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Math 185

Homework #4

Chapter 2, Sec. 5

Ex. 2 Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v

Suppose v is a harmonic conjugate for u.

This means that u is harmonic and v is harmonic such that $u + iv$ is analytic.

so: $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2}$

We need to show that $-u$ is harmonic and $v - iu$ is analytic

and since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$and $\lim_{\Delta x \to 0} \frac{(-u)(x+\Delta x, y) - (-u)(x,y)}{\Delta x} = -\frac{\partial u}{\partial x}$

and $\lim_{\Delta y \to 0} \frac{(-u)(x, y+\Delta y) - (-u)(x,y)}{\Delta y} = -\frac{\partial u}{\partial y}$

and the partial derivatives of u are continuous, so are the ones for $-u$.

So we have:

$\frac{\partial(-u)}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial(-u)}{\partial y} = \frac{\partial v}{\partial x}$, this shows $v - iu$ is analytic.

And: $\frac{\partial^2(-u)}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial(-u)}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2(-u)}{\partial y^2}$, this shows $-u$ is harmonic

Ex. 7 Show that $\log|z|$ has no conjugate harmonic function on the punctured plane $\mathbb{C}\backslash${0}, though it does have a conjugate harmonic function on the slit plane $\mathbb{C} \setminus (-\infty, 0]$

Over $D = \mathbb{C} \setminus (-\infty, 0]$

letting $u = \log|z|$ so $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$ and $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

and $\frac{\partial^2 u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$ and $\frac{\partial^2 u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$, so $u = \log|z|$ is harmonic

Taking the integral of $\frac{x}{x^2+y^2}$ w.r.t. to y,

we get $\arctan(y/x) + C(x)$ and then taking the derivative of this with respect to x it can be easily shown that $C'(x) = 0$ if v is a harmonic conjugate to u

it must be equal to $\arctan(y/x) + C$ where C is a constant $\in \mathbb{R}$

and $\arctan(y/x) = Arg(z)$

so $u + iv = Log(z) + iC$

If we were to extend u to $D = \mathbb{C}\backslash$ {0}

u is still continuous, since $|z|$ is continuous over all points in $\mathbb{C}$ and $\log(x)$ is continuous over all $x \in \mathbb{R}$ s.t. $x > 0$ and therefore harmonic.

however, $Arg(z)$ = $\arctan(y/x)$ is not continuous over $\mathbb{C}\backslash\{0\}$ since

for $x < 0$:

$\lim_{y\to 0^+} Arg(x+iy) = \pi$

$\lim_{y\to 0^-} Arg(x+iy) = -\pi$

Chapter 2, Sec. 6

Ex. 1 Sketch the families of level curves of u and v for the following functions $f = u + iv$

(a) $f(z) = 1/z$

$f(x+iy) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

$u(x,y) = \frac{x}{x^2+y^2}$, $v(x,y) = \frac{-y}{x^2+y^2}$

$u(x^2+y^2) = x$ and $v = \frac{-y}{x^2+y^2}$

$x^2 - \frac{x}{u} + y^2 = 0$ $\quad x^2 + y^2 + \frac{y}{v} = 0$

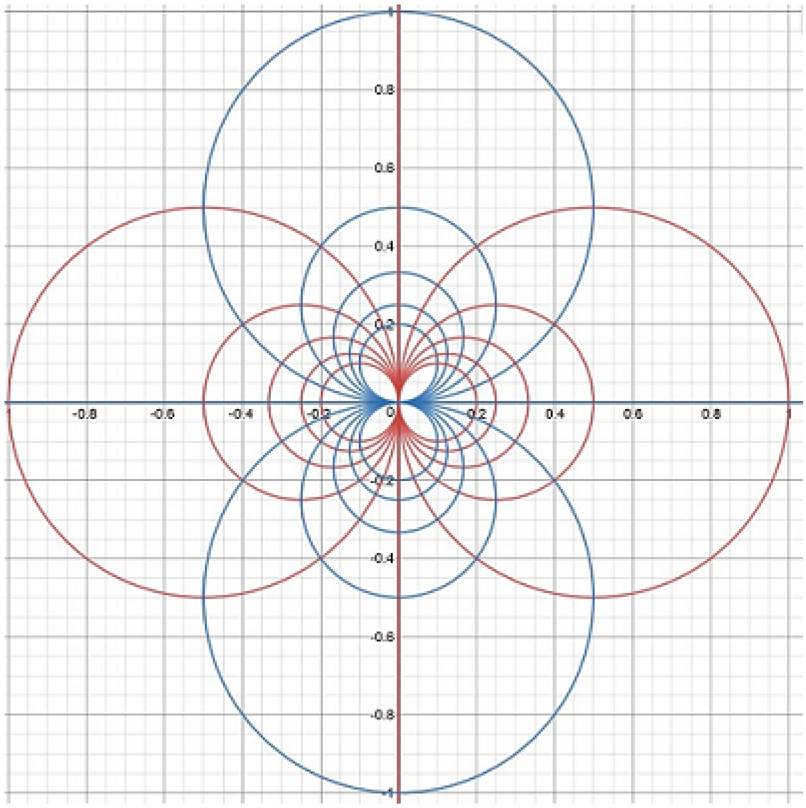
$(x - \frac{1}{2u})^2 - \frac{1}{4u^2} + y^2 = 0$ $\quad x^2 + (y + \frac{1}{2v})^2 = \frac{1}{4v^2}$

$(x - \frac{1}{2u})^2 + y^2 = \frac{1}{4u^2}$

So the families of level curves of u are going to be circles centered at $(\frac{1}{2u}, 0)$ with radius $|\frac{1}{2u}|$

and families of level curves of v are going to be circles centered at $(0, -\frac{1}{2v})$ with radius $|\frac{1}{2v}|$

Except at $u = 0, v = 0$, $u = 0$ represents the line $x = 0$ and $v = 0$ represents the line $y = 0$



red curves represent u level curves, from $-5 \leq u \leq 5$ increasing by 1

blue curves represent v level curves from $-5 \leq v \leq 5$ increasing by 1 and when $u < 0$ they are circles in the left plane, and when $u > 0$ they are circles in the right plane

when $v < 0$ they are circles in the upper plane and when $v > 0$ they are circles in the lower plane

also smaller magnitudes of u, v correspond to larger circles

So f is conformal whenever $z \neq 0$, since $1/z$ is not defined at $z = 0$ and $f'(z) = -\frac{1}{z^2}$

(b) $f(z) = 1/z^2$

$f(x+iy) = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+i2xy} = \frac{x^2-y^2}{(x^2-y^2)^2+(2xy)^2} + i\frac{-2xy}{(x^2-y^2)^2+(2xy)^2}$

In terms of $|z|$ and $\arg z = \theta$, and $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 = |z|^4$

$u = \frac{|z|^2(\cos^2\theta - \sin^2\theta)}{|z|^4} = \frac{\cos 2\theta}{|z|^2}$

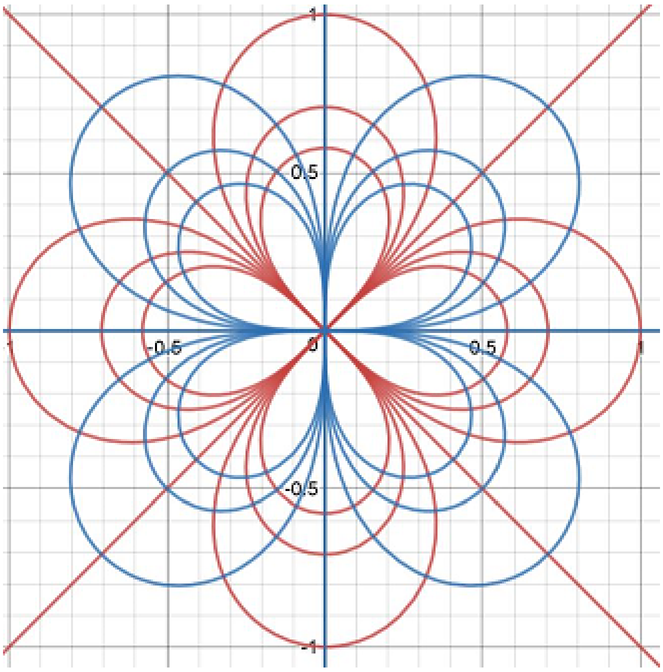
$v = \frac{-2|z|^2 \sin\theta\cos\theta}{|z|^4} = -\frac{\sin 2\theta}{|z|^2}$

$u = c, \cos 2\theta = c|z|^2 \equiv |z| = \sqrt{\frac{1}{c}\cos 2\theta}$, if $c > 0$ not defined for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ and $-\frac{3\pi}{4} < \theta < -\frac{\pi}{4}$

if $c < 0$ not defined for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$

$v = c, |z|^2 = -\frac{1}{c}\sin 2\theta$, if $c > 0$ not defined for $0 < \theta < \frac{\pi}{2}$ and $-\pi < \theta < -\frac{\pi}{2}$

if $c < 0$ not defined for $\frac{\pi}{2} < \theta < \pi$ and $-\frac{\pi}{2} < \theta < 0$



red curves correspond to u, blue curves correspond to v

and each curve represents an integer from -3 to 3

When $u = 0$, we must have $x^2 = y^2$, so the lines $y = x$ and $y = -x$

When $v = 0$ we must have either $x = 0$ or $y = 0$, so the lines $y = 0$ and $x = 0$ ($f(z)$ not defined at $(0, 0)$)

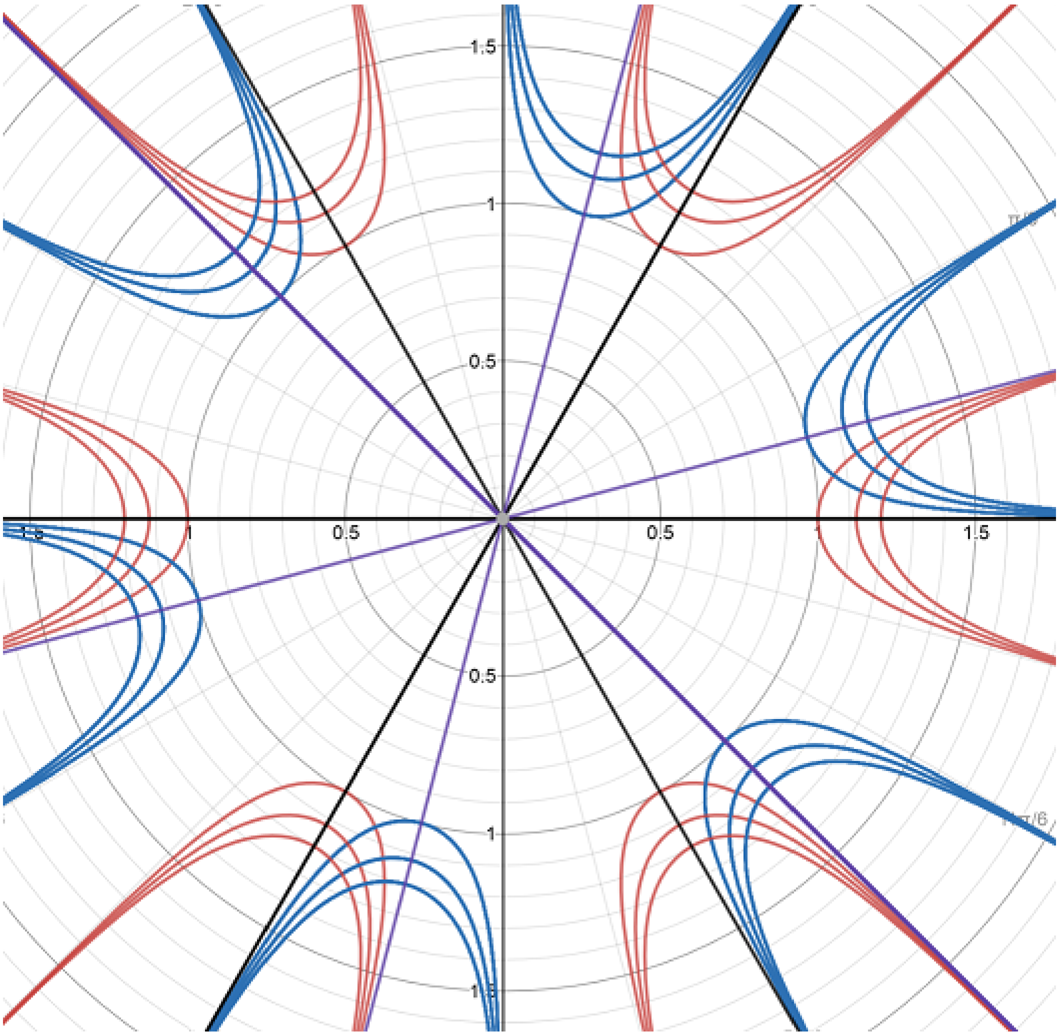
Conformal everywhere except $z = 0$

(c) $f(z) = z^6$

$f(|z|e^{i\theta}) = |z|^6 e^{i6\theta}$

$u = |z|^6 \cos(6\theta)$
$u = c \implies$
$|z|^6 \cos(6\theta) = c$
$|z| = (\frac{c}{\cos(6\theta)})^{1/6}$

$v = |z|^6 \sin(6\theta)$
$v = c \implies |z|^6 \sin(6\theta) = c$
$|z| = (\frac{c}{\sin(6\theta)})^{1/6}$



Red, u curves. Blue, v curves

The radius of the circle at which 6 curves (corresponding to the same u or v value) intersects only once = the value of $u^{1/6}$ or $v^{1/6}$

and the points (x, y) at which these curves intersect the circle represent a 6th root of u or iv

The black lines represent the angles $k\frac{\pi}{3}$ where $k \in Z$

and where it intersects with the curves for v as $v \to 0$,

The purple lines represent the angles $\frac{\pi}{12} + k\frac{pi}{3}$ where $k \in \mathbb{Z}$

and the curves for u as $u \to 0$

and where the black lines intersect with the circle of radius 1 represents 6th roots of 1

the second set of red curves these black lines intersect with represent 6th roots of 2

the third set of red curves these black lines intersect with represent 6th roots of 3

and similarly for the purple lines and 6th roots of ki where $k = 1, 2$ or 3

$f(z) = z^6$ is conformal everywhere except for when $f'(z) = 0$ @ $z = 0$

Ex. 7 For the function $f(z) = z + 1/z = u + iv$ sketch the families of level curves of u and v Determine the images under $f(z)$ of

so $f(z) = (x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$

and $u = 0 \implies x = 0$

If we have $y^2 + x^2 = 1 - \epsilon$, for some $0 \leq \epsilon \leq 1$ (meaning, (x,y) are on or inside the unit circle)

$u(x, y) = x + \frac{x}{x^2+y^2} = x + \frac{x}{1-\epsilon}$

and as $\epsilon \to 0^+$, we get $u \to 2x$, so $x = \frac{u}{2}$

so for any $|c| \leq 2$, we can find points (x, y) within or on the unit circle that satisfy $u(x, y) = c$

if we restrict $x^2 + y^2 = 1/(c - 1)$, with $c > 2$, we will always have points (x, y) in the unit disk

and $u(x, y) = x + (c - 1)x = cx$

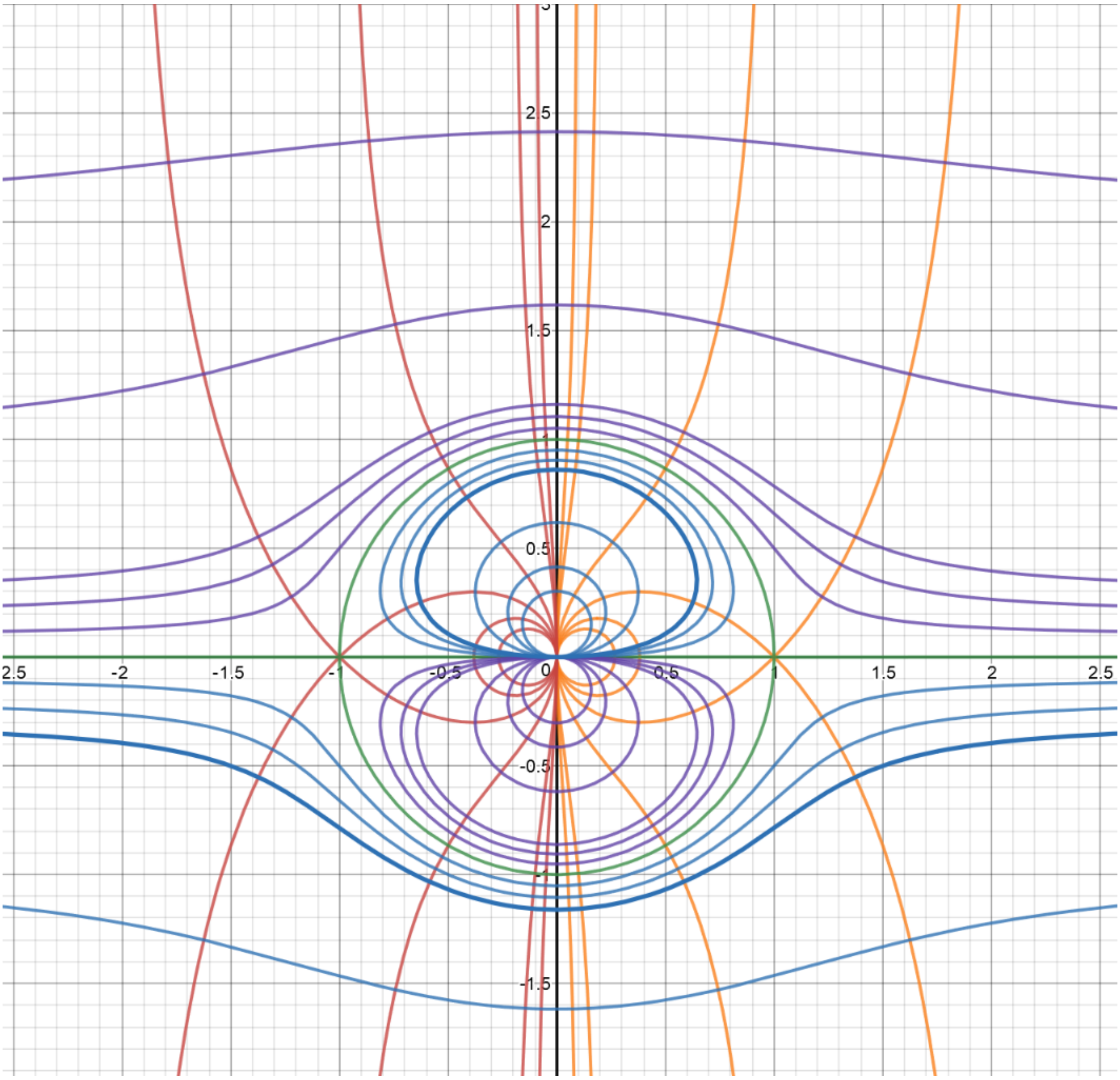
$x = u/c$, if $u > 0$, $x > 0$

$y = \pm\sqrt{\frac{1}{c-1} - \frac{u^2}{c^2}}$

and $y = 0$ when $\frac{1}{c-1} - \frac{u^2}{c^2} = 0$

$c^2 - u^2c + u^2$

so when $c = \frac{u^2 \pm \sqrt{u^4 - 4u^2}}{2}$ real when $u^2 \geq 4$



orange lines correspond to positive u values

not pictured: for large u values, we have lines looking like $x = u$ corresponding to this u value

because as x grows larger, $\frac{x}{x^2+y^2} \to 0$

red lines correspond to negative u values

and $x = 0$ is the line corresponding to $u = 0$

purples lines correspond to positive v values

blue lines correspond to negative v values

the green line corresponds to $v = 0$

As can be seen in the image, for any $u = c$ curve, it will intersect any curve v in and out of the unit circle.

the top half of the unit disk $(x^2 + y^2 \leq 1), y \geq 0$

Corresponds to $v \leq 0$

and $u \in (-\infty, +\infty)$

so the lower plane

the bottom half of the unit disk

Corresponds to $v \geq 0$

$u \in (-\infty, +\infty)$

the upper plane

the part of the upper half plane outside the unit disk

$v > 0$, and $u \in (-\infty, +\infty)$

the upper plane

the part of the lower half-plane outside the unit disk

$v < 0$, and $u \in (-\infty, +\infty)$

the lower plane

Chapter 2, Sec. 7

Ex. 2 Consider the fractional linear transformation $(1+i, 2, 0) \mapsto (0, \infty, i-1)$. Without referring to an explicit formula, determine the image of the circle $|z-1| = 1$, the image of the disk $|z-1| < 1$ and the image of the real axis

$1+i, 2, 0$ all lie on the circle $|z-1| = 1$

so this fractional linear transformation maps the circle to the line passing through origin and $(i-1)$ $(y = -x)$

Since orientation is preserved, all points within the circle are mapped under the line $y = -x$

The real line is a circle containing 0, 2 and is orthogonal to the circle $|z-1| = 1$

Its image will be the circle that contains the points ∞ and $i-1$, so a line passing through $i-1$ that is orthogonal to the line $y = -x$, so a line with slope 1 containing the point (-1, 1)

so $(y-1) = (x+1)$, $y = x+2$

Ex. 8 Show that any fractional linear transformation can be represented in the from $f(z) = (az+b)/(cz+d)$ where $ad - bc = 1$. Is this representation unique?

Suppose $g(z) = (Az+B)/(Cz+D)$ is a fractional linear transformation

with $AD - BC = \alpha$ where $\alpha \neq 0$

$(AD - BC)/\alpha = 1$, $(A/\sqrt{\alpha})(D/\sqrt{\alpha}) - (B/\sqrt{\alpha})(C/\sqrt{\alpha}) = 1$

so we let $a = A/\sqrt{\alpha}, b = B/\sqrt{\alpha}, c = C/\sqrt{\alpha}, d = D/\sqrt{\alpha}$

and this still gives the same function:

$g(z) \times 1 = g(z) \times \frac{1/\sqrt{\alpha}}{1/\sqrt{\alpha}} = \frac{(A/\sqrt{\alpha})z + B/\sqrt{\alpha}}{(C/\sqrt{\alpha})z + D/\sqrt{\alpha}}$

Not unique: we could also do $a = -A/\sqrt{\alpha}$, etc...

so multiplying $g(z)$ by $1 = \frac{-1/\sqrt{\alpha}}{-1/\sqrt{\alpha}}$ will give the same result, different representation and

$(-A/\sqrt{\alpha})(-D/\sqrt{\alpha}) - (-B/\sqrt{\alpha})(-C/\sqrt{\alpha}) = 1$

Ex. 11 Two maps f and g are conjugate if there is h such that $g = h \circ f \circ h^{-1}$. Here the conjugating map is assumed to be one-to-one, with appropriate domain and range. We can think of f and g as the "same" map, after the change of variable $w = h(z)$. A point z_0 is a fixed point of f if $f(z_0) = z_0$ Show the following

(a) If f is conjugate to g, then g is conjugate to f.

Let f be conjugate to g then $\exists$ a map h such that

$g = h \circ f \circ h^{-1}$ then:

$h^{-1} \circ g \circ h = h^{-1} \circ h \circ f \circ h^{-1} \circ h = f$

thus, g is conjugate to f, with h^{-1} being the conjugating map

(b) If f_1 is conjugate to f_2 and f_2 to f_3, then f_1 is conjugate to f_3

We have:

$f_2 = h \circ f_1 \circ h^{-1}$

$f_3 = g \circ f_2 \circ g^{-1}$

Substituting $h \circ f_1 \circ h^{-1}$ for f_2 :

$f_3 = g \circ h \circ f_1 \circ h^{-1} \circ g^{-1} = (g \circ h) \circ f_1 (g \circ h)^{-1}$

this proves the statement.

(c) If f is conjugate to g, then $f \circ f$ is conjugate to $g \circ g$, and more generally, the m-fold composition $f \circ \cdots \circ f$ (m times) is conjugate to $g \circ \cdots \circ g$ (m times).

$g = h \circ f \circ h^{-1}$

$g \circ g = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} = h \circ f \circ f \circ h^{-1}$

We can use induction, as we've proven $n = 2$, and assuming $(g \circ \ldots \circ g) = h \circ f \ldots \circ f \circ h^{-1}$ k times $k > 2$, we prove for $k + 1$

$g \circ \cdots \circ g = (g \circ \cdots \circ g) \circ g$, (k times)(1 time)

= $(h \circ f \circ \ldots \circ f \circ h^{-1})(\circ h \circ f \circ h^{-1})$
$= h \circ f \circ \ldots \circ f \circ f \circ h^{-1}$

So, for any natural number m, we have proven the statement

(d) If f and g are conjugate, then the conjugate function h maps fixed points of f to the fixed points of g. In particular f and g have the same number of fixed points

Let z_0 be any point such that $f(z_0) = z_0$

$g = h \circ f \circ h^{-1}$

Let w_0 be the point such that $h^{-1}(w_0) = z_0 \equiv h(z_0) = w_0$

then $g(w_0) = h(f(z_0)) = h(z_0) = w_0$

So we have that h maps fixed point of z_0 to fixed point of g

and h^{-1} maps fixed point of g to fixed points of f

Ex. 12 Classify the conjugacy classes of fractional linear transformations by establishing the following:

(a) A fractional linear transformation that is not the identity has either 1 or 2 fixed points, that is points satisfying $f(z_0) = z_0$

$f(z) = \frac{az+b}{cz+d}$

$\frac{az+b}{cz+d} = z$
$az + b = cz^2 + dz$
$cz^2 + (d-a)z - b = 0$

if every z made this equal to 0, we'd have $f(z) = z$ for all z, but f is not the identity so:

By solving for z we find fixed points,

if $c \neq 0$: since polynomials of degree 2 have 2 finite roots, we get 2 finite fixed points.

$z = \frac{a-d \pm \sqrt{(d-a)^2+4cb}}{2c}$

If $(d-a)^2 + 4cb = 0$, we have one root with multiplicity 2

if $c = 0$, we have one finite fixed point, $z = b/(d-a)$ and $z = \infty$ is also a fixed point

if $d - a = 0$ and $b \neq 0$ then we only have one fixed point at ∞

(b) If a fractional linear transformation $f(z)$ has two fixed points, then it is conjugate to the dilation $z \mapsto az$ with $a \neq 0$, $a \neq 1$, that is, there is a fractional linear transformation $h(z)$ such that $h(f(z)) = ah(z)$. Is a unique?

Let $f(z)$ have two fixed points: z_0, z_1, $f(z_0) = z_0, f(z_1) = z_1$

If $f(0) = 0$, $f(\infty) = \infty$ so $f(0) = \frac{b}{d} \implies b = 0$ and $\lim_{z\to\infty} \frac{az}{cz+d} = \frac{a}{c}$, but we need it to be ∞, so $c = 0$ and so we have: $f(z) = \frac{az+b}{cz+d} = \frac{az}{d}$, which is a dilation $z \mapsto \frac{a}{d}z$

$f(z) = \frac{az+b}{cz+d}$ with $\frac{az_i+b}{cz_i+d} = z_i \; i = 0, 1$

So, if we define h to map fixed points of f to 0 and ∞, $h^{-1}(0) = z_0$ so $h(z_0) = 0, h^{-1}(\infty) = z_1, h(z_1) = \infty$

We have so $h \circ f \circ h^{-1}(0) = 0$ and $h \circ f \circ h^{-1}(\infty) = \infty$

so $g = h \circ f \circ h^{-1}$ is a dilation, and f is conjugate to g

Suppose $g : z \mapsto az$

If g is conjugate with another dilation, j, then f is conjugate to j, meaning, a is not unique

So suppose this is true and $j : z \mapsto Az$, where $A \neq a$

then there exists a map k s.t. $j = k \circ g \circ k^{-1}$

And k must map fixed points of g to a fixed point of j

so either $k(0) = 0$ and $k(\infty) = \infty$ so k is a dilation, $k : z \mapsto cz$

or $k(0) = \infty$ and $k(\infty) = 0$, so k is an inversion, $k : z \mapsto \frac{c}{z}$

suppose $k(z) = cz$ so $k^{-1}(z) = c^{-1}z$

$j = k(g(k^{-1}(cz))) = k(g(z)) = k(az) = caz = Acz$ so $a = A$

suppose $k(z) = c/z$ and $k^{-1} = c/z$

$j = k(g(k^{-1}(cz^{-1}))) = k(g(z)) = k(az) = c/(az) = Ac/z$ so $\frac{1}{a} = A$

so there exists a mapping k s.t. g is conjugate to $j : z \mapsto 1/a$

a is not unique

(c) If a fractional linear transformation $f(z)$ has exactly one fixed point, then it is conjugate to the translation $\zeta \mapsto \zeta + 1$. In other words, there is a fractional linear transformation $h(z)$ such that $h(f(h^{-1}(\zeta))) = \zeta + 1$ or equivalently, such that $h(f(z)) = h(z) + 1$

if a fractional linear transformation only has one fixed point at ∞, then we must have (from (a))

$f(z) = z + b$,

Let h map the fixed point z_0 of f to ∞

so we have $h^{-1}(\infty) = z_0$

so $g = h \circ f \circ h^{-1}(\infty) = \infty$ is the only fixed point

so we must have $g(z) = z + b$, $b \neq 0$ (else 0 would be a fixed point)

we need to show there exists a map, k s.t.

$k(g(k^{-1}(z))) = z + 1$

this means $k(k^{-1}(z) + b) = k^{-1}(z)/b + 1$ so $k^{-1}(z) = bz$

and $k(z) = z/b$

so $(k \circ h \circ f \circ h^{-1} \circ k^{-1}(z) = k \circ g \circ k^{-1}(z) = z + 1$

So, $(k \circ h)f(k \circ h)^{-1}(z) = z + 1$

$k \circ h$ is the linear mapping that takes $f(z)$ to $k(h(z)) + 1$