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Math 123

Homework #10

Chapter 10, 1 and 4

1. For each of the following systems, identify all points that lie in either an ω- or an α- limit set

(a) $r' = r - r^2, \theta' = 1$

We have no equilibria except at origin

$r' = 0$ when $r = r^2$,

so $r = 1$

The unit circle is a periodic solution

when $r > 1$:

$r^2 > r \implies r - r^2 < 0$, so $r' < 0$

which means solutions that begin outside the unit disk radially decrease until they get to $r = 1$,

when $r < 1$:

$r^2 < r \implies r - r^2 > 0$, so $r' > 0$

so solutions inside the unit disk radially increase until $r = 1$

so we have that all solutions tend towards the unit circle (periodic solution)

$\omega-$limit of all solutions is the periodic solution at the unit circle

if solutions start on the unit circle, their α-limits are also the periodic solution on the unit circle

The α limit for solutions that start outside of the unit disk: ∞ since when moving backwards, $r' > 0$

the α limit for solutions that start inside of the unit disk: origin, since $r' < 0$ when $r < 1 \implies r \to 0$

(b) $r' = r^3 - 3r^2 + 2r, \theta' = 1$

again, no equilibria except at origin

$r' = 0$ when $r^3 - 3r^2 + 2r = 0$

so $r(r^2 - 3r + 2) = 0 \equiv r(r-2)(r-1) = 0$

so at $r = 1, r = 2$ we have periodic solutions

for ω-limits:

if $0 < r < 1$:

$(r-2)(r-1) > 0$, so $r' > 0$

so solutions that begin inside the unit circle tend towards the unit circle if $1 < r < 2$

$r - 2 < 0$, but $r - 1 > 0$

so $r' < 0$, so all solutions that begin inside the circle of radius 2 tend towards the unit circle

if $r > 2$, $r' > 0$, so solutions tend towards infinity

When we go backwards in time:

solutions that begin inside the unit circle tend towards origin

solutions that begin outside the unit circle but inside the circle with radius 2 tend towards the circle of radius 2

solutions that begin outside the circle of radius 2 tend towards the circle of radius 2

(c) $r' = \sin r$, $\theta = -1$

$r' = 0$ whenever $r = \pi + 2\pi k$ or $r = 2\pi k$ for some nonnegative integer k

the only equilibrium solution is at origin

we have periodic solutions at $\pi + 2\pi k$ and $2\pi k$ for any nonnegative integer k

if $2\pi k < r < \pi + 2\pi k$,

$r' > 0$

so solutions that start in this annulus all tend towards the circle with radius $\pi + 2\pi k$

if $\pi + 2\pi k < r < 2\pi(k + 1)$

$r' < 0$

so solutions that begin in this annulus tend towards $\pi + 2\pi k$

ω limits are the circles of radius $\pi + 2\pi k$

going backwards in time

all solutions tend to some circle with radius $2\pi k$

α limits are the circles of radius $2\pi k$ and origin

(d) $x' = \sin x \sin y$, $y' = -\cos x \cos y$

$x' = 0$:

$x = 0$ mod 2π or $y = 0$ mod 2π or $x = \pi$ mod 2π of $y = \pi$ mod 2π

likewise, $y' = 0$ whenever x or y equal $\pi/2$ mod 2π or $3\pi/2$ mod 2π

we have equilibrium at

$(0, \pi/2)$, $(0, 3\pi/2)$, $(\pi, \pi/2)$, $(\pi, 3\pi/2)$

$(\pi/2, 0)$, $(\pi/2, \pi)$, $(3\pi/2, 0)$, $(3\pi/2, \pi)$

(adding $2\pi k$ for any $k \in \mathbb{Z}$)

equilibrium points are their own ω, α sets

$x' < 0$

$0 < x < \pi$ AND $\pi < y < 2\pi$

Or

$\pi < x < 2\pi$ AND $0 < y < \pi$

$x' > 0$

$0 < x < \pi$ AND $\pi < y < 2\pi$

Or

$\pi < x < 2\pi$ AND $\pi < y < 2\pi$

(mod 2π)

$y' < 0$

$-\pi/2 < x < \pi/2$ AND $-\pi/2 < y < \pi/2$

OR

$\pi/2 < x < 3\pi/2$ AND $\pi/2 < y < 3\pi/2$

$y' > 0$

$-\pi/2 < x < \pi/2$ AND $\pi/2 < y < 3\pi/2$

OR

$\pi/2 < x < 3\pi/2$ AND $-\pi/2 < y < \pi/2$

(mod 2π)

$(0, \pi/2)$ is a source (α)

$(0, 3\pi/2)$ is a sink (ω)

$(\pi, \pi/2)$ is a sink (ω)

$(\pi, 3\pi/2)$ is a source (α)

$(\pi/2, 0)$ is a saddle (ω and α)

solutions tend towards it *from* the northwest or the southeast

$(\pi/2, \pi)$ is a saddle (ω and α)

solutions tend towards it from the northeast or the southwest

$(3\pi/2, 0)$ is a saddle (ω and α)

solutions tend towards it from the northeast or the southwest

$(3\pi/2, \pi)$ is a saddle (ω and α)

solutions tend towards it *from* the northwest or the southeast