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Math 185

Homework #13

Chapter VIII, Sec. 1, ex. 1; Sec. 2, ex. 2, 8; Sec. 3, ex. 2; Sec. 4, ex. 4, 6

Chapter VIII, Sec. 1, ex. 1

Show that $z^4 + 2z^2 - z + 1$ has exactly one root in each quadrant:

This is a polynomial with real coefficients and no real zeros,

$z^4 + 2z > 0$ for all $z \in \mathbb{R}$

and for all $z < 0$, $p(z) > 0$

for all $z \geq 1$, $z^2 \geq z$, so $2z^2 - z > 0$, meaning $p(z) > 0$ for $z \geq 1$

for $z = 0$, $p(z) = 1 > 0$

so for $0 < z < 1$:

$-1 < -z < 0 \implies 0 < -z + 1 < 1$, so $p(z) > 0$

so in the end: $p(z) > 0$ for all $z \in \mathbb{R}$

so we have 2 conjugate pairs of zeros. ($a_i \pm b_i, i = 1, 2$)

So we need only to check two quadrants: the first and second

We have that there are 2 zeros in the 1st and 2nd quadrant and 2 zeros in the 3rd and 4th quadrant

possibilities: 2 in either the first or second, or one in each:

First Quadrant:

estimate the increase of $\arg p(z)$ around the boundary of the quarter disk D_R of a large radius R

The increase in the argument of $p(z)$ around $\partial D_R = 2\pi(N_0)$ (no singularities)

where N_0 is the number of zeros in D_R

∂D_R is made up of three paths:

1. The real axis from 0 to R

   $p(x) > 0$, so there is no increase in argument, i.e., $= 0$
2. The quarter-circle Γ_R defined by $|z| = R$ and $0 \leq \arg z \leq \pi/2$,

   the term z^4 "dominates"

   and $\arg p(z) \approx 4 \arg z$

   so the increase is approximately 2π along this path
3. along the imaginary axis from iR to 0

let $z = iy$ with $0 \leq y \leq R$

$p(iy) = y^4 - 2y + 1 - iy$

$\Re(p(iR)) \approx R^4, \Im(p(iR)) = -R$

so $p(iy)$ starts in the 4th quadrant with argument ≈ 0

At 0, $p(0) = 1$

So starts at about the real axis, ends at about the real axis

Need to know k for $2\pi k$ for increase in argument here

so how many times does it cross the real axis?

crosses once at $y = 0$, the terminal point

so the increase is approximately 0 here

So in the first quadrant, the increase in argument around ∂D_R is approx. 2π and therefore exactly 2π by the theorem on page 226

the number of zeros we have in the first quadrant is 1, so that means

1. there's a conjugate zero in the 4th quadrant
2. a zero in the 2nd quadrant with a conjugate zero in the 3rd

Therefore, one zero in each quadrant.

Chapter VIII, Sec. 2 ex. 2

How many roots does $z^9 + z^5 - 8z^3 + 2z + 1$ have between the circles $\{|z| = 1\}$ and $\{|z| = 2\}$

Let D be the region bounded by the two circles

Goal: find $f(z), h(z)$ analytic on D and ∂D with $|h(z)| < |f(z)|$ for $z \in \partial D$

$f(z) = z^9 + z^5 - 8z^3, h(z) = 2z + 1$

I want to use the property:

$|a - b| \geq ||a| - |b||$ but to prove it I need to show:

if $a, b \in \mathbb{R}$ with $b \geq 0$ then:

$|a| \leq b \iff -b \leq a \leq b$

Proof:

let $|a| \leq b \equiv -|a| \geq -b$

$\implies$ since $-|a| \leq a \leq |a|$

we immediately get $-b \leq a \leq b$

Let $-b \leq a \leq b$:

a is either $= -|a|$ or $= |a|$

if $a = |a|$:

$|a| \leq b$ as desired

if $a = -|a|$

then $-|a| \geq -b \equiv |a| \leq b$

so proving $|a-b| \geq ||a|-|b||$:

using triangle inequality:

$|a-b+b| = |a| \leq |a-b| + |b| \equiv |a| - |b| \leq |a-b|$

$|b-a+a| = |b| \leq |b-a| + |a| = |a-b| + |a| \equiv |b| - |a| \leq |a-b| \equiv |a| - |b| \geq -|a-b|$

so $-|a-b| \leq |a| - |b| \leq |a-b| \implies ||a|-|b|| \leq |a-b|$

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for $|z| = 1$:

$|h(z)| = |2z+1| \leq 2|z| + 1 = 3$

$|f(z)| = |z^9 + z^5 - 8z^3| \geq ||z^9 + z^5| - |8z^3||$

since $|z| = 1$, $|z^k| = 1$ for any $k \in \mathbb{N}$

so z^9 and z^5 are on the unit circle still and the longest length the resultant vector from adding any two vectors of length one gets is 2

since $|z^9 + z^5| - 8$ gets smaller the closer $|z^9 + z^5|$ gets to 8, and the closest it gets to 8 is 2:

so $||z^9 + z^5| - |8z^3|| = ||z^9 + z^5| - 8| \geq 6$

so $|f(z)| > |h(z)|$ on $|z| = 1$

$|z| = 2$:

$|h(z)| = |2z+1| \leq 2|z| + 1 = 5$

$|f(z)| \geq ||z^9 + z^5| - |8z^3|| = ||z^9 + z^5| - 64|$

z^9 will have length 2^9 and z^5 will have length 2^5

again, $|z^9 + z^5| - 64$ gets smaller in magnitude to the closer $|z^9 + z^5|$ gets to 64

the smallest we can get from adding two vectors of these lengths, 2^9 and 2^5 is 480

and $480 - 64 = 416$

so $|f(z)| \geq 416$ for $|z| = 2$ so $|f(z)| > |h(z)|$ on $|z| = 2$

so we may use Rouche's Theorem: $f(z)$ has the same number of zeros as $f(z) + h(z)$

the number of zeros $z^9 + z^5 - 8z^3$ has in D:

$z^3(z^6 + z^2 - 8)$

so 0 is a root of multiplicity 3, and we have 6 remaining to find:

$|z^6 + z^2 - 8|$ when $|z| \geq 2$:

$|z^6 + z^2 - 8| \geq |z^6 + z^2| - 8$, and $|z^6 + z^2|$ is at least $2^6 - 2^2 = 60$ (if we consider adding two vectors, one of at least length 2^6 and the other at least 4, the shortest a resultant vector can get is if they are going the opposite direction, thus the length of this vector = absolute value subtracting one length from the other)

so we have $|z^6 + z^2 - 8| > 0$ when $|z| \geq 2$, which means $z^6 + z^2 - 8 \neq 0$ when $|z| \geq 2$

so the roots must lie in the circle of radius 2

and when $|z| \leq 1$:

if $z = re^{i\theta}$

$z^6 = r^6(\cos(6\theta) + i\sin(6\theta))$

$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$

$z^6 + z^2 - 8 = (r^6\cos(6\theta) + r^2\cos(2\theta) - 8) + i(r^6\sin(6\theta) + r^2\sin(2\theta))$

The real part will be at least -10 and at most -8, which means we never obtain 0 within the unit circle or on its boundary

so we must have 6 roots lying in between the circle of radius 1 and the circle of radius 2

Chapter VIII, Sec. 2 ex. 8

Let D be a bounded domain, and let $f(z)$ and $h(z)$ be meromorphic functions on D that extend to be analytic on ∂D. Suppose that $|h(z)| < |f(z)|$ on ∂D. Show by example that $f(z)$ and $f(z) + h(z)$ can have different numbers of zeros on D. What can be said about $f(z)$ and $f(z) + h(z)$? Prove your assertion.

$f(z) = z$

$h(z) = \frac{1}{2z}$

D is the region inside the unit circle

$f(z)$ has one zero in D at 0

$h(z)$ has an isolated singularity at $z = 0$ and is otherwise analytic on ∂D

on ∂D, $|z| = 1$

$|f(z)| = |z| = 1$

$|h(z)| = \frac{1}{2|z|} = \frac{1}{2} < |f(z)| = 1$

$f(z) + h(z) = z + \frac{1}{2z}$ for $|z| < 1$ has 2 zeros:

$z + \frac{1}{2z} = 0 \equiv z^2 + \frac{1}{2} = 0$

$z = \sqrt{-1/2} = \pm i/\sqrt{2}$, which are both in the unit circle

We still have that the increase in argument for $f(z)$ is the same as for $f(z) + h(z)$

so $\int d\arg(f(z)) = \int d\arg(f(z) + h(z))$

and these are each equal to $2\pi(N_0 - N_\infty)$ (for respective N_0, N_∞)

so while they may not have the same number of zeros, they have the same value for $N_0 - N_\infty$:

$f(z)$: $N_0 = 1, N_\infty = 0$

$f(z) + h(z) : N_0 = 2, N_\infty = 1$

Chapter VIII, Sec. 3 ex. 2

Let S be the family of univalent functions $f(z)$ defined on the open unit disk $\{|z| < 1\}$ that satisfy $f(0) = 0$ and $f'(0) = 1$. Show that S is closed under normal convergence, that is, if a sequence in S converges normally to $f(z)$, then $f \in S$.

Remark: It is also true, but more difficult to prove, that S is a compact family of analytic functions, that is, every sequence in S has a normally convergent subsequence.

Univalent: analytic and one-to-one on a domain D (in this case the open unit disk)

let $\{f_k(z)\}$ be a sequence of univalent functions on the open disk.

We know that the sequence converges normally to a function $f(z)$ that is either univalent or constant

for each $f_k(z)$, $f_k(0) = 0$, $f'_k(0) = 1$

for any k, $f_k(z)$ has at least one zero at 0 and we must have that the zeros of the functions in the sequence converge to the zeros of the limit function,

so $f(z)$ must have at least one zero at 0

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also, by the theorem on page 137,

the sequence of first derivatives $\{f'_k(z)\}$ also converges normally to $f'(z)$

if we were to take $f'_k(z) - 1$ as a new sequence

since converging normally means to converge uniformly on any closed disk in D:

if $\{f_k(z)\}$, $\{g_k(z)\}$ each converge uniformly to $f(z), g(z)$ respectively

$\implies |f_j(x) - f(x)| \leq \epsilon_j$ for all $x \in D$, $|g_j(x) - g(x)| \leq \delta_j$

with $\epsilon_j, \delta_j \to 0$ as $j \to \infty$

and so we have $\epsilon_j + \delta_j \to 0$

taking $|f_j(x) + g_j(x) - (f(x) + g(x))| = |f_j(x) - f(x) + g_j(x) - g(x)| \leq \epsilon_j + \delta_j$ by triangle inequality, so $f_k + g_k \to f + g$

therefore:$f'_k(z) - 1$ would converge normally to $f'(z) - 1$

and since these functions have a zero at 0, by the same logic as above

since $f'_k(0) - 1 = 0$ for all k and the zeros of $f'_k - 1$ converge to the zeros of $f' - 1$

0 is a zero for $f'(z) - 1 \implies f'(0) = 1$

Chapter VIII, Sec 4, ex. 4

Let $f(z)$ be an analytic function on the open unit disk $D = \{|z| < 1\}$. Suppose there is an annulus $U = \{r < |z| < 1\}$ such that the restriction of $f(z)$ to U is one-to-one. Show that $f(z)$ is one-to-one on D

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let $z_0 \in U$ and $f(z_0) = w_0$

then $f(z)$ attains w_0 only at z_0 some number $m(\geq 1)$ times

in some open disk centered on z_0, we have that there is an open disk centered on w_0 such that

each w in this open disk centered on w_0 is attained exactly m times only by one point in the open disk centered on z_0

so the boundary of U is included in these open disks, and keeping with this, all of D is in these open disks, meaning each point in $f(D)$ has m preimages (counting multiplicities)

If we were to take some ρ with $r < \rho < 1$ and take the image of the circle with radius ρ

with $\gamma(\theta) = \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$

we'd get a curve that starts and ends at the same point (closed)

and since $f(z)$ is one-to-one on the annulus, simple

let z_0 be a point such that $|z| < \rho$

we have that $f(z) - f(z_0)$ has a zero of order m,

also, $f(z) - f(z_0)$ is $1-1$ on the annulus:

$f(z_1) - f(z_0) = f(z_2) - f(z_0) \equiv f(z_1) = f(z_2)$

so similarly, the curve $f(\gamma) - f(z_0)$ is simple and closed

(since on $|z| = \rho$, $f(z)$ is one-to-one, we have $f(z) - f(z_0) \neq 0$, else a contradiction to one-to-one)

$\int_{|z|=\rho} \frac{f'(z)}{f(z)-f(z_0)} dz = 2\pi(m)$ (since $f(z) - f(z_0)$ is analytic in D, there's no singularities)

since the curve of $f(\gamma) - f(z_0)$ is a simple closed curve, it must only travel around origin once.

The increase in argument is therefore 2π

so $m = 1$, which means each point in $f(D)$ has exactly 1 preimage

Chapter VIII, Sec 4, ex. 6

Let $f(z)$ be a meromorphic function on the complex plane, and suppose there is an integer m such that $f^{-1}(w)$ has at most m points for all $w \in \mathbb{C}$. Show that $f(z)$ is a rational function.

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let w_0 be a point in $\mathbb{C}$ such that there are in $f^{-1}(w_0)$ the maximum number of points z such that $f(z) = w_0$

since all other values of w can only be achieved at a number of points less than the number of points that achieve w_0, we have that only near the points in $f^{-1}(w_0)$ does $f(z) = w$

so since there are finitely many points such that $f(z) - w_0 \to 0$

$1/(f(z) - w_0)$ as $z \to \infty$ is bounded, which means $1/(f(z) - w_0)$ has a removable singularity at ∞

so $1/(f(z) - w_0)$ is analytic at ∞

which means that ∞ is a pole of $f(z) - w_0$, so $f(z) - w_0$ is meromorphic on $\mathbb{C}^*$

and since sums of meromorphic functions are meromorphic:

w_0 is meromorphic on $\mathbb{C}^*$ since it is bounded everywhere and at infinity

so $f(z) - w_0 + w_0 = f(z)$ is meromorphic on $\mathbb{C}^*$ and therefore, rational by the theorem on page 179