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THE REFORMULATION OF EDELBAUM'S LOW-THRUST TRANSFER PROBLEM USING OPTIMAL CONTROL THEORY*

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ABSTRACT

Edelbaum's problem of low-thrust transfer between inclined orbits is reformulated within the framework of optimal control theory. The original treatment considered the time-constrained inclination maximization with velocity as independent variable allowing the use of the theory of maxima. Since the independent variable is double valued for some transfers, two expressions for the inclination change involving inverse sine functions are needed to describe all possible transfers. The present analysis casts this problem as a minimum-time transfer between given noncoplanar circular orbits and obtains a single analytic expression for the orbital inclination involving a single inverse tangent function, uniformly valid for all transfers.

I. INTRODUCTION

An elegant discussion of the problem of low-thrust transfer between inclined circular orbits was presented by Edelbaum in the early sixties.[1] Assuming constant acceleration and constant thrust vector yaw angle within each revolution, Edelbaum linearized the Lagrange planetary equations of orbital motion about a circular orbit and using the velocity as the independent variable, reduced the transfer optimization problem to a problem in the theory of maxima. The variational integral involved a single constant Lagrange multiplier since it involved a single integral constraint equation for the transfer time or velocity change, while maximizing the change in orbital inclination. The control variable being the yaw angle, the necessary condition for a stationary solution was obtained by simply setting the partial derivative of the integrand of the variational integral with respect to the control to zero. This optimum control was then used in the right hand sides of the original equations of motion which were integrated analytically to provide expressions for the time and inclination in terms of the independent variable, the orbital velocity. However, the analytic expression for the inclination is given in terms of a certain inverse sine function which is a double-valued function in the interval $(0,2\pi)$, for a given value of its argument which includes the velocity. It is for this reason that two expressions for the inclination must be provided to cover the case of large inclination change transfers. For these transfers, it is necessary to use one of these expressions first to describe the transfer up to a certain point in time and switch to the other expression for the balance of the transfer. A certain condition involving current time, initial velocity and initial yaw angle must be monitored in order to evaluate the time at which the switch from one expression to the other must take place for a continuous description of the overall inclination change history. This complication was introduced as a result of the formulation used by Edelbaum who adopted orbital velocity as the independent variable which led to the integration of the differential equation for the inclination with respect to the velocity.

This unnecessary difficulty is removed, and a single expression for the inclination change, uniformly valid throughout any desired transfer is obtained by the present analysis which casts the original Edelbaum problem into a minimum-time problem using the formalism of optimal control theory.

In this analysis, inclination and velocity are the dependent variables and time the independent variable such that the original linearized equations of orbital motion are used as the constraining differential equations to form the variational Hamiltonian involving the two inclination and velocity adjoint variables.

As expected, the inclination adjoint is constant and the velocity adjoint is given simply in terms of the cosine of the single control variable, the yaw angle.

All the Edelbaum formulas are recovered including the ΔV formula which relates the total velocity change to the initial and final velocities as well as the total inclination change. This formula is valid for the range of total inclination changes $0 < \Delta i < 114.6°$ since for $\Delta i > 114.6°$, $\Delta V = V_o + V_f$ the sum of the boundary velocities regardless of the magnitude of the inclination change. In this latter case, the transfer expands the orbit radius to infinity with cost V_o where the rotation is carried out at no cost, and then it shrinks the radius to the final condition with cost V_f such that the total velocity change to be imparted to the transferring vehicle is the sum of the initial and final orbital velocities. We believe our expressions for all the pertinent state and control variables are simple and straightforward to use and a simple algorithm shows how to apply them equally well for both the problems of transfer to higher orbit or lower orbit irrespective of whether the initial inclination is smaller or larger than the final inclination. It is also established that for autonomous on-board use, an Electric Orbit Transfer Vehicle (EOTV) can make use of this analytic theory even though shadowing can somewhat complicate the transfer. In this latter case, the yaw profile must be modified and eccentricity buildup prevented by introducing the pitch control angle and shifting the yaw switch points from the antinodes to new locations along the orbit in order to carry out the transfer to the final desired orbit.

In Section II, Edelbaum's analysis is presented with all the necessary details. Section III presents the formulation of Edelbaum's problem using optimal control theory with several transfer examples shown in Section IV.

II. EDELBAUM'S FORMULATION

The full set of the Gaussian form of the Lagrange planetary equations for near-circular orbits is given by

$$\dot{a} = \frac{2af_t}{V} \tag{1}$$

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$$\dot{e}_x = \frac{2f_t c_\alpha}{V} - \frac{f_n s_\alpha}{V} \quad (2)$$

$$\dot{e}_y = \frac{2f_t s_\alpha}{V} + \frac{f_n c_\alpha}{V} \quad (3)$$

$$\dot{i} = \frac{f_h c_\alpha}{V} \quad (4)$$

$$\dot{\Omega} = \frac{f_h s_\alpha}{(V s_i)} \quad (5)$$

$$\dot{\alpha} = n + \frac{2f_n}{V} - \frac{f_h s_\alpha}{(V \tan i)} \quad (6)$$

where s_α and c_α stand for $\sin\alpha$ and $\cos\alpha$ respectively, and where a stands for the orbit semi-major axis, i for inclination, Ω for the right ascension of the ascending node, $e_x = e\cos\omega$, $e_y = e\sin\omega$ with e and ω standing for orbital eccentricity and argument of perigee.

Finally, $\alpha = \omega + M$ represents the mean longitude, M the mean anomaly and $n = (\mu/a^3)^{1/2}$ the orbit mean motion with μ standing for the earth gravity constant. For near-circular orbits, $V = na = (\mu/a)^{1/2}$. The components of the thrust acceleration vector along the tangent, normal and out-of-plane directions are depicted by f_t, f_n and f_h with the normal direction oriented toward the center of attraction.

Assuming only tangential and out-of-plane acceleration, and assuming the orbit remains circular during the transfer, the above equations reduce to

$$\dot{a} = \frac{2af_t}{V} \quad (7)$$

$$\dot{i} = \frac{f_h c_\alpha}{V} \quad (8)$$

$$\dot{\Omega} = \frac{f_h s_\alpha}{(V s_i)} \quad (9)$$

$$\dot{\alpha} = n - \frac{f_h s_\alpha}{(V \tan i)} \quad (10)$$

If f represents the magnitude of the acceleration vector, and β the out-of-plane or yaw angle, then $f_t = f c_\beta$ and $f_h = f s_\beta$. Furthermore, $\alpha = \omega + M = \omega + \theta^* = \theta$ the angular position when $e = 0$, with $\theta = nt$ and θ^* the true anomaly.

If the angle β is held piecewise constant switching sign at the orbital antinodes, then the $f_h s_\alpha$ terms above will have a net zero contribution such that the system of differential equations further reduces to

$$\dot{a} = \frac{2af_t}{V} \quad (11)$$

$$\dot{i} = \frac{c_\theta f_h}{V} \quad (12)$$

$$\dot{\theta} = n \quad (13)$$

Edelbaum averages out the angular position θ in Eq. (12) by integrating with respect to θ and by holding f, β, and V constant

$$\int_o^{2\pi} \left(\frac{di}{dt}\right) d\theta = \frac{2fs_\beta}{V} \int_{-\pi/2}^{\pi/2} c_\theta \, d\theta$$

such that

$$2\pi \frac{di}{dt} = \frac{4fs_\beta}{V}$$

and

$$\frac{di}{dt} = \frac{2fs_\beta}{\pi V} \quad (14)$$

From the energy equation $V^2/2 - \mu/r = -\mu/2a$, with $r = a$, and using Eq. (11) to eliminate the semi-major axis,

$$dV = -\left[\frac{\mu}{2Va^2}\right] da$$

$$= -fc_\beta \, dt$$

so that

$$\frac{dV}{dt} = -fc_\beta \quad (15)$$

Equation (14) can also be obtained by dividing Eq. (12) by Eq. (13)

$$\frac{di}{d\theta} = \frac{c_\theta f_h}{Vn}$$

$$\Delta i = \frac{2f_h}{Vn} \int_{-\pi/2}^{\pi/2} c_\theta d\theta = \frac{4f_h}{Vn}$$

and since $\Delta t = 2\pi \, a/V$,

$$\frac{di}{dt} = \frac{\Delta i}{\Delta t} = \frac{2f_h}{V\pi}$$

Equations (14) and (15) can be replaced by the following set where V is now the independent variable

$$\frac{di}{dV} = -\frac{2\tan\beta}{\pi V} \quad (16)$$

$$\frac{dt}{dV} = -\frac{1}{fc_\beta} \quad (17)$$

Let I represent the functional to be maximized

$$I = \int_{V_o}^{V_f} \left(\frac{di}{dV}\right) dV = -\int_{V_o}^{V_f} \frac{2}{\pi V} \tan\beta dV \quad (18)$$

and let J represent the integral constraint given by

$$J = \int_{V_o}^{V_f} \left(\frac{dt}{dV}\right) dV = constant \quad (19)$$

Edelbaum adjoins J to I by way of a constant Lagrange multiplier λ such that the optimization problem is reduced to a succession of ordinary maximum problems for each value of V between V_o and V_f the initial and final velocities respectively. The necessary condition for a stationary solution of the augmented integral

$$K = I + \lambda J = \int_{V_o}^{V_f} \left[-\frac{2}{\pi V}\tan\beta - \frac{\lambda}{fc_\beta} \right] dV \quad (20)$$

is then simply given by

$$\frac{\partial}{\partial\beta}\left[\frac{2}{\pi V}\tan\beta + \frac{\lambda}{fc_\beta}\right] = 0 \qquad (21)$$

The optimization problem consists therefore of the maximization of the inclination change subject to the constraint of given total transfer time since

$$\Delta V = ft \qquad (22)$$

This constraint is equivalent to the fixed ΔV constraint for constant acceleration f. Furthermore, V_o and V_f being given, the initial and final radii are therefore given too since the orbits are assumed circular. The acceleration being applied continuously, this problem is equivalent to minimizing the total transfer time for a given change in the inclination and velocity. This is also equivalent to minimizing the total ΔV or propellant usage because the thrust is always on and no coasting arcs are allowed. In this latter case, I and J are simply interchanged to yield the optimality condition

$$\frac{\partial}{\partial\beta}\left[\frac{1}{fc_\beta} + \lambda_i\frac{2}{\pi V}\tan\beta\right] = 0 \qquad (23)$$

From Eq. (21), it follows that

$$Vs_\beta = -\frac{2f}{\pi\lambda} = constant = V_o s_{\beta o} \qquad (24)$$

such that

$$\lambda = -\frac{2f}{\pi V_o s_{\beta o}} \qquad (25)$$

The optimal ß steering law given by Eq. (24) can be used in Eq. (17) for dV/dt in order to obtain the expression for the velocity as a function of time t or ΔV = ft.

$$\frac{dV}{dt} = -fc_\beta$$

$$fdt = -\frac{dV}{c_\beta} = \frac{-dV}{\pm(1-s_\beta^2)^{1/2}}$$

$$\int_o^t fdt = -\int_{V_o}^{V} -\frac{VdV}{\pm\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}}$$

$$\Delta V = ft = -\frac{1}{\pm}\left[\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2} - (\pm)V_o c_{\beta o}\right]$$

$$\Delta V = V_o c_{\beta o} \mp \left(V^2 - V^2 s_\beta^2\right)^{1/2} = V_o c_{\beta o} \mp (\pm)Vc_\beta \qquad (26)$$

$$\Delta V = V_o c_{\beta o} - Vc_\beta \qquad (27)$$

From Eq. (26),

$$\Delta V - V_o c_{\beta o} = \mp\left(V^2 - V^2 s_\beta^2\right)^{1/2}$$
$$= \mp\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}$$

and after squaring

$$V^2 = V_o^2 + \Delta V^2 - 2\Delta V\, V_o c_{\beta o} \qquad (28)$$

This then represents V as a function of time since ΔV = ft. The initial yaw angle ßo must still be determined

In a similar way, Eq. (16) for di/dV can be integrated to provide an expression for the evolution of the inclination in time.

$$\frac{di}{dV} = -\frac{2}{\pi V}\tan\beta = -\frac{2}{\pi V}\frac{s_\beta}{c_\beta}$$

$$= -\frac{2}{\pi V}\frac{Vs_\beta}{\left(V^2 - V^2 s_\beta^2\right)^{1/2}}$$

such that with the use of $Vs_\beta = V_o s_{\beta o}$,

$$\int_{i_o}^{i} di = -\frac{2}{\pi}V_o s_{\beta o}\int_{V_o}^{V}\frac{dV}{V\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}}$$

$$\Delta i = -\frac{2}{\pi}\sin^{-1}\left[\frac{-V_o s_{\beta o}}{V}\right]\Bigg|_{V_o}^{V}$$

$$\Delta i = -\frac{2}{\pi}\left[\sin^{-1}(s_{\beta o}) - \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right)\right]$$

$$\Delta i = -\frac{2}{\pi}\beta o + \frac{2}{\pi}\sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) \qquad (29)$$

and since $V_o s_{\beta o} = V s_\beta$,

$$\Delta i = \frac{2}{\pi}(\beta - \beta o)$$

Now, since the inverse sine function in Eq. (29) is double valued in the interval (0, 2π), it is necessary to write this function as

$$\sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) \quad \text{if} \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) < \frac{\pi}{2}$$

and

$$\frac{\pi}{2} + \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right)\right]$$

$$= \pi - \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) \quad \text{if} \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) > \frac{\pi}{2}$$

since the function is symmetrical with respect to π/2. In the second condition above, Δi can be written as

$$\Delta i = \frac{2}{\pi}\left[\pi - \sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right)\right] - \frac{2}{\pi}\beta o \qquad (30)$$

or

$$\Delta i = 2 - \frac{2}{\pi}\sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) - \frac{2}{\pi}\beta o \qquad (31)$$

This is equivalent to writing Eqs. (29) and (31) as

$$\Delta i = \frac{2}{\pi}(\beta - \beta o) \qquad \text{if } \beta < \frac{\pi}{2} \tag{32}$$

$$\Delta i = 2 - \frac{2}{\pi}(\beta + \beta o) \quad \text{if } \beta > \frac{\pi}{2} \tag{33}$$

Of course the number 2 appearing in Eq. (33) is given in radians and it corresponds to 114.6°.

Finally, from Eq. (26),

$$\Delta V = V_o c_{\beta o} - \left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2} \quad \text{if } \Delta V - V_o c_{\beta o} < 0 \tag{34}$$

$$\Delta V = V_o c_{\beta o} + \left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2} \quad \text{if } \Delta V - V_o c_{\beta o} > 0 \tag{35}$$

It is clear from $\Delta V = V_o c_{\beta o} - V c_\beta$ that the condition $\Delta V - V_o c_{\beta o} < 0$ is identical to $c_\beta > 0$ or $\beta < \pi/2$ or $\sin^{-1}(V_o s_{\beta o}/V) < \pi/2$ and that the condition $\Delta V - V_o c_{\beta o} > 0$ is identical to $\beta > \pi/2$ or $\sin^{-1}(V_o s_{\beta o}/V) > \pi/2$ such that the Edelbaum analysis leads to the following set of equations

1) if $\Delta V - V_o c_{\beta o} < 0$, then

$$\left.\begin{aligned} V &= \left(V_o^2 - 2V_o \Delta V c_{\beta o} + \Delta V^2\right)^{1/2} \\ \Delta i &= \frac{2}{\pi}\sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) - \frac{2}{\pi}\beta o = \frac{2}{\pi}(\beta - \beta o) \end{aligned}\right\} \tag{36}$$

2) if $\Delta V - V_o c_{\beta o} > 0$, then

$$\left.\begin{aligned} V &= \left(V_o^2 - 2V_o \Delta V c_{\beta o} + \Delta V^2\right)^{1/2} \\ \Delta i &= 2 - \frac{2}{\pi}\sin^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) - \frac{2}{\pi}\beta o = 2 - \frac{2}{\pi}(\beta + \beta o) \end{aligned}\right\} \tag{37}$$

It is clear from the above equations that one must monitor the condition $\Delta V - V_o c_{\beta o}$ and use Eq. (36) to describe the transfer starting from time 0 and later switch to Eqs. (37) as soon as $t = \Delta V/f$ exceeds $V_o c_{\beta o}/f$ which will take place for large transfers as will be shown later by some examples.

Finally, for those large transfers, ΔV as given in Eq. (26) could become double valued in V such that one must use $\Delta V = V_o c_{\beta o} - (V^2 - V_o^2 s_{\beta o}^2)^{1/2}$ from V_o to $V_o s_{\beta o}$ and $\Delta V = V_o c_{\beta o} + (V^2 - V_o^2 s_{\beta o}^2)^{1/2}$ from $V = V_o s_{\beta o}$ to V_f where $V_o s_{\beta o} < V_f$. This minimum velocity takes place when $\Delta V = V_o c_{\beta o}$ indicating that the orbit will grow to become larger than the final desired orbit, and later shrink to that desired orbit. This will happen when larger inclination changes are required since then the orbit plane rotation will be carried out mostly at those higher intermediate altitudes. This of course, is the result of the trade between inclination and radius or velocity. From

$$\Delta V = V_o c_{\beta o} \mp \left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}$$

$$\frac{\partial \Delta V}{\partial V} = \mp \frac{V}{\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}}$$

which is equal to ∞ for $V = V_o s_{\beta o}$ the minimum velocity reached.

The initial yaw angle βo can be obtained from the terminal conditions at time t_f. At $t = t_f$, $V = V_f$ and $\Delta i = \Delta i_f$. Using Eq. (36) for Δi, it is clear that

$$\frac{\pi}{2}\Delta i_f + \beta o = \sin^{-1}\left(\frac{V_o s_{\beta o}}{V_f}\right)$$

or

$$\sin\left(\beta o + \frac{\pi}{2}\Delta i_f\right) = \frac{V_o s_{\beta o}}{V_f}$$

$$\sin\beta o \cos\frac{\pi}{2}\Delta i_f + \cos\beta o \sin\frac{\pi}{2}\Delta i_f = \frac{V_o s_{\beta o}}{V_f}$$

Dividing both sides by $c_{\beta o}$, yields

$$\left[\cos\frac{\pi}{2}\Delta i_f - \frac{V_o}{V_f}\right]\tan\beta o = -\sin\frac{\pi}{2}\Delta i_f$$

$$\tan\beta o = \frac{\sin\frac{\pi}{2}\Delta i_f}{\frac{V_o}{V_f} - \cos\frac{\pi}{2}\Delta i_f} \tag{38}$$

Now carrying out the same manipulations using Eq. (37), we get

$$\frac{\pi}{2}\left(\Delta i_f - 2 + \frac{2}{\pi}\beta o\right) = -\sin^{-1}\left(\frac{V_o s_{\beta o}}{V_f}\right)$$

$$\pi - \left(\beta o + \frac{\pi}{2}\Delta i_f\right) = \sin^{-1}\left(\frac{V_o s_{\beta o}}{V_f}\right)$$

$$\sin\left(\beta o + \frac{\pi}{2}\Delta i_f\right) = \frac{V_o s_{\beta o}}{V_f}$$

$$\sin\beta o \cos\frac{\pi}{2}\Delta i_f + c_{\beta o}\sin\frac{\pi}{2}\Delta i_f = \frac{V_o s_{\beta o}}{V_f}$$

Dividing by $c_{\beta o}$

$$\tan\beta o = \frac{\sin\frac{\pi}{2}\Delta i_f}{\frac{V_o}{V_f} - \cos\frac{\pi}{2}\Delta i_f} \tag{39}$$

Equations (38) and (39) indicate that βo is given by

$$\beta o = \tan^{-1}\left[\frac{\sin\frac{\pi}{2}\Delta i_f}{\frac{V_o}{V_f} - \cos\frac{\pi}{2}\Delta i_f}\right] \tag{40}$$

regardless of whether $\Delta V - V_o c_{\beta o} < 0$ or $\Delta V - V_o c_{\beta o} > 0$, and from $\Delta V = V_o c_{\beta o} - V c_\beta$, the yaw angle β at future times is given by

$$\beta = \cos^{-1}\left[\frac{V_o c_{\beta o} - \Delta V}{\left(V_o^2 - 2V_o \Delta V c_{\beta o} + \Delta V^2\right)^{1/2}}\right] \tag{41}$$

where $\Delta V = ft$ and where $0 \le \beta \le \pi$. This expression is better than $V s_\beta = V_o s_{\beta o}$ which would yield $\beta = \sin^{-1}(V_o s_{\beta o}/V)$ since β could, for large tranfers exceed $\pi/2$.

As stated earlier, if the evolution of Δi as a function of time or velocity or ΔV is desired, one uses Eq. (36) for Δi until $\Delta V = V_o c_{\beta o}$. As soon as $\Delta V - V_o c_{\beta o} > 0$, then one must use Δi as given in Eq. (37). However, in Eq. (37), the inverse sine function will always return a β angle that is always less than $\pi/2$, and this value for β is the correct value to be used in $\Delta i = 2-(2/\pi)(\beta+\beta o)$. This β angle is clearly not the real yaw angle since in this case it would be given by $\pi-\beta$ with $\beta<\pi/2$, such that the yaw angle is now larger than $\pi/2$. If the real β angle is used in $\Delta i = 2-(2/\pi)(\beta+\beta o)$ one gets

$$\Delta i = 2 - \frac{2}{\pi}(\pi - \beta + \beta o)$$

$$= \frac{2}{\pi}(\beta - \beta o) \tag{42}$$

This then shows that Eq. (42) is universally valid for all yaw angles $0 \leq \beta < 180°$ and should be the only one used. Equation (42) will effectively replace Eqs. (36) and (37) provided that the angle β is computed from Eq. (41). Since the sign of $\Delta V - V_o c_{\beta o}$ is effectively accounted for in Eq. (41), it will return the yaw angle β to be used in Eq. (42) for the unambiguous evaluation of Δi.

Expressions for βo and β can also be obtained by using the identity in Eq. (42) since

$$\cos\frac{\pi}{2}\Delta i = c_\beta c_{\beta_o} + s_\beta s_{\beta_o} \tag{43}$$

$$\sin\frac{\pi}{2}\Delta i = s_\beta c_{\beta_o} - s_{\beta_o} c_\beta \tag{44}$$

If we multiply Eq. (43) by VV_o and replace Vs_β by $V_o s_{\beta o}$ and Vc_β by $V_o c_{\beta o} - \Delta V$, then after regrouping terms we get

$$c_{\beta_o} = \frac{V_o - V\cos\frac{\pi}{2}\Delta i}{\Delta V} \tag{45}$$

In a similar manner, from Eq. (43), if we replace this time $V_o s_{\beta o}$ by Vs_β and $V_o c_{\beta o}$ by $\Delta V + Vc_\beta$ we get an expression for c_β

$$c_\beta = \frac{V_o\cos\frac{\pi}{2}\Delta i - V}{\Delta V} \tag{46}$$

Equation (44) can also be written as

$$VV_o\sin\frac{\pi}{2}\Delta i = Vs_\beta \ V_o c_{\beta_o} - V_o s_{\beta_o} V c_\beta$$

if we replace Vs_β by $V_o s_{\beta o}$ and $V_o c_{\beta o}$ by $\Delta V + Vc_\beta$ then

$$s_{\beta_o} = \frac{V\sin\frac{\pi}{2}\Delta i}{\Delta V} \tag{47}$$

If on the other hand, we replace $V_o c_{\beta o}$ by $\Delta V + Vc_\beta$ then the identity will yield

$$s_\beta = \frac{V_o\sin\frac{\pi}{2}\Delta i}{\Delta V} \tag{48}$$

Now these expressions will readily yield

$$\tan\beta o = \frac{V\sin\frac{\pi}{2}\Delta i}{V_o - V\cos\frac{\pi}{2}\Delta i} \tag{49}$$

$$\tan\beta = \frac{V_o\sin\frac{\pi}{2}\Delta i}{V_o\cos\frac{\pi}{2}\Delta i - V} \tag{50}$$

These last two expressions can provide the initial βo and current β provided that the appropriate Δi expression of Eqs. (36) or (37) is used according to whether $\Delta V - V_o c_{\beta o}$ is positive or negative.

Although Eqs. (49) and (50) are valid for any optimal $(V, \Delta i)$ pair during the transfer, there is clearly a singularity at time 0 when $V = V_o$ and $\Delta i = 0$. The angle βo is best obtained by setting $V = V_f$ and $\Delta i = \Delta i_f$ in Eq. (49).

It is better to use Eq. (41) for the control time history instead of Eq. (50) since we do not have to switch between two Δi expressions to describe that evolution in the first case.

Finally, Edelbaum's ΔV equation in terms of the velocities and inclination is obtained from the velocity equation

$$V = \left(V_o^2 - 2V_o\Delta V \ c_{\beta_o} + \Delta V^2\right)^{1/2}$$

If we square this expression, and replace ΔV in the product term by $V_o c_{\beta o} - Vc_\beta$ and then use the identity

$$c_\beta c_{\beta_o} = \frac{1}{2}c_{\beta-\beta_o} + \frac{1}{2}c_{\beta+\beta_o}$$

with $c_{\beta-\beta o} = \cos \pi/2\ \Delta i$ from Eq. (42), then we get with $Vs_\beta = V_o s_{\beta o}$

$$V^2 = -V_o^2 + V_o^2 s_{\beta_o}^2 + VV_o\cos\frac{\pi}{2}\Delta i + V_o c_{\beta_o} V c_\beta + \Delta V^2$$

However $V_o^2 s_{\beta o}^2 = V_o s_{\beta o} Vs_\beta$ and if this term is combined with $V_o c_{\beta o} Vc_\beta$, the result will be $VV_o\, c_{\beta-\beta o}$ which can be replaced by $VV_o \cos \pi/2\ \Delta i$.

The final result is given by

$$V^2 = -V_o^2 + 2VV_o\cos\frac{\pi}{2}\Delta i + \Delta V^2$$

from which

$$\Delta V = \left(V_o^2 - 2VV_o\cos\frac{\pi}{2}\Delta i + V^2\right)^{1/2} \tag{51}$$

This of course is Edelbaum's ΔV equation for constant acceleration circle to inclined circle transfer. It is valid for any $(V, \Delta i)$ pair along the transfer provided once again that the appropriate Δi expression is used i.e., Eqs. (36) or (37) according to whether $\Delta V - V_o c_{\beta o}$ is < 0 or > 0 respectively. As shown earlier, ΔV is double valued in the velocity since Δi itself is double valued in that same variable.

However, Eq. (51) is mainly used to obtain the total ΔV_t required to achieve a given transfer between V_o and V_f with a relative inclination change of Δi_f

It is valid for any $0 < \Delta i_f < 114.6°$ or $0 < \Delta i_f < 2$ radians since this is the limiting Δi in Eq. (37). It is evident that the transfer time t_f is simply obtained from

$$t_f = \frac{\Delta V_t}{f} \tag{52}$$

III. The Formulation Using Optimal Control

Let the system equations be given by Eqs. (14) and (15) with time as the independent variable and i and V as the state variables. The yaw angle β is the control variable.

$$\frac{di}{dt} = \frac{2}{\pi}\frac{f}{V}s_\beta \tag{53}$$

$$\frac{dV}{dt} = -fc_\beta \tag{54}$$

This problem is now cast as a minimum time transfer problem between initial and final parameters i_o, V_o and i_f, V_f respectively. The variational Hamiltonian is then given by

$$H = 1 + \lambda_i\left(\frac{2}{\pi}\frac{f}{V}s_\beta\right) + \lambda_V(-fc_\beta) \tag{55}$$

since the performance index is simply given by

$$J = \int_{t_o}^{t_f} L\ dt$$

with L = 1

The Euler-Lagrange differential equations are given by

$$\dot{\lambda}_V = -\frac{\partial H}{\partial V} = \frac{2}{\pi}\frac{fs_\beta}{V^2}\lambda_i \tag{56}$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial i} = 0 \tag{57}$$

Therefore λ_i is a constant. The optimality condition is given by

$$\frac{\partial H}{\partial \beta} = \lambda_i\frac{2}{\pi}\frac{f}{V}c_\beta + f\lambda_V s_\beta = 0 \tag{58}$$

which yields the optimal control law

$$\tan\beta = -\frac{2}{\pi}\frac{\lambda_i}{V\lambda_V} \tag{59}$$

There is no need to integrate $\dot{\lambda}_V$ since we can use the transversality condition $H_f = 0$ at the final time. The Hamiltonian is a constant of the motion since it is not an explicit function of time. Therefore it is equal to zero all the time. We can therefore solve for λ_V and λ_i from

$$H = 0 = 1 + \frac{2}{\pi}\frac{f}{V}s_\beta\lambda_i - fc_\beta\lambda_V$$

$$\frac{\partial H}{\partial \beta} = 0 = \frac{2}{\pi}\frac{f}{V}c_\beta\lambda_i + fs_\beta\lambda_V$$

This results in

$$\lambda_i = -\frac{\pi s_\beta V}{2f} = constant \tag{60}$$

$$\lambda_V = \frac{c_\beta}{f} \tag{61}$$

Equation (60) reveals that $Vs_\beta = V_o s_{\beta o}$ since the acceleration f is assumed to be a constant.

We can now take advantage of this constancy of Vs_β in order to integrate the velocity Eq. (54).

$$fdt = -\frac{dV}{c_\beta} = \frac{-dV}{\pm\left(1 - s_\beta^2\right)^{1/2}}$$

$$f\int_o^t dt = -\int_{V_o}^{V}\frac{VdV}{\pm\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}} = ft = \Delta V$$

This yields as in the previous section

$$V^2 = V_o^2 + \Delta V^2 - 2\Delta V V_o c_{\beta o}$$

or

$$V = \left(V_o^2 + f^2t^2 - 2ftV_o c_{\beta o}\right)^{1/2} \tag{62}$$

We can also obtain the above equation without integrating dV/dt by simply writing

$$V = \frac{V_o s_{\beta o}}{s_\beta} = V_o s_{\beta o}\frac{(1 + \tan^2\beta)}{\tan\beta} \tag{63}$$

However, an expression for tanß is needed first. If we differentiate Eq. (59)

$$\frac{d}{dt}(\tan\beta) = \frac{2}{\pi}\lambda_i\frac{(\dot{V}\lambda_V + V\dot{\lambda}_V)}{V^2\lambda_V^2} \tag{64}$$

Replacing $\dot{V}$ and $\dot{\lambda}_V$ by Eqs. (54) and (56) and using Eq. (60) and (61) to eliminate λ_i and λ_V, the above derivative can be written as

$$\frac{d}{d\beta}(\tan\beta)\dot{\beta} = \frac{\dot{\beta}}{c_\beta^2} = \frac{fs_\beta}{Vc_\beta^2}$$

which yields

$$\dot{\beta} = \frac{fs_\beta}{V} \tag{65}$$

Since $Vs_\beta = V_o s_{\beta o}$, this can also be written as

$$\dot{\beta} = \frac{fs_\beta^2}{V_o s_{\beta o}}$$

such that

$$\int_{\beta o}^{\beta}\frac{d\beta}{s_\beta^2} = \frac{f}{V_o s_{\beta o}}\int_o^t dt$$

or

$$\cot\beta o - \cot\beta = \frac{ft}{V_o s_{\beta o}}$$

and finally the control law

$$\tan\beta = \frac{V_o s_{\beta o}}{V_o c_{\beta o} - ft} \tag{66}$$

Going back to Eq. (63) and replacing tanß with the above expression results in

$$V = \left(V_o^2 + f^2t^2 - 2V_o c_{\beta o} ft\right)^{1/2} \tag{67}$$

Now the inclination time history can be obtained by direct integration of Eq. (53) by using the expression for V in Eq. (67)

$$\frac{di}{dt} = \frac{2}{\pi}\frac{f}{V^2}Vs_\beta = \frac{2}{\pi}\frac{f}{V^2}V_o s_{\beta o}$$

$$\int_o^i di = \frac{2}{\pi}V_o s_{\beta o} f\int_o^t\frac{dt}{V_o^2 + f^2t^2 - 2V_o c_{\beta o} ft}$$

which yields

$$\Delta i = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{ft - V_o c_{\beta o}}{V_o s_{\beta o}}\right) - \tan^{-1}(-\cot\beta o)\right]$$

Since $\tan^{-1}x = -\tan^{-1}(-x)$ and $\tan^{-1}(\cot x) = \pi/2 - \tan^{-1}x$, the final result can be written as

$$\Delta i = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{ft - V_o c_{\beta o}}{V_o s_{\beta o}}\right) + \frac{\pi}{2} - \beta o\right] \tag{68}$$

This formula is uniformly valid for all t unlike the formulation of the previous section which resulted in a set of two expressions for Δi due to the fact that Δi was double valued in the velocity. This simplification is achieved because time was selected as the independent variable instead of the velocity.

If we integrate Eq. (56) for $\dot{\lambda}_V$ and use Eq. (60) to eliminate the constant λ_i, then

$$\dot{\lambda}_V = -\frac{V_o^2 s_{\beta o}^2}{V^3} = -V_o^2 s_{\beta o}^2\left(V_o^2 + f^2t^2 - 2V_o c_{\beta o} ft\right)^{-3/2}$$

which upon integration yields

$$\lambda_V = \frac{V_o c_{\beta o} - ft}{fV} \tag{69}$$

with $(\lambda_V)_o = c_{\beta o}/f$ and in view of Eq. (61)

$$c_\beta = \frac{V_o c_{\beta o} - ft}{V} \tag{70}$$

From the definition of the influence functions λ_i and λ_V we have

$$(\lambda_V)_o = \frac{c_{\beta o}}{f} = \frac{\partial t_f}{\partial V_o}$$

or

$$\delta t_f = \frac{c_{\beta o}}{f}\delta V_o \tag{71}$$

and

$$(\lambda_i)_o = -\frac{\pi V_o s_{\beta o}}{2f} = \frac{\partial t_f}{\partial(\Delta i)_o}$$

or

$$\delta t_f = -\frac{\pi V_o s_{\beta o}}{2f}\delta(\Delta i)_o \tag{72}$$

Equations (71) and (72) show how the total transfer time will vary for small variations in the initial velocity and inclination

Let us obtain an expression for ΔV in terms of βo, V_o and Δi. From Eq. (68) we have

$$\frac{ft - V_o c_{\beta o}}{V_o s_{\beta o}} = \tan\left(\frac{\pi}{2}\Delta i + \beta o - \frac{\pi}{2}\right)$$

$$= \frac{-1}{\tan\left(\frac{\pi}{2}\Delta i + \beta o\right)}$$

and since $ft = \Delta V$

$$\Delta V = V_o c_{\beta o} - \frac{V_o s_{\beta o}}{\tan\left(\frac{\pi}{2}\Delta i + \beta o\right)} \tag{73}$$

This expression can be used to evaluate the total ΔV for the desired transfer once βo the initial yaw angle is known. To obtain βo, let us first observe that during the integration of the $\dot{V}$ Eq. (54), the same intermediate results shown in the previous section and resulting from that particular integration are valid here too. They are

$$\Delta V = ft = V_o c_{\beta o} - V c_\beta$$

$$\Delta V = V_o c_{\beta o} \pm \left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2} \tag{74}$$

From the Δi Eq. (68) and using the control law for $\tan\beta$ given in Eq. (66) we get

$$\Delta i = \frac{2}{\pi}(\beta - \beta o) \tag{75}$$

We can now obtain βo by using the identity in Eq. (74) and writing Δi as

$$\Delta i = \frac{2}{\pi}\left\{\tan^{-1}\left[\frac{\pm\left(V^2 - V_o^2 s_{\beta o}^2\right)^{1/2}}{V_o s_{\beta o}}\right] + \frac{\pi}{2} - \beta o\right\}$$

Since $\tan^{-1}x = \pm\cos^{-1}[1/(x^2+1)^{1/2}]$ according to whether x is > 0 or < 0, the above expression for Δi can be cast as

$$\Delta i = \frac{2}{\pi}\left[\pm\cos^{-1}\left(\frac{V_o s_{\beta o}}{V}\right) + \frac{\pi}{2} - \beta o\right]$$

or

$$\cos\left\{\mp\left[\frac{\pi}{2} - \left(\beta o + \frac{\pi}{2}\Delta i\right)\right]\right\} = \frac{V_o s_{\beta o}}{V}$$

or since $\cos(-x) = \cos x$

$$\sin\left(\beta o + \frac{\pi}{2}\Delta i\right) = \frac{V_o s_{\beta o}}{V}$$

which after expansion and division by $c_{\beta o}$ yields

$$\tan\beta o = \frac{\sin\frac{\pi}{2}\Delta i}{\frac{V_o}{V} - \cos\frac{\pi}{2}\Delta i} \tag{76}$$

For given V_o, V_f, and $(\Delta i)_f$, βo can be obtained from above which then allows us to describe V and Δi as well as β as a function of time from Eqs. (67), (68) and (66), respectively.

It is clear from Eq. (76) that as Δi approaches the 2 radians value or $\Delta i = 114.59°$, $\sin \pi/2\, \Delta i$ will approach zero so that $\beta o \to 0$ indicating that the initial phase of the transfer will be coplanar. The ΔV equation of Edelbaum given by Eq. (51) will then approach $\Delta V = V_o + V_f$ which is the sum of the initial and final velocities. Since V_o represents the ΔV needed to transfer from V_o to ∞ or escape, and since V_f represents the ΔV needed to transfer from ∞ back to V_f, then the transfer is initially coplanar until escape. At infinity $V_\infty = 0$ and the inclination change is achieved at zero cost, after which the return leg to V_f is also coplanar resulting in $\Delta V = V_o + V_f$. This is clear from

$$\Delta V_1 = \left(V_o^2 - 2V_o V_\infty + V_\infty^2\right)^{1/2} = V_o$$

$$\Delta V_2 = \left(V_\infty^2 - 2V_\infty V_f + V_f^2\right)^{1/2} = V_f$$

$$\Delta V = \Delta V_1 + \Delta V_2 = V_o + V_f \tag{77}$$

For any inclination larger than $\Delta i = 114.59°$, the cost of the orbit rotation is zero and the ΔV remains stationary at $V_o + V_f$.

If the transfer is purely coplanar, i.e., $\Delta i = 0$ then

$$\Delta V = |V_o - V_f| \quad (78)$$

the difference of the boundary velocities. For given Vo and V_f, ΔV reaches a maximum if we set

$$\frac{\partial \Delta V}{\partial \Delta i} = 0$$

where $\Delta V = (V_o^2 - 2V_f V_o \cos \pi/2\, \Delta i + V_f^2)^{1/2}$. This results in $\sin \pi/2\, \Delta i = 0$ or $\Delta i = 114.59°$ as discussed above.

Therefore, Edelbaum's Eq. (51) is to be used for $0 \le \Delta i \le 114.59°$ only. For $\Delta i > 114.59°$ Eq. (77) must be used instead, as the use of Eq. (51) in this case will yield the wrong ΔV. Furthermore, from Eq. (74), it is clear that as Δi approaches 114.59° from below, ΔV must approach $V_o + V_f$ which implies that $\beta o \rightarrow 0$ as shown earlier, and $\beta_f \rightarrow 180°$. Our ΔV Eq. (73) also approaches $V_o + V_f$ since with $\beta o = \epsilon$, $s_{\beta o} \simeq \beta o$, $c_{\beta o} \sim 1$, $s_{\beta f} \sim -s_{\beta o} \sim -\beta o$ and $\pi/2\, \Delta i \sim 180°$.

$$\Delta V \simeq V_o - \frac{V_f s_{\beta f}}{\tan(180 + \epsilon)}$$

$$\simeq V_o + \frac{V_f \epsilon}{\epsilon} = V_o + V_f$$

IV. NUMERICAL RESULTS

First, one computes V_o and V_f from the knowledge of initial and final semi-major axes a_o and a_f

$$V_o = \left(\frac{\mu}{a_O}\right)^{1/2}$$

$$V_f = \left(\frac{\mu}{a_f}\right)^{1/2}$$

Given $\mu = 398601.3$ km^3/s^2, earth's gravity constant, and Δi the total inclination change desired and f the low-thrust acceleration, one computes βo from Eq. (76) and the total ΔV_{tot} from Eq. (73) such that the transfer time is known from $t_f = \Delta V_{tot}/f$. The variation with time of the various variables of interest is obtained from

$$\Delta V = ft$$

$$\beta = \tan^{-1}\left(\frac{V_o s_{\beta o}}{V_o c_{\beta o} - ft}\right)$$

$$V = \left(V_o^2 - 2V_o ft\, c_{\beta o} + f^2 t^2\right)^{1/2}$$

$$\lambda_V = \frac{c_\beta}{f}$$

$$\Delta i = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{ft - V_o c_{\beta o}}{V_o s_{\beta o}}\right) + \frac{\pi}{2} - \beta o\right]$$

λ_i is of course constant and given by $-\pi V_o s_{\beta o}/(2f)$.

The total $(\Delta i)_f$ is obtained from $|i_o - i_f|$ where i_o and i_f are the initial and final inclination respectively. If $i_f > i_o$, the current inclination i is given by

$$i = i_o + \Delta i$$

If $i_f < i_o$, then

$$i = i_o - \Delta i$$

this is needed since we assumed $\Delta i > 0$ so that $\beta > \beta o$ too.

Figures 1 through 5 show the variation of the β angle, velocity V, ΔV, i and λ_V as a function of time for a low thrust transfer between $a_o = 7000$ km, $i_o = 28.5°$ and $a_f = 42166$ km, $i_f = 0°$ geostationary orbit. The acceleration $f = 3.5 \times 10^{-7}$ km/s^2 for all the examples presented in this paper. The total transfer time $t_f = 191$ days with a $\Delta V_{tot} = 5.77584$ km/s. The yaw angle increases from $\beta o = 21.98°$ to $\beta_f = 66.61°$ as it is more efficient to rotate the orbit at higher altitudes.

Figures 6 through 8 show some of the same plots except that $i_o = 90°$. $\beta o = 10.92°$ stays almost stationary for the first 100 days before surging to the final $\beta f = 152.28°$. It goes through 90° at day 245 where it starts to decelerate the vehicle since the orbit radius has exceeded the desired final altitude, and thus must be shrinked until the final transfer time of $t_f = 335$ days for a total $\Delta V = 10.13$ km/s. This is clear from Figure 7, where the intermediate velocity is much less than the final desired velocity of 3.07 km/s at GEO.

Figures 9 through 11 are for $i_o = 114.59°$ the uppermost limit of Edelbaum's theory. The β angle stays at zero indicating a coplanar transfer up to t = 250 days where $V = 1.39 \times 10^{-2}$ km/s and a = 2,063,046,944 km or infinity before flipping to 180° for the return leg to GEO. The inclination change is carried out instantaneously at infinity with zero cost. The total transfer time is $t_f = 351$ days with a ΔV of 10.61 km/s.

In Figures 12 through 14, the transfer is from $a_o = 42166$ km, $i_o = 0°$ to $a_f = 7000$ km, $i_f = 28.5°$ indicating a return from GEO to inclined LEO.

These examples show that this algorithm generates the exact transfer regardless of whether the transfer is to a higher orbit or a lower orbit irrespective of the direction of the inclination change.

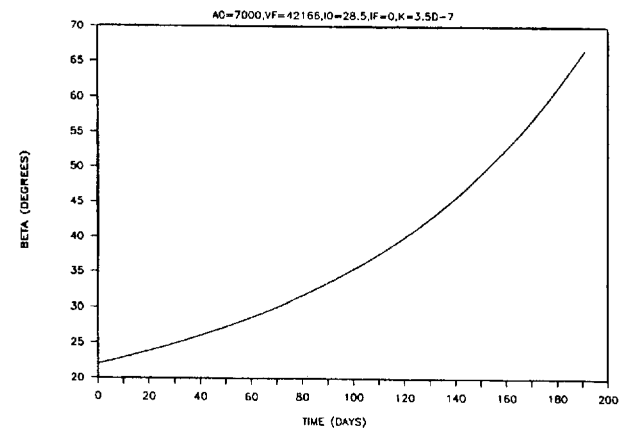

Figure 1. Yaw Angle versus Time

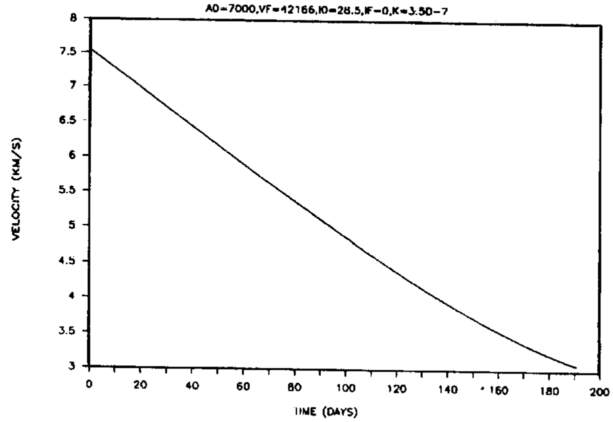

Figure 2. Velocity versus Time

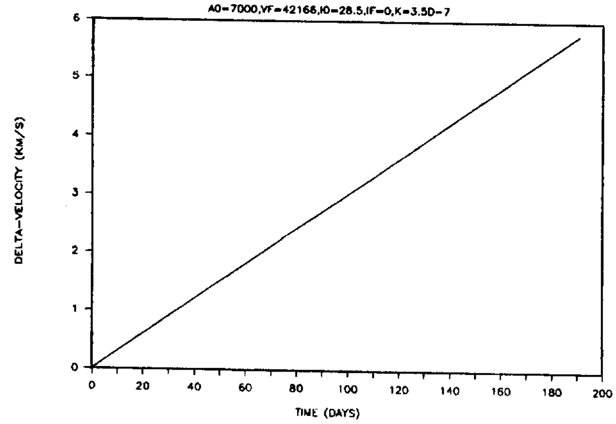

Figure 3. Delta—Velocity versus Time

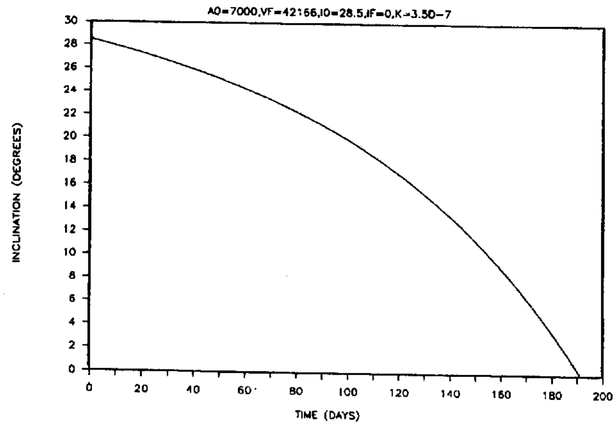

Figure 4. Inclination versus Time

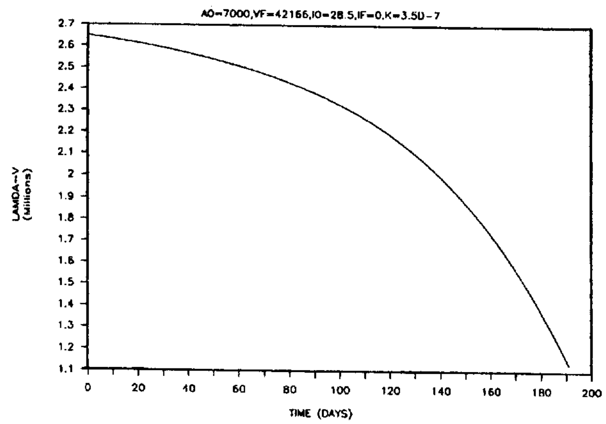

Figure 5. Lambda-V versus Time

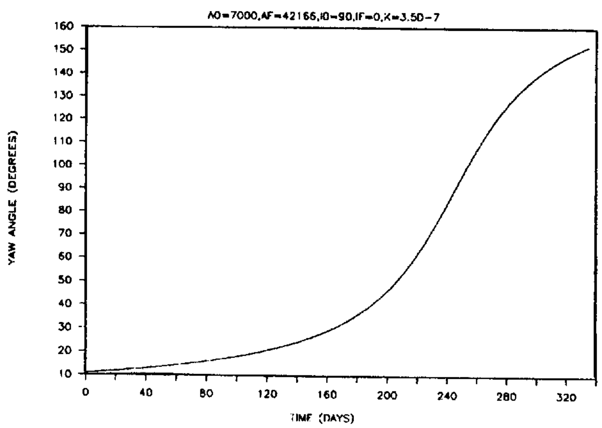

Figure 6. Yaw Angle versus Time

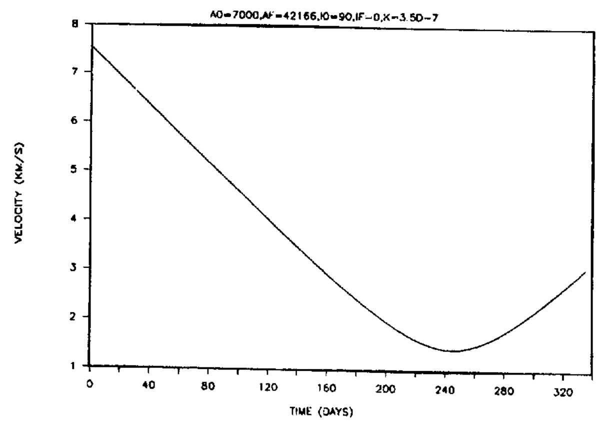

Figure 7. Velocity versus Time

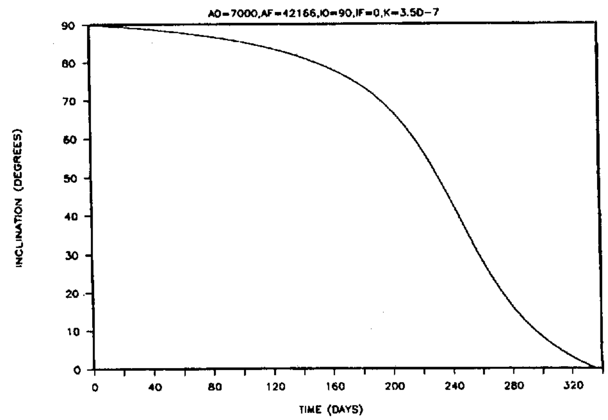

Figure 8. Inclination versus Time

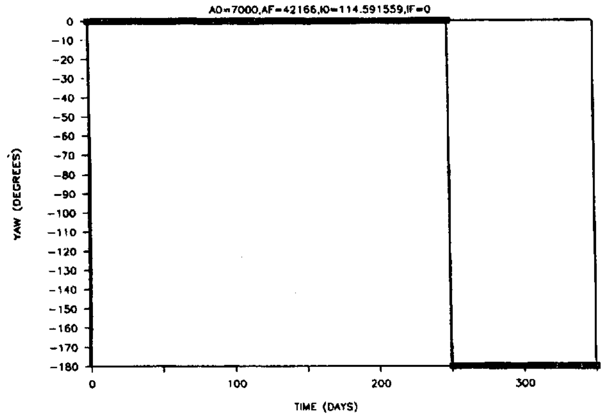

Figure 9. Yaw Angle versus Time

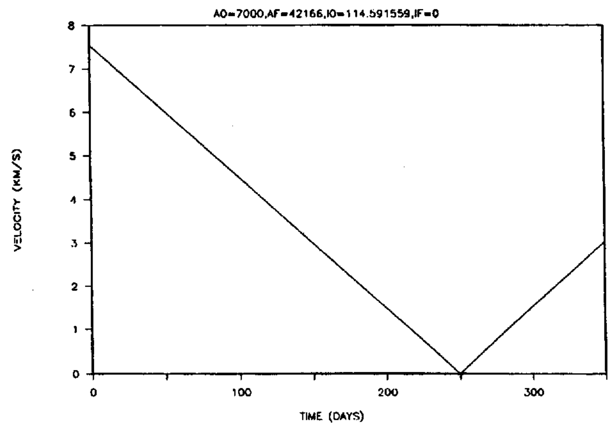

Figure 10. Velocity versus Time

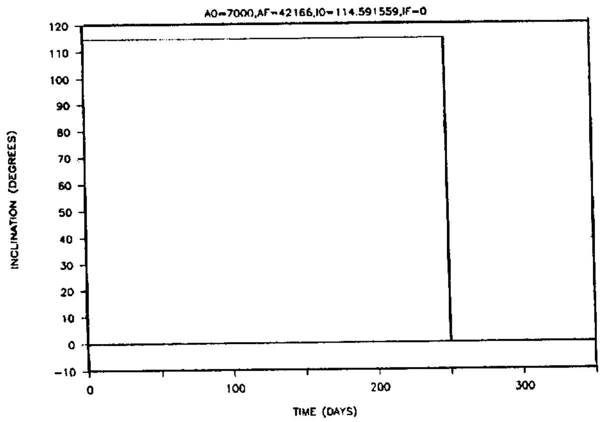

Figure 11. Inclination versus Time

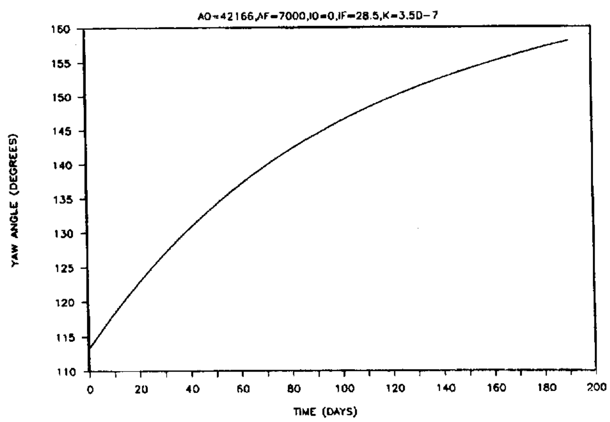

Figure 12. Yaw Angle versus Time

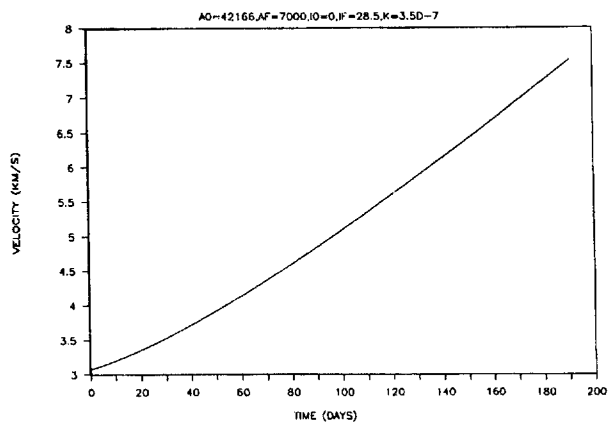

Figure 13. Velocity versus Time

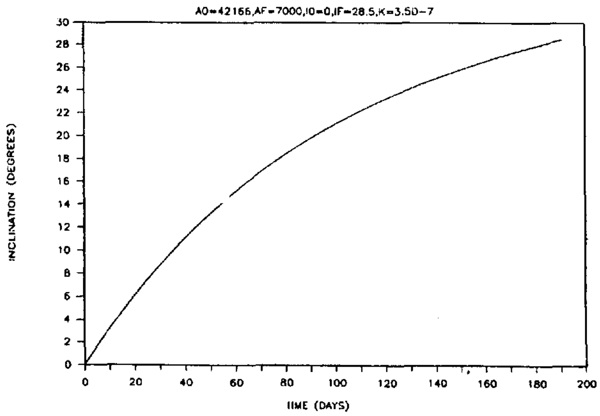

Figure 14. Inclination versus Time

V. CONCLUDING REMARKS

Edelbaum's analytic low-thrust circle to inclined circle orbit transfer theory is reformulated as a minimum-time transfer problem. An essential simplification is obtained by replacing a set of two expressions for the orbital inclination by a single analytic expression uniformly valid for all transfers.

Additional expressions for the initial value of the control parameter needed for a given transfer as well as its functional dependency on time are also presented using inverse tangent functions. Finally, the applicability of this simplified transfer in an autonomous mode is also discussed.

REFERENCES

1. Edelbaum, T. N., "Propulsion Requirements for Controllable Satellites," *ARS J.*, Aug 1961, pp. 1079-89.