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Math 185

Homework #10

Chapter VI, Sec. 1, ex. 1, 3, 5

1. Find all possible Laurent expansions centered at 0 of the following functions:

(a) $\frac{1}{z^2-z}$

$\frac{1}{z^2-z} = \frac{1}{z(z-1)}$

$0 < |z| < 1: \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$, $f_0(z) = \frac{1}{z-1}$ is analytic for $|z| < 1$, and $f_1(z) = -\frac{1}{z}$ is analytic for $|z| > 0$

$f_0(z) = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k$

$f_1(z) = -\frac{1}{z}$

$f(z) = -\sum_{k=-1}^\infin z^k$

$1 < |z| < \infty$: The function is analytic at ∞ and vanishes there, so its Laurent decomposition with respect to this exterior domain is $f(z) = f_1(z)$, $f_0(z) = 0$

$f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\sum_{k=0}^{\infty}(\frac{1}{z})^k = \sum_{k=2}^{\infty}(\frac{1}{z})^k$

$= \sum_{k=-\infty}^{-2} z^k$

(b) $\frac{z-1}{z+1}$

singularity at $z = -1$

$|z| < 1$: $f_0(z) + 0 = \frac{z-1}{z+1} = \frac{z+1-2}{z+1} = 1 - \frac{2}{z+1} = 1 - 2\frac{1}{1-(-z)}$

$= 1 - 2\sum_{k=0}^{\infty}(-z)^k = 1 - 2\sum_{k=0}^{\infty}(-1)^k z^k = 1 - 2 - 2\sum_{k=1}^{\infty}(-1)^k z^k = -1 - 2\sum_{k=1}^{\infty}(-1)^k z^k$

$1 < |z| < \infty$, $f(z)$ is analytic, but doesn't disappear at ∞

so we obtain $f(z) = 1 - \frac{2}{z+1}$ again with $f_0(z) = 1$, $f_1(z) = -\frac{2}{z+1}$

$f_1(z) = -2\frac{1}{z+1} = -\frac{2}{z}\frac{1}{1+(1/z)} = -\frac{2}{z}\sum_{k=0}^{\infty}(-1)^k(\frac{1}{z})^k$

$2\sum_{k=1}^\infin (-1)^{k}(\frac{1}{z})^k$

$f(z) = 1 + 2\sum_{k=1}^{\infty}(-1)^k(\frac{1}{z})^k$

(c) $\frac{1}{(z^2-1)(z^2-4)}$

$f(z) = \frac{A}{z^2-4} + \frac{B}{z^2-1}$

$A + B = 0$

$-A - 4B = 1$, $A = -4B - 1$

$-4B - 1 + B = 0 \equiv B = -1/3$

$A = 1/3$

$f(z) = \frac{1}{3}\frac{1}{z^2-4} - \frac{1}{3}\frac{1}{z^2-1}$

$|z| < 1$:

$f(z)$ is analytic for all $|z| < 1$

so $f(z) = -\frac{1}{12}\frac{1}{1-\frac{z^2}{4}} + \frac{1}{3}\frac{1}{1-z^2}$

$= \frac{-1}{12}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k} + \frac{1}{3}\sum_{k=0}^{\infty} z^{2k} = \frac{-1}{12}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k} + \frac{1}{3*4}\sum_{k=0}^{\infty} 4z^{2k}$

$\frac{1}{12}\sum_{k=0}^{\infty}(4-4^{-k})z^{2k}$

$1 < |z| < 2$

$f_0(z) = \frac{1}{3}\frac{1}{z^2-4}$

$f_1(z) = -\frac{1}{3}\frac{1}{z^2-1}$

$f_0(z) = -\frac{1}{12}\frac{1}{1-\frac{z^2}{4}} = -\frac{1}{12}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k}$

$f_1(z) = -\frac{1}{3}\frac{1}{z^2-1} = -\frac{1}{3z^2}\frac{1}{1-(1/z^2)} = \frac{-1}{3z^2}\sum_{k=0}^{\infty}\frac{1}{z^{2k}} = \frac{-1}{3}\sum_{k=1}^{\infty}\frac{1}{z^{2k}}$

$= \frac{-1}{3}\sum_{k=-\infty}^{-1} z^{2k}$

$f(z) = \sum_{k=-\infty}^{\infty} a_k z^{2k}$

where $a_k = -1/3$ for $k < 0$ and $a_k = \frac{-1}{12*4^k}$ for $k \geq 0$

$2 < |z| < \infty$

$f(z) = \frac{1}{3z^2}\frac{1}{1-\frac{4}{z^2}} - \frac{1}{3z^2}\frac{1}{1-\frac{1}{z^2}}$

$f(z) = \frac{1}{3z^2}\sum_{k=0}^{\infty}\frac{4^k}{z^{2k}} - \frac{1}{3z^2}\sum_{k=0}^{\infty}\frac{1}{z^{2k}}$

$f(z) = \frac{1}{3}\sum_{k=-\infty}^{-1}(4^{-k-1}-1)z^{2k}$

3. Recall the power series for the Bessel function $J_n(z), n \geq 0$, given in Exercise V.4.11, and define $J_{-n} = (-1)^n J_n(z)$. For fixed $w \in C$, establish the Laurent series expansion $\exp[\frac{w}{2}(z-1/z)] = \sum_{n=-\infty}^{\infty} J_n(w)z^n, 0 < |z| < \infty$. From the coefficient formula (1.4) deduce that $J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - z\sin\theta)}d\theta, z \in \mathbb{C}$

$J_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{n+2k}}{k!(n+k)!(2^{n+2k})}$

$J_{-n} = \sum_{k=0}^{\infty}\frac{(-1)^{n+k} z^{n+2k}}{k!(n+k)!(2^{n+2k})}$

$a_n = \frac{1}{2\pi i}\oint_{|z-z_0|=r}\frac{e^{\frac{w}{2}(z-1/z)}}{(z-z_0)^{n+1}}dz$

$z = z_0 + re^{-i\theta}$

The negative sign is for the circle going clockwise, since $\exp[\frac{w}{2}(z-1/z)]$ is analytic on the punctured complex plane, so

$dz = ire^{i\theta} d\theta$

$a_n = \large \frac{1}{2\pi }\int_0^{2\pi}\frac{e^{\frac{w}{2}(z_0 + re^{i\theta} - \frac{1}{z_0 + re^{i\theta}})}}{r^{n}e^{in\theta}} d\theta$

$a_n = \large {\frac{1}{2\pi i}\oint_{|z| = 1}\frac{e^{\frac{w}{2}(z-1/z)}}{(z)^{n+1}}} \normalsize dz $

$z = e^{-i\theta}$, $dz = -ie^{-i\theta} d\theta$

$a_n = -\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{w}{2}(e^{-i\theta} - \frac{1}{e^{-i\theta}})}}{e^{-in\theta}}$

and using $\cos(-\theta) + i\sin(-\theta) - \frac{1}{\cos(-\theta)+i\sin(-\theta)} =$

$\cos(\theta) - i\sin(\theta) - \frac{1}{\cos(\theta) - i\sin(\theta)} =$

$\frac{\cos^2\theta - 2i\cos\theta\sin\theta - \sin^2\theta -1}{\cos\theta - i\sin\theta}$

$= \frac{-2i\cos\theta\sin\theta - 2\sin^2\theta}{\cos\theta - i \sin\theta}$

$ = \frac{-2i\sin\theta(\cos\theta -i\sin\theta)}{\cos\theta - i\sin\theta}$

$= -2i\sin\theta$

we obtain $a_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-iw\sin\theta}}{e^{-in\theta}} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - w\sin\theta)} d\theta$

and since $J_n(w) = a_n$, we have shown $J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - z\sin\theta)} d\theta, z \in \mathbb{C}$

replacing w with z

5. Suppose $f(z)$ is analytic on the punctured plane $D = \mathbb{C}\backslash${0}. Show that there is a constant c such that $f(z) - c/z$ has a primitive in D. Give a formula for c in terms of an integral of $f(z)$

we need $f(z) - c/z$ to be analytic on D

on D, we can express $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$

and from (1.4)

$a_{-1} = \frac{1}{2\pi i}\oint_{|z-z_0|r} \frac{f(z)}{z^0} dz$, where $r > 0$

so $f(z) - \frac{a_{-1}}{z} = \sum_{k\neq -1, -\infty<k<\infty} a_k z^k$

so let $c = a_{-1}$

a function is analytic on D if and only if $f(z)dz$ is closed

which means on any closed path containing a region in D,

$\int_\gamma f(z) - c/z \ \ dz = 0$

and we've learned before that any closed path in an annulus can be continuously deformed to a circular path C in the annulus, so

$\int_{\gamma} f(z) - c/z dz = \int_{C} \sum_{k\neq -1}a_kz^k$

and since the power series converges uniformly on any circle strictly smaller than ∞

$\int_C \sum_{k\neq -1} a_k z^k = \sum_{k\neq -1} a_k \int_C z^k$

and from earlier, we know that $\int_{|z-z_0|=r} z^m = 0$ whenever $m \neq -1$ so the above is equal to 0

so $f(z) - c/z$ is analytic on a star-shaped domain D, so it has a primitive in D