Renee Senining

Math 185

Homeowrk #6:

Chapter III, Sec. 3, ex. 2; Sec. 4, ex. 2; Sec. 5, ex. 3, 7;

Chapter IV, Sec. 1, ex. 4, 6, 8; Sec. 2, ex. 5.

Chapter III, Sec 3

Ex. 2: Show that a complex-valued function $h(z)$ on a star-shaped domain D is harmonic if and only if $h(z) = f(z) + \overline{(g(z))}$, where $f(z)$ and $g(z)$ are analytic

Suppose $h(x+iy) = u(x,y) + iv(x,y)$ where u, v are its real and imaginary parts, is a complex-valued function on a star-shaped domain is harmonic

u, v are harmonic.

u has a harmonic conjugate u_c and so does $v(x,y)$, v_c

$F(x+iy) = u(x,y) + iu_c(x,y)$ and $G(x+iy) = v(x,y) + iv_c(x,y)$ are analytic

$u = \frac{F+\overline{F}}{2}$, $v = \frac{G+\overline{G}}{2}$

$h = \frac{1}{2}(F + \overline{F} + i(G + \overline{G})) = \frac{1}{2}(F + iG + \overline{F} + i\overline{G})$

and linear combinations of analytic functions are analytic

$f(x+iy) = \frac{1}{2}(F+iG)$ is analytic and $g(x+iy) = F - iG$ is analytic,

$\overline{g} = \overline{F - iG} = \overline{F} + \overline{-iG} = \overline{F} + i\overline{G}$

$h = f + \overline{g}$, where f, g are analytic

Suppose $h(z) = f(z) + \overline{g(z)}$, where f, g are analytic

so $h(z) = \text{Re } f + \text{Re } g + i(\text{Im } f - \text{Im } g)$

$\text{Re } f, \text{Re } g, \text{Im } f, \text{Im } g$ are all harmonic since f, g are analytic

so $\Delta \text{Re} f = \frac{\partial^2 \text{Re } f}{\partial x^2} + \frac{\partial^2 \text{Re } f}{\partial y^2} = 0$

and $\Delta \text{Re } g = \frac{\partial^2 \text{Re } g}{\partial x^2} + \frac{\partial^2 \text{Re } g}{\partial y^2} = 0$

$\Delta \text{Im } f = \frac{\partial^2 \text{Im } f}{\partial x^2} + \frac{\partial^2 \text{Im } f}{\partial y^2} = 0$

$\Delta \text{Im } g = \frac{\partial^2 \text{Im } g}{\partial x^2} + \frac{\partial^2 \text{Im } g}{\partial y^2} = 0$

so $\frac{\partial^2 (\text{Re } f + \text{Re } g)}{\partial x^2} + \frac{\partial^2 (\text{Re } f + \text{Re } g)}{\partial y^2} = \Delta \text{Re } f + \Delta \text{Re } g = 0 + 0 = 0$

and $\frac{\partial^2(\text{Im } f - \text{Im } g)}{\partial x^2} + \frac{\partial^2(\text{Im } f - \text{Im } g)}{\partial y^2} = \Delta\text{Im } f - \Delta\text{Im } g = 0 - 0 = 0$

Chapter III, Sec 4

Ex. 2: Derive (4.2) from the polar form of the Cauchy-Riemann equations (Exercise II.3.8)

(4.2) $0 = r\int_0^{2\pi}[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta]d\theta = r\int_0^{2\pi}\frac{\partial u}{\partial r}(z_0 + re^{i\theta})$

polar form: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Laplace's equation in polar coordinates

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Proof: Given that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$

$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}(r^2\sin^2\theta) + \frac{\partial^2 u}{\partial y^2}(r^2\cos^2\theta) + \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta)$

and $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$

it easily follows that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

and since we are looking at an open disk for the theorem, the harmonic conjugate v for u exists

and $u + iv$ is an analytic, so we may use the CR equations for the polar form:

$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial v}{\partial r} = -\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

so we have that $\frac{1}{r}\frac{\partial u}{\partial r} = 0$ from the Laplace's equation (in polar coordinates)

so $r^2\frac{1}{r}\frac{\partial u}{\partial r} = 0 * r^2 \implies r\frac{\partial u}{\partial r} = 0$

Now consider the line integral:

$\oint_{|z-z_0|=r} r\frac{\partial u}{\partial r}d\theta$

since $P = r\frac{\partial u}{\partial r} = 0$ and $Q = 0$

we have $\frac{\partial P}{\partial \theta} = \frac{\partial}{\partial \theta}(r\frac{\partial u}{\partial r}) = \frac{\partial}{\partial \theta}(0) = 0$ and so by Green's Theorem:

$0 = r\int_0^{2\pi}\frac{\partial u}{\partial r}d\theta$, which is equation (4.2)

Chapter III, Sec 5

Ex. 3: Use the maximum principle to prove the fundamental theorem of algebra, that any polynomial $p(z)$ of degree $n \geq 1$ has a zero, by applying the maximum principle to $1/p(z)$ on a disk of large radius.

$p(z)$ is a polynomial $\implies$ $p(z)$ is analytic $\implies$ $p(z)$ is harmonic since it's real and imaginary parts are harmonic

so $1/p(z)$ is analytic and harmonic whenever $p(z) \neq 0$

Suppose $p(z)$ doesn't have a zero

This means $1/p(z)$ is defined for all $z \in \mathbb{C}$, and is harmonic on all points in $\mathbb{C}$

and if $p(z) = a_n z^n + ... + a_1 z + a_0$

as $|z| \to \infty$, $|p(z)| \geq |a_n||z^n| \to \infty$

so $|p(z)|$ attains its minimum in $\mathbb{C}$, meaning $|1/p(z)|$ attains its max, M at some point z_0 in $\mathbb{C}$

If we fix a (large) disk around this point, z_0 we have that $1/p(z)$ is a harmonic function on a bounded domain (the disk), and it extends continuously to the boundary (since $1/p(z)$ is defined and continuous for all $z \in \mathbb{C}$ since $p(z)$ doesn't have a zero)

so we have that for all z in this disk, $|1/p(z)| \leq M$, and $|1/p(z_0)| = M$, so $1/p(z)$ is constant on this disk

and if we keep extending the disk, we can keep applying the strict maximum principle, and have that $1/p(z)$ is constant and $|1/p(z)| = M$ for all $z \in \mathbb{C}$

and since $1/p(z)$ is constant for all $z \in \mathbb{C}$, this means $p(z)$ must be constant for all $z \in \mathbb{C}$

So we have that if $p(z)$ doesn't have a zero, it must be a constant function.

$\equiv$ if $p(z)$ isn't a constant function, it has a zero.

Ex. 7: Let $f(z)$ be a bounded analytic function on the open unit disk $\mathbb{D}$. Suppose there are a finite number of points on the boundary such that $f(z)$ extends continuously to the arcs of $\partial\mathbb{D}$ separating the points and satisfies $|f(e^{i\theta})| \leq M$ there. Show that $|f(z)| \leq M$ on $\mathbb{D}$.

Hint: In the case that there is only one exceptional point $z = 1$, consider the function $(1 - z)^\epsilon f(z)$

so $|f(z)| \leq C$ since $f(z)$ is bounded

Suppose there exists a δ, satisfying $0 < \delta \leq C - M$, s.t. at some point $z_0 \in \mathbb{D}$ $|f(z_0)| = M + \delta$

Define $g(z) = (z_1 - z)^\epsilon (z_2 - z)^\epsilon \cdots (z_n - z)^\epsilon f(z)$

$g(z)$ is continuous for all points of z where $f(z)$ is continuous

and as $z \to z_i$ for $i = 1, .., n$, $g(z) \to 0$, and $g(z_i) = 0$ so g is continuous on $\mathbb{D} \cup \partial\mathbb{D}$

and $|(z_1 - z)^\epsilon (z_2 - z)^\epsilon \cdots (z_n - z)^\epsilon|^\epsilon M \leq |(z_1 - z)^\epsilon (z_2 - z)^\epsilon \cdots (z_n - z)^\epsilon f(z)|$
$\leq |(z_1 - z)^\epsilon (z_2 - z)^\epsilon \cdots (z_n - z)^\epsilon|(M + \delta)$

we can choose an value of $\epsilon > 0$ s.t.

$1 \leq |(z_1 - z)^\epsilon (z_2 - z)^\epsilon \cdots (z_n - z)^\epsilon| \leq \frac{M+\delta/2}{M}$

and $g(z)$ is analytic, (at the points where $f(z)$ is not continuous, the derivative of $g(z)$ is ∞ which is in the extended complex plane) and therefore harmonic, on $\mathbb{D} \cup \partial\mathbb{D}$ so we may apply the Maximum Principle

on the boundary, $|f(z)| \leq M$ where it is defined,

so on the boundary, $|g(z)| \leq (M + \delta/2)$ so for all $z \in \mathbb{D}$, $|g(z)| \leq M + \delta/2$

and at z_0, $|g(z_0)| \geq (M + \delta)$

but since $z_0 \in \mathbb{D}$ we must also have $|g(z_0)| \leq M + \delta/2 < M + \delta$

a contradiction.

Chapter IV, Sec 1

Ex. 4: Show that if D is a bounded domain with smooth boundary, then $\int_{\partial D} \bar{z} dz = 2i \text{Area}(D)$

So $\int_{\partial D} \bar{z} dz = \int_{\partial D} x - iy dx + \int_{\partial D} y + ix dy$

$P(x, y) = x - iy, Q(x, y) = y + ix$ and so

Green's Theorem: $= \int\int_D i - (-i) dxdy = 2i \int\int_D dA = 2i(\text{Area}(D))$

the last equality comes from multivariable calculus

Ex. 6: Show that $|\oint_{|z|=R} \frac{\text{Log } z}{z^2} dz| \leq 2\sqrt{2}\pi \frac{\log R}{R}, R > e^\pi$

$|\oint_{|z|=R} \frac{\text{Log } z}{z^2} dz| \leq \oint_{|z|=R} |\frac{\text{Log } z}{z^2}||dz|$

$z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$

$|\text{Log } z| = |\log R + i\theta| = \sqrt{(\log R)^2 + \theta^2} \sqrt{(\log R)^2 + \pi^2} \leq \sqrt{2(\log R)^2} = \sqrt{2} \log R$

since $R > e^\pi$, $\log R > \pi$

$|z^2| < R^2$

$|dz| = | - R \sin \theta + iR \cos \theta| = R \, d\theta$

$|\oint_{|z|=R} \frac{\text{Log } z}{z^2} dz| \leq \int_{-\pi}^{\pi} \frac{\sqrt{2} \log R}{R^2} * R \, d\theta = 2\pi \frac{\sqrt{2} \log R}{R}$

Ex. 8: Suppose the continuous function $f(e^{i\theta})$ on the unit circle satisfies $|f(e^{i\theta})| \leq M$ and $|\int_{|z|=1} f(z) dz| = 2\pi M$. Show that $f(z) = c\bar{z}$ for some constant c with modulus $|c| = M$

$\int_{|z|=1} |f(z)||dz| = \int_0^{2\pi} |f(e^{i\theta})| d\theta$

if we parameterize $z = e^{i\theta}$

and $|dz| = |ie^{i\theta}| d\theta = d\theta$

From the hypothesis and using the triangle inequality:

$2\pi M = |\int_{|z|=1} f(z) dz| \leq \int_0^{2\pi} |f(e^{i\theta})| d\theta \leq 2\pi M$

so $|f(e^{i\theta})| = M$

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taking $g(e^{i\theta}) = ie^{i\theta} f(e^{i\theta})$

$|g(e^{i\theta})| = |i||e^{i\theta}||f(e^{i\theta})| = M$

we may multiply g by a unimodular constant λ s.t.

$\lambda g(e^{i\theta}) = M,$

so we have $\lambda ie^{i\theta} f(e^{i\theta}) = M$

$f(e^{i\theta}) = -i\lambda^{-1} Me^{-i\theta}$

and $| - i\lambda^{-1} M| = M$, since $| - i| = 1, |\lambda^{-1}| = 1/|\lambda| = 1$

Chapter IV, Sec 2

Ex. 5: Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\,dz = 0$ for every closed path γ in D

Let $f(z)$ have a primitive, $F(z)$ in D

since $f(z)$ is analytic, it is continuous on D,

so $\int_A^B f(z)\,dz = F(B) - F(A)$

for any path in D from A to B

for a closed path γ from A to B, $A = B$, so $\int_\gamma f(z)\,dz = \int_A^A f(z)\,dz = F(A) - F(A) = 0$

since γ was an arbitrary closed path, we have for any closed path $\int_\gamma f(z)\,dz = 0$

Now suppose for every closed path γ in D, $\int_\gamma f(z)\,dz = 0$

so $\int_\gamma f(z)(dx + idy) = \int_\gamma f(z)\,dx + if(z)\,dy$

so since it is independent of path, the differential $f(z)\,dx + if(z)\,dy$ is exact

so there exists a functin $F(z)$ s.t. $dF = f(z)dx + if(z)dy$

$\frac{\partial F}{\partial x} = f(z)$ and $\frac{\partial F}{\partial y} = if(z)$

for any function $g = u + iv$, where u, v are the real, imaginary parts of g

$\frac{\partial}{\partial x}\mathrm{Re}\; g = \frac{\partial u}{\partial x}$, $\mathrm{Re}\;\frac{\partial g}{\partial x} = \mathrm{Re}\;(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}) = \frac{\partial u}{\partial x}$

and similarly for $\frac{\partial}{\partial y}$ and/or $\mathrm{Im}\; g$. Finding the partial derivative of a real/imaginary part of a function is the same as finding the real/imaginary part of the partial derivative

so $\frac{\partial}{\partial x}\mathrm{Re}\; F = \mathrm{Re}\; f = \mathrm{Im}\; if = \mathrm{Im}\;\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\mathrm{Im}\; F$

and $\frac{\partial}{\partial y}\mathrm{Re}\; F = \mathrm{Re}\;\frac{\partial F}{\partial y} = \mathrm{Re}\; if = -\mathrm{Im}\; f = -\mathrm{Im}\;\frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}\mathrm{Im}\; F$

so F is analytic since it satisfies the CR equations

and $F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y} = f(z)$

so f has a primitive in D, F