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Math 123 Fall 2018

Homework #1

Ex. 5 Consider the family of differential equations

$x' = ax + \sin x$

where a is a parameter.

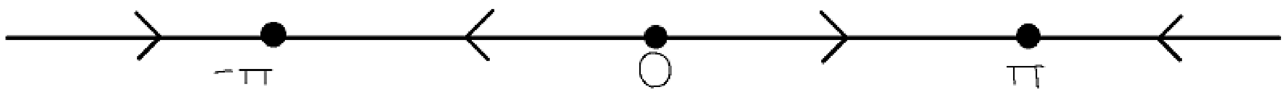
a. Sketch the phase line when $a = 0$.

$x' = \sin x$

Equilibrium points at πk , where $k \in \mathbb{Z}$

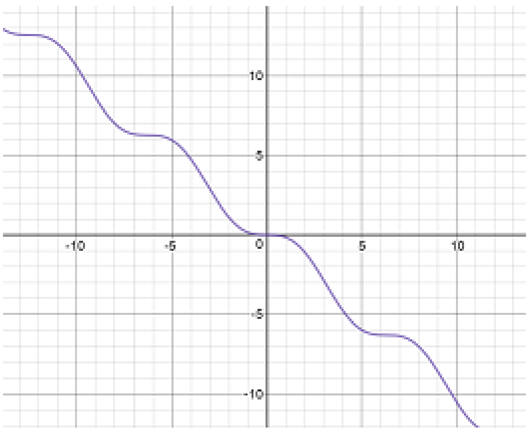
$x'' = \cos x = \begin{cases} 1 & x = 2k\pi \\ -1 & x = (2k+1)\pi \end{cases}$

So if $x = 2k\pi$ it is a source and if $x = (2k+1)\pi$ it is a sink



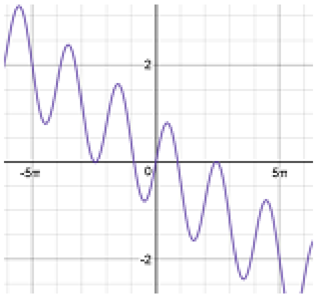
b. as a increases from -1 to 1

When $a = -1$, we have 1 equilibrium point at x = 0, and it seems to be a sink.



We keep gaining equilibrium points in quantities of 2 as a increases towards 0, since

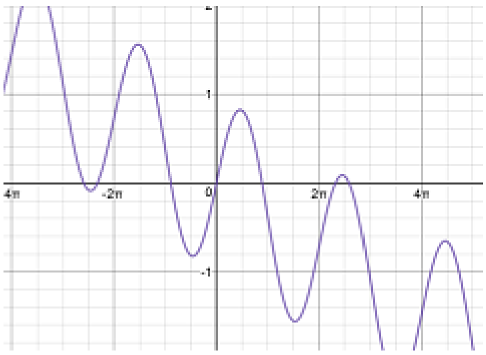
1. For every $a = -(\frac{2}{(4m+1)\pi})$, with m an integer ≥ 0, we are "adding" one more equilibrium point at $x = \frac{(4m+1)\pi}{2}$, since $ax = -1$ and $\sin \frac{(4m+1)\pi}{2} = 1$
    1. Example: $a = -\frac{2}{5\pi}$



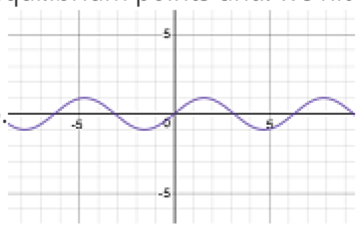
2. for any additional equilibrium point x, with $ax + \sin x = 0$, we have $-x$ as an equilibrium point as well: $a(-x) + \sin(-x) = -ax - \sin x = -(ax + \sin x) = 0$, since ax and $\sin x$ are both odd functions.

3. Then once we have $a \in (-\frac{2}{(4m+1)\pi,} - \frac{2}{(4m+5)\pi})$, the equilibrium points at $x = \pm\frac{(4m+1)\pi}{2}$ each split into 2 more equilibrium points.

   Example: $a = -\frac{2}{5\pi} + .01$



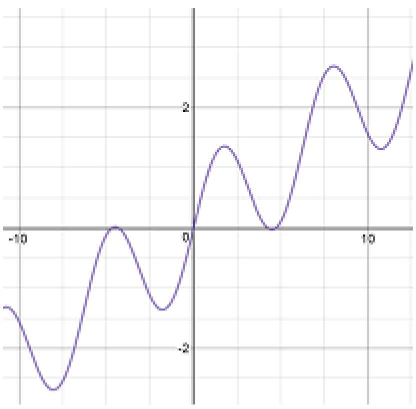
   And so we keep gaining equilibrium points until we hit $a = 0$, where we have infinitely many points, as can be seen from part (a).

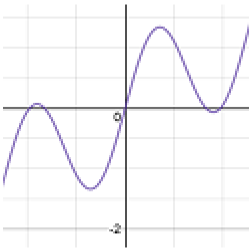


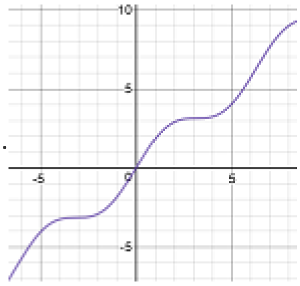
As we go from $a = 0$ to $a = 1$, we observe a similar phenomenon as that of when a increased from -1 to 0, except that we lose equilibrium points as we a increases, and whenever $a = \frac{2}{(4m+3)\pi}$, with m a nonnegative integer, we have that $x = \pm\frac{(4m+3)\pi}{2}$ are equilibrium points: at these x, $ax = \pm 1$ and $\sin x = \mp 1$ so $ax + \sin x = 0$.

Examples:

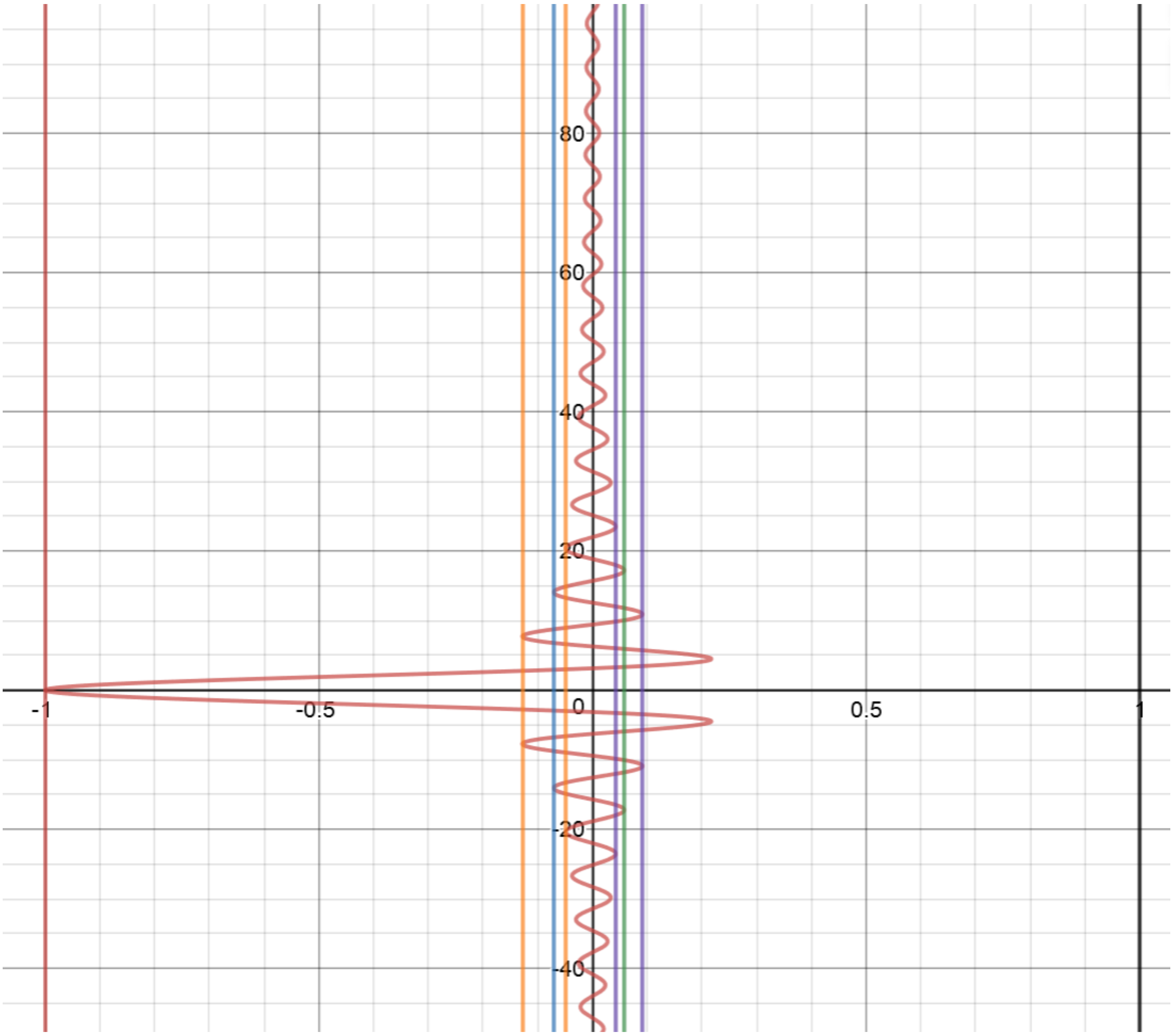
$a = \frac{2}{3\pi}$



$a = \frac{2}{3\pi} + .01$ 

When $a = 1$ only 1 equilibrium point at $x = 0$ and is a source. 

(c) The bifurcation diagram:



Ex. 8 Consider a first-order linear equation of the form $x' = ax + f(t)$ where $a \in \mathbb{R}$. Let $y(t)$ be any solution of this equation. Prove that the general solution is $y(t) + c\exp(at)$ where $c \in \mathbb{R}$ is arbitrary.

Let $w(t)$ be some solution to this equation.

If we took $\frac{w(t)-y(t)}{e^{at}}$ and took the derivative w.r.t. t, since $(f \pm g)' = f' \pm g'$

we get, using product rule,

$$(aw(t) + f(t) - ay(t) - f(t))(e^{-at}) - ae^{-at}(w(t) - y(t))$$
$$= ae^{-at}(w(t) - y(t)) - ae^{-at}(w(t) - y(t)) = 0$$

Which means $\frac{w(t)-y(t)}{e^{at}}$ is equal to some constant $c \in \mathbb{R}$

so $\frac{w(t)-y(t)}{e^{at}} = c \implies w(t) = y(t) + ce^{at}$

Ex. 9 Consider a first-order, linear, nonautonomous equation of the form $x'(t) = a(t)x$

(a) Find a formula involving integrals for the solution of this system.

$\frac{dx}{dt} = a(t)x$
$\frac{1}{x}dx = a(t)dt$
$\int \frac{1}{x}dx = \int a(t)dt$

$\ln|x| = \int a(t)dt + c$

letting $\exp(c) = C$

$x = C\exp[\int a(t)dt]$

(b) Prove that your formula gives the general solution of this system.

let $w(t)$ be a solution to this differential equation.

Finding the derivative w.r.t. t of $w(t)\exp[-\int a(t)dt]$:

Using chain rule and product rule, noting the derivative of $\int a(t)dt$ is $a(t)$

$[w(t)\exp[-\int a(t)dt]]' = a(t)w(t)\exp[-\int a(t)dt] - a(t)\exp[-\int a(t)dt]w(t) = 0$

so $w(t)\exp[-\int a(t)dt] = C$, where C is an arbitrary real constant,

$\implies w(t) = C\exp[\int a(t)dt]$, which was my formula from (a).

Ex. 11 First-order differential equations need not have solutions that are defined for all times

(a) Find the general solution of the equation $x' = x^2$

$\frac{dx}{dt} = x^2 \equiv \frac{1}{x^2}dx = dt$

$\int \frac{1}{x^2}dx = t + C$
$-\frac{1}{x} = t + C$
$x = -\frac{1}{t+C}$

$x_0 = -\frac{1}{C} \equiv C = -\frac{1}{x_0}$

$x = -\frac{1}{t - \frac{1}{x_0}} = -\frac{x_0}{x_0 t - 1}$

(b) Discuss the domains over which each solution is defined

defined over $(-\infty, \frac{1}{x_0}) \cup (\frac{1}{x_0}, \infty)$

(c) Give an example of a differential equation for which the solution satisfying x(0) = 0 is defined only for -1 < t < 1

$x' = -2t\exp[-x]\; x(0) = 0$

A solution for this:

$\int \exp[x]dx = -\int 2tdt$

$\exp[x] = -t^2 + C$

$x(t) = \ln(-t^2 + C)$

$x(0) = \ln(C) = 0 \equiv C = 1$

and so $x(t) = \ln(1-t^2)$, indeed, $\ln(1-t^2)$ is undefined when $1 - t^2 < 0 \equiv t^2 > 1$

which happens when $t < -1$ or when $t > 1$

Ex. 12. First-order differential equations need not have unique solutions satisfying a given initial condition

(a) Prove that there are infinitely many different solutions of the differential equations $x' = x^{1/3}$ **satisfying** $x(0) = 0$

$\int x^{-1/3}dx = dt$

letting $u = x^{1/3}$ we get $\frac{du}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3}u^{-2}$

so $dx = 3u^2du$

$3\int u^{-1}u^2du = 3\int u\,du = \frac{3}{2}u^2 = \frac{3}{2}x^{2/3}$

$\frac{3}{2}x^{2/3} = t + C$

$x(t) = (\frac{2}{3}(t+C))^{3/2}$

Notice: not defined for $t < -C$

We extend $x(t) = \begin{cases} (\frac{2}{3}(t+C))^{3/2} & t \geq -C \\ 0 & t < -C \end{cases}$

This extended function is a solution: if $t \geq -C, x'(t) = (\frac{2}{3}(t+C))^{1/2} = x(t)^{1/3}$

if $t < -C\; x(t) = 0, x'(t) = 0 = 0^{1/3}$

It is also differentiable: we know the derivative for $t > -C$ and $t < -C$, but need to make sure limit on the left matches limit on the right.

$\lim_{h\to 0} \frac{x(-C+h) - x(-C)}{h}$

if $h < 0$, $\frac{0 - (\frac{2}{3}\times 0)^{3/2}}{h} = 0$

if $h > 0$ $\frac{(\frac{2}{3}h)^{3/2}}{h} = (2/3)^{3/2} \times h^{1/2} \to (2/3)^{3/2} \times 0 = 0$

As long as $C < 0$, we have that $t = 0 < -C$, and so $x(0) = 0$ by definition.

(b) Discuss the corresponding situation that occurs for $x' = x/t \qquad x(0) = x_0$

$\frac{dx}{dt} = \frac{x}{t}$

$\int x^{-1} dx = \int t^{-1} dt$

$\ln|x| = \ln|t| + c_1$

define $C = e^{c_1}$

$|x| = C|t|$

$x = Ct$ is the general solution. (if $x = C|t|$ x is not differentiable at $t = 0$)

$x(0) = C(0) = 0$, no matter what C equals. so $\forall C, x = Ct$ is a solution.

(c) Discuss the situation that occurs for $x' = x/t^2 \;\; x(0) = 0$

$\frac{dx}{dt} = x/t^2$

$\int x^{-1} dx = \int t^{-2} dt$

$\ln|x| = -1/t + c_1$

again $e^{c_1} = C$

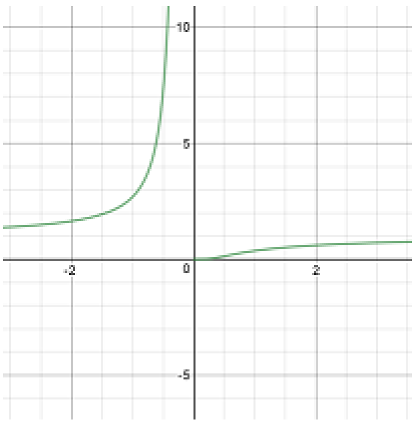
$|x| = C\exp[-1/t]$

$x = C\exp[-1/t]$

As it is, the function is not continuous at $t = 0$

$\lim_{t \to 0^+} 1/t = \infty$ and so the limit of $\exp[-1/t]$ approaches 0 as $t \to 0^+$

However, as $t \to 0^-$, the limit of $\exp[-1/t]$ approaches positive infinity.



We may be able to remedy this by setting $x(t) = 0$ whenever $t \leq 0$, and when $t > 0$, $x(t) = C\exp[-1/t]$, the function will be differentiable and a solution:

$t \leq 0 \; x'(t) = 0 = 0/t^2 = x(t)/t^2$

$t > 0 \; x'(t) = (1/t^2)(C\exp[-1/t]) = x(t)/t^2$

So this new function is a solution and differentiable at $t < 0$ and $t > 0$

To show differentiability at $t = 0$:

$\lim_{h\to 0^-} \frac{x(h)-x(0)}{h} = \frac{0-0}{h} = 0$

$\lim_{h\to 0^+} \frac{x(h)-x(0)}{h} = \lim_{h\to 0^+} \frac{C\exp[-1/h]}{h} = \lim_{h\to 0^+} \frac{C}{h\exp[1/h]}$

$\lim_{h\to 0^+} h\exp[1/h] = \exp[1/h]/(1/h) = \infty^+/\infty^+$

Using L'hospital's rule: $\lim_{h\to 0^+} \frac{(-1/h^2)\exp[1/h]}{(-1/h^2)} = \exp[1/h] \to \infty^+$

so $\lim_{h\to 0^+} C/(h\exp[1/h]) \to C/\infty^+ = 0$

So the limit for $\frac{x(h)-x(0)}{h}$ as h approaches 0 is 0.

Also, $x(0) = 0$ by definition, so $x(t) = \begin{cases} 0 & t \le 0 \\ C\exp[-1/t] & t > 0 \end{cases}]$ a solution for any $C \in \mathbb{R}$, so we have infinitely many solutions for this IVP.

Ex. 13.

(a) Suppose $f'(x_0) = 0$. What can you say about the behavior of solutions near x_0? Give examples.

Can't say anything:

$x = 0$ is an equilibrium point or which $f'(0) = 0$ for each of the following

$x' = -x^3$, x_0 is a sink/stable

$x' = x^3$, x_0 is a source/unstable

$x' = x^2$, solutions to the left of x_0 tend towards x_0 and solutions to the right tend away

$x' = 0$ solutions neither tend towards, nor away from x_0

(b) Suppose $f'(x_0)$ and $f''(x_0) \neq 0$, what can you say now?

Rules out $x' = 0$ and $x' = x^3$

if the slope is negative to the left of x_0 then the slope needs to increase for $f'(x_0)$ to become 0 at x_0, so we have $f''(x) > 0$ to the left of x_0 . But if we try to make $f'(x)$ negative again when x is to the the right, we need $f''(x) < 0$ when x is to the right of x_0 so this means at x_0, $f''(x_0) = 0$, a contradiction. So we must have $f'(x)$ go from negative to 0 at x_0 to positive, or positive to 0 at x_0 to negative using the same logic as before. In other words, as x passes through x_0, $f'(x)$ must change signs as it goes through x_0

so x_0 will be an equilibrium point that looks like the case for $x' = x^2$ i.e, it attracts on the left, repels on the right or vice versa.

(c) Suppose $f'(x_0) = f''(x_0) = 0$ but $f'''(x_0) \neq 0$

If $f'''(x_0) \neq 0$, then as $f''(x)$ passes through x_0, it is going from negative to positive or positive to negative.

Suppose $f''(x)$ is going from negative to positive ($f'''(x_0) > 0$)

This means for $f'(x_0) = 0$, $f'(x)$ must be decreasing from a positive number at the left of x_0 then hits 0 at x_0, but since $f''(x)$ is positive on the right of x_0, $f'(x)$ increases on the right of x_0 to become positive again.

meaning, $f(x)$ must be increasing from a negative number at the left of x_0 to hit 0 at x_0 then it continues to increase. //looks more like a source

And if $f''(x)$ is going from positive to negative ($f'''(x_0) < 0$), using the same logic as earlier, $f(x)$ must b decreasing from the left of x_0 and decreasing to the right of x_0 //looks more like a sink

Ex. 14. Consider the first-order nonautonomous equation $x' = p(t)x$, where $p(t)$ is differentiable and periodic with period T. Prove that all solutions of this equation are periodic with period $T \iff \int_0^T p(s)ds = 0$

In general:

$dx/dt = p(t)x$

$\int x^{-1}dx = \int p(t)dt$

$\ln |x| = \int p(t)dt + c_1$

$|x| = \exp[c_1]\exp[\int p(t)]dt$

with $c_2 = \exp[c_1]$

$x(t) = \pm C\exp[\int p(t)]$, since C is any constant, we can include $\pm$ in it.

Define $P(t) = \int p(t)$ then $P(t) = \int_0^t p(s)ds + P(0)$

$x(t) = C\exp[P(0)]\exp[\int_0^t p(s)ds]$

and $x(0) = C\exp[P(0)]$, so $x(t) = x_0\exp[\int_0^t p(s)ds]$

From exercise 9, this is the general solution.

1. Proving $\rightarrow$

   Suppose all solution $x(t)$ that satisfy $x' = p(t)x$ also satisfies $x(t+T) = x(t)$

   then: $x(T) = x(0)$

   $x(T) - x(0) = 0 = x_0(\exp[\int_0^T p(s)ds] - \exp[\int_0^0 p(s)ds]) =$

   $x_0(\exp[\int_0^T p(s)ds] - 1)$

   so $\exp[\int_0^T p(s)ds] = 1 \implies \int_0^T p(s)ds = 0$

2. Proving $\leftarrow$

   Since $p(t)$ is periodic: $\int_t^{t+T} p(s)ds = \int_0^T p(s)ds$ for all t

   b/c: $P'(t+T) - P'(t) = p(t+T) - p(t) = 0$

   so: $P(t+T) - P(t)$ is a constant. And we're assuming $P(T) - P(0) = \int_0^T p(s)ds = 0$

   So $P(t+T) - P(t) = 0$

   also from the general solution of the equation, we have that $x(T) = x(0)$

   so if we multiply by $x(t)$ on both sides we get:

   $x(t)x(T) = x(0)x(t)$

   $x_0^2\exp[\int_0^t p(s)ds + \int_0^T p(s)ds] = x_0x(t)]$

   since $\int_t^{t+T} p(s)ds = \int_0^T p(s)ds$ for all t and cancelling out x_0 The right side becomes $x(t)$, and the left side:

   $x_0\exp[\int_0^t p(s)ds + \int_t^{t+T} p(s)ds] = x_0\exp[\int_0^{t+T} p(s)ds] = x(t+T) \implies x(t+T) = x(t)$

Ex. 15: Consider the differential equation $x' = f(t,x)$, where $f(t,x)$ is continuously differentiable in t and x. Suppose that $f(t+T,x) = f(t,x)$ for all t. Suppose there are constants p, q such that $f(t,p) > 0, f(t,q) < 0$ for all t. Prove that there is a periodic solution $x(t)$ for this equation with $p < x(0) < q$

I want to prove that the Poincare map is continuous, maps $[p,q]$ to itself, and therefore has a fixed-point, which means there exists a periodic solution with $x(0) \in (p,q)$

Poincare map $P(x_0) = \phi(T, x_0)$ and from section 1.5, we have:

$P'(x_0) = \exp[\int_0^T \frac{\partial f}{\partial x_0}(s, \phi(s,x_0))ds]$, and since f is continuously differentiable in x, P' is defined for all x_0, meaning P is differentiable and therefore continuous.

To show P maps $[p,q]$ to itself, I will need to show $\forall x_0 \in [p,q], P(x_0) \in [p,q]$

since the derivative of $P'(x_0)$is positive ($\exp$ takes positive values), and $f(t,p) > 0$ and $f(t,q) < q$ for all t

we must have not only that $P(p) > p$ and $P(q) < q$, $P(p) < q$ and $P(q) > p$ since $P'(x_0)$ is positive, P is increasing: so $P(q) - P(p) > 0$ AND for $x_0 \in (p,q)$ $p < P(p) < P(x_0) < P(q) < q$

Therefore, P maps $[p,q]$ to itself.

Taking the function $Q(x_0) = P(x_0) - x_0$,

$Q(p) = P(p) - p > 0$

$Q(q) = P(q) - q < 0$

Since the endpoint have different signs, using intermediate value theorem,

there exists an $x_0 \in (p,q)$ s.t. $Q(x_0) = 0 = P(x_0) - x_0 \equiv \exists x_0 \in (p,q)$ s.t. $P(x_0) = x_0$

Thus, there exists a fixed point, and therefore a periodic solution $x(t)$ with $p < x(0) < q$.