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Math 123

Homework #7

Chapter 17, exercises 3, 4, 8, 12, and 14.

Ex. 3:

Let

$x' = y$

$y' = -x$

$X_0 = (x(0), y(0)) = (1, 0)$

We now have the system:

$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$, with $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We have i as an eigenvalue,

and so the general solution is

$X(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

and using the initial condition, the solution to the initial value problem:

$(\cos t, -\sin t)$

Using the Picard Iteration:

$u_0(t) = (1, 0)$

$u_1(t) = (1, 0) + \int_0^t F(1, 0) ds = (1, 0) + \int_0^t (0, -1) ds = \begin{pmatrix} 1 \\ -t \end{pmatrix}$

$u_2(t) = (1, 0) + \int_0^t (-s, -1) ds = \begin{pmatrix} 1 - \frac{t^2}{2} \\ -t \end{pmatrix}$

$u_3(t) = (1, 0) + \int_0^t (-s, -1 + s^2/2) ds = \begin{pmatrix} 1 - t^2/2 \\ -t + t^3/3! \end{pmatrix}$

$u_4(t) = (1, 0) + \int_0^t (-s + s^3/3!, -1 + s^2/2) ds = \begin{pmatrix} 1 - t^2/2 + t^4/4! \\ -t + t^3/3! \end{pmatrix}$

I want to show $u_k = \begin{pmatrix} \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=1}^{\lceil k/2 \rceil} (-1)^i \frac{t^{2i-1}}{(2i-1)!} \end{pmatrix}$

We know this is true for the case $k = 0$, $u_0(t) = (1, 0)$

so $u_{k+1}(t) = u_0(t) + \int_0^t F(u_k(s))ds$

Using Inductive Step, we assume $u_k(t) = u_k = \begin{pmatrix} \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=1}^{\lceil k/2 \rceil} (-1)^i \frac{t^{2i-1}}{(2i-1)!} \end{pmatrix}$

and we end up with $u_{k+1}(t) = u_0(t) + \int_0^t \begin{pmatrix} \sum_{i=1}^{\lceil k/2 \rceil} (-1)^i \frac{s^{2i-1}}{(2i-1)!} \\ -\sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{s^{2i}}{(2i)!} \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^{\lceil k/2 \rceil} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+1} \frac{t^{2i+1}}{(2i+1)!} \end{pmatrix}$

$= u_{k+1}(t) = \begin{pmatrix} \sum_{i=0}^{\lceil k/2 \rceil} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=1}^{\lfloor k/2 \rfloor + 1} (-1)^i \frac{t^{2i-1}}{(2i-1)!} \end{pmatrix}$

and $\lfloor (k+1)/2 \rfloor = \lceil k/2 \rceil$, $\lceil (k+1)/2 \rceil = \lfloor k/2 \rfloor + 1$

so we have $u_{k+1}(t) = \begin{pmatrix} \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=1}^{\lceil (k+1)/2 \rceil} (-1)^i \frac{t^{2i-1}}{(2i-1)!} \end{pmatrix}$

so as $k \to \infty$,

$u_k(t) \to (\cos t, -\sin t)$, by the Existence and Uniqueness Theorem, since $F(x, y) = (-y, x)$ is C^1

so $\begin{pmatrix} \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i}}{(2i)!} \\ \sum_{i=1}^{\infty} (-1)^i \frac{t^{2i-1}}{(2i-1)!} \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

Ex 4: For each of the following functions, find a Lipschitz constant on the region indicated, or prove there is none:

To find a Lipschitz constant means to find a constant K s.t.

$|F(Y) - F(X)| \leq K|Y - X|$ for all X, Y in the region

this is equivalent to $|\frac{F(Y)-F(X)}{Y-X}| \leq K$

(a) $f(x) = |x|, -\infty < x < \infty$

Using the triangle inequality:

$|(a-b) + b| \leq |a-b| + |b| \implies |a| - |b| \leq |a-b|$

$|(b-a) + a| \leq |b-a| + |a| \implies |b| - |a| \leq |b-a| \equiv |a| - |b| \geq -|a-b|$

so $||a| - |b|| \leq |a-b|$

meaning: $||y| - |x|| \leq |y-x| \equiv |F(y) - F(x)| \leq |y-x|$, so $K = 1$

(b) $f(x) = x^{1/3}, -1 \leq x \leq 1$

Let $x = 0$, and $y \to 0$,

I will show: $|\frac{f(y)-f(0)}{y-0}| \to \infty$ as $y \to 0$, which would mean there is no constant K

s.t. $|f(y) - f(0)| \leq K|y-0|$, so there is no Lipschitz constant on the whole region.

as $y \to 0$, $|\frac{f(y)-f(0)}{y-0}| \to \lim_{y \to 0} |\frac{y^{1/3}}{y}| = \lim_{y \to 0} \frac{1}{y^{2/3}} \to \infty$

(c) $f(x) = 1/x, 1 \leq x \leq \infty$

Need to find a constant K s.t.

$|\frac{1}{y} - \frac{1}{x}| \leq K|y - x|$, for all $y, x \in [1, \infty]$

$|\frac{1}{y} - \frac{1}{x}| = |\frac{x-y}{yx}|$

and since $y, x \geq 1$, $yx \geq 1$, $|\frac{x-y}{yx}| = \frac{|x-y|}{xy}$

and $\frac{|x-y|}{yx} \leq |x - y|$, since $\frac{1}{xy} \leq 1$

(d) $f(x, y) = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}$, $(x, y) \in \mathbb{R}^2$

so let $X_1, X_2 \in \mathbb{R}^2$,

$|f(X_2) - f(X_1)| = |\begin{pmatrix} x_2 + 2y_2 \\ -y_2 \end{pmatrix} - \begin{pmatrix} x_1 + 2y_1 \\ -y_1 \end{pmatrix}|$

$= |(x_2, -y_2) + (2y_2, 0) - (x_1, -y_1) - (2y_1, 0)|$

and using the triangle inequality, ($\mathbb{R}^2$ is a metric space, metrics have the triangle inequality)

$\leq |(x_2, -y_2) - (x_1, -y_1)| + |(2y_2 - 2y_1, 0)|$

Looking at each of the summands

1. $|(x_2, -y_2) - (x_1, -y_1)| = |(x_2 - x_1, y_1 - y_2)|$
    1. and since $(y_2 - y_1)^2 = (y_1 - y_2)^2$ we have $|(x_2 - x_1, y_1 - y_2)| = |X_2 - X_1|$
2. $|(2(y_2 - y_1), 0)| = 2|(y_2 - y_1, 0)|$
    1. $(y_2 - y_1)^2 \leq (y_2 - y_1)^2 + (x_2 - x_1)^2 \equiv |(y_2 - y_1, 0)| \leq |X_2 - X_1|$
    2. $2|(y_2 - y_1, 0)| \leq 2|X_2 - X_1|$

So we have $|f(X_2) - f(X_1)| \leq 3|X_2 - X_1|$, $K = 3$

(e) $f(x, y) = \frac{xy}{1+x^2+y^2}, x^2 + y^2 \leq 4$

I want to show f is C^1, so I may use that an upper bound for $|Df_x| = |\nabla f|$ is a Lipchitz constant for $f(x, y)$ on $x^2 + y^2 \leq 4$, which is convex (it is a closed ball of radius 2)

$f(x, y)$ has both of its partial derivatives:

$f_x(x, y) = \frac{y - x^2y + y^3}{(1+x^2+y^2)^2}$, and $f_y(x, y) = \frac{x - xy^2 + x^3}{(1+x^2+y^2)^2}$

sums, products, and quotients of continuous functions are continuous,

and $g(x, y) = x, h(x, y) = y, \text{and } k(x, y) = c$ are continuous

polynomials are therefore continuous as they are sums and products of f, g, h

and f_x, f_y are each a quotient of polynomials of two variables, so they are continuous (and the denominator is never 0 on the region $x^2 + y^2 \leq 4$)

so f is C^1

$|\nabla f| = \sqrt{f_x^2 + f_y^2}$

this is equal to the square root of

$\frac{x^2+y^2-x^4y^2-x^2y^4+x^6y^6+2(x^4+y^4)}{(1+x^2+y^2)^2}$

$|x|, |y| \leq 2$, so roughly, we have

$x^2+y^2-x^4y^2-x^2y^4+x^6+y^6+2(x^4+y^4) \leq 4+2^7+2^6=196$

and $1 \leq (1+x^2+y^2)^2 \leq 25$

so $|\nabla f|^2 \leq 196 \equiv |\nabla f| \leq 14$, $K=14$ is a Lipschitz Constant

Ex. 8 Let $A(t)$ be a continuous family of $n \times n$ matrices and let $P(t)$ be the matrix solution to the IVP $P' = A(t)P$, $P(0) = P_0$ Show that $\det P(t) = (\det P_0)\exp(\int_0^t \text{Tr } A(s)ds)$

$\frac{1}{h}(P(t+h)-P(t)) \to P'(t) = A(t)P$ as $h \to 0$

$hA(t)P(t)- (P(t+h) - P(t)) = o(h)$,

since $\frac{A(t)P(t)h-P(t+h)+P(t)}{h} \to 0$ as $h \to 0$

$P(t+h) = hP'(t)+P(t)+o(h) = hA(t)P(t)+P(t)+o(h) = (I+hA(t))P(t)+o(h)$

$\det P(t+h) = \det(I+hA(t))\det P(t)+o(h)$

if $M = \begin{pmatrix} a+o(h) & b+o(h) \\ c+o(h) & d+o(h) \end{pmatrix}$, we have the determinant = $ad-bc+do(h)+ao(h)-bo(h)-co(h)$
$= \det M + o(h)$

so with induction, it would be easy to set that this is the case for $\det P(t+h)$

and $I + hA(t) = \begin{pmatrix} 1+ha_{11}(t) & \cdots & ha_{1n}(t) \\ \vdots & \ddots & \vdots \\ ha_{n1}(t) & \cdots & 1 + ha_{nn}(t) \end{pmatrix}$

so the determinant will be $= 1+ha_{11}(t)+ha_{22}(t)+\ldots+ha_{nn}(t)+O(h^2)$

$= 1+h\text{Trace } A+O(h^2)$

Proof:

Induction on size of square matrix,

obvious for case 1×1, where the determinant is $1+ha_{11}(t)+0$,

and $0 \leq h^2$ whenever $0 \leq |h| < \delta$, for any $\delta > 0$

and the case 2×2 where the determinant is $(1+ha_{11})(1+ha_{22})-h^2a_{12}a_{21}$
$= 1+ha_{11}+ha_{22}+h^2(a_{11}a_{22}-a_{12}a_{21})$

Suppose this is true for $k \times k$ matrices, and $(k-1) \times (k-1)$ matrices

let $B = I + hA$ be a $k+1 \times k+1$ matrix ,

and let B_{ij} be the matrix without the ith row and jth column

then $\det B = (1+ha_{11}(t))\det B_{11} + \sum_{j=2}^{k+1}(-1)^{1+j}ha_{1j}\det B_{1j}$

$\det B_{11} = 1+h\text{Trace } A_{11}+O(h^2)$ by inductive hypothesis and each $\det B_{1j}$ is going to be a linear combination of the determinant of $(k-1) \times (k-1)$ matrices that have $k-1$ diagonal entries of the form $1 + ha_{ii}(t)$, where $i \neq 1, i \neq j$

and all the other entries will be of the form ha_{il}, where $i \neq 1, l \neq 1, j$, and using the inductive hypothesis again, the determinant of this matrix is $(1 + h\text{Trace } A_{12,jl} + O(h^2))$, where $l \neq j$ and these terms get multiplied by $h^2 a_{1j} a_{2l}$,

so the sum will be $O(h^2)$, and we have altogether

$1 + h\text{Trace } A_{11} + ha_{11}(t) + O(h^2) = 1 + h\text{Trace } A + O(h^2)$

so $\det P(t+h) = (1 + h\text{Trace } A + O(h^2) \det P(t)) + o(h)$

and $\frac{d}{dt} \det P(t) = \lim_{h \to 0} \frac{\det P(t+h) - \det P(t)}{h} - \lim_{h \to 0} \frac{h\text{Trace } A \cdot \det P(t) + O(h^2) \det P(t) + o(h)}{h}$

which $= \lim_{h \to 0} \text{Trace A} \cdot \det P(t) + O(h) + o(h)/h$, and $O(h), o(h)/h \to 0$ as $h \to 0$

so $\frac{d}{dt} \det P(t) = \text{Trace A} \cdot \det P(t)$, so solving this differential equation, we get

$\det P(t) = C \exp(\int_0^t \text{Trace } A(s) ds)$, and when $t = 0$, $C = \det P(0) = \det P_0$, which proves the statement.

Ex. 12 Prove the following general fact: If $C \geq 0$ and $u, v : [0, \beta] \to \mathbb{R}$ are continuous and nonnegative, and $u(t) \leq C + \int_0^t u(s)v(s)ds$ for all $t \in [0, \beta]$, then $u(t) \leq Ce^{V(t)}$ where $V(t) = \int_0^t v(s)\, ds$

Suppose $C > 0$,

Let $U(t) = C + \int_0^t u(s)v(s)ds$, which is greater than 0 since u, v are nonnegative

$U'(t) = u(t)v(t)$

$\frac{U'(t)}{U(t)} = \frac{u(t)v(t)}{U(t)} \leq v(t)$ since $u(t) \leq U(t)$

and $\frac{d}{dt} \log U(t) = (1/U(t))U'(t)$ so

$\frac{d}{dt}(\log U(t)) \leq v(t)$

$\implies \int_0^t \frac{d}{dt} \log U(s) ds = \log U(t) - \log U(0) \leq \int_0^t v(s) ds$

$\log U(t) \leq \log U(0) + \int_0^t v(s) ds$

and $U(0) = C + \int_0^0 u(s)v(s)ds = C$

$U(t) \leq \exp(\log C + \int_0^t v(s)ds) = C \exp(\int_0^t v(s)ds)$

and $u(t) \leq C \exp(\int_0^t v(s)ds)$

For the case of $C = 0$

We can construct a sequence of positive real numbers $c_n \to 0$ as $n \to \infty$, like $c_n = 1/n$

this gives us a sequence $u(t) \leq u_n(t) = c_n + \int_0^t u(s)v(s)ds$,

and since $c_n > 0$, using the argument above, this means each $u_n(t) \leq c_n e^{V(t)}$ for each $n \in \mathbb{N}$,

and as $n \to \infty$, we have $c_n e^{V(t)} \to 0$, and so $u_n(t) \leq 0$, meaning $u(t) \leq 0$

Ex. 14 Let $A(t)$ be a continuous family of $n \times n$ matrices. Let $(t_0, X_0) \in J \times \mathbb{R}^n$. Then the initial value problem $X' = A(t)X, X(t_0) = X_0$ has a unique solution on all of J

This the Corollary of the Theorem on page 399:

Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be open and $F : \mathcal{O} \to \mathbb{R}^n$ a function that is C^1 in X and continuous in t. If $(t_0, X_0) \in \mathcal{O}$, there is an open interval J containing t_0 and a unique solution of $X' = F(t, X)$ defined on J and satisfying $X(t_0) = X_0$

The function $F(t, X) = A(t)X$ is a function that takes $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

F is continuous in t, since $A(t)$ is continuous

Showing F is C^1 in X:

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

$$A(t)X = \begin{pmatrix} \sum_{k=1}^n a_{1k}(t)x_k(t) \\ \vdots \\ \sum_{k=1}^n a_{nk}(t)x_k(t) \end{pmatrix}$$

$$DF_X = (\frac{\partial f_i}{\partial x_j}) = \frac{\partial}{\partial x_j}(\sum_{k=1}^n a_{ik}(t)x_k) = \sum_{k=1}^n \frac{\partial}{\partial x_j}(a_{ik}(t)x_k)$$

the last equality coming from the definition of a derivative being a limit, and the limit of a sum is the sum of the limits.

$= a_{ij}(t)$

$DF_X = A(t)$

and $A(t)$ is continuous, so DF_X is continuous.

$A(t)$ is continuous everywhere, and so $DF_X = A(t)$ is continuous everywhere,

so on any $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$, $F(t, X) = A(t)X$ is C^1 in X and continuous in t.

J is an open interval containing t_0, and so, if we define $\mathcal{O}$ to be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing J, then using the theorem, we have a unique solution for $X' = A(t)X$ defined on J satisfying $X(t_0) = X_0$