MATH 111 (FALL 2013) - ASSIGNMENT 1
SELECTED SOLUTION

PROBLEM 2: (Ch. 2, #6) Consider the harmonic oscillator system $x'' + bx' + kx = 0$ where $k \neq 0$.
(a) Rewrite this second-order equation into a first-order linear system of the form $\mathbf{X}' = A\mathbf{X}$.
(b) Find all values of b and k for which this system has real, distinct eigenvalues, and find the general solution of this system in these cases. (Use of software such as Mathematica is allowed provided that you indicate which part was done by the machine.)
(c) Find the solution of the system that satisfies the initial condition $\mathbf{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
(d) Describe the motion of the mass in (c).

Sketch of solutions:

(a) Let $v = x'$ and $\mathbf{X} = \begin{bmatrix} x \\ v \end{bmatrix}$, then $\mathbf{X}' = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \mathbf{X}$.

(b) Characteristic equation is: $\lambda^2 + b\lambda + k = 0$, roots are:

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4k}}{2}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4k}}{2}.$$

Real distinct if $b^2 - 4k > 0$.

To find eigenvectors for λ_1, we solve $\begin{cases} 0x + v & = \lambda_1 x \\ -kx - bv & = \lambda_1 v \end{cases}$. We can avoid lengthy computations by noting that **one of the equations must be redundant** as we expect to get infinitely many solutions. Ignore the more difficult second equation and we get $v = \lambda_1 x$. Similarly for λ_2.

Corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

(c) It amounts to solving $\begin{cases} c_1 + c_2 & = 1 \\ \lambda_1 c_1 + \lambda_2 c_2 & = 0 \end{cases}$ for c_1, c_2. Use whatever method you like but in my opinion the Cramer's Rule works the best here. One should get:

$$c_1 = -\frac{\lambda_2}{\lambda_1 - \lambda_2}, \quad c_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}.$$

$$\mathbf{X}(t) = -\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are the eigenvalues found in (b).

(d) $x(t) = -\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$. To be physically relevant, you may assume $b \geq 0$ and $k > 0$ (see Textbook P. 26, I should have stated that in the homework). From (b) we have $\lambda_1 < 0$ and $\lambda_2 < 0$. Also by observing that $\lambda_2 < \lambda_1$, one can check (from elementary calculus) that $x(t) > 0$. Therefore the mass will go towards the origin as t increases, and there is no oscillation.

PROBLEM 4: (Ch. 2, #11 modified) Prove that two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ are linearly dependent **if and only if**

$$\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = 0.$$

Solution: ($\Rightarrow$) Suppose $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. WLOG, assume $\mathbf{v} = c\mathbf{w}$ for some real scalar c. Then we have $v_1 = cw_1$ and $v_2 = cw_2$.

$$\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1 = cw_1 w_2 - cw_2 w_1 = 0.$$

($\Leftarrow$) Conversely, suppose $0 = \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1$, then $v_1 w_2 = v_2 w_1$. If both $v_1 = 0$ and $v_2 = 0$ then $\mathbf{v} = 0$ and therefore $\mathbf{v} = 0\mathbf{w}$. Now assume either one of $\{v_1, v_2\}$ is non-zero, and WLOG assume $v_1 \neq 0$, then we have

$$w_2 = \frac{v_2}{v_1} w_1.$$

One can then verify that

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \frac{v_2}{v_1} w_1 \end{bmatrix} = \frac{w_1}{v_1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{w_1}{v_1} \mathbf{v}.$$

Therefore $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

PROBLEM 5: Consider the following system $\mathbf{Y}'(t) = A\mathbf{Y}(t) + \mathbf{b}$ where A is a 2×2 matrix with constant entries and $\mathbf{b} \in \mathbb{R}^2$ is a fixed vector. Given that $\mathbf{Y}_0(t)$ is a particular solution to the system. Prove that any solution $\mathbf{Y}(t)$ to the system $\mathbf{Y}'(t) = A\mathbf{Y}(t) + \mathbf{b}$ can be written in the form:

$$\mathbf{Y}(t) = \mathbf{Y}_0(t) + \mathbf{X}(t)$$

where $\mathbf{X}(t)$ is a solution to the system $\mathbf{X}'(t) = A\mathbf{X}(t)$.

Solution: Given any solution $\mathbf{Y}(t)$ to the system $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}$, we first verify that $\mathbf{Y} - \mathbf{Y}_0$ is a solution to the homogeneous system $\mathbf{X}' = A\mathbf{X}$:

$$\begin{aligned} (\mathbf{Y} - \mathbf{Y}_0)' &= \mathbf{Y}' - \mathbf{Y}_0' \\ &= (A\mathbf{Y} + \mathbf{b}) - (A\mathbf{Y}_0 + \mathbf{b}) \qquad \text{since both } \mathbf{Y} \text{ and } \mathbf{Y}_0 \text{ are solutions to the } \mathbf{b}\text{-system.} \\ &= A\mathbf{Y} - A\mathbf{Y}_0 \\ &= A(\mathbf{Y} - \mathbf{Y}_0). \end{aligned}$$

Now it shows $\mathbf{Y} - \mathbf{Y}_0$ satisfies the equation $\mathbf{X}' = A\mathbf{X}$. Letting $\mathbf{X} = \mathbf{Y} - \mathbf{Y}_0$ yields

$$\mathbf{Y} = \mathbf{Y}_0 + \mathbf{X}$$

and from above, this $\mathbf{X}$ is a solution to the homogeneous system $\mathbf{X}' = A\mathbf{X}$. It completes the proof.

Remark: This problem tells us that to find the general solution to the system $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}$, we just need to solve $\mathbf{X}' = A\mathbf{X}$ and find ONE particular solution $\mathbf{Y}_0$ (maybe by trial-and-error or other DE techniques) to system $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}$. Then you may just add this $\mathbf{Y}_0$ to the solution set of $\mathbf{X}' = A\mathbf{X}$ to get the general solution of $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}$.