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Math 185

Homework #9

Chapter V, Sec. 4. ex. 1d, 1f, 3; Sec. 5, ex. 1c; Sec. 6, ex. 3; Sec. 7, ex. 1a, 1f, 1i; Sec. 7, ex. 9, 11;

Extra Problem (by the link)

Sec. 4. ex 1: Find the radius of convergence of the power series for the following functions, expanded about the indicated point

(d) $\mathrm{Log}\ z$, **about** $z = 1 + 2i$

The nearest singularity is 0, since $(1, 2)$ is in the first quadrant

$R = |1 + 2i| = \sqrt{1+4} = \sqrt{5}$

(f) $\frac{z-i}{z^3-z}$ **about** $z = 2i$

$z^3 - z = z(z^2 - 1) = z(z-1)(z+1)$

singularities at $0, -1, 1$

nearest is at 0

$R = |2i| = \sqrt{2^2} = 2$

Sec. 4. ex. 3 Find the power series expansion of $\mathrm{Log}\ z$ **about the point** $z = i - 2$. **Show that the radius of convergence of the series is** $R = \sqrt{5}$. **Explain why this does not contradict the discontinuity of** $\mathrm{Log}\ z$ **at** $z = -2$

$\mathrm{Log}\ z = \sum_{k=0}^{\infty} a_k (z - (i-2))^k$

$a_0 = \mathrm{Log}(i-2) = \log(\sqrt{5}) + \mathrm{Arg}\ (i-2)$

$a_1 = \frac{1}{i-2}$

$a_2 = -\frac{1}{2(i-2)^2}$

$a_3 = \frac{2}{6(i-2)^3} = \frac{1}{3(i-2)^3}$

$a_k = (-1)^{k+1} \frac{1}{k(i-2)^k}, k \geq 1$

$|a_k / a_{k+1}| = |\frac{(k+1)(i-2)^{k+1}}{k(i-2)^k}| = |\frac{(k+1)(i-2)}{k}| \to |i-2|$ as $k \to \infty$

and $|i-2| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

so $R = \sqrt{5}$

$\mathrm{Log}\ z = \mathrm{Log}(i-2) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(i-2)^k} (z - (i-2))^k$

This power series has the same radius of convergence as that for $\frac{1}{z}$, which is analytic on any disk contained in $\mathbb{C}\backslash\{0\}$.

In the part of the disk that goes beyond the cut (the negative real axis and origin), the series expansion does not agree.

$\text{Log } z$ extends to be analytic for $|z-(i-2)|<\sqrt{5}$, though the extension does not coincide with $\text{Log } z$ in the part of the disk in the lower half plane

Sec. 5. ex. 1: Expand the following functions in power series about ∞:

(c) e^{1/z^2}

$g(w)=f(1/w)=e^{w^2}$

using $e^z=\sum_{k=0}^{\infty} z^k/k!$

$e^{w^2}=\sum_{k=0}^{\infty}\frac{w^{2k}}{k!}$

$e^{1/z^2}=\sum_{k=0}^{\infty}\frac{1}{z^{2k}k!}$

Sec. 6. ex. 3: Show that $\frac{e^z}{1+z}=1+\frac{1}{2}z^2-\frac{1}{3}z^3+\frac{3}{8}z^4-\frac{11}{30}z^5+\ldots$. Show that the general term of the power series is given by $a_n=(-1)^n[\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}]$. What is the radius of convergence of the series?

$1/(1+z)=\sum_{k=0}^{\infty}(-1)^k z^k=b_k z^k$

$e^z=\sum_{k=0}^{\infty} z^k/k!=\sum a_k z^k$

$e^z/(1+z)=\sum c_k z^k$

so $a_k=1/k!$, $b_k=(-1)^k \implies$

$c_k=a_k b_0+a_{k-1}b_1+\ldots+a_0 b_k=\frac{1}{k!}-\frac{1}{(k-1)!}+\ldots+(-1)^{k-1}+(-1)^k=\frac{1}{k!}-\frac{1}{(k-1)!}+\ldots+\frac{(-1)^{k-2}}{2}$

so $c_k=(-1)^k[\frac{(-1)^{-2}}{2}+\frac{(-1)^{-3}}{3!}+\ldots+\frac{(-1)^{-k}}{k!}]$

which is equivalent to $c_k=(-1)^k[\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^k}{k!}]$

and $c_0=1, c_1=(1*1)+(1*(-1))=0$

and using the above equation:

$c_2=1/2$

$c_3=-1/2+1/6=-2/6=-1/3$

$c_4=1/2-1/6+1/24=\frac{12-4+1}{24}=\frac{9}{24}=\frac{3}{8}$

$c_5=-1/2+1/6-1/24+1/120=-\frac{3}{8}+\frac{1}{120}=\frac{-44}{120}=-11/30$

Since we are looking at the power series about $z_0=0$, and the nearest singularity is at $z=-1$, The radius of convergence is $R=|-1|=1$

Sec 7. ex. 1: Find the zeros and orders of zeros of the following functions

(a) $\frac{z^2+1}{z^2-1}$

$f(i) = 0$, $f(-i) = 0$

and $f'(z) = \frac{2z(z^2-1)-2z(z^2+1)}{(z^2-1)^2} = \frac{-4z}{(z^2-1)^2}$, which is not zero at $z = \pm i$

so each of the zeros, $\pm i$ are simple zeros

(f) $\frac{\cos z-1}{z^2}$

zeros at $z = 2\pi n$

$f'(z) = \frac{-z^2 \sin z-2z(\cos z-1)}{z^4}$

$f'(2\pi n) = \frac{-(2\pi n)^2(0)-4\pi n(0)}{16n^4\pi^4} = 0$

$f''(z) = \frac{(-2z \sin z-z^2 \cos z-2(\cos z-1)+2z(\sin z))(z^4)-4z^3(-z^2 \sin z-2z(\cos z-1))}{z^8}$

$f''(2\pi n) = \frac{(2\pi n)^4(-(2\pi n)^2}{(2\pi n)^8} \neq 0$

so double zeros at $z = 2\pi n$ when $n \neq 0$,

$\cos z - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$

and $\frac{\cos z-1}{z^2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2(k+1))!}$

so $z = 0$ is not a zero, since $a_0 = -1/2$

(i) $\frac{\text{Log } z}{z}$ **(principal value)**

$\text{Log } 1 = \log(1) + \text{Arg } (1) = 0 + 0 = 0$, since $\text{Arg}(1) \in (-\pi, \pi]$

so $z = 1$ is a zero

$f'(z) = \frac{1-\text{Log } z}{z^2}$, which is not 0 at $z = 1$, so $z = 1$ is a simple zero

Sec. 7. ex. 9: Show that if the analytic function $f(z)$ has a zero of order N at z_0, then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z_0) \neq 0$

we can write $f(z)$ as $f(z) = (z - z_0)^N h(z)$, where $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$

$g(z) = (z - z_0)e^{\log(h(z))/N}$, for a branch of logarithm containing $h(z_0)$ so: $f_m(z) = \text{Log } z + 2\pi i m$, where $m \in (-\infty, \infty)$

$g'(z_0) = e^{\log(h(z_0))/N} + (z_0 - z_0)\frac{d}{dz}[e^{\log(h(z))/N}]_{z=z_0} \neq 0$

since $e^z \neq 0$ for all $z \in \mathbb{C}$

and obviously $g(z)^N = (z - z_0)^N h(z) = f(z)$

also e^z is analytic, $(z - z_0)$ is analytic

and $\log(h(z))/N$ is analytic for the branch containing $h(z_0)$

so $g(z)$ is a composition of functions that are analytic near z_0 , so it is analytic near z_0

Sec. 7. ex. 11: Show that if $f(z)$ is a nonconstant analytic function on a domain D, then the image under $f(z)$ of any open set is open.

at any point $z_0 \in D$, where $f'(z_0) \neq 0$,

there is a small disk $U \subset D$ containing z_0 such that $f(z)$ is one-to-one on U, and the image $V = f(U)$ of U is open

f maps open disks to open sets, so for any z_0 where $f'(z_0) \neq 0$, $f(D)$ contains an open disk centered at $f(z_0)$, so $f(D)$ is open

and since any open set in D is a union of open disks, for any z_0 in an open set,

there is a disk that is contained in U and D s.t. a disk centered at $f(z_0)$ is contained in the image of this set under f, so the image is open.

if $f'(z_0) = 0$:

What I will use:

1. $z \mapsto az + b$ is open when $a \neq 0$

   let $a = Ae^{i\phi}$, where $A \in \mathbb{R}^+$ and $\phi \in \mathbb{R}$

   for any open disk centered on z_0 with radius r,

   z in this disk satisfies $|z| - |z_0| \leq |z - z_0| < |re^{i\theta}|$, for any $\theta \in \mathbb{R}$

   let $w_0 = az_0 + b$, and for any z in the disk, let $w = az + b$

   $|w - w_0| = |a(z - z_0)| = |Ae^{i\phi}||z - z_0| < |Ae^{i\phi}||re^{i\theta}| = Ar$

   so the image of any open disk is also an open disk

   $\implies$ since open sets are unions of open disks, this mapping is open

2. z^k where $k \in \mathbb{N}$ is open

   Show for a disk centered at origin with radius r

   the image of this disk will be an open disk centered at origin with radius r^k

3. The composition of open mappings is open

   if f is open on D_1 and g is open on D_2 and $g(D_2) \subset D_1$

   then $f(g(D_2))$ is open

if $f(z_0) = 0$

then $f(z) = g(z)^N$ (as in exercise 9)

and from 2., $f(z)$ is open

if $f(z_0) \neq 0$

then we can define $f^*(z) = f(z) - f(z_0)$, so $f^*(z_0) = 0$

and so $f^*(z) = g(z)^N$ (as in exercise 9)

and $f(z) = g(z)^N + f(z_0)$

and since $f(z)$ is a composition of $g(z)$, z^N, and $z + f(z_0)$ (each are open maps)

$f(z)$ is open

Extra Problem (by the link)

1. Let C be the right half of the unit circle centered at origin. Does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ with the specified values on C :

$f(e^{i\theta}) = e^{i\theta/2}$ for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

so C is a set that has a non isolated point (every point on the right half of the unit circle gets arbitrarily close to each other)

and any function $f(z)$ s.t. f maps $e^{i\theta}$ to $e^{i\theta/2}$ for all $\theta \in [-\pi/2, \pi/2]$

is equal to $g(z) = \sqrt{z}$ on C, which is analytic on the entire slit plane $\mathbb{C} \setminus (-\infty, 0]$

and by the Uniqueness Principle, if $f(z)$ is also analytic on the slit plane,

$f(z) = g(z)$ for all z in the slit plane

however, $g(z)$ is not analytic for any point in $(-\infty, 0]$ (not continuous)

if $f(z)$ were entire, it would also need to be analytic on $(-\infty, 0)$

but as $z \to$ the negative axis, $f(z)$ has no limit, (approaching the $-r \in \mathbb{R}$ from the top, we have $f(z) \to i\sqrt{r}$ but coming up from the bottom, $f(z) \to -i\sqrt{r}$)

so $f(z)$ is not continuous on $(-\infty, 0]$

so f cannot be entire.