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Math 123

Homework #12

Chapter 13: Exercises 1, 2, 3, 5, 8, 9, and 10

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1. Which of the following force fields on $\mathbb{R}^2$ are conservative?

(a) $F(x, y) = (-x^2, -2y^2)$

conservative: there is a function U s.t. $F(x, y) = -\text{grad}\, U(x, y)$

$U = \frac{x^3}{3} + \frac{2y^3}{3}$

$\frac{\partial U}{\partial x} = x^2$, $\frac{\partial U}{\partial y} = 2y^2$

(b) $F(x, y) = (x^2 - y^2, 2xy)$

need $\frac{\partial U}{\partial x} = -(x^2 - y^2)$, $\frac{\partial U}{\partial y} = -2xy$

$\frac{\partial U}{\partial x} = -x^2 + y^2 \implies U(x, y) = -\frac{x^3}{3} + xy^2 + C(y)$

$\frac{\partial U}{\partial y} = 2xy + C'(y)$,

but we need $\frac{\partial U}{\partial y} = -2xy$, and no choice of $C(y)$ makes this happen as it only depends on y so

not conservative

(c) $F(x, y) = (x, 0)$

$\frac{\partial U}{\partial x} = -x$, $\frac{\partial U}{\partial y} = 0$

$\implies U = -\frac{x^2}{2} + C(y)$

$\frac{\partial U}{\partial y} = C'(y) = 0$

so $U = -\frac{x^2}{2}$ is a function s.t. $F(x, y) = -\text{grad}\, U(x, y)$

conservative

2. Prove that the equation $\frac{1}{r} = \frac{1}{h}(1 + \epsilon \cos\theta)$ determines a hyperbola, parabola, and ellipse when $\epsilon > 1, \epsilon = 1$, and $\epsilon < 1$ respectively$r = \frac{h}{1+\epsilon\cos\theta}$

h is the latus rectum, which $= p\epsilon$, where p is the distance from the focus to the directrix

let the focus be origin, and the directrix be $x = d$, for some $d \in \mathbb{R}^+$, so $p = d$

so $r + r\epsilon\cos\theta = h \equiv r = h - r\epsilon\cos\theta = \epsilon(d - r\cos\theta)$

squaring both sides: $r^2 = \epsilon^2(d - r\cos\theta)^2$

subbing $x^2 + y^2$ for r^2 and x for $r\cos\theta$: $x^2 + y^2 = \epsilon^2(d-x)^2 = \epsilon^2(d^2 - 2dx +x^2)$

$\equiv x^2 + y^2 + 2\epsilon^2 dx - \epsilon^2 x^2 = \epsilon^2 d^2$

if $\epsilon = 1$: $x^2 + y^2 + 2dx - x^2 = d^2 \equiv y^2 + 2dx = d^2$**, which is an equation for a parabola.**

However, if $\epsilon \neq 1$, we keep going:

$(1-\epsilon^2)x^2 +2\epsilon^2dx + y^2 = \epsilon^2d^2$

$x^2 + \frac{2\epsilon^2d}{1-\epsilon^2}x + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2d^2}{1-\epsilon^2}$ $\implies$

$(x + \frac{\epsilon^2d}{1-\epsilon^2})^2 - \frac{\epsilon^4d^2}{(1-\epsilon^2)^2} + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2d^2}{1-\epsilon^2}$ $\implies$

$(x + \frac{\epsilon^2 d}{1-\epsilon^2})^2 + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 d^2}{(1-\epsilon^2)^2} \implies$

if $\epsilon < 1 \implies 1 - \epsilon^2 > 0$

and with $h = -\frac{\epsilon^2 d}{1-\epsilon^2}$, $a^2 = \frac{\epsilon^2 d^2}{(1-\epsilon^2)^2}$, $b^2 = \frac{\epsilon^2 d^2}{1-\epsilon^2}$

we have $\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse

if $\epsilon > 1$, $1 - \epsilon^2 < 0$

we have h the same, a^2 the same, but $b^2 = -\frac{\epsilon^2 d^2}{1-\epsilon^2}$

and we obtain $\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1$, a hyperbola

3. Consider the case of a particle moving directly away from the origin at time $t = 0$ in the Newtonian central force system. Find a specific formula for this solution and discuss the corresponding motion of the particle. For which initial conditions does the particle eventually reverse direction?

so I'm assuming "directly away" means the particle is moving in a straight line away from origin

so $\theta' = 0 \implies v_\theta = 0$ so $v_\theta' = 0$ as well and θ is fixed.

so our system becomes (when $V' = -\frac{X}{|X|^3}$, normalized the constants)

however, when $V' = -gmX/|X|^3$,where m is the mass of the sun, since

$m_p X'' = -gm_s m_p \frac{X}{|X|^3} \implies X'' = -gm_s \frac{X}{|X|^3}$ so:

$\frac{-gm}{r^2}(\cos\theta, \sin\theta) = -gmX/|X|^3 = (v_r'-v_\theta\theta')(\cos\theta, \sin\theta) + (v_rv_\theta/r + v_\theta')(-\sin\theta, \cos\theta)$

$(v_r' - v_\theta \theta') = -gm/r^2$

so:

$\frac{dr}{dt} = v$

$\frac{dv}{dt} = -\frac{gm}{r^2} \equiv r^2 dv = -dt \implies r^2 v = -t + C$

$\frac{dt}{dv} = -r^2/gm$

$\frac{dr}{dt}\frac{dt}{dv} = \frac{dr}{dv} = v(-r^2)/gm$

$\frac{gm}{-r^2} dr = v dv$

$\frac{gm}{r} = \frac{v^2}{2} + C \implies \frac{gm}{v^2/2+C} = r$

$r = \frac{2gm}{v^2+C}$

$r(v(t)) = \frac{2gm}{v(t)^2+C}$

$r(v(0)) = \frac{2gm}{v_0^2 + C}$

$r(v_0) = \frac{2gm}{v_0^2 + C}$

$C = \frac{2gm}{r(v_0)} - v_0^2$ (I suppose when $r(v_0) = r_0$ as this is the radius when we are at our initial velocity at $v_0 = v(0)$)

$C = \frac{2gm}{r_0} - v_0^2$ $\implies r(v) = \frac{2}{v^2 + \frac{2gm}{r_0}-v_0^2}$

so our $v_0 > 0$, since we are initially moving away from origin at $t = 0$

and since $\frac{dv}{dt} = -\frac{1}{r^2}$, v is decreasing but also noting that once $r \to \infty$, $\frac{dv}{dt} \to 0$

so one of two things may happen: 1. velocity decreases to below zero before $dv/dt \approx 0$ and the particle begins moving towards origin

2. velocity is still > 0 when $dv/dt \approx 0$, and the particle doesn't reverses direction

so we look at $v = 0$, $r(0) = \frac{2}{\frac{2gm}{r_0} - v_0^2}$

so $\frac{2gm}{r_0} -v_0^2> 0$ for the particle to eventually reverse direction (because r must be > 0)

if $\frac{2gm}{r_0} - v_0^2 > 0 \implies v^2 + \frac{2gm}{r_0} - v_0^2 > 0$, for all v and $r(v)$ won't $\to \infty$

if $\frac{2gm}{r_0} - v_0^2 \leq 0 \implies v_0^2 - \frac{2gm}{r_0} \geq 0$ and $v_0^2 > v_0^2 - \frac{2gm}{r_0}$, so

because v starts at $v_0 > 0$ and decreases as time increases, v will eventually $= v_0^2 - \frac{2gm}{r_0} \implies r(v) \to \infty$ as $v \rightarrow v_0^2 - \frac{2gm}{r_0}$

5. Let $F(X)$ be a force field on $\mathbb{R}^3$. Let X_0, X_1 be points in $\mathbb{R}^3$ and let $Y(s)$ be a path in $\mathbb{R}^3$ with $s_0 \leq s \leq s_1$, parametrized by arc length s, from X_0 to X_1. The *work* done in moving a particle along this path is defined to be the integral $\int_{s_0}^{s_1} F(y(s)) \cdot y'(s) ds$, where $Y'(s)$ is the unit tangent vector to the path. Prove that the force field is conservative if and only if the work is independent of path. In fact, if $F = -\text{grad V}$, then the work done is $V(X_1) - V(X_0)$

Assume the force field is conservative:

then there exists a smooth function $U : \mathbb{R}^n \to \mathbb{R}$ such that

$F(X) = -\text{grad } U(X) = -(\frac{\partial U}{\partial x_1}(X), \frac{\partial U}{\partial x_2}(X), \frac{\partial U}{\partial x_3}(X))$

$F(y(s)) \cdot y'(s) = -\text{grad } U(y(s)) \cdot y'(s)$

$= -\frac{d}{ds} U(y(s))$

$\int_{s_0}^{s_1} F(y(s))\cdot y'(s)ds = \int_{s_0}^{s_1} -\frac{d }{d s}U(y(s))ds$

$= \int_{s_0}^{s_1} -dU(y(s))$ = $\int_{X_0}^{X_1} -dU = -[U(X_1) - U(X_0)]$, so independent of path, as it doesn't matter which path we take from X_0 to X_1 since the integral is the same for any path from X_0 to X_1

Assuming independence of path

$\int_{s_0}^{s_1} F(y(s)) \cdot y'(s)ds$, as long as X_0 and X_1 are the initial and terminal points of any path $y(s)$, the integral is the same

so $\int_{s_0}^{s_1} F(y(s)) \cdot y'(s) = \int_{X_0}^{X_1} F(X)dX$

fixing X_0, define a function $h(X_1) = \int_{X_0}^{X_1} F(X)dX$

we fix a path $\gamma(s)$ from X_0 to (a, b, c) $y(s) = (s, b, c)$, where $s_0 = a$, $s_1 = x_1$,for x_1 near a

we have then $h(x_1, b, c) = \int_\gamma F(X)dX + \int_a^{x_1} F_1(s, b, c)ds$

the first integral on the right is a constant, since we fixed γ, so

$h(x_1, b, c) = \int_a^{x_1} F_1(s, b, c)ds + C$, and by FTC, $\frac{\partial h}{\partial x_1}(a, b, c) = F_1(a, b, c)$

and similarly for the other coordinates

so $\text{grad }h = (F_1, F_2, F_3) = F$ and we let $U = -h$, so $-\text{grad } U = \text{grad } h = F$

8. The following three problems deal with the two-body problem. Let the potential energy be $U = \frac{gm_1m_2}{|X_2 - X_1|}$ and $\text{grad}_j(U) = (\frac{\partial U}{\partial x_1^j}, \frac{\partial U}{\partial x_2^j}, \frac{\partial U}{\partial x_3^j})$. Show that the equations for the two-body problem may be written $m_j X_j'' = -\text{grad}_j(U)$

For the two -body problem, the equations of motion are

$m_1 X_1'' = gm_1m_2 \frac{X_2 - X_1}{|X_2 - X_1|^3}$

$m_2 X_2'' = gm_1m_2 \frac{X_1 - X_2}{|X_1 - X_2|^3}$

We have $|X_2 - X_1|$ = $\sqrt{\sum_{i=1}^3 (x_i^2 - x_i^1)^2}$

and $|X_1 - X_2| = \sqrt{\sum_{i=1}^3 (x_i^1 - x_i^2)^2}$

so $U = \frac{gm_1m_2}{\sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2 + (x_3^2 - x_3^1)^2}}$

and $\frac{\partial U}{\partial x_i^1} = gm_1gm_2(-\frac{1}{2|X_2 - X_1|^3})(2(x_i^2 - x_i^1))(-1)$ $ = gm_1m_2\large \frac{x_i^2 - x_i^1}{|X_2 - X_1|^3}$

and $\frac{\partial U}{\partial x_i^2} = -gm_1m_2 \frac{x_i^2 - x_i^1}{|X_2 - X_1|^3}$ $ = gm_1m_2 \large \frac{x_i^1 - x_i^2}{|X_1 - X_2|^3}$ (because $|X_1 - X_2| = |X_2 - X_1|$)

I think either $U = \frac{gm_1m_2}{|X_1 - X_2|}$ so that the signs are switched. If we do that, then we have the proof.

9. Show that the total energy $K + U$ of the system is a constant of motion, where $K = \frac{1}{2}(m_1|V_1|^2 + m_2|V_2|^2)$

so $E = K + U = \frac{1}{2}(m_1|V_1|^2 + m_2|V_2|^2) + \frac{gm_1m_2}{|X_2-X_1|}$

$\dot{E} = m_1V_1 \cdot V_1' + m_2V_2 \cdot V_2'$ $+\text{grad}_1 U \cdot X_1'$ $+\text{grad}_2 U \cdot X_2'$

$= V_1 \cdot (-\text{grad}_1 U)$ $+V_2 \cdot (-\text{grad}_2 U)$ $+\text{grad}_1 U \cdot V_1 + + \text{grad}_2 U \cdot V_2$

$= 0$

so E is constant

10. Define the angular momentum of the system by $l = m_1(X_1 \times V_1) + m_2(X_2 \times V_2)$ and show that l is also a first integral

$l' = m_1(X_1' \times V_1 + X_1 \times V_1') + m_2(X_2' \times V_2 + X_2 \times V_2')$

so $X_i' = V_i$, so $X_i' \times V_i = 0$

$l' = m_1(X_1 \times X_1'') + m_2(X_2 \times X_2'')$

X_i'' are scalar multiples of $X_2 - X_1$ if $i = 1$, $X_1 - X_2$, if $i = 2$

$X_1 \times (X_2 - X_1) = (x_2^1(x_3^2 - x_3^1) - x_3^1(x_2^2 - x_2^1), x_3^1(x_1^2 - x_1^1) - x_1^1(x_3^2 - x_3^1), x_1^1(x_2^2 - x_2^1) - x_2^1(x_1^2 - x_1^1))$
$= X_1 \times X_2 + X_1 \times (-X_1)$ = $X_1 \times X_2 - X_1 \times X_1 = X_1 \times X_2$

and similalry, $X_2 \times (X_1 - X_2) = X_2 \times X_1 = -X_1 \times X_2$

and we have

$l' = \frac{g m_1 m_2}{|X_2 - X_1|^3}(X_1 \times X_2 - X_1 \times X_2) = 0$

so since $l' = 0$, l is constant and thus a first integral