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Math 123

Homework #9

Chapter 8, exercise 3.

Chapter 9, exercises 6, 7(a, b), 8(b), 9, 10, 11, 16.

Chapter 8, exercise 3: Consider a first-order differential equation $x' = f_a(x)$ for which $f_a(x_0) = 0$ and $f'_a(x_0) \neq 0$. Prove that the differential equation $x' = f_{a+\epsilon}(x)$ has an equilibrium point $x_0(\epsilon)$ where $\epsilon \to x_0(\epsilon)$ is a smooth function satisfying $x_0(0) = x_0$ for ϵ sufficiently small

rewrite $f_a(x)$ as $f(a, x)$

we have $f(a, x_0) = 0$ and $\frac{\partial f}{\partial x}(a, x_0) \neq 0$

since $\frac{\partial f}{\partial x}(a, x_0) \neq 0$ (invertible)

Using Implicit Function Theorem:

There exists an open set U of $\mathbb{R}$ containing a such that there is a unique continuously differentiable function $g : U \to \mathbb{R}$ with $g(a) = x_0$

and $f(a, g(a)) = 0$ for all $a \in U$

so let $\epsilon > 0$ be sufficiently small so that $a + \epsilon \in U$

we have that $f(a + \epsilon, g(a + \epsilon)) = 0$

so $x_0(\epsilon) = g(a + \epsilon)$ is a point such that $f_{a+\epsilon}(x_0(\epsilon)) = 0$

with $x_0(\epsilon) = g(a + \epsilon)$ being the smooth function (since g is continuously differentiable)

satisfying $x_0(0) = g(a + 0) = g(a) = x_0$

Chapter 9

6: Find a strict Liapunov function for the equilibrium point $(0, 0)$ of $\begin{array}{l} x' = -2x - y^2 \\ y' = -y - x^2 \end{array}$. Find $\delta > 0$ as large as possible so that the open disk of radius δ and center $(0, 0)$ is contained in the basin of $(0, 0)$

$L(x, y) = ax^2 + by^2$

$\dot{L} = 2ax * x' + 2byy' = -4ax^2 - 2axy^2 - 2by^2 - 2byx^2$

with $a = b = 1$ is a strict Lyapunov function:

$L(0, 0) = 0$, $\dot{L} < 0$ when (x, y) is near origin

$\dot{L} < 0$ when

$-4x^2 - 2xy^2 - 2y^2 - 2yx^2 < 0$

$\equiv$

$2x^2 + xy^2 + y^2 + yx^2 > 0$

$\equiv$

$(2 + y)x^2 + (x + 1)y^2 > 0$

polar coordinates:

$(2 + r\sin\theta)r^2\cos^2\theta + (r\cos\theta + 1)r^2\sin^2\theta > 0$

$2r^2\cos^2\theta + r^3\sin\theta\cos^2\theta + r^3\cos\theta\sin^2\theta + r^2\sin^2\theta > 0$

$r^2(1 + \cos^2\theta + r\sin\theta\cos\theta(\sin\theta + \cos\theta)) > 0$

$r^3(1 + \cos^2\theta)(1/r + \frac{(\sin\theta\cos\theta(\sin\theta+\cos\theta))}{1+\cos^2\theta})$

since $r^3(1 + \cos^2\theta) > 0$ for all $\theta > 0$

we turn our attention toa

$\frac{(\sin\theta\cos\theta(\sin\theta+\cos\theta))}{1+\cos^2\theta}$

$1 + \cos^2\theta \leq 2$

since $-\sin(\pi/4)\cos(\pi/4) < \sin\theta + \cos\theta < \sin(\pi/4)\cos(\pi/4) \approx 1.414$

and $|\sin(\theta)\cos(\theta)| < 0.5$

we have $\frac{(\sin\theta\cos\theta(\sin\theta+\cos\theta))}{1+\cos^2\theta} \geq -1/2$

$\implies 1/r > 1/2 \equiv r < 2$

so $\dot{L} < 0$ in the open disk with radius 2 centered at origin

which means no solution (that enters the disk) exists such that L is constant except for the equilibrium point at origin, since L is strictly decreasing

so by Lasalle's Invariance Principle, the disk in contained in the basin of attraction of $(0, 0)$

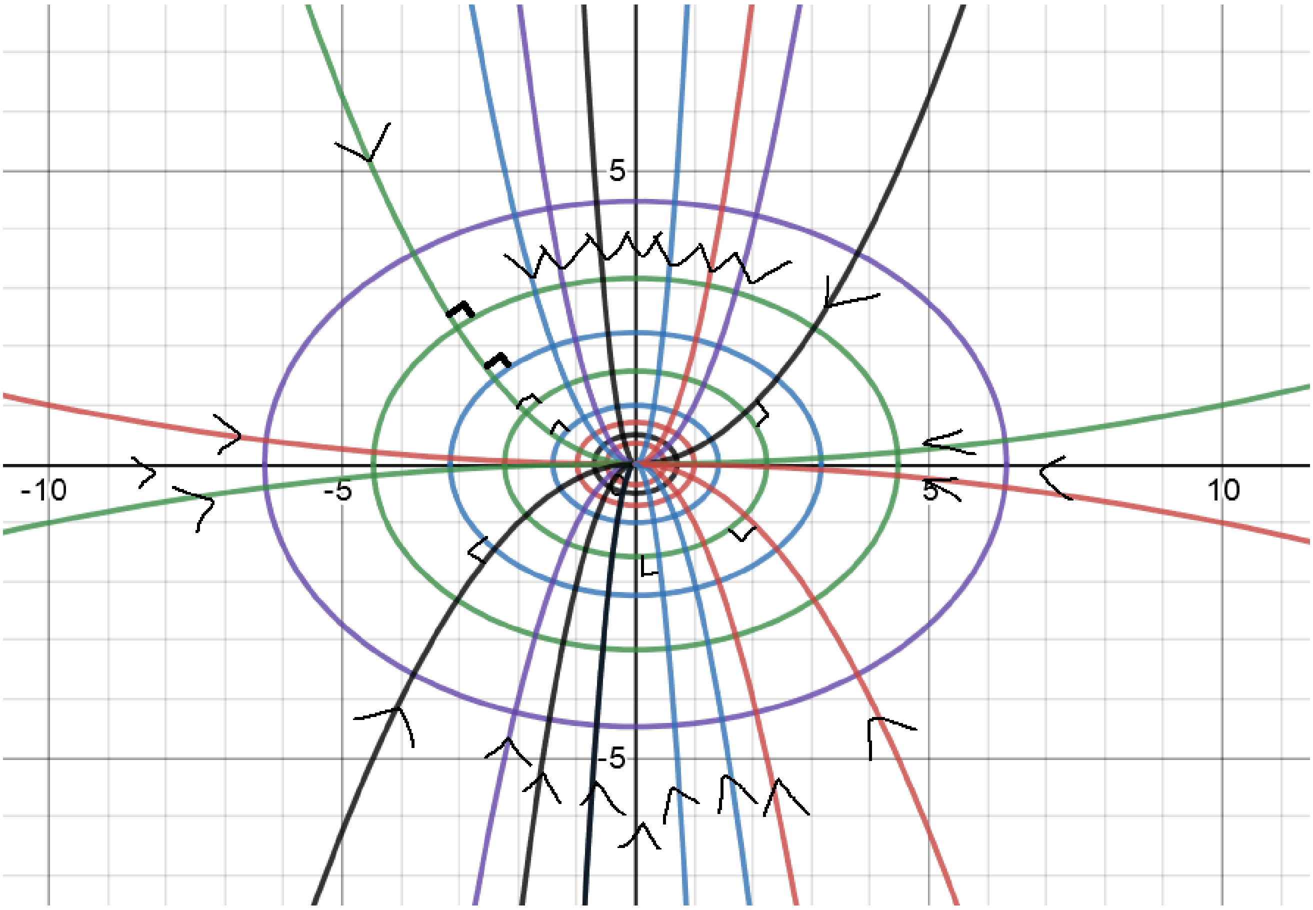
*since for any closed disk with radius $r < 2$, L is strictly decreasing for any solution entering this closed disk

7: For each of the following functions $V(X)$ sketch the phase portrait of the gradient flow $X' = -\text{grad } V(X)$. Sketch the level surfaces of V on the same diagram. Find all of the equilibrium points and determine their type.

(a) $x^2 + 2y^2$

$\text{grad } V(X) = (2x, 4y)$

$x' = -2x$
$y' = -4y$



The only equilibrium point: $(0,0)$, and it is a real sink

(b) $x^2 - y^2 - 2x + 4y + 5$

$\mathrm{grad}\, V(X) = (2x-2, -2y+4)$

$x' = -2x + 2$
$y' = 2y - 4$

$\frac{1}{-2x+2} dx = dt$

$-\log|-2x+2|/2 = t + C$

$-2x + 2 = e^{-2t+C}$

$x = Ce^{-2t} + 1\, x' = -2Ce^{-2t} = -2Ce^{-2t} - 2 + 2 = -2x + 2$

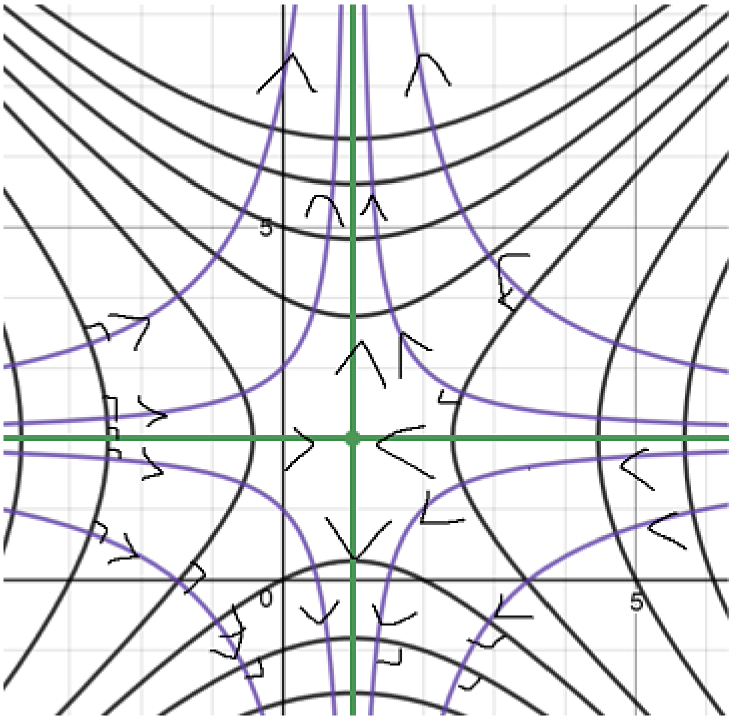
$y = Ce^{2t} + 2, y' = 2Ce^{2t} = 2Ce^{2t} + 4 - 4 = 2y - 4$

equilibrium points: $(1,2)$ saddle

the line $y = 2$ is invariant, the line $x = 1$ is invariant

and solutions on $y = 2$ tend towards equilibrium

solutions on $x = 1$ tend away



8: Sketch the phase portraits for the following systems. Determine if the system is Hamiltonian or gradient along the way.

(b) $x' = y^2 + 2xy$
$y' = x^2 + 2xy$

$x' = 0$

$y^2 = -2xy \equiv -2x = y$ (if $y \neq 0$)

or $y = 0$

$y' = 0$

$x^2 = -2xy \equiv -2y = x$ (if $x \neq 0$)

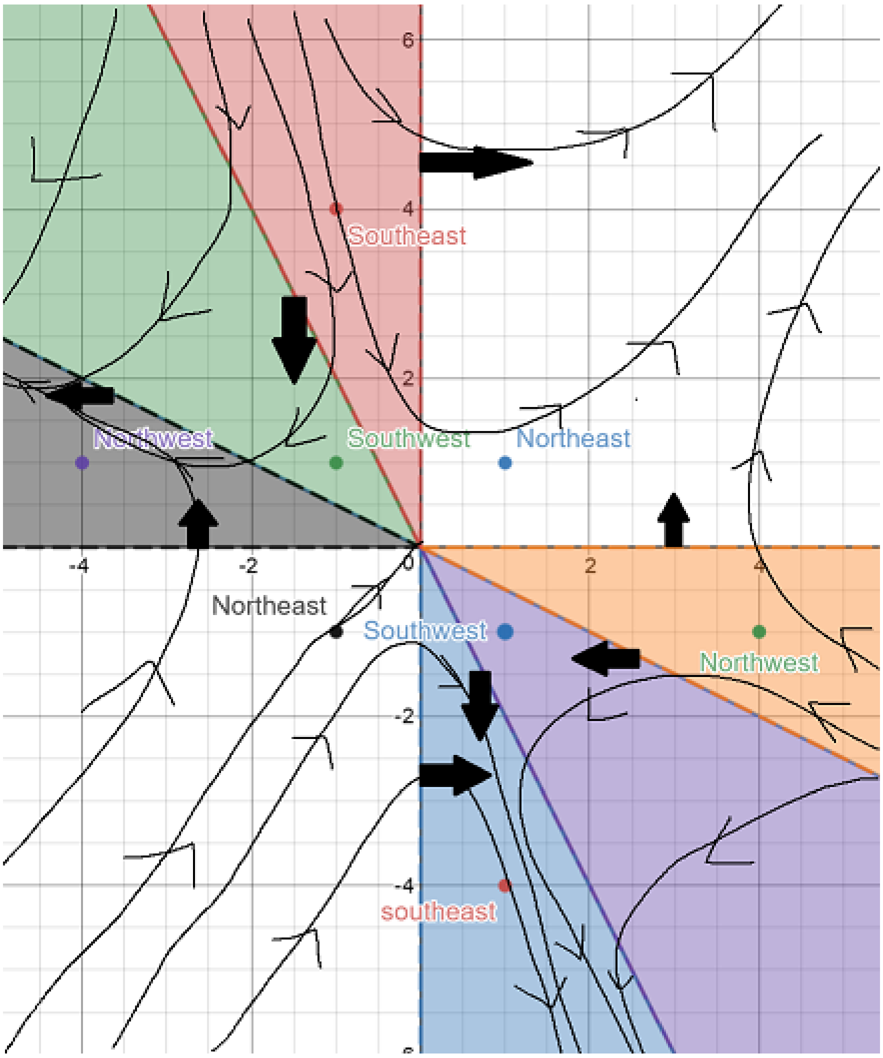
or $x = 0$

the only point these two lines intersect at is $(0, 0)$

8 basic regions: 1st quadrant

$-y/2 < x < 0$ $-x/2 < y < -2x$, $0 < y < -x/2$

3rd quadrant, $0 < x < -y/2$ $-2x < y < -x/2$ $-x/2 < y < 0$



All solutions either tend towards infinity or towards $(0, 0)$

If gradient then:

$\frac{\partial V}{\partial x} = -y^2 - 2xy \implies V(x, y) = -y^2x - x^2y + C(y)$

$\frac{\partial V}{\partial y} = -x^2 - 2xy = -2xy - x^2 + C'(y)$

$\implies C(y) =$ real constant and $V(x, y) = -y^2x - x^2y + C$

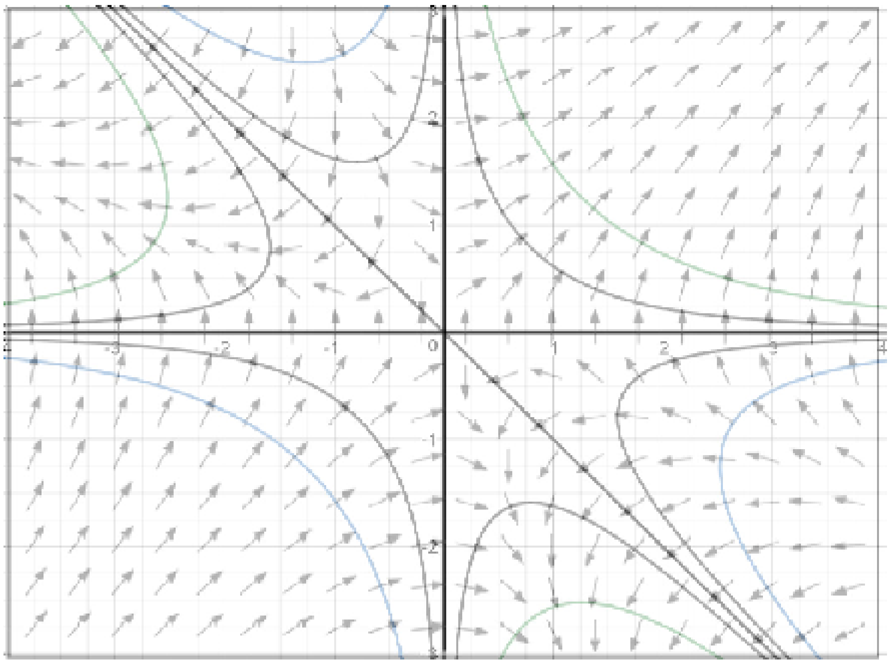
and along the level curve $-y^2x - x^2y = 0$, which is the union of the $y-$axis, the x-axis, and $y = -x$

on $y = -x$, $x' = y^2 - 2y^2 = -y^2$, so $x' = y'$ and we have the solutions on $y = -x$ going perpendicular to

$y' = y^2 - 2y^2 = -y^2$

$y = -x$

and similarly for other level curves:



If Hamiltonian

$\frac{\partial H}{\partial y} = y^2 + 2xy \implies H(x,y) = \frac{y^3}{3} + xy^2 + C(x)$

$-y^2 - C'(x) = x^2 + 2xy$, which implies that $C(x)$ is a function of x and y

so this is a gradient system with $V = -y^2x - x^2y + C$

9: Let $X' = AX$ be a linear system where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(a) Determine conditions on a, b, c and d that guarantee that this system is a gradient system. Give a gradient function explicitly

$x' = ax + by$
$y' = cx + dy$

$\frac{\partial V}{\partial x} = -ax - by \; \frac{\partial V}{\partial y} = -cx - dy$

$V(x,y) = \frac{-ax^2}{2} - bxy + C(y)$

$-bx + C'(y) = -cx - dy$

so $b = c$, $C'(y) = -dy$, so $C(y) = -\frac{dy^2}{2}$

$V(x,y) = \frac{-ax^2}{2} - bxy - \frac{dy^2}{2} + C$, with $b = c$, which makes sense since the linearized system needs to be a symmetric matrix

(b) Repeat the previous question for a Hamiltonian system

$\frac{\partial H}{\partial y} = ax + by$, $\frac{\partial H}{\partial x} = -cx - dy$

$H(x,y) = axy + \frac{by^2}{2} + C(x)$

$ay + C'(x) = -cx - dy$

$a = -d$, $C'(x) = -cx \implies C(x) = -cx^2/2$

$H(x,y) = axy + \frac{by^2}{2} - \frac{cx^2}{2}$ where $a = -d$

which makes sense since eigenvalues are of the form $\pm\lambda$ or $\pm\lambda i$

and for that to happen we'd need $a + d = 0$

10. Consider the planar system $x' = f(x,y) y' = g(x,y)$.

Determine the explicit conditions on f and g that guarantee that this system is a gradient system or a Hamiltonian system.

Gradient System

There needs to be a function $V(x,y)$ s.t.

$\frac{\partial V}{\partial x} = -f(x,y)$ and $\frac{\partial V}{\partial y} = -g(x,y)$

For a gradient system, the mixed partial derivatives of $V(x,y)$ are equal (it is C^∞) so:

$\frac{\partial f}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$

$\frac{\partial f}{\partial y}(X^*) = \frac{\partial g}{\partial x}(X^*)$

So if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, we have a gradient system

also: f must be continuously differentiable and g must be continuously differentiable (need to be C^∞ since V must be C^∞)

if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$, then $f(x,y), g(x,y)$ are functions of x and y respectively and we rewrite

$f(x,y)$ as $f(x)$ and similarly for $g(x,y$)

then we may set $C'(y) = g(y)$ and therefore: $C(y) = \int_0^y g(t)dt$

$V(x,y) = -\int_0^x f(t)dt - \int_0^y g(t)dt$ is a function whose associated gradient system is the above.

Otherwise,

we let $V(x,y) = -\int f(x,y)dx$ (negative the integral of f w.r.t. to x)

$\frac{\partial V}{\partial x} = -f(x,y)$ by FTC (the partial deriv. of V w.r.t. to x is -f)

and $\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}[\int f(x,y)dx]$ (the partial deriv. of V w.r.t. to y is negative the integral of f w.r.t. to x)

$\frac{\partial V}{\partial y} = -\int \frac{\partial f}{\partial y}(x,y)dx$ (can switch the order of differentiation and integration since f is continuously differentiable)

$\frac{\partial V}{\partial y} = -\int \frac{\partial g}{\partial x}(x,y)dx$ (replacing partial deriv. f w.r.t. y with partial deriv. g w.r.t. to x from hypothesis)

$\frac{\partial V}{\partial y} = -g(x,y)$ (we obtain partial deriv. of V w.r.t. to y is -g)

So we have that there exists a function $V(x,y)$ s.t. $\frac{\partial V}{\partial x} = -f(x,y)$, $\frac{\partial V}{\partial y} = -g(x,y)$ and the system above is a gradient system for V

For a Hamiltonian System

The linearized system at an equilibrium point X^*

$$\begin{pmatrix} \frac{\partial f}{\partial x}(X^*) & \frac{\partial f}{\partial y}(X^*) \\ \frac{\partial g}{\partial x}(X^*) & \frac{\partial g}{\partial y}(X^*) \end{pmatrix}$$

Since Hamiltonian systems have eigenvalues of $\pm\lambda$ or $\pm i\lambda$

we need $\frac{\partial f}{\partial x}(X^*) + \frac{\partial g}{\partial y}(X^*) = 0$

so $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$

$f(x,y) = -\int_0^x \frac{\partial g}{\partial y}(t,y)dt$

$g(x,y) = -\int_0^y \frac{\partial f}{\partial x}(x,t)dt$

So we have a Hamiltonian system if f, g are C^∞ and $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$

To see this let: $H(x,y) = \int f(x,y)dy$

$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}[\int f(x,y)dy] = \int \frac{\partial f}{\partial x}(x,y)dy$ since f is C^∞

$= \int -\frac{\partial g}{\partial y}(x,y)dy = -g(x,y)$

which means $x' = \frac{\partial H}{\partial y}, y' = -\frac{\partial H}{\partial x}$, the definition of a Hamiltonian system

11. Prove that the linearization at an equilibrium point of a planar Hamiltonian system has eigenvalues that are either $\pm\lambda$ or $\pm i\lambda$ where $\lambda \in \mathbb{R}$

the linearization:

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y}(X^*) & \frac{\partial^2 H}{\partial y^2}(X^*) \\ -\frac{\partial^2 H}{\partial x^2}(X^*) & -\frac{\partial^2 H}{\partial y \partial x}(X^*) \end{pmatrix}$$

and since $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$

so $T = \frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 H}{\partial y \partial x} = 0$

so $\implies$ the eigenvalues $\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

so $\lambda_{1,2} = \pm\frac{\sqrt{-4D}}{2}$, where if $D < 0$, λ's are real otherwise pure imaginary

16. A solution $X(t)$ of a system is called recurrent if $X(t_n) \to X(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solutions.

So need to show if $X(t)$ is recurrent, it must be constant

so let $X(t)$ be a solution such that

$\lim_{n\to\infty} X(t_n) = X(0) = X_0$ for some sequence $t_n \to \infty$

Using the definition: ω-limit points of a given solutions are points Z such that

$\lim_{n\to\infty} X(t_n) = Z$ for some sequence $t_n \to \infty$

so X_0 is an ω-limit point

and from the proposition at the bottom page 204, X_0 is an equilibrium point and since the ω-set is invariant

$\phi_t(X_0)$ remains in ω-set for all t, which means $\phi_t(X_0)$ is an equilibrium point for all t since ω-limits in gradient systems are equilibrium points

which means $\phi_t(X_0)$ is constant for all t