Renee Senining

Math 185

Homework #12

Chapter VII, Sec. 1, ex. 1e, 1g, 1h, 3b, 3e; Sec. 2, ex. 5; Sec. 3. ex. 4; Sec. 4, ex. 8

Sec. 1

Ex. 1e $\mathrm{Res}[\frac{\cos z}{z^2}, 0]$

$\frac{\cos z}{z^2} = \frac{1}{z^2} - 1 + z^2 - z^4 + \cdots$

$a_{-1} = 0$

also rule 2:

$\frac{d}{dz}[z^2 * \cos z/z^2] = -\sin z$, which $= 0$ at $z = 0$

Ex. 1g $\mathrm{Res}[\frac{z}{\mathrm{Log}\ z}, 1]$

Can use Rule 3:

$\mathrm{Log}\ z$ has a simple zero at $z = 1$ since

$\frac{d}{dz}[\mathrm{Log}\ z] = \frac{1}{z}$ and at $z = 1$ is equal to 1

so $f(z) = z$, $g(z) = \mathrm{Log}\ z$

$f(z)/g'(z) = \frac{z}{1/z} = z^2$ and at $z = 1$ is equal to 1

Ex. 1h $\mathrm{Res}[\frac{e^z}{z^5}, 0]$

$\frac{e^z}{z^5} = \frac{1}{z^5}\sum_{k=0}^{\infty}\frac{z^k}{k!} = \sum_{k=-5}^{\infty}\frac{z^k}{(k+5)!}$

and at $k = -1$, $a_{-1} = \frac{1}{4!} = \frac{1}{24}$

Ex. 3b

$\oint_{|z|=2}\frac{e^z}{z^2-1}dz = \oint_{|z|=2}\frac{e^z}{(z+1)(z-1)}dz$

Using Residue Theorem:

We have isolated singularities: $z = \pm 1$

We may use Rule 3 for both:

$f(z) = e^z, g(z) = (z+1)(z-1)$, since $g(z)$ has simple zeros at ± 1 and both are analytic over the entire complex plane

$g'(z) = 2z$

$\mathrm{Res}[\frac{e^z}{z^2-1}, 1] = \frac{e}{2}$

$\text{Res}[\frac{e^z}{z^2-1}, -1] = -\frac{e^{-1}}{2}$

so $\oint_{|z|=2} \frac{e^z}{z^2-1} dz = 2\pi i(\frac{e^1-e^{-1}}{2}) = 2\pi i \sinh(1)$

or just $\pi i(e^1 - e^{-1})$

Ex. 3e

$\oint_{|z-1|=1} \frac{1}{z^8-1} dz$

$z^8 - 1 = 0$ for $z^8 = 1$

at the 8 roots of unity

however, we only need to consider the singularities in $|z-1| \leq 1$, a disk that is fully contained in the right half-plane except at $z = 0$

so only the roots of unity in the right half-plane should be considered:

$e^{-i\pi/4}, 1, e^{i\pi/4}$

$z^8 - 1 = \prod_{k=0}^{7} (z - e^{i\pi k/4})$

so $g(z) = z^8 - 1$ has simple zeros at each root, we may use rule 4

$g'(z) = 8z^7$

$g'(e^{i\pi k/4}) = 8e^{i7\pi k/4}$

the integral is equal to

$2\pi i \sum_{k=-1}^{1} \frac{1}{8e^{i7\pi k/4}}$

$= \frac{\pi i}{4}[\frac{1}{e^{-7\pi/4}} + 1 + \frac{1}{e^{7\pi/4}}]$

$\frac{7\pi}{4} = 2\pi - \pi/4 \equiv -\pi/4$

$-\frac{7\pi}{4} = -2\pi + \pi/4 \equiv \pi/4$

so $= \frac{\pi i}{4}[e^{-\pi/4} + 1 + e^{-(-\pi/4)}]$

and $e^{-\pi/4} = \frac{1-i}{\sqrt{2}}$, $e^{\pi/4} = \frac{1+i}{\sqrt{2}}$

$= \frac{\pi i}{4}[1 + \frac{2}{\sqrt{2}}] = \frac{\pi i}{4}[1 + \sqrt{2}]$

Sec. 2

Ex. 5 Using the residue theory, show that $\int_0^\infty \frac{x^2}{x^4+1} dx = \frac{\pi}{2\sqrt{2}}$

Using (2.4) on page 200:

the poles of $z^2/(z^4+1)$ in the upper half-plane:

$z^4 = -1$

$e^{i(\pi+2\pi k)/4} = e^{i\pi/4+\pi k/2}$

$e^{i\pi/4}$, $e^{i3\pi/4}$ are the (simple) poles in the upper half-plane

Finding the residue at each of these poles

using Rule 3:

with $g(z) = z^4 + 1$, we have $g'(z) = 4z^3$

we have $f(z)/g(z) = z^2/4z^3 = 1/4z$

and so $\frac{f(e^{i\pi/4})}{g'(e^{i\pi/4})} = \frac{1}{4z}|_{z=e^{i\pi/4}} = \frac{\sqrt{2}}{4(1+i)}$

and at $z = e^{i3\pi/4}$

$\frac{f(e^{i3\pi/4})}{g(e^{i3\pi/4})} = \frac{1}{4z}|_{z=e^{i3\pi/4}} = \frac{\sqrt{2}}{4(-1+i)}$

so $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx =$

$2\pi i(\frac{\sqrt{2}(1+i-1+i)}{4(-2)}) = 2\pi(\frac{\sqrt{2}(2)}{8}) = \frac{\pi}{\sqrt{2}}$

since $\frac{x^2}{x^4+1}$ is an even function, it is symmetric about the x-axis, and we have

$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = 2\int_0^{\infty} \frac{x^2}{x^4+1} dx$

so we have $\int_0^{\infty} \frac{x^2}{x^4+1} = \frac{\pi}{2\sqrt{2}}$ as desired

Sec. 3

Ex. 4 Show using residue theory that $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}$

$z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$\sin^2\theta = (\frac{z-1/z}{2i})^2 = \frac{z^2-2+1/z^2}{-4}$

$1 + \sin^2\theta = 1 - \frac{z^2-2+1/z^2}{4} = \frac{4-z^2+2-1/z^2}{4} =- \frac{z^2-6+1/z^2}{4}$

$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \oint_{|z|=1} \frac{4}{-z^2+6-1/z^2} \frac{dz}{iz}$

$= \frac{4}{i} \oint_{|z|=1} \frac{dz}{-z^3+6z-1/z}$

$= \frac{4}{i} \oint_{|z|=1} \frac{z}{-z^4+6z^2-1} dz$

$-z^4 + 6z^2 - 1 = 0$

$w = z^2$

$-w^2 + 6w - 1 = 0$

$w^2 - 6w + 1$

$\frac{6\pm\sqrt{36-4}}{2} = 3 \pm \sqrt{8}$

$z^2 = 3 \pm \sqrt{8}$

roots:

zeros are at

$z = \pm\sqrt{2} \pm 1$

with $-\sqrt{2}+1$, $\sqrt{2}-1$ being the only roots inside the unit circle

each of these roots are simple poles

Using rule 3:

$\frac{f(z)}{g'(z)} = \frac{z}{-4z^3+12z} = \frac{1}{-4z^2+12}$

so at $-\sqrt{2}+1$ and $\sqrt{2}-1$

$(-\sqrt{2}+1)^2 = 2-2\sqrt{2}+1 = 3-2\sqrt{2} = (\sqrt{2}-1)^2$

The residues are $\frac{1}{8\sqrt{2}}$

so the integral is

$= \frac{4}{i} 2\pi i (\frac{2}{8\sqrt{2}}) = \pi\sqrt{2}$

Sec. 4

Ex. 8

By integrating a branch of $(\log z)/(z^3+1)$ around the boundary of an indented sector of aperture $2\pi/3$, show that

$\int_0^\infty \frac{\log x}{x^3+1} dx = \frac{-2\pi^2}{27}$, $\int_0^\infty \frac{1}{x^3+1} dx = \frac{2\pi}{3\sqrt{3}}$

$\log z = \frac{\log|z| + i \arg z}{(z-e^{i\pi/3}(z-e^{i\pi})(z-e^{i5\pi/3}))}$

the sector is from $0 < \theta < 2\pi/3$ and we consider the branch cut : $\mathbb{C} \setminus (-\infty, 0]$

The only pole contained in this sector is at $e^{i\pi/3}$ (is a simple pole)

$\text{Res}[\frac{\log|z| + i \arg z}{z^3+1}, e^{\pi i/3}] = \frac{\log|z| + i \arg z}{3z^2}|_{z=e^{\pi i/3}} = \frac{\log(1) + i\pi/3}{3e^{2\pi i/3}} = \frac{i\pi}{9} e^{-2\pi i/3}$

we have four integrals:

γ_1 :one from ϵ to R along the real axis,

$\int_\epsilon^R \frac{\log x}{x^3+1} dx$

Γ_R is the path along the circle of radius $R > 0$ from $(R, 0)$ to $Re^{i2\pi/3}$, $(-\frac{R}{2}, \frac{\sqrt{3}R}{2})$, going counterclockwise

γ_2 from $(-\frac{R}{2}, \frac{\sqrt{3}R}{2})$ to $(-\frac{\epsilon}{2}, \frac{\sqrt{3}\epsilon}{2})$

γ_ϵ is the path along the circle of radius $\epsilon > 0$ (small) from $(-\frac{\epsilon}{2}, \frac{\sqrt{3}\epsilon}{2})$ $(\epsilon e^{i2\pi/3})$ clockwise to $(\epsilon, 0)$

$|\log|z| + i \arg z| \leq \sqrt{\log^2 R + (\frac{2\pi}{3})^2}$

$z^3 + 1 \leq R^3 + 1$

and the length of Γ_R is $2\pi R/3$

$|\int_{\Gamma_R} \frac{\log|z| + i \arg z}{z^3+1} dz| \leq \frac{\sqrt{\log^2 R + (\frac{2\pi}{3})^2}}{R^3-1} \cdot \frac{2\pi R}{3} \sim \frac{2\pi \log R}{3R^2} \to 0$ as $R \to \infty$

and along γ_2 (straight line, so can treat "like" the real line)

letting $z = xe^{2\pi i/3}$ and $dz = e^{2\pi i/3}dx$

$\int_R^\epsilon \frac{\log x + 2\pi i/3}{x^3+1} e^{2\pi i/3} dx$

$= -e^{2\pi i/3} \int_\epsilon^R \frac{\log x + 2\pi i/3}{x^3+1} dx$

$= -e^{2\pi i/3} [\int_\epsilon^R \frac{\log x}{x^3+1} dx + \frac{2\pi i}{3} \int_\epsilon^R \frac{1}{x^3+1} dx]$

and along γ_ϵ :

$| \int_{\gamma_\epsilon} \frac{\log|z| + i \arg z}{z^3+1} dz| \leq \frac{\sqrt{\log^2 \epsilon + (\frac{2\pi}{3})^2}}{1-\epsilon^3} \cdot \frac{2\pi\epsilon}{3} \sim \frac{2\pi\epsilon |\log \epsilon|}{3}$ and as $\epsilon \to 0^+$

$\epsilon \log \epsilon = \frac{\log \epsilon}{1/\epsilon}$

using L'hospital's

$\frac{1/\epsilon}{-1/\epsilon^2} = \frac{1}{-1/\epsilon} \to 0$ as $\epsilon \to 0^+$

so we have

$(1 - e^{2\pi i/3}) \int_\epsilon^R \frac{\log x}{x^3+1} dx - \frac{2\pi i e^{2\pi i/3}}{3} \int_\epsilon^R \frac{1}{x^3+1} dx = 2\pi i \frac{i\pi}{9} e^{-2\pi i/3}$

multiplying both sides by $e^{-\pi i/3}$ we have:

(using $(e^{-\pi i/3} - e^{\pi i/3}) = 2i \sin(\pi/3)$)

$-2i \sin(\pi/3) \int_\epsilon^R \frac{\log x}{x^3+1} dx - \frac{2\pi i e^{\pi i/3}}{3} \int_\epsilon^R \frac{1}{x^3+1} dx = \frac{2\pi^2}{9}$

$\equiv$

$-\sqrt{3}i \int_\epsilon^R \frac{\log x}{x^3+1} - \frac{\pi i}{3} \int_\epsilon^R \frac{1}{x^3+1} dx + \frac{\pi\sqrt{3}}{3} \int_\epsilon^R \frac{1}{x^3+1} dx = \frac{2\pi^2}{9}$

so

$\int_\epsilon^R \frac{1}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$

and

$-\sqrt{3}i \int_\epsilon^R \frac{\log x}{x^3+1} - \frac{\pi i}{3} \int_\epsilon^R \frac{1}{x^3+1} dx = 0$

so $\int_\epsilon^R \frac{\log x}{x^3+1} = \frac{\pi}{3} \frac{2\pi}{3\sqrt{3}} \frac{-1}{\sqrt{3}} = -\frac{2\pi^2}{27}$