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Math 185

Homework #8

Chapter V, Sec. 1, ex. 7; Sec. 2, ex. 3, 8, 9; Sec. 3, ex. 1b, 1e, 1i

Chapter V, Sec. 1, Ex. 7:

Suppose $\sum a_k$ converges

This means the sequence of partial sums, $\{S_k\}$ converges

and that $a_k \to 0$ as $k \to \infty$

The sequence of partial sums is a sequence of complex numbers

and convergent sequence of complex numbers are Cauchy Sequences

so, as $m, n \to \infty$, $|S_n - S_m| \to 0$

$|S_n - S_m| = |a_0 + \ldots + a_n - (a_0 + \ldots + a_m)| = |a_{m+1} + \ldots + a_n|$

and so for any $\epsilon > 0$, $\exists\, N_1 \geq 1$ s.t. if $m, n \geq N_1$, $|S_n - S_m| < \epsilon/2$

and also, $\exists N_2 \geq 1$ s.t. if $m \geq N_2$, $|a_m| < \epsilon/2$

so taking $N = \max(N_1, N_2)$, if $m, n \geq N$,

$|S_n - S_m + a_m| = |a_m + \ldots + a_n| \leq |a_m| + |a_{m+1} + \ldots + a_n| < \epsilon/2 + \epsilon/2 = \epsilon$

this means that $a_m + \ldots + a_n \to 0 \equiv \sum_{k=m}^{k=n} a_k$ converges to 0 as $m, n \to \infty$

Now suppose $\sum_{k=m}^{k=n} a_k \to 0$ as $m, n \to \infty$

this means $a_m + \ldots + a_n \to 0$ as $m, n \to \infty$

so for any $\epsilon > 0$, $\exists N \geq 1$ s.t. $|a_m + \ldots + a_n| < \epsilon$ for all $m, n \geq N$

and $|a_{m+1} + \ldots + a_n| \leq |a_m + a_{m+1} + \ldots + a_n| < \epsilon$ for all $m, n \geq N$

$\implies |S_n - S_m| < \epsilon$ for all $m, n \geq N$ (ϵ was arbitrary)

so $\{S_k\}$ is a cauchy sequence $\implies$ it is a convergent sequence

which means $\sum a_k$ converges

Chapter V, Sec. 2, Ex. 3

1. $\frac{z^k}{k}$ converges uniformly for $|z| < 1$, $E =$$\{z : |z| < 1\}$
   1. I will show $\{f_k\}$ converges to 0 uniformly as $k \to \infty$
   2. $|z^j| = |z|^j$ and $|z| < 1$, so $|z|^j < 1^j = 1$ for all $j \in \mathbb{N}$
   3. so for all $z \in E$, $j \in \mathbb{N}$:
   4. $|f_j(z)| = |\frac{z^j}{j}| < \frac{1}{j}$ and as $j \to \infty$, $\frac{1}{j} \to 0$,

    5. which means $|f_j(z) - 0| \leq \frac{1}{j}$, and $\frac{1}{j} \to 0$, for all $z \in E$
    6. so $\{f_k\}$ converges to 0 uniformly.
2. $f_k'(z)$ doesn't converge uniformly for $|z| < 1$
    1. Suppose $f_k'(z)$ does converge uniformly to $g(z)$for $|z| < 1$
    2. so let z be any point in E . $f_k(z)$ is analytic everywhere assuming, $k \geq 1$, so $f_k'(z)$ is analytic by the corollary on page 115 (E is a domain) and E is a star-shaped domain so:
    3. $f_k(z) = \int_0^z f_k'(\zeta)d\zeta$, where the integral can be taken over any path from 0 to z
        1. since $z^k/k - 0^k/k = \int_0^z \zeta^{k-1}d\zeta$
    4. let γ be a broken line segment contained in E (which is possible since E is a domain and star-shaped with respect to 0) that goes from 0 to z, $z \in E$
    5. and since f_k' is analytic on E, it is continuous over E
    6. and $\gamma \in E$, so f_k' is continuous on γ
    7. and assuming $\{f_k'\}$ converges uniformly to $g(z)$ on E (and therefore γ)
    8. $\int_\gamma f_k'(z)dz = \int_0^z f_k'(\zeta)d\zeta = f_k(z) \to \int_\gamma g(z)dz$
    9. and since $\{f_k'\}$ is a sequence of analytic functions, g is analytic (on E)
    10. and $\int_\gamma g(z)dz = \int_0^z g(\zeta)d\zeta = G(z) - G(0)$
        1. where G is a primitive for g
    11. so $\{f_k\}$ converges uniformly to $G(z) - G(0)$, but it also converges to 0
    12. so $G(z) - G(0) = 0$
    13. which means $G(z) = G(0)$ for any $z \in E$, so $G(z)$ is constant over E
    14. which means $G'(z) = g(z) = 0$
    15. so $f_k' \to 0$
    16. but for every $j \geq 1$, $|f_j'| = |z^{j-1}| \leq 1 = \epsilon_j$, so ϵ_j does not converge to 0 as $j \to \infty$, which is a contradiction: f_k' does not converge uniformly for $|z| < 1$
3. However, using the theorem on page 136
    1. since $f_k(z)$ is analytic for $|z| \leq R$ for each $R < 1$ and converges uniformly to 0 for $|z| \leq R$
    2. For each $r < R < 1$, $\{f_k'(z)\}$ converges uniformly to 0 for $|z| \leq r$
        1. for any $\epsilon > 0$, taking $R = 1 - \epsilon/2$, we have that the above is true for $r = 1 - \epsilon$

Chapter V, Sec. 2, Ex. 8:

Show that $\sum \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$

define $g_k(z) = \frac{z^k}{k^2}$

Let $M_k = \frac{1}{k^2} \geq 0$

for all z such that $|z| < 1$,

$|g_k(z)| \leq |\frac{z^k}{k^2}| = \frac{|z|^k}{k^2} < \frac{1}{k^2} = M_k$

so $\sum g_k(z)$ converges uniformly for $|z| < 1$

Chapter V, Sec. 2, Ex. 9:

Show that $\sum \frac{z^k}{k}$ does not converge uniformly for $|z| < 1$ $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

and the terms $\frac{z^k}{k} \to \frac{1}{k}$ as $z \to 1$ from the left side (and in this case, z is increasing towards 1 along the positive real axis)

so letting $z \to 1$ as above, $\sum \frac{z^k}{k} \to \sum \frac{1}{k} \to \infty$

Chapter V, Sec. 3, Ex 1b.

$\sum_{k=0}^{\infty} \frac{k}{6^k} z^k$

define $w = \frac{z}{6}$

so $w^k = \frac{z^k}{6^k}$

$\sum_{k=0}^{\infty} kw^k$

from the example on page 142,

the ratio test gives the radius of convergence $R = 1$ for $\sum_{k=0}^{\infty} kw^k$

so $|w| = |\frac{z}{6}| < 1 \equiv |z| < 6$

so $R = 6$

Chapter V, Sec. 3, Ex 1e

$\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2+k}$

$w = 2z^2$

$\sum_{k=1}^{\infty} \frac{w^k}{k^2+k}$

so $a_k = \frac{1}{k^2+k}$

$|a_k/a_{k+1}| = \frac{(k+1)^2+k+1}{k^2+k} = \frac{k^2+2k+1+k+1}{k^2+k} = \frac{k^2+3k+2}{k^2+k} = \frac{1+3/k+2/k^2}{1+1/k} \to 1$ as $k \to \infty$

so $|2z^2| < 1 \equiv |z| < 1/\sqrt{2}$, so $R = \frac{1}{\sqrt{2}}$

Chapter V, Sec. 3, Ex 1i

$\sum_{k=1}^{\infty} \frac{k! z^k}{k^k}$

using ratio test:

$|a_k/a_{k+1}| = \frac{k!/k^k}{(k+1)!/(k+1)^{k+1}} = \frac{k!(k+1)^{k+1}}{(k+1)k!k^k}$

$R = \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = e$

Extra Problems: https://math.berkeley.edu/~art/data/F18-185/HW8.pdf

1. Give an example of a power series (centered at $z_0 = 0$) with radius of convergence $R = 1$ which converges at $z = i$ and diverges at $z = -i$. Justify your answer

$\sum a_k z^k$

where $\sum a_k (i)^k$ converges but $\sum a_k (-i)^k$ diverges

so consider the series $\sum_{k=1}^{\infty} \frac{i^k}{k} z^k$

at $z = i$, we obtain $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which converges (conditionally)

at $z = -i$, we obtain $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

2.

$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$

$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}$

taking the differential equation:

$z^2 f''(z) + z f'(z) + (z^2 - 1) f(z) = 0$

and substituting each $f^{(m)}(z)$ with the corresponding series:

$\sum_{k=2}^{\infty} k(k-1) a_k z^k + \sum_{k=1}^{\infty} k a_k z^k + \sum_{k=0}^{\infty} a_k z^{k+2} - \sum_{k=0}^{\infty} a_k z^k = 0$

Letting $\sum_{k=2}^{\infty} k(k-1) a_k z^k = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^{k+2}$

$\sum_{k=1}^{\infty} k a_k z^k = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^{k+1}$

$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^{k+2} + \sum_{k=0}^{\infty} (k+1) a_{k+1} z^{k+1} + \sum_{k=0}^{\infty} a_k z^{k+2} - \sum_{k=0}^{\infty} a_k z^k = 0$

$-a_0 - a_1 z + a_1 z + \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^{k+2} + \sum_{k=1}^{\infty} (k+1) a_{k+1} z^{k+1} + \sum_{k=0}^{\infty} a_k z^{k+2} - \sum_{k=2}^{\infty} a_k z^k = 0$

Letting:

$\sum_{k=1}^{\infty} (k+1) a_{k+1} z^{k+1} = \sum_{k=0}^{\infty} (k+2) a_{k+2} z^{k+2}$

$\sum_{k=2}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_{k+2} z^{k+2}$

then substituting and simplifying:

$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} + (k+2) a_{k+2} + a_k - a_{k+2}] z^{k+2} = a_0$

$(k+2)(k+1) a_{k+2} + (k+2) a_{k+2} + a_k - a_{k+2} = (k+3)(k+1) a_{k+2} + a_k$

so $\sum_{k=0}^{\infty} [(k+3)(k+1) a_{k+2} + a_k] z^{k+2} = a_0$

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plugging in $z = 0$ to the differential equation:

$(0-1) f(0) = 0 \implies -f(0) = 0 \implies f(0) = 0 = a_0$

$(k+3)(k+1) a_{k+2} = -a_k$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

$k = 1$: $a_3 = -\frac{1/2}{4*3} = -\frac{1}{24}$

$k = 3: a_5 = -\frac{a_3}{6*5} = \frac{-1}{24} * \frac{-1}{30} = \frac{1}{720}$