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Math 185

Homework #11

Chapter VI,

Sec. 2, ex. 1g, 1e, 1i, 3, 12;

1g. $\text{Log}\,(1-\frac{1}{z})$

Singularities occur when

$1-\frac{1}{z}\leq 0$

so $\frac{1}{z}\geq 1$

$0\leq z\leq 1$

there are no isolated singularities

1e. $z^2\,\sin(\frac{1}{z})$

Singularity at $z=0$

$\sin z=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!}$

so $\sin(\frac{1}{z})=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k+1}}$

so $z^2\,\sin(\frac{1}{z})=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k-1}}$

has infinitely many negative powers of z

so $z=0$ is an essential singularity

1i. $e^{1/(z^2+1)}$

singularities when

$z^2+1=0$, so $z=\pm i$

$z=re^{i\pi/2}$

as $r\rightarrow 1^+$, we have $1/(z^2+1)\rightarrow-\infty$

so $e^{1/(z^2+1)}\rightarrow 0$ (not a pole)

as $r\rightarrow 1^-$, we have $1/(z^2+1)\rightarrow+\infty$,

so $e^{1/z^2+1}\rightarrow\infty$ (not removable)

and similarly for $z=-i$ with $z=re^{-i\pi/2}$

essential

3. Consider the function $f(z) = \tan z$ in the annulus {$3 < |z| < 4$} Let $f(z) = f_0(z) + f_1(z)$ be the Laurent decomposition of $f(z)$, so that $f_0(z)$ is analytic for $|z| < 4$, and $f_1(z)$ is analytic for $|z| > 3$ and vanishes at ∞

(a) Obtain an explicit expression for $f_1(z)$

Singularities: $\frac{\pi}{2} + \pi m = \frac{(2m+1)\pi}{2}$ $m \in \mathbb{Z}$

$1/f(z) = \frac{\cos z}{\sin z}$, which is analytic at each $\frac{(2m+1)\pi}{2}$

$(1/f(z))' = \frac{-\sin^2 z - \cos^2 z}{\sin^2 z} = -\frac{1}{\sin^2 z} \neq 0$ at $z = \frac{(2m+1)\pi}{2}$

so $1/f(z)$ has a zero of order 1 at each $z = \frac{(2m+1)}{2}$

so each $\frac{(2m+1)\pi}{2}$ is a simple pole

$\pm\frac{\pi}{2}$ are the only isolated singularities in {$|z|$ <4}

$\cos z = \sum_{k=0}^{\infty} a_k (z - \pi/2)^k$, if k even, $a_k = 0$, and $a_1 = -1$, $a_3 = 1$, etc...

since $a_0 = \cos(\pi/2) = 0$, $a_1 = -\sin(\pi/2) = -1$, $a_2 = 0$, $a_3 = \sin(\pi/2)$

$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(z-\pi/2)^{2k+1}}{(2k+1)!} = (z - \pi/2) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z - \pi/2)^{2k}$

$= (z - \pi/2)(-1 + \frac{(z-\pi/2)^2}{3!} - \ldots)$

$= -(z - \pi/2) + (z - \pi/2) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z - \pi/2)^{2k}$

$\frac{1}{\cos z} = -\frac{1}{(z-\pi/2)} \frac{1}{1 - \sum_{k=1}^{\infty} \frac{(-1)^{k+1})}{(2k+1)!}(z-\pi/2)^{2k}}$

$\frac{1}{\cos z} = -\frac{1}{z-\pi/2} + \text{analytic}$, the analytic part is $\sum_{l=1}^{\infty}(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(z - \pi/2)^{2k-1})^l$

$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(z - \pi/2)^{2k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}(z - \pi/2)^{2k}$

since $a_0 = \sin(\pi/2) = 1$, $a_1 = \cos(\pi/2) = 0$, $a_2 = -\sin(\pi/2)/2! = -1/2!$, $a_3 = 0$

so $\sin z / \cos z = \tan z = -\frac{1}{z-\pi/2} + \text{analytic}$

for $-\pi/2$,

$a_k = 0$ still when $k = 0$, but this time $a_1 = 1$, $a_3 = -1$, so

$\cos z = (z + \pi/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z + \pi/2)^{2k}$

so we have $\frac{1}{\cos z} = \frac{1}{z+\pi/2} + \text{analytic}$

$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!}(z + \pi/2)^{2k} = -1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!}(z - \pi/2)^{2k}$

$\tan z = -\frac{1}{z+\pi/2} + \text{analytic}$

so since $\tan z + \frac{1}{z-\pi/2} + \frac{1}{z+\pi/2}$ is analytic for {$|z| < 4$}

$f_1(z) = -(\frac{1}{z-\pi/2} + \frac{1}{z+\pi/2})$, and is obviously analytic for $|z| > 3$

and as $z \to \infty$, $f_1(z) \to 0$, so with $f_0 = \tan z - f_1$

$f_0(z) + f_1(z)$ is unique Laurent Decomposition

(b) Write down the series expansion for $f_1(z)$, and determine the largest domain on which it converges

$-\frac{1}{z-\pi/2} = -\frac{1}{z}\frac{1}{1-\pi/2z}$

and this is equal to a geometric series: $-\frac{1}{z}\sum_{k=0}^{\infty}\frac{\pi^k z^{-k}}{2^k}$

$-\frac{1}{z+\pi/2} = -\frac{1}{z}\sum_{k=0}^{\infty}(-1)^k\frac{\pi^k z^{-k}}{2^k}$

adding the two:

$f_1(z) = -\frac{1}{z}\sum_{k=0}^{\infty}[(-1)^k + 1]\frac{\pi^k z^{-k}}{2^k}$

so when k is odd, $a_k = 0$, so

$f_1(z) = -2\sum_{k=0}^{\infty}\frac{\pi^{2k} z^{-2k-1}}{4^k}$

$f_1(z) = -2\sum_{k=-\infty}^{0}\frac{\pi^{-2k}}{4^{-k}}z^{2k-1}$

converges for all $|z| > \pi/2$

(c) Obtain the coefficients $a_0, a_1,$ and a_2 of the power series expansion of $f_0(z)$

since $\tan z$ is an odd function,

$a_0 = a_2 = 0$

Since $f_0(z)$ is a power series centered at 0 and

$f_0(z) = \tan z + \frac{1}{z-\pi/2} + \frac{1}{z+\pi/2}$

$f_0'(z) = \frac{1}{\cos^2(z)} - \frac{1}{(z-\pi/2)^2} - \frac{1}{(z+\pi/2)^2}$

and $f_0'(0) = 1 - 8/\pi^2$

so $a_1 = f_0'(0)/1 = 1 - 8/\pi^2$

I'm currently using the 2nd edition, and the back of the book says $1 + 8/\pi^2$, but a list of errata changes it to $1 - 8/\pi^2$

http://www.math.ucla.edu/~twg/errata.pdf see the 3rd page

(d) What is the radius of convergence of the power series expansion for $f_0(z)$?

$f_0(z) = \tan z - f_1(z)$ is analytic except at $z = \pm 3\pi/2$, so the radius of convergence is $3\pi/2$

12. Show that if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity for $e^{f(z)}$

if z_0 is essential for $f(z)$

then for any $w_0 \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that

$f(z_n) \to w_0$

so there are two sequences $z_n \to z_0$ and $z'_n \to z_0$

with $f(z_n) \to w_0$, $f(z'_n) \to w'_0$

which means $e^{f(z)}$ as $z \to z_0$ has no limit

so $|e^{f(z)}|$doesn't $\to \infty$ as $z \to z_0$

but also $e^{f(z)}$ isn't bounded near z_0

for any $M > 0$, we may choose w_0 with $\log |w_0| > \log M$ such that $f(z_n) \to w_0$ for some sequence $z_n \to z_0$

and there exists some N, such that for $n \geq N$, $|f(z_n)| > \log M$

which means $|e^{f(z_n)}| > M$ for $n \geq N$

so z_0 is essential

if z_0 is a pole of order N for $f(z)$

then $f(z) = g(z)/(z - z_0)^N$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$

and $|f(z)| \to \infty$

we're always able to approach z_0 in a way that $(z - z_0)^N$ is positive/negative or neither (for example, approaching z_0 from the positive real direction so that $(z - z_0)^N > 0$ or we may build a sequence $z - z_0 \to 0$ so that every other term is real or pure imaginary)

so depending on the value of $g(z_0)$, we may choose a way to approach z_0 so that $f(z) \to +\infty$ or $f(z) \to -\infty$

which means $e^{f(z)}$ isn't bounded at z_0, so z_0 isn't removable

but also, $e^{f(z)}$ doesn't have a limit, so $|e^{f(z)}|$ doesn't approach ∞ as $z \to z_0$ so it isn't a pole

so z_0 is essential

Sec. 3, ex. 1 (for (g),(e),(i));

1g. $\mathrm{Log}\,(1 - \frac{1}{z})$

by definition $f(z)$ has an isolated singularity at ∞ if $f(z)$ is analytic outside some bounded set,

so $f(z)$ is analytic for $|z| > 1$, since for $z \in [0, 1]$, it isn't analytic

so there is an isolated singularity at ∞

From earlier problem:

if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity for $e^{f(z)}$

$\equiv$ if z_0 is not essential for $e^{f(z)}$ then z_0 is a removable singularity of $f(z)$ $g(w) = \mathrm{Log}(1 - w)$ is analytic at $w = 0$ (if there were an isolated singularity at $w = 0$, it is removable)

$e^{g(w)} = 1 - w$, so since $a_k = 0$ for all $k < 0$, $w = 0$ is not essential for $e^{g(w)}$,

which means 0 is a removable singularity for $g(w)$

$\equiv \infty$ is removable for $\mathrm{Log}\,(1 - \frac{1}{z})$

1e. $z^2 \sin(\frac{1}{z})$

from its Laurent series, $z^2\sin(\frac{1}{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!z^{2k-1}}$

we have $N = 1$, where $b_1 = 1$, and for all $k > 1$, $b_k = 0$

so we have a simple pole at ∞

1i. $e^{1/(z^2+1)}$

$g(w) = f(1/w) = e^{1/(\frac{1}{w^2}+1)}$

has an isolated singularity at $w = 0$, so $f(z)$ has an isolated singularity at ∞

as $w \to 0$, $\frac{1}{w^2} \to \infty$, so $\frac{1}{\frac{1}{w^2}+1} \to 0$ and $e^{1/(\frac{1}{w^2}+1)} \to 1$

which means near 0, $g(w)$ is bounded, so $g(w)$ has a removable singularity at 0

which means $f(z)$ has a removable singularity at ∞

Sec. 4, ex. 1c, 1f, 2c, 3

1c. $\frac{1}{(z+1)(z^2+2z+2)}$

$\frac{A}{z+1} + \frac{B}{z+1-i} + \frac{C}{z+1+i}$

since: $(z+1+i)(z+1-i) = z^2+2z+2$

$A(z^2+2z+2) + B(z+1)(z+1+i) + C(z+1)(z+1-i) = 1$

$(z+1)(z+1+i) = z^2+z+iz+z+1+i = z^2+(2+i)z+(1+i)$

$(z+1)(z+1-i) = z^2+z-iz+z+1-i = z^2+(2-i)z+(1-i)$

$A+B+C = 0$

$2A+(2+i)B+(2-i)C = 0$

$2A+(1+i)B+(1-i)C = 1$

$(2+i)B+(2-i)C = (1+i)B+(1-i)C-1$

$2B+2C = (2+i)B+(2-i)C$

$B+C = -1$

$B-C = 0$

$B = C$

$B = -\frac{1}{2} = C$

$A = 1$

$\frac{1}{(z+1)(z^2+2z+2)} = \frac{1}{z+1} - \frac{1/2}{z+1-i} - \frac{1/2}{z+1+i}$

1f. $\frac{z^2-4z+3}{z^2-z-6}$

$z^2-4z+3 = (z^2-z-6)-3z+9$

$\frac{z^2-4z+3}{z^2-z-6} = 1 + \frac{-3z+9}{z^2-z-6}$

$z^2 - z - 6 = (z-3)(z+2)$

poles at $-2, 3$

$\frac{A}{z-3} + \frac{B}{z+2}$

$A + B = -3$

$-3B + 2A = 9$

$-5B = 15$

$B = -3$, $A = 0$

also $\frac{-3(z-3)}{(z-3)(z+2)} = \frac{-3}{z+2}$

so $\frac{z^2-4z+3}{z^2-z-6} = 1 - \frac{3}{z+2}$

2c $\frac{z^6}{(z^2+1)(z-1)^2}$

$(z^2+1)(z-1)^2 = (z^2+1)(z^2-2z+1) = z^4 - 2z^3 + z^2 + z^2 - 2z + 1 = z^4 - 2z^3 + 2z^2 - 2z + 1$

$z^6 = z^2(z^4 - 2z^3 + 2z^2 - 2z + 1) + 2z^5 - 2z^4 + 2z^3 - z^2$

$2z^5 - 2z^4 + 2z^3 - z^2 = 2z(z^4 - 2z^3 + 2z^2 - 2z + 1) + 2z^4 - 2z^3 + 3z^2 - 2z$

$2z^4 - 2z^3 + 3z^2 - 2z = 2(z^4 - 2z^3 + 2z^2 - 2z + 1) + 2z^3 - z^2 + 2z - 2$

$z^6 = (z^2 + 2z + 2)(z^4 - 2z^3 + 2z^2 - 2z + 1) + 2z^3 - z^2 + 2z - 2$

$\frac{z^6}{(z^2+1)(z-1)^2} = z^2 + 2z + 2 + \frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2}$

$\frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{(z-1)^2} + \frac{D}{z-1}$

$A(z-i)(z-1)^2 = A(z-i)(z^2-2z+1) = A(z^3 - 2z^2 + z - iz^2 + 2iz - i)$
$= A(z^3 - (2+i)z^2 + (1+2i)z - i\,)$

$B(z+i)(z^2-2z+1) = B(z^3 - 2z^2 + z + iz^2 - 2iz + i) = B(z^3 + (-2+i)z^2 + (1-2i)z + i)$

$C(z^2+1)$

$D(z^2+1)(z-1) = D(z^3 - z^2 + z - 1)$

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(1) $A + B + D = 2$

(2) $-(2+i)A + (-2+i)B + C - D = -1$

(3) $(1+2i)A + (1-2i)B + D = 2$

(4) $-iA + iB + C - D = -2$

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(1) and (3) $\implies A + B = (1+2i)A + (1-2i)B$

$\equiv -A + B = 0$, $A = B$ (5)

(5) and (1) $\implies 2A + D = 2$

$D = 2 - 2A$ (6)

(5) and (4) and (6): $C - D = -2 \implies C - 2 + 2A = -2 \implies 2A + C = 0$

so $C = -2A$ (7)

(2) and (5) and (7) and (6): $-(2+i)A + (-2+i)A - 2A - 2 + 2A = -1$

$-2A - 2A = 1$, $A = -\frac{1}{4} = B$

$D = 2 + \frac{1}{2} = \frac{5}{2}$

$C = \frac{1}{2}$

$\frac{z^6}{(z^2+1)(z-1)^2} = \frac{-1/4}{z+i} + \frac{-1/4}{z-i} + \frac{1/2}{(z-1)^2} + \frac{5/2}{z-1}$

3. Let V be the complex vector space of functions that are analytic on the extended complex plane except possibly at the points 0 and i, where they have poles of order at most two. What is the dimension of V? Write down explicitly a vector space basis for V

$V =$\{$f(z) : f(z) = f_\infty(z) + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z-i} + \frac{E}{(z-i)^2}$\}

so $f_\infty(z) = A$, where we let A be a complex constant

this is equivalent to finding all vectors:

$(A, B, C, D, E) \in \mathbb{C}^5$

The dimension is therefore equal to 5

and using standard bases in $\mathbb{C}^5$:

$(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)$

$1, 1/z, 1/z^2, 1/(z-i), 1/(z-i)^2$

is a basis for V