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Math 185

Homework #2

Chapter 1, Sec. 3

Ex. 1 Sketch the image under the spherical projection of the following sets on the sphere:

(a) the lower Hemisphere $Z < 0$

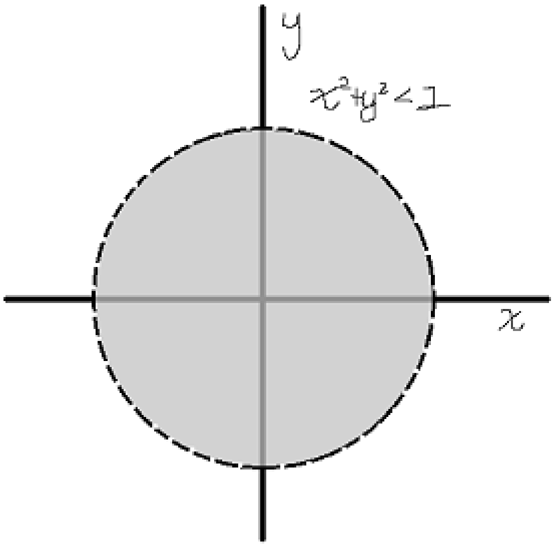
$x = X/(1-Z)$, $y = Y/(1-Z)$

We have that $1 - Z > 1$, since $Z < 0$

so $|x| < |X|$ and $|y| < |Y|$

and also because $Z \neq 0$, $X^2 + Y^2 < 1$

meaning, $x^2 + y^2 < X^2 + Y^2 < 1$



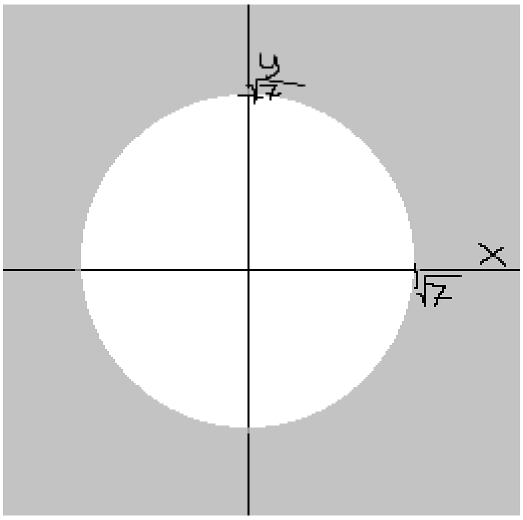
(b) the polar cap $\frac{3}{4} \leq Z \leq 1$

When $Z = \frac{3}{4}$, $1 - Z = \frac{1}{4}$, so we have: $x = 4X$ and $y = 4Y$

And $1 - Z^2 = 1 - \frac{9}{16} = \frac{7}{16}$

so: $x^2 + y^2 = 16(X^2 + Y^2) = 16 \times \frac{7}{16} = 7$

The image will be the grey section.



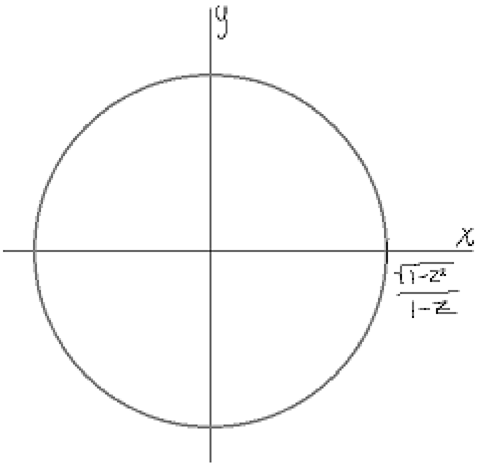
and as Z goes towards 1, the radii of the circles on the plane centered at 0 tend towards ∞, so we have that $x^2 + y^2 \geq 7$ when $Z \in [\frac{3}{4}, 1]$

(c) lines of latitude $X = \sqrt{1 - Z^2}\cos\theta$ **,** $Y = \sqrt{1 - Z^2}\sin\theta$**, for** Z **fixed and** $0 \leq \theta \leq 2\pi$

so this will correspond to a circle on the plane centered at 0.

the radius will be $= \frac{\sqrt{1-Z^2}}{1-Z}$, since $x = \frac{X}{1-Z} = (\frac{\sqrt{1-Z^2}}{1-Z})\cos\theta = r\cos\theta$

So for each Z:



(d) lines of longitude $X = \sqrt{1 - Z^2}\cos\theta, Y = \sqrt{1 - Z^2}\sin\theta$**, for** θ **fixed and** $-1 \leq Z \leq 1$

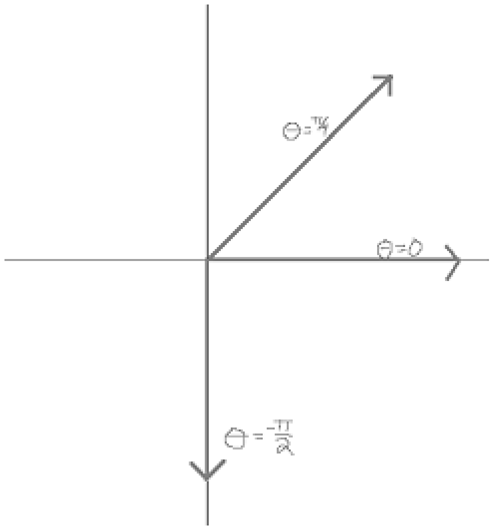
These will correspond to lines on the plane that pass through origin

and slope $m = y/x = \tan\theta$, $x = \frac{\sqrt{1-Z^2}}{1-Z}\cos\theta, y = \frac{\sqrt{1-Z^2}}{1-Z}\sin\theta$.

For a fixed θ, when x is nonzero, x has the same sign as $\cos\theta$, because $\sqrt{1 - Z^2} \geq 0$ and $1 - Z \geq 0$, and similarly for y and $\sin\theta$

when $Z = -1, 1, x = y = 0$, regardless of the value of θ, so the origin is always included in the image.

as $Z \to 1$, x and y approach either positive or negative infinity depending on θ.



(e) the spherical cap $A \leq X \leq 1$, with center lying on the equator for fixed A. Separate into cases, according to various ranges of A.

$A = 1$: We only have the point $(1, 0, 0)$, which means $x = 1$, $y = 0$

$A = 0$: We have the entire right half-plane. $X \geq 0 \implies x \geq 0$

$A = -1$: We get the whole sphere as the domain, which means the whole plane.

$A \in (0, 1)$:

$-\sqrt{1-A^2} \leq Y, Z \leq \sqrt{1-A^2}$,

Setting $Y = 0$:

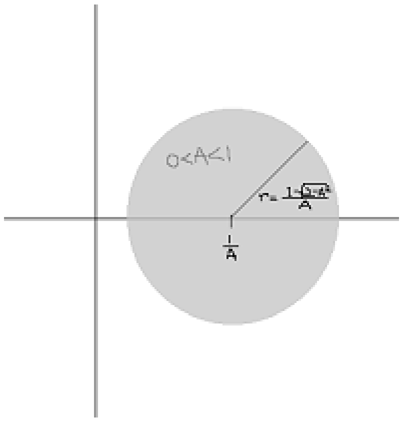
$\frac{A}{1+\sqrt{1-A^2}} \leq x \leq \frac{A}{1-\sqrt{1-A^2}}$, as Z goes from $-\sqrt{1-A^2}$ to $\sqrt{1-A^2}$

In the plane $X = A$, the intersection with the unit sphere is the circle in the plane centered at $(0, 0)$ with radius $\sqrt{1-A^2}$

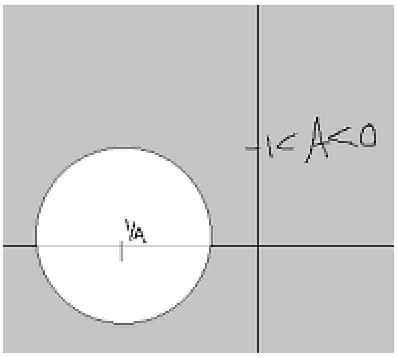
this gives us the circle: $(x - \frac{1}{A})^2 - y^2 = \frac{1}{A^2} - 1 = \frac{1-A^2}{A^2}$

It is centered at $(\frac{1}{A}, 0)$ with radius $\frac{\sqrt{1-A^2}}{A}$.

and if we continue to look at circles that come from the intersection of the sphere with the planes of the form $X = D$, with $D \in (A, 1)$. As D increases, we get circles that are contained within the one before, each with center $(1/D, 0)$ and radius $\frac{\sqrt{1-D^2}}{D}$. Our image is therefore the section within the circle centered at $(1/A, 0)$ with radius $\frac{\sqrt{1-A^2}}{A}$



When $A \in (-1,0)$, If we looked at the spherical cap $-1 \leq X \leq A$, we'd get a similar situation as the one before. Since we're looking at the projection from the rest of the sphere, we'd get the image being the exterior of the circle.



Ex. 6 We define the chordal distance $d(z,w)$ between two points $z, w \in \mathbb{C}^*$ to be the length of the straight line segment joining the points P and Q on the unit sphere whose stereographic projections are z and w respectively.

(a) Show that the chordal distance is a metric, that is, it is symmetric, $d(z,w) = d(w,z)$; it satisfies the triangle inequality $d(z,w) \leq d(z,\zeta) + d(\zeta,w)$, and $d(z,w) = 0$ if and only if $z = w$

let $P = (X_1, Y_1, Z_1)$ and $Q = (X_2, Y_2, Z_2)$, and let $z = x_1 + iy_1$ and $w = x_2 + iy_2$

then we have the length of the line segment $\overline{PQ}$ $= \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} =$ the length of $\overline{QP}$ so $d(z,w) = d(w,z)$

For the Triangle Inequality: the length of the segment is the distance between the two points, where the distance is the usual metric on $\mathbb{R}^3$, and by the definition of a metric, the triangle inequality is satisfied. Also, if we let R be a point on the unit sphere with the stereographic projection ζ, we could find the angles between $\overline{PQ}, \overline{QR}$, and $\overline{RP}$ by identifying the vectors, say u_1 from P to Q, u_2 from Q to R, and u_3 from P to R, using these vectors to find the cosine of the angle θ_1 between u_1 and u_3, and θ_2 between $-u_1$ and u_2, and use:

$||u_1|| = ||u_3|| \cos\theta_1 + ||u_2|| \cos\theta_2$ since these angles are going to be less than π, $\cos\theta_1, \cos\theta_2 \in (0,1)$

$d(z,w) = ||u_1|| \leq ||u_3|| + ||u_2|| = d(z,\zeta) + d(\zeta,w)$

For $d(z,w)$ = 0, we'd need $X_1 - X_2 = 0, Y_1 - Y_2 = 0, Z_1 - Z_2 = 0$, which means

$P = (X_1, Y_1, Z_1) = (X_2, Y_2, Z_2) = Q$, which means they have the same stereographic projection $z = w$

(b) Show that the chordal distance from z to w is given by $d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$, $z, w \in \mathbb{C}$

We rewrite $d(z,w)$ in terms of x_1, y_1, x_2, y_2 and $|z|, |w|$

$$\sqrt{(\frac{2x_1}{|z|^2+1} - \frac{2x_2}{|w|^2+1})^2 + (\frac{2y_1}{|z|^2+1} - \frac{2y_2}{|w|^2+1})^2 + (\frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1})^2}$$

$$(\frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1})^2 = \frac{(|z|^2-1)^2}{(|z|^2+1)^2} - 2\frac{(|w|^2-1)(|z|^2-1)}{(|z|^2+1)(|w|^2+1)} + \frac{(|w|^2-1)^2}{(|w|^2+1)^2}$$

$$(\frac{2x_1}{|z|^2+1} - \frac{2x_2}{|w|^2+1})^2 = 4(\frac{x_1^2}{(|z|^2+1)^2} - 2\frac{x_1x_2}{(|z|^2+1)(|w|^2+1)} + \frac{x_2^2}{(|w|^2+1)^2})$$

For the part containing y_1, y_2 can replace the x_i's' with the corresponding y_i's

And using $|z|^2 = x_1^2 + y_1^2$, when combining terms with denominator $(|z|^2+1)^2$

When we combine$(|z|^2-1)^2 + 4|z|^2 = |z|^4 - 2|z|^2 + 1 + 4|z|^2 = (|z|^2+1)^2$ (and similarly for $|w| = x_2^2 + y_2^2$

Under the radical we now have:

$$2(\frac{-4x_1x_2 - 4y_1y_2 - (|w|^2-1)(|z|^2-1) + (|w|^2+1)(|z|^2+1)}{(|z|^2+1)(|w|^2+1)}) = 2\frac{-4x_1x_2 - 4y_1y_2 + 2|w|^2 + 2|z|^2}{(|z|^2+1)(|w|^2+1)} = 2\frac{2(x_1-x_2)^2 + 2(y_1-y_2)^2}{(|z|^2+1)(|w|^2+1)}$$

squaring this we get: $2\frac{\sqrt{(x_1-x_2)-(y_1-y_2)}}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ and the numerator is, by definition, $|z-w|$

(c) What is $d(z,\infty)$?

as $w \to \infty$,

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} = \frac{2|z/w-1||w|}{\sqrt{1+|z|^2}(\sqrt{1/|w|^2+1})|w|} = \frac{2|z/w-1|}{\sqrt{1+|z|^2}\sqrt{1/|w|^2+1}} \to \frac{2|-1|}{\sqrt{1+|z|^2}} = \frac{2}{\sqrt{1+|z|^2}}$$

Chapter 1, Sec. 5

Ex. 4 Show that the only periods of e^z are the integral multiples of $2\pi i$, that is, if $e^{z+\lambda} = e^z$ for all z, then \lambda is an integer times $2\pi i$

Let $e^{z+\lambda} = e^z$

$e^{z+\lambda} = e^z e^\lambda = e^z$ (because of the addition formula property)

multiplying by e^{-z} : $e^\lambda = 1$ = $\cos(2\pi k) + i\sin(2\pi k) = e^{i2\pi k}$, where k is an integer.

multiplying by $e^{-i2\pi k}$ we have $e^\lambda e^{-i2\pi k} = 1$,

$e^{\lambda - i2\pi k} = e^0$

$\lambda - i2\pi k = 0 \equiv \lambda = i2\pi k$

Chapter 1, Sec. 6

Ex. 1 Find and plot $\log z$ for the following complex numbers z. Specify the principal value.

(a) 2, principal value = $\log|2|$

$x = \log|2|$, $y = 0$

(b) i, **principal value** $= 0 + i\pi/2$

$|i|$ = 1

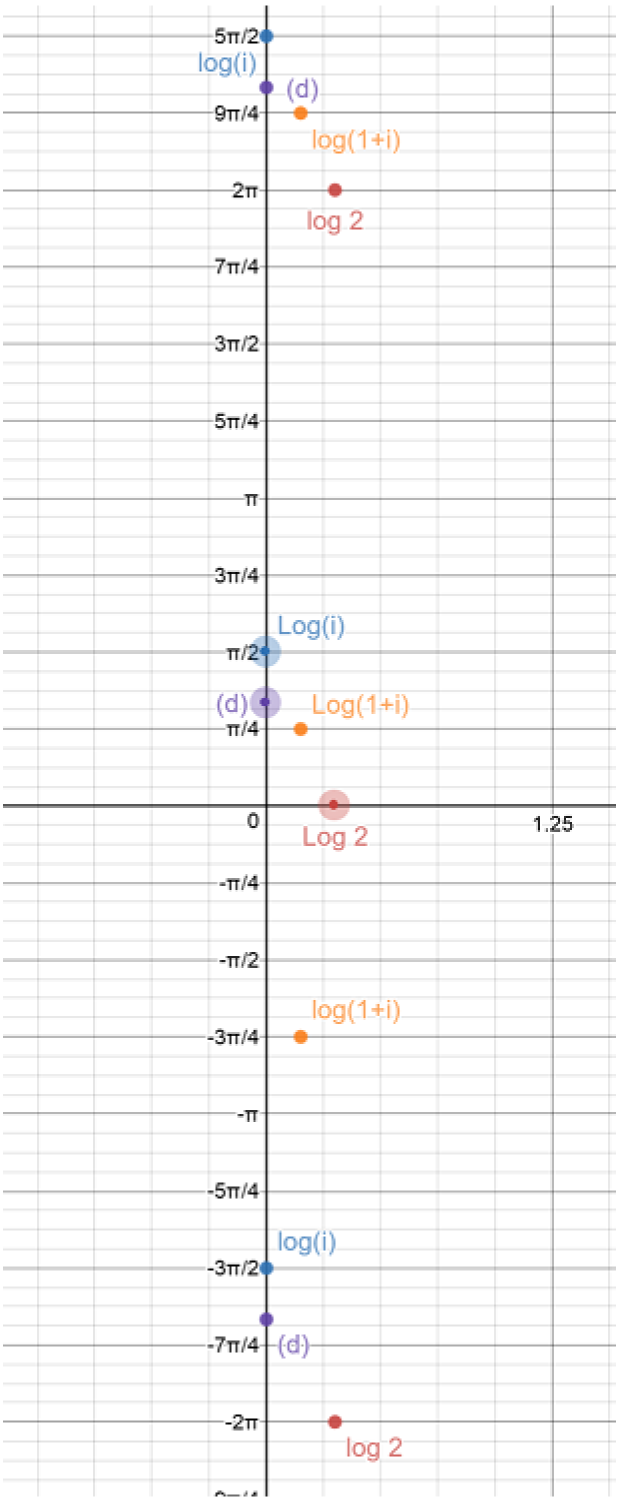
$x = 0, y = \pi/2$

(c) $1 + i$, **principal value** $= \log(\sqrt{2}) + i\pi/4 = (1/2)\log(2) + i\pi/4$

$x = \log\sqrt{2}, y = \pi/4$

(d) $(1 + i\sqrt{3})/2$, **principal value** $= \log(\sqrt{1/4 + 3/4} = 1) + i\pi/3$

$x = 0, y = \pi/3$



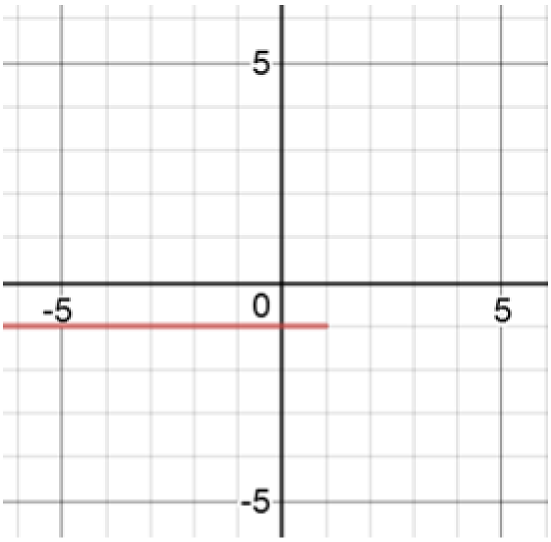
Ex. 4 How would you make a branch cut to define a single-value branch of function $\log(z+i-1)$? How about $\log(z-z_0)$

Each branch of $f_m(z) = Log(z+i-1) + 2\pi im$

We want to make a slit in the z-plane where $z+i-1 \in (-\infty, 0]$, and this will allow us to define a single-value branch of the function similar to the way we define one for $\log(z)$

so any $z = x - i$, where $x \in (-\infty, 1]$

So:



Where the red line is the slit in the complex plane.

Basically, started at the point $(1, -1)$ and drew a horizontal line towards $-\infty$ from there.

It's like treating $(1, -1)$ as a new origin and moving the axes over to the right by 1 and down by 1.

For example: For any point on the red line, $(x, -1)$the function maps this point to $(\log(\sqrt{(x-1)^2}, \pi)$

In general, for any z_0, we start at z_0 and draw a horizontal line towards $-\infty$

Chapter 2, Sec. 1

Ex. 10 At what points are the following functions continuous? Justify your answer.

(a) z

Let $z_0 \in \mathbb{C}$ and let $\epsilon > 0$

Let $\delta = \epsilon$

then for any z s.t. $|z - z_0| < \delta \implies |f(z) - f(z_0)| = |z - z_0| < \epsilon$

so as z approaches z_0, $f(z)$ approaches $f(z_0)$, and since z_0 was an arbitrary point in $\mathbb{C}$, z is continuous at any point in $\mathbb{C}$

(b) $z/|z|$

from (a), z is continuous everywhere. $|z|$ is a function that maps $\mathbb{C}$ to $\mathbb{R}$

From triangle inequality: $|(z-z_0) + (z_0)| = |z| \leq |z - z_0| + |z_0| \implies |z| - |z_0| \leq |z - z_0|$

so let $z_0 \in \mathbb{C}$ and let $\delta = \epsilon$, then:

for any z in $\mathbb{C}$ s.t. $|z - z_0| < \delta \implies ||z| - |z_0|| < \delta = \epsilon$

so $|z|$ is continuous everywhere.

However, $z/|z|$ must have $|z| \neq 0$ for this to be continuous, and $|z| = 0$ when $z = 0$

so continuous everywhere except $z = 0$

(c) $z^2/|z|$

From page 37, and (a), z^2 is continuous on every point in $\mathbb{C}$

From (b), we have that $1/|z|$ is not continuous at $z = 0$

so continuous everywhere except at 0.

If we were to define this function to be 0 at $z = 0$, then it would be continuous since the limit as z approaches 0 is 0:

for any $\epsilon > 0$, we have that if $|z| < \epsilon$

Using the fact that for any 2 complex numbers, $|zw| = |z||w|$

then $|f(z) - 0| = |\frac{z^2}{|z|}| = \frac{|z|^2}{|z|} = |z| < \epsilon$

(d) $z^2/|z|^3$

Continuous everywhere except at 0.

There is no limit as $z \to 0$, since $|z^2/|z|^3| = |z|^2/|z|^3 = 1/|z| \to \infty$

Ex. 15 Which of the following sets are open subsets of $\mathbb{C}$? Which are closed? Sketch the sets.

(a) The punctured plane $\mathbb{C} \backslash \{0\}$

Let $z = x + iy$ be a point in $\mathbb{C}$, (x, y)

Let $\rho = |z| > 0$, since $z \neq 0$

let w be a point in the open disk centered at z with radius p

then $|w - z| < \rho$

by triangle inequality: $|z| \leq |z - w| + |w| \equiv |w| \geq |z| - |z - w| > 0$, since $|z - w| = |w - z| < \rho$

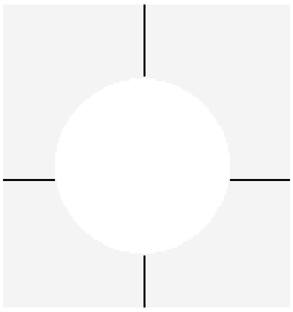
so $w \neq 0$, and since w was arbitrary, the disk is contained in the punctured plane.

It is open.

(b) the exterior of the open unit disk in the plane, $\{|z| \geq 1\}$

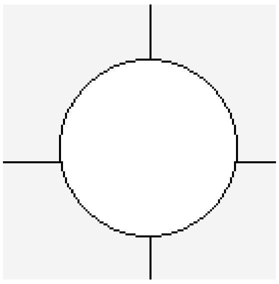
Not open, and also:

Closed, because the complement of the set is open.



(c) the exterior of the closed unit disk in the plane, {$|z| > 1$}

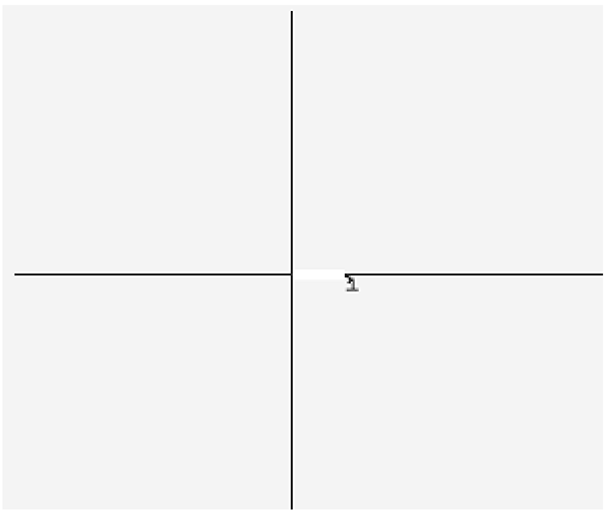
Open, you can take any point z and the open disk centered around it with radius $|z| - 1$ is contained in the set. Also it is the complement of a closed set.



(d) the plane with the open unit interval removed, $\mathbb{C} \setminus (0, 1)$

Not Open, if you take $z = 0$ or $z = 1$, then any open disk with radius > 0 will contain points in the open unit interval, such as $(0, .9999999. \ldots)$. (if the radius is ≥ 1, then all of the open unit interval will be contained in the disk, if the $\rho =$ radius is < 1, then the point $(0, 1 - \rho/2)$ is a point in the interval and in the disk.

Not closed: $(0, 1)$ is not an open subset: let $z = (x, 0)$ be in $(0, 1)$. Any disk centered at the point with radius ρ will contain the point $(x, \rho/2)$, which is not in the open unit interval



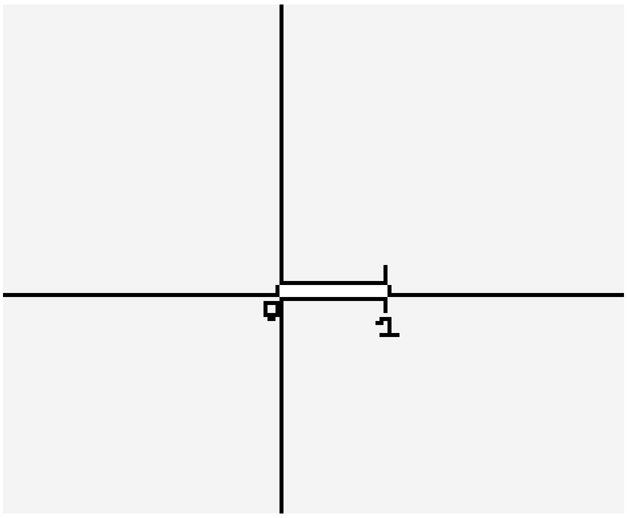
(e) the plane with the closed unit interval removed, $\mathbb{C} \setminus [0, 1]$

open: let $z \notin [0, 1]$. there exists a point $w \in [0, 1]$ such that for all $w' \in [0, 1]$

$|z - w'| > |z - w|$. In other words, w is the closets point to z

We define the radius for the open disk to be $< |z-w|$, which is possible since $|z-w| > 0$, and there are infinitely many real numbers between any two real numbers, in this case: (0, $|z-w|$) .

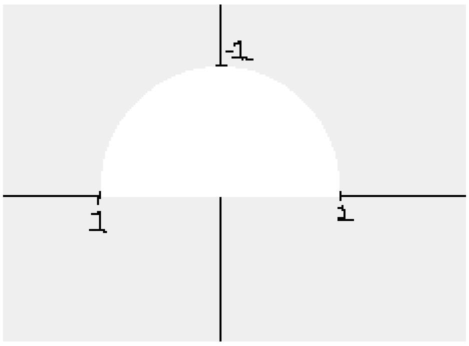
So, for every point in the set there is an open disk centered around it that is contained in the set.



(f) the semidisk {$|z| < 1, Im(z) \geq 0$}

not closed: the complement of this set contains the points on the boundary of the disk, such as $(0,1)$. And any open disk centered around $(0,1)$ with radius $\rho < 1$ will contain $(0, 1-\rho/2)$in the semidisk, so the complement is not open, meaning this set is not closed.

not open:Let z be in the semidisk with $x \in (-1,1), y = 0$. Any open disk centered on z with radius ρ will contain the point $(x, -\rho/2)$, which is not in the semidisk, since $Im(z) = -\rho/2 < 0$, so there exists points where no open disk centered on them will be contained in the set.



(g) the complex plane $\mathbb{C}$

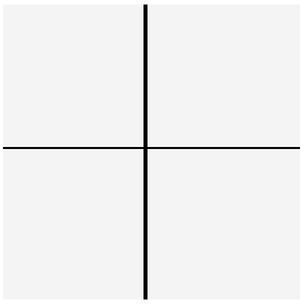
Both open and closed.

Any disk centered on any point in $\mathbb{C}$ is in $\mathbb{C}$, so open.

The empty set is considered open, so $\mathbb{C}$ is closed.

Also, there are no points in the empty set, so it does satisfy

whenever a point is in the set, there exists an open disk such that the disk is contained in the set.



Ex. 19 Give a proof of the fundamental theorem of algebra along the following lines. Show that if $p(z)$ is a nonconstant polynomial, then $|p(z)|$ attains its minimum at some point $z_0 \in \mathbb{C}$ Assume that the minimum is attained at $z_0 = 0$, and that $p(z) = 1 + az^m + \ldots$, where $m \geq 1$ and $a \neq 0$. Contradict the minimality by showing that $|P(\epsilon e^{i\theta_0})| < 1$ for an appropriate choice of θ_0

Let $p(z) = a_n z^n + \ldots + a_1 z + a_0$, $a_n \neq 0$ and $n \geq 1$

If we were to take the limit of $|p(z)|/|z|^n$ as $|z| \to \infty$

We would get $\frac{|z^n||(a_0/z^n + a_1/z^{n-1} + \ldots + a_n)|}{|z|^n}$ and since $|z^n| = |z|^n$ we have

$|(a_0/z^n + a_1/z^{n-1} + \ldots + a_n)| \to |a_n|$ as $z \to \infty$

So for any $\delta > 0$

so $\exists N > 0$ such that for $|z| > N$, $|p(z)|/|z|^n \in (|a_n| - \delta, |a_n| + \delta)$

Restricting $\delta < |a|$, so $|a| - \delta > 0$, we have for all $|z| > N$

$|p(z)|/|z|^n > |a_n| - \delta \equiv |p(z)| > (|a_n| - \delta)(|z|^n) > 0$

So we know that a minimum for $|p(z)|$ is obtained when $|z| \leq R$, and this minimum is at most $(|a_n| - \delta)(|z|^n)$

Now looking at $p(z) = 1 + az^m + \ldots$ higher order terms,

Assuming $z_0 = 0$ is when $|p(z)|$ has its minimum, $|p(0) = 1| = 1$

let $|a|e^{i\phi}$ be the polar representation of $a \neq 0$

Then $a\epsilon^m e^{i\phi} = |a|e^{i\phi}\epsilon^m e^{im\theta_0} = |a|\epsilon^m e^{i(\phi + m\theta_0)}$

if we choose θ_0 such that $\phi + m\theta_0 = \pi$, then $e^{i(\phi + m\theta_0)} = -1$

so we get

$p(z) = 1 - |a|\epsilon^m + C_1\epsilon^{m+1}e^{i(m+1)\theta_0} + \ldots$ and if we choose ϵ to be sufficiently small, we'll have $|p(z)| = |1 - |a|\epsilon^m + C_1\epsilon^{m+1}e^{i(m+1\theta_0)} + \ldots| < 1$

this is because no matter how large the coefficient, C_i, for any number $n > 0$ there always exists an $\epsilon > 0$ s.t. $|C_i|\epsilon < n$.

We could for example have a condition on ϵ to make $|1 - |a|\epsilon^m| < 1/2$

and $|C_1\epsilon^{m+1}e^{i(m+1\theta_0)} + \ldots| < 1/2$

so that by triangle inequality:

$|p(z)| = |1 - |a|\epsilon^m + C_1\epsilon^{m+1}e^{i(m+1\theta_0)} + \ldots| < 1/2 + 1/2 = 1$