Renee Senining

Math 185

Homework #7

Chapter IV Sec. 3, ex. 1, 2; Sec. 4, ex. 1f, 1g, 1h; Sec. 5, ex. 2; Sec. 6, ex. 2

Sec. 3

Ex. 1 By integrating $e^{-z^2/2}$ around a rectangle with vertices $\pm R, it \pm R$ and sending R to ∞, show that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{itx} dx = e^{-t^2/2}$, $\infty < t < \infty$. Use the known value of the integral for $t = 0$.

$f(z) = e^{-z^2/2}$ is analytic on the complex plane, so it is analytic on $D =$ the rectangle with vertices $\pm R, it \pm R$ So:

$\int_{(-R,0)}^{(R,0)} f(z)dz + \int_{(R,0)}^{(R,t)} f(z)dz + \int_{(R,t)}^{(-R,t)} f(z)dz + \int_{(-R,t)}^{(-R,0)} f(z)dz = 0$

$\equiv$

$\int_{-R}^{R} e^{-x^2/2} dx + i\int_0^t e^{(-R^2+y^2)/2} e^{-iRy} dy + \int_R^{-R} e^{(-x^2+t^2)/2} e^{-ixt} dx + i\int_t^0 e^{(-R^2+y^2)/2} e^{iRy} dy = 0$

with $dy = 0$, moving from $(-R, 0)$ to $(R, 0)$ and from (R, t) to $(-R, t)$

and $dx = 0$ moving from $(R, 0)$ to (R, t) and from $(-R, t)$ to $(-R, 0)$

and $e^{(-R^2+y^2)/2-iRy} = e^{-(R+iy)^2/2}$ and $e^{(-R^2+y^2)/2+iRy} = e^{-(-R+iy)^2/2}$

and since $0 \leq y \leq t$, $(\infty < t < \infty)$ and as $R \to \infty$, $|e^{-(\pm R+iy)^2/2}| \to 0 \implies$

by the ML estimate, $|\int_0^t e^{-(\pm R+iy)^2/2} dy| \leq ML \to 0$ as $R \to \infty$

so: $\int_0^t e^{-(\pm R+iy)^2/2} = 0$

so: $\int_{-R}^{R} e^{-x^2/2} dx = \int_{-R}^{R} e^{(-x^2+t^2)/2} e^{-ixt} dx = \int_{-R}^{R} e^{-(x+it)^2/2} dx$

and when $t = 0$, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1$

so: $\int_{-R}^{R} e^{-x^2/2} e^{-ixt} e^{t^2/2} dx = \int_{-R}^{R} e^{-x^2/2} \to \sqrt{2\pi}$ as $R \to \infty$

$\implies \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ixt} dx = e^{-t^2/2}$

Ex. 2 We define the Hermite polynomial $H_n(x)$ and Hermite orthogonal functions $\phi_n(x)$ for $n \geq 0$ by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$, $\phi_n(x) = e^{-x^2/2} H_n(x)$

(a) Show that $H_n(x) = 2^n x^n + \cdots$ is a polynomial of degree n that is even when n is even, and odd when n is odd

the derivative of an odd function is even, and the derivative of an even function is odd:

$f(x) = f(-x)$

letting $g(x) = -x$, we have $f(g(x)) = f(x)$

$\frac{d}{dx}f(g(x)) = \frac{df}{dx}(g(x)) * \frac{d}{dx}g(x) = -\frac{d}{dx}f(-x)$

but also $\frac{d}{dx}f(g(x)) = \frac{d}{dx}f(x)$

so: $\frac{d}{dx}f(-x) = -\frac{d}{dx}f(x)$

if $f(-x) = -f(x)$, again letting $g(x) = -x$, $f(g(x)) = -f(x)$

so $\frac{d}{dx}f(g(x)) = \frac{d}{dx}(-1)f(x) = -\frac{d}{dx}f(x)$

and $\frac{d}{dx}f(g(x)) = \frac{df}{dx}(g(x)) * \frac{d}{dx}g(x) = -\frac{d}{dx}f(-x)$

so $\frac{d}{dx}f(x) = \frac{d}{dx}f(-x)$

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e^{-x^2} is an even function: $e^{-(-x)^2} = e^{-x^2}$

so the $n = 1$ derivative is going to be odd

and the $n = 2$ derivative is the derivative of an odd function, so it will be even

and so on: if n is odd the nth derivative is odd, and if n is even, the nth derivative is even

the product of even functions is even: $f(-x)g(-x) = f(x)g(x)$, and the product of an even and odd function is odd: $f(-x)g(-x) = -f(x)g(x)$ (either $f(-x) = -f(x)$ and $g(-x) = g(x)$ or vice versa)

so $(-1)^n$ is a constant, so it is even, and e^{x^2} is even: $(-1)^n e^{x^2}$ is even

and if n is even: $\frac{d^n}{dx^n}(e^{-x^2})$ is even, so $H_n(x)$ is even

and if n is odd: $\frac{d^n}{dx^n}(e^{-x^2})$ is odd, so $H_n(x)$ is odd

(b) By integrating the function $e^{(z-it)^2/2}\frac{d^n}{dz^n}(e^{-z^2})$ around a rectangle with vertices $\pm R, it \pm R$ and sending R to ∞, show that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_n(x)e^{-itx}dx = (-i)^n\phi_n(t)$, $-\infty < t < \infty$

$f(z) = e^{(z-it)^2/2}\frac{d^n}{dz^n}(e^{-z^2}) = e^{(z^2-2zit+t^2)/2}\frac{d^n}{dz^n}(e^{-z^2})$

$\int_{(-R,0)}^{(R,0)} f(z)dz + \int_{(R,0)}^{(R,t)} f(z)dz + \int_{(R,t)}^{(-R,t)} f(z)dz + \int_{(-R,t)}^{(-R,0)} f(z)dz = 0$

$\equiv$

$\int_{-R}^{R} e^{(x-it)^2/2}\frac{d^n}{dx^n}(e^{-x^2})dx + i\int_0^t e^{(R+iy-it)^2/2}e^{-(R+iy)^2} - \int_{-R}^{R} e^{x^2/2}\frac{d^n}{dx^n}e^{-(x+it)^2}dx$
$-i\int_0^t e^{(-R+iy-it)^2/2}\frac{d^n}{dy^n}(e^{-(-R+iy)^2})dy = 0$

as seen before, for $0 \le y \le t$, $R \to \infty$ $|e^{-(\pm R+iy)^2}| \to 0$ and similarly $|e^{(\pm R+iy-it)^2/2}| \to 0$

so: $\int_{-R}^{R} e^{(x-it)^2/2}\frac{d^n}{dx^n}(e^{-x^2})dx = \int_{-R}^{R} e^{x^2/2}\frac{d^n}{dx^n}e^{-(x+it)^2}dx$

Left side:

$\int_{-R}^{R} e^{(x-it)^2/2}\frac{d^n}{dx^n}(e^{-x^2})dx = \int_{-R}^{R} e^{x^2/2}e^{-itx}e^{-t^2/2}\frac{d^n}{dx^n}(e^{-x^2})dx$

$= e^{-t^2/2}\int_{-R}^{R} e^{-itx}(-1)^n\phi_n(x)dx$

since $\phi_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2})$

Right side:

$\int_{-R}^{R} e^{x^2/2} \frac{d^n}{dx^n} e^{-(x+it)^2} dx = \int_{-R}^{R} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2-2itx+t^2}) dx$

$= e^{t^2} \int_{-R}^{R} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2-2itx}) dx$

justifying the hint: $\frac{d^n}{dx^n} e^{-(x+it)^2} = \frac{1}{i^2} \frac{d^n}{dt^n} e^{-(x+it)^2}$

since: $\frac{dF}{dx} = \frac{1}{i} \frac{dF}{dy}$ and F in this case has continuous partial derivatives of all orders

so: $\frac{\partial^n F}{\partial x^n} = (\frac{1}{i})^n \frac{\partial^n F}{\partial y^n}$, and in this integral, $y = t$ since it goes from $(-R, t)$ to (R, t)

So using the hint we get:

$= \int_{-R}^{R} e^{x^2/2} \frac{1}{i^n} \frac{d^n}{dt^n} (e^{-x^2-2itx+t^2}) dx$ and using $1/i = -i$:

$= (-i)^n \frac{d^n}{dt^n} e^{t^2} \int_{-R}^{R} e^{-x^2/2} e^{-2itx} dx =$

as $R \to \infty$... $\int_{-R}^{R} e^{-x^2} e^{-i(2t)x} dx \to \sqrt{2\pi} e^{-(2t)^2/2}$ (from exercise 1 with 2t in place of t)

$(-i)^n \frac{d^n}{dt^n} e^{t^2} \int_{-R}^{R} e^{-x^2/2} e^{-2itx} dx \to \sqrt{2\pi} (i)^n (-1)^n \frac{d^n}{dt^n} e^{-t^2}$

$= \sqrt{2\pi} i^n e^{-t^2/2} [(-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2}] = \sqrt{2\pi} i^n e^{-t^2/2} \phi_n(t)$

Equating both sides:

$e^{-t^2/2} \int_{-R}^{R} e^{-itx} (-1)^n \phi_n(x) dx = \sqrt{2\pi} i^n e^{-t^2/2} \phi_n(t)$

and sending R to ∞

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \phi_n(x) dx = (-i)^n \phi_n(t)$

(c) Show that $\phi_n'' - x^2\phi_n + (2n+1)\phi_n = 0$

$\phi_n'' = e^{-x^2/2} H_n''(x) + (-x) e^{-x^2/2} H_n'(x) + (-x) e^{-x^2/2} H_n'(x) + (x^2 e^{-x^2/2} - e^{-x^2/2}) H_n(x)$

$= e^{-x^2/2} H_n''(x) - 2x e^{-x^2/2} H_n'(x) + x^2\phi_n - \phi_n(x)$

$\phi_n'' - x^2\phi_n + \phi_n = e^{-x^2/2} (H_n''(x) - 2x H_n'(x))$ $(*)$

$H_n'(x) = (-1)^n 2x e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) = 2x H_n(x) - H_{n+1}(x)$

$2x H_n'(x) = (-1)^n [4x^2 e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) + 2x e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2})]$

$H_n''(x) = (-1)^n [(2e^{x^2} + 4x^2 e^{x^2}) \frac{d^n}{dx^n} (e^{-x^2}) + 4x e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) + e^{x^2} \frac{d^{n+2}}{dx^{n+2}} (e^{-x^2})]$

$\implies$

$H_n''(x) - 2x H_n'(x) = (-1)^n [2e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) + 2x e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) + e^{x^2} \frac{d^{n+2}}{dx^{n+2}} (e^{-x^2})]$

$H_n''(x) - 2x H_n'(x) = (-1)^n 2e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) - (-1)^{n+1} 2x e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) + (-1)^{n+2} e^{x^2} \frac{d^{n+2}}{dx^{n+2}} (e^{-x^2})$

$H_{n+2}(x) = (-1)^{n+2} e^{x^2} \frac{d^{n+2}}{dx^{n+2}} (e^{-x^2}) = 2x H_{n+1}(x) - (2(n+1)) H_n(x)$

This comes from $H_n'(x) = 2x H_n(x) - H_{n+1}(x)$ (shown earlier)

and that Hermite polynomials constitute an Appell sequence, so

$H_n'(x) = 2n H_{n-1}(x)$

$H_{n+1}(x) = (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2})$

$H_n''(x) - 2xH_n'(x) = 2H_n(x) - 2xH_{n+1}(x) + 2xH_{n+1}(x) - 2(n+1)H_n(x)$

$\implies H_n''(x) - 2xH_n'(x) = 2H_n(x) - 2(n+1)H_n(x) = -2nH_n(x)$

So plugging this back in $(*)$

$\phi_n'' - x^2\phi_n + \phi_n = -2n\phi_n \equiv \phi_n'' - x^2\phi_n + (2n+1)\phi_n = 0$

(d) Using $\int \phi_n''\phi_m dx = \int \phi_n\phi_m'' dx$ and (c) show that $\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)dx = 0, n \neq m$

$\int_{-\infty}^{\infty} \phi_n\phi_m dx =$ (using (c))

$\int_{-\infty}^{\infty} (-\phi_n'' + x^2\phi_n - 2n\phi_n)\phi_m dx$

=$\int_{-\infty}^{\infty} (-\phi_n''\phi_m + x^2\phi_n\phi_m - 2n\phi_n\phi_m)dx$

$= \int_{-\infty}^{\infty} (-\phi_n''\phi_m)dx + \int_{-\infty}^{\infty} x^2\phi_n\phi_m dx - 2n\int_{-\infty}^{\infty} \phi_n\phi_m dx$

= $\int_{-\infty}^{\infty} -\phi_n\phi_m'' dx + \int_{-\infty}^{\infty} x^2\phi_n\phi_m dx - 2n\int_{-\infty}^{\infty} \phi_n\phi_m dx$

using: $-\phi_n\phi_m'' = -\phi_n(x^2\phi_m - (2m+1)\phi_m) = (2m - x^2 + 1)\phi_n\phi_m$

$= 2m\int_{-\infty}^{\infty} \phi_n\phi_m dx - \int_{-\infty}^{\infty} x^2\phi_n\phi_m dx + \int_{-\infty}^{\infty} \phi_n\phi_m dx + \int_{-\infty}^{\infty} x^2\phi_n\phi_m dx - 2n\int_{-\infty}^{\infty} \phi_n\phi_m dx$

$\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)dx = (2m-2n)\int_{-\infty}^{\infty} \phi_m\phi_n dx + \int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)$

$\implies (2m-2n)\int_{-\infty}^{\infty} \phi_m\phi_n dx = 0$, and $2m - 2n \neq 0$, since $m \neq n$

so $\int_{-\infty}^{\infty} \phi_m\phi_n dx = 0$

Sec. 4

$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw$, $z \in D, m \geq 0$

D is a bounded domain with piecewise smooth boundary

$f(z)$ is analytic on D that extends smoothly to the boundary of D

1f $\int_{|z-1-i|=5/4} \frac{\text{Log } z}{(z-1)^2}dz = 2\pi i$

$1 \in \{|z-1-i| < 5/4\}$, since $|1-1-i| = |i| = 1 < 5/4$

$\text{Log } z = f(z)$ is analytic on D, since for any $w \in (-\infty, 0]$

$|w-1-i| = |-(|w|+1+i)| = ||w|+1+i| \geq |1+i| = \sqrt{2} > 5/4$, w is not in D and not on the boundary of D

so: $\frac{1}{1} = \frac{1}{2\pi i}\int_{|z-1-i|=5/4} \frac{\text{Log } z}{(z-1)^2}dz$, which gives the final answer $2\pi i$

1g $\oint_{|z|=1} \frac{dz}{z^2(z^2-4)(e^z)}$

$f(z) = e^{-z}$ is analytic everywhere

$\oint_{|z|=1} \frac{e^{-z}}{z^2(z^2-4)}dz$

$f(z) = \frac{e^{-z}}{z^2-4}$

$\oint_{|z|=1} f(z)/z^2 dz = 2\pi i[\frac{-e^{-z}(z^2-4)-e^{-z}(2z)}{(z^2-4)^2}]_{z=0}$

$= 2\pi i[\frac{-(-4)}{(-4)^2}] = 2\pi i\frac{1}{4} = \frac{\pi i}{2}$

1h $\oint_{|z-1|=3} \frac{e^{-z}}{z(z^2-4)} dz$

$0, 2 \in \{|z-1| < 3\}$

Let D_ϵ be the domain obtained from punching out small disks around $0, 2$ from $\{|z-1| < 3\}$

so we obtain

$\oint_{|z-1|=3} \frac{e^{-z}}{z(z^2-4)} dz = \oint_{|z|=\epsilon} \frac{e^{-z}/(z^2-4)}{z} dz + \oint_{|z-2|=\epsilon} \frac{e^{-z}/(z(z+2))}{z-2} dz$

$= 2\pi i(\frac{1}{-4} + \frac{e^{-2}}{8}) = -\frac{\pi i}{2} + \frac{\pi i}{4e^2}$

Sec 5

Ex. 2 Show that if $f(z)$ is an entire function, and there is a nonempty disk such that $f(z)$ does not attain any values in the disk, then $f(z)$ is a constant

$f(z)$ is entire $\implies$ $f(z)$ is analytic on the entire complex plane

and $\exists$ a disk D with center z_0 and radius ρ s.t. $f(z) \notin D$ for all z

so $1/(f(z) - z_0)$ is an entire function, $f(z) \neq z_0$, and $f(z)$ and z_0 are analytic everywhere

and is bounded:

for all z, since $f(z)$ doesn't attain any value in the disk D $|f(z) - z_0| \geq \rho \implies \frac{1}{|f(z)-z_0|} \leq 1/\rho$ for all z

so $1/(f(z) - z_0)$ is equal to some constant C by Liouville's theorem

which means $f(z) = 1/C + z_0$, a constant

Sec 6

Ex. 2 Let $h(t)$ be a continuous function on the interval $[a, b]$. Show that the Fourier transform $H(z) = \int_a^b h(t)e^{-itz} dt$ is an entire function that satisfies $|H(z)| \leq Ce^{A|y|}$, $z = x + iy \in \mathbb{C}$, for some constants $A, C > 0$

$H(z)$ is an entire function if it is analytic on the entire complex plane

We have that $h(t)e^{-itz}$ is continuous for $t \in [a, b]$ since the product of continuous functions are continuous.

Also, for each fixed t, $h(t)e^{-itz}$ is an analytic of $z \in \mathbb{C}$, as $h(t)$ is a constant, and e^{az} is analytic over all of $\mathbb{C}$, and in this case $a = -it$,

so we have $H(z)$ is analytic over all of $\mathbb{C}$ by the theorem on page 121

since $h(t)$ is continuous on the interval

$|H(z)| \leq (b-a)|h(t)||e^{-itz}|$

$|e^{-itz}| \leq |e^{-it(x+iy)}| \leq |e^{yt-itx}| = |e^{yt}e^{-itx}| = |e^{yt}|$

let $A = \max(|a|, |b|)$

let $C = M(b-a)$, where $|h(t)| \leq M$ for all t

$|H(z)| \leq Ce^{A|y|}$

Extra Problems

https://math.berkeley.edu/~art/data/F18-185/HW7.pdf

Ex 1

If we can continuously deform γ_1 to a small circular path around z_1 contained in γ_3

and continuously deform γ_2 to a small circular path around z_2 contained in γ_3

(so w is not in these disks)

We can define a region D to be the region inside γ_3 with punched out disks at z_1 and z_2

the disk centered at z_1 has the same orientation as γ_1, clockwise around z_1

the disk centered at z_2 doesn't have the same orientation as γ_2, which goes counterclockwise around z_2

and using the Cauchy Integral formula

$f(w) = \frac{1}{2\pi i}(I_3 - I_2 + I_1)$

When deforming closed paths, we can allow the starting point to move (page 81)

Let $\gamma_0'(t) = z_i + r(t)e^{i\theta(t)}$, $0 \leq t \leq 1$

We move the starting point such that $r(t) \geq r(0) = r(1)$ for all $t \in (0,1)$

Define $\gamma_1'(t) = z_i + r(0)e^{i\theta(0)+i2\pi mt}$, for some integer m

we define subdivisions of $[0,1]$, $0 = t_0 < \cdots < t_n = 1$ s.t. t_j is a point where we have intersections

1. $\theta_j(t_j) = \theta_{j+1}(t_j)$ since the path is continuous
2. $\theta_j(t_j) - \theta_j(t_{j-1}) = \begin{cases} 2\pi & \text{if the path went around counterclockwise once} \\ 0 & \text{if the path came back to } \gamma(t_{j-1}) \text{ from where it left} \\ -2\pi & \text{if the path went around clockwise once} \end{cases}$

So: $\sum_{i=1}^{n} \theta_i(t_i) - \theta_i(t_{i-1}) = m2\pi$, where m is an integer

also: $\sum_{i=1}^{n} \theta_i(t_i) - \theta_i(t_{i-1}) = \theta_n(t_n) - \theta_1(t_0) = \theta(1) - \theta(0) = m2\pi$

so $\gamma_1'(0) = \gamma_0'(0)$ and $\gamma_1'(1) = \gamma_0'(1)$

Define $\gamma_s'(t) = z_i + ((1-s)r(t) + sr(0))e^{i((1-s)\theta(t)+s(\theta(0)+2\pi m))}$

$\gamma_s'(t)$ depends continuously on s, t for $0 \leq s, t \leq 1$

let M be the max value of $r(t)$

we have that $z_i + Me^{i\theta(t)}$ is fully contained in the region bounded by γ_3

$r(0) \leq (1-s)r(t) + sr(0) \leq (1-s)M + sr(0) \leq (1-s)M + sM = M$

so $\gamma_s(t)$ remains inside the region bounded by γ_3

Ex 2

Determine if B is a domain

Domains must be open sets, but $0 \in B$

and for every open disk around 0, with radius $\epsilon > 0$,

$-\epsilon/2$ is contained in this disk, so not every element in B has an open disk centered on them contained in B, so B is not open.

B is not a domain.

How many times is $g(z)$ is complex diff. at $z = 0$

$g(z)$ is differentiable at 0 if

$\frac{g(z)-g(0)}{z}$ has a limit as $z \to 0$

if the limit exists, then the limit coming from the left must be the same as the limit coming from the right, but $z \to 0^-$ is not defined

so none.

How many times is $g(z)$ complex diff. at $z = 1$

restricting the function to an open disk D at 1 with an appropriately small radius,

$g(z) = z^2 \text{Log } z$ for all z in D

z^2 is analytic on D and so is $\text{Log } z$, so $g(z)$ is analytic on D

thus by the corollary on 115, $g(z)$ is infinitely differentiable at $z = 1$