1, vR = 0}
Solve ODEs with IC for t=0 to t=1
Plot t vs R, Plot t vs theta,
Polarplot theta vs R
```

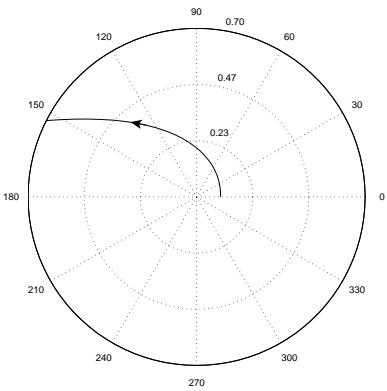


Figure 10.46: (Filename:sfig11.3.newgunRth)

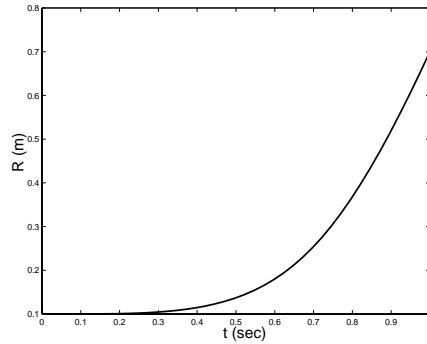
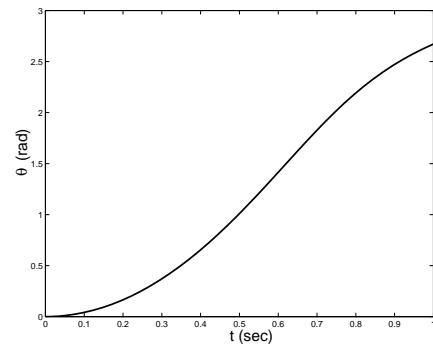


Figure 10.47: (Filename:sfig11.3.newgunSol)



The  $R(t)$  and  $\theta(t)$  plots obtained from the numerical solution are shown in Fig. 10.47 and the polar plot of  $R(\theta)$  is shown in Fig. 10.46.

- (b) We do not know apriori the value of  $t$  at which the collar leaves the rod. So, we have to carry out the solution for some assumed  $t_f$  which gives us  $R(t_f) > \ell$  so that we know the collar has gone past the end of the rod. We then plot  $R(t)$ , including the unreal value of  $R(t_f) > \ell$ , and find the time  $t$  at which  $R(t) = \ell$ , either by zooming into the graph or by interpolation (although, there are various sophisticated algorithms to find this  $t$ ). Following the method of zooming into the graph (see Fig. ??) we find the terminal value of  $t$  to be 1.147 s. We carry out the numerical solution again from  $t = 0$  to  $t_f = 0.147$  s and find that  $R(t_f) = 1$  m,  $v_R(t_f) = 2.13$  m/s, and  $\omega(t_f) = 1.03$  rad/s, so that  $v_f = \sqrt{\dot{R}^2 + (R\dot{\theta})^2} = 2.37$  m/s. This is the terminal speed of the collar.

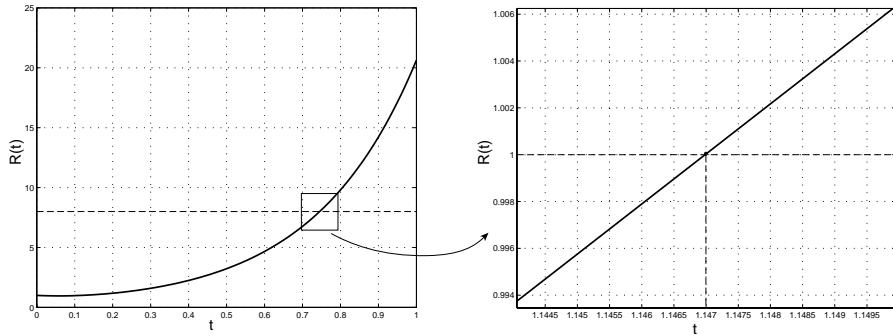


Figure 10.48: Finding the time  $t$  at which the collar leaves the rod from the graph of  $R(t)$ .

(Filename:sfig11.3.newgunRf)

- (c) The work done by the torque is  $W = T\theta$  at any instant. The system only possesses kinetic energy. So the energy of the system at any instant is  $E = E_1 + E_2$  where  $E_1 = \frac{1}{2}I_1\omega^2$  and  $E_2 = \frac{1}{2}m_2(\dot{R}^2 + R^2\omega^2)$ . Computing these quantities for the solution obtained above, we plot  $W$  and  $E$  vs  $t$  as shown in Fig. 10.50. Clearly,  $W = E$ .

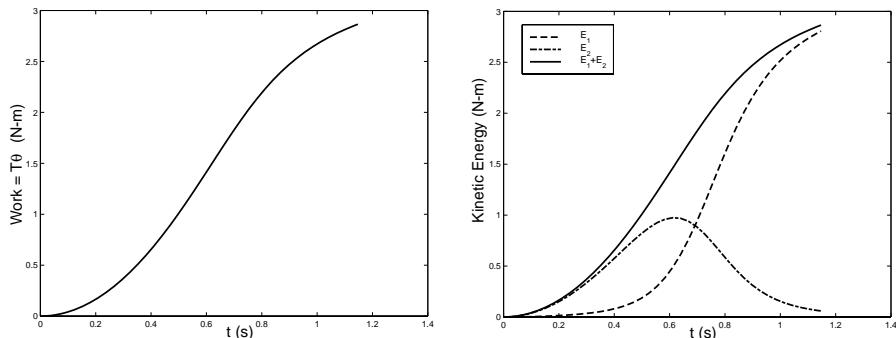


Figure 10.50: Work done by the torque and the kinetic energy of the system.

(Filename:sfig11.3.newgunWE)

- (d) Now we take several values of  $T$  (0.1, 0.5, 1, 1.5, 2, 2.5, and 3) and carry out the numerical solutions for each  $T$ . We note the terminal values of  $\theta$ ,  $\omega (= \dot{\theta})$ , and  $v_R (= \dot{R})$ . By plotting the terminal value of  $\theta$  against  $T$  (Fig. 10.49), we see that the collar leaves the rod at exactly the same  $\theta = 2.86$  rad for each  $T$ ! But this is possible only if  $\dot{R}$  and  $\dot{\theta}$  both change in the same proportion for each  $T$ . So, plot the ratio  $\dot{R}/\dot{\theta}$  just for the terminal values against  $T$  and find that the ratio is indeed constant (see Fig. 10.51).

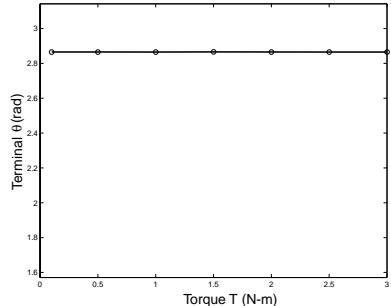


Figure 10.49: Angle  $\theta$  at which the collar leaves the rod vs torque  $T$ .

(Filename:sfig11.3.newgunTvsθ)

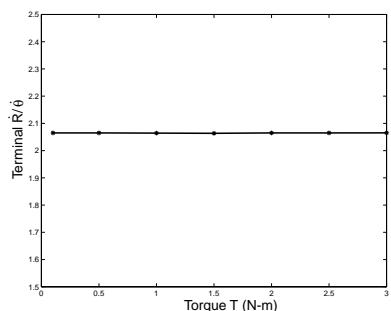


Figure 10.51: The ratio of terminal  $\dot{R}/\dot{\theta}$  vs torque  $T$ .

(Filename:sfig11.3.newgunTvsRθ)

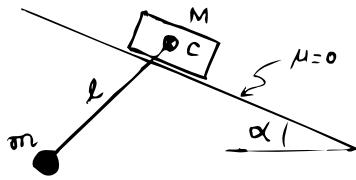


Figure 10.52: (Filename:sfig11.3.slidingpend)

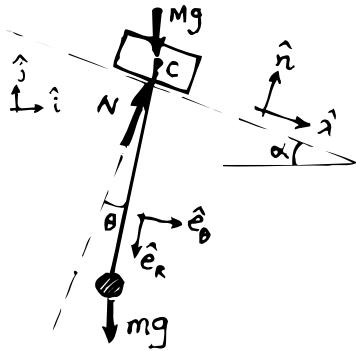


Figure 10.53: (Filename:sfig11.3.slidingpend.a)

**SAMPLE 10.16** *Dynamics of a sliding-base pendulum.* A cart of mass  $M$  slides down a frictionless inclined plane as shown in the figure. A simple pendulum of mass  $m$  and length  $\ell$  hangs from the center of mass of the cart. Find the equation of motion of the pendulum.

**Solution** Let us measure the angular displacement of the pendulum with respect to the cart with angle  $\theta$  measured anticlockwise from the normal to the inclined plane. Let  $s$  be the position of the cart along the inclined plane from some reference point. Then, the acceleration of the cart can be written as  $\vec{a}_C = \ddot{s}\hat{\lambda}$  and the acceleration of the pendulum mass as  $\vec{a} = \vec{a}_C + \vec{a}_{\text{rel}} = \ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R$ .

The free body diagram of the cart and the pendulum system is shown in Fig. 10.53. Writing angular momentum balance of the system about point C, we get

$$\begin{aligned} \vec{M}_C &= \dot{\vec{H}}_C \\ \ell\hat{e}_R \times (-mg\hat{j}) &= \ell\hat{e}_R \times m(\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R) \\ -mgl \sin(\theta - \alpha)\hat{k} &= m\ddot{s}\ell \cos \theta \hat{k} + m\ell^2 \ddot{\theta} \hat{k} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{\ell} \sin(\theta - \alpha) - \frac{\ddot{s}}{\ell} \cos \theta. \end{aligned} \quad (10.58)$$

To find  $\ddot{s}$ , we write the linear momentum balance for the whole system in the  $\hat{\lambda}$  (so that we do not involve the unknown normal reaction  $N$ ) direction.

$$\begin{aligned} \hat{\lambda} \cdot (-Mg\hat{j} - mg\hat{j}) &= \hat{\lambda} \cdot [M\ddot{s}\hat{\lambda} + m(\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R)] \\ \Rightarrow (M+m)g \sin \sin \alpha &= (M+m)\ddot{s} + m\ell\ddot{\theta} \cos \theta - m\ell\dot{\theta}^2 \sin \theta \\ \Rightarrow \ddot{s} &= g \sin \alpha - \frac{m\ell}{M+m} (\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) \end{aligned} \quad (10.59)$$

Substituting eqn. (10.59) in eqn. (10.58) and rearranging terms, we get

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta \frac{(1 + \frac{m}{M}) \cos \alpha}{1 + \frac{m}{M} \sin^2 \theta} + \frac{\frac{m}{M} \dot{\theta}^2 \sin \theta \cos \theta}{1 + \frac{m}{M} \sin^2 \theta}$$

$$\boxed{\ddot{\theta} = -\frac{g}{\ell} \sin \theta \frac{(1 + \frac{m}{M}) \cos \alpha}{1 + \frac{m}{M} \sin^2 \theta} + \frac{\frac{m}{M} \dot{\theta}^2 \sin \theta \cos \theta}{1 + \frac{m}{M} \sin^2 \theta}}$$

Note that if we set  $\alpha = 0$  and let  $M \rightarrow \infty$  so that the cart behaves like a fixed ground, then we recover the equation of simple pendulum,  $\ddot{\theta} = -\frac{g}{\ell} \sin \theta$ , from the equation of motion above. It is a good practice to carry out such simple checks wherever possible.

**Remarks:** We could write the equations of motion, eqn. (10.58) and eqn. (10.59) in the coupled form as

$$\begin{bmatrix} \ell & \cos \theta \\ m\ell \cos \theta & M+m \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} -\frac{g}{\ell} \sin(\theta - \alpha) \\ (M+m)g \sin \alpha - m\ell\dot{\theta}^2 \sin \theta \end{pmatrix}$$

and leave it at that, since for most computational purposes, it is enough. It is not so hard to find expressions for  $\ddot{\theta}$  and  $\ddot{s}$  from here by solving the matrix equation, even by hand.

**SAMPLE 10.17 Resonant capture.** A slightly unbalanced motor mounted on an elastic machine part is modeled as a spring mass system with a simple pendulum of mass  $m$  and length  $\epsilon$  driven by a constant torque  $T$  as shown in the figure. The spring has stiffness  $k$  and the motor has mass  $M$ . There is no friction between mass  $M$  and the horizontal surface.

- Find the equation of motion of the system.
- Take  $M = m = 1 \text{ kg}$ ,  $k = 1 \text{ N/m}$ ,  $T = 15 \times 10^{-3} \text{ N}\cdot\text{m}$ . Solve (numerically) the equations of motion with zero initial conditions and plot  $x(t)$  and  $\theta(t)$  for  $t = 0$  to 100 s.

### Solution

(a) The free body diagram of the system is shown in Fig. 10.55. Let the angular displacement of the eccentric mass  $m$  at some instant  $t$  be  $\theta$ . At the same instant, let the displacement of the motor be  $x$  from the relaxed state of the spring. Then we can write the acceleration of the motor as  $\ddot{x}\hat{i}$  and that of the eccentric mass as  $\ddot{\mathbf{a}}_P = \ddot{x}\hat{i} + \epsilon\dot{\theta}\hat{\mathbf{e}}_\theta - \epsilon\theta^2\hat{\mathbf{e}}_R$ . Now, we can write the angular momentum balance for the system about point C (fixed in the stationary frame of reference but instantly coincident with the center of mass of motor M) as

$$\begin{aligned} T\hat{k} &= \epsilon\hat{\mathbf{e}}_R \times m(\ddot{x}\hat{i} + \epsilon\dot{\theta}\hat{\mathbf{e}}_\theta - \epsilon\theta^2\hat{\mathbf{e}}_R) \\ &= m\epsilon\ddot{x}(\hat{\mathbf{e}}_R \times \hat{i}) + m\epsilon^2\ddot{\theta}(\hat{\mathbf{e}}_R \times \hat{\mathbf{e}}_\theta) \\ &= m\epsilon(-\ddot{x}\sin\theta + \epsilon\ddot{\theta})\hat{k} \\ \Rightarrow \epsilon\ddot{\theta} - \sin\theta \ddot{x} &= \frac{T}{m\epsilon} \end{aligned} \quad (10.60)$$

This is just one scalar equation in  $\ddot{\theta}$  and  $\ddot{x}$ . We need one more independent equation  $\ddot{\theta}$  and  $\ddot{x}$  without involving any other unknowns. So, we write the linear momentum balance for the system in the  $x$ -direction:

$$\begin{aligned} -kx &= M\ddot{x} + m(\ddot{x}\hat{i} + \epsilon\dot{\theta}\hat{\mathbf{e}}_\theta - \epsilon\theta^2\hat{\mathbf{e}}_R) \cdot \hat{i} \\ &= (M+m)\ddot{x} - m\epsilon\ddot{\theta}\sin\theta - m\epsilon\dot{\theta}^2\cos\theta \\ \Rightarrow \epsilon\sin\theta\ddot{\theta} - \left(\frac{M+m}{m}\right)\ddot{x} &= \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \end{aligned} \quad (10.61)$$

Thus we have the required equations of motion. We can write eqn. (10.60) and (10.61) compactly as

$$\begin{bmatrix} \epsilon & -\sin\theta \\ \epsilon\sin\theta & -\frac{m+M}{M} \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \frac{T}{m\epsilon} \\ \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta \end{pmatrix}$$

◀

- We use the following pseudocode to solve the equations of motion. Note that we first convert the two second order ODEs into four first order ODEs by introducing new variables  $\omega = \dot{\theta}$  and  $u = \dot{x}$ .

```
Set T = 0.015, m = 1, M = 1, k = 1, e = 1
A=[e -sin(theta); e*sin(theta) -(1+M/m)];
b = [T/(m*e); k/m*x-e*omega^2*cos(theta)];
solve A*acln = b for acln % acln = accelerations
ODEs = {omega = thetadot, u = xdot,
        omegadot = acln(1), udot = acln(2)}
IC = {theta = 0, x = 0, omega = 0, u = 0}
Solve ODEs with IC for t=0 to t=100
```

The plots of  $x(t)$  and  $\theta(t)$  obtained from the numerical solution are shown in Fig. 10.56 and Fig. 10.57, respectively. Note the resonance of  $M$  for the given values of the system.

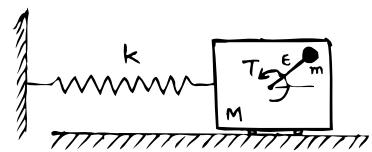


Figure 10.54: (Filename:sfig11.3.eccentricmotor)

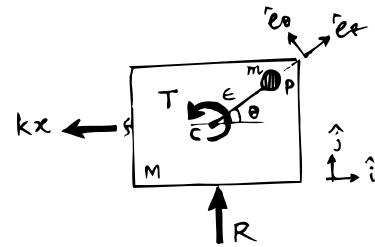


Figure 10.55:  
(Filename:sfig11.3.eccentricmotor.a)

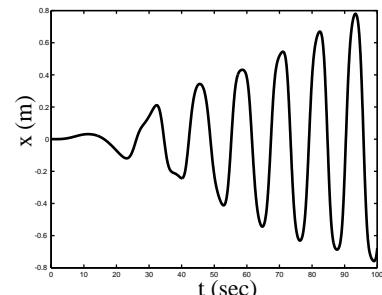


Figure 10.56: (Filename:sfig11.3.eccmotor.x)

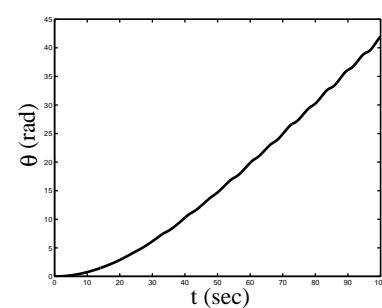


Figure 10.57: (Filename:sfig11.3.eccmotor.th)

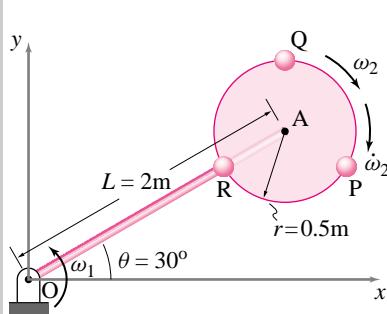
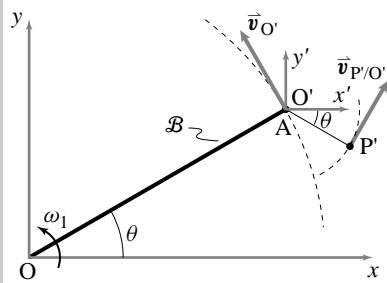


Figure 10.58: (Filename:sfig8.2.2again)

Figure 10.59: The velocity of point  $P'$  is the sum of two terms: the velocity of  $O'$  and the velocity of  $P'$  relative to  $O'$ .  
(Filename:sfig8.2.2d)

**SAMPLE 10.18** *Dynamics using a rotating and translating coordinate system.* Consider the rotating wheel of Sample 9.11 which is shown here again in Figure 10.58. At the instant shown in the figure find

- (a) the linear momentum of the mass P and
- (b) the net force on the mass P.

For calculations, use a frame  $\mathcal{B}$  attached to the rod and a coordinate system in  $\mathcal{B}$  with origin at point A of the rod OA.

**Solution** We attach a frame  $\mathcal{B}$  to the rod. We choose a coordinate system  $x'y'z'$  in this frame with its origin  $O'$  at point A. We also choose the orientation of the primed coordinate system to be parallel to the fixed coordinate system  $xyz$  (see Fig. 10.59), i.e.,  $\hat{i}' = \hat{i}$ ,  $\hat{j}' = \hat{j}$ , and  $\hat{k}' = \hat{k}$ .

- (a) **Linear momentum of P:** The linear momentum of the mass P is given by

$$\vec{L} = m \vec{v}_P.$$

Clearly, we need to calculate the velocity of point P to find  $\vec{L}$ . Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = \underbrace{\vec{v}_{O'} + \vec{v}_{P'/O'}}_{\vec{v}_{P'}} + \vec{v}_{\text{rel}}.$$

Note that  $O'$  and  $P'$  are two points on the same (imaginary) rigid body OAP'. Therefore, we can find  $\vec{v}_{P'}$  as follows:

$$\begin{aligned} \vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}} + \underbrace{\vec{v}_{P'/O'}}_{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}} \\ &= \omega_1 \hat{k} \times L(\cos \theta \hat{i} + \sin \theta \hat{j}) + \omega_1 \hat{k} \times r(\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= \omega_1 [(L+r) \cos \theta \hat{j} - (L-r) \sin \theta \hat{i}] \\ &= 3 \text{ rad/s} \cdot [2.5 \text{ m} \cdot \cos 30^\circ \hat{j} - 1.5 \text{ m} \cdot \sin 30^\circ \hat{i}] \\ &= (6.50 \hat{j} - 2.25 \hat{i}) \text{ m/s} \quad (\text{same as in Sample 9.11.}), \end{aligned}$$

$$\begin{aligned} \vec{v}_{\text{rel}} &= \vec{v}_{P/\mathcal{B}} \\ &= -\omega_2 \hat{k}' \times r(\cos \theta \hat{i}' - \sin \theta \hat{j}') \\ &= -\omega_2 r(\cos \theta \hat{j}' + \sin \theta \hat{i}') \\ &= -(2.16 \hat{j}' + 1.25 \hat{i}') \text{ m/s} \\ &= -(2.16 \hat{j} + 1.25 \hat{i}) \text{ m/s}. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{v}_P &= \vec{v}_{P'} + \vec{v}_{\text{rel}} \\ &= (4.34 \hat{j} - 3.50 \hat{i}) \text{ m/s} \quad \text{and} \\ \vec{L} &= m \vec{v}_P \\ &= 0.5 \text{ kg} \cdot (4.34 \hat{j} - 3.50 \hat{i}) \text{ m/s} \\ &= (-1.75 \hat{i} + 2.17 \hat{j}) \text{ kg} \cdot \text{m/s}. \end{aligned}$$

$$\boxed{\vec{L} = (-1.75 \hat{i} + 2.17 \hat{j}) \text{ kg} \cdot \text{m/s}}$$

(b) **Net force on P:** From the

$$\sum \vec{F} = m\vec{a}$$

for the mass P we get  $\sum \vec{F} = m\vec{a}_P$ . Thus to find the net force  $\sum \vec{F}$  we need to find  $\vec{a}_P$ . The calculation of  $\vec{a}_P$  is the same as in Sample 9.11 except that  $\vec{a}_{P'}$  is now calculated from

$$\vec{a}_{P'} = \vec{a}_{O'} + \vec{a}_{P'/O'}$$

where

$$\begin{aligned}\vec{a}_{O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}) \\ &= -\omega_1^2 \vec{r}_{O'/O} \\ &= -\omega_1^2 L (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= -(3 \text{ rad/s})^2 \cdot 2 \text{ m} \cdot (\cos 30^\circ \hat{i} + \sin 30^\circ \hat{j}) \\ &= -(15.59 \hat{i} + 9.00 \hat{j}) \text{ m/s}^2,\end{aligned}$$

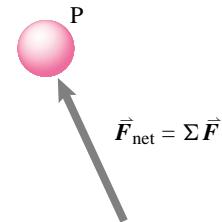


Figure 10.60: (Filename:sfig8.2.2again1)

$$\begin{aligned}\vec{a}_{P'/O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\ &= -\omega_1^2 \vec{r}_{P'/O'} \\ &= -\omega_1^2 r (\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= -(3 \text{ rad/s})^2 \cdot 0.5 \text{ m} \cdot (\cos 30^\circ \hat{i} - \sin 30^\circ \hat{j}) \\ &= -(3.90 \hat{i} - 2.25 \hat{j}) \text{ m/s}^2.\end{aligned}$$

Thus,

$$\vec{a}_{P'} = -(19.49 \hat{i} + 6.75 \hat{j}) \text{ m/s}^2$$

which, of course, is the same as calculated in Sample 9.11. The other two terms,  $\vec{a}_{\text{cor}}$  and  $\vec{a}_{\text{rel}}$ , are exactly the same as in Sample 9.11. Therefore, we get the same value for  $\vec{a}_P$  by adding the three terms:

$$\vec{a}_P = -(17.83 \hat{i} + 3.63 \hat{j}) \text{ m/s}^2.$$

The net force on P is

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_P \\ &= 0.5 \text{ kg} \cdot (-17.83 \hat{i} - 3.63 \hat{j}) \text{ m/s}^2 \\ &= -(8.92 \hat{i} + 1.81 \hat{j}) \text{ N}.\end{aligned}$$

$$\sum \vec{F} = -(8.92 \hat{i} + 1.81 \hat{j}) \text{ N}$$

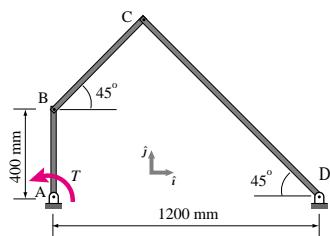


Figure 10.61: (Filename:sfig11.4.fourbar)

**SAMPLE 10.19** *Inverse dynamics of a four bar mechanism.* A four bar mechanism ABCD consists of three uniform bars AB, BC, and CD of length  $\ell_1$ ,  $\ell_2$ ,  $\ell_3$ , and mass  $m_1$ ,  $m_2$ ,  $m_3$ , respectively. The mechanism is driven by a torque  $T$  applied at A such that bar AB rotates at constant angular speed. Write equations to find the torque  $T$  at some instant  $t$ .

**Solution** This is an inverse dynamics problem, that is, we are given the motion and we are supposed to find the forces (torque  $T$  in this case) that cause that motion. We are given that rod AB rotates at constant angular speed, say  $\dot{\theta}$ . From kinematics, we can find out angular velocities and angular accelerations of the other two bars as well as the accelerations of center of mass of each rod. Then we can write the momentum balance equations and compute the forces and moments required to generate this motion. So, in contrast to what we usually do, let us do the kinematics first. Please see Sample 9.5 on page 577. We found the angular velocities,  $\dot{\beta}$  (eqn. (9.73)) and  $\dot{\phi}$  (eqn. (9.72)), of rods BC and CD, respectively, in terms of  $\dot{\theta}$ . We can rewrite those equations as

$$\begin{bmatrix} -\ell_2 \sin \beta & \ell_3 \sin \phi \\ -\ell_2 \cos \beta & \ell_3 \cos \phi \end{bmatrix} \begin{pmatrix} \dot{\beta} \\ \dot{\phi} \end{pmatrix} = \dot{\theta} \begin{pmatrix} \ell_1 \sin \theta \\ \ell_1 \cos \theta \end{pmatrix} \quad (10.62)$$

We wrote this equation in matrix form to make it easier for us to find the angular accelerations which we do by simply differentiating this equation once:

$$\begin{bmatrix} -\ell_2 \cos \beta \dot{\beta} & \ell_3 \cos \phi \dot{\phi} \\ \ell_2 \sin \beta \dot{\beta} & -\ell_3 \sin \phi \dot{\phi} \end{bmatrix} \begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} + \begin{bmatrix} -\ell_2 \sin \beta & \ell_3 \sin \phi \\ -\ell_2 \cos \beta & \ell_3 \cos \phi \end{bmatrix} \begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} = \ell_1 \begin{pmatrix} \ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta \\ \ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta \end{pmatrix}$$

Rearranging terms we get

$$\begin{bmatrix} -\ell_2 \sin \beta & \ell_3 \sin \phi \\ -\ell_2 \cos \beta & \ell_3 \cos \phi \end{bmatrix} \begin{pmatrix} \ddot{\phi} \\ \ddot{\phi} \end{pmatrix} = - \begin{bmatrix} -\ell_2 \cos \beta & \ell_3 \cos \phi \\ \ell_2 \sin \beta & \ell_3 \sin \phi \end{bmatrix} \begin{pmatrix} \ddot{\beta}^2 \\ \ddot{\beta}^2 \end{pmatrix} + \ell_1 \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\theta} \end{pmatrix} \quad (10.63)$$

Thus, we can find the angular accelerations of BC and CD,  $\ddot{\beta}$  and  $\ddot{\phi}$ , because the quantities on the right hand side are known ( $\ddot{\theta} = 0$ ) and  $\dot{\theta}$  are given, and  $\dot{\beta}$  and  $\dot{\phi}$  are determined by eqn. (10.62)). Now, we can find the accelerations of center of mass of each rod as follows.

$$\vec{a}_{G_1} = -\frac{\ell_1}{2} \dot{\theta}^2 \hat{\lambda}_1 \quad (10.64)$$

$$\vec{a}_{G_2} = \vec{a}_B + \vec{a}_{G_2/B} = -\ell_1 \dot{\theta}^2 \hat{\lambda}_1 - \frac{\ell_2}{2} \dot{\beta}^2 \hat{\lambda}_2 + \ell_2 \ddot{\beta} \hat{n}_2 \quad (10.65)$$

$$\vec{a}_{G_3} = -\frac{\ell_3}{2} \dot{\phi}^2 \hat{\lambda}_3 \quad (10.66)$$

We are now ready to write momentum balance equations. Since we are only interested in finding the torque  $T$ , we should try to write equations involving minimum number of unknown forces. So, we draw free body diagrams of the whole mechanism, of part BCD, and of bar CD alone; and write angular momentum balance equations about appropriate points so that we involve only the unknown torque  $T$  and the unknown reaction  $\vec{R}_D$  at D. Thus, we will have only three scalar unknowns  $T$ ,  $R_{D_x}$  and  $R_{D_y}$  (since  $\vec{R}_D = R_{D_x} \hat{i} + R_{D_y} \hat{j}$ ). So, we will need only three independent equations.

Consider the free body diagram of the whole mechanism. We can write angular momentum balance about point A for the whole mechanism as

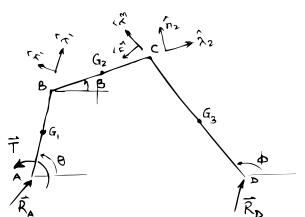


Figure 10.62: (Filename:sfig11.4.fourbar.fbd1)

$$T\hat{k} + \vec{r}_{D/A} \times \vec{R}_D = \dot{\vec{H}}_A = \dot{\vec{H}}_{1/A} + \dot{\vec{H}}_{2/A} + \dot{\vec{H}}_{3/A} \quad (10.67)$$

where

$$\begin{aligned}\dot{\vec{H}}_{1/A} &= I_1\ddot{\theta}\hat{k} + \vec{r}_{G_1} \times m_1\vec{a}_{G_1} \\ \dot{\vec{H}}_{2/A} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{G_2} \times m_2\vec{a}_{G_2} = I_2\ddot{\beta}\hat{k} + (\vec{r}_B + \vec{r}_{G_2/B}) \times m_2\vec{a}_{G_2} \\ \dot{\vec{H}}_{3/A} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3} \times m_3\vec{a}_{G_3} = I_3\ddot{\phi}\hat{k} + (\vec{r}_D + \vec{r}_{G_3/D}) \times m_3\vec{a}_{G_3}.\end{aligned}$$

Similarly, the angular momentum balance about point B for BCD gives

$$\vec{r}_{D/B} \times \vec{R}_D = \dot{\vec{H}}_{2/B} + \dot{\vec{H}}_{3/B} \quad (10.68)$$

where

$$\begin{aligned}\dot{\vec{H}}_{2/B} &= I_2\ddot{\beta}\hat{k} + \vec{r}_{G_2/B} \times m_2\vec{a}_{G_2} \\ \dot{\vec{H}}_{3/B} &= I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3/B} \times m_3\vec{a}_{G_3}\end{aligned}$$

and angular momentum balance of bar CD about point C gives

$$\vec{r}_{D/C} \times \vec{R}_D = \dot{\vec{H}}_{3/C} = I_3\ddot{\phi}\hat{k} + \vec{r}_{G_3/C} \times m_3\vec{a}_{G_3}. \quad (10.69)$$

Note that we can easily write the position vectors in terms of  $\ell_1, \ell_2, \ell_3$  and the unit vectors  $(\hat{\lambda}_1, \hat{n}_1), (\hat{\lambda}_2, \hat{n}_2)$  and  $(\hat{\lambda}_3, \hat{n}_3)$  where

$$\begin{aligned}\hat{\lambda}_1 &= \cos\theta\hat{i} + \sin\theta\hat{j}, & \hat{n}_1 &= -\sin\theta\hat{i} + \cos\theta\hat{j} \\ \hat{\lambda}_2 &= \cos\beta\hat{i} + \sin\beta\hat{j}, & \hat{n}_2 &= -\sin\beta\hat{i} + \cos\beta\hat{j} \\ \hat{\lambda}_3 &= \cos\phi\hat{i} + \sin\phi\hat{j}, & \hat{n}_1 &= -\sin\phi\hat{i} + \cos\phi\hat{j}.\end{aligned}$$

We can put all the three angular momentum balance equations, (10.67), (10.68), and (10.69), in one matrix equation by dotting both sides of the equations with  $\hat{k}$  and assembling them as follows.

$$\left[ \begin{array}{ccc} 1 & 0 & \ell_4 \\ 0 & \hat{k} \cdot (\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3) \times \hat{i} & \hat{k} \cdot (\ell_2\hat{\lambda}_2 - \ell_3\hat{\lambda}_3) \times \hat{j} \\ 0 & \hat{k} \cdot (-\ell_3\hat{\lambda}_3 \times \hat{i}) & \hat{k} \cdot (-\ell_3\hat{\lambda}_3 \times \hat{j}) \end{array} \right] \begin{pmatrix} T \\ R_{D_x} \\ R_{D_y} \end{pmatrix} = \begin{pmatrix} \dot{H}_{123/A} \\ \dot{H}_{23/B} \\ \dot{H}_{3/C} \end{pmatrix} \quad (10.70)$$

where  $\dot{H}_{123/A} = \hat{k} \cdot (\dot{\vec{H}}_{1/A} + \dot{\vec{H}}_{2/A} + \dot{\vec{H}}_{3/A})$ ,  $\dot{H}_{23/B} = \hat{k} \cdot (\dot{\vec{H}}_{2/B} + \dot{\vec{H}}_{3/B})$ , and  $\dot{H}_{3/C} = \hat{k} \cdot \dot{\vec{H}}_{3/C}$ .

Note that we know the  $\dot{\vec{H}}$ 's on the right hand side and the matrix on the left side can be evaluated for any given  $(\theta, \beta, \phi)$ . Thus we can solve for  $T$ ,  $R_{D_x}$ , and  $R_{D_y}$ .

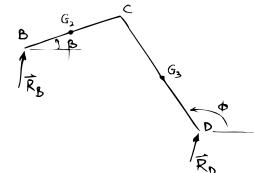


Figure 10.63: (Filename:sfig11.4.fourbar.fbd2)

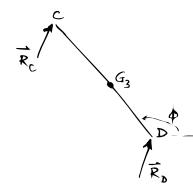


Figure 10.64: (Filename:sfig11.4.fourbar.fbd3)

**SAMPLE 10.20** Numerical solution of the inverse dynamics problem. Consider Sample 10.3 again. Using numerical solutions on a computer, find and plot torque  $T$  against  $\theta$  for one complete cycle of the drive arm AB. Take  $m_1 = m_2 = m_3 = 1 \text{ kg}$  and  $\ell_1 = 400 \text{ mm}$ ,  $\ell_2 = 400\sqrt{2} \text{ mm}$ ,  $\ell_3 = 800\sqrt{2} \text{ mm}$ , and  $\ell_4 = 1200 \text{ mm}$ .

### Solution

Since we have to plot  $T$  against  $\theta$  for one complete revolution, we need to find angular velocities, angular accelerations and center of mass accelerations for several values of  $\theta$  and then solve for  $T$  for each of those  $\theta$ 's. We can do this several ways. One way would be to first solve kinematic equations to find  $\theta(t)$ ,  $\beta(t)$ , and  $\phi(t)$  at discrete times over one complete cycle and then compute all other quantities at each  $(\theta(t_i), \beta(t_i), \phi(t_i))$  where  $t_i$  represents a discrete time. So, let us follow this method step by step with pseudocodes. Here, we assume that we have vector functions called `dot` and `cross` that compute the dot product and the cross product of two vectors that are given as input arguments.

**Step-1: solve for angular positions.** Specify the given geometry

```
L1=0.4, L2=0.4*sqrt(2), L3=0.8*sqrt(2), L4=1.2
```

and use the pseudocode of Sample 9.5 to find  $\theta(t_i)$ ,  $\beta(t_i)$ ,  $\phi(t_i)$  for, say, 100 values of  $t_i$  between 0 and 1 sec. Now, for each triad of  $(\theta(t_i), \beta(t_i), \phi(t_i))$ , follow all the steps below.

**Step-2: solve for angular velocities.** Since  $\dot{\theta} = 2\pi \text{ rad/s}$  is given, we only need to solve for  $\dot{\beta}$  and  $\dot{\phi}$ . We use eqn. (9.73) and eqn. (9.72) to compute  $\dot{\beta}$  and  $\dot{\phi}$  as follows (or modify the pseudocode of Sample 9.5 to save  $\dot{\beta}$  and  $\dot{\phi}$  along with the values for  $\beta$  and  $\phi$ ).

```
define thdot=thetadot, bdot=betadot, pdot=phidot
thdot = 2*pi % this is given
set th = theta(ti), b = beta(ti), p = phi(ti)
set unit vectors
  l1=[cos(th) sin(th) 0]', n1=[-sin(th) cos(th) 0]'
  l2=[cos(b) sin(b) 0]', n2=[-sin(b) cos(b) 0]'
  l3=[cos(p) sin(p) 0]', n3=[-sin(p) cos(p) 0]'
  bdot = -(L1/L2)*(cross(n1,l3)/cross(n2,l3))*thdot
  pdot = (L1/L3)*(cross(n1,l2)/cross(n3,l2))*thdot
```

**Step-3: solve for angular accelerations.** Now that we have  $(\theta, \beta, \phi)$  and the corresponding values of  $(\dot{\theta}, \dot{\beta}, \dot{\phi})$ , we can use eqn. (10.63) to calculate  $\ddot{\beta}$  and  $\ddot{\phi}$  (we are given  $\ddot{\theta} = 0$ ).

```
define thddot=thetaddot, bddot=betaddot,
      pdot=phiddot
thddot = 0 % this is given
B = [-L2*sin(b) L3*sin(p); -L2*cos(b) L3*cos(p)]
C = L1*[sin(th) cos(th); cos(th) -sin(th)]
D = [-cos(b) cos(p); sin(b) -sin(p)]
c = [thddot thdot^2]', d = [L2*bddot^2 L3*pdot^2]',
assume w = [bddot pdot]'
solve B*w = C*c + D*d for w
```

So, now we know  $\ddot{\theta}$ ,  $\ddot{\beta}$ ,  $\ddot{\phi}$  also. We are now ready to compute  $\hat{H}$ 's required for dynamic calculations.

**Step-4: set up equations and solve for unknown forces.** We need to set up and solve eqn. (10.70). Note that we need to compute several quantities for this equation but the vector computations are more or less straightforward.

```
% set mass and inertia properties
m1 = 1, m2 = 1, m3 = 1
I1 = m1*L1^2/12, I2 = m2*L2^2/12, I3 = m3*L3^2/12

% set fixed unit vectors
i = [1 0 0]', j = [0 1 0]', k = [0 0 1]'

% compute position vectors
rA = [0;0;0], rB = rA+L1*i1
rC = rB+L2*i2, rD = L4*k4
rG1 = L1/2*i1
rG2 = rB+L2/2*i2
rG3 = rD+L3/2*i3

% compute center of mass accelerations
aG1 = 0.5*L1*(tddot*n1-tdot^2*i1)
aG2 = 2*aG1+0.5*L2*(bddot*n2-bdot^2*i2)      % aB = 2*aG1
aG3 = 0.5*L3*(pddot*n3-pdot^2*i3)

% compute Hdot_cms

Hdot_cm1 = I1*tddot*uk
Hdot_cm2 = I2*bddot*uk
Hdot_cm3 = I3*pddot*uk

% compute Hdots
Hdot_123_A = Hdot_cm1 + cross(rG1, m1*aG1)
+ Hdot_cm2 + cross(rG2, m2*aG2)
+ Hdot_cm3 + cross(rG3, m3*aG3)
Hdot_23_B = Hdot_cm2 + cross(rG2-rB, m2*aG2)
+ Hdot_cm3 + cross(rG3-rB, m3*aG3)
Hdot_3_C = Hdot_cm3 + cross(rG3-rC, m3*aG3)

% set up the linear eqns for torque and RD

b = [dot(Hdot_123_A,k) dot(Hdot_23_B,k) dot(Hdot_3_C,k)]
A = [1 dot(k,cross(rD,i)) dot(k,cross(rD,j))
0 dot(k,cross(rD-rB,i)) dot(k,cross(rD-rB,j))
0 dot(k,cross(rD-rC,i)) dot(k,cross(rD-rC,j))]
% let forces = [T RDx RDy]'
solve A*forces = b for forces
```

**Step-5, repeat calculations.** Now repeat Step-2 – Step-4 for each triad  $(\theta, \beta, \phi)$  obtained in Step-1 and save the corresponding values of  $T$  in a vector. Finally,

plot T vs theta

The plot thus obtained is shown in Fig. 10.65. We can also plot  $T$  vs time (as shown in Fig. 10.66), and, of course, expect to see the same graph of  $T$  since  $\theta$  is just a linear function of  $t$ . Note that the area under the graph of  $T$  over one complete cycle must equal zero since the net impulse must be zero over one cycle.

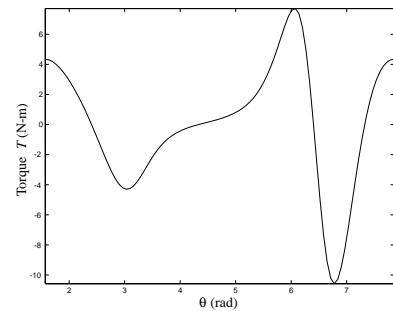


Figure 10.65: Torque  $T$  as a functions of  $\theta$  over one complete cycle of motion ( $\theta(0) = \pi/2, \theta(1) = 5\pi/2$ ).  
(Filename:fig11.4.fourbar.torque)

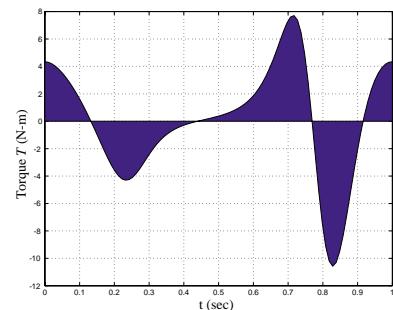


Figure 10.66: Torque  $T$  as a functions of time over one complete cycle of motion.  
(Filename:fig11.4.fourbar.Tvst)



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# 11

# Introduction to three dimensional rigid body mechanics

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In chapter 7 we discussed the motions of particles and rigid bodies that lie in a plane and rotate about a fixed axis perpendicular to that plane. In this chapter we again are going to think about fixed axis rotation, but now in three dimensions. The axis of rotation might be in any skew direction and the rotating bodies might be arbitrarily complicated three-dimensional shapes. As a cartoon, imagine a rigid body skewered with a rigid rod and then turned by a motor that speeds up and slows down. More practically think of the crankshaft in a car engine (Fig. 11.1). Other applications include accelerating or decelerating shafts of all kinds, gears, turbines, flywheels, pendula, and swinging doors.

To understand this motion we need to take a little more care with the kinematics because it now involves three dimensions, although in some sense the basic ideas are unchanged from the previous two-dimensional chapter. The three dimensional mechanics naturally gets more involved.

This one special motion, rotation about a fixed axis, serves as our introduction to three dimensional rigid body mechanics.

As for all motions of all systems, the momentum balance equations apply to any system or any part of a system that has fixed axis rotation. So our mechanics results will be based on these familiar equations:

$$\text{Linear momentum balance: } \sum \vec{F}_i = \dot{\vec{L}},$$



Figure 11.1: A car crankshaft is a complex three-dimensional object which is well approximated for many purposes as rotating about a fixed axis. The relative timing of the reciprocating pistons is controlled by this complex shape. “Connecting rods” are pinned to the cylinders at one end and to the short offset cylinders on the crankshaft at the other.

(Filename:tfigure.crankshaft)

$$\text{Angular momentum balance: } \sum \vec{M}_{i/O} = \dot{\vec{H}}_O.$$

and

$$\text{Power balance: } P = \dot{E}_K.$$

As always, we will evaluate the left hand sides of the momentum equations using the forces and moments in the free body diagram. We evaluate the right hand sides of these equations using our knowledge of the velocities and accelerations of the various mass points.

The chapter starts with a discussion of kinematics. Then we consider the mechanics of systems with fixed axis rotation. The moment of inertia matrix is then introduced followed by a section where the moment of inertia is used as a shortcut in the evaluation of  $\dot{\vec{H}}_O$ . Finally we discuss dynamic balance, an important genuinely three-dimensional topic in machine design.

## 11.1 3-D description of circular motion

Let's first assume each particle is going in circles around the  $z$  axis, as in the previous chapter. The figure below shows this two dimensional situation first two dimensionally (left) and then as a two-dimensional motion in a three dimensional world (right).

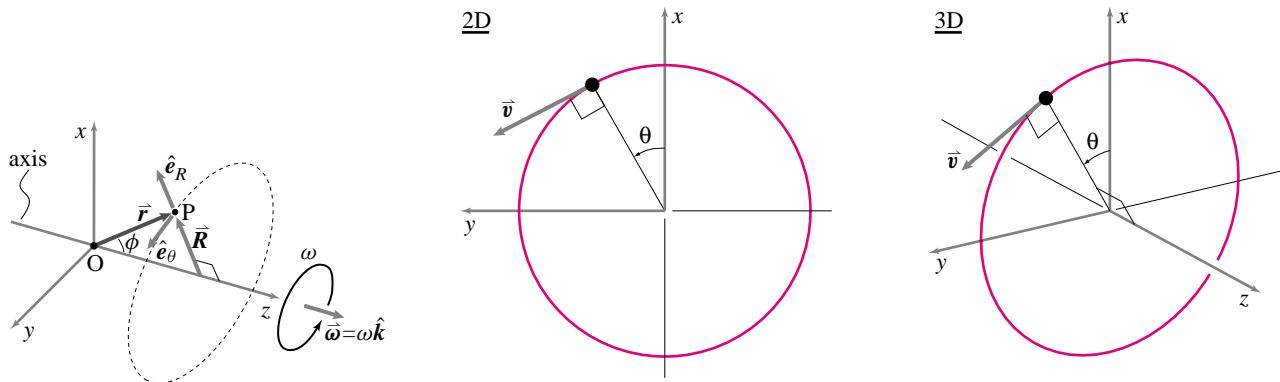


Figure 11.2: When a point P is going in circles about the  $z$  axis, we define the unit vector  $\hat{e}_R$  to be pointed from the axis to the point P. We define the unit vector  $\hat{e}_\theta$  to be tangent to the circle at P. Both of these vectors change in time as the point moves along its circular path.

(Filename:tfigure4.2)

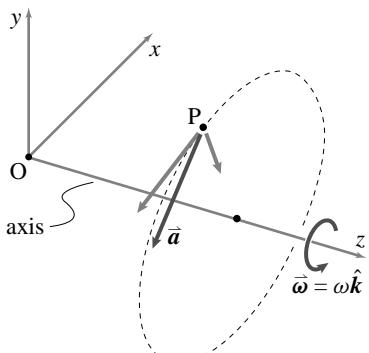


Figure 11.3: The acceleration is the sum of two components. One directed towards the center of the circle in the  $-\hat{e}_R$  direction, and one tangent to the circle in the  $\hat{e}_\theta$  direction.

(Filename:tfigure5.3)

Either way, the velocity and acceleration are the same:

- the velocity is tangent to the circle it is going around and is proportional in magnitude to the radius of the circle and also to its angular speed. That is, the direction of the velocity is in the direction  $\hat{e}_\theta$  and has magnitude  $\omega R$ , where  $\omega$  is the angular rate of rotation and  $R$  is the radius of the circle that the particle is going around.
- the acceleration can be constructed as the sum of two vectors. One is pointed to the center of the circle and proportional in magnitude to both the square of the angular speed and to the radius. The other vector is tangent to the circle and equal in magnitude to the rate of increase of speed.

These two ideas are summarized by the following formulas:

$$\vec{v} = \omega R \hat{e}_\theta \quad \text{and} \quad \vec{a} = -\underbrace{\omega^2 R \hat{e}_R}_{v^2/R} + \underbrace{R \ddot{\theta} \hat{e}_\theta}_{\vec{v}} \quad (11.1)$$

with

$$v = \omega R. \quad (11.2)$$

The axis of rotation might not be the  $z$ -axis of a convenient  $xyz$  coordinate system. So the  $xy$  plane of circles might not be the  $xy$  plane of the coordinate system you

might want to use for some other reasons. Fortunately, we can write the formulas 11.1 in a way that rids us of these problems.

Here are some formulas which are equivalent to the formulas 11.1 but which do not make use of the polar coordinate base vectors.

$$\vec{v} = \vec{\omega} \times \vec{r}, \quad (11.3)$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}. \quad (11.4)$$

To check that equations 11.3 and 11.4 are really equivalent to 11.1 we need to verify that the vector  $\vec{\omega} \times \vec{r}$  is equal to  $\omega R \hat{e}_\theta$ , that the vector  $\vec{\omega} \times (\vec{\omega} \times \vec{r})$  is equal to the vector  $-\omega^2 R \hat{e}_R$ , and that  $\dot{\vec{\omega}} \times \vec{r}$  is equivalent to  $R \ddot{\theta} \hat{e}_\theta$ .

First, define  $\vec{r}$  as the position of the point of interest relative to any point on the axis of rotation. If this point happens to be the center of the circle then  $\vec{r} = \vec{R}$ . But, in general,  $\vec{r} \neq \vec{R}$ .

Now look at  $\vec{v} = \vec{\omega} \times \vec{r}$  with respect to Fig. 11.2. Using the right hand rule, it is clear that the direction of the cross product  $\vec{\omega} \times \vec{r}$  is in fact the  $\hat{e}_\theta$  direction. What about the magnitude? The magnitude  $|\vec{\omega} \times \vec{r}| = |\vec{\omega}| |\vec{r}| \sin \phi$ . But  $|\vec{r}| \sin \phi = R$ . So the magnitude of  $\vec{\omega} \times \vec{r}$  is  $\omega R$ . That is,

$$\vec{\omega} \times \vec{r} = (|\vec{\omega} \times \vec{r}|) \cdot (\text{unit vector in the direction of } \vec{\omega} \times \vec{r}) \quad (11.5)$$

$$= (\omega R) \cdot \hat{e}_\theta \quad (11.6)$$

$$= \vec{v} \quad (11.7)$$

So  $\vec{v} = \vec{\omega} \times \vec{r}$  is correct. The check of the second term in the acceleration formula follows the same reasoning. But the check of the first term involves the triple cross product.

### Triple cross product

The formula for acceleration of a point on a rigid body includes the centripetal term  $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ . This expression is a special case of the general vector expression

$$\vec{A} \times (\vec{B} \times \vec{C})$$

which is sometimes called the ‘vector triple product’ because its value is a vector (as opposed to the scalar value of the ‘scalar triple product’). The primary useful identity with vector triple products is:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}. \quad (11.8)$$

This formula may be remembered by the semi-mnemonic device ‘cab minus bac’ since  $\vec{A} \cdot \vec{C} = \vec{C} \cdot \vec{A}$  and  $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ . This formula is discussed in box 11.1 on page 643.

So now we can write

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times (\omega R) \cdot \hat{e}_\theta = -\omega^2 \underbrace{R}_{\vec{R}} \hat{e}_R = -\omega^2 \vec{R} = \vec{a}. \quad (11.9)$$

Note that equations 11.3 and 11.4 are *vector* equations. They do not make use of any coordinate system. So, for example, we can use them even if  $\vec{\omega}$  is not in the  $z$  direction.

## Angular velocity of a rigid body in 3D

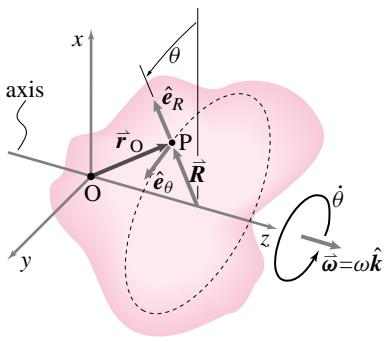


Figure 11.4: A rigid body spinning about the  $z$  axis. Every point on the body, like point  $P$  at  $\vec{r}_P$ , is going in circles. All of these circles have centers on the axis of rotation. All the points are going around at the same angular rate,  $\dot{\theta} = \omega$ .

(Filename:tfigure4.3D)

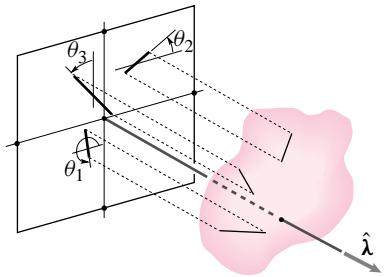


Figure 11.5: The shadows of lines marked in a 3-D rigid body are shown on a plane perpendicular to the axis of rotation. The shadows rotate on the plane at the rate  $\dot{\theta}_1 = \dot{\theta}_2 = \omega$ . The angular velocity vector is  $\vec{\omega} = \omega \hat{\lambda}$ .

(Filename:tfigure.shadowlines.4.3)

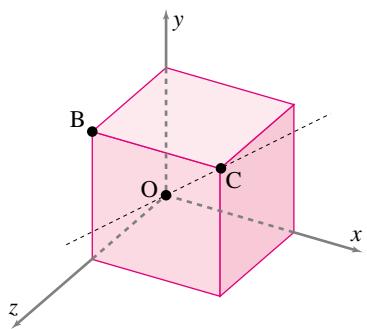


Figure 11.6: A spinning cube.

(Filename:tfigure4.cube)

If a rigid body is constrained to rotate about an axis then all points on the body have the same angular rate about that axis. Hence one says that the *body* has an angular velocity. So the measure of rotation rate of a three-dimensional rigid body is the body's angular velocity vector  $\vec{\omega}$ . At any instant in time a given body has one and only one angular velocity  $\vec{\omega}$ . Although we only discuss fixed axis rotation in this chapter, a given body has a unique angular velocity for general motion.

For rotations about the  $z$ -axis,  $\vec{\omega} = \omega \hat{k}$ .  $\omega$  is the  $\dot{\theta}$  shown in figure 11.4. Since all points of a rigid body have the same  $\dot{\theta}$ , even if they have different  $\theta$ 's, the definition is not ambiguous. We would like to make this idea precise enough to be useful for calculations. Why, one may ask, do we talk about rotation rate  $\omega$  or  $\vec{\omega}$  instead of just using the derivative of an angle  $\theta$ , namely  $\dot{\theta}$ ? The answer is that for a rigid body one would have trouble deciding what angle  $\theta$  to measure.

First recall the situation for a two dimensional rigid body. Consider all possible  $\theta_1, \theta_2, \theta_3, \dots$ , the angles that all possible lines marked on a body could make with the positive  $x$ -axis, the positive  $y$  axis, or any other fixed line that does not rotate. As the body rotates all of these angles increment by the same amount. Therefore, each of these angles increases at the same rate. Because all these angular rates are the same, one need not define  $\dot{\theta}_1 = \omega_1, \dot{\theta}_2 = \omega_2, \dot{\theta}_3 = \omega_3$ , etc. for each of the lines. Every line attached to the body rotates at the same rate and we call this rate  $\omega$ . So  $\dot{\theta}_1 = \omega, \dot{\theta}_2 = \omega, \dot{\theta}_3 = \omega$ , etc. Rather than say the lengthy phrase ‘the rate of rotation of every line attached to the rigid body is  $\omega$ ’, we instead say ‘the rigid body has angular velocity  $\omega$ ’. For use in vector equations, we define the angular velocity vector of a two-dimensional rigid body as  $\vec{\omega} = \omega \hat{k}$  and for a 3-D body rotating about an axis in the  $\hat{\lambda}$  direction as  $\vec{\omega} = \omega \hat{\lambda}$ .

What do we mean by these angles  $\theta_i$  for crooked lines in a three-dimensional body? We simply look at shadows of lines drawn in or on the body of interest onto a plane perpendicular to the axis of rotation; *i.e.*, perpendicular to  $\hat{\lambda}$ . See figure 11.5. The rate of change of their orientation ( $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3$ ) is  $\omega$ , and  $\vec{\omega}$  is therefore  $\omega \hat{\lambda}$ . This intuitive geometric definition of  $\omega$  in terms of the rotation of shadows has run its course. It gives you a picture but is not very convenient for developing formulas.

### Example: What are the velocity and acceleration of one corner of a cube that is spinning about a diagonal?

A one foot cube is spinning at 60 rpm about the diagonal  $OC$ . What are the velocity and acceleration of point B? First let's find the velocity using  $\vec{v} = \vec{\omega} \times \vec{r}$ :

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r} \\ &= (60 \text{ rpm } \lambda_{OC}) \times \vec{r}_{OB} \\ &= \left(2\pi \text{ s}^{-1} \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}\right) \times (1 \text{ ft}(\hat{j} + \hat{k})) \\ &= (2\pi/\sqrt{3})(-\hat{j} + \hat{k}) \text{ ft/s.}\end{aligned}$$

Now of course this equation could have been worked out with the first of equations ?? but it would have been quite tricky to find the vectors  $\hat{e}_\theta$ ,  $\hat{R}$ , and  $\hat{e}_R$ ! To find the acceleration we just plug in the formula  $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$  as follows:

$$\vec{a} = \vec{\omega} \times [\vec{\omega} \times \vec{r}]$$

$$\begin{aligned}
&= (60 \text{ rpm } \lambda_{OC}) \times [(60 \text{ rpm } \lambda_{OC}) \times \vec{r}_{OB}] \\
&= (2\pi \text{ s}^{-1} \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}) \\
&\quad \times \left[ \frac{(2\pi \text{ s}^{-1} (\hat{i} + \hat{j} + \hat{k}))}{\sqrt{3}} \times (1 \text{ ft} (\hat{j} + \hat{k})) \right] \\
&= (2\pi/\sqrt{3})^2 (2\hat{i} - \hat{j} - \hat{k}) \text{ ft/s}^2.
\end{aligned}$$

The last line of calculation is eased by the calculation of velocity above where the term in square brackets, the velocity, was already calculated.

□

## Relative motion of points on a rigid body

The relative velocity of two points  $A$  and  $B$  is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A$$

So, the relative velocity of two points glued to one rigid body, as observed from a Newtonian frame, is given by

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad (11.10)$$

$$= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} \quad (11.11)$$

$$= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) \quad (11.12)$$

$$= \vec{\omega} \times \vec{r}_{B/A}, \quad (11.13)$$

where point  $O$  is a point in the Newtonian frame on the fixed axis of rotation. Clearly, since points  $A$  and  $B$  are fixed in the body  $\mathcal{B}$  their velocities and hence their relative velocity as observed in a reference frame fixed to  $\mathcal{B}$  is  $\vec{0}$ . But, point  $A$  has some absolute velocity that is different from the absolute velocity of point  $B$ , as viewed from point  $O$  in the fixed frame. The relative velocity of points  $A$  and  $B$ , the difference in absolute velocity of the two points, is due to the difference in their positions relative to point  $O$ . Similarly, the relative acceleration of two points glued to one rigid body spinning about a fixed axis is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) + \vec{\dot{\omega}} \times \vec{r}_{B/A}. \quad (11.14)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point  $O$  fixed on the axis. Like their junior 2D cousins, these kinematics results, 11.13 and 11.14, are useful for calculating angular momentum relative to the center of mass as well as for the understanding of the motions of machines with moving connected parts.

To repeat, for two points on one rigid body we have that

$$\vec{r}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \quad (11.15)$$

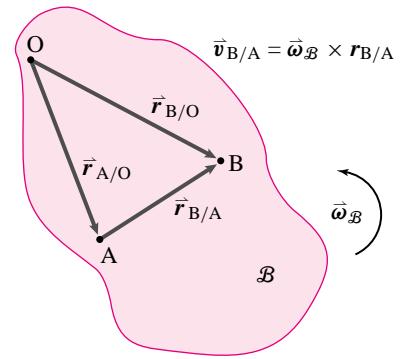


Figure 11.7: Two points on a rigid body.

(Filename:tfigure4.vel.accel.rel)

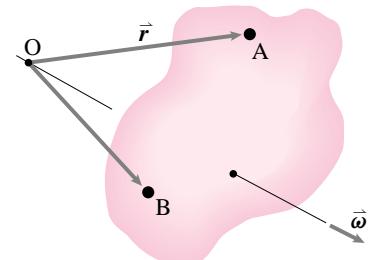


Figure 11.8: Two points,  $A$  and  $B$  on one body that has a fixed axis of rotation.

(Filename:tfigure.twopointsonabody.4.3)

Equation (11.15) is the perhaps the most fundamental equation for those desiring an understanding of the motions of rigid bodies. Unless one desires to pursue matrix representations of rotation, equation (11.15) is *the* defining equation for  $\vec{\omega}$ . There is always exactly one vector  $\vec{\omega}$  so that equation (11.15) is true for every pair of points on a rigid body.

Equation (11.15) is not so simple a defining equation as one would hope for such an intuitive concept as spinning. But, besides the pictorial definition with shadows, its the simplest definition we have.

## Relative velocity and acceleration using rotating frames

If we glued a coordinate system  $x'y'$  to a rotating rigid body  $\mathcal{C}$ , we would have what is called a rotating frame as shown in figure 11.9. The base vectors in this frame change in time the same way as did  $\hat{\mathbf{e}}_R$  and  $\hat{\mathbf{e}}_\theta$  in section 7.1. That is

$$\frac{d}{dt}\hat{\mathbf{i}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{i}}' \quad \text{and} \quad \frac{d}{dt}\hat{\mathbf{j}}' = \vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{j}}'.$$

If we now write the relative position of B to A in terms of  $\hat{\mathbf{i}}'$  and  $\hat{\mathbf{j}}'$ , we have

$$\vec{\mathbf{r}}_{B/A} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}'.$$

Since the coordinates  $x'$  and  $y'$  rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned} \frac{d}{dt}(\vec{\mathbf{r}}_{B/A}) &= \frac{d}{dt}(x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}') \\ &= \underbrace{\dot{x}'}_0 \hat{\mathbf{i}}' + x' \frac{d}{dt}\hat{\mathbf{i}}' + \underbrace{\dot{y}'}_0 \hat{\mathbf{j}}' + y' \frac{d}{dt}\hat{\mathbf{j}}' \\ &= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{i}}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\mathbf{j}}') \\ &= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}')}_{{\vec{\mathbf{r}}}_{B/A}} \\ &= \vec{\omega}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}. \end{aligned}$$

If we now try to calculate the rate of change of  $\vec{\mathbf{v}}_{B/A}$ ,

$$\begin{aligned} \frac{d}{dt}(\vec{\mathbf{v}}_{B/A}) &= \frac{d}{dt}(\vec{\omega}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}) \\ &= \frac{d\vec{\omega}_{\mathcal{C}}}{dt} \times \vec{\mathbf{r}}_{B/A} + \vec{\omega}_{\mathcal{C}} \times \frac{d\vec{\mathbf{r}}_{B/A}}{dt} \\ \vec{\mathbf{a}}_{B/A} &= \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A} + \vec{\omega}_{\mathcal{C}} \times (\vec{\omega}_{\mathcal{C}} \times \vec{\mathbf{r}}_{B/A}). \end{aligned}$$

## Mechanics

Now that we know the velocity and acceleration of every point in the system we are ready, in principle, to find  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{H}}_O$  in terms of the angular velocity vector  $\vec{\omega}$ , its rate of change  $\dot{\vec{\omega}}$ , and the position of all the mass in the system. This we do in the next section.

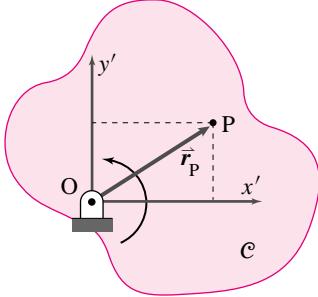


Figure 11.9: A rotating rigid body  $\mathcal{C}$  with rotating frame  $x'y'$  attached.

(Filename:tfigure4.intro.rot.frames)

## 11.1 THEORY

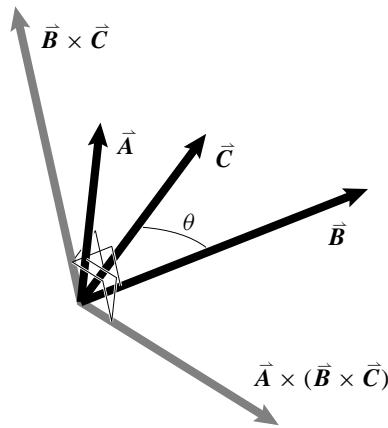
### The triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$

The formula

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \quad (11.16)$$

can be verified by writing each of the vectors in terms of its orthogonal components (e.g.,  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ ) and checking equality of the 27 terms on the two sides of the equations (only 12 are non-zero). If this 20 minute proof seems tedious it can be replaced by a more abstract geometric argument partly presented below that surely takes more than 20 minutes to grasp.

### Geometry of the vector triple product



Because  $\vec{B} \times \vec{C}$  is perpendicular to both  $\vec{B}$  and  $\vec{C}$ , it is perpendicular to the plane of  $\vec{B}$  and  $\vec{C}$ , that is, it is ‘normal’ to the plane  $BC$ .  $\vec{A} \times (\vec{B} \times \vec{C})$  is perpendicular to both  $\vec{A}$  and  $\vec{B} \times \vec{C}$ , so it is perpendicular to the normal to the plane of  $BC$ . That is, it must be in the plane of  $\vec{B}$  and  $\vec{C}$ . But any vector in the plane of  $\vec{B}$  and  $\vec{C}$  must be a combination of  $\vec{B}$  and  $\vec{C}$ . Also, the vector triple product must be proportional in magnitude to each of  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$ . Finally, the triple cross product of  $\vec{A} \times (\vec{B} \times \vec{C})$  must be the negative of  $\vec{A} \times (\vec{C} \times \vec{B})$  because  $\vec{B} \times \vec{C} = -\vec{C} \times \vec{B}$ . So the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

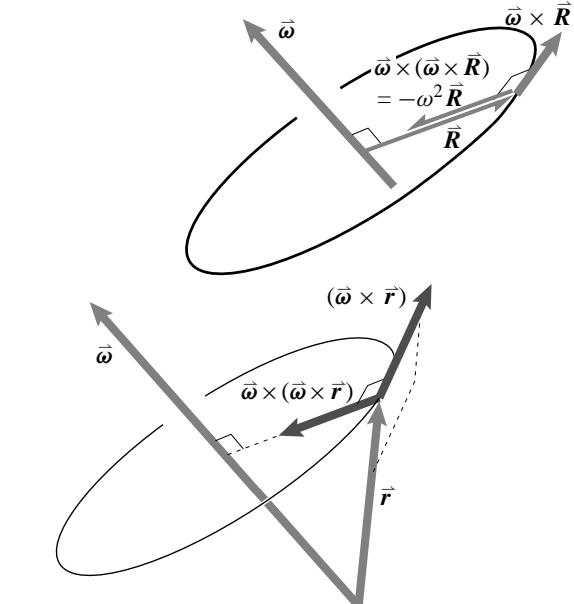
is almost natural: The expression above is almost the only expres-

sion that is a linear combination of  $\vec{A}$  and  $\vec{B}$  that is linear in both, also linear in  $\vec{C}$  and switches sign if  $\vec{B}$  and  $\vec{C}$  are interchanged. These properties would be true if the whole expression were multiplied by any constant scalar. But a test of the equation with three unit vectors shows that such a multiplicative constant must be one. This reasoning constitutes an informal derivation of the identity 11.8.

### Using the triple cross product in dynamics equations

We will use identity 11.8 for two purposes in the development of dynamics equations:

- (a) In the 2D expression for acceleration, the centripetal acceleration is given by  $\vec{\omega} \times (\vec{\omega} \times \vec{R})$  simplifies to  $-\omega^2 \vec{R}$  if  $\vec{\omega} \perp \vec{R}$ . This equation follows by setting  $\vec{A} = \vec{\omega}$ ,  $\vec{B} = \vec{\omega}$  and  $\vec{C} = \vec{R}$  in equation 11.8 and using  $\vec{R} \cdot \vec{\omega} = 0$  if  $\vec{\omega} \perp \vec{R}$ . In 3D  $\vec{\omega} \times (\vec{\omega} \times \vec{R})$  gives the vector shown in the lower figure below.



- (b) The term  $\vec{r} \times (\vec{\omega} \times \vec{r})$  will appear in the calculation of the angular momentum of a rigid body. By setting  $\vec{A} = \vec{r}$ ,  $\vec{B} = \vec{\omega}$  and  $\vec{C} = \vec{r}$ , in equation 11.8 and use  $\vec{r} \cdot \vec{r} = r^2$  because  $\vec{r} \parallel \vec{r}$  we get the useful result that  $\vec{r} \times (\vec{\omega} \times \vec{r}) = r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r}$ .

**SAMPLE 11.1** For a particle in circular motion, we frequently use angular velocity  $\vec{\omega}$  and angular acceleration  $\vec{\alpha}$  to describe its motion. You have probably learned in physics that the linear speed of the particle is  $v = \omega r$ , the tangential acceleration is  $a_t = \alpha r$ , and the centripetal or radial acceleration is  $a_r = \omega^2 r$ , where  $r$  is the radius of the circle. These formulae have scalar expressions. Their vector forms, as learned in Chapters 5 and 6, are  $\vec{v} = \vec{\omega} \times \vec{r}$ ,  $\vec{a}_t = \vec{\alpha} \times \vec{r}$ , and  $\vec{a}_r = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ . Using these definitions, find (i)  $\vec{v}$ , (ii)  $\vec{a}_t$ , and (iii)  $\vec{a}_r$  and show the resulting vectors for  $\vec{\omega} = 2 \text{ rad/s} \hat{k}$ ,  $\vec{\alpha} = 4 \text{ rad/s}^2 \hat{k}$  and  $\vec{r}_G = 3 \text{ m}(\cos 30^\circ \hat{i} + \sin 30^\circ \hat{j})$ , where  $\vec{r}_G$  is the position vector of the particle.

### Solution

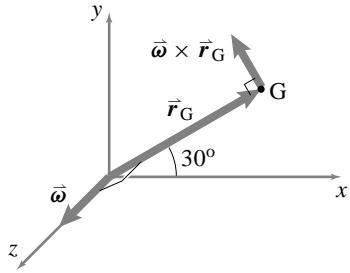


Figure 11.10:  $\vec{v} = \vec{\omega} \times \vec{r}_G$ .  
(Filename:sfig1.2.10a)

$$\begin{aligned}\vec{\omega} &= \omega \hat{k} = 2 \text{ rad/s} \hat{k} \\ \vec{\alpha} &= \alpha \hat{k} = 4 \text{ rad/s}^2 \hat{k} \\ \vec{r}_G &= r_{G_x} \hat{i} + r_{G_y} \hat{j} = 3 \text{ m}(\cos 30^\circ \hat{i} + \sin 30^\circ \hat{j}).\end{aligned}$$

(i) From the given formulae, the linear velocity

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}_G \\ &= \omega \hat{k} \times (r_{G_x} \hat{i} + r_{G_y} \hat{j}) = \omega r_{G_x} \hat{j} + \omega r_{G_y} (-\hat{i}) \\ &= 6 \text{ m/s}(\cos 30^\circ \hat{j} - \sin 30^\circ \hat{i}) \\ &= 3 \text{ m/s}(-\hat{i} + \sqrt{3} \hat{j}).\end{aligned}$$

The velocity vector  $\vec{v}$  is perpendicular to both  $\vec{\omega}$  and  $\vec{r}_G$ . These vectors are shown in Fig. 11.10. You should use the right hand rule to confirm the direction of  $\vec{v}$ .

$$\boxed{\vec{v} = 3 \text{ m/s}(-\hat{i} + \sqrt{3} \hat{j})}$$

(ii) The tangential acceleration

$$\begin{aligned}\vec{a}_t &= \vec{\alpha} \times \vec{r}_G \\ &= \alpha \hat{k} \times (r_{G_x} \hat{i} + r_{G_y} \hat{j}) = \alpha r_{G_x} \hat{j} - \alpha r_{G_y} \hat{i} \\ &= 8 \text{ m/s}^2(\cos 30^\circ \hat{j} - \sin 30^\circ \hat{i}).\end{aligned}$$

Since  $\vec{\omega}$  and  $\vec{\alpha}$  are in the same direction, calculation of  $\vec{a}_t$  is similar to that of  $\vec{v}$  and  $\vec{a}_t$  has to be in the same direction as  $\vec{v}$ . This vector is shown in Fig. 11.11. Once again, just as in the case of  $\vec{v}$  we could easily check that  $\vec{a}_t$  is perpendicular to both  $\vec{\omega}$  and  $\vec{r}_G$ .

$$\boxed{\vec{a}_t = 4 \text{ m/s}^2(3 \text{ m/s}(-\hat{i} + \sqrt{3} \hat{j}))}$$

(iii) Finally, the radial acceleration

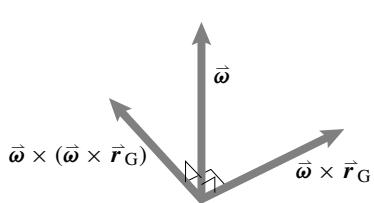


Figure 11.12:  $\vec{a}_r = \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)$ .  
(Filename:sfig1.2.10c)

$$\begin{aligned}\vec{a}_r &= \vec{\omega} \times (\vec{\omega} \times \vec{r}_G) \\ &= \omega \hat{k} \times (\omega r_{G_x} \hat{j} - \omega r_{G_y} \hat{i}) \\ &= \omega^2 r_{G_x} (-\hat{i}) - \omega^2 r_{G_y} \hat{j} = -\omega^2 \vec{r}_G \\ &= 12 \text{ m/s}^2(-\cos 30^\circ \hat{i} - \sin 30^\circ \hat{j}).\end{aligned}$$

This cross product is illustrated in Fig. 11.12. Both from the illustration as well as the calculation you should be able to see that  $\vec{a}_r$  is in the direction of  $-\vec{r}_G$ . In fact, you could show that  $\vec{a}_r = -\omega^2 \vec{r}_G$ .

$$\boxed{\vec{a}_r = 6 \text{ m/s}^2(-\sqrt{3} \hat{i} - \hat{j})}$$

**SAMPLE 11.2** Simple 3-D circular motion. A system with two point masses A and B is mounted on a rod OC which makes an angle  $\theta = 45^\circ$  with the horizontal. The entire assembly rotates about the y-axis with constant angular speed  $\omega = 3 \text{ rad/s}$ , maintaining the angle  $\theta$ . Find the velocity of point A. What is the radius of the circular path that A describes? Assume that at the instant shown, AB is in the xy plane.

**Solution** The angular velocity of the system is

$$\vec{\omega} = \omega \hat{j} = 3 \text{ rad/s} \hat{j}.$$

Let  $\vec{r}_A$  be the position vector of point A. Then the velocity of point A is

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}_A \quad (\text{and } \vec{r}_A = \vec{r}_C + \vec{r}_{A/C}) \\ &= \omega \hat{j} \times (\underbrace{l \cos \theta \hat{i} + l \sin \theta \hat{j}}_{\vec{r}_C} + \underbrace{d \cos \theta \hat{j} - d \sin \theta \hat{k}}_{\vec{r}_{A/C}}) \\ &= \omega \hat{j} \times [(l \cos \theta - d \sin \theta) \hat{i} + (l \sin \theta + d \cos \theta) \hat{j}] \\ &= -(\omega l \cos \theta - \omega d \sin \theta) \hat{k} \\ &= -[3 \text{ rad/s} (1 \text{ m} \cdot \cos 45^\circ - 0.5 \text{ m} \cdot \sin 45^\circ)] \\ &= -1.06 \text{ m/s} \hat{k}\end{aligned}$$

$$\boxed{\vec{v} = -1.06 \text{ m/s} \hat{k}.}$$

We can find the radius of the circular path of A by geometry. However, we know that the velocity of A is also given by

$$\vec{v} = \omega R \hat{e}_\theta$$

where  $R$  is the radius of the circular path. At the instant of interest,  $\hat{e}_\theta = -\hat{k}$  (see figure 11.15).

$$\text{Thus } \vec{v} = -\omega R \hat{k}.$$

Comparing with the answer obtained above, we get

$$\begin{aligned}-1.06 \text{ m/s} \hat{k} &= -\omega R \hat{k} \\ \Rightarrow R &= \frac{1.06 \text{ m/s}}{3 \text{ rad/s}} \\ &= 0.35 \text{ m.}\end{aligned}$$

$$\boxed{R = 0.35 \text{ m}}$$

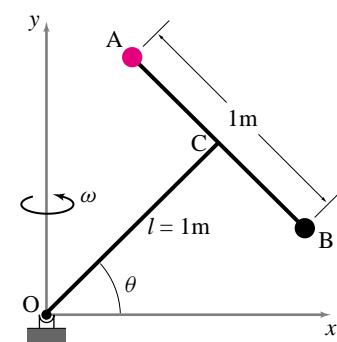


Figure 11.13: (Filename:sfig4.2.DH1)

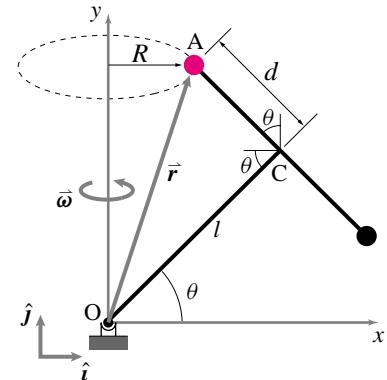


Figure 11.14: (Filename:sfig4.2.DH2)

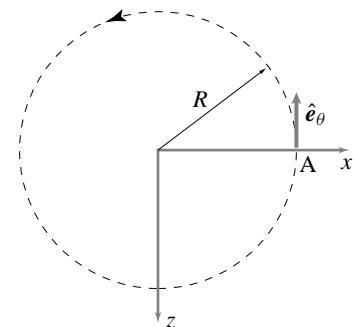


Figure 11.15: Circular trajectory of point A as seen by looking down along the y-axis. At the instant shown,  $\hat{e}_\theta = -\hat{k}$ . (Filename:sfig4.2.DH3)

**SAMPLE 11.3 Kinematics in 3-D—some basic questions:** The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r} \\ \vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\end{aligned}$$

- (a) In the formulae above, what is  $\vec{r}$ ? How is it different from  $\vec{R} = R\hat{e}_R$  used in the formulae  $\vec{v} = R\dot{\theta}\hat{e}_\theta$ ?
- (b) What is the difference between  $\dot{\theta}$  and  $\vec{\omega}$ , and  $\ddot{\theta}$  and  $\vec{\dot{\omega}}$ ?
- (c) Are the parentheses around the  $\vec{\omega} \times \vec{r}$  term necessary in the acceleration formula?
- (d) Under what condition(s) can a particle have only tangential acceleration?

### Solution

- (a) In the formulae for velocity and acceleration,  $\vec{r}$  refers to a vector from any point on the axis of rotation to the point of interest. Usually the origin of a coordinate system located on the axis of rotation is a convenient point to take as the base point for  $\vec{r}$ . You can, however, choose any other point on the axis of rotation as the base point.
- The vector  $\vec{r}$  is different from  $\vec{R}$  in that  $\vec{R}$  is the position vector of the point of interest with respect to the center of the circular path that the point traces during its motion. See Fig. 5.2 of the text.
- (b)  $\dot{\theta}$  and  $\ddot{\theta}$  are the magnitudes of angular velocity and angular acceleration, respectively, in planar motion, *i.e.*,  $\vec{\omega} = \dot{\theta}\hat{k}$  and  $\vec{\dot{\omega}} = \ddot{\theta}\hat{k}$ . We have introduced these notations to highlight the simple nature of planar circular motion. Of course, you are free to use  $\vec{\omega} = \omega\hat{k}$  and  $\vec{\dot{\omega}} = \alpha\hat{k}$  if you wish.
  - (c) Yes, the parentheses around  $\vec{\omega} \times \vec{r}$  in the acceleration formula are mandatory. The parentheses imply that this term has to be calculated before carrying out the cross product with  $\vec{\omega}$  in the formula. Since the term in the parentheses is the velocity, you may also write the acceleration formula as

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \vec{v}.$$

Even if the formula is clear in your mind and you know which cross product to carry out first, it is a good idea to put the parentheses.

- (d) First of all let us identify the tangential and the normal (or radial) components of the acceleration:

$$\vec{a} = \overbrace{\dot{\vec{\omega}} \times \vec{r}}^{\text{tangential}} + \overbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}^{\text{radial/normal}}.$$

Clearly, for a particle to have only tangential acceleration, the second term must be zero. For the second term to be zero we must have either  $\vec{r} = \vec{0}$  or  $\vec{\omega} = \vec{0}$ . But if  $\vec{r} = \vec{0}$ , then the tangential acceleration also becomes zero; the particle is on the axis of rotation and hence has no acceleration. Thus the condition that allows only tangential acceleration to survive is  $\vec{\omega} = \vec{0}$ . Now remember that  $\vec{\dot{\omega}}$  is *not* zero. Therefore, the condition we have found can be true only momentarily. This disappearance of the radial acceleration happens at start-up motions and in direction-reversing motions. ①

① In all start-up motions, the velocity is zero but the acceleration is not zero at the start up ( $t = 0$ ). In direction-reversing motions, such as that of the washing machine drum during the wash-cycle, just at the moment when the direction of motion reverses, velocity becomes zero but the acceleration is non-zero.

**SAMPLE 11.4 Velocity and acceleration in 3-D:** The rod shown in the figure rotates about the  $y$ -axis at angular speed  $10 \text{ rad/s}$  and accelerates at the rate of  $2 \text{ rad/s}^2$ . The dimensions of the rod are  $L = h = 2 \text{ m}$  and  $r = 1 \text{ m}$ . There is a small mass  $P$  glued to the rod at its free end. At the instant shown, the three segments of the rod are parallel to the three axes.

- Find the velocity of point  $P$  at the instant shown.
- Find the acceleration of point  $P$  at the instant shown.

**Solution** We are given:

$$\vec{\omega} = \omega \hat{j} = 10 \text{ rad/s} \hat{j} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega} \hat{j} = 2 \text{ rad/s}^2 \hat{j}.$$

- The velocity of point  $P$  is

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

At the instant shown, the position vector of point  $P$  (the vector  $\vec{r}_{P/O}$ ) seems to be a good choice for  $\vec{r}$ . ① Thus,

$$\vec{r} \equiv \vec{r}_{P/O} = L\hat{i} + h\hat{j} + r\hat{k}.$$

Therefore,

$$\begin{aligned} \vec{v} &= \vec{\omega} \times (L\hat{i} + h\hat{j} + r\hat{k}) \\ &= \omega(-L\hat{k} + r\hat{i}) \\ &= 10 \text{ rad/s} \cdot (-2 \text{ m}\hat{k} + 1 \text{ m}\hat{i}) \\ &= (20\hat{i} - 10\hat{k}) \text{ m/s.} \end{aligned}$$

As a check, we look down the  $y$ -axis and draw a velocity vector at point  $P$  (tangent to the circular path at point  $P$ ) without paying attention to the answer we got. From the top view in Fig. 11.17 we see that at least the signs of the components of  $\vec{v}$  seem to be correct.

$$\boxed{\vec{v} = (20\hat{i} - 10\hat{k}) \text{ m/s}}$$

- The acceleration of point  $P$  is

$$\begin{aligned} \vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= \dot{\omega} \hat{j} \times (L\hat{i} + h\hat{j} + r\hat{k}) + \omega \hat{j} \times \underbrace{\omega(-L\hat{k} + r\hat{i})}_{\vec{\omega} \times \vec{r}} \\ &= \dot{\omega}(-L\hat{k} + r\hat{i}) + \omega^2(-L\hat{i} - r\hat{k}) \\ &= 2 \text{ rad/s}^2(-2 \text{ m}\hat{k} + 1 \text{ m}\hat{i}) - 100(\text{rad/s})^2(2 \text{ m}\hat{i} + 1 \text{ m}\hat{k}) \\ &= -(98\hat{i} + 104\hat{k}) \text{ m/s}^2. \end{aligned}$$

We can check the sign of the components of  $\vec{a}$  also. Note that the tangential acceleration,  $\dot{\vec{\omega}} \times \vec{r}$ , is much smaller than the centripetal acceleration,  $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ . Therefore, the total acceleration is almost in the same direction as the centripetal acceleration, that is, directed from point  $P$  to  $A$ . If you draw a vector from  $P$  to  $A$ , you should be able to see that it has negative components along both the  $x$ - and  $z$ -axes. Thus the answer we have got seems to be correct, at least in direction.

$$\boxed{\vec{a} = -(98\hat{i} + 104\hat{k}) \text{ m/s}^2}$$

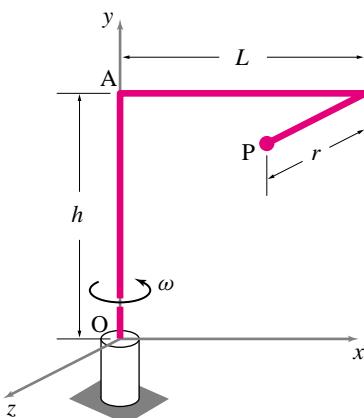
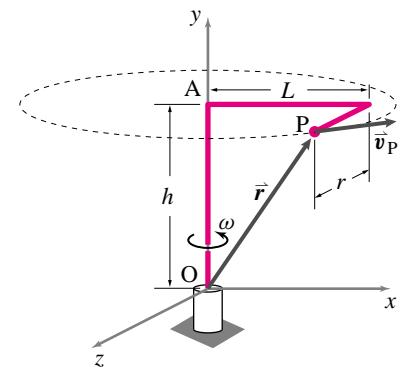


Figure 11.16: (Filename:sfig5.2.2)

① An even better choice, perhaps, is the vector  $\vec{r}_{P/A}$ . Remember, the only requirement on  $\vec{r}$  is that it must start at *some* point on the axis of rotation and must end at the point of interest.



Path of point  $P$  seen from the top:

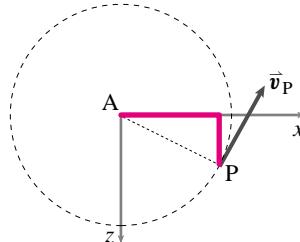


Figure 11.17: By drawing the velocity vector at point  $P$  (a vector tangent to the path) we see that  $\vec{v}_P$  must have a positive  $x$ -component and a negative  $z$ -component.

(Filename:sfig5.2.2a)

## 11.2 Dynamics of fixed-axis rotation

We now address mechanics questions concerning objects which are known *a priori* to spin about a fixed axis. We would like to calculate forces and moments if the motion is known. And we would like to determine the details of the motion, the angular acceleration in particular, if the applied forces and moments are known. Once the angular acceleration is known (as a function of some combination of time, angle and angular rate) the angular rate and angular position can be found by integration or solution of an ordinary differential equation.

The full content of the subject follows from the basic mechanics equations

$$\text{linear momentum balance, } \sum \vec{F}_i = \dot{\vec{L}}$$

$$\text{angular momentum balance, } \sum \vec{M}_{i/C} = \dot{\vec{H}}_C,$$

$$\text{and power balance: } P = \dot{E}_K + \dot{E}_P + \dot{E}_{\text{int.}}$$

The quantities  $\dot{\vec{L}}$  and  $\dot{\vec{H}}_C$  are defined in terms of the position and acceleration of the system's mass (see the second page of the inside cover). To evaluate  $\dot{\vec{L}}$  and  $\dot{\vec{H}}_C$  for fixed-axis rotation we can use the kinematics relations from the previous chapter which determine velocity and acceleration of points on a body spinning about a fixed axis in terms of the position  $\vec{r}$  of the point of interest relative to any point on the axis.

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{a} &= \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}.\end{aligned}$$

For fixed-axis rotation  $\vec{\omega} = \omega \hat{\lambda}$  and  $\dot{\vec{\omega}} = \dot{\omega} \hat{\lambda}$  with  $\hat{\lambda}$  a constant unit vector along the axis of rotation.

To solve problems we draw a free body diagram, write the equations of linear and angular momentum balance, and evaluate the terms using the kinematics relations. In general this will lead to the evaluation of a sum or an integral. A short cut, the moment of inertia matrix, will be introduced in later sections.

Before proceeding to more difficult three-dimensional problems, let's review a simple 2D problem.

### Example: Spinning disk

The round flat uniform disk in figure 11.18 is in the  $xy$  plane spinning at the constant rate  $\vec{\omega} = \omega \hat{k}$  about its center. It has mass  $m_{\text{tot}}$  and radius  $R_0$ . What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{F}_i = \dot{\vec{L}} = m_{\text{tot}} \vec{a}_{cm} = \vec{0},$$

which we could also have calculated by evaluating the integral  $\dot{\vec{L}} \equiv \int \vec{a} dm$  instead of using the general result that  $\dot{\vec{L}} = m_{\text{tot}} \vec{a}_{cm}$ . From angular momentum balance we have:

$$\sum \vec{M}_{i/O} = \dot{\vec{H}}_{/O}$$

$$\begin{aligned}\Rightarrow \vec{M} &= \int \vec{r}_{/O} \times \vec{a} dm \\ &= \int_0^{R_0} \int_0^{2\pi} (R\hat{e}_R) \times (-R\omega^2\hat{e}_R) \underbrace{\frac{m_{\text{tot}}}{\pi R_O^2} R d\theta dR}_{dm} \\ &= \int \int \vec{0} d\theta dR \\ &= \vec{0}.\end{aligned}$$

□

Power balance is of limited use for constant rate circular motion. If all parts of a system move at constant angular rate at a constant radius then they all have constant speed. Thus the kinetic energy of the system is constant. So the power balance equation just says that the net power into the system is the amount dissipated inside (assuming no energy storage).

### Example: Spinning disk with power balance

Consider the spinning disk from figure 11.18 and the previous example.

The power balance equation III gives

$$P = \dot{E}_K + \underbrace{\dot{E}_P}_{0} + \underbrace{\dot{E}_{\text{int}}}_{0} \Rightarrow P = \int \vec{v} \cdot \vec{a} dm = \int 0 dm = 0. \quad (11.17)$$

In this example there is no force or torque acting on the disk so the power  $P$  must turn out to be zero. In other constant rate problems the force and moment will *not* turn out to be zero, but the kinetic energy of the system will still be constant and so, assuming no energy storage or dissipation, we will still have  $P = 0$ . □

### A stick sweeps out a cone

Now we consider a genuinely three-dimensional problem involving fixed-axis, rigid body rotation. Consider a long narrow stick swinging in circles so that it sweeps out a cone (Fig. 11.19). Each point on the stick is moving in circles around the  $z$ -axis at a constant rate  $\omega$ . What is the relation between  $\omega$  and the angle of the stick  $\phi$ ? The approach to this problem is, as usual, to draw a free body diagram, write momentum balance equations, evaluate the left and right hand sides, and then solve for quantities of interest. The hard part of this problem is evaluating the right hand side of the angular momentum balance equations.

To simplify calculation, we look at the pendulum at the instant it passes through the  $yz$ -plane, assuming the  $xyz$  axes are fixed in space.

The free body diagram shown in figure 11.20 shows the gravity force at the center of mass, the reaction force at point  $O$ , and, consistent with the shown construction of the hinge, the moments at  $O$  perpendicular to the hinge.

Because we are interested in the relation between  $\phi$  and  $\omega$  and not the reaction force, at least for now, we look at angular momentum balance about point  $O$ .

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

First, we show and discuss the results of evaluating the equation of angular momentum balance. Then, we will show the details of calculating  $\sum \vec{M}_O$  and the details of several methods for calculating  $\dot{\vec{H}}_O$ .

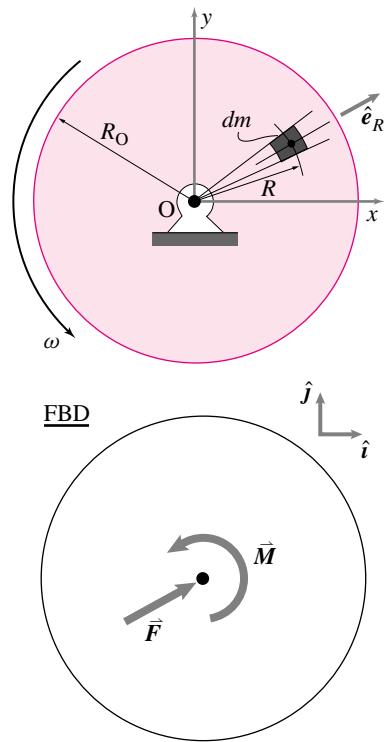


Figure 11.18: A uniform disk turned by a motor at a constant rate.

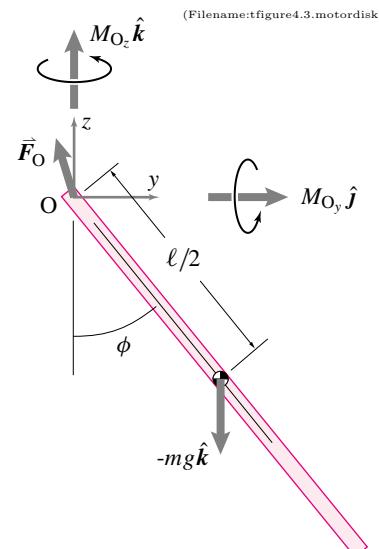


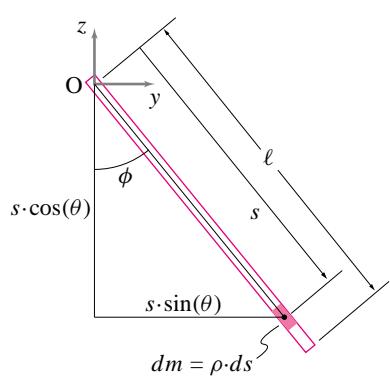
Figure 11.20: Free body diagram of the rod.

(Filename:tfigure4.3.motordisk)

### Evaluation of $\sum \vec{M}_O$

To find  $\sum \vec{M}_O$ , we had to find the moment of the gravity force. The most direct method is to use the definition

$$\begin{aligned}\sum \vec{M}_O &= \vec{r}_{/O} \times \vec{F} \\ &= \frac{\ell}{2} [\cos \phi (-\hat{k}) + \sin \phi \hat{j}] \times (-mg\hat{k}) \\ &= -\frac{mg\ell \sin \phi}{2} \hat{i}\end{aligned}$$



One could also ‘slide’ the gravity force to the level of O (a force displaced in its direction of action is mechanically equivalent). Then you can see from the figure that the force is perpendicular to its position relative to  $O$ . Moment is then force ( $mg$ ) times distance ( $\frac{\ell}{2} \sin \phi$ ) in the direction given by the right hand rule ( $-\hat{i}$ ).

Figure 11.21: The rod is shown on the  $yz$  plane. We use this figure to locate the bit of mass  $dm$  corresponding to the bit of length of rod  $ds$ .

(Filename:tfigure4.sphere.ds)

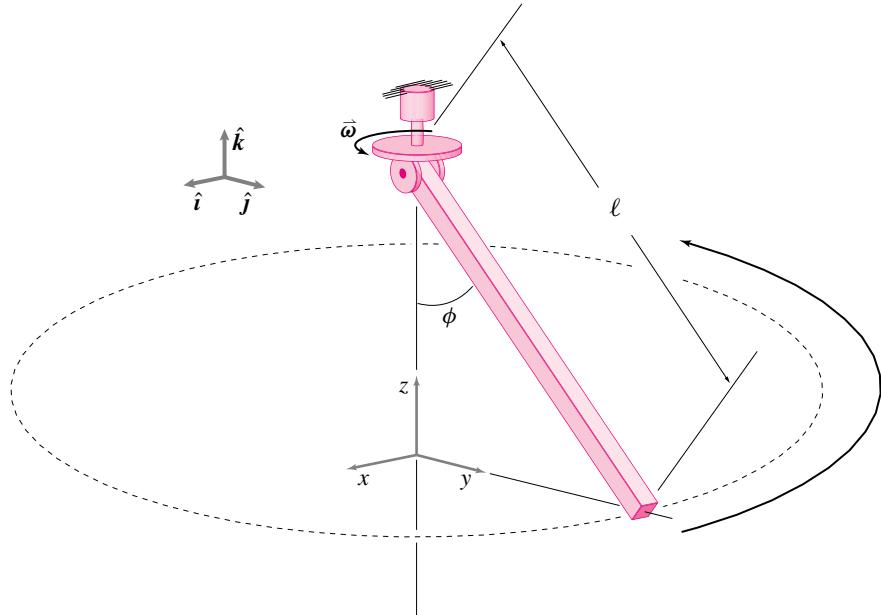


Figure 11.19: A spherical rigid body pendulum (uniform thin rod) going in circles at constant rate  $\vec{\omega}$ .

(Filename:tfigure4.spherical.pend)

*Evaluation of  $\dot{\hat{H}}_O$* 

We now evaluate  $\dot{\hat{H}}_O$  by adding up the contribution to the sum from each bit of mass.

$$\begin{aligned}
 \dot{\hat{H}}_O &= \int \overbrace{\vec{r}_{/O} \times \vec{a}}^{\text{For constant rate circular motion,}} \underbrace{dm}_{dm = \rho ds, \text{ where } \rho = \text{mass per unit length}} \\
 &= \int_0^\ell \underbrace{(-s \cos \phi \hat{k} + s \sin \phi \hat{j})}_{\vec{r}_{/O}} \times \underbrace{(-\omega^2 s \sin \phi \hat{j})}_{\vec{a}} (\rho ds) \\
 &= \int_0^\ell -s^2 \cos \phi \sin \phi \omega^2 \hat{i} \rho ds \quad (\text{evaluating the cross product}) \\
 &= -\cos \phi \sin \phi \omega^2 \rho \hat{i} \int_0^\ell s^2 ds \quad (\phi, \rho, \text{ and } \omega \text{ do not vary with } s) \\
 &= -\cos \phi \sin \phi \omega^2 (\rho \frac{\ell^3}{3}) \hat{i} \quad (\text{evaluating the integral}) \\
 \dot{\hat{H}}_O &= -\cos \phi \sin \phi \omega^2 (\frac{m\ell^2}{3}) \hat{i} \quad (\text{because } m = \rho\ell).
 \end{aligned}$$

We could have taken a short-cut in the calculation of acceleration  $\vec{a}$ . Instead of using  $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ , we could have used  $\vec{a} = -\omega^2 \vec{R}$  where  $\vec{R}$  is the radius of the circle each particle is traveling on. It is evident from the picture that the appropriate radius is  $\vec{R} = s \sin \phi \hat{j}$ , so  $\vec{a} = -\omega^2 s \sin \phi \hat{j}$ .

We will show two more methods for calculating  $\dot{\hat{H}}_O$  in section 11.4 on page 672 once you have studied the moment of inertia matrix in section 11.3.

*The results for the conically swinging stick*

We can now evaluate the terms in the angular momentum balance equation as

$$\underbrace{\sum \vec{M}_O}_{-mg \sin \phi \frac{\ell}{2} \hat{i} + M_{O_y} \hat{j} + M_{O_z} \hat{k}} \underbrace{\dot{\hat{H}}_{/O}}_{= -\sin \phi \cos \phi \frac{m\ell^2}{3} \omega^2 \hat{i}} = 0. \quad (11.18)$$

We can get three scalar equations from eqn. 11.18 by dotting it with  $\hat{j}$ ,  $\hat{k}$ , and  $\hat{i}$  to get

$$M_{O_y} = 0 \quad \text{and} \quad M_{O_z} = 0$$

and

$$\omega^2 = \frac{3g}{2\ell \cos \phi}.$$

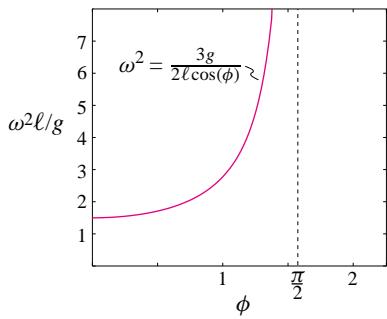


Figure 11.22: Plot of non-dimensional rotational speed  $\frac{\omega^2 \ell}{g}$  versus hang angle  $\phi$ . For  $\omega^2 \ell / g < 3/2$  the only solution is  $\phi = 0$  (hanging straight down). At or very close to  $\omega^2 \ell / g = 3/2$  a range of  $\phi$ 's is possible. As  $\phi \rightarrow \pi/2$  and the rod becomes close to horizontal, the spin rate  $\omega$  goes to infinity.

(Filename:tfigure4.spherical.vv)

① **Caution:** For more general three dimensional motion than rotation about a fixed axis the equation  $M = I\alpha$  does not apply. Trying to vectorize by underlining various terms gives the wrong answer.

Note that  $M_{O_y} = M_{O_z} = 0$ . That is, for this special motion, the hinge joint at  $O$  could be replaced with a ball-and-socket joint.

Note that the solution for a point mass spherical pendulum is  $\omega^2 = \frac{g}{\ell \cos \phi}$ . That is, this stick would rotate at the same rate and angle as a point mass at the end of a rod of length  $\frac{2\ell}{3}$ . One could not easily anticipate this result. We point it out here to emphasize that the analysis of this rigid-body problem cannot be reduced *a priori* to any simple particle mechanics problem.

In figure 11.22, non-dimensional rotational speed  $\frac{\omega^2 \ell}{g}$  is plotted versus hang angle  $\phi$ . As one might expect intuitively unless  $\omega$  is high enough, ( $\omega^2 > \frac{3g}{2\ell}$ ), the only solution is hanging straight down ( $\phi = 0$ ). At the critical speed ( $\omega^2 = \frac{3g}{2\ell}$ ), the curve is nearly flat, implying that a range of hang angles  $\phi$  is possible all with nearly the same angular velocity. As is also intuitively plausible, the bar gets close to the horizontal (close to  $\frac{\pi}{2}$ ), the spin rate goes to infinity.

## The scalar equations governing rotation about an axis

For two dimensional motion of flat hinged objects we had the simple relation “ $M = I\alpha$ ”. This formula captures our simple intuitions about angular momentum balance. When you apply torque to a body its rate of rotation increases. It turns out that, for three-dimensional motion of a rigid body about a fixed axis, the same result applies if we interpret the terms correctly.①

If the axis of rotation goes through C and is in the direction  $\hat{\lambda}$  we can define  $M = \hat{\lambda} \cdot \vec{M}_C$  as the moment about the axis of rotation. We can similarly look at the  $\hat{\lambda}$  component of  $\vec{H}_C$  (assume, for definiteness, that the system is continuous).

$$\begin{aligned}\hat{\lambda} \cdot \vec{H}_C &= \hat{\lambda} \cdot \int \vec{r} \times \vec{a} dm \\ &= \hat{\lambda} \cdot \int \vec{r} \times [(\omega \hat{\lambda}) \times ((\omega \hat{\lambda}) \times \vec{r}) + (\dot{\omega} \hat{\lambda}) \times \vec{r}] dm \\ &= \dot{\omega} \int R^2 dm,\end{aligned}\tag{11.19}$$

where  $R$  is the distance of the mass points from the axis. The last line follows from the previous most simply by paying attention to directions and magnitudes when using the right-hand rule and the geometric definition of the cross product. We thus have derived the result that

$$M = I\alpha,$$

if by  $M$  we mean moment about the fixed axis and by  $I$  we mean  $\int R^2 dm$ . Actually, the scalar we call  $I$  in the above equation is a manifestation of a more general matrix  $[I]$  that we will explore in the next section.



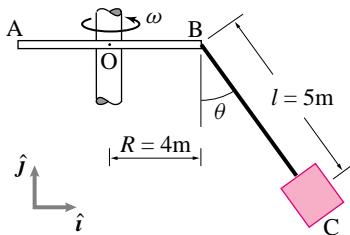


Figure 11.23: A carnival ride rotating at a constant speed

**SAMPLE 11.5** *Going on a carnival ride at a constant rate.* A carnival ride with roof AB and carriage BC is rotating about the vertical axis with constant angular velocity  $\vec{\omega} = \omega \hat{j}$ . If the carriage with its occupants has mass  $m = 100 \text{ kg}$ , find the tension in the inextensible and massless rod BC when  $\theta = 30^\circ$ . What is the required angular speed  $\omega$  (in revolutions/minute) to maintain this angle?

**Solution** The free body diagram of the carriage is shown in Fig. 11.24(a). The

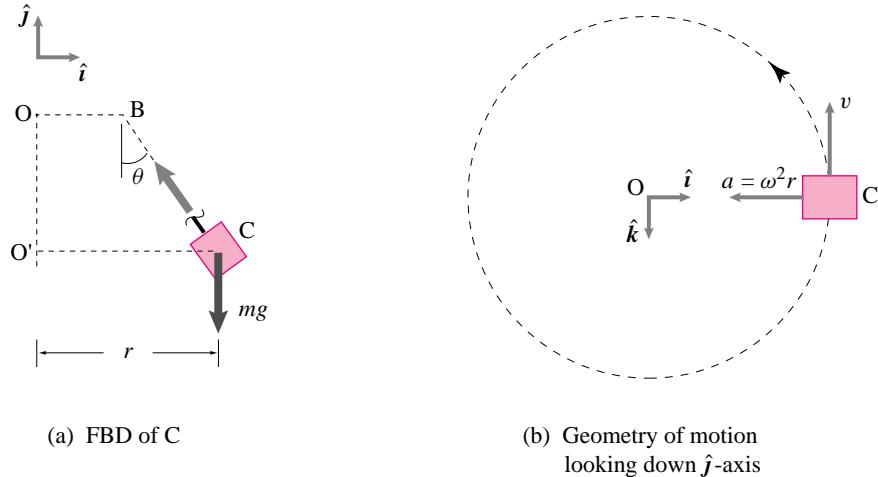


Figure 11.24: (Filename:sfig4.2.1a)

geometry of motion of the carriage is shown in Fig. 11.24(b). The carriage goes around a circle of radius  $r = O'C$  with constant speed  $v = \omega r$ . The only acceleration that the carriage has is the centripetal acceleration and at the moment of interest  $\vec{a} = -\omega^2 r \hat{i}$ .

The linear momentum balance ( $\sum \vec{F} = \dot{\vec{L}}$ ) for the carriage gives:

$$\begin{aligned} T \hat{\lambda}_{CB} - mg \hat{j} &= m \vec{a} \\ \text{or} \quad T(-\sin \theta \hat{i} + \cos \theta \hat{j}) - mg \hat{j} &= -m\omega^2 r \hat{i} \end{aligned} \quad (11.20)$$

Scalar equations from eqn. (11.20) are:

$$\begin{aligned} [\text{eqn. (11.20)}] \cdot \hat{j} &\Rightarrow T \cos \theta - mg = 0 \\ &\Rightarrow T = \frac{mg}{\cos \theta} \\ &= \frac{100 \text{ kg} \cdot 9.8 \text{ m/s}^2}{\sqrt{3}/2} \\ &= 1133 \text{ N}. \\ [\text{eqn. (11.20)}] \cdot \hat{i} &\Rightarrow -T \sin \theta = -m\omega^2 r \\ &\Rightarrow \omega^2 = \frac{T \sin \theta}{mr} \\ &= \frac{T \sin \theta}{m(R + l \sin \theta)} \\ &= \frac{1133 \text{ N} \cdot \frac{1}{2}}{100 \text{ kg}(4 \text{ m} + 5 \text{ m} \cdot \frac{1}{2})} \\ &= 0.87 \frac{1}{\text{s}^2} \end{aligned}$$

$$\begin{aligned}\Rightarrow \omega &= 0.93 \frac{\text{rad}}{\text{s}} \\ &= 0.93 \frac{1}{\text{s}} \cdot \frac{1 \text{rev}}{2\pi} \cdot \frac{60 \text{s}}{1 \text{min}} \\ &= 8.9 \text{ rpm.}\end{aligned}$$

$T = 1133 \text{ N}, \omega = 8.9 \text{ rpm}$

Alternatively,

we could also find the angular speed using angular momentum balance. The angular momentum balance about point B gives

$$\sum \vec{M}_{/B} = \dot{\vec{H}}_{/B}$$

$$\begin{aligned}\sum \vec{M}_{/B} &= \vec{r}_{C/B} \times (-mg\hat{j}) \\ &= -mgl \sin \theta \hat{k} \\ \dot{\vec{H}}_{/B} &= \vec{r}_{C/B} \times (-m\omega^2 r\hat{i}) \\ &= -m\omega^2 rl \cos \theta \hat{k}\end{aligned}$$

Equating the two quantities, we get

$$\begin{aligned}\cancel{m\omega^2 r} \cancel{\cos \theta} &= \cancel{mgl} \cancel{\sin \theta} \\ \Rightarrow \omega^2 &= \frac{g}{r} \tan \theta \\ &= \frac{g \tan \theta}{R + l \sin \theta} \\ &= \frac{9.8 \text{ m/s}^2 \cdot 0.577}{4 \text{ m} + 5 \text{ m} \frac{1}{2}} \\ &= 0.87 \frac{1}{\text{s}^2} \\ \omega &= 0.93 \text{ s}^{-1} = 8.9 \text{ rpm}\end{aligned}$$

which is the same value as we found using the linear momentum balance.

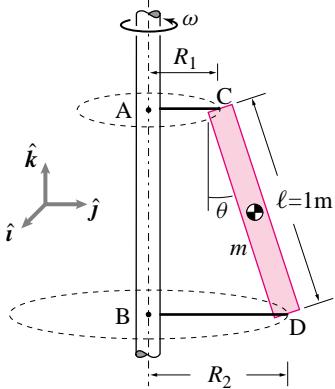


Figure 11.25: A bar held by two strings rotates in 3-D.  
(Filename:sfig4.6.2)

**SAMPLE 11.6** A crooked bar rotating with a shaft in space. A uniform rod CD of mass  $m = 2 \text{ kg}$  and length  $\ell = 1 \text{ m}$  is fastened to a shaft AB by means of two strings: AC of length  $R_1 = 30 \text{ cm}$ , and BD of length  $R_2 = 50 \text{ cm}$ . The shaft is rotating at a constant angular velocity  $\vec{\omega} = 5 \text{ rad/s} \hat{k}$ . There is no gravity. At the instant shown, find the tensions in the two strings.

**Solution** The free body diagram of the rod is shown in Fig. 11.26. The linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the rod gives:

$$T_1 + T_2 = m\omega^2 r_G. \quad (11.21)$$

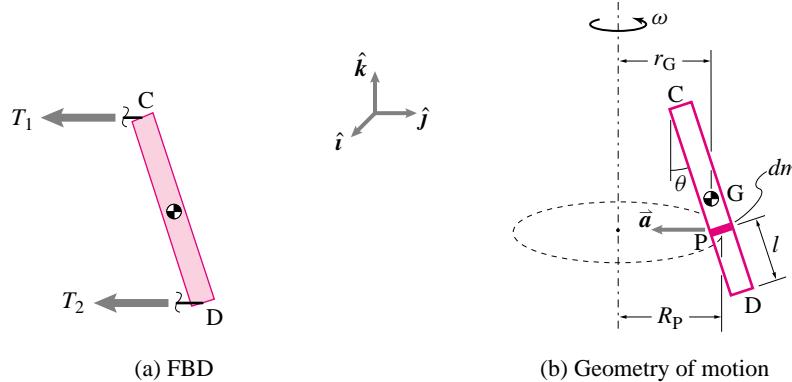


Figure 11.26: (Filename:sfig4.6.2a)

Usually, linear momentum balance gives us two scalar equations in 2-D and three scalar equations in 3-D. Unfortunately, in this case, it gives only one equation for two unknowns  $T_1$  and  $T_2$ . Therefore, we need one more equation.

The angular momentum balance about point D gives:

$$\begin{aligned} \sum \vec{M}_{/D} &= \dot{\vec{H}}_{/D}, \\ \text{where } \sum \vec{M}_{/D} &= \vec{r}_{C/D} \times (-T_1 \hat{j}) \\ &= \ell(-\sin \theta \hat{j} + \cos \theta \hat{k}) \times (-T_1 \hat{j}) \\ &= \ell T_1 \cos \theta \hat{i}, \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_{/D} &= \int_m \vec{r}_{P/D} \times (-\omega^2 R_P \hat{j}) dm \\ &= \int_0^\ell l \overbrace{(-\sin \theta \hat{j} + \cos \theta \hat{k})}^{R_P} \times (-\omega^2 \overbrace{(R_2 - l \sin \theta)}^{R_P} \hat{j}) \frac{dm}{\ell} dl \\ &= \frac{m\omega^2}{\ell} \left( R_2 \cos \theta \int_0^\ell l dl - \cos \theta \sin \theta \int_0^\ell l^2 dl \right) \hat{i} \end{aligned}$$

$$\begin{aligned}
 &= \frac{m\omega^2}{\ell} \left( R_2 \cos \theta \frac{\ell^2}{2} - \cos \theta \sin \theta \frac{\ell^3}{3} \right) \hat{i} \\
 &= m\omega^2 \ell \cos \theta \left( \frac{1}{2} R_2 - \frac{1}{3} \ell \sin \theta \right) \hat{i}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \cancel{\ell T_1 \cos \theta} &= m\omega^2 \cancel{\ell \cos \theta} \left( \frac{1}{2} R_2 - \frac{1}{3} \ell \sin \theta \right) \\
 \Rightarrow T_1 &= m\omega^2 \left( \frac{1}{2} R_2 - \frac{1}{3} \ell \sin \theta \right).
 \end{aligned}$$

Substituting in (11.21) we get

$$T_2 = m\omega^2 \left( r_G - \frac{1}{2} R_2 + \frac{1}{3} \ell \sin \theta \right).$$

Plugging in the given numerical values and noting that  $r_G = (R_1 + R_2)/2 = 40\text{ cm}$  and  $\ell \sin \theta = R_2 - R_1 = 20\text{ cm}$ , we get

$$\begin{aligned}
 T_1 &= 2\text{ kg} \cdot (5\frac{1}{s})^2 \cdot (0.4\text{ m} - \frac{1}{2}0.5\text{ m} + \frac{1}{3}0.2\text{ m}) \\
 &= 9.17 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = 9.17\text{ N}
 \end{aligned}$$

and  $T_2 = 10.83\text{ N}$ .

$T_1 = 9.1\text{ N}, \quad T_2 = 10.9\text{ N}$
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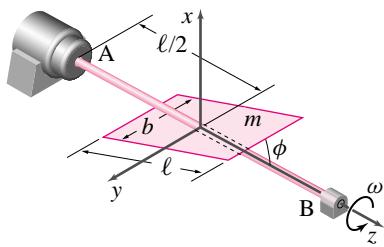


Figure 11.27: A rectangular plate, mounted rigidly at an angle  $\phi$  on a shaft, wobbles as the shaft rotates at a constant speed.

(Filename:fig4.6.6)

**SAMPLE 11.7** A crooked plate rotating with a shaft in space. A rectangular plate of mass  $m$ , length  $\ell$ , and width  $b$  is welded to a shaft AB in the center. The long edge of the plate is parallel to the shaft axis but is tipped by an angle  $\phi$  with respect to the shaft axis. The shaft rotates with a constant angular speed  $\omega$ . The end B of the shaft is free to move in the  $z$ -direction. Assume there is no gravity. Find the reactions at the supports.

**Solution** A simple line sketch and the Free Body Diagram of the system are shown in Fig. 11.28ab. The linear momentum balance equation for the shaft and the plate

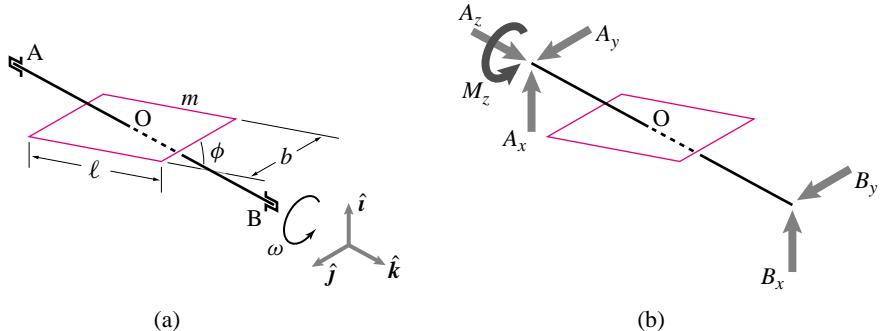


Figure 11.28: (Filename:fig4.6.6a)

system is:

$$\sum \vec{F} = m_{total} \vec{a}_{cm}.$$

Since the center of mass is on the axis of rotation,  $\vec{a}_{cm} = \vec{0}$ . Therefore,

$$(A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + A_z\hat{k} = \vec{0} \\ \Rightarrow A_x + B_x = 0, \quad A_y + B_y = 0, \quad A_z = 0. \quad (11.22)$$

The angular momentum balance about the center of mass O is:

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

• Calculation of  $\sum \vec{M}_O$ :

$$\begin{aligned} \sum \vec{M}_O &= \vec{r}_{A/O} \times \vec{F}_A + \vec{r}_{B/O} \times \vec{F}_B + M_z\hat{k} \\ &= -\frac{\ell}{2}\hat{k} \times (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + \frac{\ell}{2}\hat{k} \times (B_x\hat{i} + B_y\hat{j}) + M_z\hat{k} \\ &= \frac{\ell}{2}(A_y - B_y)\hat{i} + \frac{\ell}{2}(B_x - A_x)\hat{j} + M_z\hat{k} \end{aligned} \quad (11.23)$$

• Calculation of  $\dot{\vec{H}}_O$ :  $\dot{\vec{H}}_O$  can be computed in various ways. ① Here, to compute  $\dot{\vec{H}}_O$ , we use

$$\dot{\vec{H}}_O = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm} dm,$$

the formula which we have used so far. To carry out this integration for the plate, we take, as usual, an infinitesimal mass  $dm$  of the body, calculate its angular momentum about O, and then integrate over the entire mass of the body:

$$\dot{\vec{H}}_O = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm} dm$$

We need to write carefully each term in the integrand. Let us define an axis  $w$  (don't confuse this dummy variable  $w$  with  $\omega$ ) along the length of the plate (see Fig. 11.29(a)). We take an area element  $dA = dw dy$  on the plate as our infinitesimal mass. Fig. 11.29(b) shows this element and its coordinates.

$$\begin{aligned} dm &= \rho dA = \frac{m}{\ell b} dw dy \quad (\rho = \text{mass per unit area}) \\ \vec{a}_{dm} &= \vec{\omega} \times (\vec{\omega} \times \vec{r}_{dm/O}) \\ \vec{r}_{dm/O} &= x\hat{i} + y\hat{j} + z\hat{k} \end{aligned}$$

where

$$x = w \sin \phi, \quad y = y, \quad z = w \cos \phi. \quad (11.24)$$

Therefore,

$$\begin{aligned} \vec{a}_{cm} &= \omega\hat{k} \times (\omega\hat{k} \times (w \sin \phi \hat{i} + y\hat{j} + w \cos \phi \hat{k})) \\ &= \omega\hat{k} \times (\omega w \sin \phi \hat{j} - \omega y \hat{i}) = -\omega^2(w \sin \phi \hat{i} + y\hat{j}), \\ \vec{r}_{dm/O} \times \vec{a}_{dm} &= (w \sin \phi \hat{i} + y\hat{j} + w \cos \phi \hat{k}) \times [-\omega^2(w \sin \phi \hat{i} + y\hat{j})] \\ &= \omega^2(-w^2 \sin \phi \cos \phi \hat{j} + wy \cos \phi \hat{i}). \end{aligned}$$

Thus,

$$\begin{aligned} \dot{\vec{H}}_O &= \int_{-b/2}^{b/2} \int_{-\ell/2}^{\ell/2} \overbrace{\omega^2(-w^2 \sin \phi \cos \phi \hat{j} + wy \cos \phi \hat{i})}^{\vec{r}_{dm/O} \times \vec{a}_{dm}} \underbrace{\frac{m}{\ell b} dw dy}_{dm} \\ &= \frac{m}{\ell b} \omega^2 \int_{-b/2}^{b/2} \left( \int_{-\ell/2}^{\ell/2} (-w^2 \sin \phi \cos \phi \hat{j} + wy \cos \phi \hat{i}) dw \right) dy \\ &= \frac{m}{\ell b} \omega^2 \int_{-b/2}^{b/2} \left( -\sin \phi \cos \phi \frac{w^3}{3} \Big|_{-\ell/2}^{\ell/2} + y \cos \phi \frac{w^2}{2} \Big|_{-\ell/2}^{\ell/2} \right) dy \\ &= -\frac{m \omega^2 \ell^2}{12} \sin \phi \cos \phi \hat{j}. \end{aligned} \quad (11.25)$$

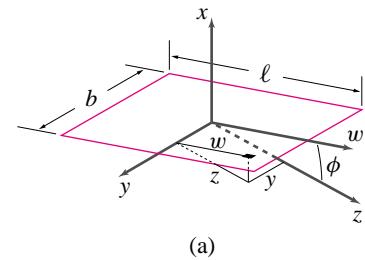
- **Now, back to angular momentum balance:** Now equating (11.23) and (11.25) and dotting both sides with  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  we get

$$A_y - B_y = 0, \quad B_x - A_x = -\frac{m \omega^2 \ell}{6} \sin \phi \cos \phi, \quad M_z = 0, \quad (11.26)$$

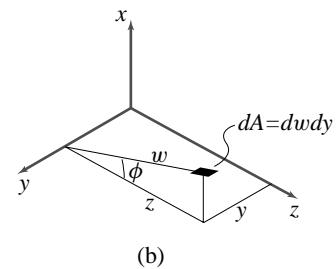
respectively. Solving (11.22) and (11.26) simultaneously we get

$$A_y = B_y = 0, \quad A_x = \frac{m \omega^2 \ell}{12}, \quad \sin \phi \cos \phi B_x = -\frac{m \omega^2 \ell}{12} \sin \phi \cos \phi.$$

$$A_x = \frac{m \omega^2 \ell}{12} \sin \phi \cos \phi, \quad B_x = -\frac{m \omega^2 \ell}{12} \sin \phi \cos \phi, \quad A_y = B_y = A_z = M_z = 0$$



(a)



(b)

Figure 11.29: Calculation of  $\dot{\vec{H}}_O$ : (a) mass element  $dm$  is shown on the plate, (b) the mass element as an area element and its geometry.

(Filename:sfig4.6.6b)

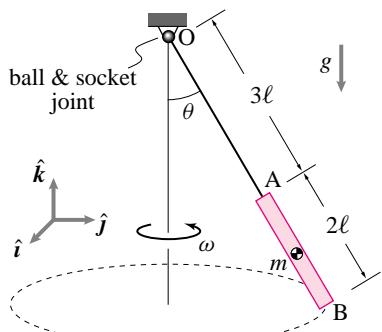
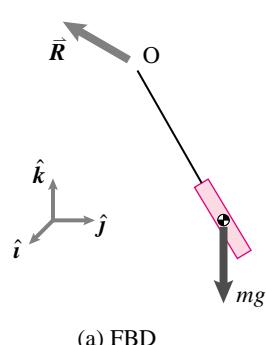
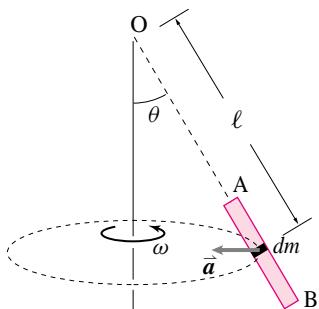


Figure 11.30: A short rod swings in 3-D.

(Filename:fig4.6.1)



(a) FBD



(b) Calculation of H-dot

Figure 11.31: (Filename:fig4.6.1a)

**SAMPLE 11.8** *A short rod as a 3-D pendulum.* A uniform rod AB of mass  $m$  and length  $2\ell$  is welded to a massless, inextensible thin rod OA at point A. Rod OA is attached to a ball and socket joint at point O. The rods are going around in a circle with constant speed maintaining a constant angle  $\theta$  with the vertical axis. Assume that  $\theta$  is small.

- (a) How many revolutions does the system make in one second?

### Solution

- (a) The free body diagram of the system (rod OA + rod AB) is shown in Fig. 11.31. Let  $\omega$  be the angular speed of the system. Then, the number of revolutions in one second is  $n = \omega/(2\pi)$ . Therefore, to find the answer we need to calculate  $\omega$ .

The angular momentum balance about point O gives  $\sum \vec{M}_O = \dot{\vec{H}}_O$ . Now,

$$\begin{aligned} \sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{k}) \\ &= 4\ell(\sin \theta \hat{j} - \cos \theta \hat{k}) \times (-mg\hat{k}) \\ &= -4\ell mg \sin \theta \hat{i}. \\ \text{and } \dot{\vec{H}}_O &= \int_m \vec{r}_{dm/O} \times \vec{a}_{dm} dm \\ &= \int_{3\ell}^{5\ell} \overbrace{\ell(\sin \theta \hat{j} - \cos \theta \hat{k})}^{\vec{r}_{dm/O}} \times \underbrace{(-\omega^2 l \sin \theta \hat{j})}_{\vec{a}_{dm}} \frac{dm}{2\ell} d\ell \\ &= -\frac{m}{2\ell} \omega^2 \sin \theta \cos \theta \hat{i} \int_{3\ell}^{5\ell} \ell^2 d\ell \\ &= -\frac{49}{3} \ell^2 m \omega^2 \sin \theta \cos \theta \hat{i}. \end{aligned}$$

By equating the two quantities ( $\sum \vec{M}_O = \dot{\vec{H}}_O$ ), we get

$$\begin{aligned} -4\ell mg \sin \theta \hat{i} &= -\frac{49}{3} \ell^2 m \omega^2 \sin \theta \cos \theta \hat{i} \\ \Rightarrow \quad \omega^2 &= \frac{12g}{49\ell \cos \theta}. \end{aligned}$$

But for small  $\theta$ ,  $\cos \theta \approx 1$ . Therefore,

$$\omega = \sqrt{\frac{12g}{49\ell}} = \frac{2\sqrt{3}}{7} \sqrt{\frac{g}{\ell}}$$

and the number of revolutions per unit time is  $n = \frac{2\sqrt{3}}{14\pi} \sqrt{\frac{g}{\ell}}$ .

$$n = \frac{2\sqrt{3}}{14\pi} \sqrt{\frac{g}{\ell}}$$

- (b) Note, the natural frequency of this rod swinging back and forth as a simple pendulum turns out to be the same as the angular speed  $\omega$  of the rotating system above for small  $\theta$ .

## 11.3 Moment of inertia matrices

We now know how to find the velocity and acceleration of every bit of mass on a rigid body as it spins about a fixed axis. It is just a matter of doing integrals or sums to calculate the various motion quantities (momenta, energy) of interest. As the body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut. A big sum (over all atoms, say), or a difficult integral is reduced to a simple multiplication using the moment of inertia. In three-dimensions this multiplication is a matrix multiplication.

[I], the moment of inertia matrix<sup>①</sup>, is defined for the purpose of simplifying the expressions for the angular momentum, the rate of change of angular momentum, and the energy of *a system which moves like a rigid body*.

First review the situation for flat objects in planar motion. A flat object spinning with  $\vec{\omega} = \omega \hat{k}$  in the  $xy$  plane has a mass distribution which gives a polar moment of inertia  $I_{zz}^{cm}$  or just ' $I$ ' so that:

$$\vec{H}_{cm} = I\omega \hat{k} \quad (11.27)$$

$$\dot{\vec{H}}_{cm} = \vec{0} \quad (11.28)$$

$$E_{K/cm} = \frac{1}{2}\omega^2 I. \quad (11.29)$$

Now, for a rigid body spinning in 3-D about a fixed axis with the angular velocity  $\vec{\omega}$  we need matrix multiplication, where the determination of the needed matrix is the central topic of this section.

$$\vec{H}_{cm} = [I^{cm}] \cdot \vec{\omega} \quad (11.30)$$

$$\dot{\vec{H}}_{cm} = \vec{\omega} \times \underbrace{[I^{cm}] \cdot \vec{\omega}}_{\vec{H}_{cm}} + [I^{cm}] \cdot \dot{\vec{\omega}} \quad (11.31)$$

$$E_{K/cm} = \frac{1}{2}\vec{\omega} \cdot ([I^{cm}] \cdot \vec{\omega}) = \frac{1}{2}\vec{\omega} \cdot \vec{H}_{cm}. \quad (11.32)$$

In detail, for example,

$$\begin{bmatrix} H_{x/cm} \\ H_{y/cm} \\ H_{z/cm} \end{bmatrix} = \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}_{xyz} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (11.33)$$

where  $\vec{H}_{cm} = H_{x/cm}\hat{i} + H_{y/cm}\hat{j} + H_{z/cm}\hat{k}$ . Note that the 2-D results are a special case of the 3-D results because, as you will soon see, for 2-D objects  $I_{xz} = I_{yz} = 0$ . We postpone the use of these equations till section 11.4.

<sup>①</sup> In fact the moment of inertia matrix for a given object depends on what reference point is used. Most commonly when people say 'the' moment of inertia they mean to use the center of mass as the reference point. For clarity this moment of inertia matrix is often written as  $[I^{cm}]$  in this book. If a different reference point, say point  $O$  is used, the matrix is noted as  $[I^O]$ .

### The moment of inertias in 3-D: $[I^{cm}]$ and $[I^O]$

For the study of three-dimensional mechanics, including the simple case of constant rate rotation about a fixed axis, one often makes use of the moment of inertia matrix, defined below and motivated by the box 11.3 on page 668.

The distances  $x, y, z$  in the formulas below are the  $x, y, z$  components of the position of mass relative to a coordinate system which has either the center of mass ( $cm$ ) or the point  $O$  as its origin. <sup>②</sup>

<sup>②</sup>Caution: While we have  $I_{xy} = -\int xy dm$ , some old books define  $I_{xy} = \int xy dm$ . They then have minus signs in front of the off-diagonal terms in the moment of inertia matrix. They would say  $I_{12} = -I_{xy}$ . The numerical values in the matrix they write is the same as in the one we write. They just have a different sign convention in the definition of the components.

$$[\mathbf{I}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \quad (11.34)$$

$$= \begin{bmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{bmatrix} \quad (11.35)$$

If all mass is on the  $xy$  plane then it is clear that  $I_{xz} = I_{yz} = 0$  since  $z = 0$  for the whole  $xy$  plane. If rotation is also about the  $z$ -axis then  $\vec{\omega} = \omega \hat{k}$ .

Applying the formulas above we find that, if all of the mass is in the  $xy$ -plane and rotation is about the  $z$ -axis, the only relevant non-zero term in  $[\mathbf{I}]$  is  $I_{zz} = \int (x^2 + y^2) dm$ . And,  $I_{xx}$ ,  $I_{yy}$ , and  $I_{xy}$  don't contribute to  $\vec{H}$ ,  $\dot{\vec{H}}$ , or  $E_K$ . In this manner you can check that the three dimensional equations, when applied to two-dimensional bodies, give the same results that we found directly for two-dimensional bodies.

#### Example: Moment of inertia matrix for a uniform sphere

A sphere is a special shape which is, naturally enough, spherically symmetric. Therefore,

$$I_{xx}^{cm} = I_{yy}^{cm} = I_{zz}^{cm}$$

and

$$I_{xy}^{cm} = I_{xz}^{cm} = I_{yz}^{cm} = 0.$$

So, all we need is  $I_{xx}^{cm}$  or  $I_{yy}^{cm}$  or  $I_{zz}^{cm}$ . Here is the trick:

$$\begin{aligned} I_{xx}^{cm} &= \frac{1}{3}(I_{xx}^{cm} + I_{yy}^{cm} + I_{zz}^{cm}) \\ &= \frac{1}{3} \left[ \int (y^2 + z^2) dm + \int (x^2 + z^2) dm + \int (x^2 + y^2) dm \right] \\ &= \frac{2}{3} \int (x^2 + y^2 + z^2) dm \\ &= \frac{2}{3} \int r^2 dm \\ &= \frac{2}{3} \int_0^R r^2 (4\rho\pi r^2 dr) \\ &= \frac{8}{3}\rho\pi \int_0^R r^4 dr \\ &= \frac{8}{15}\rho\pi R^5 \\ &= \frac{2}{5}mR^2 \quad (m = \frac{4}{3}\rho\pi R^3). \end{aligned}$$

So,

$$[\mathbf{I}^{cm}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

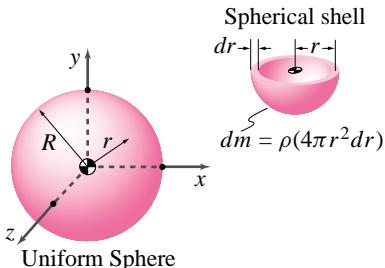


Figure 11.32: (filename:tfigure4.3Dsphere)

## The parallel axis theorem for rigid bodies in three dimensions

The 3-D parallel axis theorem is stated below and in the table on the inside back cover. It is derived in box 11.4 on page 670. The parallel axis theorem for rigid bodies in three dimensions is the equation

$$[\mathbf{I}^O] = [\mathbf{I}^{cm}] + m \begin{bmatrix} y_{cm/o}^2 + z_{cm/o}^2 & -x_{cm/o}y_{cm/o} & -x_{cm/o}z_{cm/o} \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 + z_{cm/o}^2 & -y_{cm/o}z_{cm/o} \\ -x_{cm/o}z_{cm/o} & -y_{cm/o}z_{cm/o} & x_{cm/o}^2 + y_{cm/o}^2 \end{bmatrix} \quad (11.36)$$

In this equation,  $x_{cm/o}$ ,  $y_{cm/o}$ , and  $z_{cm/o}$  are the  $x$ ,  $y$ , and  $z$  coordinates, respectively, of the center of mass defined with respect to a coordinate system whose origin is located at some point O not at the center of mass  $cm$ . That is, if you know  $[\mathbf{I}^{cm}]$ , you can find  $[\mathbf{I}^O]$  without doing any more integrals or sums. Like the 2-D parallel axis theorem. The primary utility of the 3-D parallel axis theorem is for the determination of  $[\mathbf{I}]$  for an object that is a composite of simpler objects. Such are not beyond the scope of this book in principle. But in fact, given the finite time available for calculation, we do not leave much time for practice of this tedious but routine calculation.

## Matrices and tensors

We have just introduced the 3 by 3 moment of inertia matrix  $[\mathbf{I}]$ . We will find it in expressions having to do with angular momentum sitting next to either a vector  $\vec{\omega}$  or a vector  $\vec{\alpha}$ :  $[\mathbf{I}] \cdot \vec{\omega}$  or  $[\mathbf{I}] \cdot \vec{\alpha}$ . What we mean by this expression is the three element column vector that comes from matrix multiplication of the matrix  $[\mathbf{I}]$  and

the column vector for  $\vec{\omega}$ ,  $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ , an expression that only makes sense if everyone knows what bases are being used.

More formally, and usually only in more advanced treatments, people like to define a coordinate-free quantity called the tensor  $\underline{\underline{\mathbf{I}}}$ . Then we would have

$$\underline{\underline{\mathbf{I}}} \cdot \vec{\omega}$$

by which we would mean the vector whose components would be found by  $[\mathbf{I}] \cdot$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

## Eigenvectors and Eigenvalues

A square matrix  $[\mathbf{A}]$  when multiplied by a column vector  $[\mathbf{v}]$  yields a new vector  $[\mathbf{w}]$ . A given matrix has a few special vectors, somehow characteristic of that matrix, called *eigenvectors*. The vector  $\vec{v}$  is an eigenvector of  $[\mathbf{A}]$  if

$$[\mathbf{A}] \cdot [\mathbf{v}] \text{ is parallel to } [\mathbf{v}].$$

In other words, if

$$[\mathbf{A}] \cdot [\mathbf{v}] = \lambda [\mathbf{v}]$$

for some  $\lambda$ .

The scalar  $\lambda$  is called the *eigenvalue* associated with the eigenvector  $[\mathbf{v}]$  of the matrix  $[\mathbf{A}]$ . The eigen-values and eigen-vectors of a matrix are found with a single command in many computer math programs. In statics you had little or no use for eigen-values and eigen-vectors. In dynamics, eigenvectors and eigenvalues are useful for understanding dynamic balance, 3-D rigid body rotations, and normal mode vibrations.

### *Eigenvectors of $[I]$*

The moment of inertia matrix is always a symmetric matrix. This symmetry means that  $[I]$  always has a set of three mutually orthogonal eigenvectors. The importance of the eigenvectors of  $[I]$  will be discussed in section 11.5 on dynamic balance. Sometimes a pair of the eigenvalues are equal to each other implying that any vector in the plane of the corresponding eigenvectors is also an eigenvector.

If the physical object has any natural symmetry directions these directions will usually manifest themselves in the dynamics of the body as being in the directions of the eigenvectors of object's moment of inertia matrix. For example, the dotted lines on figure ?? are all in directions of eigenvectors for the objects shown. But even if an object is wildly asymmetric in shape, its moment of inertia matrix is always symmetric and thus all objects have moment of inertia matrices with at least three different eigenvectors at least three of which are mutually orthogonal.

### *Properties of $[I]$*

For those with experience with linear algebra various properties of the moment of inertia matrix  $[I]$  are worth noting (although not worth proving here). Unless all mass is distributed on one straight line, the moment of inertia matrix is invertible (it is non-singular and has rank 3). Further, when invertible it is positive definite. In the special case that all the mass is on some straight line, the moment of inertia matrix is non-invertible and only positive semi-definite. The positive (semi) definiteness of the moment of inertia matrix is equivalent to the statement that the rotational kinetic energy of a body is always equal to or greater than zero. Finally, the eigenvalues of the moment of inertia matrix are all positive and have the property that no one can be greater than the sum of the other two (the same inequalities are satisfied the lengths of the sides of a triangle, the “triangle inequality”).

**SAMPLE 11.9** For the dumbbell shown in Figure 11.33, take  $m = 0.5 \text{ kg}$  and  $\ell = 0.4 \text{ m}$ . Given that at the instant shown  $\theta = 30^\circ$  and the dumbbell is in the  $yz$ -plane, find the moment of inertia matrix  $[\mathbf{I}^O]$ , where  $O$  is the midpoint of the dumbbell.

**Solution** The dumbbell is made up of two point masses. Therefore we can calculate  $[\mathbf{I}^O]$  for each mass using the formula from the table on the inside back cover of the text and then adding the two matrices to get  $[\mathbf{I}^O]$  for the dumbbell.

Now, from Table 4.9 of the text,

$$[\mathbf{I}^O] = m \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

For mass 1 (shown in Figure 11.34)

$$x = 0, \quad y = -\frac{\ell}{2} \cos \theta, \quad z = \frac{\ell}{2} \sin \theta.$$

Therefore,

$$\begin{aligned} [\mathbf{I}^O]_{\text{mass1}} &= m \begin{bmatrix} \frac{\ell^2}{4} & 0 & 0 \\ 0 & \frac{\ell^2}{4} \sin^2 \theta & \frac{\ell^2}{4} \cos \theta \sin \theta \\ 0 & \frac{\ell^2}{4} \cos \theta \sin \theta & \frac{\ell^2}{4} \cos^2 \theta \end{bmatrix} \\ &= 0.5 \text{ kg} \begin{bmatrix} 0.04 \text{ m}^2 & 0 & 0 \\ 0 & 0.04 \text{ m}^2 \cdot \frac{1}{4} & 0.04 \text{ m}^2 \cdot \frac{\sqrt{3}}{4} \\ 0 & 0.04 \text{ m}^2 \cdot \frac{\sqrt{3}}{4} & 0.04 \text{ m}^2 \cdot \frac{3}{4} \end{bmatrix} \\ &= 0.02 \text{ kg} \cdot \text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \end{aligned}$$

Similarly for mass 2,

$$\begin{aligned} x &= 0, \quad y = \frac{\ell}{2} \cos \theta, \quad z = -\frac{\ell}{2} \sin \theta \\ \Rightarrow [\mathbf{I}^O]_{\text{mass2}} &= 0.02 \text{ kg} \cdot \text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} [\mathbf{I}^O] &= [\mathbf{I}^O]_{\text{mass1}} + [\mathbf{I}^O]_{\text{mass2}} \\ &= 0.04 \text{ kg} \cdot \text{m}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

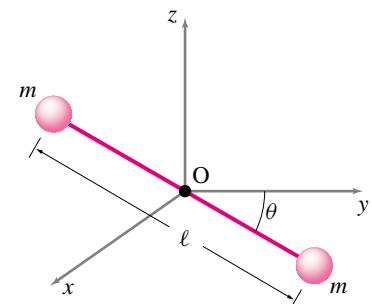


Figure 11.33: (Filename:sfig4.6.4)

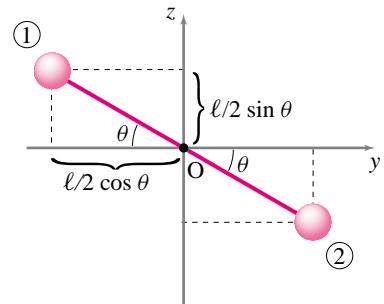


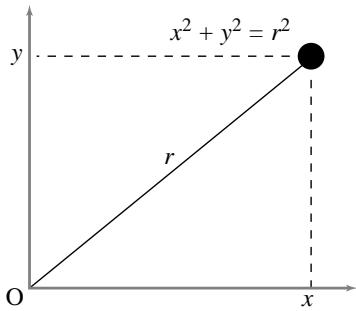
Figure 11.34: (Filename:sfig4.6.4a)

◀

## 11.2 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

### One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I_{zz}^o = \int r_{i/o}^2 dm$$

reduces to the sum

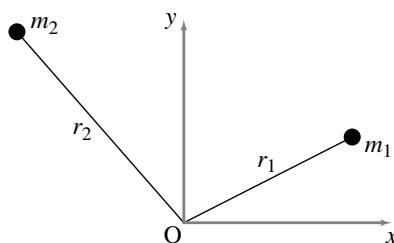
$$I_{zz}^o = \sum r_{i/o}^2 m_i$$

which reduces to one term if there is only one mass,

$$I_{zz}^o = r^2 m = (x^2 + y^2)m.$$

So, if  $x = 3$  in,  $y = 4$  in, and  $m = 0.1$  lbm, then  $I_{zz}^o = 2.5$  lbm in<sup>2</sup>. Note that, in this case,  $I_{zz}^{cm} = 0$  since the radius from the center of mass to the center of mass is zero.

### Two point masses

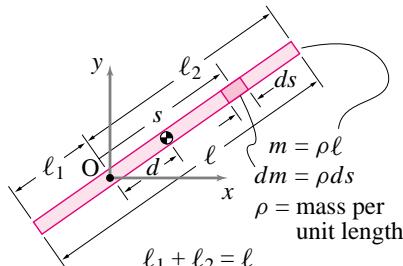


In this case, the sum that defines  $I_{zz}^o$  reduces to two terms, so

$$I_{zz}^o = \sum r_{i/o}^2 m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if  $r_1 = r_2 = r$ , then  $I_{zz}^o = m_{tot}r^2$ .

### A thin uniform rod



Consider a thin rod with uniform mass density,  $\rho$ , per unit length, and length  $\ell$ . We calculate  $I_{zz}^o$  as

$$\begin{aligned} I_{zz}^o &= \int r^2 \overbrace{dm}^{\rho ds} \\ &= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\ &= \frac{1}{3} \rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad (\text{since } \rho \equiv \text{const.}) \\ &= \frac{1}{3} \rho (\ell_1^3 + \ell_2^3). \end{aligned}$$

If either  $\ell_1 = 0$  or  $\ell_2 = 0$ , then this expression reduces to  $I_{zz}^o = \frac{1}{3} m \ell^2$ . If  $\ell_1 = \ell_2$ , then  $O$  is at the center of mass and

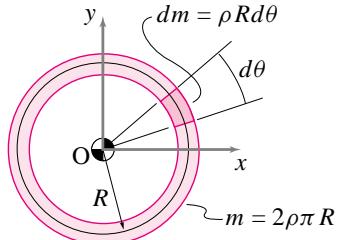
$$I_{zz}^o = I_{zz}^{cm} = \frac{1}{3} \rho \left( \left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3 \right) = \frac{m \ell^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for  $I_{zz}^o$  can be arranged as follows:

$$\begin{aligned} I_{zz}^o &= \frac{1}{3} \rho (\ell_1^3 + \ell_2^3) \\ &= \underbrace{\rho (\ell_1 + \ell_2)}_m \left( \underbrace{\frac{\ell_2 - \ell_1}{2}}_d \right)^2 + \underbrace{\rho \frac{(\ell_1 + \ell_2)^3}{12}}_{m \ell^2 / 12} \\ &= m d^2 + m \frac{\ell^2}{12} \\ &= m d^2 + I_{zz}^{cm} \end{aligned}$$

That is, the moment of inertia about point  $O$  is greater than that about the center of mass by an amount equal to the mass times the distance from the center of mass to point  $O$  squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

### A uniform hoop



For a hoop of uniform mass density,  $\rho$ , per unit length, we might consider all of the points to have the same radius  $R$ . So,

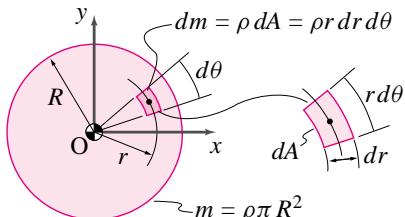
$$I_{zz}^o = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I_{zz}^o &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi \rho R^3 = \underbrace{(2\pi \rho R)}_m R^2 = m R^2. \end{aligned}$$

This  $I_{zz}^o$  is the same as for a single point mass  $m$  at a distance  $R$  from the origin  $O$ . It is also the same as for two point masses if they both are a distance  $R$  from the origin. For the hoop, however,  $O$  is at the center of mass so  $I_{zz}^o = I_{zz}^{cm}$  which is not the case for a single point mass.

### A uniform disk

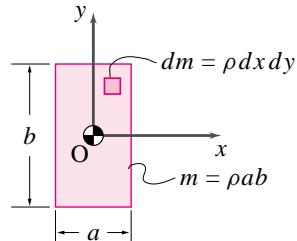


Assume the disk has uniform mass density,  $\rho$ , per unit area. For a uniform disk centered at the origin, the center of mass is at the origin so

$$\begin{aligned} I_{zz}^o = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r dr d\theta \\ &= \int_0^R 2\pi \rho r^3 dr \\ &= 2\pi \rho \frac{r^4}{4} \Big|_0^R = \pi \rho \frac{R^4}{2} = (\pi \rho R^2) \frac{R^2}{2} \\ &= m \frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

### Uniform rectangular plate



For the special case that the center of the plate is at point  $O$ , the center of mass of mass is also at  $O$  and  $I_{zz}^o = I_{zz}^{cm}$ .

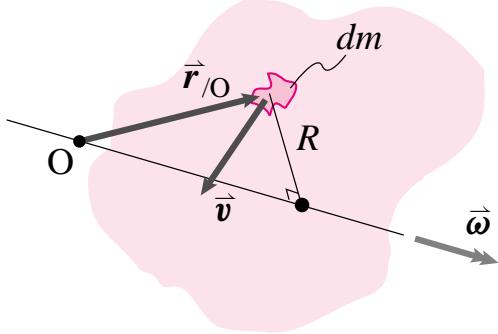
$$\begin{aligned} I_{zz}^o = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) \rho dx dy \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho \left( \frac{x^3}{3} + xy^2 \right) \Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho \left( \frac{x^3 y}{3} + \frac{xy^3}{3} \right) \Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} \Big|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho \left( \frac{a^3 b}{12} + \frac{ab^3}{12} \right) \\ &= \frac{m}{12} (a^2 + b^2). \end{aligned}$$

Note that  $\int r^2 dm = \int x^2 dm + \int y^2 dm$  for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle,  $\int y^2 dm = \rho \int y^2 dA$ . But the integral  $y^2 dA$  is just the term often used for  $I$ , the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You may recall that  $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$  for a rectangle. Similarly,  $\int x^2 dA = \frac{Aa^2}{12}$ . So, the polar moment of inertia  $J = I_{zz}^o = m \frac{1}{12}(a^2 + b^2)$  can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

### 11.3 Discovering the moment of inertia matrix

#### Derivation 1

Here we present a direct derivation of the moment of inertia matrix; that is, a derivation in which the moment of inertia matrix arises as a convenient short hand. Assume a rigid body is moving in such a way that point  $O$  is fixed (i.e., It is either on the line of a hinge or a ball-and-socket joint).



The most basic kinematic relation for a rigid body is that

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where  $\vec{r}_{/O} = x\hat{i} + y\hat{j} + z\hat{k}$  is the position of a point on the body relative to  $O$  and  $\vec{\omega}$ , the angular velocity of the body.

Now, we tediously calculate and arrange the terms in the angular momentum about point  $O$ ,

$$\begin{aligned} \vec{H}_O &= \int \vec{r}_{/O} \times \vec{v} dm \\ &= \int \vec{r}_{/O} \times (\vec{\omega} \times \vec{r}_{/O}) dm \\ &= \int (x\hat{i} + y\hat{j} + z\hat{k}) \times \\ &\quad [(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})] dm \\ &= \int (x\hat{i} + y\hat{j} + z\hat{k}) \times \\ &\quad [(\omega_yz - \omega_zy)\hat{i} + (\omega_zx - \omega_xy)\hat{j} + (\omega_xy - \omega_yx)\hat{k}] dm \\ &= \int [(y(\omega_xy - \omega_yx) - z(\omega_zx - \omega_xy))\hat{i} \\ &\quad + (z(\omega_yz - \omega_zy) - x(\omega_xy - \omega_yx))\hat{j} \\ &\quad + (x(\omega_zx - \omega_xy) - y(\omega_yz - \omega_zy))\hat{k}] dm \end{aligned}$$

Since the integral is over the mass and  $\vec{\omega}$  is constant over the body, we can pull  $\vec{\omega}$  out of the integral so that we may write the equation in matrix form. Writing  $\vec{H}_O$  as a column vector, we can rewrite the last equation as

$$\begin{bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{bmatrix} = \underbrace{\begin{bmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{bmatrix}}_{[\mathbf{I}^O]} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Finally, defining  $[\mathbf{I}^O]$  by the matrix above, we can compactly write

$$\vec{H}_O = [\mathbf{I}^O] \cdot \vec{\omega}$$

assuming  $O$  is a fixed point on the body where we represent  $\vec{H}_O$  and  $\vec{\omega}$  in terms of  $x$ ,  $y$ , and  $z$  components.

#### Center-of-mass inertia matrix

For any system moving, distorting, and rotating any crazy way, we have the general result that

Contribution of the system to  $\vec{H}_O$   
if treated as a particle at the system  
center of mass

$$\vec{H}_O = \overbrace{\vec{r}_{cm/O} \times m_{tot} \vec{v}_{cm}} + \overbrace{\int (\vec{r}_{/cm} \times \vec{v}_{/cm}) dm}$$

$\vec{v}_{/cm} = (\vec{v} - \vec{v}_{cm})$

as you can verify by substituting  $\vec{v} = \vec{v}_{cm} + \vec{v}_{/cm}$  and  $\vec{r} = \vec{r}_{cm} + \vec{r}_{/cm}$  into the general definition of  $\vec{H}_O = \int \vec{r}_{/O} \times \vec{v} dm$ . For a rigid body, we have

$$\vec{v}_{/cm} = \vec{\omega} \times \vec{r}_{/cm}.$$

So, by a derivation essentially identical to that for  $\vec{H}_O$ , we get

$$\vec{H}_{cm} = [\mathbf{I}^{cm}] \cdot \vec{\omega}$$

with  $[\mathbf{I}^{cm}]$  being defined using  $x$ ,  $y$ , and  $z$ , as the distances from the center of mass rather than from point  $O$ . So, for a rigid body in general motion, we can find the angular momentum by

$$\vec{H}_O = \vec{r}_{cm/O} \times \vec{v}_{cm} m_{tot} + [\mathbf{I}^{cm}] \cdot \vec{\omega} \quad (11.37)$$

#### Comment (aside)

In the special case that the body is rotating about point  $O$ , we also have

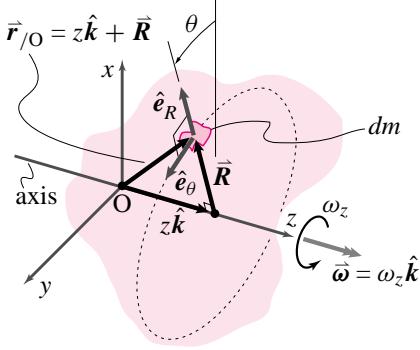
$$\vec{H}_O = [\mathbf{I}^O] \cdot \vec{\omega}. \quad (11.38)$$

You will see, if you look at the parallel axis theorem, that these two expressions 11.37 and 11.38 do in fact agree.

## Derivation 2

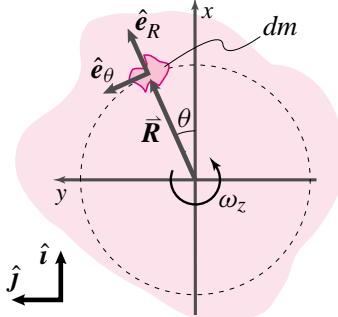
Here, we present a less direct but perhaps more intuitive derivation of the moment of inertia matrix. We start with the special case of a 3-D rigid body spinning in circles at constant rate about a fixed axis.

To see from where the moment of inertia matrix comes, we will first calculate the angular momentum about point  $O$  of a general 3-D rigid body spinning about the  $z$ -axis with constant rate  $\dot{\theta} \equiv \text{const.} = \omega_z$  or  $\vec{\omega} = \omega_z \hat{k}$ . We will refer to this case as (1).



Starting with the definition of angular momentum, we get

$$\begin{aligned}\vec{H}_O &= \int \vec{r}_{/O} \times \vec{v} dm \\ &= \int (z\hat{k} + \vec{R}) \times (\dot{\theta} R \hat{e}_\theta) dm.\end{aligned}$$



Looking down the  $z$ -axis in the figure, we see that

$$\begin{aligned}\vec{R} &= R\hat{e}_R = \underbrace{R \cos \theta}_{x} \hat{i} + \underbrace{R \sin \theta}_{y} \hat{j}, \quad \text{or} \\ \vec{R} &= x\hat{i} + y\hat{j}.\end{aligned}$$

where  $R = \sqrt{x^2 + y^2}$ . To compute the cross product in the integrand, we need

$$\begin{aligned}\hat{k} \times \hat{e}_\theta &= \hat{k} \times (-\sin \theta \hat{i} + \cos \theta \hat{j}) = -\hat{e}_R \quad \text{and} \\ \vec{R} \times \hat{e}_\theta &= R\hat{e}_R \times \hat{e}_\theta = R\hat{k}.\end{aligned}$$

Therefore, we now have

$$\begin{aligned}\vec{H}_O &= \int -z\dot{\theta} R \hat{e}_R dm + \int \dot{\theta} R^2 \hat{k} dm \\ &= \dot{\theta} \left[ - \int z \left( \underbrace{R \cos \theta}_{x} \hat{i} + \underbrace{R \sin \theta}_{y} \hat{j} \right) dm + \int (x^2 + y^2) \hat{k} dm \right] \\ &= \dot{\theta} \left[ \left( - \int zx dm \right) \hat{i} + \left( - \int zy dm \right) \hat{j} + \left( \int (x^2 + y^2) dm \right) \hat{k} \right].\end{aligned}$$

To ‘un-clutter’ this expression, let’s define the following:

$$\begin{aligned}I_{xz}^O &= - \int xz dm \\ I_{yz}^O &= - \int yz dm \\ I_{zz}^O &= - \int (x^2 + y^2) dm\end{aligned}$$

So, now, we have for case (1)

$$(\vec{H}_O)_1 = I_{xz}^O \omega_z \hat{i} + I_{yz}^O \omega_z \hat{j} + I_{zz}^O \omega_z \hat{k}.$$

The substitutions we have defined form the elements of the third column of the inertia matrix, as we will see below in the general case. Let’s now move on to general 3-D rigid body motion and infer the first and second columns of the inertia matrix.

In general, the angular velocity of a rigid body is given by

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

So far, we have considered the special case above,  $\vec{\omega} = \omega_z \hat{k}$ . But, we could have looked at  $\vec{\omega} = \omega_x \hat{i}$ , case (2), and, similarly, would obtain instead the following angular momentum about point  $O$

$$(\vec{H}_O)_2 = I_{xx}^O \omega_x \hat{i} + I_{yx}^O \omega_x \hat{j} + I_{zx}^O \omega_x \hat{k}.$$

Likewise, for  $\vec{\omega} = \omega_y \hat{j}$ , case (3), we would obtain

$$(\vec{H}_O)_3 = I_{xy}^O \omega_y \hat{i} + I_{yy}^O \omega_y \hat{j} + I_{zy}^O \omega_y \hat{k}.$$

Finally, for  $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$ , we obtain

$$\begin{aligned}\vec{H}_O &= (\vec{H}_O)_1 + (\vec{H}_O)_2 + (\vec{H}_O)_3 \\ &= I_{xz}^O \omega_z \hat{i} + I_{yz}^O \omega_z \hat{j} + I_{zz}^O \omega_z \hat{k} \\ &\quad + I_{xx}^O \omega_x \hat{i} + I_{yx}^O \omega_x \hat{j} + I_{zx}^O \omega_x \hat{k} \\ &\quad + I_{xy}^O \omega_y \hat{i} + I_{yy}^O \omega_y \hat{j} + I_{zy}^O \omega_y \hat{k}.\end{aligned}$$

Collecting components, we get

$$\vec{H}_O = H_{Ox} \hat{i} + H_{Oy} \hat{j} + H_{Oz} \hat{k}$$

where

$$H_{Ox} = I_{xx}^O \omega_x + I_{xy}^O \omega_y + I_{xz}^O \omega_z$$

$$H_{Oy} = I_{yx}^O \omega_x + I_{yy}^O \omega_y + I_{yz}^O \omega_z$$

$$H_{Oz} = I_{zx}^O \omega_x + I_{zy}^O \omega_y + I_{zz}^O \omega_z.$$

We can combine the above results into a matrix representation. Representing  $\vec{H}_O$  as a column vector, we can re-write the above set of three equations as a product of a matrix and the angular velocity written as a column vector.

$$\begin{bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{bmatrix} = \begin{bmatrix} I_{xx}^O & I_{xy}^O & I_{xz}^O \\ I_{yx}^O & I_{yy}^O & I_{yz}^O \\ I_{zx}^O & I_{zy}^O & I_{zz}^O \end{bmatrix} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We define the coefficient matrix above to be the *moment of inertia matrix* about point  $O$

$$[\mathbf{I}^O] = \begin{bmatrix} I_{xx}^O & I_{xy}^O & I_{xz}^O \\ I_{yx}^O & I_{yy}^O & I_{yz}^O \\ I_{zx}^O & I_{zy}^O & I_{zz}^O \end{bmatrix}.$$

whose components are

$$\begin{aligned}I_{xx}^O &= \int (y^2 + z^2) dm & I_{xy}^O &= - \int xy dm & I_{xz}^O &= - \int xz dm \\ I_{yx}^O &= - \int yx dm & I_{yy}^O &= \int (x^2 + z^2) dm & I_{yz}^O &= - \int yz dm \\ I_{zx}^O &= - \int zx dm & I_{zy}^O &= - \int zy dm & I_{zz}^O &= \int (x^2 + y^2) dm.\end{aligned}$$

By inspection, one can see that  $I_{xy}^O = I_{yx}^O$ ,  $I_{xz}^O = I_{zx}^O$ , and  $I_{yz}^O = I_{zy}^O$ . Thus, the inertia matrix is symmetric; i.e.,  $[\mathbf{I}^O]^T = [\mathbf{I}^O]$ . So, there are always at most only six, not nine, independent components in the inertia matrix to compute.

## 11.4 THEORY

### 3-D parallel axis theorem

In three dimensions, the two matrices  $[I^O]$  and  $[I^{cm}]$  are related to each other in a way similar to the two-dimensional case. Since the inertia matrix has six independent entries in it, the derivation involves six integrals. Let's look at a typical term on the diagonal, say,  $I_{zz}^O$  and a typical off-diagonal term, say,  $I_{xy}^O$ . The calculation for the other terms is similar with a simple change of letters in the subscript notation.

First,  $I_{zz}^O = \int (x_{/O}^2 + y_{/O}^2) dm = I_{zz}^{cm} + m(x_{cm/O}^2 + y_{cm/O}^2)$  by exactly the same reasoning used to derive the 2-D parallel axis theorem. We *cannot* do the last line in that derivation, however, since  $r_{cm/O}^2 = x_{cm/O}^2 + y_{cm/O}^2 + z_{cm/O}^2 \neq x_{cm/O}^2 + y_{cm/O}^2$ , because now, for three-dimensional objects,  $z_{cm/O} \neq 0$ .

Now, let's look at an off-diagonal term.

$$\begin{aligned} I_{xy}^O &= - \int x_{/O} y_{/O} dm \\ &= - \int \underbrace{(x_{cm/O} + x_{/cm})}_{x_{/O}} \underbrace{(y_{cm/O} + y_{/cm})}_{y_{/O}} dm \\ &= -x_{cm/O} y_{cm/O} \underbrace{\int dm}_{m} - y_{cm/O} \underbrace{\int x_{/cm} dm}_{0} \end{aligned}$$

$$\begin{aligned} &-x_{cm/O} \underbrace{\int y_{/cm} dm}_{0} - \underbrace{\int x_{/cm} y_{/cm} dm}_{-I_{xy}^{cm}} \\ &= -mx_{cm/O} y_{cm/O} + I_{xy}^{cm} \end{aligned}$$

Similarly, we can calculate the other terms to get the whole 3-D parallel axis theorem.

$$[I^O] = [I^{cm}] +$$

$$m \begin{bmatrix} y_{cm/O}^2 + z_{cm/O}^2 & -x_{cm/O} y_{cm/O} & -x_{cm/O} z_{cm/O} \\ -x_{cm/O} y_{cm/O} & x_{cm/O}^2 + z_{cm/O}^2 & -y_{cm/O} z_{cm/O} \\ -x_{cm/O} z_{cm/O} & -y_{cm/O} z_{cm/O} & x_{cm/O}^2 + y_{cm/O}^2 \end{bmatrix}.$$

Again, one can think of this result as follows. The moment of inertia matrix about point  $O$  is the same as that for parallel axes through the center of mass *plus* the moment of inertia matrix for a point mass at the center of mass.

### Relation between 2-D and 3-D parallel axis theorems

The (3,3) (lower right corner) element in the matrix of the 3-D parallel axis theorem is the 2-D parallel axis theorem.

**SAMPLE 11.10** A uniform rod of mass  $m = 2 \text{ kg}$  and length  $\ell = \frac{1}{2} \text{ m}$  is pivoted at one of its ends. At the instant shown, the rod is in the  $xy$ -plane and makes an angle  $\theta = 45^\circ$  with the  $x$ -axis. Find the moment of inertia matrix  $[\mathbf{I}^0]$  for the rod.

**Solution** The moment of inertia matrix  $[\mathbf{I}^0]$  for a continuous system is given by

$$[\mathbf{I}^0] = \int_{\text{over all mass}} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} dm.$$

Thus we need to carry out the integrals for the rod to find each component of the inertia matrix  $[\mathbf{I}^0]$ . Let us consider an infinitesimal length element  $dl$  of the rod at distance  $l$  from O (see Fig 11.36). The mass of this element is  $dm = \underbrace{\frac{m}{\ell}}_{\text{mass/length}} \cdot dl$ ,

where  $m$  is the total mass.

The coordinates of this element are

$$x = l \cos \theta, \quad y = l \sin \theta, \quad z = 0 \quad (\text{since the rod is in the } xy\text{-plane}).$$

Therefore,

$$\begin{aligned} I_{xx} &= \int_m (y^2 + \underbrace{z^2}_0) dm = \int_0^\ell l^2 \sin^2 \theta \cdot \underbrace{\frac{m}{\ell} dl}_{dm} \\ &= \frac{m}{\ell} \sin^2 \theta \int_0^\ell l^2 dl = \frac{m}{\ell} \sin^2 \theta \cdot \frac{\ell^3}{3} = \frac{m\ell^2}{3} \sin^2 \theta. \end{aligned}$$

Similarly,

$$\begin{aligned} I_{yy} &= \int_m (x^2 + \underbrace{z^2}_0) dm = \int_0^\ell l^2 \cos^2 \theta \cdot \frac{m}{\ell} dl = \frac{m\ell^2}{3} \cos^2 \theta. \\ I_{zz} &= \int_m (x^2 + y^2) dm = \int_0^\ell l^2 \cdot \frac{m}{\ell} dl = \frac{m\ell^2}{3}. \\ I_{xy} &= - \int_m xy dm = - \int_0^\ell l^2 \cos \theta \sin \theta \cdot \frac{m}{\ell} dl = - \frac{m\ell^2}{3} \cos \theta \sin \theta. \\ I_{xz} &= - \int_m x \underbrace{z}_0 dm = 0. \quad I_{yz} = - \int_m y \underbrace{z}_0 dm = 0. \end{aligned}$$

Thus,

$$[\mathbf{I}^0] = \frac{m\ell^2}{3} \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta & \cos^2 \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting the values of  $m$ ,  $\ell$  and  $\theta$ , we get

$$[\mathbf{I}^0] = 0.083 \text{ kg} \cdot \text{m}^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

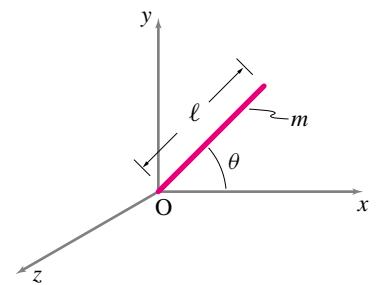


Figure 11.35: (Filename:sfig4.6.3)

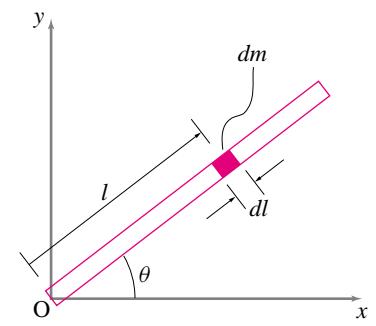


Figure 11.36: (Filename:sfig4.6.3a)

△

## 11.4 Mechanics using the moment of inertia matrix

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from the inside cover. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. Linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center of mass of the system of interest. One would like something so simple for the calculation of angular momentum.

We are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies, the scalar moment of inertia from Chapter 7 is all we need. The luck is not so great for 3-D rigid bodies but still there is some simplification<sup>①</sup>. The simplification is to use the moment of inertia matrix  $[I^{cm}]$  from the previous section. One may have to do a sum or integral to find  $I \equiv I_{zz}^{cm}$  or  $[I^{cm}]$  if an adequate table is not handy. But once you know  $[I^{cm}]$ , say, you need not work with the integrals to evaluate angular momentum and its rate of change. Assuming that you are comfortable calculating and looking-up moments of inertia, we proceed to use it for the purposes of studying mechanics.

Lets now consider again the conically swinging rod of Fig. 11.19 on page 650. Method 1 for evaluating  $\dot{\vec{H}}_O$  was evaluating the sums directly. If we accept the formulae presented for rigid bodies in Table I at the back of the book, we can find all of the motion quantities by setting  $\vec{\omega} = \omega \hat{k}$  and  $\vec{\alpha} = \vec{0}$ .

*Method 2 of evaluating  $\dot{\vec{H}}_O$ : using the xyz moment of inertia matrix about point O*

In section 11.1 we examined the conical swinging of a straight uniform rod. Now let's look at that example again using its moment of inertia matrix. For a rigid body in constant rate circular motion about an axis through  $O$ ,

$$\dot{\vec{H}}_O = \vec{\omega} \times \vec{H}_O$$

because the  $\vec{H}_O$  vector rotates with the body. For a rigid body rotating about point  $O$ ,

$$\vec{H}_O = [I^O] \cdot \vec{\omega}.$$

We assumed at the outset that

$$\vec{\omega} = \omega \hat{k} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$

So, the only trick is to find  $[I^O]$  for a rod in the configuration shown. We recall, look up, or believe for now that

$$[I^O] = \begin{bmatrix} I_{xx}^O & I_{xy}^O & I_{xz}^O \\ I_{yx}^O & I_{yy}^O & I_{yz}^O \\ I_{zx}^O & I_{zy}^O & I_{zz}^O \end{bmatrix}.$$

$$I_{xy}^O = - \int_0^x y dm = 0 \quad (x = 0, \text{all mass in the } yz \text{ plane})$$

$$\begin{aligned}
I_{xz}^O &= - \int_0^{\ell} \underbrace{x}_0 z dm = 0 && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
I_{yy}^O &= \int_0^{\ell} (\underbrace{x^2}_0 + z^2) dm && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int_0^{\ell} \underbrace{(-s \cdot \cos \phi)^2}_z \rho ds \\
&= \cos^2 \phi \rho \frac{\ell^3}{3} = \cos^2 \phi m \frac{\ell^2}{3} \\
I_{zz}^O &= \int_0^{\ell} (\underbrace{x^2}_0 + y^2) dm && (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int_0^{\ell} \underbrace{(s \cdot \sin \phi)^2}_y \rho ds \\
&= \sin^2 \phi \rho \frac{\ell^3}{3} = \sin^2 \phi m \frac{\ell^2}{3} \\
I_{xx}^O &= \int (y^2 + z^2) dm && (\text{both integrals have been evaluated above}) \\
&= I_{yy}^O + I_{zz}^O && (\text{perpendicular axis theorem}) \\
&= (\cos^2 \phi + \sin^2 \phi)m \frac{\ell^2}{3} \\
&= m \frac{\ell^2}{3} \\
I_{yz}^O &= - \int yz dm \\
&= - \int (s \cdot \sin \phi)(-s \cdot \cos \phi)^2 \rho ds \\
&= \sin \phi \cos \phi \cdot \rho \frac{\ell^3}{3} = \sin \phi \cos \phi \cdot m \frac{\ell^2}{3}.
\end{aligned}$$

Putting these terms all together in the matrix, we get

$$[\mathbf{I}^O] = m \frac{\ell^2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2 \phi & \cos \phi \sin \phi \\ 0 & \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}.$$

Now we can calculate  $\vec{H}_O$  as

$$\begin{aligned}
\vec{H}_O &= [\mathbf{I}^O] \cdot \vec{\omega} \\
&= m \frac{\ell^2}{3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2 \phi & \cos \phi \sin \phi \\ 0 & \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}}_{[\mathbf{I}^O]} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}}_{\vec{\omega}} \\
&= m \frac{\ell^2}{3} \begin{bmatrix} 0 \\ \omega \cos \phi \sin \phi \\ \omega \sin^2 \phi \end{bmatrix} \\
&= m \omega \frac{\ell^2}{3} \sin \phi (\cos \phi \hat{j} + \sin \phi \hat{k}).
\end{aligned}$$

You may notice, by the way, that for this problem,  $\vec{H}_O$  (in the direction of  $\cos \phi \hat{j} + \sin \phi \hat{k}$ ) is perpendicular to the rod (in the direction of  $\sin \phi \hat{j} - \cos \phi \hat{k}$ ). So  $\vec{H}_O$  is not in the direction of  $\vec{\omega}$ . ①

①  $\vec{H}_O$  is only parallel to  $\vec{\omega}$  if rotation is about one of the principal axes (eigen-vector directions) of the inertia matrix.

Now we calculate  $\dot{\vec{H}}_O$  as

$$\begin{aligned}\dot{\vec{H}}_O &= \vec{\omega} \times \vec{H}_O \\ &= (\omega \hat{k}) \times \left[ \omega m \frac{\ell^2}{3} (\cos \phi \sin \phi \hat{j} + \sin^2 \phi \hat{k}) \right] \\ &= -m \frac{\ell^2}{3} \cos \phi \sin \phi \omega^2 \hat{i},\end{aligned}$$

the same result we got before.

*Method 3 of calculating  $\dot{\vec{H}}_O$ : using  $[I^O]$  and a coordinate system lined up with the rod*

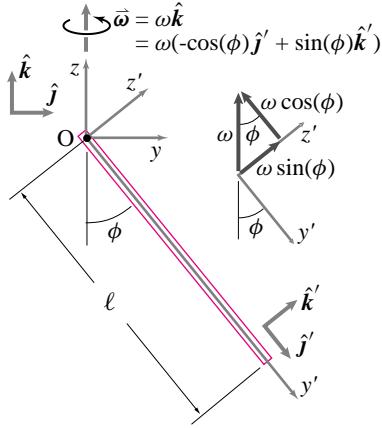


Figure 11.37: The spherical pendulum using  $x'y'z'$  axes aligned with the rod.

(Filename:tfigure4.spherical.rotaxis)

Here, as suggested above, we redo the problem using a rotated set of axes better aligned with the rod. This method makes calculation of the moment of inertia matrix quite a bit easier — we can even look it up in a table — but makes the determination of  $\vec{\omega}$  a little harder. We are stuck finding the components of  $\vec{\omega}$  in a rotated coordinate system. Referring to the table of moment of inertias on the inside of the back cover and taking care because different coordinates are used, the moment of inertia matrix about point  $O$  for a thin uniform rod, in terms of the rotated coordinates is

$$[I^O]_{x'y'z'} = m \frac{\ell^2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and the angular velocity in rotated coordinates is  $\vec{\omega} = -\omega \cos \phi \hat{j}' + \omega \sin \phi \hat{k}'$ , which can be written in component form as

$$[\vec{\omega}]_{x'y'z'} = \omega \begin{bmatrix} 0 \\ -\cos \phi \\ \sin \phi \end{bmatrix}.$$

We calculate the angular momentum about point  $O$  as

vector equation, independent of coordinates

$\vec{H}_O = [I^O] \cdot \vec{\omega}$

$\begin{bmatrix} H_{Ox'} \\ H_{Oy'} \\ H_{Oz'} \end{bmatrix} = m \frac{\ell^2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{x'y'z'} \cdot \omega \begin{bmatrix} 0 \\ -\cos \phi \\ \sin \phi \end{bmatrix}_{x'y'z'}$

all components in  $x'y'z'$  coordinate system

$\begin{bmatrix} H_{Ox'} \\ H_{Oy'} \\ H_{Oz'} \end{bmatrix} = m \frac{\ell^2}{3} \omega \begin{bmatrix} 0 \\ 0 \\ \sin \phi \end{bmatrix}_{x'y'z'}$

Finally, we calculate  $\dot{\vec{H}}_O$  as

$$\dot{\vec{H}}_O = \vec{\omega} \times \vec{H}_O$$

$$\begin{aligned}
 &= [\omega(-\cos \phi \hat{\mathbf{j}}' + \sin \phi \hat{\mathbf{k}}')] \times (\omega m \frac{\ell^2}{3} \sin \phi \hat{\mathbf{k}}') \\
 &= -\sin \phi \cos \phi m \omega^2 \frac{\ell^2}{3} \hat{\mathbf{i}}'.
 \end{aligned}$$

This answer is again the same as what we got before because the  $x$  and  $x'$  axis are coincident so  $\hat{\mathbf{i}}' = \hat{\mathbf{i}}$ .

### The multitude of ways to calculate $\dot{\mathbf{H}}_O$

We just showed three ways to calculate  $\dot{\mathbf{H}}_O$  for a conically swinging stick, but there are many more. You can get a sense of the possibilities by studying table I summarizing momenta and energy in the back of the book. Here are three basic choices.

$$(1) \dot{\mathbf{H}}_O = \int \vec{r}_{/O} \times \vec{a} dm,$$

$$(2) \dot{\mathbf{H}}_O = \vec{\omega} \times \bar{\mathbf{H}}_O, \quad \text{or}$$

$$(3) \dot{\mathbf{H}}_O = \vec{r}_{cm/O} \times \vec{a}_{cm} m_{tot} + \dot{\mathbf{H}}_{cm}.$$

Choice (3) is the safest choice to make if you are in doubt, since it is the only one of the three choices that does not depend on point  $O$  being a fixed point (which it was for this example). For option (2) above, we can calculate  $\dot{\mathbf{H}}_O$  various ways as

$$(a) \dot{\mathbf{H}}_O = \int \vec{r}_{/O} \times \vec{v} dm \quad \text{or},$$

$$(b) \dot{\mathbf{H}}_O = [I^O] \cdot \vec{\omega} \quad \text{or},$$

$$(c) \dot{\mathbf{H}}_O = \vec{r}_{cm/O} \times \vec{v}_{cm} m_{tot} + \dot{\mathbf{H}}_{cm}.$$

For option (3) above, we can calculate  $\dot{\mathbf{H}}_{cm}$  as

$$(d) \dot{\mathbf{H}}_{cm} = \int \vec{r}_{/cm} \times \vec{a}_{/cm} dm \quad \text{or},$$

$$(e) \dot{\mathbf{H}}_{cm} = \vec{\omega} \times \bar{\mathbf{H}}_{cm}.$$

For (c) and (e) above, we can calculate  $\dot{\mathbf{H}}_{cm}$  as

$$(f) \dot{\mathbf{H}}_{cm} = \int \vec{r}_{/cm} \times \vec{v}_{/cm} dm \quad \text{or},$$

$$(g) \dot{\mathbf{H}}_{cm} = [I^{cm}] \cdot \vec{\omega}.$$

For either of options (b) or (g), we can calculate  $[I]$  relative to the  $xyz$  axes (as we did for the second method on the previous pages) or relative to some rotated axes better aligned with the rod (as we did for the third method on the previous pages. )

As you can surmise from all the choices above, the list of options for the calculation of  $\dot{\mathbf{H}}_O$  is too long and boring to show here.

The pros and cons of these methods depend on the problem at hand. If you want to avoid integration you are pretty much stuck using either  $[I^O]$  or  $[I^{cm}]$  with (e) and (g) above). Avoiding the integrals depends on your having a table of moments of inertia (like table IV in the back of this book).

## Energy of things going in circles at variable rate

The energy only depends on the speeds of the parts of a system, not their accelerations. So, as with constant rate motion,

$$\begin{aligned}
 E_K &= \frac{1}{2} \int \underbrace{\vec{v}^2}_{\vec{v} \cdot \vec{v}} dm \\
 &= \frac{1}{2} \int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) dm \\
 &= \frac{1}{2} \omega^2 \int R^2 dm \\
 &\quad \boxed{R \text{ is the distance from the axis to the mass}} \\
 &= \frac{1}{2} \vec{\omega} \cdot [I^O] \cdot \vec{\omega} \\
 &= \underbrace{\frac{1}{2} \omega^2 I_{zz}^O}_{\text{For rotation about the } z \text{ axis the 9 terms in the matrix formula reduce to this one simple term.}}
 \end{aligned}$$

## The safest bet

The following are the most reliable (least prone to error) formulas for evaluating the motion quantities for a rigid body rotating about a fixed axes (they also apply to arbitrary motion of a rigid body).

$$\vec{H}_O = \vec{r}_{cm/o} \times m_{tot} \vec{v}_{cm} + [I^O] \cdot \vec{\omega} \quad (11.39)$$

$$\dot{\vec{H}}_O = \vec{r}_{cm/o} \times m_{tot} \vec{a}_{cm} + \vec{\omega} \times \underbrace{[I^{cm}] \cdot \vec{\omega}}_{\dot{\vec{H}}_{cm}} + [I^{cm}] \cdot \dot{\vec{\omega}} \quad (11.40)$$

$$E_K = \frac{1}{2} m_{tot} v_{cm}^2 + \vec{\omega} \cdot ([I^{cm}] \cdot \vec{\omega}) \quad (11.41)$$

Please survey table I at the back of the book which summarizes the ways of evaluating momenta and energy.

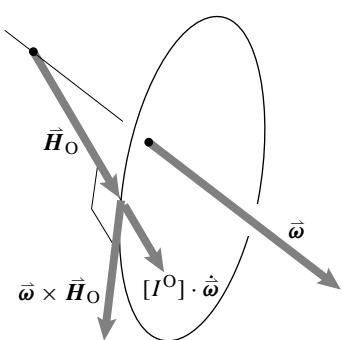


Figure 11.38: The two terms in the rate of change of angular momentum are shown.  
(Filename:tfigure5.7)

## The geometry of $\dot{\vec{H}}_O$

It is possible to understand the formula for the change of angular momentum geometrically. Here is one way of looking at it. For this example it is most direct to look at  $\dot{\vec{H}}_O$  calculated using  $[I^O]$ , but the same discussion would work with  $\dot{\vec{H}}_{cm}$  calculated using  $[I^{cm}]$ .

$$\begin{aligned}
 \dot{\vec{H}}_O = & \underbrace{\vec{\omega} \times \vec{H}_O}_{\text{Contribution from rotation of } \vec{H}_O} + \underbrace{[I^O] \cdot \dot{\vec{\omega}}}_{\text{Due to the changing length of } \vec{H}_O}
 \end{aligned}$$

The first term in the equation describes the rotation of the angular momentum vector. The second term describes its rate of change of length.

Since the body is spinning about a fixed axis, the orientation of the axis of rotation is not only fixed relative to the Newtonian frame, the room environment, but also relative to the body. At one instant of time we use a coordinate system to calculate  $\vec{H}_O$ . After the body has rotated a little, if we now used a coordinate system that rotated the same amount as the body, a coordinate system ‘glued’ to the body, we could calculate  $\vec{H}_O$  again.

This new calculation of  $\vec{H}_O$  will be almost identical to the calculation before the small rotation, however, because the moment of inertia matrix does not change in time relative to a coordinate system that moves with the body. Also, the coordinates of  $\vec{\omega}$  will be unchanged except for possibly a multiplication by a constant because the direction of  $\vec{\omega}$  doesn’t change. Thus the only change of  $\vec{H}_O$  as represented in this rotated coordinate system, is a possible change in its length due to a change in the spinning rate.

But this new coordinate system is a rotated coordinate system. To find the actual change in  $\vec{H}_O$  we need to take this rotation into account. To picture this rotation consider the special case when the rotation rate is constant. Then the vector  $\vec{H}_O$  is constant in the rotating coordinate system. That is, the vector  $\vec{H}_O$  rotates with the body.

So the net change in  $\vec{H}_O$  is a change due to rotation of the body added to a change due to the change in the rotation rate. For small angular changes, the direction of the first term is the same as the tangent to the circle that is traced by the tip of the angular momentum vector  $\vec{H}_O$  as drawn on the body. To approximate the first term, consider the following reasoning. First, let’s denote the first term by  $(\dot{\vec{H}}_O)_{\text{rot}} = \vec{\omega} \times \vec{H}_O$ , where the subscript ‘rot’ indicates that this term is the contribution to  $\dot{\vec{H}}_O$  from the rotation of  $\vec{H}_O$ . For small angular changes,  $(\dot{\vec{H}}_O)_{\text{rot}} = (d\vec{H}_O/dt)_{\text{rot}} = \vec{\omega} \times \vec{H}_O \approx (\Delta \vec{H}_O)_{\text{rot}}/\Delta t$ . Thus, the term due to rotation is approximately, for small  $\Delta t$ ,  $(\Delta \vec{H}_O)_{\text{rot}} = \Delta t(\vec{\omega} \times \vec{H}_0)$ .

The contribution to  $\dot{\vec{H}}_O$  due to change of length of  $\vec{H}_O$  is  $[\mathbf{I}^O] \cdot \vec{\dot{\omega}}$ . Similarly, this term is approximately, for small  $\Delta t$ ,  $\Delta t([\mathbf{I}^O] \cdot \vec{\dot{\omega}})$ .

## Rotation of a rigid body about a fixed axis: the general case

Consider a general rigid body of mass  $m$  and moment of inertia matrix with respect to the center of mass  $[\mathbf{I}^{\text{cm}}]$  rotating about a fixed axis. Without loss of generality, let the axis of rotation be the  $\hat{k}$  axis.

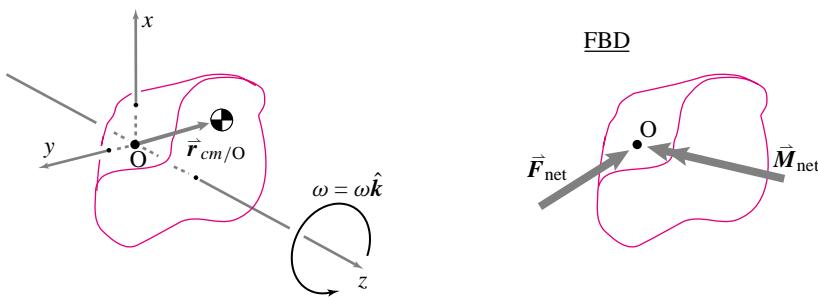


Figure 11.39: Free body diagram of a general rigid body rotating about the  $z$ -axis.

(Filename:tfigure5.gen.rigid.body)

### Linear momentum balance

Referring to the free body diagram of the rigid body, linear momentum balance gives

$$\begin{aligned}\sum \vec{F} &= \dot{\vec{L}} \\ \vec{F}_{net} &= m\vec{a}_{cm} \\ &= m[\omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O}) + \dot{\omega}\hat{k} \times \vec{r}_{cm/O}].\end{aligned}\quad (11.42)$$

The first term on the right hand side of equation 11.42, the centripetal term, is directed from the center of mass ( $\bullet$ ) through the axis of rotation; that is, it lies in the  $xy$ -plane (or has no  $\hat{k}$  component). The second term on the right hand side of equation 11.42, the tangential term, is normal to the plane determined by the axis and the center of mass. It is tangent to the circle that the center of mass travels on. It is zero if the center of mass is on the axis.

### Angular momentum balance

Angular momentum balance about point  $O$  gives

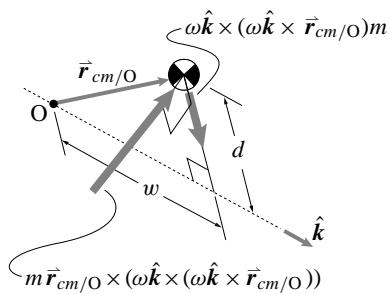


Figure 11.40: The first part of term (i)

(Filename:tfigure5.term1.a)

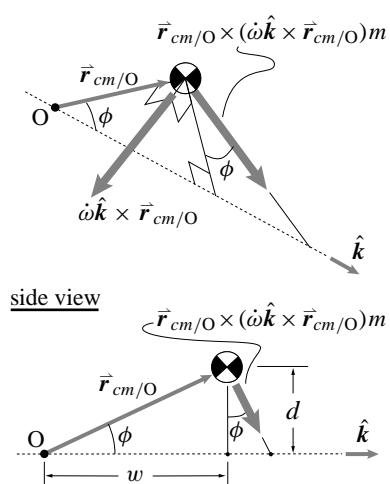


Figure 11.41: The second part of term (i)

(Filename:tfigure5.term1.b)

Let's look at each of the three terms on the right hand side in turn.

#### term (i)

The first term (i) on the right hand side of equation 11.43 is

$$\boxed{\vec{a}_{cm} = \omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O}) + \dot{\omega}\hat{k} \times \vec{r}_{cm/O}}$$

↓

$$\begin{aligned}\text{term (i)} &= \vec{r}_{cm/O} \times m \underbrace{\vec{a}_{cm}}_{\text{term (i)}} \\ &= m[\vec{r}_{cm/O} \times (\omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O})) \\ &\quad + \vec{r}_{cm/O} \times (\dot{\omega}\hat{k} \times \vec{r}_{cm/O})]\end{aligned}$$

Now, let's consider the two parts of term (i) in turn. The first part of term (i) is in the direction of  $-\hat{k} \times \vec{r}_{cm/O}$ . For example, if the center of mass is in the  $xz$  plane, this contribution to  $\vec{M}_{net}$  is in the  $-\hat{j}$  direction and could be accommodated by reaction forces on the axis in the  $\hat{i}$  and  $-\hat{i}$  directions. Now, the second part of term (i) can be decomposed into a part along the  $\hat{k}$  direction and a part perpendicular to  $\hat{k}$  (along  $d$ ). The part along  $\hat{k}$  is

$$\vec{r}_{cm/O} \times (\dot{\omega}\hat{k} \times \vec{r}_{cm/O}) \cdot \hat{k} = md^2\dot{\omega}\hat{k}.$$

The part of term (i) perpendicular to  $\hat{k}$  has magnitude  $m\dot{\omega}dw$ .

### term (ii)

Now, let's look at the second term in equation 11.43, term (ii), and expand it.

$$\begin{aligned}\text{term (ii)} &= \omega \hat{k} \times [\mathbf{I}^{cm}] \cdot \omega \hat{k} \\ &= \omega \hat{k} \times \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\ &= \omega \hat{k} \times [I_{xz}^{cm} \omega \hat{i} + I_{yz}^{cm} \omega \hat{j} + I_{zz}^{cm} \omega \hat{k}] \\ &= I_{xz}^{cm} \omega^2 \hat{j} - I_{yz}^{cm} \omega^2 \hat{i}\end{aligned}$$

So, the second term in equation 11.43, term (ii), has *no*  $\hat{k}$  component. The  $\hat{i}$  and  $\hat{j}$  components are due to the off-diagonal terms in  $[\mathbf{I}^{cm}]$ . The off-diagonal terms are responsible for dynamic imbalance.

### term (iii)

Now, the third term in equation 11.43, term (iii), is

$$\begin{aligned}[\mathbf{I}^{cm}] \cdot \dot{\omega} \hat{k} &= \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \\ &= I_{xz}^{cm} \dot{\omega} \hat{i} + I_{yz}^{cm} \dot{\omega} \hat{j} + I_{zz}^{cm} \dot{\omega} \hat{k}.\end{aligned}$$

### Putting terms (i), (ii), and (iii) back together

Now, let's put the three terms back together into the equation of angular momentum balance about point  $O$ , equation 11.43. To help with the interpretation, assume the center of mass is in the  $xz$  plane. So, we start with

$$\vec{M}_{net} = \dot{\vec{H}}_O.$$

Breaking this vector equation into components, we get

$$\begin{aligned}M_{xx} &= \vec{M}_{net} \cdot \hat{i} = \dot{\vec{H}}_O \cdot \hat{i} \\ M_{yy} &= \vec{M}_{net} \cdot \hat{j} = \dot{\vec{H}}_O \cdot \hat{j} \\ M_{zz} &= \vec{M}_{net} \cdot \hat{k} = \dot{\vec{H}}_O \cdot \hat{k}.\end{aligned}$$

Now, using all the results so far, we find the torques about the  $x$ ,  $y$ , and  $z$  axes. For the  $z$ -axis,

$d = \text{distance of cm from axis of rotation}$

$M_z = \underbrace{[m d^2 + I_{zz}^{cm}]}_{\substack{\text{sometimes called } 'I^0' \\ \text{ }} \cdot \dot{\omega}.}$

(11.44)

The only torque about the  $z$ -axis, the axis of rotation, is due to the acceleration about that axis.

Next, for the  $x$ -axis,

$$M_x = -I_{yz}^{cm}\omega^2 + I_{xz}^{cm}\dot{\omega} + m\dot{\omega}dw. \quad (11.45)$$

There is a torque about the  $x$ -axis due to off-diagonal terms in  $[I^{cm}]$ . One term is the dynamic imbalance term  $I_{yz}^{cm}$  and the other is associated with angular acceleration. (Recall, the center of mass is on the  $xz$  plane.) There is also a term due to the center of mass tangential acceleration since the center of mass is on a plane that does not contain point  $O$ .

Finally, the torque about the  $y$ -axis is

$$M_y = I_{xz}^{cm}\omega^2 + I_{yz}^{cm}\dot{\omega} + (-m\omega^2dw). \quad (11.46)$$

Here, we have nearly the same types of terms as in equation 11.45. In this case, though, the centripetal acceleration causes the center of mass motion to contribute to the torque.

So, if

$$\vec{\omega} = \omega \hat{k} \text{ and } \vec{\dot{\omega}} = \dot{\omega} \hat{k}$$

then

$$\begin{aligned} \vec{M}_O = \dot{\vec{H}}_O &= \left( -I_{yz}^{cm}\omega^2 + (I_{xz}^{cm} + mdw)\dot{\omega} \right) \hat{i} \\ &\quad + ((I_{xz}^{cm} - mdw)\omega^2 + I_{yz}^{cm}\dot{\omega}) \hat{j} \\ &\quad + (md^2 + I_{zz}^{cm})\dot{\omega} \hat{k}. \end{aligned}$$

**SAMPLE 11.11** A scalar times a vector is a vector. A matrix times a vector is a vector. What is the difference? Find the vectors  $\vec{H}_1 = I_G \vec{\omega}$  and  $\vec{H}_2 = [\mathbf{I}_G] \vec{\omega}$ , if  $\vec{\omega} = 2 \text{ rad/s} \hat{i} + 3 \text{ rad/s} \hat{j}$ ,  $I_G = 10 \text{ kg m}^2$  and  $[\mathbf{I}_G] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix} \text{ kg m}^2$ .

Draw  $\vec{\omega}$ ,  $\vec{H}_1$  and  $\vec{H}_2$ .

### Solution

$$\begin{aligned}\vec{H}_1 &= I_G \vec{\omega} \\ &= 10 \text{ kg m}^2 (2 \text{ rad/s} \hat{i} + 3 \text{ rad/s} \hat{j}) \\ &= (20\hat{i} + 30\hat{j}) \text{ kg m}^2/\text{s}.\end{aligned}$$

$$\begin{aligned}\vec{H}_2 &= [\mathbf{I}_G] \cdot \vec{\omega} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix} \text{ kg m}^2 \cdot \begin{Bmatrix} 2 \\ 3 \\ 0 \end{Bmatrix} \text{ rad/s} \\ &= (10\hat{i} + 15\hat{j} - 6\hat{k}) \text{ kg m}^2/\text{s}.\end{aligned}$$

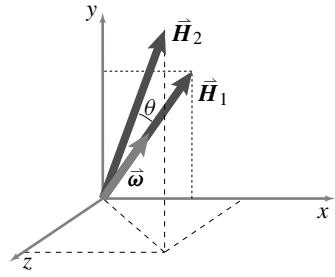


Figure 11.42: (Filename:sfig1.2.12)

These two vectors,  $\vec{H}_1$  and  $\vec{H}_2$ , are shown along with  $\vec{\omega}$  in Fig. 11.42. Note that  $\vec{H}_1$  has the same direction as  $\vec{\omega}$  but  $\vec{H}_2$  does not.  $\vec{H}_1$  and  $\vec{\omega}$  are both in the  $xy$ -plane but  $\vec{H}_2$  is not; it is in 3-D.

**Comments:** Multiplying a vector by a scalar does not change the direction of the vector but multiplying by a matrix does change the direction, in general. Find the angles between  $\vec{\omega}$  and  $\vec{H}_1$  and between  $\vec{\omega}$  and  $\vec{H}_2$  to convince yourself.

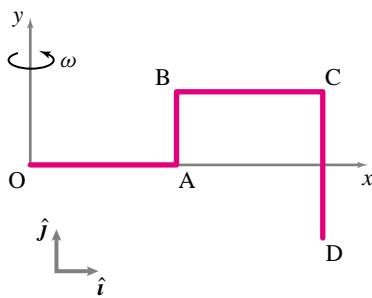


Figure 11.43: (filename:sfig4.6.5)

**SAMPLE 11.12** The composite rod OABCD shown in Figure 11.43 is made up of three identical rods OA, BC, CD of mass 0.5 kg and length 20 cm each, and the rod AB which is half of rod CD. The composite rod goes in circles about the y-axis at a constant rate  $\omega = 5 \text{ rad/s}$ . Find the angular momentum and the rate of change of angular momentum of the rod at the instant shown (*i.e.*, when the rod is in the  $xy$ -plane).

- Are all the components of  $[I^O]$  necessary to compute  $\vec{H}_O$  and  $\dot{\vec{H}}_O$ ? Find  $[I^O]$  or the necessary components of  $[I^O]$ .
- Find the angular momentum  $\vec{H}_O$ .
- Find the rate of change of angular momentum  $\dot{\vec{H}}_O$ .
- If the rod were rotating about the z-axis instead, (*i.e.*, if the motion were in the  $xy$ -plane) which components of  $[I^O]$  would be required to find  $\vec{H}_O$ ? What would be the value of  $\dot{\vec{H}}_O$  in that case?

**Solution** Since the rod rotates about the y-axis, and O is a fixed point on this axis,

$$\vec{\omega} = \omega \hat{j} \quad \text{and} \quad \vec{H}_O = [I^O] \cdot \vec{\omega}.$$

(a) Since

$$\begin{aligned} \vec{H}_O &= \begin{bmatrix} I_{xx}^O & I_{xy}^O & I_{xz}^O \\ I_{xy}^O & I_{yy}^O & I_{yz}^O \\ I_{xz}^O & I_{yz}^O & I_{zz}^O \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\ &= I_{xy}^O \omega \hat{i} + I_{yy}^O \omega \hat{j} + I_{yz}^O \omega \hat{k} \\ &= (I_{xy}^O \hat{i} + I_{yy}^O \hat{j} + I_{yz}^O \hat{k}) \omega, \end{aligned} \quad (11.47)$$

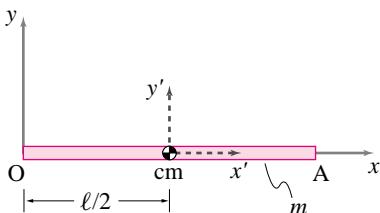


Figure 11.44: (filename:sfig4.6.5a)

we only need to find three components of  $[I^O]$  to compute  $\vec{H}_O$ , namely  $I_{xy}^O$ ,  $I_{yy}^O$  and  $I_{yz}^O$ .

We can compute these components by considering each rod individually. For any rod, the components can be calculated using the values of the components about the center of mass of the rod (see table IV in the back of the book) and then using the parallel axis theorem.

$$\begin{aligned} \text{Rod OA: } I_{yy}^O &= I_{y'y'}^{\text{cm}} + m \frac{l^2}{4} = \frac{1}{12} ml^2 + \frac{1}{4} ml^2 \\ I_{xy}^O &= \underbrace{I_{x'y'}^{\text{cm}}}_{0} + m(-x_{\text{cm}/O} \underbrace{y_{\text{cm}/O}}_{0}) = 0 \\ I_{yz}^O &= \underbrace{I_{y'z'}^{\text{cm}}}_{0} + m(-y_{\text{cm}/O} \underbrace{z_{\text{cm}/O}}_{0}) = 0. \end{aligned}$$

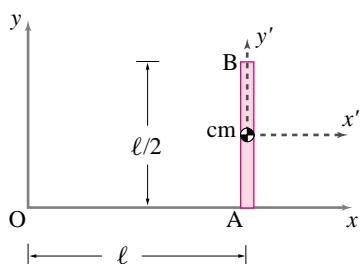


Figure 11.45: (filename:sfig4.6.5b)

$$\begin{aligned} \text{Rod AB: } I_{yy}^O &= \underbrace{I_{y'y'}^{\text{cm}}}_{0} + \frac{m}{2}(x_{\text{cm}/O}^2 + z_{\text{cm}/O}^2) = \frac{ml^2}{2} \\ I_{xy}^O &= \underbrace{I_{x'y'}^{\text{cm}}}_{0} + \frac{m}{2}(-x_{\text{cm}/O} \underbrace{y_{\text{cm}/O}}_{l/4}) = -\frac{ml^2}{8} \\ I_{yz}^O &= \underbrace{I_{y'z'}^{\text{cm}}}_{0} + m(-y_{\text{cm}/O} \underbrace{z_{\text{cm}/O}}_{0}) = 0. \end{aligned}$$

$$\begin{aligned}\text{Rod BC: } I_{yy}^O &= I_{y'y'}^{\text{cm}} + m(x_{\text{cm}/O}^2 + z_{\text{cm}/O}^2) = \frac{ml^2}{12} + m\frac{9l^2}{4} = \frac{7}{3}ml^2 \\ I_{xy}^O &= I_{x'y'}^{\text{cm}} + m(-x_{\text{cm}/O}y_{\text{cm}/O}) = -m\frac{3l}{2} \cdot \frac{l}{2} = -\frac{3ml^2}{4} \\ I_{yz}^O &= 0.\end{aligned}$$

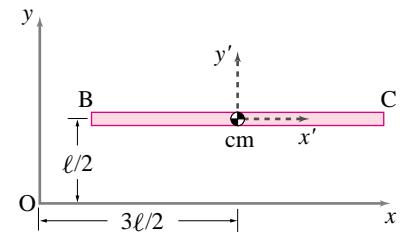


Figure 11.46: (Filename:sfig4.6.5c)

$$\begin{aligned}\text{Rod CD: } I_{yy}^O &= I_{y'y'}^{\text{cm}} + m(x_{\text{cm}/O}^2 + z_{\text{cm}/O}^2) = m(2l)^2 = 4ml^2 \\ I_{xy}^O &= 0 \\ I_{yz}^O &= 0.\end{aligned}$$

Thus for the entire rod,

$$\begin{aligned}I_{yy}^O &= \frac{1}{3}ml^2 + \frac{1}{2}ml^2 + \frac{7}{3}ml^2 + 4ml^2 = \frac{43}{6}ml^2 \\ I_{xy}^O &= -\frac{1}{8}ml^2 - \frac{3}{4}ml^2 = -\frac{7}{8}ml^2 \\ I_{yz}^O &= 0.\end{aligned}$$

$$I_{yy}^O = \frac{43}{6}ml^2, \quad I_{xy}^O = -\frac{7}{8}ml^2, \quad I_{yz}^O = 0.$$

(b)

$$\begin{aligned}\vec{H}_O &= (I_{yy}^O \hat{i} + I_{xy}^O \hat{j} + I_{yz}^O \hat{k})\omega \quad (\text{from Eqn 11.47}) \\ &= ml^2\omega(\frac{43}{6} \hat{j} - \frac{7}{8} \hat{i}) \\ &= 0.5 \text{ kg}(0.2 \text{ m})^2 \cdot 5 \text{ rad/s}(7.17 \hat{j} - 0.87 \hat{i}) \\ &= (0.717 \hat{j} - 0.087 \hat{i}) \text{ kg} \cdot \text{m}^2/\text{s}.\end{aligned}$$

$$\vec{H}_O = (0.717 \hat{j} - 0.087 \hat{i}) \text{ kg} \cdot \text{m}^2/\text{s}.$$

(c)

$$\begin{aligned}\dot{\vec{H}}_O &= \vec{\omega} \times \vec{H}_O = \omega \hat{j} \times ([I^O] \cdot \vec{\omega}) \\ &= 5 \text{ rad/s} \hat{j} \times (0.717 \hat{j} - 0.087 \hat{i}) \text{ kg} \cdot \text{m}^2/\text{s} \\ &= 0.435 \text{ N} \cdot \text{m} \hat{k}.\end{aligned}$$

$$\dot{\vec{H}}_O = 0.435 \text{ N} \cdot \text{m} \hat{k}.$$

(d) If the rod were rotating in the  $xy$ -plane, it would be planar circular motion.

The only component of  $[I^O]$  required for the calculation of  $\vec{H}_O = I_{zz}^O \vec{\omega}$  will be  $I_{zz}^O$ . Also,

$$\dot{\vec{H}}_O = \vec{\omega} \times \vec{H}_O = \vec{\omega} \times I_{zz}^O \vec{\omega} = 0$$

since  $\vec{\omega}$  is parallel to  $I_{zz}^O \vec{\omega}$ .

◀

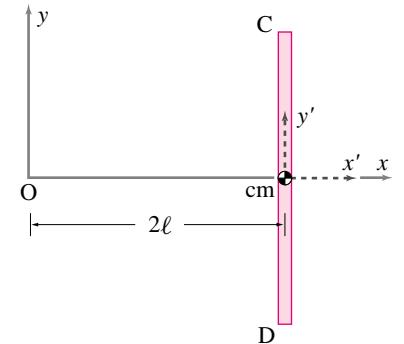


Figure 11.47: (Filename:sfig4.6.5d)

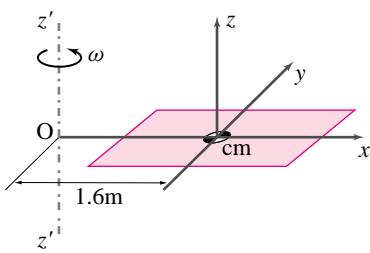


Figure 11.48: (Filename:sfig4.5.6)

**SAMPLE 11.13** A 0.5 kg uniform rectangular solar panel rotates about an off-centered axis  $z'z'$  with constant angular speed  $\omega = 1.5 \text{ rad/s}$ . Axis  $z'z'$  is parallel to the transverse  $z$ -axis of the plate and is 1.6 m away from the center of mass of the plate. The in-plane moments of inertia  $I_{xx}^{\text{cm}}$  and  $I_{yy}^{\text{cm}}$  of the plate are given:  $I_{xx}^{\text{cm}} = 4.0 \text{ kg}\cdot\text{m}^2$  and  $I_{yy}^{\text{cm}} = 12.4 \text{ kg}\cdot\text{m}^2$ . At the instant shown in Fig 11.48, calculate

- the linear momentum of the panel,
- the angular momentum of the panel, and
- the kinetic energy of the panel.

**Solution** Since the plate rotates about the  $z'z'$ -axis which is parallel to the  $z$ -axis,

$$\vec{\omega} = \omega \hat{k} = 1.5 \text{ rad/s} \hat{k}$$

(a) **Linear momentum:**

$$\begin{aligned} \vec{L} &= m_{\text{tot}} \vec{v}_{\text{cm}} = m(\vec{\omega} \times \vec{r}_{\text{cm}/O}) \\ &= 0.5 \text{ kg} \cdot (1.5 \text{ rad/s} \hat{k} \times 1.6 \text{ m} \hat{i}) \\ &= 1.2 \text{ kg}\cdot\text{m/s} \hat{j} \end{aligned}$$

$$\boxed{\vec{L} = 1.2 \text{ kg}\cdot\text{m/s} \hat{j}}$$

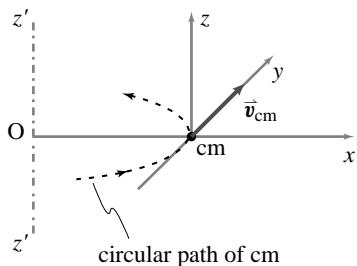


Figure 11.49: (Filename:sfig4.5.6a)

We can easily check the direction of  $\vec{L}$  since  $\vec{L} = m \vec{v}_{\text{cm}}$ , it has to be in the same direction as  $\vec{v}_{\text{cm}}$ . The center of mass goes in circles about O, therefore,  $\vec{v}_{\text{cm}}$  is tangential to the circular path, i.e. in the  $y$ -direction (see Fig 11.49).

(b) **Angular momentum:**

$$\vec{H} = I_{z'z'}^O \omega \hat{k}$$

Thus to find  $\vec{H}$ , we need to find  $I_{z'z'}^O$ . Since  $z'z' \parallel zz$ , we can use the parallel axis theorem to find  $I_{z'z'}^O$  if we know  $I_{zz}^{\text{cm}}$ . We are given the in-plane moments of inertia  $I_{xx}^{\text{cm}}$  and  $I_{yy}^{\text{cm}}$ . Therefore, from the perpendicular axis theorem:

$$I_{zz}^{\text{cm}} = I_{xx}^{\text{cm}} + I_{yy}^{\text{cm}} = (4.0 + 12.4) \text{ kg}\cdot\text{m}^2 = 16.4 \text{ kg}\cdot\text{m}^2.$$

Now using the parallel axis theorem,

$$\begin{aligned} I_{z'z'}^O &= I_{zz}^{\text{cm}} + M r_{\text{cm}/O}^2 \\ &= 16.4 \text{ kg}\cdot\text{m}^2 + 0.5 \text{ kg} (1.6 \text{ m})^2 = 17.68 \text{ kg}\cdot\text{m}^2. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \vec{H} &= 17.68 \text{ kg}\cdot\text{m}^2 \cdot (1.5 \text{ rad/s} \hat{k}) \\ &= 26.52 \text{ kg}\cdot\text{m}^2 / \hat{s} \hat{k}. \end{aligned}$$

$$\boxed{\vec{H} = 26.52 \text{ kg}\cdot\text{m}^2 / \hat{s} \hat{k}.}$$

(c) **Kinetic energy:**

$$\begin{aligned} E_K &= \frac{1}{2} I_{z'z'}^O \omega^2 = \frac{1}{2} (17.68 \text{ kg}\cdot\text{m}^2) (1.5 \text{ rad/s})^2 \\ &= 19.89 \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} = 19.89 \text{ N}\cdot\text{m} \\ &= 19.89 \text{ J} \end{aligned}$$

$$\boxed{E_K = 19.89 \text{ J}}$$



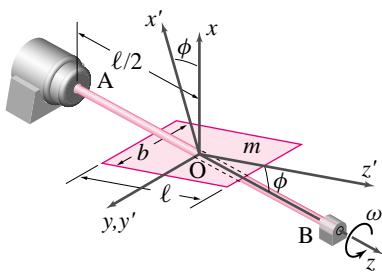


Figure 11.50: A rectangular plate of mass  $m$  rotates with shaft AB at a constant speed  $\vec{\omega} = \omega \hat{k}$ . A coordinate system  $x'y'z'$  is aligned with the *principal axes* of the plate.

(Filename:sfig4.6.8)

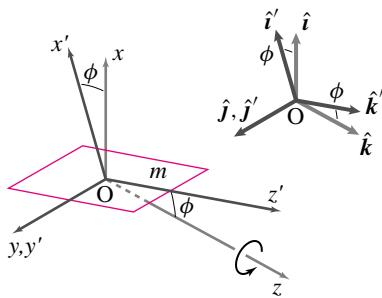


Figure 11.51: A line sketch of the rotating plate along with the two coordinate systems  $x'y'z'$  and  $xyz$ . The basis vectors associated with the two systems are also shown.

(Filename:sfig4.6.8a)

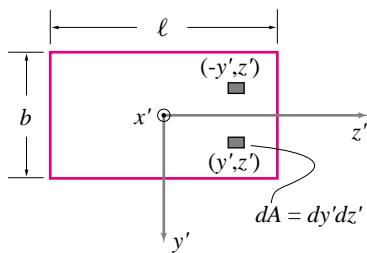


Figure 11.52: Symmetry about the  $z'$  axis implies that  $I_{y'z'} = 0$ . Same result holds if we consider the symmetry about the  $y'$  axis.

(Filename:sfig4.6.8b)

① You can use similar argument to find the off-diagonal terms in  $[I]$  whenever there is such symmetry with respect to the coordinate axes.

**SAMPLE 11.14** *The rotating crooked plate again.* For the crooked plate considered in Sample 11.7 and shown again in Fig. 11.50,

- compute the moment of inertia matrix  $[I]$  in the  $x'y'z'$  coordinate system,
- compute the rate of change of angular momentum  $\dot{\vec{H}}_O$  using the moment of inertia  $[I]_{x'y'z'}$  computed above,
- express  $\dot{\vec{H}}_O$  as a vector in the  $xyz$  coordinate system.

**Solution** A line sketch of the plate and the two coordinate systems attached to it are shown in Fig. 11.51. The set of basis vectors  $(\hat{i}, \hat{j}, \hat{k})$  and  $(\hat{i}', \hat{j}', \hat{k}')$  associated with coordinate systems  $xyz$  and  $x'y'z'$  respectively are shown separately for the sake of clarity. From the diagram of the two basis vector sets, we may write

$$\left. \begin{aligned} \hat{i}' &= \cos \phi \hat{i} - \sin \phi \hat{k} \\ \hat{j}' &= \hat{j} \\ \hat{k}' &= \sin \phi \hat{i} + \cos \phi \hat{k} \end{aligned} \right\}. \quad (11.48)$$

**(a) Calculation of  $[I]_{x'y'z'}$ :**

$$[I]_{x'y'z'} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix}.$$

Let us first consider the off diagonal terms of  $[I]_{x'y'z'}$ .

Since  $x' = 0$  on the entire plate (the plate is in the  $y'z'$  plane and the origin is on the plate.),

$$I_{x'y'} = \int_m \underbrace{x'}_0 y' dm = 0, \quad I_{x'z'} = \int_m \underbrace{x'}_0 z' dm = 0.$$

How about  $I_{y'z'}$ ? Well, you can calculate it two ways: (a) carry out the integration  $I_{y'z'} = \int_m y'z' dm$  over the entire plate mass and find that  $I_{y'z'} = 0$ , or (b) realize that for every mass element  $dm$  ( $= \frac{m}{lb} dA$ ) with coordinates  $(+y'z')$ , there exists another element  $dm$  at  $(-y'z')$  such that the sum of their contributions to the integral is zero. Therefore,  $I_{y'z'} = \int_m y'z' dm = 0$  ① Now the other terms:

$$\begin{aligned} I_{z'z'} &= \int_m (\underbrace{x'^2}_{0} + y'^2) dm = \int_{-\ell/2}^{\ell/2} \int_{-b/2}^{b/2} y'^2 \cdot \frac{m}{lb} dy' dz' \\ &= \frac{m}{lb} \int_{-\ell/2}^{\ell/2} \left( \frac{y'^3}{3} \Big|_{-b/2}^{b/2} \right) dz' = \frac{m}{lb} \cdot \frac{b^3}{12} \int_{-\ell/2}^{\ell/2} dz' \\ &= \frac{m}{lb} \cdot \frac{b^3}{12} \cdot \ell = \frac{mb^2}{12}, \end{aligned}$$

and similarly,

$$\begin{aligned} I_{y'y'} &= \int_m (\underbrace{x'^2}_{0} + z'^2) dm = \frac{m\ell^2}{12}, \\ I_{x'x'} &= \int_m (y'^2 + z'^2) dm = I_{y'y'} + I_{z'z'} = \frac{m}{12}(\ell^2 + b^2). \end{aligned}$$

Thus,

$$[\mathbf{I}]_{x'y'z'} = \frac{m}{12} \begin{bmatrix} \ell^2 + b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}.$$

(b) Calculation of  $\dot{\vec{H}}_O$ :

$$\dot{\vec{H}}_O = \vec{\omega} \times \vec{H}_O \quad \text{and} \quad \vec{H}_O = [\mathbf{I}]_{x'y'z'} \{\vec{\omega}\}_{x'y'z'}.$$

The subscripts  $x'y'z'$  in  $[\mathbf{I}]$  and  $\vec{\omega}$  have been used to denote that both  $[\mathbf{I}]$  and  $\vec{\omega}$  are expressed in  $x'y'z'$  coordinate system. ① We have calculated  $[\mathbf{I}]_{x'y'z'}$  above. Now we need to find  $\{\vec{\omega}\}_{x'y'z'}$ .

Let  $\vec{\omega} = \omega_{x'}\hat{i}' + \omega_{y'}\hat{j}' + \omega_{z'}\hat{k}'$ . But, we can also write,  $\vec{\omega} = \omega\hat{k}$ . So,

$$\omega\hat{k} = \omega_{x'}\hat{i}' + \omega_{y'}\hat{j}' + \omega_{z'}\hat{k}' \quad (11.49)$$

Dotting both sides of Eqn. (11.49) with  $\hat{i}'$ ,  $\hat{j}'$ , and  $\hat{k}'$  and using the relationships in (11.48) we get

$$\omega_{x'} = \omega(\underbrace{\hat{k} \cdot \hat{i}'}_{-\sin\phi}) = -\omega \sin\phi, \omega_{y'} = \omega(\underbrace{\hat{k} \cdot \hat{j}'}_{0}) = 0, \omega_{z'} = \omega(\underbrace{\hat{k} \cdot \hat{k}'}_{\cos\phi}) = \omega \cos\phi.$$

Thus,

$$\{\vec{\omega}\}_{x'y'z'} = -\omega \sin\phi\hat{i}' + \omega \cos\phi\hat{k}'$$

So,

$$\begin{aligned} \vec{H}_O &= \frac{m}{12} \begin{bmatrix} \ell^2 + b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix} \begin{Bmatrix} -\omega \sin\phi \\ 0 \\ \omega \cos\phi \end{Bmatrix} \\ &= \begin{Bmatrix} -\frac{m}{12}(\ell^2 + b^2)\omega \sin\phi \\ 0 \\ \frac{m}{12}b^2\omega \cos\phi \end{Bmatrix} \\ \text{or} \quad \vec{H}_O &= \frac{m\omega}{12}[-(\ell^2 + b^2)\sin\phi\hat{i}' + b^2\cos\phi\hat{k}']. \end{aligned}$$

Now we can easily compute  $\dot{\vec{H}}_O$  as follows:

$$\begin{aligned} \dot{\vec{H}}_O &= \vec{\omega} \times \vec{H}_O \\ &= (-\omega \sin\phi\hat{i}' + \omega \cos\phi\hat{k}') \times \frac{m\omega}{12}[-(\ell^2 + b^2)\sin\phi\hat{i}' + b^2\cos\phi\hat{k}'] \\ &= \frac{m\omega^2}{12}b^2\sin\phi\cos\phi\hat{j}' - \frac{m\omega^2}{12}(\ell^2 + b^2)\sin\phi\cos\phi\hat{j}' \\ &= -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{j}'. \end{aligned}$$

(c) Now, back to the  $xyz$  coordinate system: Since  $\hat{j}' = \hat{j}$  [Eqn (11.48)], we have

$$\dot{\vec{H}}_O = -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{j}$$

which is the same result as obtained in Sample 11.7.

① We can express  $[\mathbf{I}]$  and  $\vec{\omega}$  in any coordinate system of our choice but both of them must be in the same system for their product to be valid.

**SAMPLE 11.15** *The calculation of  $\dot{\vec{H}}$  using the moment of inertia matrix.* For the crooked plate considered in Sample 11.7, find the rate of change of angular momentum  $\dot{\vec{H}}_O$ , using the moment of inertia  $[I^O]$ , calculated in the  $xyz$  coordinate system

**Solution** We calculate  $\dot{\vec{H}}_O$  using the following formula:

$$\dot{\vec{H}}_O = [I^O] \cdot \underbrace{\dot{\vec{\omega}}}_{\vec{0}} + \vec{\omega} \times \widetilde{[I^O] \cdot \vec{\omega}} \vec{H}_O.$$

Note that point O is the center of mass of the plate. Let us write the expression for the angular momentum  $\vec{H}_O$  in matrix form.

$$\begin{Bmatrix} H_{x/O} \\ H_{y/O} \\ H_{z/O} \end{Bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix}.$$

It is clear from the components of  $\vec{\omega}$  that we only need the last column of  $[I]$  matrix since other elements of  $[I]$  will multiply with zeros of  $\vec{\omega}$  vector. Carrying out the multiplication we get

$$\begin{Bmatrix} H_{0x} \\ H_{0y} \\ H_{0z} \end{Bmatrix} = \begin{Bmatrix} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{Bmatrix}$$

or, in vector form

$$\vec{H}_O = I_{xz}\omega\hat{i} + I_{yz}\omega\hat{j} + I_{zz}\omega\hat{k}.$$

Therefore,

$$\begin{aligned} \dot{\vec{H}}_O &= \omega\hat{k} \times \omega(I_{xz}\hat{i} + I_{yz}\hat{j} + I_{zz}\hat{k}) \\ &= -\omega^2 I_{yz}\hat{i} + \omega^2 I_{xz}\hat{j}. \end{aligned}$$

But,

$$I_{xz} = - \int_m xz dm \quad \text{and} \quad I_{yz} = - \int_m yz dm$$

and  $x = w \sin \phi$ ,  $y = y$ ,  $z = w \cos \phi$  (see Fig. 11.29), hence

$$\begin{aligned} \dot{\vec{H}}_O &= -\omega^2 \int_m (-wy \cos \phi \hat{i} + w^2 \sin \phi \cos \phi \hat{j}) dm \\ &= \int_{-b/2}^{b/2} \int_{-\ell/2}^{\ell/2} \omega^2 (-w^2 \sin \phi \cos \phi \hat{j} + wy \cos \phi \hat{i}) \frac{m}{\ell b} dw dy \end{aligned}$$

which is the same integral as obtained in Sample 11.7 for  $\dot{\vec{H}}_O$ . Therefore, the result is also the same:

$$\dot{\vec{H}}_O = -\frac{m\omega^2 \ell^2}{12} \sin \phi \cos \phi \hat{j}.$$

$$\boxed{\dot{\vec{H}}_O = -\frac{m\omega^2 \ell^2}{12} \sin \phi \cos \phi \hat{j}}$$

**SAMPLE 11.16 Direct application of formula:** A rectangular plate is mounted on a massless shaft with the center of mass of the plate on the shaft axis. The shaft rotates about its axis with angular acceleration  $\dot{\omega} = 0.5 \text{ rpm/s}(\hat{i} + \hat{j})$ . At the instant of interest, the angular velocity is  $\vec{\omega} = 100 \text{ rpm}(\hat{i} + \hat{j})$  and the components of the moment of inertia matrix of the plate are  $I_{xx} = 2 \text{ kg}\cdot\text{m}^2$ ,  $I_{yy} = 4 \text{ kg}\cdot\text{m}^2$ ,  $I_{zz} = 6 \text{ kg}\cdot\text{m}^2$  and  $I_{xy} = I_{yz} = I_{xz} = 0$ .

- Find the angular momentum of the plate about its mass-center and show that it is not in the same direction as the angular velocity.
- Find the net moment acting on the plate.

**Solution** We are given the angular velocity, the angular acceleration, and the moment of inertia matrix of the plate:

$$\begin{aligned}\vec{\omega} &= 100 \text{ rpm}(\hat{i} + \hat{j}) = 10.47 \text{ rad/s}(\hat{i} + \hat{j}) \\ \dot{\omega} &= 5 \text{ rpm/s}(\hat{i} + \hat{j}) = 0.52 \text{ rad/s}^2(\hat{i} + \hat{j}) \\ [\mathbf{I}^{\text{cm}}] &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg}\cdot\text{m}^2.\end{aligned}$$

- The angular momentum:** The angular momentum of the plate is

$$\begin{aligned}\vec{H}_{\text{cm}} &= [\mathbf{I}^{\text{cm}}] \cdot \vec{\omega} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg}\cdot\text{m}^2 \cdot \begin{Bmatrix} 10.47 \\ 10.47 \\ 0 \end{Bmatrix} \text{ rad/s} \\ &= (20.94\hat{i} + 41.88\hat{j}) \underbrace{\text{kg}\cdot\text{m}^2 \cdot \text{s}^{-1}}_{\text{N}\cdot\text{m}\cdot\text{s}} \\ &= (20.94\hat{i} + 41.88\hat{j}) \text{ N}\cdot\text{m} \cdot \text{s}.\end{aligned}$$

$$\boxed{\vec{H}_{\text{cm}} = (20.94\hat{i} + 41.88\hat{j}) \text{ N}\cdot\text{m} \cdot \text{s}}$$

There are many ways of showing that  $\vec{H}_{\text{cm}}$  is not parallel to  $\vec{\omega}$ . We can simply draw the two vectors and show that they are not parallel. We can, alternatively, take the cross product of the two vectors:

$$\begin{aligned}\vec{H}_{\text{cm}} \times \vec{\omega} &= (20.94\hat{i} + 41.88\hat{j}) \text{ N}\cdot\text{m} \cdot \text{s} \times 10.47 \text{ rad/s}(\hat{i} + \hat{j}) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 20.94 & 41.88 & 0 \\ 10.47 & 10.47 & 0 \end{vmatrix} \text{ N}\cdot\text{m} \\ &= \hat{k}(219.24 - 438.48) \text{ N}\cdot\text{m} = -20.94 \text{ N}\cdot\text{m}\hat{k}\end{aligned}$$

which is not zero, implying that the two vectors are not parallel.

- The net moment:** The net moment on the plate can be found by applying angular momentum balance:

$$\begin{aligned}\vec{M}_{/\text{cm}} &= \dot{\vec{H}}_{\text{cm}} = \vec{\omega} \times \underbrace{[\mathbf{I}^{\text{cm}}] \cdot \vec{\omega}}_{\vec{H}_{\text{cm}}} + [\mathbf{I}^{\text{cm}}] \cdot \dot{\vec{\omega}} \\ &= \underbrace{20.94 \text{ N}\cdot\text{m}\hat{k}}_{\vec{\omega} \times \vec{H}_{\text{cm}}} + \underbrace{1.04 \text{ N}\cdot\text{m}\hat{i} + 2.08 \text{ N}\cdot\text{m}\hat{j}}_{[\mathbf{I}^{\text{cm}}] \cdot \dot{\vec{\omega}}}\end{aligned}$$

$$\boxed{\vec{M}_{/\text{cm}} = (1.04\hat{i} + 2.08\hat{j} + 20.94\hat{k}) \text{ N}\cdot\text{m}}$$

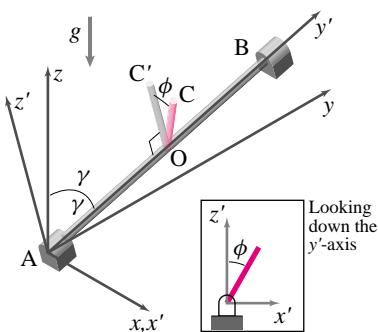


Figure 11.53: A rod, welded to a tipped shaft AB, swings around the shaft axis if it is tipped slightly from the vertical plane  $yz$ .  
 (Filename: sfig5.5.2)

**SAMPLE 11.17** A rod swings around a tipped axis in 3-D. A rigid shaft of negligible mass is connected to frictionless hinges at A and B. The shaft is tipped from the horizontal plane ( $xy$ -plane) such that the shaft axis makes an angle  $\gamma$  with the vertical axis. A uniform rod  $OC$  of mass  $m$  and length  $\ell$  is welded to the shaft. At time  $t = 0$ , the rod is tipped by a small angle  $\phi$  from its position  $OC'$  in the  $yz$  (or  $y'z'$ ) plane. Find the equation of motion of the rod.

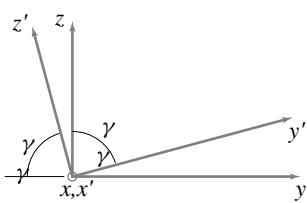


Figure 11.54:  $y'z'$  axes are obtained by rotating  $yz$  axes counterclockwise about the  $x$ -axis by an angle  $90^\circ - \gamma$ .  
 (Filename: sfig5.5.2a)

**Solution** Let  $\hat{i}'$ ,  $\hat{j}'$ ,  $\hat{k}'$  be the basis vectors associated with the  $x'y'z'$  coordinate system. Since  $y'z'$  axes can be obtained by rotating  $yz$  axes counterclockwise about the  $x$ -axis by an angle  $90^\circ - \gamma$ , we can relate the basis vectors of the two coordinate systems with the help of Fig. 11.54:

$$\begin{aligned}\hat{i}' &= \hat{i}, \\ \hat{j}' &= \sin \gamma \hat{j} + \cos \gamma \hat{k}, \\ \hat{k}' &= -\cos \gamma \hat{j} + \sin \gamma \hat{k}.\end{aligned}\quad (11.50)$$

The free body diagram of the shaft with rod  $OC$  is shown in Fig. 11.55. We can write angular momentum balance for this system about any point on the axis of rotation AB. However, rather than writing angular momentum balance about a point, let us write angular momentum balance about axis AB. This ‘trick’ will eliminate reactions  $\vec{R}_A$  and  $\vec{R}_B$  from our equations. Angular Momentum Balance about axis AB is:

$$\begin{aligned}\hat{\lambda}_{AB} \cdot [\sum \vec{M}_O] &= \dot{\vec{H}}_O \\ \text{or} \quad \hat{j}' \cdot \sum \vec{M}_O &= \hat{j}' \cdot \dot{\vec{H}}_O.\end{aligned}$$

**Calculation of  $(\hat{j}' \cdot \sum \vec{M}_O)$ :** Since  $\vec{R}_A$  and  $\vec{R}_B$  pass through axis AB, they do not produce any moment about this axis. Therefore,

$$\begin{aligned}\hat{j}' \cdot \sum \vec{M}_O &= \hat{j}' \cdot [\vec{r}_{G/O} \times mg(-\hat{k})] \\ &= \hat{j}' \cdot \left[ \frac{L}{2} (\sin \phi \hat{i}' + \cos \phi \hat{k}') \times mg(-\hat{k}) \right] \\ &= \underbrace{(\sin \gamma \hat{j} + \cos \gamma \hat{k})}_{\hat{j}'} \cdot \underbrace{[\frac{L}{2} mg (\sin \phi \hat{j} + \cos \phi \cos \gamma \hat{i})]}_{\text{using Eqn.(11.50)}} \\ &= \frac{\ell}{2} mg \sin \phi \sin \gamma\end{aligned}$$

**Calculation of  $\hat{j}' \cdot \dot{\vec{H}}_O$ :** Since the rod rotates about axis AB, we may write

$$\begin{aligned}\vec{\omega} &= \dot{\phi} \hat{j}', \\ \dot{\vec{\omega}} &= \ddot{\phi} \hat{j}'.\end{aligned}$$

Now,

$$\dot{\vec{H}}_O = [I^O] \dot{\vec{\omega}} + \vec{\omega} \times [I^O] \vec{\omega}$$

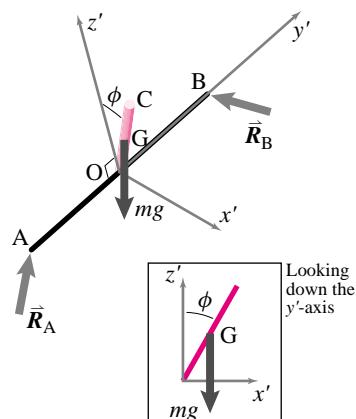


Figure 11.55: Free body diagram of the bar and shaft system. In the inset, only the weight of the bar,  $mg$ , is shown for clarity of geometry.  
 (Filename: sfig5.5.2b)

where

$$\begin{aligned} [\mathbf{I}^0]\{\dot{\vec{\omega}}\} &= [\mathbf{I}^0]_{x'y'z'} \{\dot{\vec{\omega}}\}_{x'y'z'} \\ &= \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{\phi} \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} I_{x'y'}\ddot{\phi} \\ I_{y'y'}\ddot{\phi} \\ I_{y'z'}\ddot{\phi} \end{Bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} [\mathbf{I}^0]\{\vec{\omega}\} &= \begin{Bmatrix} I_{x'y'}\dot{\phi} \\ I_{y'y'}\dot{\phi} \\ I_{y'z'}\dot{\phi} \end{Bmatrix}, \\ \Rightarrow \quad \dot{\vec{\omega}} \times [\mathbf{I}^0]\{\vec{\omega}\} &= \dot{\phi} \hat{\mathbf{j}}' \times [I_{x'y'}\dot{\phi} \hat{\mathbf{i}}' + I_{y'y'}\dot{\phi} \hat{\mathbf{j}}' + I_{y'z'}\dot{\phi} \hat{\mathbf{k}}'] \\ &= -I_{x'y'}\dot{\phi}^2 \hat{\mathbf{k}}' + I_{y'z'}\dot{\phi}^2 \hat{\mathbf{i}}'. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\mathbf{j}}' \cdot \hat{\mathbf{H}}_O &= \hat{\mathbf{j}}' \cdot [(I_{x'y'}\ddot{\phi} \hat{\mathbf{i}}' + I_{y'y'}\ddot{\phi} \hat{\mathbf{j}}' + I_{y'z'}\ddot{\phi} \hat{\mathbf{k}}') + (-I_{x'y'}\dot{\phi}^2 \hat{\mathbf{k}}' + I_{y'z'}\dot{\phi}^2 \hat{\mathbf{i}}')] \\ &= I_{y'y'}\ddot{\phi}. \end{aligned}$$

Now, setting  $\hat{\mathbf{j}}' \cdot \sum \hat{\mathbf{M}}_O = \hat{\mathbf{j}}' \cdot \hat{\mathbf{H}}_O$ , we get

$$\begin{aligned} \frac{L}{2}mg \sin \phi \sin \gamma &= I_{y'y'}\ddot{\phi} \\ \text{or} \quad \ddot{\phi} &= \frac{mg(L/2) \sin \gamma}{I_{y'y'}} \sin \phi \\ \text{or} \quad \ddot{\phi} &= C \sin \phi \end{aligned}$$

where

$$C = \frac{mg(L/2) \sin \gamma}{I_{y'y'}}.$$

For rod OC,

$$I_{y'y'} = \int_m (x'^2 + z'^2) dm = \int_0^L l^2 \frac{m}{L} dl = \frac{1}{3}mL^2.$$

Therefore, the equation of motion of the rod is

$$\begin{aligned} \ddot{\phi} - \frac{mg(L/2) \sin \gamma}{(1/3)mL^2} \sin \phi &= 0 \\ \ddot{\phi} - \frac{3g \sin \gamma}{2L} \sin \phi &= 0. \end{aligned}$$

$$\ddot{\phi} - \frac{3g \sin \gamma}{2L} \sin \phi = 0$$

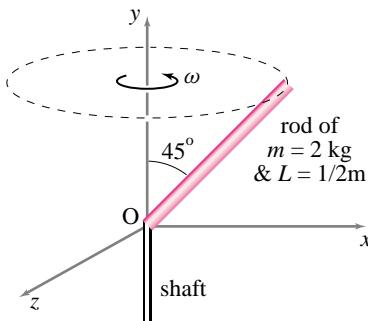


Figure 11.56: (Filename:fig7.4.2)

**SAMPLE 11.18 Kinetic energy in 3-D rotation.** A thin rod of mass 2 kg and length  $L = \frac{1}{2} \text{ m}$  is welded to a massless shaft at an angle  $\theta = 45^\circ$ . The shaft rotates about its longitudinal axis ( $y$ -axis) at 100 rpm. The moment of inertia matrix of the rod about the weld point O is

$$[I^O] = 0.08 \text{ kg} \cdot \text{m}^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the kinetic energy of the rod.

**Solution** The rod rotates about the fixed point O with angular velocity

$$\vec{\omega} = \omega \hat{j} \quad \text{where} \quad \omega = 100 \text{ rpm} = 10.47 \text{ rad/s.}$$

The kinetic energy of a rigid body rotating about a fixed point O is given by

$$E_K = \frac{1}{2} \vec{\omega} \cdot [I^O] \cdot \vec{\omega}$$

① This step is simply to facilitate computation. We carry out the multiplication and at the end substitute the value of  $K_0$ .

For the given problem, let us write the moment of inertia matrix  $[I^O]$  of the rod as ①

$$[I^O] = K_0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where  $K_0 = 0.08 \text{ kg} \cdot \text{m}^2$ . Now

$$\vec{\omega} = \omega \hat{j} = \{0 \ \ \omega \ \ 0\}^T.$$

Therefore,

$$\begin{aligned} [I^O] \cdot \vec{\omega} &= K_0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\ &= K_0 \begin{Bmatrix} -\omega \\ \omega \\ 0 \end{Bmatrix} \\ &= -K_0 \omega \hat{i} + K_0 \omega \hat{j}. \end{aligned}$$

Substituting the expression in the formula for  $E_K$  we get

$$\begin{aligned} E_K &= \frac{1}{2} \omega \hat{j} \cdot (-K_0 \omega \hat{i} + K_0 \omega \hat{j}) \\ &= \frac{1}{2} K_0 \omega^2 \\ &= \frac{1}{2} \cdot 0.08 \text{ kg} \cdot \text{m}^2 \cdot (10.47 \text{ rad/s})^2 \\ &= 4.38 \text{ N} \cdot \text{m} = 4.38 \text{ J} \end{aligned}$$

$$E_K = 4.38 \text{ J}$$

## 11.5 Dynamic balance

Sometimes when something spins it can shake the structure that holds it. A familiar example is an ‘unbalanced’ car tire which makes a car shake. But rotor balance is important in all kinds of machines. Most often one seeks to eliminate or minimize the imbalance. For example the strange design of car crank-shafts is due in large part to an attempt to minimize its imbalance. But what does it mean, in the language of mechanics, to say a rotating body is ‘unbalanced’? It means that non-zero forces and/or torques are required to hold the axis still while the object spins at constant rate. Going back to the basic momentum balance equations:

$$\text{Linear momentum balance: } \sum \vec{F} = \dot{\vec{L}}$$

and

$$\text{Angular momentum balance: } \sum \vec{M}_C = \dot{\vec{H}}_C,$$

we see that forces and torques are required if  $\dot{\vec{L}} \neq \vec{0}$  or if  $\dot{\vec{H}}_C \neq \vec{0}$ .

So imbalance means  $\dot{\vec{L}}$  and/or  $\dot{\vec{H}}_C$  is not zero. We break the concept of balance into the following two concepts:

- (1) An object is said to be *statically balanced* with respect to a given axis of rotation if  $\dot{\vec{L}} = \vec{0}$  when it spins at constant rate about that axis.
- (2) An object is said to be *dynamically balanced* with respect to a given axis of rotation if both  $\dot{\vec{L}} = \vec{0}$  and  $\dot{\vec{H}} = \vec{0}$  for constant rate rotation about that axis.

The origin of the words ‘static’ balance and ‘dynamic’ balance is in how the imbalance can be measured. Static imbalance can be measured with a static test, dynamic imbalance requires a dynamic test.

### Static balance

If the center of mass of a rigid body is on the fixed axis of rotation then it will not accelerate. Thus,  $\dot{\vec{L}} = \vec{a}_{cm}m_{tot} = \vec{0}$  and the object is statically balanced. An equivalent definition of static balance is that the *net* force on the spinning body is zero. Whether or not this net force is so can be tested with a statics experiment. Put the axis of rotation on good bearings and see if the mass hangs down in any preferred direction. In a tire shop, this kind of balancing is something called bubble balancing because of the bubble in the level measuring device.

### Dynamic balance

The first condition for dynamic balance of an object with respect to spinning about an axis is that the object be statically balanced. The center of mass must lie on the axis of rotation. The second condition for dynamic balance, that  $\dot{\vec{H}} = \vec{0}$ , is a little more subtle. To make things more specific, let’s calculate the rate of change of angular momentum  $\dot{\vec{H}}_O$  with respect to a point O that is on the axis of rotation. So, for constant rate rotation about a fixed axis we have:

$$\dot{\vec{H}}_O = \vec{\omega} \times \vec{H}_O \quad (11.51)$$

because  $\vec{H}_O$  spins with the body.  $\dot{\vec{H}}_O$  is evidently zero if the angular momentum,

① When you go to ACME garage and get your car tires balanced you can politely ask the mechanic: “Ma’am, could you make sure that one of the eigenvectors of my wheel’s moment of inertia matrix is parallel to the car axle? Thanks.” The mechanic, if she is a decent person, will make sure that it is the appropriate eigenvector that is parallel to the axle. Otherwise the wheel would point straight sideways with the axle piercing the rubber.

① **Caution:** A common misperception about angular momentum is that it is always parallel to angular velocity. In general, angular momentum is *not* parallel to angular velocity. When is angular momentum about the center of mass, say, parallel to angular velocity? When  $\vec{\omega}$  is an *eigenvector* of  $[\mathbf{I}^{\text{cm}}]$ . This is always the case for planar objects rotating about an axis perpendicular to the plane, but not generally in 3D.

$\vec{H}_O$ , is parallel to the angular velocity,  $\vec{\omega}$ . However,  $\vec{H}_O = [\mathbf{I}^O] \cdot \vec{\omega}$  which means  $\vec{H}_O$  is parallel to  $\vec{\omega}$  only when  $\vec{\omega}$  is an eigenvector of  $[\mathbf{I}^O]$ ①.

*So an object which is spinning about a fixed axis is dynamically balanced if its center of mass is on the axis (static balance) and the angular velocity vector  $\vec{\omega}$  is an eigenvector of the moment of inertia matrix.*

If we restrict ① our attention to cases where the axis of rotation is the  $z$ -axis then this condition is easy to recognize. It is when  $I_{xz} = I_{yz} = 0$ , that is, when the only non-zero element in the third column and in the third row of the  $[\mathbf{I}^O]$  matrix is  $I_{zz}^O$  in the lower right corner.

Since these terms cause wobbling if one of the coordinate axis is an axis of rotation, the terms that are off the main diagonal in the moment of inertia matrix are sometimes called the ‘imbalance’ terms. These terms lead to dynamic imbalance when the object is spun about the  $x$ ,  $y$ , or  $z$ -axes. The off diagonal terms are also sometimes called the ‘centrifugal’ terms. This naming has an intuitive basis. One way to understand dynamic imbalance is to think of it being due to the unbalanced centrifugal pull of bits of mass that are spinning in circles.

Often, but not always, you can tell by inspection if something is dynamically balanced for rotation about a certain axis. If, for every bit of mass that is not on the axis there is another equal bit of mass that is exactly opposite, with respect to the axis, then the object is balanced. Some objects that do not meet this symmetry condition are also dynamically balanced, however.

#### *Example: Some common objects*

All of the figures in figure ?? on page ?? are dynamically balanced about all the axes shown and about any axis perpendicular to any pair of these axes.  $\square$

#### *Example: A cube = a sphere*

Both a cube and a sphere have moment of inertia matrices proportional to the identity matrix

$$[\mathbf{I}^{\text{cm}}]_{\text{cube}} = \frac{m\ell^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{I}^{\text{cm}}]_{\text{sphere}} = \frac{mD^2}{10} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, for any  $\vec{\omega}$ , we get that  $\vec{H}_{\text{cm}}$  is parallel to  $\vec{\omega}$  and, thus, both of these objects are dynamically balanced for rotation about any axis through their centers of mass.

Here is an example where the mathematics contradicts intuition; it does not seem like a cube should be balanced for rotation about a random skewed axis through its center of mass!  $\square$

**SAMPLE 11.19** *Static and dynamic balance in 2-D motion.* A system with two equal masses at A and B rotates about point O at a constant rate  $\omega = 10 \text{ rpm}$ . The system is shown in the figure. The rod connecting the two masses has negligible mass.

- Is the system statically balanced? If not, suggest a way to balance it.
- Is the system dynamically balanced? If not, suggest a way to balance it.

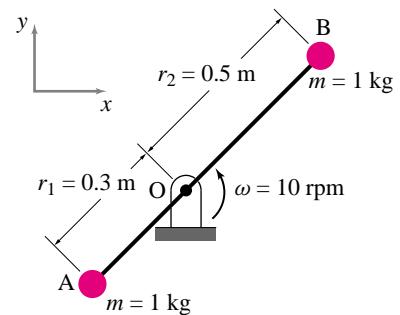


Figure 11.57: (Filename:sfig4.7.DH1)

### Solution

(a) Since the two masses are equal, the center of mass of the system is at the geometric center of rod AB, i.e., at a point O', halfway between A and B. Clearly, O' is not on the axis of rotation which passes through O and is perpendicular to the plane of motion. Therefore, the system is *not* statically balanced.

To balance the system, we must move the center of mass to O or pivot the system at O'. Say we cannot move the pivot point. So, to move the center of mass O, we add mass  $m'$  to m at A. From the definition of center of mass:

$$\begin{aligned} (m + m')r_1 &= mr_2 \\ \Rightarrow m' &= \frac{mr_2 - mr_1}{r_1} \\ &= \frac{1 \text{ kg}(0.5 \text{ m} - 0.3 \text{ m})}{0.3 \text{ m}} \\ &= 0.67 \text{ kg}. \end{aligned}$$

Add 0.67 kg to mass at A.

(b) The system, as given, is *not* dynamically balanced since it is not statically balanced. Let us check if it is dynamically balanced after adding  $m'$  to A as suggested above.

$$\sum \vec{M}_O = \dot{\vec{H}}_O.$$

Let us calculate  $\dot{\vec{H}}_O$  to see if it is zero:

$$\begin{aligned} \dot{\vec{H}}_O &= \vec{r}_1 \times m_1 \vec{a}_1 + \vec{r}_2 \times m_2 \vec{a}_2 \\ &= (-r_1 \hat{e}_R) \times m_1 (\omega^2 r_1 \hat{e}_R) + (r_2 \hat{e}_R) \times (-\omega^2 r_2 \hat{e}_R) m_2 \\ &= -m_1 \omega^2 r_1^2 (\hat{e}_R \times \hat{e}_R) \overset{\text{0}}{\underset{\text{0}}{\text{—}}} - m_2 \omega^2 r_2^2 (\hat{e}_R \times \hat{e}_R) \overset{\text{0}}{\underset{\text{0}}{\text{—}}} \\ &= \vec{0}. \end{aligned}$$

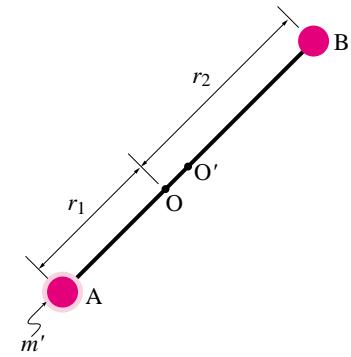


Figure 11.58: (Filename:sfig4.7.DH2)

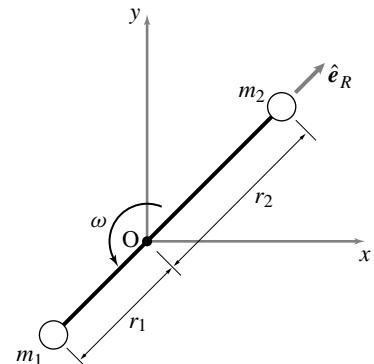


Figure 11.59: (Filename:sfig4.7.DH3)

The net torque on the system is zero, therefore it is dynamically balanced. Note that  $\dot{\vec{H}}_O = \vec{0}$  irrespective of the values of  $m_1$ ,  $m_2$ ,  $r_1$ , and  $r_2$ . Thus  $\dot{\vec{H}}_O = \vec{0}$  even for the system as given. But the system, as given, is not dynamically balanced because *static balance is a necessary condition for dynamic balance*.

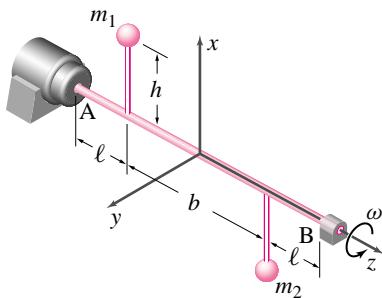


Figure 11.60: Imbalance of a rotor due to point masses spinning at a constant rate.  
(Filename:sfig4.7.1)

**SAMPLE 11.20** *Imbalance of a rotor due to rotating masses.* Two masses  $m_1$  and  $m_2$  are attached to a massless shaft AB by massless rigid rods of length  $h$  each. The two masses are separated by distance  $b$  along the shaft and are in the same plane but on the opposite sides of the shaft. The shaft is rotating with a constant angular speed  $\omega$ . The shaft is free to move along the  $z$ -axis at point B. Ignore gravity.

- Is the system *statically balanced*?
- What is the torque required to keep the motion going about the  $z$ -axis (i.e.,  $M_z$ )?
- What are the reactions at the support points of the shaft?

### Solution

- A simple line sketch and the free body diagram of the system are shown in Fig. 11.61. The linear momentum balance ( $\sum \vec{F} = m\vec{a}$ ) for the system gives:

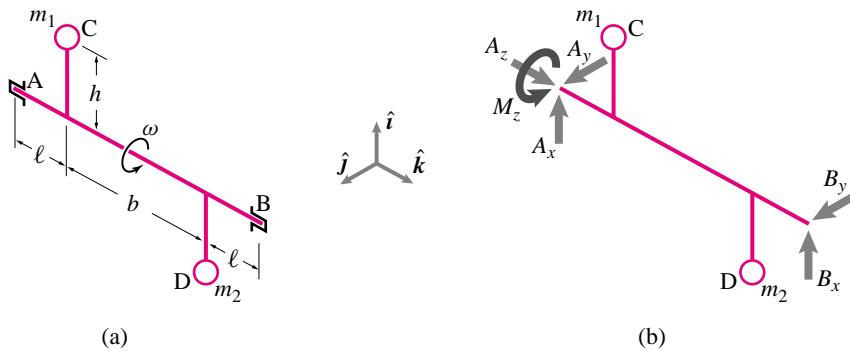


Figure 11.61: (a) A simple line diagram of the system. (b) Free Body Diagram of the system.  
(Filename:sfig4.7.1a)

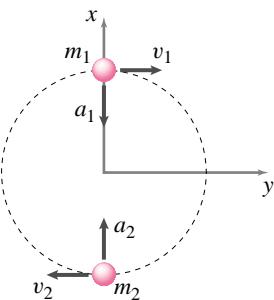


Figure 11.62: Accelerations of  $m_1$  and  $m_2$ .  
(Filename:sfig4.7.1b)

$$(A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + A_z\hat{k} = m_1\vec{a}_1 + m_2\vec{a}_2$$

$$= m_1\omega^2 h(-\hat{i}) + m_2\omega^2 h\hat{i}$$

$$\Rightarrow A_x + B_x = \omega^2 h(m_2 - m_1) \quad (11.52)$$

$$A_y + B_y = 0 \quad (11.53)$$

$$A_z = 0$$

- Clearly,  $\sum \vec{F} \neq \vec{0}$  which means the system is not statically balanced.
- $M_z$  can be easily calculated by writing angular momentum balance about axis AB. In fact, we do not even need to write the equation in this case; since all the forces pass through this axis and the accelerations of  $m_1$  and  $m_2$  also pass through it,

$$\sum M_{/AB}(\text{due to reaction forces}) = 0 \quad \text{and} \quad \dot{H}_{/AB} = 0.$$

But

$$\sum M_{/AB} = \dot{H}_{/AB}$$

$$\text{or} \quad M_z + \sum M_{/AB}(\text{due to reaction forces}) = \dot{H}_{/AB}$$

$$\Rightarrow M_z = 0.$$

$$\boxed{M_z = 0}$$

- (c) For the four unknown reactions at A and B (we have already found  $A_z$  and  $M_z$ ) we have two scalar equations so far (from Linear Momentum Balance). Angular Momentum Balance about point A gives:

$$\sum \vec{M}_{/A} = \dot{\vec{H}}_{/A}$$

Now,

$$\begin{aligned}\sum \vec{M}_{/A} &= \vec{r}_{B/A} \times \sum \vec{F}_B \\ &= (2\ell + b)\hat{k} \times (B_x\hat{i} + B_y\hat{j}) \\ &= B_x(2\ell + b)\hat{j} - B_y(2\ell + b)\hat{i} \\ \dot{\vec{H}}_{/A} &= \vec{r}_{C/A} \times m_1\vec{a}_1 + \vec{r}_{D/A} \times m_2\vec{a}_2 \\ &= (\ell\hat{k} + h\hat{i}) \times m_1(-\omega^2 h\hat{i}) + ((\ell + b)\hat{k} - h\hat{i}) \times m_2(\omega^2 h\hat{i}) \\ &= \omega^2 h(m_2(\ell + b) - m_1\ell)\hat{j}\end{aligned}$$

Equating  $\sum \vec{M}_{/A}$  and  $\dot{\vec{H}}_{/A}$  we get

$$B_x(2\ell + b)\hat{j} - B_y(2\ell + b)\hat{i} = \omega^2 h(m_2(\ell + b) - m_1\ell)\hat{j}$$

Dotting both sides of the equation with  $\hat{i}$  and  $\hat{j}$ , we get

$$\begin{aligned}B_y &= 0 \\ \text{and } B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell].\end{aligned}$$

Substituting  $B_x$  and  $B_y$  in (11.52) and (11.53) we find

$$\begin{aligned}A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)] \\ A_y &= 0\end{aligned}$$

$A_x$	$=$	$\frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)]$
$A_y$	$=$	$A_z = 0$
$B_x$	$=$	$\frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell]$
$B_y$	$=$	$0$
$M_z$	$=$	$0$

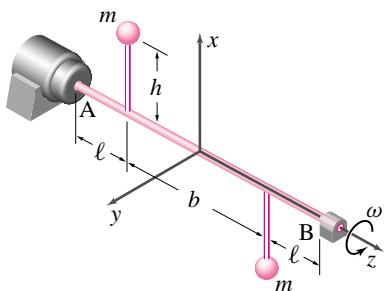


Figure 11.63: Imbalance of a rotor due to point masses spinning at a constant rate.  
(Filename:sfig4.7.2)

**SAMPLE 11.21** *Balancing a rotor with spinning masses.* Consider the same system as in the previous sample problem (Sample 11.20). Assume that  $m_1 = m_2 = m$ .

- Set  $m_1 = m_2$  in the reactions calculated. Is the system statically balanced now? Explain.
- Is the system *dynamically balanced*? Explain.
- Balance the system dynamically by (a) adjusting the geometry, (b) adding two masses to the system.

### Solution

- Recall from the solution of Sample 11.20 that

$$\begin{aligned}A_x &= \frac{\omega^2 h}{(2\ell + b)} [m_2\ell - m_1(\ell + b)] \\B_x &= \frac{\omega^2 h}{(2\ell + b)} [m_2(\ell + b) - m_1\ell] \\A_y &= B_y = A_z = M_z = 0\end{aligned}$$

Setting  $m_1 = m_2 = m$  in the above expressions for the reactions, we get

$$A_x = \frac{\omega^2 h}{(2\ell + b)} (-mb) \quad \text{and} \quad B_x = \frac{\omega^2 h}{(2\ell + b)} (mb).$$

Thus,  $A_x + B_x = 0$ . Therefore,  $\sum \vec{F} = \vec{0}$  which means the system is statically balanced.

**Static Balance:** When the two masses are equal, the center of mass of the system is on the axis of rotation. Therefore, the static reactions are zero irrespective of the orientations of the two masses.

The requirement for static balance is  $\sum \vec{F} = \vec{0}$  in any static orientation of the system. As long as the center of mass of the system lies on the axis of rotation, the system will be in static balance.

**Dynamic Balance:** When the masses are rotating at constant speed, reaction forces at A and B are required to keep the motion going. Clearly,  $\sum \vec{M} \neq \vec{0}$  now. Therefore, the system is not in dynamic balance.

At the instant shown the reaction forces are  $A_x$  and  $B_x$  as shown in Fig. 11.64, but they change directions as the position of the masses changes. For example, if the masses are in the  $y-z$  plane the reaction forces will be  $A_y$  and  $-B_y$ . The magnitudes of these forces will remain the same. Thus, we have rotating reaction forces at A and B. These rotary forces cause wear in the bearings and induce vibrations in the frame of the machine and the supporting structure. Therefore, these forces are undesirable.

If we can somehow make these *dynamic reactions* zero, the bearings, the machine frame, the supporting structure, the machine operator and the company will all be very happy. So, how do we do it? Read on.

- There are many ways in which the rotor can be dynamically balanced:

- **Trivial solution:** Remove both the masses. Of course, there is nothing left to produce any nonzero  $\vec{H}$ . Thus,

$$\sum \vec{M}_{\text{any point}} = \vec{0}. \quad \text{Also, } \sum \vec{F} = \vec{0}$$

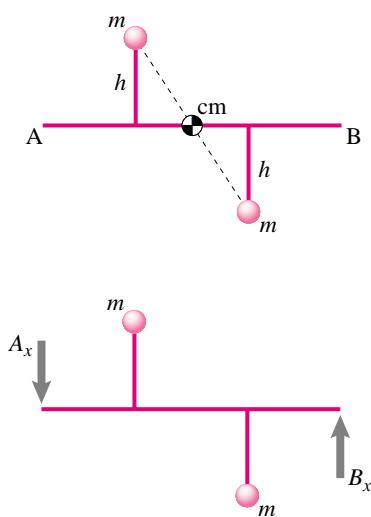


Figure 11.64: Reaction forces at A and B are required to keep the motion going.  
(Filename:sfig4.7.2a)

- **Adjust geometry:** Set  $b = 0$ . The center of mass is still on the axis of rotation.

$$\Rightarrow \sum \vec{F} = \vec{0}$$

In addition,

$$\begin{aligned} \hat{\vec{H}}_{/A} &= \vec{0} \\ \Rightarrow \sum \vec{M}_{/A} &= \vec{0}. \end{aligned}$$

- **Add two masses:** If  $b$  is required to be nonzero, we can balance the rotor by adding two masses in any two selected transverse (to the shaft-axis) planes to make  $\hat{\vec{H}}_{/A} = \vec{0}$ . ① For example:

- We can take two equal masses  $m$  and  $m$ , and place them in the opposite directions of the masses already on the shaft as shown in Fig. 11.65(a). It should be clear that the net angular momentum about any point on the shaft will be zero now.
- We can add two masses  $m'$  and  $m'$  different from the masses attached to the shaft and place them a distance  $c$  apart on the shaft. Let the length of the connecting rods of the new masses be  $h'$ . For the net angular momentum to be zero we need

$$\begin{aligned} \underbrace{\hat{\vec{H}}_{\text{due to } m'_s}}_{m' \omega^2 h' c} &= \underbrace{\hat{\vec{H}}_{\text{due to } m_s}}_{m \omega^2 h b} \\ \Rightarrow m' h' c &= m h b \end{aligned}$$

Thus, we have the freedom ② of selecting any combination of  $m'$ ,  $h'$  and  $c$  to give the required product. Here are two examples:

- Let  $m' = m/2$ ,  $c = b/2$ , then  $h' = \frac{mhb}{m'c} = 4h$ .
- Let  $m' = m/2$ ,  $c = 2b$ , then  $h' = \frac{mhb}{m'c} = h$ .

① Balancing a rotor by adding two masses in two selected transverse (to the shaft axis) planes is quite a general method. Even if the rotor has more than two spinning masses the net angular momentum due to all masses can be reduced to the angular momentum produced by an equivalent system with just two masses such as the one under consideration.

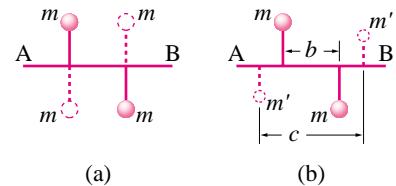


Figure 11.65: Dynamic balancing of the rotor by adding two masses. (a) The two added masses (shown by dotted circles) are the same as the original masses on the shaft. (b) The two added masses are different from the original masses ( $m' \neq m$ ).

(Filename:sfig4.7.2b)

② In practice, this freedom is usually restricted by geometric and space constraints.

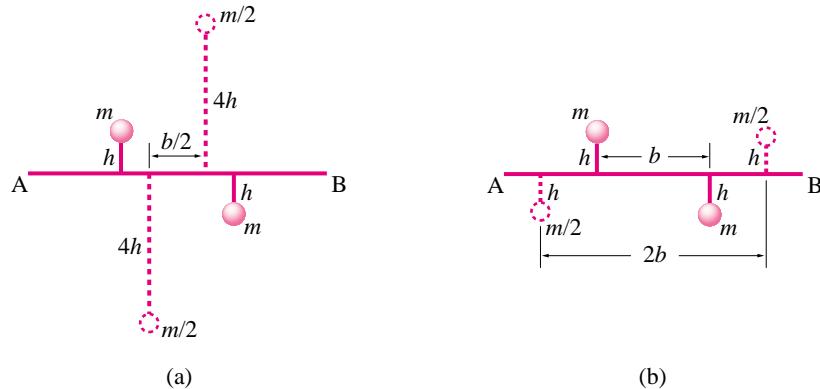


Figure 11.66: Dynamic balancing of the rotor by adding two masses. (a)  $m' = m/2$ ,  $c = b/2$  and  $h' = 4h$ . (b)  $m' = m/2$ ,  $c = 2b$  and  $h' = h$ .

(Filename:sfig4.7.2c)



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# A Units and dimensions

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## *A.1 Units and dimensions*

Many engineering texts have, somewhere near the start, a tedious and pedantic section about units and dimensions. This book is different. That section is here at the end; not to diminish the importance of the topic but because students are immune to preaching. The only way a student will get good at managing units is by imitation, or in time of panic or idle curiosity. As for imitation, we have tried to set a good example in the whole of the book. As for panic and curiosity, this section is here. The central message, mentioned in the preface, is this:

*balance your units and carry your units.*

### **Balance your units**

Every line of every calculation should be dimensionally sensible. That is, the dimensions on the left of the equal sign should be consistent with the dimensions on the right the same way numbers have to balance. Otherwise the equations are not equations. For example, if two bicycles tied in a race you could say they were in

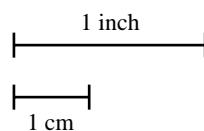


Figure A.1: Relative size of an inch and a centimeter.  
(Filename:tfigure1.a)

some way equal. But even if you noticed that the weight difference between them was 10% over 2 pounds you would not write

$$8 \text{ kg} = 9 \text{ kg}.$$

The equivalence between the two bikes does not make eight kilograms equal to nine kilograms. In this same way it would be wrong to write

$$1 \text{ in} = 1 \text{ s.}$$

if you noticed that it takes a bug about a minute (60 seconds) to walk the length of your body (say about 60 inches). That the passing of a second corresponds to the passing of an inch, so for some purposes an inch is equivalent to a second, and  $1 = 1$ , does not mean that an inch is a second. An inch has dimensions of length which cannot be equal to a second with dimensions of time. Length can equal time no more than 8 can equal 9.

Of course it is correct to write that

$$5.08 \text{ cm} = 2 \text{ in}$$

whether or not you have noticed anything. Both centimeters and inches have dimensions of length and one inch is equivalent to 2.54 centimeters always (figure A.1). An equation where the units on both sides of the equation are the same physical quantities (length in the example above) is *balanced* with regard to units.

## Carry your units

When you go from one line of a calculation to the next you should carry (keep written track of) the units with as much care as any other numerical or algebraic quantities. This written presentation of your units will help you as well as the people to whom you show your work. The rest of this section is, more or less, a discussion of how and why to ‘carry your units.’

Most physical quantities are dimensional and are represented by a number multiplying a unit: 7 m means 7 times (one meter). Thus, the ‘m’ and the 7 are of equal status in any equations in which they are used. When you do arithmetic and don’t forget any terms you have ‘carried’ the numbers. Similarly, *carrying* the units just means not forgetting them *in* your calculations (not just next to your calculations).

## Dimensions, units and changing units

Distance has dimensions of length [ $L$ ] that can be measured with various units — centimeters (cm), yards (yd), or furlongs (an obsolete unit equal to  $1/8$  mile). A meter is the standard unit of length in the SI system. In answer to the question ‘What is the length of a bicycle crank  $\ell$ ?’ we say ‘ $\ell$  is seven inches’ and write  $\ell = 7$  in or say ‘ $\ell$  is seventeen point seven centimeters’ and write  $\ell = 17.7$  cm. In each case, a number multiplies a dimensional unit.

Force has dimension of mass times acceleration [ $m \cdot a$ ]. Because acceleration itself has dimensions of length over time squared [ $L/T^2$ ], force also has dimensions of mass times length divided by time squared [ $M \cdot L/T^2$ ]. Because force has such a central role in mechanics, it is often convenient to think of force as having its own units. Force then has dimensions of, simply, force [ $F$ ]. The most common units for force are Newton (N) and the pound (lbf). The ‘f’ in the notation for the pound lbf is to distinguish a pound force lbf from the pound mass lbm,  $1 \text{ lbf} = \text{lbm} \cdot g \approx 32.2 \text{ lbm} \cdot \text{ft/s}^2$ . Some people use lb to mean pound force or pound mass, depending on context. We use lbm for pound mass and lbf for pound force to avoid confusion.

### Changing units

We can say ‘The typical force of a seated racing bicyclist on a bicycle pedal is only thirty pounds,’ and write any of the following:

$$\begin{aligned} F &= 30 \text{ lbf} \\ F &= 30 \text{ lbf} \cdot (1) \\ F &= 30 \text{ lbf} \cdot \underbrace{\left( \frac{4.45 \text{ N}}{1 \text{ lbf}} \right)}_1 \\ F &= 133.5 \text{ N.} \end{aligned}$$

Here we have shown one way to change units. Multiply the expression of interest by one (1) and then make an appropriate substitution for one. Any table of units will tell us that 1 lbf is approximately 4.45 N. So we can write  $1 = (4.45 \text{ Newtons}/1 \text{ lbf})$  and multiply any part of an equation by it without affecting the equation’s validity. See figure A.2 to get a sense of the relation between a pound force, a Newton, and the less used force units, the poundal and the kilogram-force.

What if we had made a mistake and instead multiplied the right hand side by  $1 = (1 \text{ lbf}/4.45 \text{ Newton})$ ? No problem. We would then have

$$F = 30 \text{ lbf} = 30 \text{ lbf} \cdot \frac{1 \text{ lbf}}{4.45 \text{ Newton}} = \frac{30}{4.45} \text{ lbf}^2/\text{N.}$$

This expression is admittedly weird, but it is correct. If you should end up with such a correct but weird solution you can compensate by multiplying by one again and again until the units cancel in a way that you find pleasing. In this case we could get an answer in a more conventional form by multiplying the right hand side by  $1^2$  using  $1 = (4.45 \text{ N}/\text{lbf})$ :

$$F = \frac{30}{4.45} \frac{\text{lbf}^2}{\text{N}} \cdot 1^2 = \frac{3.0}{4.45} \frac{\text{lbf}^2}{\text{N}} \cdot \left( \frac{4.45 \text{ N}}{1 \text{ lbf}} \right)^2 = 133.5 \text{ N} \quad (\text{as expected}).$$

A trivial but surprisingly useful observation is that  $F = F$ . A quantity is equal to itself no matter how it is represented. That is,  $30 \text{ lbf} = 133.5 \text{ N}$  even though  $30 \neq 133.5$ . To summarize:

*Units are manipulated in any and all calculations as if they were numbers or algebraic symbols. For example, canceling equal units from the top and bottom of a fraction is the same as canceling numbers or algebraic symbols.*

### An advertisement for careful use of units

Units and dimensions are part of scientific notation just as spelling, punctuation, and grammar are parts of English composition. If used properly, they aid both thinking and the communication of these thoughts to others. If units and dimensions are used improperly they can impede communication, even with oneself, and convey the wrong meaning.

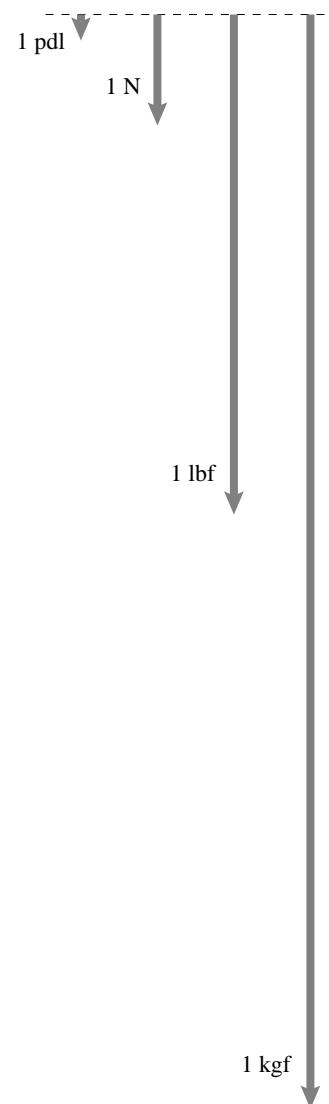


Figure A.2: Relative size of a poundal pdl, a Newton N, a pound force lbf, and a kilogram force kgf.  $1 \text{ N} = 7.24 \text{ pdl}$ ,  $1 \text{ lbf} = 4.45 \text{ N}$ , and  $1 \text{ kgf} = 1 \text{ kilopond} = 9.81 \text{ N} = 2.2 \text{ lbf}$ .

(Filename: tfigure1.b)

### *Example: Breaking load*

A gadget that breaks with a 300 N (300 Newtons) load instead of a needed 300 lbf (300 pounds force) load is exactly as bad as one that breaks with a 67 lbf load instead of a needed 300 lbf load. An unsatisfied consumer will not be placated by learning that the engineer's calculation was 'numerically correct'.  $\square$

If anybody is ever to use your calculation, giving them the wrong units is just as bad as giving them the wrong numerical value.

Although using units properly often seems annoyingly tedious, it also often pays. If units are carried through honestly, *not just tagged on to the end of an equation for appearance*, you can check your work for dimensional consistency. If you are trying to find a speed and your answer comes out  $13 \text{ kg} \cdot \text{m} / \text{s}$ , you know you have made a mistake —  $\text{kg} \cdot \text{m} / \text{s}$  just isn't a speed. Such dimensional errors in a calculation often reveal corresponding algebraic or conceptual mistakes. Also, if a problem is based on data with mixed units, such as cm and m, or pound force and pound mass, you may often not know the units of your answer unless you properly 'carry' your units<sup>①</sup>.

① You can easily generate errors of approximately a factor of 1000 with English units if you wishfully multiply or divide answers by 32.2 (the value of  $g$  in  $\text{ft}/\text{s}^2$ ) at the end of a sloppy calculation. If you do it wrong you get an error of a factor of 32.2<sup>2</sup> which is 3% greater than 1000. Following sloppiness with unscrupulousness, some are tempted to then slide a decimal point three places to the right or left to 'fix' things. The decrepit insecurity that provokes such crimes is precluded by carrying units.

### **Three ways to be fussy about units.**

People are most pleased if you speak their language, speak correctly, and make sense. Similarly, scientists and engineers with whom you communicate will be most comfortable if you use the units they use and use them with correct notation. But most importantly, you should use units in a way that makes physical sense. Just as the United Nations argues over which language to use for communication, educators, editors, and makers of standards have argued for decades over conventions for units: whether they should come in multiples of 10, whether they should use the standard international scientific conventions, and whether they will be clear to someone who has worked in the stock room of a supplier of  $\frac{1}{2}$ -inch bolts for 35 years and thinks SI might be a friend of his cousin Amil.

Even if you are not fluent in someone's favorite language, you can still say sensible things. Similarly, no matter what you or your work place's choice of units (SI, English, or hodge-podge), no matter whether you use upper case and lower case correctly, you should make sense. Physically sensible units — that is, balanced units — should be used to make your equations dimensionally correct. Then you should work on refining your notation so as to be more professional.

So, in order of importance,

1. use balanced units.
2. use units of the type that are liked by your colleagues.
3. spell and punctuate these units correctly.

If you are in a situation where your only problem is the third item on the list you are doing fine, unless you are *really* fussy, or work for someone who is really fussy. (That is, you are doing as well as the authors of this book, anyway.)

Not everyone will take the care that we advise for you. You will find that, in both school and work, there are a variety of ways in which people use and abuse units, all within the context of productive engineering. So you will have to be aware and tolerant of the various conventions, even if they sometimes seem somewhat vague and imprecise.

### **Units with calculators and computers**

Calculators and computers generally do not keep track of units for you. In order for your numerical calculations to make sense you have the following choices.

**Use dimensionless variables.** Using dimensionless variables is the preferred method of scientists and theoretical engineers. The approach requires that you define a new set of dimensionless variables in terms of your original dimensional variables.

**Use a consistent unit system.** Express all quantities in terms of units that are consistent. ①For example, all lengths should be in the same units and the unit of force should equal the unit of mass times the unit of distance divided by the unit of time squared. Each row of the table below defines a consistent set of units for mechanics.

Name	length	mass	force	time	angle
mks	meter	kilogram	Newton	second	radian
cgs	centimeter	gram	dyne	second	radian
English	foot	lbm	poundal	second	radian
English2	foot	slug	lbf	second	radian

The radian is the unit of angle in all consistent unit systems. Whether or not a radian is a proper unit or not is an issue of some philosophical debate. Practically speaking, you can generally replace 1 radian with the number 1.

①**Caution:** Doing a computer calculation using quantities from an inconsistent unit system can easily lead to wrong results. To be safe make sure that all quantities are expressed in terms of only one row of the table shown.

**Use numerical equations.** If you are using the computer to evaluate a formula that you trust, and you have balanced the units in a way that makes you secure, you can have the computer do the arithmetic part of the calculation. It is easy to make mistakes, however, unless the formula is expressed in consistent units.

#### Example: Force units conversion

What, in the SI system, is the net braking force when a 2000 lbm car skids to a stop on level ground? For this units problem we skip the careful mechanics and just work with the formula

$$F = \mu mg$$

where  $m$  is the mass of the car,  $g$  is the local gravitational constant and  $\mu$  is the coefficient of friction for sliding between the tire and the road. We won't be off by more than a quarter of a percent using the standard rather than the local value of the gravitational constant,  $g = 32.2 \text{ ft/s}^2$ . The coefficient of friction for rubber and dry road is about one, so we use  $\mu = 1$ . We proceed by plugging in values into the formula and then multiplying by 1 until things are in standard SI (Système Internationale) form. We use a table of units to make the various substitutions for 1. A few of the detailed steps could be contracted. The approach below is only one, albeit an awkward one, of many routes to the answer.

$$\begin{aligned} F &= \mu mg \\ F &= 1 \cdot 2000 \text{ lbm} \cdot (32.2 \text{ ft/s}^2) \\ &= (2000 \cdot 32.2) \frac{\text{lbm} \cdot \text{ft}}{\text{s}^2} \\ &= (2000 \cdot 32.2) \underbrace{\frac{\text{lbm} \cdot \text{ft}}{\text{s}^2}}_{1} \cdot \underbrace{\left( \frac{1 \text{ kg}}{2.2 \text{ lbm}} \right)}_1 \\ &\quad \cdot \underbrace{\left( \frac{30.48 \text{ cm}}{1 \text{ ft}} \right)}_1 \cdot \underbrace{\left( \frac{1 \text{ m}}{100 \text{ cm}} \right)}_1 \end{aligned}$$

$$= 8917 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \cdot \underbrace{\left( \frac{1 \text{N}}{1 \text{kg} \cdot \text{m} / \text{s}^2} \right)}_1 \cdot \underbrace{\left( \frac{1 \text{kN}}{1000 \text{N}} \right)}_1 \\ = 8.92 \text{ kN}$$

The net braking force is 8.92 kN. □

<sup>①</sup> An excellent description of good practice is the “Guide for the Use of the International System of Units (SI)” by Barry Taylor, 1998. This is NIST (National Institute of Standards and Technology) publication # 811.

## Use of units in old-style handbooks.

Many standard empirical formulas, formulas based on experience and not theory, are presented in an undimensional or numerical form. The units are not part of the equations. We present the approach here, not because we want to promote it, but because we don’t want your more formal approach to units to stop you from reading and using empirical sources.

### A.1 Advised and ill-advised use of units

**Good use of units** Say a car has a constant speed of  $v = 50 \text{ mi/hr}$  for half an hour. The following is true and expressed correctly.

The distance traveled in time  $t$  is  $x = vt$ , so

$$\begin{aligned} x &= vt \\ &= (50 \text{ mi/hr})(30 \text{ min}) = 50 \cdot 30 \text{ mi} \cdot \text{min/hr} \\ &\quad (\text{Awkward but true!}) \\ &= 50 \cdot 30 \text{ mi} \cdot \text{min} / \text{hr} \underbrace{\left( \frac{1 \text{ hr}}{60 \text{ min}} \right)}_1 \\ &= 25 \text{ mi} \end{aligned}$$

That is, unsurprisingly, the distance covered in half an hour is 25 mi.

**Another good use of units.** If we start with the dimensionally correct formula  $x = (50 \text{ mi/hr})t$  we can differentiate to get

$$v = \frac{dx}{dt} = 50 \text{ mi/hr}.$$

The answer is dimensionally correct without having to think about the units.  $v$  is speed and contains its units,  $x$  is distance and contains its units. In any formula that contains  $t$ ,  $x$  or  $v$  we can substitute *any* time, distance or speed. How far does the car go in one minute? As in the previous example,

$$\begin{aligned} x &= vt \\ &= (50 \text{ mi/hr})(1 \text{ min}) \end{aligned}$$

$$\begin{aligned} &= (50 \text{ mi/hr})(1 \text{ min}) \underbrace{\left( \frac{1 \text{ hr}}{60 \text{ min}} \right)}_1 \\ &= \frac{5}{6} \text{ mi} \end{aligned}$$

**Not such good use of units** It is common practice to write sentences like ‘the distance the car travels is

$$x = 50t,$$

where  $x$  is the distance in miles and  $t$  is the time of travel in hours’, although we discourage it. Why? Because the variables  $x$  and  $t$  are ambiguously defined. We would like to use the fact that speed  $v$  is the derivative of distance with respect to time:

$$v = \frac{dx}{dt} = \frac{d}{dt}(50t) = 50.$$

But now we have a speed equal to a pure number, 50, rather than a dimensional quantity. In this simple example, common sense tells us that the speed  $v$  is measured in mi/hr. But if we want to think of  $v$  as a speed, a variable with dimensions of length divided by time, the formula misleads us and requires us to add the units. For this simple example it is not much of a problem to determine what units to add.

But better is if units are included correctly in the equations; then they take care of themselves whenever they are needed. The ‘not such good’ use of units above is sometimes called using numerical equations, that is equations that have numbers in them only. The good use of units uses quantity equations, that is equations that use dimensional quantities.

For example, Mark's *Handbook for Mechanical Engineers* (8th edition, page 8-138) presents the following useful formula to describe the working life of commercially manufactured ball bearings:

$$L_{10} = \frac{16,700}{N} \left( \frac{C}{P} \right)^K,$$

where

- $L_{10}$  = the number of hours that pass before 10% of the bearings fail,
- $N$  = the rotational speed in revolutions per minute
- $C$  = the rated load capacity of the bearing in lbf,
- $P$  = the actual load on the bearing in lbf, and
- $K$  = 3 for ball bearings, 10/3 for roller bearings.

In this approach the idea of dimensional consistency has been disguised for the sake of brevity.  $L_{10}$ ,  $N$ ,  $C$ , and  $P$  are just numbers. Such an equation is sometimes called a 'numerical equation'. It is a relation between numerical quantities. If you happen to know the rotation speed of the shaft in radians per second instead of revolutions per minute you will have to first convert before plugging in the formula. Unlike a dimensional formula, the formula does not help you to convert these units.

## Units with calculators and computers

Unfortunately, most calculators and computers are not equipped to carry units. They are only equipped to carry numbers. How do we handle this problem? The best and clearest option is only to do calculations with dimensionless variables.

The simplest way to use dimensionless variables, though not necessarily the best, is to do something that involves notational compromise. For example, let  $x$  represent dimensionless distance rather than distance. That is,  $x$  represents distance divided by 1 mi. Similarly,  $t$  is time divided by 1 hr. And  $dx/dt$  is dimensionless distance differentiated with respect to dimensionless time, which is, evidently, dimensionless speed. In this example, recovering the dimensional speed is common sense: speed is in mi/hr. The notational compromise is that  $v$  is being used to represent both dimensional and dimensionless speed, with the precise meaning depending on context

## A.2 An improvement to the old-style handbook approach

An alternative to the standard approach to empirical formulas is to write a formula that makes sense with any dimensional variables. The bearing life formula would be replaced with the formula below:

$$l_{10} = \frac{16,700}{n} \left( \frac{c}{p} \right)^K \text{ hr} \cdot \text{rev/min}$$

where

- $l_{10}$  = the time that passes before 10% of the bearings fail,
- $n$  = the rotational speed,
- $c$  = the rated load capacity of the bearing,
- $p$  = the actual load on the bearing, and
- $K$  = 3 for ball bearings, 10/3 for roller bearings.

and the variables  $l_{10}$ ,  $n$ ,  $c$ , and  $p$  are dimensional quantities. One can use any dimensions one wants for all of the variables. For example, using

$$n = 50 \text{ rev/sec}$$

$$c = 1 \text{ kN}$$

$$p = 100 \text{ lbf}, \text{ and}$$

$$K = 3 \text{ for the given ball bearing,}$$

we can calculate the life of the bearing by plugging these values in

to the formula directly.

$$\begin{aligned} l_{10} &= \frac{16,700}{50 \frac{\text{rev}}{\text{sec}}} \left( \frac{1 \text{ kN}}{100 \text{ lbf}} \right)^3 \text{ hr} \cdot \text{rev/min} \\ &= \frac{16,700}{50 \text{ rev/sec}} \underbrace{\left( \frac{60 \text{ sec}}{\text{min}} \right)}_1 \\ &\quad \left( \frac{1 \text{ kN}}{100 \text{ lbf}} \underbrace{\left( \frac{1 \text{ lbf}}{4.448 \text{ N}} \right)}_1 \underbrace{\left( \frac{1 \text{ N}}{1000 \text{ kN}} \right)}_1 \right)^3 \text{ hr} \cdot \text{min/rev} \\ &\approx 45000 \text{ hr} \end{aligned}$$

This approach has the advantage of precision if mixed units are used. Any of the quantities can be measured with any units and the answer always comes out right.

- **Example:** Using notational compromise we can use the formula  $x = vt$  with  $v = 50 \text{ mi/hr}$  to do a set of calculations. Say we want to know the distance  $x$  every quarter of an hour for two hours. So we multiply 50 by .25, .5, .75, ... and thus make a table with two columns labeled  $t$  (hr) and  $x$  (mi).

$t$ (hr)	$x$ (mi)
0	0
.25	12.5
.5	25
.75	37.5

- **Example:** The exact meaning of the columns in the above example are a little ambiguous. We can make it more precise by labeling the columns as follows

$t$ / (hr)	$x$ / (mi)
0	0
.25	12.5
.5	25
.75	37.5

That is, the columns of numbers are dimensionless. The first column, is the time divided by one hour the second is distance divided by one mile.

- **Example:** If we take  $x$  to be dimensional distance,  $t$  to be dimensional time, and  $v$  to be dimensional speed, we can define new dimensionless variables.  $t^* = t/(1 \text{ hr})$ ,  $x^* = x/(1 \text{ mi})$ , and  $v^* = v/(1 \text{ mi/hr})$ . Now there is no ambiguity:  $x$  is dimensional and  $x^*$  is dimensionless. This approach is more precise, if cumbersome, than using  $x$  to be both dimensional and dimensionless depending on context. Dividing the equation  $x = vt$  on both sides by one mile, and multiplying the right side by 1, in the form of  $1 = (1 \text{ hr}/1 \text{ hr})$  we get:

$$\frac{x}{1 \text{ mi}} = \frac{v}{1 \text{ mi/hr}} \cdot \frac{t}{1 \text{ hr}}$$

which is, using the dimensionless variables,

$$x^* = v^* t^*.$$

Because  $v$  is  $50 \text{ mi/hr}$ ,  $v^* = 50$ . We can show this reasoning somewhat formally as follows.

$$v^* = v/(1 \text{ mi/hr}) = (50 \text{ mi/hr})/(1 \text{ mi/hr}) = 50.$$

The dimensionless speed  $v^*$  is just the dimensionless number 50. Now we can make a table by multiplying 50 by .25, .5, .75, .... The columns of the table can be labeled  $t^*$  and  $x^*$  and all variables are clearly defined.

$t^*$	$x^*$
0	0
.25	12.5
.5	25
.75	37.5

Most often, most people will not go to such trouble unless they have confused themselves by not being careful. But it is easy to get in doubt if problems get complicated, if you loose track of what the difference is between a pound force and a pound mass, or if some variables are measured in meters and others in feet, etc.

### A.3 Force, Weight and English Units

The force of gravity on an object is its weight — well, almost. A given object has different weight on different parts of the earth, with up to 0.5% variation. That is,  $g$ , the earth's gravitational 'constant,' varies from about  $9.78 \text{ m/s}^2$  at the equator to about  $9.83 \text{ m/s}^2$  at the North Pole. The official value of the 'constant'  $g$  is in between at exactly  $9.80665 \text{ m/s}^2$  (about  $32.1740486 \text{ ft/s}^2$ ). Multiplying the official  $g$  by the mass  $m$  will give you exactly the force it takes to hold it up if you are in exactly the official place, somewhere in Potsdam. Outside of Potsdam you have to accept an error of up to 1/4% when calculating gravitational forces — unless you know the value of  $g$  in your neighborhood very accurately.

Historically, people understood weight before they understood mass: bigger things are harder to hold up so they have more weight. This relationship is easier to perceive than that bigger things are harder to accelerate, i.e., have more mass. So people defined the quantity of matter by weight. 'How much flour?' one would ask. 'A pound of flour,' meaning one pound weight, might be the answer. A one pound weight is pulled with a 1 lbf by gravity, or in the older notation where one did not worry about mass, by 1 lb. People didn't notice that it was a little harder, i.e., would stretch a given spring more, to hold something up on the north pole than at the top of Mount Everest, so the earth's gravity force on an object was a fine measure of quantity.

When it became important to talk about mass, as opposed to weight, the pound mass was defined as the mass of something that weighed a pound. That is,

$$1 \text{ lbm} \equiv 1 \text{ lbf/g.}$$

Then people thought 'what is the mass that accelerates one foot per second squared if a one-pound force is applied?' They found

$$\begin{aligned} m &= F/a \\ &= (1 \text{ lbf})/(1 \text{ ft/s}^2) = 1 \left( \frac{\text{lbf}}{\text{ft/s}^2} \right) \underbrace{\left( \frac{32.174 \text{ lbm ft/s}^2}{\text{lbf}} \right)}_1 \\ &= 32.174 \text{ lbm.} \end{aligned}$$

But this 32.174 was awkward. People felt that if a unit force causes something to accelerate at a unit rate that thing should have a unit mass. So they invented the slug.  $1 \text{ slug} \equiv 1 \text{ lbf/(1 ft/s}^2)$ . So what do we get for the mass in the previous equation?

$$\begin{aligned} m &= F/a \\ &= (1 \text{ lbf})/(1 \text{ ft/s}^2) = 1 \left( \frac{\text{lbf}}{\text{ft/s}^2} \right) \underbrace{\left( \frac{32.174 \text{ lbm ft/s}^2}{\text{lbf}} \right)}_1 \\ &\stackrel{\text{def}}{=} 1 \text{ slug} \end{aligned}$$

That is, 1 slug accelerates  $1 \text{ ft/s}^2$  when 1 lbf is applied. How much does a slug weigh? The force of gravity on a slug, in Potsdam, is 32.174 lbf.

Now the invention of the slug did not make people happy enough. They thought, 'what is the force required to accelerate 1 lbm at an acceleration of  $1 \text{ ft/s}^2$ ?' It is

$$\begin{aligned} F &= ma \\ &= (1 \text{ lbm})(1 \text{ ft/s}^2) = 1 \left( \frac{\text{lbf m ft/s}^2}{\text{lbm ft/s}^2} \right) \underbrace{\left( \frac{1 \text{ lbf}}{32.174 \text{ lbm ft/s}^2} \right)}_1 \\ &= \left( \frac{1}{32.174} \right) \text{ lbf.} \end{aligned}$$

People found this  $1/32.174$  awkward also, so in order to simplify some arithmetic and confuse many generations of engineers, they invented the poundal. They defined the poundal to be the force it takes to accelerate one pound mass at one foot per second squared. So they got

$$\begin{aligned} F &= ma \\ &= (1 \text{ lbm})(1 \text{ ft/s}^2) \stackrel{\text{def}}{=} 1 \text{ pdl.} \end{aligned}$$

So, because scientists and engineers of old liked the number 1 better than both the number 32.174 and the number  $1/32.174$  they left us two new units to worry about: the poundal =  $1 \text{ lbm ft/s}^2 = (1/32.174) \text{ lbf}$ , and the slug =  $1 \text{ lbf/(ft/s}^2) = 32.174 \text{ lbm}$ . If you are used to the internationally acceptable units for force and mass  $1 \text{ pdl} = .138255 \text{ N}$  and  $1 \text{ slug} = 14.5939 \text{ kg}$ . Fortunately, the slug and the poundal are used less and less as the decades roll by. Certainly there are more people who laugh at their confusion about slugs and poundals than there are people who use them seriously.

**Don't laugh if you are from Europe** Unfortunately for dimensional purists, engineers using the SI system have copied one of the confusing traditions that the SI system was designed to avoid. They invented the kilogram-force, kgf, also called a kilopond, which is  $1 \text{ kg}$  times the official value of  $g$ . That is  $1 \text{ kgf} = 1 \text{ kilopond} = 9.80665 \text{ N}$ . A kilopond is the force of gravity on a kilogram, exactly so somewhere in Potsdam — well, almost.

**Well, almost** Why do we say 'well, almost' about ' $g$ ' being the acceleration due to gravity? Because, unfortunately and confusingly,  $mg$  is not the force due to gravity, it is the force of the spring which holds up the mass on a rotating earth! What is called  $g$  is the 'effective' gravity which is the acceleration due to gravity minus a centripetal term due to the earth's rotation.



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# B

# Contact: friction and collisions

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The primary interaction between reasonably sized objects, say much smaller than the earth and much larger than an atom, is through contact<sup>①</sup>. Things cause contact forces on each other when and where they touch. Contact between two bodies restricts their possible motions and also determines the forces on the bodies. In order to study the dynamics (or statics) of such systems we need a model for the relation between the contact forces and the motions between the contacting objects. Of special concern are sliding, rolling, and collisions.

The subjects of friction, rolling and collisions have two major components: the *rules* of interaction and the motion of systems that follow these rules.

- The first part of contact mechanics is the description of the forces and kinematic constraints in contact interactions, the rules of interactions. The fancy name for such rules is *constitutive* laws. Thus we need a constitutive law for friction, a constitutive law for collisions, and a constitutive law for rolling contact. Since all of these behaviors are complex and no-one understands any of them very well, there are many candidate constitutive laws for all three of the cases: friction, rolling, and collisions. They vary in their conceptual simplicity, their ease of use in analytical or numerical calculations, and their accuracy and applicability. We will present the simplest rules, describe some of the shortcomings and then give some guidance towards more sophisticated rules.
- The second part of contact mechanics is the analysis of motion of various mechanical systems that involve contact. Since most mechanical systems involve contact in some way or another, this second part of contact mechanics is a huge subject. It nearly encompasses all of dynamics since most mechanical systems involve contact of some kind or another.

<sup>①</sup> The most-often encountered non-contact force is gravity, especially the gravity force from the earth on terrestrial objects like cars, people, rocks, and ants. In one way of looking at things, forces mediated by massless springs, strings, dashpots, and rods are also not contact forces. Since the mass of springs, strings, rods, and dashpots are often neglected in dynamics, they are two-force members and the forces at their two ends are equal and opposite. So rather than saying body  $\mathcal{A}$  interacts with a spring which interacts with body  $\mathcal{B}$ , for example, one might say that the spring between body  $\mathcal{A}$  and  $\mathcal{B}$  mediates a force between them. In this case the spring force would not be a contact force.

Our primary intent in this chapter is to communicate some of the simplest contact models and also highlight their short-comings. We start with a discussion of the apparently unavoidable inaccuracies of contact laws before discussing some popular laws in more detail. As far as the consequences of these laws for real systems go, we leave that for the other chapters.

## B.1 Contact laws are all rough approximations

Unfortunately, there are no simple and accurate general rules for describing contact forces. When we study the dynamics of a system that involves the interaction of bodies we are forced to use one or another approximate description for finding the forces of interaction in terms of the bodies positions and velocities. Such a description is called a *constitutive law* or *constitutive relation*. Generally people write separate constitutive laws after categorizing the motion into being one of the three major types of contact interaction: friction, rolling, or collision. ①

We must emphasize at the outset:

Constitutive laws for contact interaction are generally only rough approximations, with theory and practice differing by 5-50% for at least some of the quantities of interest.

Equations for forces of contact are of a lower class than the fundamental equations in mechanics. At the scale of most engineering, the momentum balance equations are extremely accurate, with error of well less than a part per billion. Newton's law of gravitational attraction is a similarly accurate law. And the laws of Euclidean (non-Riemannian) geometry and calculus (the kinds you studied) are also extremely accurate. Less accurate are the laws for spring's and dashpots. But still, accuracies of one part per thousand are possible for measuring spring stiffness, say, and perhaps parts per hundred for dashpot constants.

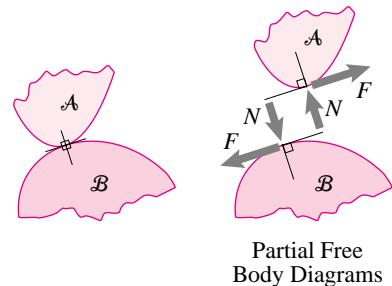
But the laws for the contact interactions of solids are much less accurate. Not only is it difficult to know the coefficient of friction between two pieces of steel with any certainty, you also can't trust even the *concept* of a coefficient of friction to have any great accuracy. It is easy to forget this inaccuracy in contact laws when one starts to do engineering calculations because you will see contact-force equations in books. Once we see an equation in print, we are too-easily tempted into believing it is 'true.' An easy mistake to make is to use contact constitutive equations with confidence, as if accurate, to get results that are in-fact only rough approximations at best.

① In practice it is not always clear how to make the distinctions between sliding and rolling or between sliding and collision. But at least for a first pass it is a useful conceptual distinction to think of sliding, rolling, and collisions as three different kinds of contact.

## B.2 Friction

When two objects are in contact and one is sliding with respect to the other, we call the force which resists this sliding *friction*. Frictional contact is usually assumed to be either ‘lubricated’ or ‘dry.’ When bodies are in lubricated contact they are not in real contact at all, a thin layer of liquid or gas separates them. Most of the metal to metal contact in a car engine is supposed to be lubricated. The contact of the car tires with the road is ‘dry’ unless the car is ‘hydroplaning’ on worn-smooth tires on a very wet road. The friction forces in lubricated contact are very small compared forces of unlubricated contact. For many purposes lubricated friction forces are neglected. There is no quick way to estimate these small lubricated slip forces. The accurate estimation of lubricated friction forces requires use of lubrication theory, a part of fluid mechanics. We will drop the discussion of lubricated friction forces because they are often negligible and because estimating them is too hard.

Dry friction forces are not small and thus cannot be sensibly neglected in dynamics problems involving sliding contact. The simplest model for friction forces is called *Coulomb’s law of friction* or just *Coulomb friction*. But, use of even this law is full of subtleties.



Partial Free Body Diagrams

Figure B.1: Two bodies in contact. The forces between them satisfy the law of action and re-action. It is often convenient to decompose the force of interaction into a part  $F$  tangent to the surface of interaction and a part  $N$  perpendicular to the surface of interaction.

(Filename: tfigure11.contact)

### ‘Rough’ and ‘smooth’

People sometimes use the words *rough* and *smooth* to denote dry friction and lubricated friction. This language seems to make sense because, when wet, very smooth things get slippery. However, despite the dominance of this intuition, in fact dry solids do not generally have less friction when polished than when rough. So

we do not use the words rough and smooth in this book to indicate high and low friction.

### Coulomb friction

Coulomb’s law of friction, also attributed to Amonton and DaVinci, is summarized by the simple equation:

$$F = \mu N. \quad (\text{B.1})$$

This equation, like many other simple equations, is not really a complete description of Coulomb’s law of friction. Some words are required.

First of all the direction of the force  $F$  on body  $\mathcal{A}$  is in the opposite direction of the slip velocity of  $\mathcal{A}$  relative to  $\mathcal{B}$ . By the principle of action and reaction we deduce that the force on body  $\mathcal{B}$  is in the opposite direction. This force is also opposite to the relative slip velocity of  $\mathcal{B}$  relative to  $\mathcal{A}$ . That is,  $F$  resists relative motion of  $\mathcal{A}$  and  $\mathcal{B}$ .

The friction force  $F$  is proportional to the normal force  $N$  with the proportionality constant  $\mu$ . The constant  $\mu$  is assumed to be independent of the area of contact between bodies  $\mathcal{A}$  and  $\mathcal{B}$ . In the simplest renditions of Coulomb’s law  $\mu$  is assumed to be independent of slip distance, slip velocity, time of contact, etc. When contacting bodies are not sliding the role of friction changes somewhat. In some sense the friction still resists slip, in fact it is the presence of the friction force that prevents slip. But another way to think of friction is that it puts an upper limit on the size of the force

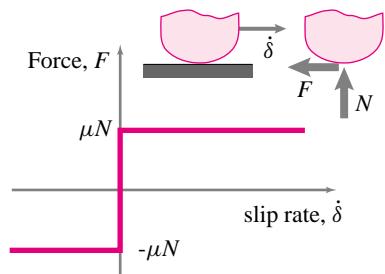


Figure B.2: **Coulomb friction.** The relation between friction force  $F$  and relative slip rate  $\delta$  is described by the dark line. Since there is a jump from  $-\mu N$  to  $\mu N$  in the friction force when the slip rate goes from negative to positive the relation is not a proper mathematical function between  $F$  and  $\delta$ . Instead the relation is a curve in the  $F, \delta$  plane.

(Filename: tfigure11.coulomb)

of interaction between two bodies which seem stuck to each other. The friction force must be less than or equal to  $\mu N$  in magnitude during contact.

$$|F| \leq \mu N \quad (\text{B.2})$$

All of the discussion above can be summarized with the following equations for the friction force

$$\begin{aligned} & \boxed{\text{The friction force, the part of the force of interaction which is tangent to the surface.}} & & \boxed{\text{The relative slip velocity of the contacting points.}} \\ & \downarrow & & \downarrow \\ \overbrace{\vec{F}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}}}^{} & = -\mu \overbrace{\frac{\vec{v}_{\mathcal{A}/\mathcal{B}}}{|\vec{v}_{\mathcal{A}/\mathcal{B}}|}}^{} N & & \text{during slip} \\ \underbrace{|\vec{F}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}}|} & \leq \underbrace{\mu N}_{\text{An upper bound on the tangential part of the contact force}} & & \text{during stationary contact} \\ & \uparrow & & \uparrow \\ & \boxed{\text{The magnitude of the tangential part of the contact force}} & & \boxed{\text{An upper bound on the tangential part of the contact force}} \end{aligned}$$

For two-dimensional problems where slip can only be in one direction (or the opposite) this pair of functions describes the dark line in the friction graph of figure B.2 in which  $\delta$  is the speed of relative slip.

In practice, to use these equations you (or your computer) may have to solve mechanics problems two or three times. First assume no-slip, solve the problem, calculate the friction force and then make sure that it is less than  $\mu N$ . If not then the assumption of no-slip leads to a violation of the friction law. Then you solve the problem again assuming there is slip to the right and therefore a known friction force to the left. The solution of the problem will tell you the direction of slip for this applied force. If this slip is to the left you have found a contradiction. You have to start again assuming slip in the left and a force to the right; then make sure that the predicted slip is indeed to the left. Most often you will find a paradox in two of the three possibilities (slip to the left, no slip, and slip to the right), thus leaving you with only one solution that satisfies both Newton's laws and the equations of friction.

### B.1 Another expression for Coulomb friction: an advanced aside

The law of Coulomb friction is both simple and confusing. Part of the confusion comes from the requirement of calculating the friction force with one equation during slip and then not being able to find the friction force, at least from the friction law, when there is no slip.

We would like to confuse the issue a little further now for the case of slip on a plane. That is, slip can be in any direction on, say, the  $xy$ -plane, not just to the right or to the left. If we define  $\vec{v}$  to be the sliding velocity of the point of contact of the body of interest relative to its partner in friction, and  $\vec{F}$  to be the tangential contact force that it causes on its partner then we could write the friction law with a pair of inequalities which must both be satisfied at all times.

$$\begin{aligned}\vec{v} \cdot (\vec{F} - \vec{F}^*) &\geq 0 \\ |\vec{F}| &\leq \mu N\end{aligned}$$

where  $\vec{v}$  and  $\vec{F}$  are such that these inequalities are satisfied for every possible  $\vec{F}^*$  that is tangent to the slip surface and has magnitude  $|\vec{F}^*| \leq \mu N$ . The force  $\vec{F}^*$  is not any actual force in the problem. It is just a label for the set of all possible friction forces consistent with the friction law.

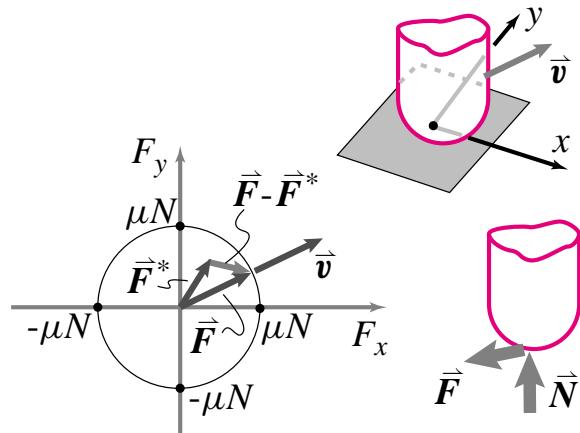
The meaning of these inequalities can be seen in the figure below. If you are a normal reader you will have two questions:

- How is this inequality the same as the Coulomb friction law described in the text? And,
- why bother to write the friction law in this strange way?

The answer to the first question, how are these inequalities an expression of Coulomb friction, is found by considering the various cases of slip and no-slip. To show that these inequalities imply that friction directly opposes motion during slip takes a little thought. The reason is that  $\vec{F}^*$  can be any point on or inside the circle shown. By

considering the two cases that  $\vec{F}^*$  is on the limit-circle just clockwise and just counterclockwise from the actual friction force you can see that for the inequality  $(\vec{F} - \vec{F}^*) \cdot \vec{v} \geq 0$  for both cases,  $\vec{v}$  must be perpendicular to the circle.

The answer to the second question, why bother, is: The pair of inequalities shown allow the proof of various theorems about frictional sliding, allow a simple description of friction on distributed contacts, and also allows a simple generalization to friction that is anisotropic, that is, of different magnitude in different directions of slip. For those who are going on to study advanced solid mechanics, this expression for the friction law shows one of the connections between friction and classical plasticity.



## B.3 A short critique of Coulomb friction

In short, Coulomb's law of friction is good because

- Coulomb's law of friction is simple.
- Coulomb's law of friction usefully predicts many phenomena.
- It has the right trends in many regards, in that
  - sliding friction *is* roughly independent of slip rate, and
  - the friction force *is* roughly proportional to the normal force.
- Other candidate laws (generally) cost more in complexity than they gain in accuracy or usefulness.

On the other hand,

- The friction coefficient is not stable, it may vary from day to day or between samples of nearly identical materials.
- Coulomb's law, without a separate static coefficient of friction or an explicit dependence on rate of slip, cannot be used to explain frictional phenomena such as
  - the squeaking of doors,
  - the excitement of a violin string by a bow, and
  - earthquakes from sliding rocks.
- For some materials the dependence of friction on normal force is noticeably different from linear. Rubber on road, for example, has more friction force per unit normal force when the normal force is low. In other words the friction force for a given normal force is greater when the area of contact is greater. This dependence of friction on normal stress is presumably why racing cars have fat tires.

We expand on some of these points below.

### The friction coefficient is not a stable property

Jaeger, a famous rock mechanician, is said to have presented the following empirical friction law:

*A friction experiment will make a monkey out of you.*

For any pair of objects and any given experiment to measure the friction coefficient, the measured value will likely vary from day to day. This observation seems to violate our common notions of determinacy. Why does this apparent indeterminacy happen? Probably because friction involves the interaction of surfaces. The chemistry of a surface can be dramatically changed by very small quantities of material (a surface is a very small volume!). So any change in humidity, or perhaps a random finger touch, or a slight spray from here or there can dramatically change the surface chemistry and hence the friction.

This problem of the non-constancy of friction from day to day or sample to sample cannot be overcome by a better friction law. So unless one understands one's materials and their chemical environment extremely well, all friction laws, however sophisticated are doomed to large inaccuracy.

## Coulomb's friction law neglects the drop in the friction force at the start of sliding

Most simple treatments of friction immediately introduce two coefficients of friction. The sliding coefficient is also sometimes called the dynamic coefficient  $\mu_d$  or the kinetic coefficient  $\mu_k$ . The other coefficient of friction is the ‘static’ coefficient of friction  $\mu_s$ .

According to standard lore, each pair of bodies has friction which is described by the static and dynamic coefficients of friction  $\mu_s$  and  $\mu_d$  with the understanding that the static coefficient of friction is greater than the dynamic coefficient of friction,  $\mu_s > \mu_d$ .

According to this description, the relation between friction velocity and friction force is as given in figure B.3. This description is useful for roughly characterizing the following phenomenon:

*It is harder to start something sliding than it is to keep it sliding.*

If the dynamics problem you are working on depends on this phenomenon the static-dynamic friction law is one way of treating it. But you should be forewarned that, though this law is great for qualitatively explaining how a bow excites a violin string, or why anti-lock brakes work better than all out skidding, it is not very accurate.

If one does careful experiments to try to understand in more detail how the friction force drops from a higher value to a lower one as slip starts, one discovers a world of phenomena that are not well captured with two simple coefficients of friction. Further, using two coefficients of friction leads to various paradoxes and indeterminacies when one studies slightly more complex problems. (See box B.2 on page 718.)

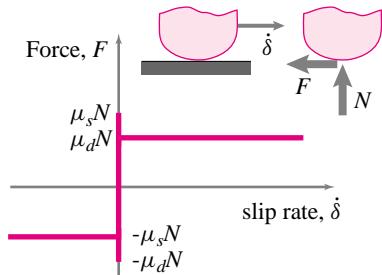


Figure B.3: **Static-Dynamic Friction.** The relation between friction velocity and friction force is such that at all times the pair of values is found on the dark line shown.

(Filename:tfigure11.static)

## Friction is not always proportional to normal force

The Coulomb friction equation, applicable during slip or at impending slip,

$$F = \mu N$$

is most directly translated into English as: *the friction force is proportional to the normal force*. This proportionality is, as far as we know, not fundamental, but rather an often reasonable approximation to many experiments. Why the interaction of so many solids obeys this proportionality so well is not known, though there are a few explanations that make this experimental result theoretically plausible.

In some books you will see an additional law of friction stated as:

*The friction force is independent of the area of contact.*

By ‘area of contact’ is meant the area you would measure macroscopically. For a 4 in  $\times$  8 in brick sliding on a pavement the area of contact is 32 in<sup>2</sup><sup>①</sup>. The independence of force with area is actually equivalent to the proportionality of friction force with normal force. Let’s explain, or at least let’s give the gist of the argument. Imagine two identical blocks side by side on a plane as in figure B.4. The force pushing down on each is  $N$  and the friction force to cause slip is  $F = \mu N$ . The act of glueing the two together side-by-side should have no effect. Now we have one bigger block with twice the normal force, twice the friction force and twice the area of contact. If we assume that friction force is proportional to normal force, we know that if we now cut the normal force in half then the friction force will be cut in half. But now we have a new block with twice the area of contact as each of the original blocks and it carries the same normal force and the same friction force. Thus the friction force is unchanged by doubling the area of contact.

But in fact, some materials have friction force which does depend on the normal force, or for a given normal force, does depend on the area of contact. The most prominent example is the friction between rubber and pavement. For a given weight car, a larger friction force can be generated with a fat tire than a narrow one. That is, the ratio of the friction force to normal force decreases as the normal force increases.

① Another concept of area of contact is the actual area of contact at all the little asperities. This definition of area of contact is useful for tribologists (people who study friction) but is of little concern to people interested in the mechanics of macroscopic things.

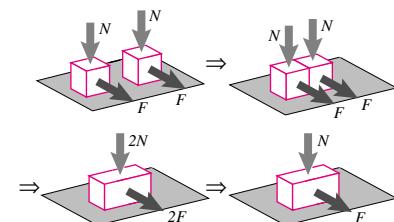


Figure B.4: Considering two blocks side by side as one block shows how friction being proportional to normal force means friction is independent of area of contact. The two blocks have twice the area as one block, but a given normal force causes the same friction force.

(Filename:tfigure11.area)

## B.2 A problem with the concept of static friction

The commonly used static friction law assumes that the friction force instantly jumps from the static value  $\mu_s N$  to the dynamic value  $\mu_d N$  when slip starts. If the contacting surfaces have more than one contact point this jump from static to dynamic friction implicitly makes use of two simultaneous limits.

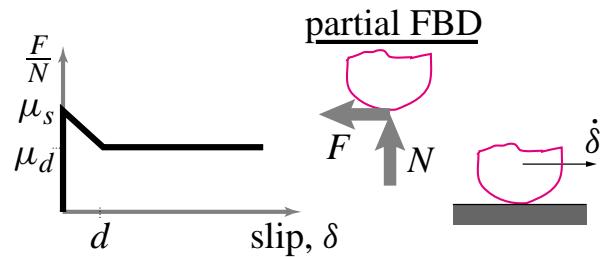
1. One limit is that the body is *infinitely* stiff.
2. The other limit is that the coefficient of friction *instantly* drops from the static value  $\mu_s$  to the dynamic value  $\mu_k$  when slip starts.

That there is a problem with simultaneous use of these limits is highlighted by considering a body that has finite stiffness and for which the friction gradually drops as slip starts. We should then *hope* to recover the concept of static friction and rigid body slip as a limit of this model. But, there is trouble.

Let's get a little more specific.

### Slip weakening friction law

The friction law we will employ is a 'slip-weakening' friction law described by the graph below. Although this description is obviously incomplete if slip occurs more than once or reverses direction, it suffices for our considerations.



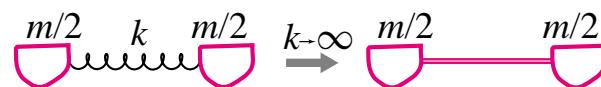
Here the describing equations are:

$$\begin{aligned} F/N &\leq \mu_s && \text{if no slip has occurred} \\ F/N &= \mu_s - (\mu_s - \mu_d)\delta/d && \text{if } \delta \leq d \\ F/N &= \mu_d && \text{if } \delta \geq d \end{aligned}$$

If we keep  $\mu_s$  and  $\mu_d$  constant and look at the limit as  $d \rightarrow 0$  this friction law becomes the classic 'static-dynamic' friction law which we are now critiquing. There are other friction laws, such as those with rate dependence, we could use that reduce to static-dynamic friction in some limit, but these laws also would lead to problems something like those we discuss below.

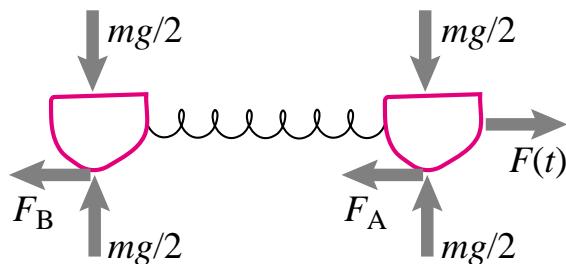
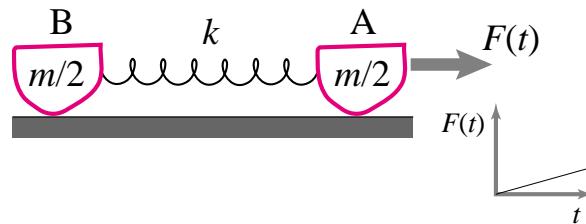
### Model of a rigid body

The model for a rigid body that we employ is: two point masses are connected with a spring. In the limit as the spring constant  $k$  goes to infinity this model becomes, at least in the common sense of the words, a 'rigid body.'



### A sliding problem

Now let's consider the problem of initial slip of the system shown. For simplicity and definiteness let's assume the spring is relaxed in the configuration shown.



Now, we very slowly increase the applied force  $F$  until both masses end up sliding. Here is the question:

*What force does it take to start the pair of masses sliding?*

There is no force causing block B to slide except the spring tension. But the spring tension does not build until the spring stretches due to the slip of A. So  $F$  increases until  $F = \mu_s mg/2$  at which time block A starts to slide. When the tension in the spring reaches  $\mu_s mg/2$  then block B starts to slide. What is the value of  $F$  at this time?

We could work out the details for every value of the parameters, but we need not do this generality to make our point. First let's take the limit that the body is rigid  $k \rightarrow \infty$ .

### Rigid body, gradual friction drop

In this case the two masses always move together. As slip starts the two masses both have a friction force of  $\mu_s mg/2$  and the force required to cause slip of both masses is

$$F = \mu_s mg \text{ at slip,}$$

as expected. As motion progresses this force gradually reduces to  $\mu_d mg$  at a rate that depends on the frictional slip weakening distance  $d$ . But for any finite value of  $d$  the applied force must first reach  $\mu_s mg$  before slip proceeds.

### Compliant body, sudden friction drop

Now let's take the limit the other way. Let's assume that the spring has fixed stiffness, possibly very high, and look at the limit  $d \rightarrow 0$ , the limit which reduces the friction law to the classical law. In this case block A breaks entirely free before there is any tension in the spring. Exactly when block B will start to slip depends on the details of all the parameters, so it turns out that finding the start of slip of block B is a genuinely complicated problem. But, no matter what,

the spring stretches some before block A comes to rest. Block A may slip several times before the spring stretch is enough to cause the slip of block B, again the details depend on the relative values of  $\mu_s$  and  $\mu_d$ . But eventually block B will be excited into sliding. This slip will most likely start when block A is already sliding. Thus the applied force need only overcome the *dynamic* friction of block A  $\mu_d mg/2$  and the static friction  $\mu_s mg/2$  of block B. Due to the complex dynamics of the situation, it turns out that the two blocks can sometimes end up sliding if the applied force is just a hair above  $\mu_d mg$ , even when  $K$  is very large (but still finite).

$$F = ? \text{ (something less than } \mu_s mg)$$

## The rigid-body static-friction paradox

If we take the limit  $k \rightarrow \infty$  and then  $d \rightarrow 0$  we get an overall effective coefficient of static friction  $\mu_s$  for the whole body. If instead we take the limit  $d \rightarrow 0$  and then  $k \rightarrow \infty$  the effective static friction limit does not exist, but for some arbitrarily large values of  $k$  it can be as low as  $\mu_d$ . That is,

*the problem of initial slip of a rigid body with more than one point of contact and with static-dynamic friction is ill-defined*

This paradox can be resolved a number of ways. One is to assume it away, effectively taking the  $k \rightarrow \infty$  limit first. Another more complex solution, beyond this book and beyond the level of detail that most people want to deal with, is to only use more sophisticated friction laws and to keep track of solid deformation.

## Compromise

To avoid these issues by users of this text we just use one coefficient of friction  $\mu_s = \mu_d = \mu$ . May the user beware if using a more complex law than this one.

### All things considered, Coulomb's law is alright

In this book we generally assume that  $\mu_d = \mu_s = \mu$ ; there is just one coefficient of friction. Most often it is reasonable to assume that static friction is close enough to dynamic friction that it is not worth the trouble to distinguish them. Of course there are situations which one may want to understand where the transition from static to dynamic friction is essential. For these cases a static-dynamic friction model might provide some insight, but it may also cause basic modeling problems. Coulomb's law with one coefficient of friction is the simplest dry friction constitutive law. It is the appropriate description for most purposes. It is reasonably accurate and more elaborate laws are not particularly more accurate.

## B.4 Collision mechanics

When two solids bump into each other there must be a nearly discontinuous change in their velocities and/or angular velocities to keep the bodies from interpenetrating. This sudden change in velocity can only come about when there are very large forces of interaction. The estimation of these short-lived yet large forces and their effects on the motions of rigid bodies is the central problem in collision mechanics.

### Why is it hard to find a good collision law

Ideally one would like a rule to determine how bodies move after a collision from how they move before the collision. Such a rule would be called a collision law or a constitutive relation for collisions. That accurate collision laws are rare at best might be surmised from a basic problem. Just the phrase *rigid body collisions* is in some sense a contradiction in terms, an oxymoron. The force generated in the contact comes from material deformation, and deformation is just what we generally try to neglect when doing rigid body mechanics.

There is a temptation to say that one wants to continue to neglect deformation during the collision. And most collision laws are formulated with this approach. But such an approach is likely to be doomed to inaccuracy in some situation or another because the actual mechanics of the force generation at the contact comes from material deformation, deformation that is not necessarily restricted to a small neighborhood near the contacting points.

For simple situations, like a ball dropping vertically on a concrete floor, reasonably approximate collision laws may be formulated and used. But for complex shaped bodies touching at various points that are generally not known *a priori*, there is generally no collision law that can be expected to be accurate.

With this caveat, we now introduce the concept of coefficient of restitution  $e$  for use in the simplest collisions.

### The coefficient of restitution $e$ .

The most commonly used collision law can be summarized with this simple equation,

$$\overbrace{(v'_b - v'_a)} = \underbrace{e}_{\text{The coefficient of restitution, assumed to be a constant for given materials.}} \overbrace{(v_a - v_b)}, \quad (\text{B.3})$$

The speed with which colliding points are separating after the collision.

The speed at which colliding points are approaching before collision.

which can be summarized as, *the rate of separation is proportional to the rate of approach*. The coefficient  $e$  is called Newton's or Poisson's coefficient of restitution.

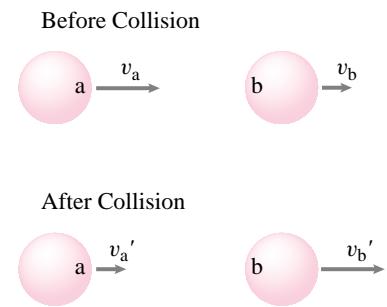


Figure B.5: The rate of approach of two about-to-collide points is  $v_a - v_b$ . The rate of separation after collision is  $v'_b - v'_a$ . The simplest collision law says  $v'_b - v'_a = e(v_a - v_b)$ , where  $e$  is the coefficient of restitution.

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TABLE I  
Momenta and energy

What system	Linear Momentum		Angular Momentum		Kinetic Energy				
	$\bar{L}$	(a)	$\dot{\bar{L}} = \frac{d}{dt}\bar{L}$	(b)	$\bar{H}_C$	(c)	$\dot{\bar{H}}_C = \frac{d}{dt}\bar{H}_C$	(d)	$E_K$
In General	$L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$ $= m_{\text{tot}}\vec{v}_{\text{cm}}$ $= \frac{d}{dt}(m_{\text{tot}}\vec{r}_{\text{cm}})$	$\dot{L}_x\hat{i} + \dot{L}_y\hat{j} + \dot{L}_z\hat{k}$ $= m_{\text{tot}}\vec{a}_{\text{cm}}$ $“F = ma”$	$H_{Cx}\hat{i} + H_{Cy}\hat{j} + H_{Cz}\hat{k}$ $= \vec{r}_{\text{cm}/C} \times \vec{v}_{\text{cm}}m_{\text{tot}} + \vec{H}_{\text{cm}}$ $= \frac{d}{dt}(\text{no such thing})$	$\dot{H}_{Cx}\hat{i} + \dot{H}_{Cy}\hat{j} + \dot{H}_{Cz}\hat{k}$ $= \vec{r}_{\text{cm}/C} \times \vec{a}_{\text{cm}}m_{\text{tot}} + \dot{\vec{H}}_{\text{cm}}$ $= (\text{no simple general expression})$	$\frac{1}{2}m_{\text{tot}} v_{\text{cm}}^2 + E_{K/cm}$	(1)			
One Particle P	$m_P \vec{v}_P$	$m_P \vec{a}_P$	$\vec{r}_{P/C} \times \vec{v}_P m_P$	$\vec{r}_{P/C} \times \vec{a}_P m_P$	$\frac{1}{2}m_P v_P^2$	(2)			
System of Particles	$\sum_{\text{all particles } i} m_i \vec{v}_i$	$\sum_{\text{all particles } i} m_i \vec{a}_i$	$\sum_{\text{all particles}} \vec{r}_{i/C} \times \vec{v}_i m_i$	$\sum_{\text{all particles}} \vec{r}_{i/C} \times \vec{a}_i m_i$	$\frac{1}{2} \sum_{\text{all particles}} v_i^2 m_i$	(3)			
Continuum	$\int_{\text{all mass}} \vec{v} dm$	$\int_{\text{all mass}} \vec{a} dm$	$\int_{\text{all mass}} \vec{r}_{/C} \times \vec{v} dm$	$\int_{\text{all mass}} \vec{r}_{/C} \times \vec{a} dm$	$\frac{1}{2} \int_{\text{all mass}} v^2 dm$	(4)			
System of Systems (eg. rigid bodies)	$\sum_{\text{all sub-systems}} m_i \vec{v}_i$	$\sum_{\text{all sub-systems}} m_i \vec{a}_i$	$\sum_{\text{all sub-systems}} \bar{H}_{Ci}$	$\sum_{\text{all sub-systems}} \dot{\bar{H}}_{Ci}$	$\sum_{\text{all sub-systems}} E_{Ki}$	(5)			

#### Rigid Bodies

One rigid body (2D and 3D)	$m_{\text{tot}} \vec{v}_{\text{cm}}$	$m_{\text{tot}} \vec{a}_{\text{cm}}$	$\vec{r}_{\text{cm}/C} \times \vec{v}_{\text{cm}}m_{\text{tot}} + \underbrace{[\mathbf{I}^{cm}] \cdot \vec{\omega}}_{\bar{H}_{\text{cm}}}$	$\vec{r}_{\text{cm}/C} \times \vec{a}_{\text{cm}}m_{\text{tot}} + \underbrace{[\mathbf{I}^{cm}] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \bar{H}_{\text{cm}}}_{\dot{\bar{H}}_{\text{cm}}}$	$\frac{1}{2}m_{\text{tot}} v_{\text{cm}}^2 + \underbrace{\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^{cm}] \cdot \vec{\omega}}_{E_{K/cm}}$	(6)
2D rigid body in xy plane with $\vec{\omega} = \omega \hat{k}$	$m_{\text{tot}} \vec{v}_{\text{cm}}$	$m_{\text{tot}} \vec{a}_{\text{cm}}$	$\vec{r}_{\text{cm}/C} \times \vec{v}_{\text{cm}}m_{\text{tot}} + \underbrace{I_{zz}^{cm} \omega \hat{k}}_{\bar{H}_{\text{cm}}}$	$\vec{r}_{\text{cm}/C} \times \vec{a}_{\text{cm}}m_{\text{tot}} + \underbrace{I_{zz}^{cm} \dot{\omega} \hat{k}}_{\dot{\bar{H}}_{\text{cm}}}$	$\frac{1}{2}m_{\text{tot}} v_{\text{cm}}^2 + \underbrace{\frac{1}{2}I_{zz}^{cm} \omega^2}_{E_{K/cm}}$	(7)
One rigid body <i>if</i> C is a fixed point (2D and 3D)	$m_{\text{tot}} \vec{v}_{\text{cm}}$	$m_{\text{tot}} \vec{a}_{\text{cm}}$	$[\mathbf{I}^C] \cdot \vec{\omega} = \bar{H}_C$	$[\mathbf{I}^C] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \bar{H}_C$	$\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^C] \cdot \vec{\omega}$	(8)
2D rigid body <i>if</i> C is a fixed point with $\vec{\omega} = \omega \hat{k}$	$m_{\text{tot}} \vec{v}_{\text{cm}}$	$m_{\text{tot}} \vec{a}_{\text{cm}}$	$I_{zz}^C \omega \hat{k}$	$I_{zz}^C \dot{\omega} \hat{k}$ $“M = I\alpha”$	$\frac{1}{2}I_{zz}^C \omega^2$	(9)

The table has used the following terms:

$m_{\text{tot}}$  = total mass of system,

$m_i$  = mass of body or subsystem i,

$\vec{r}_{\text{cm}/C}$  = the position of the center of mass relative to point C,

$\vec{v}_i$  = velocity of the center of mass of sub-system or particle i,

$\vec{a}_i$  = acceleration of the center of mass of sub-system i,

$\bar{H}_{Ci}$  = angular momentum of subsystem i relative to point C.

$\dot{\bar{H}}_{Ci}$  = rate of change of angular momentum of subsystem i relative to point C.

$\bar{H}_{\text{cm}} = \sum \vec{r}_{i/\text{cm}} \times (m_i \vec{v}_i)$  angular momentum about the center of mass

$\dot{\bar{H}}_{\text{cm}} = \sum \vec{r}_{i/\text{cm}} \times (m_i \vec{a}_i)$  rate of change of angular momentum about the center of mass

$\vec{\omega}$  is the angular velocity of a rigid body,

$\dot{\vec{\omega}} = \vec{\alpha}$  is the angular acceleration of the rigid body,

$[\mathbf{I}^{cm}]$  is the moment of inertia matrix of the rigid body relative to the center of mass, and

$[\mathbf{I}^C]$  is the moment of inertia matrix of the rigid body relative to a fixed point (not moving point) on the body.

Table II  
**Summary of methods of calculating velocity and acceleration**

Method	Position	Velocity	Acceleration
In general, as measured relative to the fixed frame $\mathcal{F}$ .	$\vec{r}$ or $\vec{r}_P$ or $\vec{r}_{P/O}$	$\vec{v}$ or $\vec{v}_P$ or $\vec{v}_{P/\mathcal{F}}$	$\vec{a}$ or $\vec{a}_P$ or $\vec{a}_{P/\mathcal{F}}$
Cartesian Coordinates	$r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$	$v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ $= \dot{r}_x \hat{i} + \dot{r}_y \hat{j} + \dot{r}_z \hat{k}$	$a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$ $= \ddot{v}_x \hat{i} + \ddot{v}_y \hat{j} + \ddot{v}_z \hat{k}$ $= \ddot{r}_x \hat{i} + \ddot{r}_y \hat{j} + \ddot{r}_z \hat{k}$
Polar Coordinates/ Cylindrical Coordinates	$R \hat{e}_R + z \hat{k}$	$v_R \hat{e}_R + v_\theta \hat{e}_\theta + v_z \hat{k}$ $= \dot{R} \hat{e}_R + R \dot{\theta} \hat{e}_\theta + \dot{z} \hat{k}$	$a_R \hat{e}_R + a_\theta \hat{e}_\theta + a_z \hat{k}$ $= (\ddot{R} - R \dot{\theta}^2) \hat{e}_R + (R \ddot{\theta} + 2 \dot{R} \dot{\theta}) \hat{e}_\theta + \ddot{z} \hat{k}$
Path Coordinates	not used	$v \hat{e}_t$	$a_t \hat{e}_t + a_n \hat{e}_n$ $= \dot{v} \hat{e}_t + (v^2 / \rho) \hat{e}_n$
Using data from a moving frame $\mathcal{B}$ with origin at $O'$ and angular velocity relative to the fixed frame of $\vec{\omega}_{\mathcal{B}}$ . The point $P'$ is glued to $\mathcal{B}$ and instantaneously coincides with $P$ .	$\vec{r}_{O'/O} + \vec{r}_{P/O'}$	$\vec{v}_{P'/\mathcal{F}} + \vec{v}_{P/\mathcal{B}} =$ $\underbrace{\dot{\vec{r}}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}_{\vec{v}_{P'/\mathcal{F}}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} +$ $\underbrace{\dot{\vec{r}}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}}_{\vec{a}_{P/\mathcal{B}}}$	$\vec{a}_{P'/\mathcal{F}} + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} =$ $\overbrace{\dot{\vec{r}}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}}^{\vec{a}_{P'/\mathcal{F}}} +$ $+ \underbrace{\dot{\vec{r}}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}}_{\vec{a}_{P/\mathcal{B}}}$  'the 5-term acceleration formula'

## Some facts about path coordinates

The path of a particle is  $\vec{r}(t)$ .

$$\hat{e}_t \equiv \frac{d \vec{r}(s)}{ds}, \quad \hat{e}_t = \frac{d \vec{r}(t)}{dt} \frac{dt}{ds} = \frac{\vec{v}}{v}, \quad \vec{\kappa} \equiv \frac{d \hat{e}_t}{ds} = \frac{d \hat{e}_t}{dt} \frac{1}{v}, \quad \hat{e}_n = \frac{\vec{\kappa}}{|\vec{\kappa}|}, \quad \mathbf{e}_b \equiv \hat{e}_t \times \hat{e}_n, \quad \rho = \frac{1}{|\vec{\kappa}|}.$$

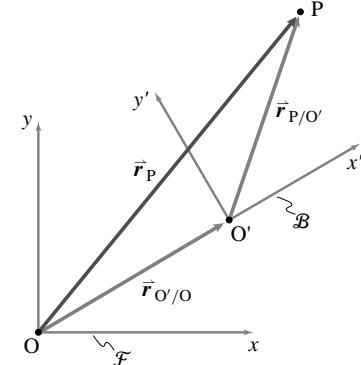
## Summary of the direct differentiation method

In the direct differentiation method, using moving frame  $\mathcal{B}$ , we calculate  $\vec{v}_P$  by using a combination of the product rule of differentiation and the facts that

$$\dot{\vec{r}}' = \vec{\omega}_{\mathcal{B}} \times \vec{r}', \quad \dot{\vec{j}}' = \vec{\omega}_{\mathcal{B}} \times \vec{j}', \text{ and } \dot{\vec{k}}' = \vec{\omega}_{\mathcal{B}} \times \vec{k}',$$

as follows:

$$\begin{aligned} \vec{v}_P &= \frac{d}{dt} \vec{r}_P \\ &= \frac{d}{dt} [\vec{r}_{O'/O} + \vec{r}_{P/O'}] \\ &= \frac{d}{dt} [(x \hat{i} + y \hat{j} + z \hat{k}) + (x' \hat{i}' + y' \hat{j}' + z' \hat{k}')] \\ &= (\dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k}) + (\dot{x}' \hat{i}' + \dot{y}' \hat{j}' + \dot{z}' \hat{k}') + \\ &\quad [x'(\vec{\omega}_{\mathcal{B}} \times \vec{r}') + y'(\vec{\omega}_{\mathcal{B}} \times \vec{j}') + z'(\vec{\omega}_{\mathcal{B}} \times \vec{k}')] \end{aligned}$$

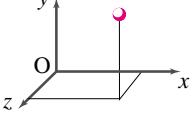
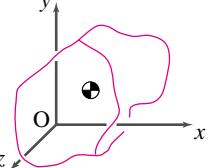
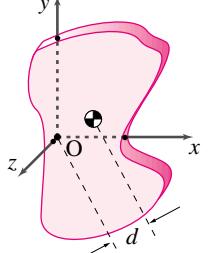


but stop short of identifying these three groups of three terms as  $\vec{v}_P = \vec{v}_{O'/O} + \dot{\vec{r}}_{rel} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}$ .

We could calculate  $\vec{a}_P$  similarly and would get a similar formula with 15 non-zero terms

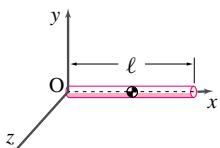
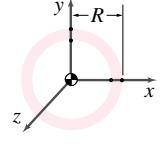
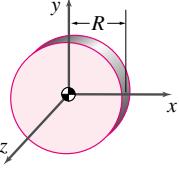
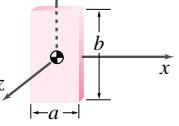
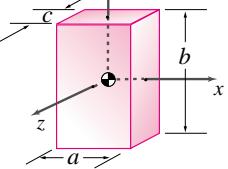
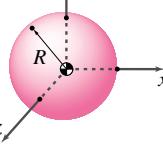
(3 for each term in the 'five-term' acceleration formula).

Table III

Object	$[I]$
<b>Point mass</b> 	$[I^{cm}] = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $[I^o] = m \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$
<b>General 3D body</b> 	$[I^{cm}] = \int \begin{bmatrix} y_{cm}^2 + z_{cm}^2 & -x_{cm}y_{cm} & -x_{cm}z_{cm} \\ -x_{cm}y_{cm} & x_{cm}^2 + z_{cm}^2 & -y_{cm}z_{cm} \\ -x_{cm}z_{cm} & -y_{cm}z_{cm} & x_{cm}^2 + y_{cm}^2 \end{bmatrix} dm$ <p>If the axes are principal axes of the body.</p> $\underbrace{[I^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}}_{\text{With } A, B, C \geq 0 \text{ and } A + B \geq C, B + C \geq A, \text{ and } A + C \geq B.}$ $[I^o] = \int \begin{bmatrix} y_o^2 + z_o^2 & -x_oy_o & -x_oz_o \\ -x_oy_o & x_o^2 + z_o^2 & -y_oz_o \\ -x_oz_o & -y_oz_o & x_o^2 + y_o^2 \end{bmatrix} dm$ $[I^o] = [I^{cm}] + m \underbrace{\begin{bmatrix} y_{cm/o}^2 + z_{cm/o}^2 & -x_{cm/o}y_{cm/o} & -x_{cm/o}z_{cm/o} \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 + z_{cm/o}^2 & -y_{cm/o}z_{cm/o} \\ -x_{cm/o}z_{cm/o} & -y_{cm/o}z_{cm/o} & x_{cm/o}^2 + y_{cm/o}^2 \end{bmatrix}}_{\text{The 3D Parallel Axis Theorem}}$
<b>General 2D Body</b> 	$[I^{cm}] = \int \begin{bmatrix} y_{cm}^2 & -x_{cm}y_{cm} & 0 \\ -x_{cm}y_{cm} & x_{cm}^2 & 0 \\ 0 & 0 & x_{cm}^2 + y_{cm}^2 \end{bmatrix} dm$ <p>If the axes are principal axes of the body.</p> $\underbrace{[I^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}}_{\text{With } A + B = C \text{ (The perpendicular axis theorem). Also, } A \geq 0, B \geq 0.}$ $[I^o] = \int \begin{bmatrix} y_o^2 & -x_oy_o & 0 \\ -x_oy_o & x_o^2 & 0 \\ 0 & 0 & x_o^2 + y_o^2 \end{bmatrix} dm$ $[I^o] = [I^{cm}] + m \underbrace{\begin{bmatrix} y_{cm/o}^2 & -x_{cm/o}y_{cm/o} & 0 \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 & 0 \\ 0 & 0 & x_{cm/o}^2 + y_{cm/o}^2 \end{bmatrix}}_{d^2}$ <p>The 3D Parallel Axis Theorem. The 2D thm concerns the lower right terms of these 3 matrices.</p>

**General moments of inertia.** The table shows a point mass, a general 3-D body, and a general 2-D body. The most general cases of the perpendicular axis theorem and the parallel axis theorem are also shown..

**Table IV**  
**Examples of Moment of Inertia**

Object	[I]
Uniform rod 	$I_{zz}^{cm} = \frac{1}{12}m\ell^2, \quad [I^{cm}] = \frac{1}{12}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_{zz}^O = \frac{1}{3}m\ell^2, \quad [I^O] = \frac{1}{3}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Uniform hoop 	$I_{zz}^{cm} = mR^2, \quad [I^{cm}] = mR^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Uniform disk 	$I_{zz}^{cm} = \frac{1}{2}mR^2, \quad [I^{cm}] = mR^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$
Rectangular plate 	$I_{zz}^{cm} = \frac{1}{12}m(a^2 + b^2), \quad [I^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$
Solid Box 	$[I^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$
Uniform sphere 	$[I^{cm}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Moments of inertia of some simple objects.** For the rod both the  $[I^{cm}]$  and  $[I^O]$  (for the end point at O) are shown. In the other cases only  $[I^{cm}]$  is shown. To calculate  $[I^O]$  relative to other points one has to use the parallel axis theorem. In all the cases shown the coordinate axes are principal axes of the objects.