Renee Senining

Math 123

# Homework #11

## 2, 10, 11, 15, 16

**2. Consider the three-dimensional system** $r' = r(1-r)$
$\theta' = 1$
$z' = -z$

Compute the Poincare map along the closed orbit lying on the unit circle given by $r = 1$ and show that this is asymptotically stable

---

if $z > 0$, $z' < 0$,

and if $z < 0$, $z' > 0$

and if $z = 0, z' = 0$

so $z \to 0$

the closed orbit is given by $(\cos t, \sin t, 0)$, with initial condition $(1, 0, 0)$

There is a local section lying along the half $xz - plane$ with $x > 0$,

since $\theta' = 1$

given any $x \in (0, \infty)$, $z \in (-\infty, \infty)$

$\theta_{2\pi}(x, 0, z)$ also lies in the half plane as described above.

so we have a Poincare map $P : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$

$P(1, 0) = (1, 0)$ since the point $x = 1, y = 0, z = 0$ is the initial condition for the periodic solution

Finding $P$:

compute the solution starting at $(x_0, 0, z_0)$

$\theta(t) = t$

$\int \frac{dr}{r(1-r)} = t + C$

$\frac{1}{r(1-r)} = \frac{1}{r} + \frac{1}{1-r}$

so $\int \frac{dr}{r(1-r)} = \log(r) - \log(1-r) = t + C$

$\equiv \log(\frac{r}{1-r}) = t + C$

$\frac{r}{1-r} = Ce^t$

$r = Ce^t - rCe^t$

$r(1 + Ce^t) = Ce^t$

$r = \frac{Ce^t}{1+Ce^t}$

and $r(0) = \frac{C}{1+C} = x_0$

so $C = x_0(1+C)$

$C(1-x_0) = x_0 \implies C = \frac{x_0}{1-x_0}$

$r = \frac{x_0 e^t}{1-x_0+x_0 e^t}$

and $z = z_0 e^{-t}$

$r(2\pi) = \frac{x_0 e^{2\pi}}{1-x_0+x_0 e^{2\pi}}$

$z(2\pi) = z_0 e^{-2\pi}$

so $P(x_0, z_0) = (r(2\pi), z(2\pi))$

$\frac{d}{dx}[r(2\pi)] = \frac{e^{2\pi}(1-x+xe^{2\pi})-(e^{2\pi}-1)xe^{2\pi}}{(1-x+xe^{2\pi})^2}$

$= \frac{e^{2\pi}}{(1-x+xe^{2\pi})^2}$

$\frac{d}{dz}z(2\pi) = e^{-2\pi}$

$\nabla P(1,0) = (e^{-2\pi}, e^{-2\pi})$

the max value of $P'(1,0)$

$||\nabla P|| = 2e^{-4\pi} < 1$

so the closed orbit is AS by the proposition 219

**10. Show that a closed orbit of a planar system meets a local section in at most one point**

Suppose $Y_1$ and $Y_2$ are distinct points on the closed orbit, $\gamma$

and $\mathcal{S}$ is a local section containing $Y_1$ and $Y_2$

we have that a closed orbit is the $\omega$-limit set for every point on it

so for any $X \in \gamma$, $\omega(X)$ contains $Y_1$ and $Y_2$

let $\mathcal{V}_k$ be the flow boxes at $Y_k$, which are defined by disjoint intervals $\mathcal{J}_k$ in the local section $\mathcal{S}$

(since $Y_1, Y_2$ are disjoint, we can find disjoint neighborhoods around these two points that contain disjoint segments of $\mathcal{S}$)

Since closed orbits are periodic,

solutions on the closed orbit enter each flow box infinitely often, crossing $\mathcal{J}_k$ infinitely

so there exists a sequence $a_1, b_1, a_2, b_2, a_3, b_3$ that is monotone along the solution through $X$ with $a_n \in \mathcal{J}_1$ and $b_n \in \mathcal{J}_2$ for $n \in \mathbb{N}$

but this sequence isn't monotone along $\mathcal{S}$ since $\mathcal{J}_1, \mathcal{J}_2$ are disjoint

so this contradicts the proposition on 221

**11. Show that a closed and bounded limit set is connected (that is, not the union of two disjoint nonempty closed sets)**

so consider the limit set $\omega(X)$ for a solution curve passing through $X$

Suppose $\omega(X)$ is disconnected

let $S_1$ and $S_2$ be disjoint open sets s.t. $\omega(X) \subset S_1 \cup S_2$

and $\omega(X) \cap S_k$ is nonempty $k = 1, 2$

Let $Y_1 \in S_1$ and $Y_2 \in S_2$

so there exists a sequence of $t_n \to \infty$ such that

$\lim_{n\to\infty} \phi_{t_n}(X) = Y_1$

and a sequence $s_n \to \infty$ such that

$\lim_{n\to\infty} \phi_{s_n}(X) = Y_2$

and $t_n < s_n < t_{n+1}$ (possibly for some subsequence)

since: for each $1/n$, $n \in \mathbb{N}$

we can find indices $N_1(n), N_2(n)$

s.t. for all $n \geq N_1(n)$, $|\phi_{t_n}(X) - Y_1| < 1/n$

for all $n \geq N_2(n)$, $|\phi_{s_n}(X) - Y_2| < 1/n$

and defining subsequences:

for each $k$: let $t_k = N_1(k)$ , and $s_k = N_2(k)$ if $N_2(k) > N_1(k)$ or let $s_k =$ some integer $> N_1(k) \geq N_2(k)$, then let $t_{k+1} =$ an integer greater than $s_k$ and greater than or equal to $N_1(k+1)$. We continue like this and obtain:

1. $|\phi_{t_k}(X) - Y_1| < 1/k$, $|\phi_{s_k}(X) - Y_2| < 1/k$, so as $k \to \infty$, $t_k, s_k \to \infty$ $\phi_{t_k}(X) \to Y_1$, $\phi_{s_k}(X) \to Y_2$
2. $t_k < s_k < t_{k+1}$

---

there exists an $N_1 \in \mathbb{N}$ s.t.

for all $n \geq N_1$, $\phi_{t_n}(X) \in \mathcal{S}_1$ since $\phi_{t_n}(X) \to Y_1$, and $S_1$ is open and $Y_1 \in S_1$ (i.e. there is an open ball centered on $Y_1$ contained in $S_1$ and since the terms of $\phi_{t_n}$ get arbitrarily close to $Y_1$, there are infinitely many terms of $\phi_{t_n}$ contained in any ball centered on $Y_1$)

and similarly, there exists an $N_2 \in \mathbb{N}$ s.t.

for all $n \geq N_2$, $\phi_{s_n}(X) \in \mathcal{S}_2$

so let $N_0 = \max(N_1, N_2)$

for all $n \geq N_0$,

$\phi_t(X)$ with $t \in (t_n, s_n)$ is the part of the solution curve connecting a point in $S_1$and a point in $S_2$, so $\exists r_n \in (t_n, s_n)$ s.t. $\phi_{r_n}(X) \notin S_1 \cup S_2$ (since they are disjoint)

and so we have a sequence $r_n \to \infty$ as $n \to \infty$

if $\phi_{r_n}(X)$ converges:

$\lim_{r_n\to\infty} \phi_{r_n}(X) = Z \in \omega(X)$

so for each open ball around $Z$, there are infinitely many terms of $\phi_{r_n}(X)$ in it

but since each $\phi_{r_n}(X) \notin S_1 \cup S_2$ and there exists an open ball centered on $Z$ fully contained in $S_1 \cup S_2$ since $\omega(X) \subset S_1 \cup S_2$ we have a contradiction since we would need infinitely many terms of $\phi_{r_n}(X)$ to be in this ball and therefore in $S_1 \cup S_2$

or $\lim_{r_n\to\infty} \phi_{r_n}(X)$ doesn't have a limit

then it is possible that

$\lim_{r_n\to\infty} \phi_{r_n}(X) \to \infty$, which means $\omega(X)$ is unbounded, a contradiction

or if the set $\{\phi_{r_n}(X)\}$ is bounded, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence, with a limit in $\omega(X)$

and we still have that each term is not in $S_1 \cup S_2$, but there are infinite terms in any open ball centered on the limit, but there is an open ball contained in $S_1 \cup S_2$, therefore we have a contradiction

(a similar argument works for $\alpha(X)$, we simply consider limits with $t_n, s_n \to -\infty$ )

**15. Let $X$ be a recurrent point of a planar system; that is, there is a sequence $t_n \to \pm\infty$ such that $\phi_{t_n}(X) \to X$**

**(a) Prove that either $X$ is an equilibrium or $X$ lies on a closed orbit**

so $X \in \omega(X)$ and in $\alpha(X)$

and $\omega(X)$ is closed and invariant, which means $\phi_t(X) \in \omega(X)$ , $\alpha(X)$ for all $t \in \mathbb{R}$

so for any $Y$ lying on the solution curve through $X$,

$\omega(Y) = \omega(X)$, $\alpha(Y) = \alpha(Y) = \alpha(X)$, which means

$\exists s_n \to \pm\infty$ such that $\phi_{s_n}(Y) \to Y$

Assuming that there exists a point $Y$ on the solution curve with $Y \neq X$, i.e., $X$ is not an equilibrium:

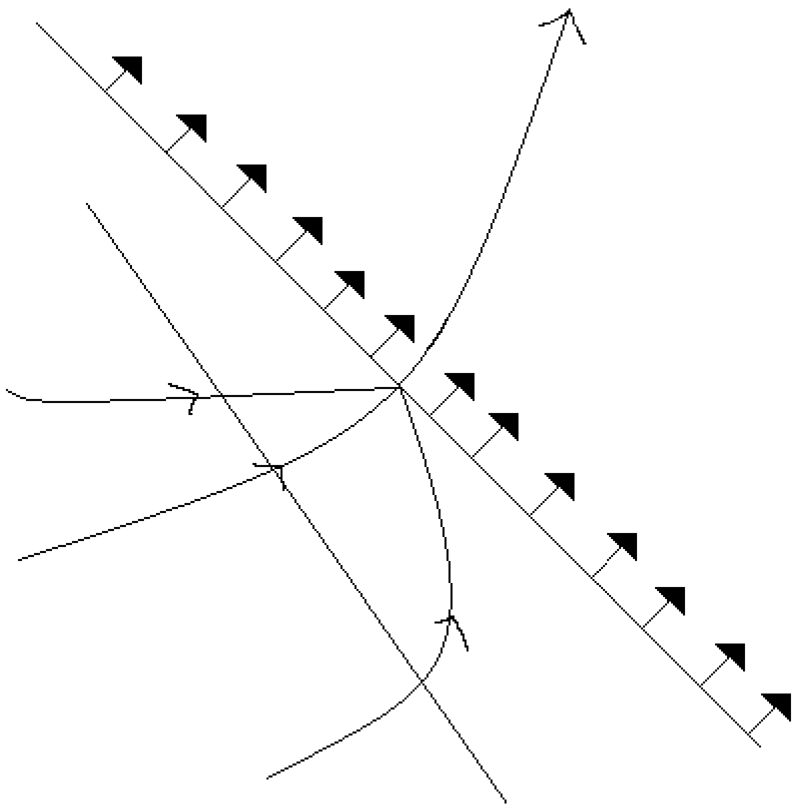
Take any $Y$ on the solution curve and consider the local section $\mathcal{S}$ at $Y$.

since $Y$ is a limit point of the solution through $X$, there are infinitely many points in time $s_n$ such that $\phi_{s_n}(X) \in$ the flow box $\mathcal{V}$. In particular, if $\phi_\tau(X) = Y$ at some time $\tau$, then there is some $s_N > \tau$ (if we're consider $s_N \to \infty$) or $s_N < \tau$, (if we're considering $s_N \to -\infty$

such that for all $n \geq N$, $\phi_{s_n}(X) \in \mathcal{V}$, which means that at infinitely many points in time, the solution curve goes into the flow box and crosses the local section at $Y$

since the entire solution curve is in $\omega(X)$, we have that the solution through $X$ crosses any local section at no more than one point, so the solution curve goes back through $Y$ infinitely many times for any point $Y$.

Note, it isn't that multiple paths of the solution curve converging to cross through $Y$ else we'd have a local section such that the solution crosses it multiple times.



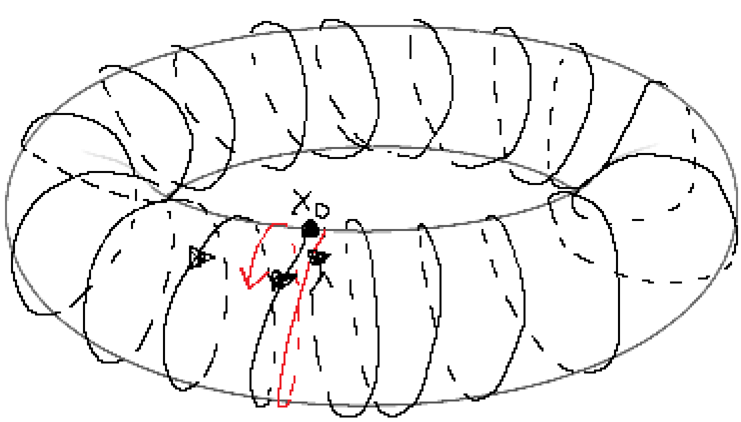
So, because the solution returns to each point more than once, there exists a time $t > 0$ s.t. $\phi_t(Y) = Y$

which means for any $s$, $\phi_s(Y)$, we have $\phi_{s+t}(Y) = \phi_s(Y)$

so the solution is a periodic function and thus a closed orbit

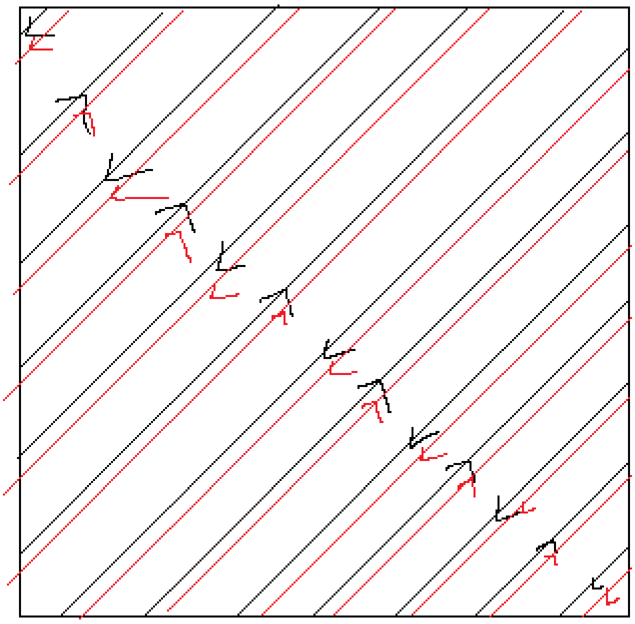
**(b) Show by example that there can be a recurrent point for a nonplanar system that is not an equilibrium and does not lie on a closed orbit.**

all points of the torus $T^2$ are recurrent with respect to an irrational flow:



The orbits are "dense", but not periodic.

projected on a two dimensional square:



where black lines are a first "lap," for lack of a better word, around and red lines are a second.

the system itself looks like

$\frac{d\theta_n}{dt} = \omega_n$, for $n = 1, 2, ..$

**16. Let $X' = F(X)$ and $X' = G(X)$ be planar systems. Suppose that $F(X) \cdot G(X) = 0$ for all $X \in \mathbb{R}^2$. If $F$ has a closed orbit, prove that $G$ has an equilibrium point**

so $F(X) = (f_1(X), \ldots, f_n(X))$, $G(X) = (g_1(X), \ldots, g_n(X))$

$F(X) \cdot G(X) = f_1(X)g_1(X) + \ldots + f_n(X)g_n(X) = 0$

so $F(X) \cdot G(X) = ||F(X)||||G(X)|| \cos(\theta) = 0$

Assuming $F$ has a closed orbit $\gamma$

at each $X \in \gamma$, $F(X) \neq 0$ so we must have $G(X) = 0$ or $\theta = \frac{\pi}{2}$

if $G(X) = 0$ for some point, $G$ has an equilibrium on the closed path and we are done.

Otherwise: Suppose the angle between $F(X)$ and $G(X)$ is $\frac{\pi}{2}$ for all $X \in \gamma$ and $G(X) \neq 0$

this means that solutions of $G(X)$ emanate from or flow into $\gamma$

$\gamma$ can be divided into two sets, since $G(X)$ is never tangent with $\gamma$:

1. points on $\gamma$ such that $G(X)$ points out
2. points on $\gamma$ such that $G(X)$ points in

Suppose both are nonempty:

On a point $X$ of the boundary of either one, we must have that any neighborhood of $X$ must contain at least one point of each set.

We consider a local section of $X$ and its flow box $\mathcal{V}$: (associated with $G$)

This vector $G(X)$ either points out from $\gamma$ or into $\gamma$, since the angle between $\gamma$ at $X$ and $G(X)$ is $\pi/2$, and so for each point $Y$ in the local section, $G(Y)$ points out from $\gamma$ if $G(X)$ points out or points in if $G(X)$ points in

a part of $\gamma$ is contained in the flow box $\mathcal{V}$, since $\gamma$ is continuous and $\mathcal{V}$ is a neighborhood of $X$, so each point of $\gamma$ in the flow box must have its vector associated with $G$ going in the same direction as $G(X)$, so there is a neighborhood of $X$ s.t. all vectors of $\gamma$ associated with $G$ point in the same direction (out or in $\gamma$). This is a contradiction, so we must have that all vectors associated with $G$ on $\gamma$ point in one direction: in or out of $\gamma$

Let $K$ be a set containing the region bounded by $\gamma$ (assuming $\gamma$ is simple closed) and $\gamma$

So we either have that solutions to $X' = G(X)$ stay in $K$ or exit $K$, making $K$ positively or negative

so $K$ contains either a limit cycle or an equilibrium point for $G(X)$ (Corollary 2)

If $K$ contains an equilibrium point, we are done.

Otherwise $K$ contains a limit cycle, $\beta$ for $X' = G(X)$, and by Corollary 4, there is an equilibrium point in the region bounded by the limit cycle