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Math 185

Homework #1

Section 1.1

Ex. 2: Verify from the definitions each of the identities

a. $\overline{z+w} = \overline{z} + \overline{w}$

If z nd w are comple numbers, they may be written in the form:

$z = x + iy$, $w = u + iv$ with $x, y, u, v \in \mathbb{R}$

From the definition of the complex conjugate:

$\overline{z} = x - iy$ and $\overline{w} = u - iv$, which are also complex numbers, so

$\overline{z} + \overline{w} = (x + u) + i(-y + (-v))$

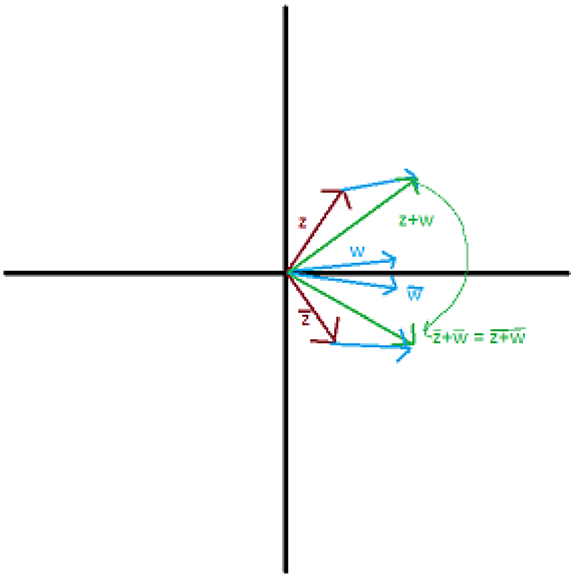
$\overline{z} + \overline{w} = (x + u) + i(-(y + v))$

= $(x + u) + (-i)(y + v) = (x + u) - i(y + v)$

since $z + w = (x + u) + i(y + v)$

We have $\overline{z+w} = (x + u) - i(y + v)$

so $\overline{z+w} = \overline{z} + \overline{w}$



b. $\overline{zw} = \bar{z}\bar{w}$

letting z and w be expressed as $z = x + iy$, $w = u + iv$ with $x, y, u, v \in \mathbb{R}$,

$zw = xu - yv + i(xv + yu)$

meaning $\overline{zw} = xu - yv - i(xv + yu)$

taking $\bar{z}$ and $\bar{w}$ and multiplying them:

$\bar{z}\bar{w} = (x - iy)(u - iv) = xu - (-y)(-v) + i(x(-v) + (-y)u)$

$= xu - yv + i(-(xv + yu)) = xu - yv - i(xv + yu) = \overline{zw}$

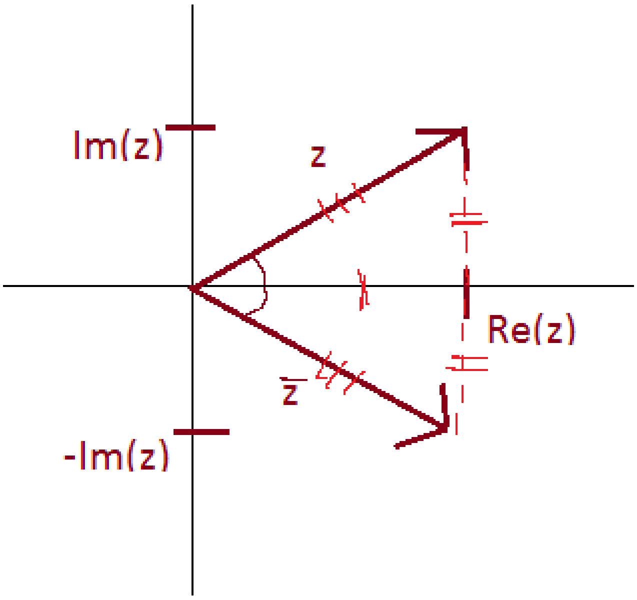
c. $|\bar{z}| = |z|$

if $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$

$\bar{z} = x - iy$ (by definition)

$|\bar{z}| = \sqrt{x^2 + (-y)^2}$ and since $(-y)^2 = y^2$

$|\bar{z}| = \sqrt{x^2 + y^2} = |z|$



Ex. 6: For Fixed $a \in \mathbb{C}$, show that $|z - a|/|1 - \bar{a}z| = 1$ if $|z| = 1$ and $|1 - \bar{a}z| \neq 0$

Since $|z|$ = 1 and $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ we have

$\frac{1}{z} = \bar{z} \equiv z\bar{z} = 1$

so we rewrite $|z - a|/|1 - \bar{a}z|$ as

$|z - a|/|z\bar{z} - \bar{a}z|$

$= |z - a|/|z(\bar{z} - \bar{a})|$

and since $|zw| = |z||w|$

= $|z - a|/(|z||\bar{z} - \bar{a}|)$

and since $|z| = 1$

$= |z-a|/|\bar{z}-\bar{a}|$

if we let $z = x + iy$ and $a = u + iv$

$z - a = (x-u) + i(y-v)$

$\bar{z} - \bar{a} = (x-u) + i(v-y)$

Meaning

$|z-a| = \sqrt{(x-u)^2 + (y-v)^2} = \sqrt{(x-u)^2 + (v-y)^2} = |\bar{z}-\bar{a}|$

so $|z-a|/|\bar{z}-\bar{a}| = 1$

Ex. 10: Let $q(z)$ be a polynomial of degree $m \geq 1$. Show that any polynomial $p(z)$ can be expressed in the form $p(z) = h(z)q(z) + r(z)$ where $h(z)$ and $r(z)$ are polynomials and the degree of the remaining $r(z)$ is strictly less than m.

Case 1: $deg(p(z)) = 0$ **and in general** $deg(p(z)) < m$

we have $p(z) = 0 \times q(z) + p(z)$

We let $h(z) = 0, r(z) = p(z)$, so $p(z)$ can be expressed in the form $h(z)q(z) + r(z)$. And since $deg(p(z)) = deg(r(z)) < m$, we have proven the statement true for this case.

Case 2: $deg(p(z)) \geq m$

we do a proof by induction on the degree of $p(z)$

Base Case: Starting with $deg(p(z)) = 1$

since $1 \leq m \leq deg(p(z)) = 1$, we have $m = 1$

$p(z) = p_1 z + p_0$, some constants p_1, p_0

$q(z) = q_1 z + q_0$

multiplying $q(z)$ by $p_1 q_1^{-1}$ we get

$p_1 q_1^{-1} q(z) = p_1 z + p_1 q_1^{-1} q_0$

and if we add $p_0 - p_1 q_1^{-1} q_0$

we get $(p_1 q_1^{-1})q(z) + (p_0 - p_1 q_1^{-1} q_0) = p_1 z + p_0 = p(z)$

So setting $h(z) = p_1 q_1^{-1}$ and $r(z) = p_0 - p_1 q_1^{-1} q_0$

we can express $p(z)$ in the form $h(z)q(z) + r(z)$.

and since $r(z)$ is a constant, it has degree 0, which is less than m = 1.

So we have proven the base case.

Assume any polynomial of degree < n can be expressed in the form $h(z)q(z) + r(z)$, where $h(z), r(z)$ are polynomials with $r(z)$ having degree less than m.

let $q(z) = q_m z^m + q_{m-1} z^{m-1} + \ldots + q_1 z + q_0$, where $q_0, \ldots, q_m$ are constants, $q_m \neq 0$ and $m \geq 1$

let $p(z) = p_n z^n + p_{n-1} z^{n-1} + \ldots p_1 z + p_0$

Since we still have $m \leq n$, $n - m \geq 0$

so we can multiply $q(z)$ by $p_n q_m^{-1} z^{n-m}$ to get:

$p_n q_m^{-1} z^{n-m} q(z) = p_n z^n + p_n q_m^{-1} q_{m-1} z^{n-1} + \ldots + p_n q_m^{-1} q_0 z^{n-m}$

and subtract it from $p(z)$:

$p(z) - p_n q_m^{-1} z^{n-m} q(z)$, which is a polynomial of degree $n-1$, since $p(z)$ and $p_n q_m^{-1} z^{n-m} q(z)$ have the same leading coefficient are of the same degree n.

So by Induction Hypothesis, we can express this polynomial in the form

$p(z) - p_n q_m^{-1} z^{n-m} q(z) = h(z)q(z) + r(z)$

this means:

$p(z) = (h(z) + p_n q_m^{-1} z^{n-m})q(z) + r(z)$

and $r(z)$ has degree less than m, and $h(z) + p_n q_m^{-1} z^{n-m}$ is a polynomial.

Section 1.2

Ex. 1 Express All values of the following expressions in both polar and Cartesian coordinates, and plot them.

Ex. 1c: $\sqrt[4]{-1}$

-1 would be expressed as $(-1, 0)$ and $e^{i\pi}$

So if $\sqrt[4]{-1} = re^{i\theta}$

$r = 1^{1/4} = 1$

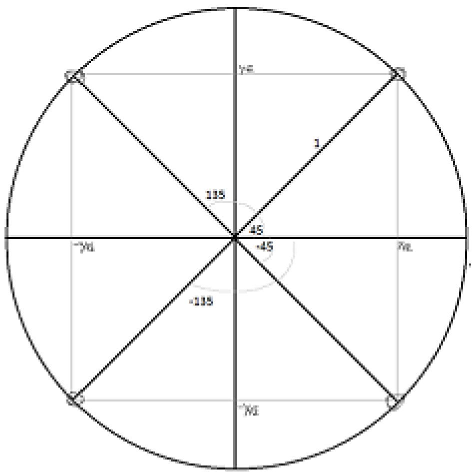
$\theta = \frac{\pi}{4} + \frac{2\pi k}{4}$

which means $\sqrt[4]{-1} = e^{i\frac{(2k+1)\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Polar coordinates, $(r, Arg\ z)$

$(1, \pi/4), (1, 3\pi/4), (1, -3\pi/4), (1, -\pi/4)$



Ex. 1g: $(1+i)^8$

$(1+i)^2 = 2i$ and $(2i)^4 = 16$

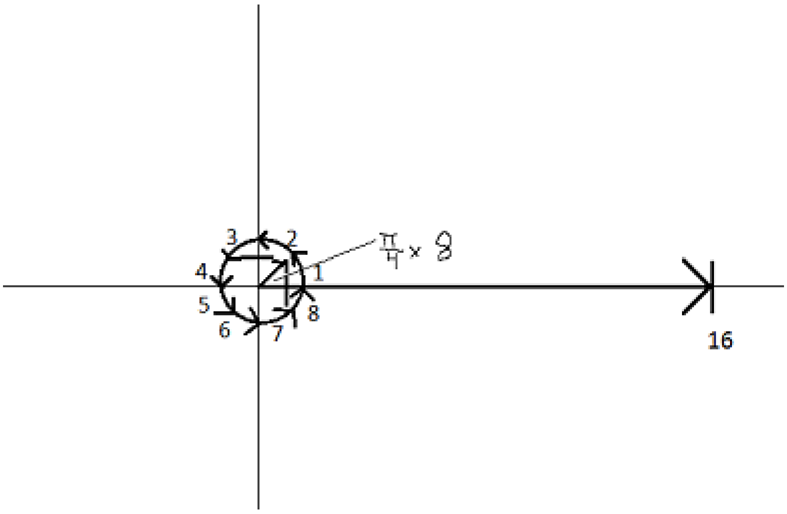
$(1+i)^8 = 16\ (+i0)$

$r = \sqrt{2}^8 = 16$

$\theta = 2\pi k$

$16e^{2\pi k}$

$(16, 0)$ for both Cartesian and Polar coordinates



Ex. 5: For $n \geq 1$, show that

(a) $1 + z + z^2 + \ldots + z^n = (1 - z^{n+1})/(1-z)$, $z \neq 1$

Proof by Induction:

Base Case: $n = 1$, $1 + z$

$(1 - z^2)/(1-z) = (1-z)(1+z)/(1-z) = 1 + z$

Assume true for n < k.

Take $1+z + . . .+z^k

- z^{k+1}$

Using Inductive Hypothesis:

$(1 + z + \ldots + z^k) + z^{k+1} = (1 - z^{k+1})/(1-z) + z^{k+1}$

multiplying z^{k+1} by $(1-z)/(1-z)$ we get:

$(1 - z^{k+1} + z^{k+1} - z^{k+2})/(1-z) = (1 - z^{k-2})/(1-z)$, proving this true for $k+1$.

Therefore, (a) is true for all $n \geq 1$

(b) $1 + cos\,\theta + cos\,2\theta + \ldots + cos\,n\theta = \frac{1}{2} + \frac{sin(n+\frac{1}{2})\theta}{2sin\theta/2}$

Starting with two equations:

1. $1+z+...+z^n$, with $z=e^{i\theta}=cos\theta+isin\theta$ to get

   $1+(cos\theta+isin\theta)+(cos2\theta+isin2\theta)+..+(cosn\theta+isin\,n\theta)=(1-e^{i(n+1)\theta})/(1-e^{i\theta})$
2. This time, with $z^{-i\theta}=cos(-\theta)+isin(-\theta)$ and since cos is even and sin is odd, $z^{-i\theta}=cos\theta-isin\theta$ to get

   $1+(cos\theta-isin\theta)+(cos2\theta-isin2\theta)+..+(cosn\theta-isin\,n\theta)=(1-e^{-i(n+1)\theta})/(1-e^{-i\theta})$

Adding the two equations:

$2(1+cos\theta+cos2\theta+...+cosn\theta)=(1-e^{i(n+1)\theta})/(1-e^{i\theta})+(1-e^{-i(n+1)\theta})/(1-e^{-i\theta})$

$=\frac{(1-e^{i(n+1)\theta})(1-e^{-i\theta})+(1-e^{-i(n+1)\theta})(1-e^{i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$

$=\frac{2-e^{i(n+1)\theta}-e^{-i\theta}+e^{in\theta}-e^{-(n+1)\theta}-e^{i\theta}+e^{-in\theta}}{2(1-cos\theta)}$, since $e^{i\theta}$ and $e^{-i\theta}$ are conjugates, adding them will result in $cos\theta$ (for any θ)

$=\frac{1-cos\theta-cos(n+1)\theta)+cosn\theta}{1-cos\theta}=1+\frac{cosn\theta-cos(n+1)\theta}{1-cos\theta}$

And using the following:

1. $1-cos\theta=2sin^2\theta/2$
2. $cos(n+1)\theta=cosn\theta cos\theta-sin\,n\theta sin\theta$
3. $sin\theta=2sin(\theta/2)cos(\theta/2)$

$=1+\frac{cosn\theta(2sin^2\theta/2)+sin\,n\theta(2sin\theta/2cos(\theta/2))}{2sin^2\theta/2}$

cancelling out 2 and $sin\theta/2$ gives:

$=1+\frac{cosn\theta sin\theta/2+sin\,n\theta cos(\theta/2)}{sin^2\theta/2}$, and $sin(n\theta+\theta/2)=sin\,n\theta cos\theta/2+sin\theta/2cosn\theta$

$=1+\frac{sin(n+1)\theta}{sin^2\theta/2}$

And dividing both sides by 2 gives:

$1+cos\theta+...+cosn\theta=1/2+\frac{sin(n+1)\theta}{2sin^2\theta/2}$