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Math 123

Homework #4

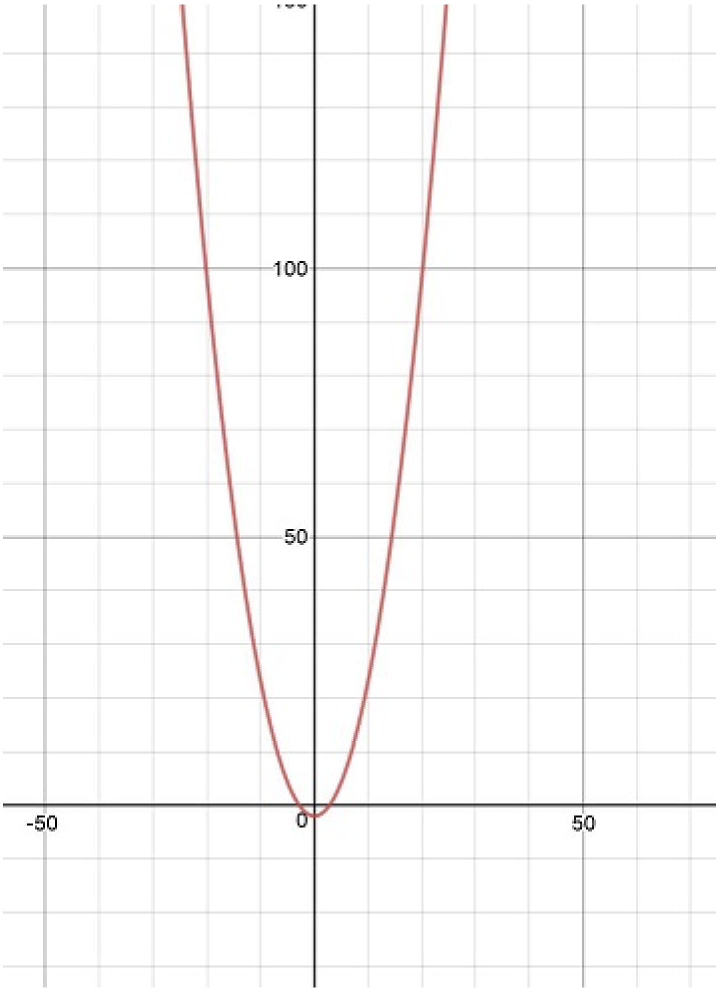
Ex. 1: Consider the one-parameter family of linear systems given by

$$X' = \begin{pmatrix} a & \sqrt{2} + (a/2) \\ \sqrt{2} - (a/2) & 0 \end{pmatrix} X$$

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies

Trace = a, Determinant = $-(2 - a^2/4) = a^2/4 - 2$ = $T^2/4 - 2$

so treating trace as x and determinant as y:



(b) Discuss any bifurcations that occur along this path and compute the corresponding values of a

since $D = T^2/4 - 2$

$D < T^2$

So the graph never crosses into complex eigenvalues.

and $T^2 - 4D = 8$, so we never get real repeated eigenvalues

When $a^2 > 8$, $a > \sqrt{8}$ or $a < -\sqrt{8}$

if $a < -\sqrt{8}$ a sink

If $a > \sqrt{8}$ a source

When $a^2 < 8$

we have a saddle, since the determinant < 0, so $\sqrt{T^2 - 4D} > |T|$

we have that $a + \sqrt{a^2 - 4D}$ is positive and $a - \sqrt{a^2 - 4D}$ is negative

And when $a = -\sqrt{8}$ or $a = \sqrt{8}$

$D = 8/4 - 2 = 2 - 2 = 0$

Ex. 2: Sketch the analog of the trace-determinant plane for the two-parameter family of systems $X' = \begin{pmatrix} a & b \\ b & a \end{pmatrix} X$ in the ab- plane. That is, identify the regions in the ab-plane where this system has similar phase portraits.

$T = 2a$, $D = a^2 - b^2$

$T^2 - 4D = 4a^2 - 4a^2 + 4b^2 = 4b^2 \geq 0$ for any value of a, b so we never have complex eigenvalues.

Our eigenvalues: $\lambda = a\pm |b|$

when $b = 0$, we have real repeated eigenvalues at a (with two linearly independent eigenvectors)

if $a > 0$ then all solutions go away from origin

if $a < 0$ then all solutions go towards origin

when $D< 0 \equiv b^2 > a^2 \equiv |b| > |a|$, we have a saddle:

$a + |b| > a + |a| \geq 0$ and $a - |b| < a - |a| \leq 0$

$a - |b| < 0 < a + |b|$

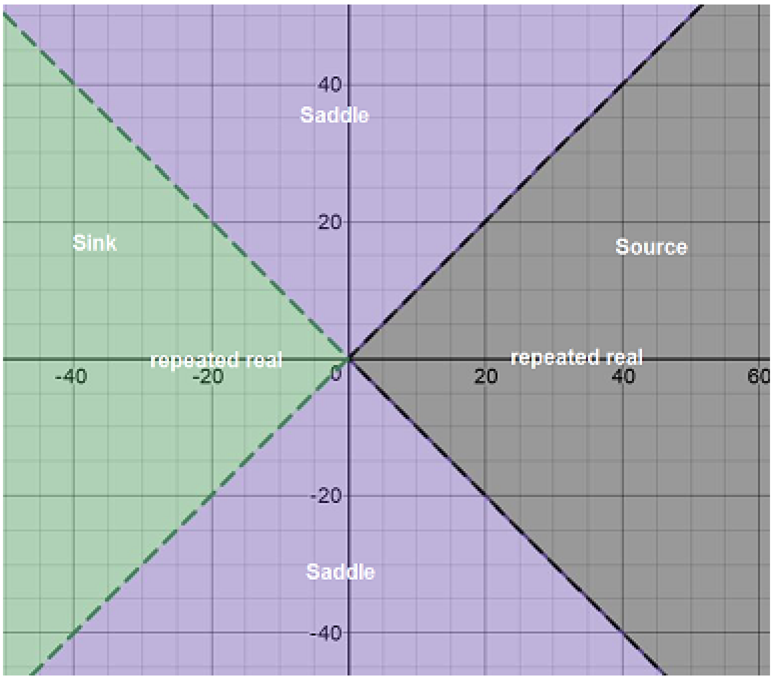
when $D>0 \equiv|b| < |a|$ we have either a source or a sink

if $a < 0$, then we have a sink ($T < 0$)

$| a - |b|\,| > |a + |b|\,| $

if $a > 0$ we have a source ($T > 0$)

$|a + |b|| > |a - |b||$



Ex. 3: Consider the harmonic oscillator equation (with m = 1) $x'' + bx' + kx = 0$ where $b \geq 0$ and $k > 0$. Identify the regions in the relevant portion of the bk-plane where the corresponding system has similar phase portraits.

letting $y = x'$, we have $X' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} X$

$T = -b$, $D = 0(-b) - (1)(-k) = 0 + k = k$

$T^2 - 4D = b^2 - 4k$

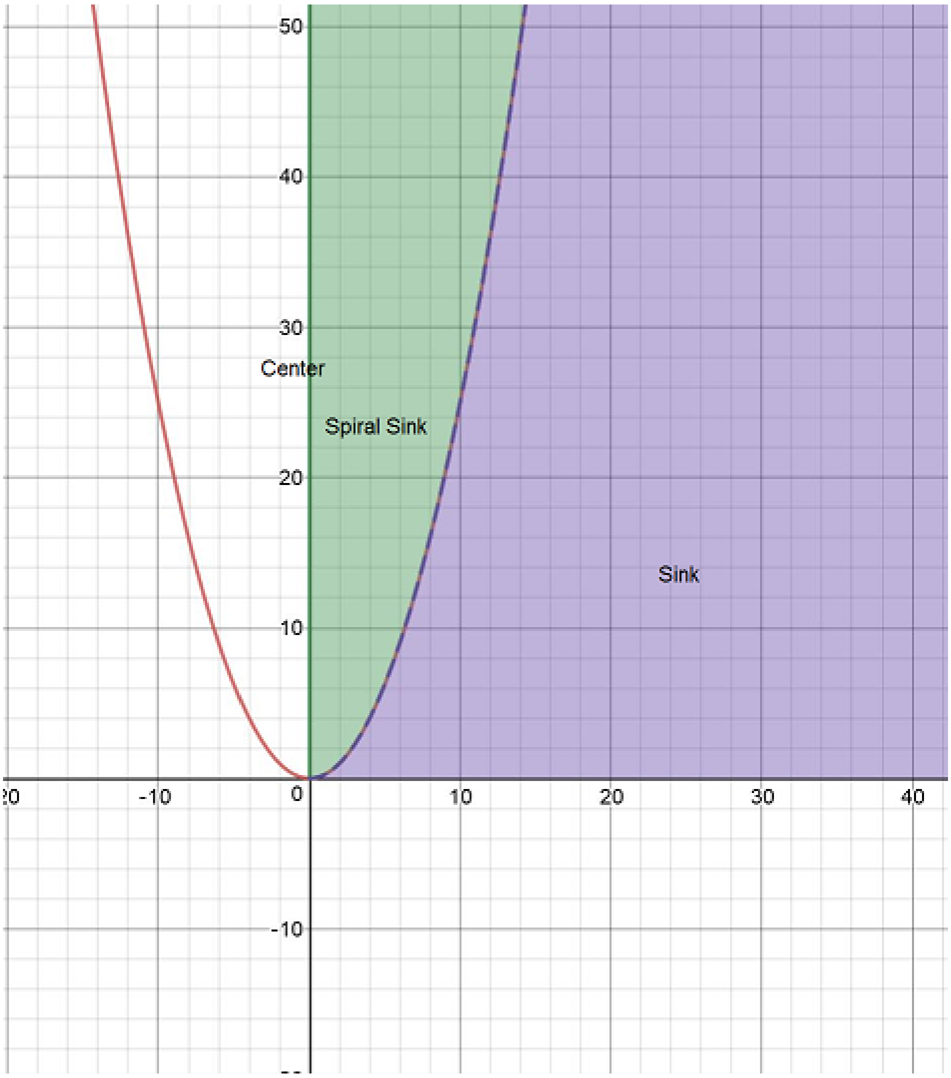
Eigenvalues: $\lambda = \frac{-b \pm \sqrt{b^2-4k}}{2}$

Plotting $D = T^2/4$ (when we have real repeated eigenvalues) is the same as looking at $k = b^2/4$

so when $b^2 - 4k < 0$ or $k > b^2/4$ we have complex eigenvalues, and since $b \geq 0$, we must have spiral sinks or a center traveling clockwise

When $b^2 - 4k > 0$ or $k < b^2/4$ we have real eigenvalues, and $b^2 - 4k > b^2$ since $k > 0$,

so we must have a sink



(the x-axis shouldn't be included, since $y = k > 0$)

Ex. 4: Prove that $H(x, y) = (x, -y)$ provides a conjugacy between $X' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X$ and

$Y' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Y$

So let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$H(x, y)$ is a conjugacy if it is one-to-one, onto, and continuous whose inverse is also continuous

and $\phi^B(t, H(X_0)) = H(\phi^A(t, X_0))$

$H(x, y)$ is one-to-one:

If $H(x_1, y_1) = H(x_2, y_2) \implies (x_1, -y_1) = (x_2, -y_2)$

so $x_1 = x_2$ and $y_1 = y_2$, so $(x_1, y_1) = (x_2, y_2)$

$H(x, y)$ is onto:

for any point (x, y), $H(x, -y) = (x, -(-y)) = (x, y)$

therefore, any point in the plane has a pre-image $(x, -y)$

To show $H(x,y)$ is continuous, we prove that for any point (x_0,y_0): for all $\epsilon>0, \exists\delta>0$ such that whenever a point (x,y) in the plane satisfies $|(x,y)-(x_0,y_0)|<\delta$ then $|H(x,y)-H(x_0,y_0)|<\epsilon$

For any point (x_0,y_0) and any $\epsilon>0$, take $\delta\le\epsilon$

so whenever $\sqrt{(x-x_0)^2-(y-y_0)^2}<\delta\le\epsilon$, we have
$\sqrt{(x-x_0)^2+(-y-(-y_0))^2}=\sqrt{(x-x_0)^2+(y_0-y)^2}=|(x,y)-(x_0,y_0)|<\epsilon$

so for any $\epsilon>0$, we have shown there exists a δ such that the statement above is satisfied $\implies$ as $(x,y)\to(x_0,y_0)$, $H(x,y)\to H(x_0,y_0)$

So $H(x,y)$ is continuous.

The H is its own inverse: $(H\circ H)(x,y)=H(H(x,y))=H(x,-y)=(x,y)$

and since we showed H is continuous, it follows that $H^{-1}=H$ is continuous.

So solutions for $X'=AX$ will be of the form:

$\phi^A_t(X_0) = e^t(x_0\cos t, -x_0\sin t) + e^t(y_0\sin t, y_0\cos t)$

And solution for $Y'=BY$ will be of the form

$\phi_t^B(X_0) = e^t(x_0\cos t, x_0\sin t) + e^t(-y_0\sin t, y_0 \cos t)$

So $H(X_0)=(x_0,-y_0)$

$\phi_t^B(H(X_0))=e^t(x_0\cos t, x_0\sin t)+e^t(y_0\sin t,-y_0\cos t)$

$H(\phi_t^A(X_0))=e^t(x_0\cos t, x_0\sin t)+e^t(y_0\sin t,-y_0\cos t\,)$

So we indeed have $\phi_t^B(H(X_0))=H(\phi_t^A(X_0))$

Ex. 5(a) Find an explicit conjugacy between the flows of $X'=\begin{pmatrix}-1 & 1\\ 0 & 2\end{pmatrix}X$ and $Y'=\begin{pmatrix}1 & 0\\ 1 & -2\end{pmatrix}Y$

Solutions for $X'=AX$

$\phi_t^A(X_0)=(x_0-\frac{y_0}{3})e^{-t}\begin{pmatrix}1\\0\end{pmatrix}+\frac{y_0}{3}e^{2t}\begin{pmatrix}1\\3\end{pmatrix}$

Putting this solution in terms of a solution for the canonical form left multiplied by $T=\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}$

$=\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}((x_0-\frac{y_0}{3})e^{-t}\begin{pmatrix}1\\0\end{pmatrix}+\frac{y_0}{3}e^{2t}\begin{pmatrix}0\\1\end{pmatrix})$

= $\phi_t^{T^{-1}AT}(x_0-\frac{y_0}{3},\frac{y_0}{3})$

noting that $\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}\begin{pmatrix}x_0-y_0/3\\ y_0/3\end{pmatrix}=(x_0,y_0)$

so $\phi_t^{T^{-1}AT}(T^{-1}X_0) = T^{-1}(\phi_t^A)$

Solution for $Y'=BY$

$\phi_t^B(X_0)=(y_0-\frac{x_0}{3})e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}+(\frac{x_0}{3})e^{t}\begin{pmatrix}3\\1\end{pmatrix}$

To put in terms of a solution for the canonical form left multiplied by $S=\begin{pmatrix}0 & 3\\ 1 & 1\end{pmatrix}$

$= \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} ((y_0 - \frac{x_0}{3})e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{x_0}{3} e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix})$

= $\phi_t^{S^{-1}BS}(y_0 - \frac{x_0}{3}, \frac{x_0}{3})$

So to find a conjugacy G s.t. $G(\phi_t^A(X_0))$ = $\phi_t^{T^{-1}AT}(G(X_0))$

$G(x, y) = T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$ (can be easily seen from above)

For the canonical forms, $H(\phi_t^{T^{-1}AT}(X_0)) = \phi_t^{S^{-1}BS}(H(X_0))$

$H = (h_1(x), h_2(y))$ where $h_1(x) = \begin{cases} x^2 & x \geq 0 \\ -|x^2| & x < 0 \end{cases}$, $h_2(y) = \begin{cases} y^{1/2} & y \geq 0 \\ -|y^{1/2}| & y < 0 \end{cases}$

And for $J(\phi_t^{S^{-1}BS}(X_0)) = \phi_t^B(J(X_0))$

$J(x, y) = S \begin{pmatrix} x \\ y \end{pmatrix}$

So define the conjugacy $K(x, y)$ s.t. $K(\phi_t^A(X_0)) = \phi_t^B(K(X_0))$

to be $K(x, y) = (J \circ H \circ G)(x, y)$

So from above, we know that we have $G(\phi_t^A(X_0)) = (x_0 - \frac{y_0}{3})e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{y_0}{3} e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and then $H(G(\phi_t^A{X_0})) = (x_0-\frac{y_0}{3})^2e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\frac{y_0}{3})^{1/2}e^{t}\begin{pmatrix} 0 \\ 1\end{pmatrix}$

and then $J(H(G(\phi_t^A(X_0)))) = (x_0-\frac{y_0}{3})^2e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\frac{y_0}{3})^{1/2}e^{t}\begin{pmatrix} 3 \\ 1\end{pmatrix}$

To show equality to $\phi_t^B(K(X_0))$: $(J \circ H \circ G)(X_0) = (3(\frac{y_0}{3})^{1/2}, (x_0 - \frac{y_0}{3})^2 + (\frac{y_0}{3})^{1/2})$

And $\phi_t^B((J \circ H \circ G)(X_0)) = ((x_0 - \frac{y_0}{3})^2 + (\frac{y_0}{3})^{1/2} - (\frac{y_0}{3})^{1/2})e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\frac{y_0}{3})^{1/2} e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

So we indeed have have that $K(\phi_t^A(X_0)) = \phi_t^B(K(X_0))$

We know that H is a homeomorphism

The matrices T^{-1} and S have columns that are linearly independent, so J and G are one-to-one and onto, and their inverses exist. And since left multiplying by a matrix is linear transformation, and we are working with finite-dimensional space $\mathbb{R}^2$ [1]

J and G are continuous (the same applies to the inverses, which use matrices T and S^{-1}. Both these matrices are one-to-one and over a finite-dimensional space)

Ex. 6: Prove that any two linear systems with the same eigenvalues $\pm i\beta, \beta \neq 0$ are conjugate.

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1. http://pfister.ee.duke.edu/courses/ecen601/notes_ch5.pdf↩