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Math 123 Fall 2018

Homework #1

5. Consider the family of differential equations

   $x' = ax + \sin x$

   where a is a parameter.

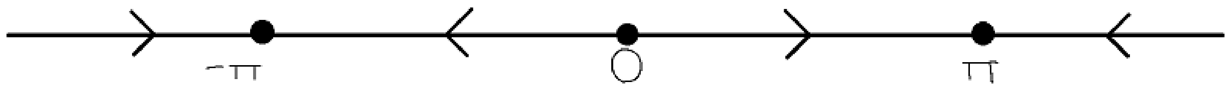
   a. Sketch the phase line when $a = 0$.

   $x' = \sin x$

   Equilibrium points at πk , where $k \in \mathbb{Z}$

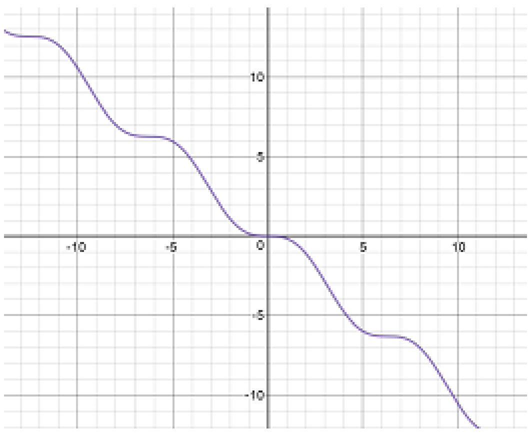
   $$x'' = \cos x = \begin{cases} 1 & x = 2k\pi \\ -1 & x = (2k+1)\pi \end{cases}$$

   So if $x = 2k\pi$ it is a source and if $x = (2k+1)\pi$ it is a sink



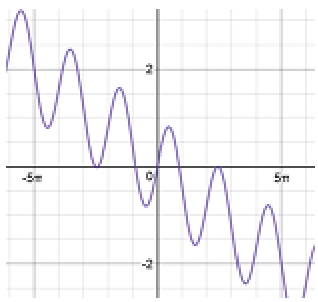
   b. as a increases from -1 to 1

   When $a = -1$, we have 1 equilibrium point at x = 0, and it seems to be a sink.



   We keep gaining equilibrium points in quantities of 2 as a increases towards 0, since

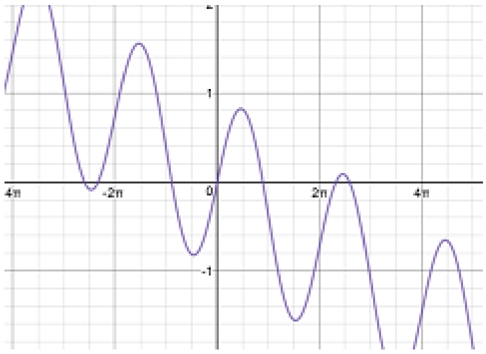
   1. For every $a = -(\frac{2}{(4m+1)\pi})$, with m an integer ≥ 0, we are "adding" one more equilibrium point at $x = \frac{(4m+1)\pi}{2}$, since $ax = -1$ and $\sin\frac{(4m+1)\pi}{2} = 1$
      1. Example: $a = -\frac{2}{5\pi}$



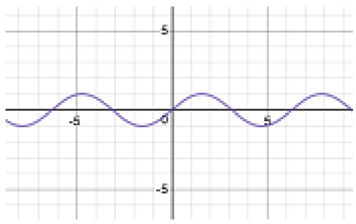
2. for any additional equilibrium point x, with $ax + \sin x = 0$, we have $-x$ as an equilibrium point as well: $a(-x) + \sin(-x) = -ax - \sin x = -(ax + \sin x) = 0$, since ax and $\sin x$ are both odd functions.

3. Then once we have $a \in (-\frac{2}{(4m+1)\pi,} - \frac{2}{(4m+5)\pi})$, the equilibrium points at $x = \pm\frac{(4m+1)\pi}{2}$ each split into 2 more equilibrium points.

   Example: $a = -\frac{2}{5\pi} + .01$



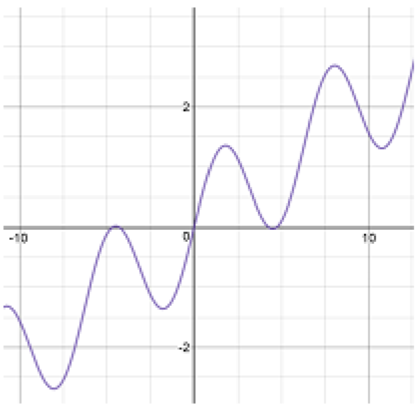
   And so we keep gaining equilibrium points until we hit $a = 0$, where we have infinitely many points, as can be seen from part (a).



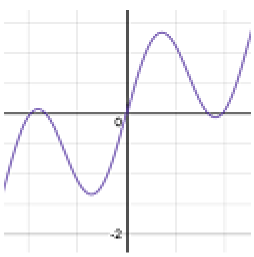
As we go from $a = 0$ to $a = 1$, we observe a similar phenomenon as that of when a increased from -1 to 0, except that we lose equilibrium points as we a increases, and whenever $a = \frac{2}{(4m+3)\pi}$, with m a nonnegative integer, we have that $x = \pm\frac{(4m+3)\pi}{2}$ are equilibrium points: at these x, $ax = \pm 1$ and $\sin x = \mp 1$ so $ax + \sin x = 0$.

Examples:

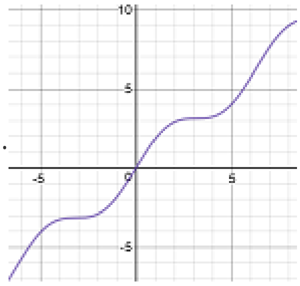
$a = \frac{2}{3\pi}$



$a = \frac{2}{3\pi} + .01$



When $a = 1$ only 1 equilibrium point at $x = 0$ and is a source.



(c) The bifurcation diagram: