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Math 185

Homework #3

Chapter 2, Section 2:

Ex. 5: Show that if f is analytic on D, then $g(z) = \overline{f(\bar{z})}$ is analytic on the reflected domain $D^* =$ {$\bar{z} : z \in D$} and $g'(z) = \overline{f'(\bar{z})}$

We check to see if $g(z)$ is differentiable at each point of D^*

Let z_0 be a point on D. Then $\overline{z_0}$ is a point in the reflected domain.

so for any $\bar{z} \in D^*$ s.t. $\bar{z} \to \overline{z_0}$

$\lim_{\bar{z}\to\overline{z_0}} \frac{g(\bar{z})-g(\overline{z_0})}{\bar{z}-\overline{z_0}} = \lim_{\bar{z}\to\overline{z_0}} \frac{\overline{f(z)}-\overline{f(z_0)}}{\bar{z}-\overline{z_0}} = \lim_{\bar{z}\to\bar{z_0}} \overline{(\frac{f(z)-f(z_0)}{z-z_0})}$

We have that for any $\delta > 0$, if $z \neq z_0$ and $0 < |z - z_0| < \delta$ then $0 < |\bar{z} - \bar{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta$, so as $z \to z_0$, $\bar{z} \to \bar{z_0}$

so the limit above is equal to

$\lim_{z\to z_0} \overline{(\frac{f(z)-f(z_0)}{z-z_0})}$

and since f is analytic on D, $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and is equal to $f'(z_0)$

and so for every $\epsilon > 0$, $\exists\ \delta > 0$ such that $\forall z \in D$, with $z \neq z_0$ if $0 < |z - z_0| < \delta$ then $|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)| < \epsilon$ this also means that if $0 < |z - z_0| < \delta$

$|\overline{(\frac{f(z)-f(z_0)}{z-z_0})} - \overline{f'(z_0)}| = |\overline{(\frac{f(z)-f(z_0)}{z-z_0}) - f'(z_0)}| = |\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)| < \epsilon$

so $\lim_{\bar{z}\to\overline{z_0}} \frac{g(\bar{z})-g(\bar{z_0})}{\bar{z}-\bar{z_0}} = \overline{f'(z_0)}$ = $g'(\overline{z_0})$

so since $\overline{z_0}$ was an arbitrary point in D^*, for every $z \in D^*$, we have that $g'(z) = \overline{f'(\bar{z})}$

Chapter 2, Section 3

Ex. 3: Show that if f and $\bar{f}$ are both analytic on a domain, then f is constant.

$Re(f) = \frac{1}{2}(f + \bar{f})$ and $Im(f) = \frac{1}{2}(f - \bar{f})$

and since $f, \bar{f}$ are analytic, $Re(f), Im(f)$ are analytic along with being real-valued, meaning they are constant, which means $f = Re(f) + iIm(f)$ is constant.

Ex. 4: Show that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant

If $f(z) = 0$ for some $z \in D$, then $|f(z)| = 0$, and since $|f|$ is constant, we must have $f \equiv 0$ on D

Otherwise, $f(z) \neq 0$ for all $z \in D$, and therefore $\bar{f} = |f|^2/f$ is defined for all $z \in D$

Since f is analytic, and it is differentiable, $1/f$ is differentiable since $f(z) \neq 0$ for all $z \in D$ and $g(z) = 1$ is differentiable.

and since $|f|^2$ is a constant, $\bar{f} = |f|^2/f$ is differentiable at all $z \in D$ and therefore analytic on D

Also, $\overline{f}' = -|f|^2/f^2$, which is continuous, since products and sums of continuous functions are continuous... so $\overline{f}$ is analytic.

and if f and $\bar{f}$ are both analytic, by exercise 3, then f is constant.

Chapter 2, Section 4

Ex. 8: Sketch the image of the circle $|z-1| \leq 1$ under the map $w = z^2$. Compute the are of the image.

the domain is any (x, y) satisfying $(x-1)^2 + y^2 = 1$

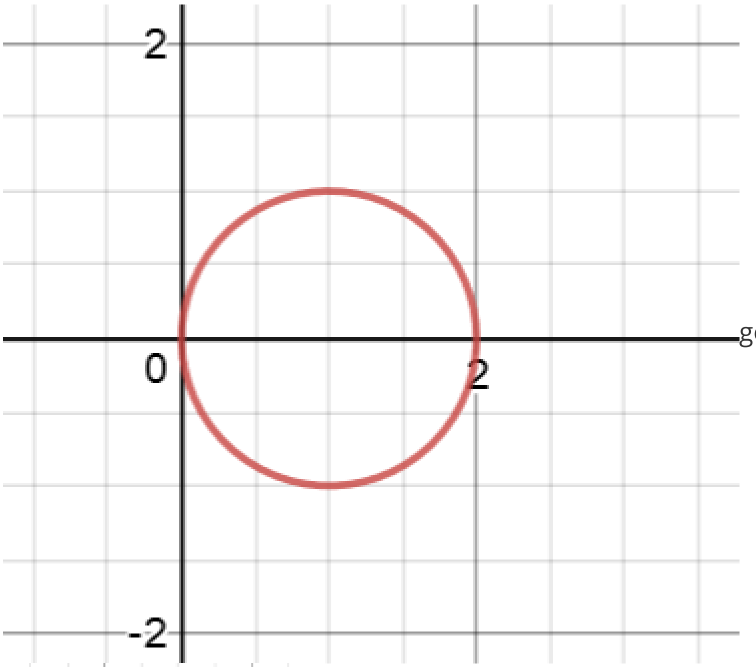
since $x = |z|\cos\theta$ and $y = |z|\sin\theta$, where $\theta = \arg z$

we have that $|z|^2 \cos^2\theta - 2|z|\cos\theta + 1 + |z|\sin^2\theta = 1$

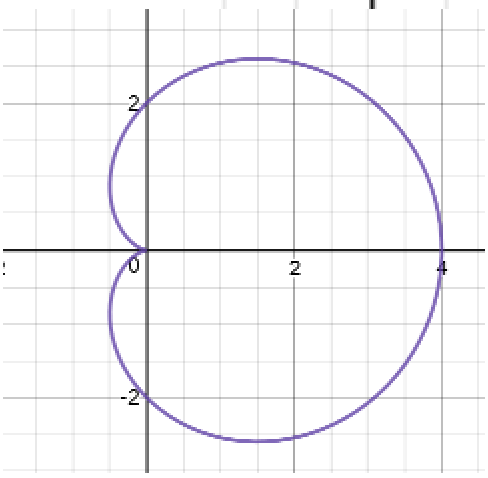
$$|z|^2 - 2|z|\cos\theta = 0$$
$$|z|^2 = 2|z|\cos\theta$$
$$|z| = 2\cos\theta$$

$z = 2\cos\theta e^{i\theta}$

$z^2 = 4\cos^2\theta e^{i2\theta}$ = $4\cos^2(\theta/2)e^{i\theta}$

So $|z-1| \leq 1:$ 

gets mapped to



From the graphs above, the function $f(z) = z^2$ is bounded. Also, it is analytic, since polynomials are differentiable, and one to one over the bounded domain D = $\{z : |z-1| \leq 1\}$: $f(w) = f(z) \implies w^2 = z^2$

$\implies w$ and z are square roots of $w^2 = z^2$

since z is a square root, another root would be $w = -z$.

But for z to be in the domain, $Re(z) > 0$, since $|z-1| = \sqrt{(Re(z)-1)^2 + Im(z)^2} \geq |Re(z)-1|$ and if $Re(z) < 0$, then $Re(z) - 1 < -1 \implies |Re(z) - 1| > 1$

so if z is in the domain, $-z$ cannot be, so if $f(z) = f(w) \implies z = w$ Therefore, $f(z) = z^2$ over D is one to one so:

The area of the image = $\int \int \det J_f dxdy$

$\det J_f = |f'(z)|^2 = |2z|^2 = 4|z|^2 = 4(x^2 + y^2)$

so D = $\{(r, \theta)| - \pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2\cos\theta\}$

So we have $4 \int \int_D (x^2 + y^2) dA = 4 \int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} r^3 \, d\theta dr$

$= 16 \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = 16 \int_{-\pi/2}^{\pi/2} (\frac{1+\cos 2\theta}{2})^2 d\theta = 4 \int_{-\pi/2}^{\pi/2} (1 + 2\cos 2\theta + (\frac{1+\cos 4\theta}{2}))$

$= 4[\theta + \sin 2\theta + \theta/2 + (\sin 4\theta)/8]_{-\pi/2}^{\pi/2} = 4[3\pi/4 + 3\pi/4] = 6\pi$

Extra Problems

1. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic function on the complex plane and $f(0) = 0$. Suppose that for all real numbers $x, y \in \mathbb{R}$ we have $Re(f(x+iy)) = e^{-y}(x\cos(x) - y\sin(x))$.

Find an expression for $f(z)$ as a function of a complex variable $z = x + iy$

so $u(x,y) = e^{-y}(x\cos(x) - y\sin(x))$

since f is analytic:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{-y}(\cos(x) - x\sin(x) - y\cos(x))$

$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -[-e^{-y}(x\cos(x) - y\sin(x)) + e^{-y}(-\sin(x))] = e^{-y}(x\cos(x) - y\sin(x)) + e^{-y}\sin(x)$

$= e^{-y}(x\cos(x) + \sin(x) - y\sin(x))$

So:

$v(x,y) = \int e^{-y}(\cos(x) - x\sin(x)) - ye^{-y}\cos(x)dy$ =
$e^{-y}(x\sin(x) - \cos(x)) + (ye^{-y} + e^{-y})(\cos(x)) + C(x)$

$= e^{-y}(x\sin(x) + y\cos(x)) + C(x)$, then we derive this w.r.t. to x and equate it to $-\frac{\partial u}{\partial y}$

$e^{-y}(x\cos(x) + \sin(x) - y\sin(x)) + C'(x) = e^{-y}(x\cos(x) + \sin(x) - y\sin(x))$

$e^{-y}(x\cos(x) + \sin(x) + \sin(x)) + (ye^{-y} + e^{-y})(\cos(x))$

Simplifying:

$C'(x) = 0$ meaning $C(x)$ is a real constant.

$u(x,y) + iv(x,y) = e^{-y}(x\cos(x) - y\sin(x)) + i[e^{-y}(x\sin(x) + y\cos(x)) + C]$

$f(0) = u(0,0) + iv(0,0) = 0 + iC = 0$

so $f(x+iy) = e^{-y}(x\cos(x) - y\sin(x)) + i[e^{-y}(x\sin(x) + y\cos(x))]$

$f(x+iy) = e^{-y}[(x+iy)(\cos(x) + i\sin(x))] = e^{-y}ze^{ix} = ze^{iz}$

2. Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. Denote the real and imaginary part of f as $u(x,y)$ and $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$

Let ∇u and ∇v denote the gradient vector fields,

1. **Find the angle between the horizontal axis and the vector (∇u)(x_0, y_0),**
2. **Find the angle between the horizontal axis and the vector (∇v)(x_0, y_0)$**

if $f'(z_0) = 5e^{i\pi/6}$ at the point $z_0 = x_0 + iy_0$

We have $f'(z_0) = f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) = 5e^{i\pi/6}$
$= 5(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) = 5\cos(\pi/6) = 5(\sqrt{3}/2)$

$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0) = 5\sin(\pi/6) = 5/2$

so $\nabla u(x_0, y_0) = (5(\sqrt{3}/2), -5/2)$ and $\nabla v(x_0, y_0) = (5/2, 5(\sqrt{3}/2))$

We can find the angle between any two vectors a, b using

$\theta = \arccos(\frac{a \cdot b}{|a||b|})$

so $|\nabla u| = |\nabla v| = |5e^{i\pi/6}| = 5$

letting the vector $w =< 1, 0 >$

1. $\theta = \arccos(\frac{5\sqrt{3}/2}{5}) = \pi/6$ specifically, $\pi/6$ below the x-axis
2. $\theta = \arccos(\frac{5/2}{5}) = \pi/3$, above the x-axis