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Math 123

# Homework #2

## Ex 2. Find the general solution of each of the following linear systems

**(a)** $X' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} X$

Finding the eigenvalues and eigenvectors:

$(1-\lambda)(3-\lambda) - 2 \times 0 = (1-\lambda)(3-\lambda)$, meaning roots at $\lambda = 1, 3$

When $\lambda = 1$, eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, One Solution: $e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

When $\lambda = 3$, eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , One solution: $e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution: $X(t) = \alpha e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**(b)** $X' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} X$

Finding eigenvalues: $(1-\lambda)(6-\lambda) - 6 = \lambda^2 - 7\lambda$, roots at $\lambda = 0, 7$

When $\lambda = 0$, eigenvector: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

When $\lambda = 7$, eigenvector: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

General solution: $X(t) = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \beta e^{7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

**(c)** $X' = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} X$

Finding eigenvalues: $(1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, roots at $\lambda = -1, 2$

When $\lambda = -1$, eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

When $\lambda = 2$, eigenvector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

General solution: $X(t) = \alpha e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

**(d)** $X' = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix} X$

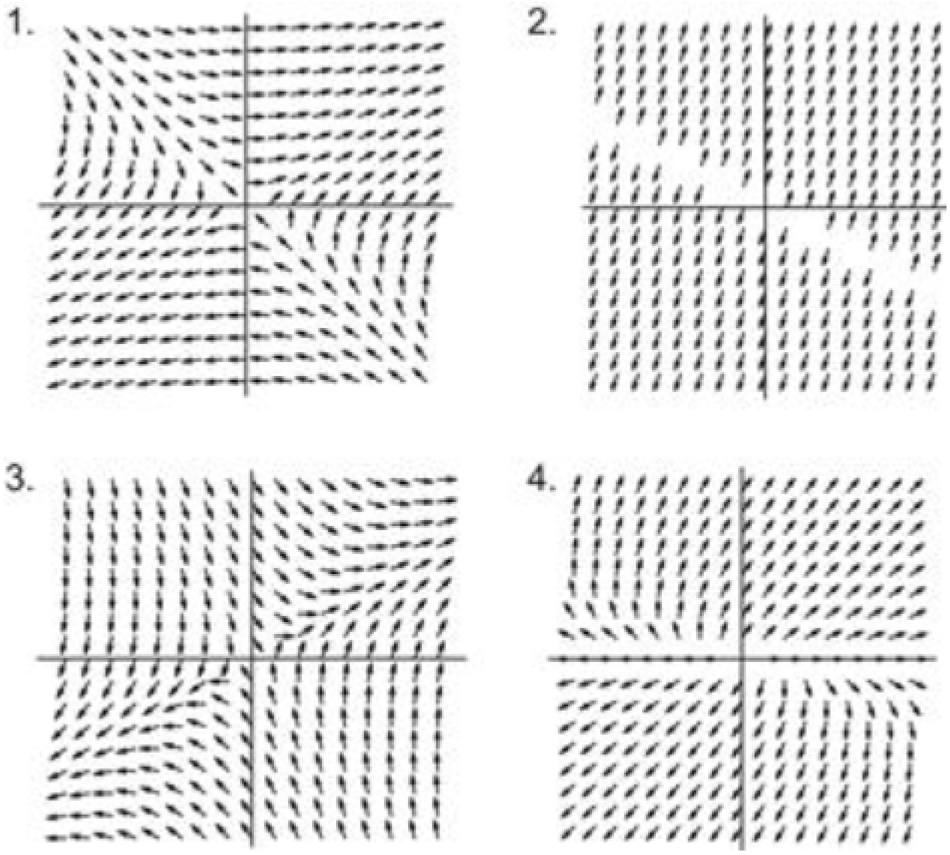
Finding eigenvalues: $(1-\lambda)(-3-\lambda)-6=\lambda^2+2\lambda-9$, roots at $\lambda=-1\pm\sqrt{10}$, (found using quadratic equation)

When $\lambda=-1+\sqrt{10}$ , eigenvector: $\begin{pmatrix} 1 \\ -1+\sqrt{10}/2 \end{pmatrix}$

When $\lambda=-1-\sqrt{10}$, eigenvector: $\begin{pmatrix} 1 \\ -1-1\sqrt{10}/2 \end{pmatrix}$

General solution: $X(t)=\alpha e^{(-1+\sqrt{10})t}\begin{pmatrix} 1 \\ -1+\sqrt{10}/2 \end{pmatrix}+\beta e^{(-1-\sqrt{10})t}\begin{pmatrix} 1 \\ -1-1\sqrt{10}/2 \end{pmatrix}$

**Ex 3. In Figure 2.2 you see four direction fields. Match each of these direction fields with one of the systems in the previous exercise.**




For (a) , when we look at $(1,0)$, we get back the tangent vector $(1,0)$, and when we look at $(1,1)$, we get $(3,3)$ as the tangent vector

For (b), on the line $y=(-1/2)x$ , we should have no vectors

For (c) at $(-1,1)$, we get the tangent vector $(1,-1)$, and at $(1,-1)$, the tangent vector $-1,1$

and at $(2,1)$, the tangent vector $(4,2)$, also along the $y-$axis, when $y>0$, we should get vectors that go towards the right horizontally, and when $y<0$, vectors that go to the left horizontally.

For (d) along the line $y=x$, we should have vectors of the form $(3x,0)$

So:

$(a) \to 4$

$(b) \to 2$

$(c) \to 1$

$(d) \to 3$

**Ex 4. Find the general solution of the system $X' = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, where $bc > 0$.**

Assuming $a, b, c$ are real:

Finding eigenvalues: $(\lambda - a)^2 - bc = \lambda^2 - 2a + (a^2 - bc)$

Using quadratic equation: we get roots at $a \pm \sqrt{bc}$, which is real, since $bc > 0$

For $\lambda = a + \sqrt{bc}$, the eigenvector is $\begin{pmatrix} \sqrt{|b|} \\ \sqrt{|c|} \end{pmatrix}$

For $\lambda = a - \sqrt{bc}$, the eigenvector is $\begin{pmatrix} \sqrt{|b|} \\ -\sqrt{|c|} \end{pmatrix}$

General solution: $X(t) = \alpha e^{(a+\sqrt{bc})t} \begin{pmatrix} \sqrt{|b|} \\ \sqrt{|c|} \end{pmatrix} + \beta e^{(a-\sqrt{bc})t} \begin{pmatrix} \sqrt{|b|} \\ -\sqrt{|c|} \end{pmatrix}$

**Ex 6. For the harmonic oscillator system $x'' + bx' + kx = 0$, find all values of $b$ and $k$ for which this system has real, distinct eigenvalues. Find the general solution of this system in these cases. Find the solution of the system that satisfies the initial condition $(0, 1)$. Describe the motion of the mass in this particular case.**

$x' = y$

$y' = -kx - by$

We now have: $X' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} X$

Finding the eigenvalues: $(-\lambda)(-b - \lambda) + k = \lambda^2 + b\lambda + k$

We get: $\frac{-b \pm \sqrt{b^2 - 4k}}{2}$

setting $\lambda_1 = (-b + \sqrt{b^2 - 4k})/2, \lambda_2 = (-b - \sqrt{b^2 - 4k})/2$

General solution: $X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

To find the solution that satisfies the initial condition $(0, 1)$, we must solve

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we row reduce the augmented matrix:

$$\begin{pmatrix} 1 & 0 & \lambda_2/(\lambda_2 - \lambda_1) \\ 0 & 1 & -\lambda_1/(\lambda_2 - \lambda_1) \end{pmatrix}$$

We end up with $\alpha = \lambda_2/(\lambda_2 - \lambda_1)$ , $\beta = -\lambda_1/(\lambda_2 - \lambda_1)$

$X(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

1. If $k > 0$, we need $b^2 - 4k > 0$ to have real, distinct values, so $b > 2\sqrt{k}$ or $b < -2\sqrt{k}$
   1. If $b > 0$

      $\lambda_2 < 0$, since $(-b - \sqrt{b^2 - 4k})/2 < 0$

      $\lambda_1 < 0$, since $b^2 - 4k < b^2$, so $\sqrt{b^2 - 4k} < b$

      for all $t > 0$, and because $\lambda_2 < \lambda_1$

      $e^{\lambda_1 t} > e^{\lambda_2} t > 0$

      $\lambda_2/(\lambda_2 - \lambda_1) > \lambda_1/(\lambda_2 - \lambda_1) > 0$

      so:

      $x(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} > 0$ for all $t$, and as $t$ increases, both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ go towards 0, so the mass moves towards the origin.
   2. If $b < 0$

      $\lambda_1 > 0$, $\lambda_2 > 0$

      and $\lambda_2 < \lambda_1$, so $\lambda_2 - \lambda_1 < 0$

      And we have $x(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t}$

      and when $e^{(\lambda_1 - \lambda_2)t} > \frac{\lambda_1}{\lambda_2}$, which will happen eventually, since $e^{(\lambda_1 - \lambda_2)t} \to \infty$ as $t \to \infty$

      we have that $x(t) < 0$, and in fact, $x(t) \to -\infty$ as $t \to \infty$

      which means the mass moves back to origin and the spring keeps compressing.
2. If $k = 0$, we have eigenvalues: $\lambda_1 = 0, \lambda_2 = -b$ with eigenvectors: $(1, 0)$ and $(1, -b)$ respectively. For distinct, real, we must have $b \in \mathbb{R} \setminus 0$.

   The solution that satisfies $X(0) = (1, 0)$ is $\alpha = 1, \beta = 0$ , since which means that $X(t) = (1, 0)$ for all $t$, so the mass isn't moving: this makes sense as the velocity is 0, and the spring constant is 0, so there's no restorative force to bring the mass back to its natural resting place, $x = 0$
3. If $k < 0$, $b$ must be real,
   1. regardless of the sign of $b$, since $b^2 - 4k > b^2$, $\sqrt{b^2 - 4k} > |b|$

      $\lambda_1 > 0$, $\lambda_2 < 0$

      $\lambda_2/(\lambda_2 - \lambda_1) > 0$, and $\lambda_1/(\lambda_2 - \lambda_1) < 0$

      so as $t \to +\infty$ : $\frac{\lambda_2}{(\lambda_2 - \lambda_1)} e^{\lambda_1 t} \to +\infty$ and $\frac{\lambda_1}{(\lambda_2 - \lambda_1)} e^{\lambda t} \to 0$

      so $x(t) \to +\infty$ , meaning the mass keeps moving away from origin

## Ex 7. Consider the $2 \times 2$ matrix $A = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$ Find the value of $a_0$ of the parameter of $a$ for which $A$ has repeated real eigenvalues. What happens to the eigenvectors of this matrix as $a$ approaches $a_0$?

the characteristic polynomial: $(\lambda - a)(\lambda - 1)$

So when $a = 1$, we have repeated eigenvalues, and the corresponding eigenvector will be:

$(1, 0)$

When $a \neq 1$, we have two eigenvectors, so as $a \to 1$, we go from 2 eigenvectors:

$(1, 0)$ corresponding to the eigenvalue $a$, and

$(1, 1 - a)$ corresponding to the eigenvalue 1.

and we notice as $a \to 1$ the eigenvector corresponding to 1 approaches the eigenvector corresponding to $a$

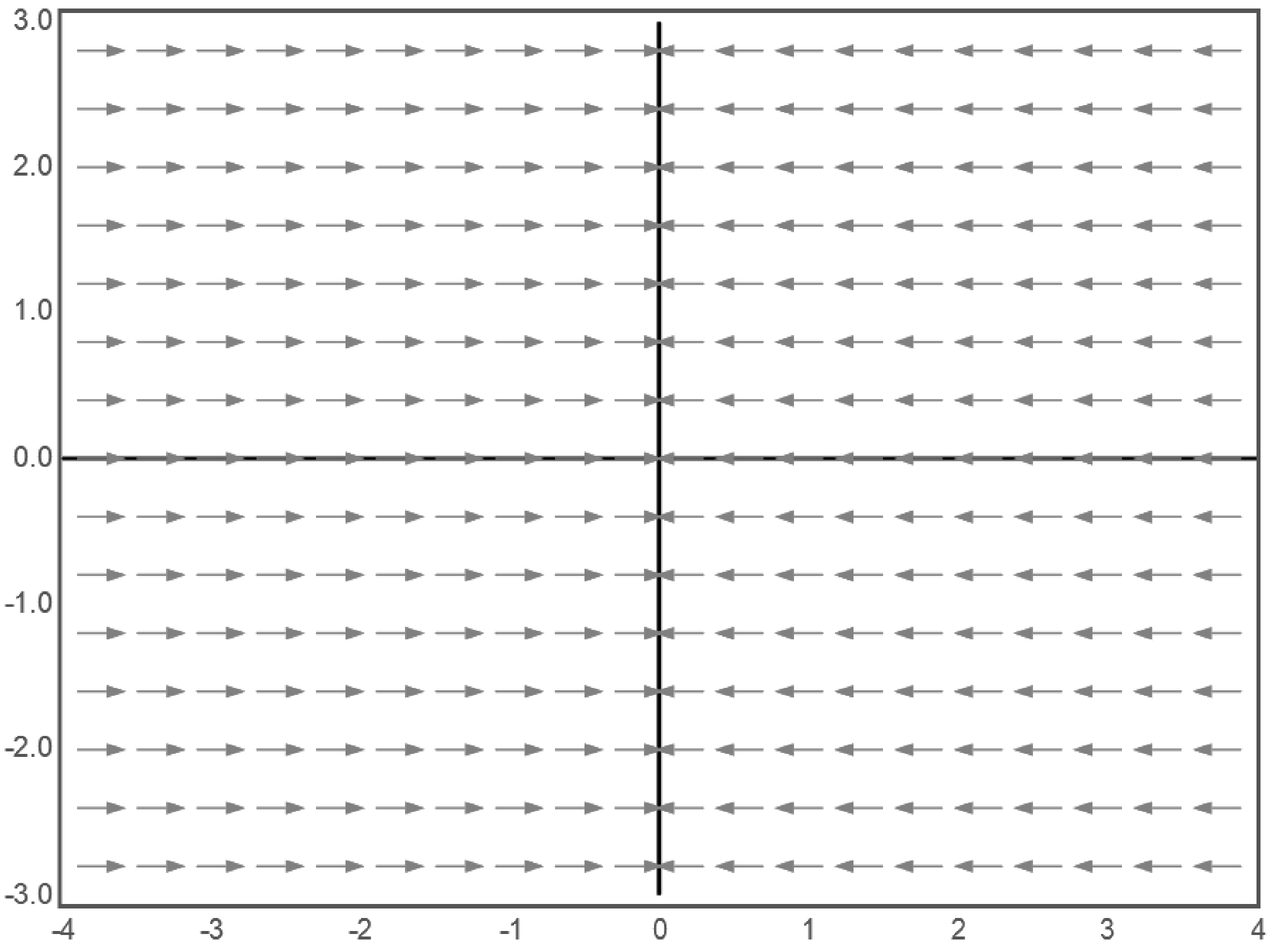
### Ex 9. Give an example of a linear system for which $(e^{-t}, \alpha)$ is a solution for every constant $\alpha$. Sketch the direction field for this system. What is the general solution of this system?

$Y(t) = e^{-t}\begin{pmatrix}1\\0\end{pmatrix} + \alpha\begin{pmatrix}0\\1\end{pmatrix}$ and $Y'(t) = \begin{pmatrix}-e^{-t}\\0\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}e^{-t}\\\alpha\end{pmatrix} = \begin{pmatrix}-e^{-t}\\0\end{pmatrix}$, for all $\alpha, t$

The general solution is $X(t) = \alpha\begin{pmatrix}0\\1\end{pmatrix} + \beta e^{-t}\begin{pmatrix}1\\0\end{pmatrix}$

$x' = -x$
$y' = 0$



## Ex 11 Prove that two vectors $V = (v_1, v_2)$ and $W = (w_1, w_2)$ are linearly independent $\iff \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0$

We can prove this by proving: $V$ and $W$ are linearly dependent $\iff \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = 0$

Let $V = \lambda W$, which means $V$ and $W$ are linearly dependent $\iff v_1 = \lambda w_1$ and $v_2 = \lambda w_2 \iff$ $\lambda = v_1/w_1 = v_2/w_2 \iff v_1 w_2 = v_2 w_1 \iff v_1 w_2 - v_2 w_1 = 0 \iff \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = 0$

## Ex 12 Prove that if $\lambda, \mu$ are real eigenvalues of a $2 \times 2$ matrix, then any nonzero column of the matrix $A - \lambda I$ is an eigenvector for $\mu$

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , since $\lambda, \mu$ are eigenvalues,

The characteristic polynomial: $x^2 - (a+d)x - (ad - bc)$

We let $\lambda = \frac{(a+d)+\sqrt{(a+d)^2-4(ad-bc)}}{2}$

and $\mu = \frac{(a+d)-\sqrt{(a+d)^2-4(ad-bc)}}{2}$ without loss of generality.

Let's take the column vector from $A - \lambda I$, $\begin{pmatrix} a-\lambda \\ c \end{pmatrix}$ and multiply $A - \mu I$ by this vector.

We get $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a-\mu \\ c \end{pmatrix}$ = $\begin{pmatrix} (a-\lambda)(a-\mu)+bc \\ c(a+d-(\mu+\lambda)) \end{pmatrix}$

If we prove this is equal to the 0 vector, then $\begin{pmatrix} a-\lambda \\ c \end{pmatrix}$is an eigenvector, provided $\mu \neq a$ and $c \neq 0$

$(a-\lambda)(a-\mu) = a^2 - a(\lambda+\mu) - \lambda\mu$

$(\lambda+\mu)$ =$(a+d)$, so the second entry of the vector is $= c(a+d-(a+d)) = 0$

and $\lambda\mu$ =$((a+d)^2 - (a+d)^2 + 4(ad-bc))/4 = ad - bc$

So $(a-\lambda)(a-\mu) = a^2 - a^2 - ad + ad - bc = -bc$,

so: $\begin{pmatrix} (a-\lambda)(a-\mu)+bc \\ c(a-\mu+d-\lambda) \end{pmatrix}$ = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So now we prove the other column vector is an eigenvector in the same way:

$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} b \\ d-\mu \end{pmatrix}$ = $\begin{pmatrix} b(a+d-(\lambda+\mu)) \\ bc+(d-\lambda)(d-\mu) \end{pmatrix}$

From earlier, we know $b(a+d-\lambda-\mu) = 0$

and replacing $(d-\lambda)(d-\mu) = d^2 - ad - d^2 + ad - bc = -bc$,

so: $\begin{pmatrix} (a-\lambda)(a-\mu)+bc \\ c(a-\mu+d-\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

As long as $\mu \neq d$ and $b \neq 0$, the vector $\begin{pmatrix} b \\ d-\mu \end{pmatrix}$ is an eigenvector

## Ex 14 Prove that the eigenvectors of a $2 \times 2$ matrix corresponding to distinct real eigenvalues are always linearly independent.

Let $A$ be a $2 \times 2$ matrix with $2$ eigenvalues: $\lambda, \mu$ where $\lambda \neq \mu$ let $v_\lambda$ and $v_\mu$ be the (nonzero) eigenvectors corresponding to $\lambda$ and $\mu$ respectively.

Suppose $v_\lambda$ and $v_\mu$ are linearly dependent $\implies v_\lambda = cv_\mu$ for some constant $c \in \mathbb{R} \setminus 0$, since both vectors must be nonzero.

$Av_\lambda = \lambda v_\lambda = \lambda(cv_\mu)$

$A(cv_\mu) = \mu(cv_\mu)$

equating the two: $\lambda(cv_\mu) = \mu(cv_\mu)$

$\lambda(cv_\mu) - \mu(cv_\mu) = 0 \equiv (\lambda - \mu)(cv_\mu) = 0$

which means, since $cv_\mu$ is nonzero, $\lambda - \mu = 0$, a contradiction. So these two eigenvectors must be linearly independent.