and magnetically actuated spherical shell is feasible and that the mass, size, and wattage requirements are reasonable when compared with other systems. This method of attitude control offers the advantages of large range of control, lack of frictional effects, negligible interacting torques (no gyroscopic effects), and ability to passively dump undesired cumulative angular momentum into the Earth's magnetic field.

The parameters for the example discussed in the text are summarized in Table 1.

Acknowledgments

The work reported here is the outgrowth of a study (5) performed for Lyman Spitzer Jr. of Princeton University. Many of the concepts and ideas expressed here are Professor Spitzer's.

Roderic Scott of Perkin-Elmer contributed significantly to this program during its conceptual phases. We wish also to thank Robert Noble of Perkin-Elmer for his help during this program and for his aid in reviewing the manuscript.

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Table 1 Summary of parameters

Radius of spherical shell	25 cm
Shell thickness	0.5 cm
Shell material	aluminum
Shell mass	10.6 kg
ω_m = relative angular speed at maximum torque	54 radians/sec
Minimum average torque in Earth's magnetic field[a] when $\omega = \omega_m$	240 dyne-cm
Maximum slewing torque	1.4×10^6 dyne-cm
Driving field intensity for maximum slewing torque	20 gauss
Driving field frequency = $1.2\omega_m$	10 cps
Power required for maximum torque	18 w
Mass of torquing coils	30 kg
Peak suspension field strength	5 gauss
Suspension field frequency	288 cps
Effective suspension "spring constant"	1000 dynes/cm
Total power for sphere suspension[b]	8.4 w
Mass of suspension coils[b]	5 kg

[a] For 400 mile orbit at 45 deg inclination.
[b] Not counting damping coils if added.

Propulsion Requirements for Controllable Satellites

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Propulsion requirements are determined for several controllable satellite missions. Both high thrust propulsion systems such as chemical rockets and low thrust propulsion systems such as ion rockets are considered. Rendezvous missions are treated by determining minimum fuel maneuvers for small, simultaneous changes in the elements of quasi-circular orbits. Orbit transfer missions are treated by determining minimum fuel maneuvers for large changes in the elements of circular orbits. Orbit maintenance missions are treated by determining the propulsion necessary to cancel perturbations due to the atmosphere, the Earth's bulge, and the sun and moon. Among the results of the study is a closed-form analytic solution for the optimum low thrust transfer between inclined circular orbits of different radii.

THIS paper is an outgrowth of a presentation on controllable satellite propulsion given at the ARS Controllable Satellites Conference in April 1959. The material for that presentation was prepared jointly by the Pratt & Whitney Aircraft Division and the Research Laboratories of United Aircraft Corp. The present paper includes that portion of the original study that dealt with propulsion requirements as well as extensions to the orbit transfer analyses.

Presented at the ARS Semi-Annual Meeting, Los Angeles, Calif., May 9–12, 1960.

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Scope of Study

Fig. 1 illustrates a typical elliptical satellite orbit which can be described in terms of various sets of orbital elements. In this report, the following elements are considered: length

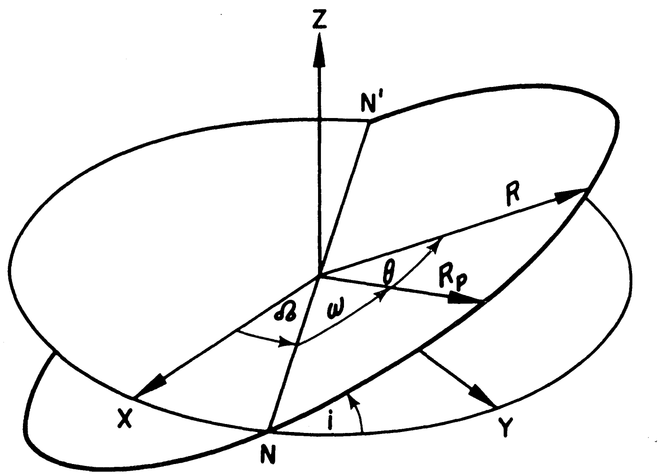

Fig. 1 Satellite orbit

of the semimajor axis a, eccentricity e, inclination to the equatorial plane i, and angle between the line of nodes and the line of apsides ω. The semimajor axis is half the distance between the point of closest approach to the Earth (perigee) and the point of furthest retreat from the Earth (apogee). The line connecting these two points (the major axis) is known as the line of apsides. The eccentricity represents the percentage difference between the semimajor axis and either the perigee or the apogee distance. The inclination is the angle between the orbital plane and the X, Y reference plane. The intersection of the orbital plane and the reference plane is known as the line of nodes (the line N-N' in Fig. 1). Angular position in the orbit is measured from either the perigee θ or a fixed reference point θ'.

This study is concerned with the propulsion requirements for varying these orbital elements. Several representative missions are treated under the following headings: (a) Precision Orbit Establishment (b) Modification of Circular Orbits (c) Orbit Maintenance.

Typical missions in the section on precision orbit establishment include the placement of a 24-hr satellite directly above a given point on the Earth's Equator and rendezvous with other vehicles in orbit. Circular orbits have been considered because they are simple to analyze and because they are desirable for many satellite missions. In all cases, the initial departure of the orbit from the desired circular orbit is assumed to be small so that the changes may be considered as linear.

In the section on modification of circular orbits, the use of propulsion to produce gross changes in the orbit is considered. Here also only circular orbits are considered. Typical missions under this heading include the establishment of equatorial orbits from nonequatorial parking orbits, changes in orbit altitude, and changes in the position in the orbit.

The section on orbit maintenance deals with cases where a perturbed orbit is controlled so as to cancel the effects of the perturbations. As particular examples, the maintenance of an orbit in the atmosphere and of a 24-hr satellite orbit in the presence of solar, lunar and bulge perturbations are considered.

Some of these missions can be accomplished either with systems (such as chemical rockets) having high thrust-weight ratios or with systems having low thrust-weight ratios (such as ion rockets or plasma jets). The proposed devices can give accelerations of from less than 10^{-4} gm to greater than 10^{2} gm. These accelerations can result in corrections being made during many revolutions of the satellite or at essentially a single point in the orbit. Most of the available propulsion systems tend to result in maneuver times that are either much shorter or much longer than the time for one revolution. Only these cases are treated herein because they constitute the primary interest and because the intermediate cases do give intermediate results. Generally, a high thrust system is taken as having a vehicle thrust-weight ratio of $\frac{1}{2}$ or greater, while a low thrust system is assumed to have a vehicle thrust-weight ratio of 10^{-2} or less. A further restriction on the low thrust system is that the maneuver must require many revolutions of the satellite. If the required maneuver becomes very small, the low thrust system need be used during only a fraction of a revolution and will behave as a high thrust system. With typical electrical propulsion systems having thrust-weight ratios on the order of 10^{-4}, this latter circumstance corresponds to minute corrections.

The high thrust systems are normally characterized by low specific impulse and high fuel consumption. The maneuvers for this system are optimized so as to minimize fuel consumption. On the other hand, the low thrust systems tend to have high specific impulse and low fuel consumption but very long maneuver times. For these systems the engine is assumed to run continuously and maneuver time is minimized. As long as the engine runs continuously, minimum-time maneuvers are also minimum-fuel maneuvers and vice versa. One unusual system is the energy storage system described in (1).[2] This system uses an electric generator to charge an energy storage unit over many revolutions of the satellite and then releases the energy in short impulses. Here, the interest is in minimizing both fuel consumption and energy consumption in order to minimize charging time as well as fuel. However, since energy minimization is equivalent to fuel minimization in this case, the high thrust system results are directly applicable to this system.

The control systems examined are not required to have variable thrust magnitude. Only burning time and thrust direction are varied. In general, the engines are assumed to be fully controllable in direction. This could be realized in practice by gimballing the nozzles or by turning the whole satellite. Some attitude control system is probably required but is not discussed.

All the results of the mission analyses are expressed in terms of a required characteristic velocity. Characteristic velocity is a useful parameter because it is independent of both specific impulse and thrust-weight ratio for either high or low thrust systems. All propulsion systems are assumed to operate at constant specific impulse so that the mass ratio required for any maneuver is equal to the exponential of the required characteristic velocity divided by the exhaust velocity.

Precision Orbit Establishment

Small Changes of Individual Elements of Orbit

Optimum High Thrust Corrections: Optimum methods of making both large and small changes in the individual elements of circular satellite orbits with high thrust have been treated in the literature, e.g., (1 through 10). For changes in the semimajor axis or eccentricity, the well-known Hohmann transfer ellipse is optimum. For small changes in inclination, the optimum maneuver is a single impulse at the line of nodes directed normal to the plane of the orbit. Changes in position in the orbit are accomplished by transferring to a larger or a smaller orbit and revolving in this orbit until the original orbit can be re-entered at the desired point. Reference (3) shows that tangential impulses are essentially indistinguishable from the true optimum for this case. For this case the characteristic velocity requirements are inversely proportional to the number of revolutions in the intermediate orbit.

[2] Numbers in parentheses indicate References at end of paper.

Optimum Low Thrust Corrections: The optimum low thrust correction maneuvers for each of these elements are derived in the Appendix. Derivations of the first three of these are also given in (11) and the fourth is derived in (1). For changes in the semimajor axis, the optimum maneuver requires directing the thrust so that it is always tangent to the velocity vector. For changes in eccentricity, the optimum maneuver closely approximates directing the thrust in one direction normal to the line of apsides. This program is more than twice as efficient as the radial thrust programs for changes in eccentricity suggested in the literature. Changes in inclination are accomplished by directing the thrust normal to the plane of the orbit for half a cycle and then reversing the direction for the other half. Rider has recently treated low thrust changes in inclination for elliptic orbits of any eccentricity (12). Changes in position in the orbit are made by applying tangential thrust in one direction until half the correction is made and then applying tangential thrust in the other direction until the original orbit is re-established. As with high thrust, the characteristic velocity is reduced by increasing the number of revolutions required for the maneuver. The characteristic velocity requirements for these maneuvers with both high and low thrust are given in Table 1. Changes in the major axis can be accomplished with the same characteristic velocity with either type of system, but changes in eccentricity with low thrust require a 30% higher characteristic velocity than with high thrust, changes in inclination require 57% more, and changes in position with the same number of revolutions, 100% more. This difference in characteristic velocity for changes in position is probably not significant because the low thrust system will normally require a large number of revolutions to make the correction, and even for only one revolution the characteristic velocity requirement for changing position is an order of magnitude smaller than that required for other changes. It is necessary to perform a complete mission analysis to determine the significance of these differences in characteristic velocity. Low thrust systems and high thrust systems are fundamentally different and cannot be compared on this basis alone.

Table 1 Characteristic velocity requirements for small changes of quasi-circular orbits

High thrust	Low thrust
$\frac{\Delta V}{V_0} = 0.5 \frac{\Delta a}{a_0}$	$\frac{\Delta V}{V_0} = 0.5 \frac{\Delta a}{a_0}$
$\frac{\Delta V}{V_0} = 0.5\ \Delta e$	$\frac{\Delta V}{V_0} = 0.649\ \Delta e$
$\frac{\Delta V}{V_0} = 1.0\ \Delta i$	$\frac{\Delta V}{V_0} = 1.571\ \Delta i$
$\frac{\Delta V}{V_0} = 0.106 \frac{\Delta\theta'}{n}$	$\frac{\Delta V}{V_0} = 0.212 \frac{\Delta\theta'}{n}$

Combination Maneuvers

High Thrust: Small changes in both the semimajor axis and the eccentricity can be made simultaneously with a total velocity increment which is equal to that for the larger correction alone. The reason for this is that whatever must be added to one of the impulses must be subtracted from the other, and the sum of the impulses remains the same. Changes in position can also be obtained free by splitting one of the impulses into two parts. For example, not quite all of the required velocity increment is applied in the second impulse, and the rest is applied in a third impulse. Several recent papers have used this concept for changes in position (13 through 15).

Analysis of the optimum maneuvers for the general three-dimensional case is quite complex even for the quasi-circular orbits considered herein (16). It is believed that the optimum maneuver generally requires three impulses for changes in major axis, eccentricity, and inclination; four impulses are required if orbital position is to be changed also. Because of the large number of variables and the complex equations needed to analyze this general maneuver, the general case is not treated herein. Instead, a simplified case is analyzed where the optimum planar maneuver is combined with a change in inclination without changing the planar maneuver. The results of this analysis, which is contained in the Appendix, are given in Fig. 2. This figure shows the combinations of planar and nonplanar changes that can be obtained for any given characteristic velocity. The planar change in semimajor axis or eccentricity per unit characteristic velocity is plotted against the nonplanar change in inclination that can be obtained with the same unit characteristic velocity. As in the planar case, small changes in both the semimajor axis and the eccentricity can be made simultaneously with a total velocity increment which is equal to that for the larger correction alone. The total characteristic velocity required for a maneuver always lies between the algebraic sum and the vector sum of the planar and nonplanar velocity impulses. For cases where the line of nodes is not coincident with the lines of apsides, it should be possible to improve these results by allowing for some variation of the planar maneuver.

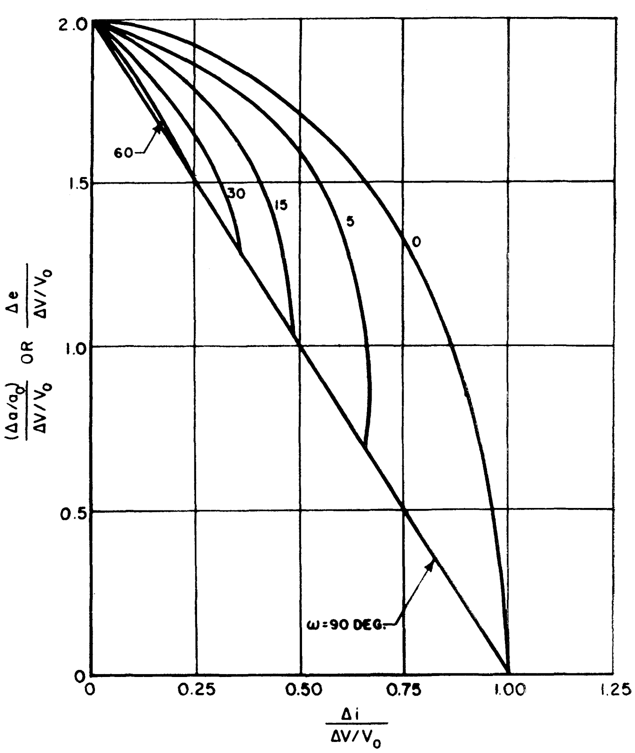

Fig. 2 Combination maneuvers for quasi-circular orbits: high thrust

Low Thrust: The optimum steering programs for combinations of changes in the semimajor axis, eccentricity and inclination are derived in the Appendix. Changes in orbital position can also be made with these programs because such variations require an alternation of period, which is equivalent to changes in semimajor axis. Some special cases of this optimum steering program for planar maneuvers are illustrated in Fig. 3. The figure shows the variation in the angle

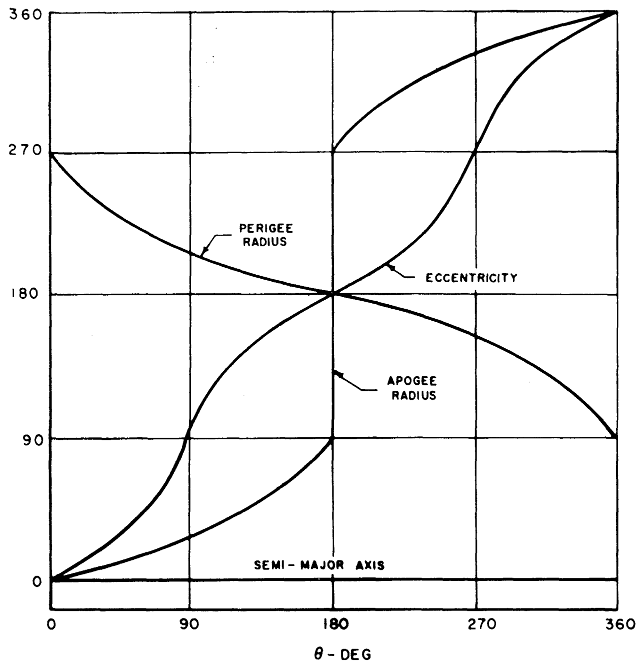

Fig. 3 Some optimum steering programs with low thrust

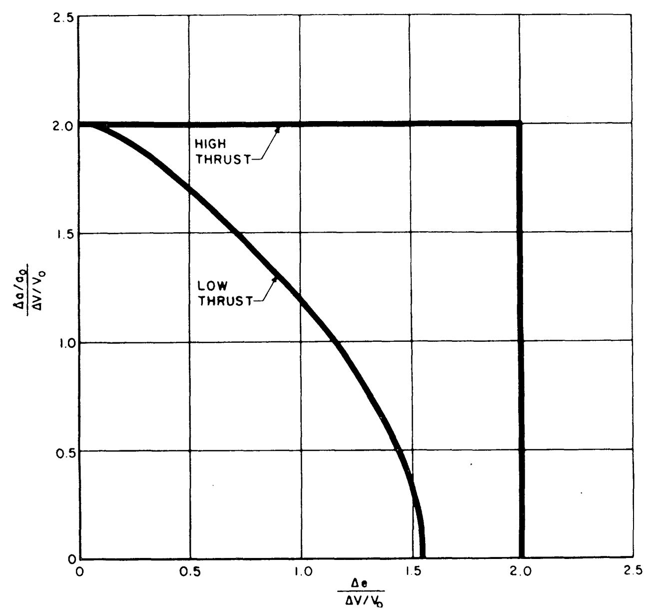

Fig. 4 Characteristic velocity requirements for planar maneuvers

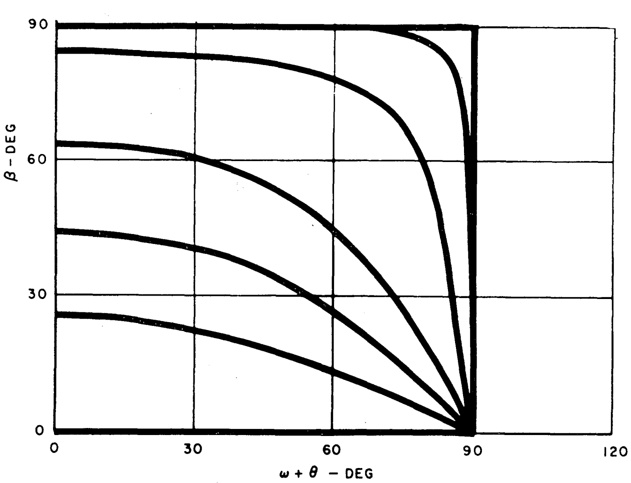

Fig. 5 Optimum steering programs for changing semimajor axis and inclination

between the thrust vector and the velocity vector required for optimum combinations of changes in semimajor axis and eccentricity. The particular programs shown in this figure maximize the semimajor axis, the eccentricity, and the apogee radius, or minimize the perigee radius. The results of these programs are shown in Fig. 4, where they are compared with the corresponding high thrust results. As in Fig. 2, the curves represent the changes that can be obtained with given characteristic velocities. The intersection of the curves with the axis represents individual maneuvers while all other points on the curves represent combination maneuvers. The steering program that maximizes semimajor axis produces no change in eccentricity while the steering program that maximizes eccentricity produces no change in semimajor axis. If the size of the changes that can be obtained with low thrust is compared to those obtainable with high thrust, it can be seen that the disparity is greater for combination maneuvers than for the individual maneuvers. This shows that planar maneuvers with high thrust can be combined more efficiently than planar maneuvers with low thrust.

The steering programs for a number of different combinations of changes in semimajor axis and inclination are shown in Fig. 5. The angle β is the angle between the thrust vector and the plane of the orbit. The results of these programs are illustrated in Fig. 6, where they are compared with the results for high thrust systems. Also shown is the result of keeping the angle between the thrust line and the orbit plane constant in magnitude. This latter steering program gives near-optimum results which can be analyzed simply in terms of trigonometric functions rather than in terms of the elliptic integrals that result from the optimum program. In contrast with the planar results, nonplanar combination maneuvers with low thrust can be combined at least as efficiently as can the maneuver with high thrust. In many cases the characteristic velocity advantage of the high thrust system will be smaller for nonplanar combination maneuvers than for individual maneuvers.

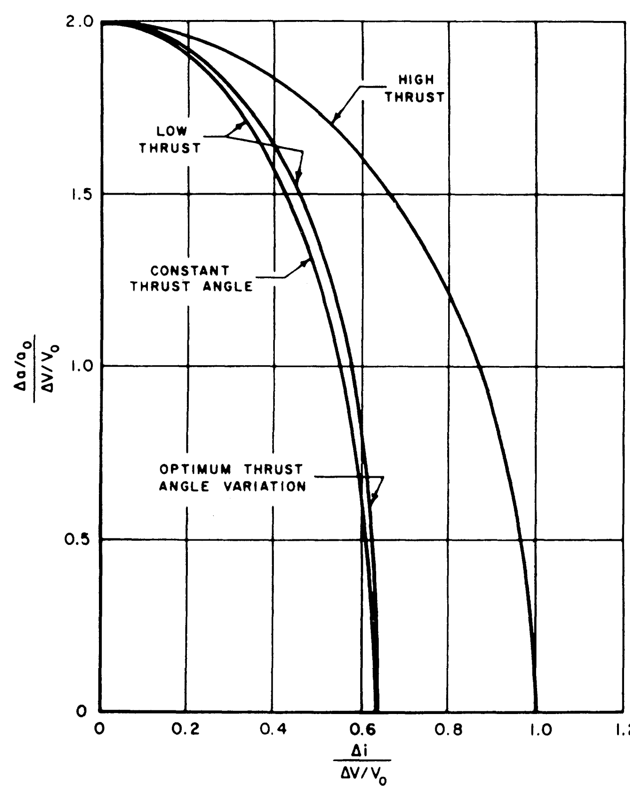

Fig. 6 Characteristic velocity requirements for changes in semimajor axis and inclination

Modification of Circular Orbits

Large changes in the individual elements of circular satellite orbits are considered in this section. The small-change assumption of the last section is dropped, but the quasi-circular assumption is retained. Because of this latter assumption, large changes in eccentricity are not considered, although changes in orbit altitude and inclination are considered. The allowance of large changes complicates the analysis because the possible fuel savings of various intermediate orbits may be larger than the amount of fuel required to enter the intermediate orbits. It becomes necessary to consider many different possibilities.

Changes in Radius

In 1925 Hohmann (2) demonstrated his now classic result that the optimum way to transfer from one circular orbit to another is via an ellipse which is tangent to both orbits. This two-impulse maneuver represents the absolute minimum fuel consumption for most cases of practical interest (5). However, for transfer to circular orbits which are more than twelve times larger (or smaller) than the original orbit, it is possible to save fuel by going to a three-impulse transfer. The three-impulse bi-elliptic transfer is illustrated in Fig. 7. It starts as a conventional Hohmann transfer except that a larger initial impulse is used to transfer beyond the radius of the final orbit. At the apogee of this transfer orbit a second impulse is used to bring the perigee of the transfer ellipse to the radius of the final orbit. The final circular orbit is then entered at the perigee of this second ellipse by means of a third impulse. The farther away the apogee of the transfer ellipse is located, the greater is the fuel saving. The absolute minimum fuel consumption represents going to infinity and applying an infinitesimal impulse there to return along a parabola which is tangent to the final orbit. The characteristic velocity savings of this high thrust maneuver are shown in Fig. 8. Going to twice the radius of the final orbit produces about half the characteristic velocity savings of going to infinity. Since the maximum possible saving in characteristic velocity is about 8%, these savings probably do not justify the increased complexity and transfer times of these maneuvers. This maneuver was apparently first derived by Shternfeld (8). Independent descriptions of this maneuver appear in (9) and (10). A comparison of the Hohmann transfer and the bi-elliptic transfer for co-apsidal elliptic orbits is contained in (6).

Also shown in Fig. 8 are the characteristic velocity requirements for low thrust transfer between circular orbits. It is demonstrated in (17) that the optimum thrust direction for escape with low thrust is halfway between the tangent to the orbit and the normal to the radius vector. This result is also largely applicable to transfer between circular orbits. For the purposes of this report, the difference between this program and tangential thrust can be considered negligible. In transferring between circular orbits with tangential thrust, the vehicle will quickly approach a mean path whose slope is twice the local thrust-weight ratio and whose velocity is equal to local circular velocity (18). As a result, the satellite will arrive at the final orbit with an eccentricity of twice the local thrust-weight ratio. This is a very small eccentricity and can be cancelled by a high thrust type maneuver after the final orbit radius has been reached, or it can be cancelled by a combination maneuver during the final cycle or cycles. In either case, the characteristic velocity requirement is very small.

Changes in Inclination

To change the inclination of a satellite orbit it is necessary to change the direction of the velocity vector. The smaller this velocity vector, the smaller the required characteristic velocity. This is a case in which it is desirable to use fuel to establish an intermediate orbit which will have a low velocity so that the change in inclination can be made economically. In (7), it is shown that for large changes in inclination, a tangential impulse which establishes an elliptic orbit, followed by a normal impulse at the apogee of this ellipse and a tangential impulse at perigee to re-establish a circular orbit, can produce fuel savings over a single normal impulse. In the Appendix it is shown that it is advantageous to do part of the change in inclination when entering and leaving the elliptic orbit. The optimum distribution of inclination changes has been treated in several papers (13,14,19,20). Distributing the inclination change produces a small saving in characteristic velocity, usually less than 5% of circular velocity (20), and makes the three-impulse maneuver optimum for all angles rather than for large angles only. The characteristic velocity requirements of these maneuvers are shown in Fig. 9. For small changes in inclination, there is an optimum period for each inclination, while for large changes, the optimum period is infinite. As a consequence, the theoretical optimum is to escape to infinity along a parabolic orbit, change the inclination of the parabolic orbit with an infinitesimal impulse at infinity, and then return and re-enter a circular orbit. Although this maneuver is clearly impractical, it is possible to save fuel by taking longer and longer times to make the maneuver. The curves of Fig. 9 show the maneuver times in terms of the initial orbital periods (P_0) that are required for various characteristic velocities. Although not shown, the optimum three-impulse maneuvers for small changes produce only small improvements over the one-impulse maneuver.

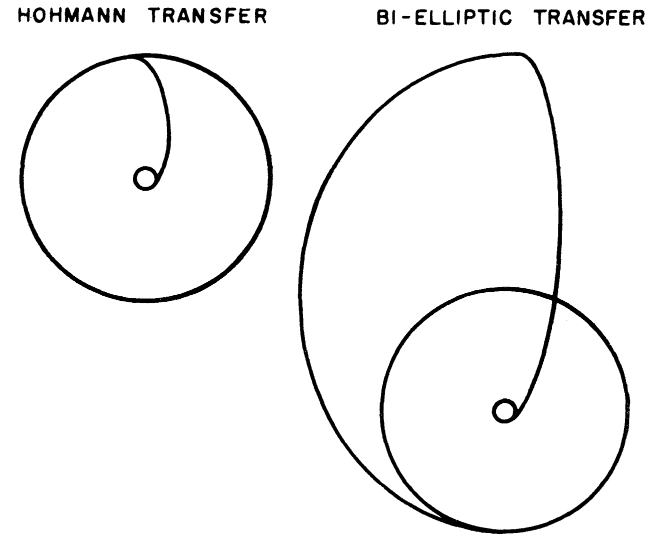

Fig. 7 Minimum fuel transfers

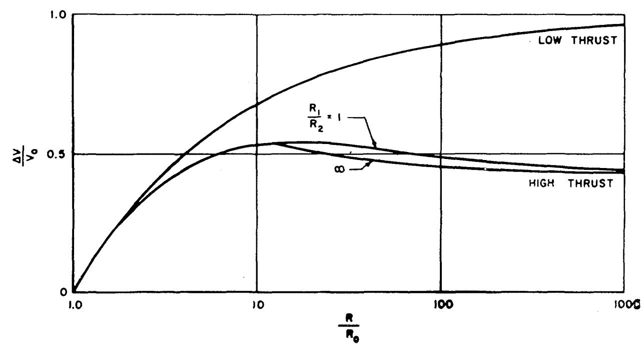

Fig. 8 Characteristic velocities for transfer between circular orbits

With low thrust it is also advantageous to expend some fuel in reducing velocity as well as in changing inclination. For this case the inclination is changed continuously while the satellite revolves in its orbit, so that it is necessary to reduce the velocity throughout the orbit rather than just at one point. This is done by maintaining the orbit as a circle and changing inclination while this circle is first expanded and then shrunk. Two optimum methods of doing this are discussed in the Appendix. The first represents the true optimum where thrust angle is varied through each cycle and also as a function of the size of the orbit. In the second case, thrust angle is maintained constant through each cycle, although it is still varied as a function of the size of the orbit. For the latter case, all the results can be expressed in closed form. The difference in characteristic velocity between the two cases varies between zero and a maximum of 3%. If large changes in inclination are required, it is once again desirable to escape to "infinity," change the inclination at "infinity," and then return. Because the motor is run continuously, this is a minimum-time as well as a minimum-fuel maneuver. It is not possible to trade these two quantities by changing the maneuver as with the high thrust case.

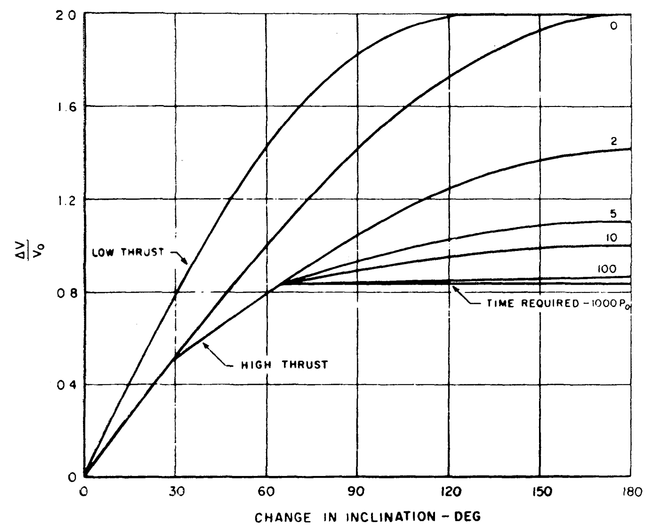

Fig. 9 Minimum characteristic velocity for orbital plane rotation of a circular orbit

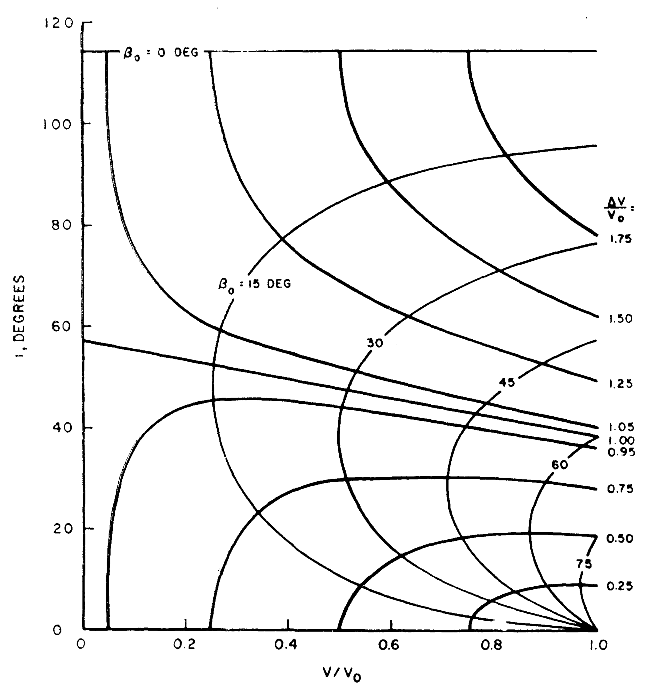

Fig. 10 Characteristic velocities for transfer between inclined circular orbits

Changes in Inclination and Radius

Rider has recently published an excellent survey of high thrust transfer between inclined circular orbits of different radii (20). He compares two- and three-impulse transfers with and without distribution of the change in inclination.

The optimum low thrust transfer between inclined circular orbits of arbitrary radius is derived in the Appendix in the course of deriving the optimum maneuver for large changes in inclination only. The variational formulation of the inclination problem is necessarily identical to the variational formulation of the more general problem. Every one of the optimum trajectories for producing changes in inclination alone is also an optimum way of getting to every combination of radius and inclination along that trajectory. This is illustrated in Fig. 10 where the trajectories corresponding to various values of the initial steering angle β_0 are plotted vs. inclination and velocity. Velocity can be used as a variable instead of radius because the orbit always remains quasi-circular so that the square of the velocity is inversely proportional to the radius. It is used in preference to radius because it does not become infinite. Fig. 10 also shows the characteristic velocity requirements for any desired change in both inclination and velocity.

Orbit Maintenance

The orbit of any real satellite will not be a precise Keplerian ellipse. Various natural forces will tend to perturb the orbit. For many applications these perturbations will be acceptable, but there are other applications for which it will be necessary to use propulsion to cancel the effects of perturbations. The most important of these perturbing forces are due to atmospheric drag, the gravitational field of the Earth's equatorial bulge, and the gravitational field of the sun and moon. Radiation pressure will be important only if the satellite has an unusually low density.

Atmospheric drag tends to decrease both the semimajor axis and the eccentricity of any elliptic orbit. The perigee altitude of the satellite will decrease at an increasing rate with time until the satellite re-enters the dense part of the atmosphere and either burns up or impacts the surface of the Earth. For altitudes above approximately 300 miles this effect is negligible. However, thrust will be required to maintain satellites at altitudes of the order of 100 miles for more than a day or so. A satellite of standard density can be maintained at about this altitude by continuous application of thrusts of from 10^{-5} to 10^{-4} times the vehicle weight.

The most important effects of the Earth's equatorial bulge are to cause a small change in the period of the satellite, a continuous rotation of the orbital plane around the polar axis of the Earth, and a continuous rotation of the line of apsides in the plane of the orbit. The sun and the moon cause similar effects except that in these cases the satellite orbital plane rotates with respect to the polar axis of the ecliptic and the moon's orbital plane.

One mission for which these perturbations are important is the stationary 24-hr equatorial satellite. If such a satellite were put precisely over a given point on the Earth's Equator, the combined effect of the equatorial bulge, the sun, and the moon would be to cause a slow rotation of the plane of the orbit in space. As seen from Earth, the satellite would appear to gradually develop a figure 8 motion with a period of one day. This motion would grow in amplitude until it reached about 15 deg north and south of the equator after 25 years. After another 25 years the amplitude would have decreased back to zero, and the cycle would repeat itself.

In the first year the maximum excursion would be about a degree. This motion could be suppressed by changing the orbital inclination when it grew too large or by continuous application of thrust. A continuous thrust of 2.5 times 10^{-7} times the weight of the satellite would suppress the motion if the direction were changed by 180 deg every half day. Alternately, thrusts of larger magnitude with characteristic velocities of 165 fps per year could be used. Because of the small amount of thrust required, even an ion rocket with a thrust-weight ratio as low as 10^{-5} could be considered a high thrust system for this application.

Appendix: Derivation of Optimum Maneuvers

Equations of Motion of Orbit Elements

The analyses of this appendix are carried out in terms of the elements of the elliptic orbit. The rates of change of each of these elements have been derived by following the techniques presented in (21). The thrust force is broken down into three mutually perpendicular components: a tangential component which is tangent to the velocity vector, an orthogonal component which is normal to the plane of the orbit, and a normal component which is normal to the other two. These components are expressed in terms of two angles, the angle between the velocity vector and the component of the thrust vector in the plane of the orbit α, and the angle between the thrust vector and the plane of the orbit β

$$F_T = F\cos\beta\cos\alpha \qquad F_N = F\cos\beta\sin\alpha \qquad F_z = F\sin\beta \quad [1]$$

For almost circular orbits where the eccentricity and inclination are small, the equations of motion are

$$\frac{di}{dt} = \frac{\cos(\omega+\theta)}{V_0}\frac{F_z}{m}$$

$$\frac{d\Omega}{dt} = \frac{\sin(\omega+\theta)}{iV_0}\frac{F_z}{m}$$

$$\frac{da}{dt} = \frac{2a}{V_0}\frac{F_T}{m}$$

$$\frac{de}{dt} = \frac{2\cos\theta}{V_0}\frac{F_T}{m} + \frac{\sin\theta}{V_0}\frac{F_N}{m}$$

$$\frac{d\omega}{dt} = \frac{2\sin\theta}{eV_0}\frac{F_T}{m} + \frac{\cos\theta}{eV_0}\frac{F_N}{m} - \frac{\sin(\omega+\theta)}{iV_0}\frac{F_z}{m} \quad [2]$$

Optimum High Thrust Maneuvers

Large Changes in Inclination

An efficient maneuver for effecting large changes in inclination is discussed in (7). This maneuver consists of applying a tangential impulse to change the original circular orbit to an ellipse, changing inclination at the apogee of the ellipse, and re-establishing a circular orbit with a tangential retrothrust at perigee. The following analysis is intended to show that this maneuver can be improved slightly by combining part of the change in inclination with the initial and final impulses.

The characteristic velocity required for finite changes in inclination with high thrust is simply the vector difference between the initial and final velocity vectors. For the present problem, the required characteristic velocity is given by

$$\frac{\Delta V}{V_0} = 2\frac{V}{V_0}\sin i/2 \quad [3]$$

The perigee velocity required to establish a specific apogee radius is given by

$$\frac{V_1}{V_0} = \sqrt{\frac{2Ra}{Ra+R_0}} \quad [4]$$

The characteristic velocity required to produce an initial (or final) change in inclination is

$$\frac{\Delta V_1}{V_0} = \sqrt{\frac{V_1^2}{V_0^2} - \frac{2V_1}{V_0}\cos i_1 + 1} \quad [5]$$

The total characteristic velocity for the maneuver is given by

$$\frac{\Delta V}{V_0} = 2\sqrt{\frac{V_1^2}{V_0^2} - \frac{2V_1}{V_0}\cos i_1 + 1} + 2\left(\frac{2V_0}{V_1} - \frac{V_1}{V_0}\right)\sin\left(\frac{i}{2} - i_1\right) \quad [6]$$

The time of the maneuver depends only upon the choice of apogee radius. The required characteristic velocity for a given maneuver time can be minimized by optimizing the initial change in inclination. An explicit expression for the optimum value of this angle can be derived by assuming that this angle is small. More exact calculations have shown that this assumption is justified in all cases of interest

$$\frac{\Delta V}{V_0} \approx 2\sqrt{\left(\frac{V_1}{V_0} - 1\right)^2 + \frac{V_1}{V_0}i_1^2} + 2\left(\frac{2V_0}{V_1} - \frac{V_1}{V_0}\right)\left(\sin\frac{i}{2} - i_1\cos\frac{i}{2}\right) \quad [7]$$

Differentiating Eq. 7 with respect to the initial change in inclination and setting the result equal to zero produces the following expression for the optimum initial angle

$$i_{1_{opt}} \approx \frac{1 - \dfrac{V_0}{V_1}}{\sqrt{\left(\dfrac{\sec i/2}{2\dfrac{V_0}{V_1} - \dfrac{V_1}{V_0}}\right)^2 - \dfrac{V_0}{V_1}}} \quad [8]$$

This optimum initial angle was used with Eq. 6 to draw up the curves of Fig. 9. The optimum initial angle given by Eq. 8 approaches zero as the inclination becomes large. As this is the case considered in (7), the results of Fig. 9 become identical with the results in (7) for large changes in inclination.

Combination of Small Planar Maneuvers with Small Changes in Inclination

As mentioned in the body of the report, the optimum combination maneuvers for high thrust systems are difficult to derive because of the large number of variables involved. A simplified approach will be considered herein, where a given planar maneuver is combined with a change in inclination. The planar components of the initial and final impulses are assumed to be tangent to the velocity vector, applied at apogee or perigee, and fixed in magnitude. The magnitude of the orthogonal components of the initial and final impulses, and the magnitude and point of application of an intermediate orthogonal impulse, are allowed to vary. These four variables are interconnected so that only two may be considered as independent. The two independent variables are optimized to produce the results shown in Fig. 2.

The derivation starts with the first and last of Eqs. 2. The first describes the rate of change of inclination due to an orthogonal force, while the last describes the rate of change of the angle between the line of apsides and the line of nodes.

Examination of these two equations shows that the rate of change of the angle between the line of nodes and the line of apsides depends upon the inclination and that the rate of change of inclination depends on this angle. These equations must be solved simultaneously to determine the effect of small finite impulses. This determination is simplified by the assumption of tangential impulses at the apsides, because such impulses have no effect on the position of the line of apsides. Using this assumption, the first and last of Eqs. 2 can be combined to yield

$$\frac{d\omega}{\tan(\omega + \theta)} = -\frac{di}{i} \qquad [9]$$

This equation can be integrated directly

$$\ln \frac{\sin(\omega+\theta)_2}{\sin(\omega+\theta)_1} = \ln \frac{i_1}{i_2} \qquad i \sin(\omega + \theta) = \text{const.} \qquad [10]$$

Eq. 10 has a simple physical interpretation: the distance of the satellite from the reference plane does not change during the impulsive maneuver. This means that the final impulse that reduces the inclination of the satellite to zero must be given at the line of nodes. Since it has also been assumed that the last impulse takes place at the line of apsides, the second of the three impulses must make the line of apsides coincident with the line of nodes. The two remaining independent variables which will be optimized are taken as the point of application of this second impulse (θ) and the inclination after this second impulse (i). The impulse required to produce a given change in inclination or in position of the line of nodes is found by combining Eq. 10 with the last of Eqs. 2 and integrating

$$d\omega = -\frac{\sin^2(\omega+\theta)}{i_1 \sin(\omega+\theta)_1}\frac{F_z dt}{V_0 m}$$

$$\frac{1}{V_0}\int \frac{F_z dt}{m} = -i_1 \sin(\omega+\theta)_1 \int \frac{d\omega}{\sin^2(\omega+\theta)}$$

$$\frac{\Delta V_z}{V_0} = i_1 \sin(\omega+\theta)_1[\cot(\omega+\theta)_2 - \cot(\omega+\theta)_1] \qquad [11]$$

The sum of the required velocity impulse is given by

$$\Sigma \frac{\Delta V}{V_0} = \sqrt{\left(\frac{\Delta V_{T1}}{V_0}\right)^2 + [i_0(\cos\omega_0 + \sin\omega_0 \cot\theta) - i]^2} + i_0 \frac{\sin\omega_0}{\sin\theta} + \sqrt{\left(\frac{\Delta V_{T2}}{V_0}\right)^2 + i^2} \qquad [12]$$

Differentiating this equation with respect to θ and setting the result equal to zero results in the following expression for the optimum value of θ

$$\cot\theta_{\text{opt}} = \frac{i_0 \cos\omega_0 - i}{\dfrac{\Delta V_{T1}}{V_0} - i_0 \sin\omega_0} \qquad [13]$$

Differentiating Eq. 12 with respect to i and setting the result equal to zero results in the following expression for the optimum value of i

$$i_{\text{opt}} = \frac{\dfrac{\Delta V_{T2}}{V_0} i_0(\cos\omega_0 + \sin\omega_0\cot\theta)}{\dfrac{\Delta V_{T1}}{V_0} + \dfrac{\Delta V_{T2}}{V_0}} \qquad [14]$$

Combining Eqs. 13 and 14 leads, after some reduction, to

$$\cot\theta_{\text{opt}} = \frac{i_0\cos\omega_0}{\dfrac{\Delta V_{T1}}{V_0} + \dfrac{\Delta V_{T2}}{V_0} - i_0\sin\omega_0} \qquad [15]$$

$$i_{\text{opt}} = \frac{\Delta V_{T2}}{V_0}\cot i_{\text{opt}} \qquad [16]$$

Introducing these equations into Eq. 12 and simplifying leads to the final result

$$\Sigma\frac{\Delta V}{V_0} = \left(\frac{\Delta V_{T1}}{V_0} + \frac{\Delta V_{T2}}{V_0} + i_0\sin\omega_0\right) \times \sqrt{1 + \left(\frac{i_0\cos\omega_0}{\dfrac{\Delta V_{T1}}{V_0} + \dfrac{\Delta V_{T2}}{V_0} - i_0\sin\omega_0}\right)^2} \qquad [17]$$

One of the interesting characteristics of this equation is that the required velocity impulse depends only upon the total magnitude of the planar velocity impulse and not upon its distribution. This means that in both the three-dimensional and two-dimensional cases, a change in eccentricity or orbital position can be obtained free if the required change in major axis is large enough. The velocity increment required for the planar part of the maneuver depends only upon the largest single planar correction that must be made.

Optimum Low Thrust Maneuvers

Steering Programs for Small Changes in Elements of Orbit

The optimum steering program for small changes in the elements of almost circular orbits can be easily derived by using the ordinary theory of maxima and minima. The general theory of the calculus of variations can offer improvements over this simple theory only when the changes are large, as in the inclination problem considered at the end of this appendix.

The problem is to optimize the steering angles of the thrust vector as a function of orbital position. Combination of Eqs. 1 and 2 yields

$$\frac{da/a_0}{dt} = \frac{F}{mV_0}[2\cos\alpha\cos\beta]$$

$$\frac{de}{dt} = \frac{F}{mV_0}[2\cos\theta\cos\alpha\cos\beta + \sin\theta\sin\alpha\cos\beta]$$

$$\frac{di}{dt} = \frac{F}{mV_0}[\cos(\omega+\theta)\sin\beta] \qquad [18]$$

The optimum steering program is found by combining Eqs. 18 by means of Lagrange multipliers and setting the partial derivatives of this expression with respect to the steering angles equal to zero

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\left[\frac{da/a_0}{dt} + \lambda_1\frac{de}{dt} + \lambda_2\frac{di}{dt}\right] = 0 \qquad [19]$$

Substituting the expression of Eqs. 18 into Eqs. 19 and carrying out the indicated operations result in the following expressions for the optimum steering angles

$$\tan\alpha_{\text{opt}} = \frac{\lambda_1\sin\theta}{2(1+\lambda_1\cos\theta)}$$

$$\tan\beta_{\text{opt}} = \frac{\lambda_2\cos(\omega+\theta)}{\sqrt{4(1+\lambda_1\cos\theta)^2 + \lambda_1^2\sin^2\theta}} \qquad [20]$$

Planar Maneuvers

When the required maneuver is a planar one, the second Lagrange multiplier of Eqs. 19 and 20 becomes zero and the thrust line always lies in the plane of the orbit. Only the first of Eqs. 20 need then be considered. This section will consider some special cases of this equation. The first case is a change in major axis only. The Lagrange multiplier is then zero and the thrust should always be tangent to the orbit

$$\frac{da}{a_0} = 2\frac{F}{mV_0}dt \qquad \frac{\Delta V}{V_0} = \frac{1}{2}\frac{\Delta a}{a_0} \tag{21}$$

If it is desired to change the eccentricity without changing the major axis, then the Lagrange multiplier should be taken as infinite and the steering program is given by

$$\tan \alpha_{\text{opt}} = 1/2 \tan \theta \tag{22}$$

This equation is plotted in Fig. 3. The thrust direction comes close to representing a constant direction in space and this latter program gives very close to optimum results.

The characteristic velocity for changes in eccentricity is found by substituting Eq. 22 into the second of Eqs. 18 and integrating

$$\frac{de}{dt} = \frac{F}{mV_0}\left[\frac{4\cos\theta}{\sqrt{4+\tan^2\theta}} + \frac{\tan\theta\sin\theta}{\sqrt{4+\tan^2\theta}}\right]$$

$$\frac{de}{d\theta} = \frac{FR}{mV_0^2}\frac{4\cos^2\theta + \sin^2\theta}{\sqrt{4\cos^2\theta+\sin^2\theta}}$$

$$\frac{de}{d\theta} = \frac{FR}{mV_0^2}2\sqrt{1 - 3/4\sin^2\theta}$$

The change in eccentricity per revolution is given by a complete elliptic integral of the second kind

$$\Delta e = \frac{FR}{mV_0^2}8\int_0^{\pi/2}\sqrt{1 - 3/4\sin^2\theta}\,d\theta$$

$$\Delta e = \frac{8FR}{mV_0^2}E(\sqrt{3/4}) = \frac{8FR}{mV_0^2}(1.2111) \tag{23}$$

The characteristic velocity for the change in eccentricity per cycle is given by

$$\frac{\Delta V}{V_0} = \frac{F}{m}\frac{R_0}{V_0^2}2\pi \qquad \frac{\Delta V}{V_0} = 0.649\Delta e \tag{24}$$

There are several other values of the Lagrange multiplier where the optimum steering program can be substituted in Eqs. 18 and integrated analytically. The details of these will not be given here as the integrations are straightforward and the results are shown as the low thrust curve in Fig. 4. The steering program for two of the more important cases, maximizing the rates of change of apogee or perigee radius, is shown in Fig. 3. For these cases the values of the Lagrange multipliers are plus and minus one, respectively.

Nonplanar Maneuvers

Nonplanar maneuvers can be treated exactly as were the planar maneuvers of the last section. For simplicity, only the cases where the eccentricity does not change will be considered. With this assumption, Eq. 20 reduces to

$$\tan \alpha_{\text{opt}} = 0 \qquad \tan \beta_{\text{opt}} = \frac{\lambda_2}{2}\cos(\omega+\theta) \tag{25}$$

The second equation is plotted in Fig. 5 for various values of the Lagrange multiplier. Changing the notation of the second of Eqs. 25 and substituting into Eq. 18 yields

$$\tan\beta = k'\cos\theta$$

$$\frac{mV_0^2}{FR_0}di = \frac{k'\cos^2\theta}{\sqrt{1+k'^2\cos^2\theta}}d\theta = \frac{k' - k'\sin^2\theta}{\sqrt{1+k'^2-k'^2\sin^2\theta}}d\theta$$

$$\frac{mV_0^2}{FR_0}\frac{da}{a_0} = \frac{2}{\sqrt{1+k'^2\cos^2\theta}}d\theta = \frac{2}{\sqrt{1+k'^2-k'^2\sin^2\theta}}d\theta \tag{26}$$

The following standard substitution

$$k = \frac{k'}{\sqrt{1+k'^2}} \tag{27}$$

changes Eqs. 26 to

$$\frac{mV_0^2}{FR_0}di = \frac{k - k\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta$$

$$= \frac{k^2-1}{k\sqrt{1-k^2\sin^2\theta}}d\theta + \frac{\sqrt{1-k^2\sin^2\theta}}{k}d\theta$$

$$\frac{mV_0^2}{FR_0}\frac{da}{a_0} = \frac{2\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2\theta}}d\theta \tag{28}$$

Integrating Eqs. 28 over a complete cycle results in complete elliptic integrals of the first and second kind

$$\frac{mV_0^2}{FR_0}\Delta i = 4\left[\frac{k^2-1}{k}K + \frac{E}{k}\right]$$

$$\frac{mV_0^2}{FR_0}\frac{\Delta a}{a_0} = 8\sqrt{1-k^2}\,K \tag{29}$$

Expressing these in terms of required characteristic velocity results in

$$\frac{\Delta V}{V_0} = \frac{\Delta i}{\frac{2}{\pi}\left[\frac{k^2-1}{k}K + \frac{E}{k}\right]} \qquad \frac{\Delta V}{V_0} = \frac{\Delta a/a_0}{\frac{4}{\pi}\sqrt{1-k^2}K} \tag{30}$$

The characteristic velocity requirements given by Eqs. 30 are plotted in Fig. 6. Also shown in Fig. 6 is the result of keeping the thrust angle β constant. This can be seen to give near optimum results.

Where only a change in inclination is desired, k should be taken as unity, and where only a change in major axis is desired k should be taken as zero. The results of these two cases are given by

$$\frac{\Delta V}{V_0} = \frac{\pi}{2}\Delta i \qquad \frac{\Delta V}{V_0} = \frac{1}{2}\frac{\Delta a}{a_0} \tag{31}$$

The second of these equations checks with Eq. 21, as it should.

Changes in Position in Orbit

The only method of producing changes in orbital position with a continuous small thrust is to change the period of the orbit. The most economical way of doing this is to use tangential thrust in one direction until half the desired change in position is accomplished, and then to reverse the thrust direction until the original orbit is re-established.

For small changes in period, the change in period for a given change in major axis is given by

$$\frac{\Delta P}{P_0} = \frac{3}{2}\frac{\Delta a}{a_0} \tag{32}$$

The change in period produced by a given characteristic velocity is found by substituting Eq. 21 into Eq. 32, as follows

$$\frac{\Delta P}{P_0} = 3\frac{\Delta V}{V_0} = \frac{3FP_0}{mV_0}n \tag{33}$$

The change in position per cycle is given by

$$\frac{\Delta\theta'}{\Delta n} = 2\pi\frac{\Delta P}{P_0} = 6\pi\frac{FP_0}{mV_0}n \tag{34}$$

Integrating Eq. 34 over n cycles yields

$$\Delta\theta' = 3\pi \frac{FP_0}{mV_0} n^2 = 3\pi \frac{\Delta V}{V_0} n \qquad [35]$$

Eq. 35 applies to only one segment of the maneuver. For the total maneuver where the period is first increased (or decreased) and then decreased (or increased), the change in θ' is given by

$$\Delta\theta' = 2\left[3\pi \frac{\Delta V}{2V_0} \frac{n}{2}\right] \qquad [36]$$

The required characteristic velocity is

$$\frac{\Delta V}{V_0} = \frac{4}{3} \frac{\Delta\theta'}{2\pi n} \qquad [37]$$

Eq. 37 agrees with the result obtained in (1).

Large Changes in Inclination

This section will consider the derivation of the optimum steering program for producing large changes in inclination with low thrust devices. This is a problem which can be solved exactly by the general methods of the calculus of variations. However this solution cannot be carried out in a closed form and would require very extensive numerical calculations. The problem will be simplified by making two assumptions which allow a closed form solution within the framework of the general theory of the calculus of variations. The first assumption is that the orbits always remain quasi-circular. It can be shown that this should be the case for very small changes, as well as for very large changes in inclination. It will be assumed that the orbits should remain quasi-circular for all intermediate changes as well. This assumption allows the use of the previous results on small changes in inclination and major axis for the individual revolutions of the satellite. The problem is basically one of how rapidly the orbit should enlarge and shrink while the inclination is being changed.

The second assumption is that the thrust angle will be held constant during each revolution. Fig. 6 shows that there is little difference between this case and the true optimum for each cycle. The more general case can be developed in exactly the manner presented here, but it leads to complicated expressions having no closed form solution. The present solution produces near optimum results having a surprisingly simple form.

When the thrust angle is kept constant, the change in inclination can be expressed by

$$di = \frac{2}{\pi} \frac{F \sin\beta}{mV} dt \qquad [38]$$

As the orbits will remain quasi-circular if the thrust angle is held constant over each revolution, the velocity will decrease as the major axis increases

$$dV = -\frac{F}{m} \cos\beta \, dt \qquad [39]$$

The variational problem is formulated using velocity as the independent variable. This is merely an artifice to simplify the analysis and does not affect the results. The variational integral to be optimized is given by

$$-\int \left(\frac{di}{dV} + \lambda \frac{dt}{dV}\right) dV = \int \left(\frac{2}{\pi} \frac{\tan\beta}{V} + \frac{\lambda m}{F \cos\beta}\right) dV \qquad [40]$$

The Euler equation of this problem is simply that the partial derivative of the integrand with respect to the thrust angle should be zero

$$\frac{\partial}{\partial\beta}\left(\frac{2}{\pi} \frac{\tan\beta}{V} + \frac{\lambda m}{F\cos\beta}\right) = 0$$

This reduces to

$$V \sin\beta = \frac{2F}{m\pi\lambda} = \text{const} = V_0 \sin\beta_0 \qquad [41]$$

Eqs. 39 and 38 can now be integrated with this steering program

$$\frac{F}{m} dt = -\frac{dV}{\cos\beta} - \frac{dV}{\sqrt{1 - \sin^2\beta \frac{V_0^2}{V^2}}}$$

$$\frac{F}{m} dt = -\frac{dV}{2\sqrt{V^2 - V_0^2 \sin\beta_0}}$$

$$ft = \Delta V = V_0 \cos\beta_0 - \sqrt{V^2 - V_0^2 \sin^2\beta_0}$$

$$V = \sqrt{V_0^2 - 2V_0\Delta V \cos\beta_0 + \Delta V^2}$$

. . . . [42]

$$di = -\frac{2}{\pi} \frac{\tan\beta}{V} dV \qquad di = -\frac{2}{\pi} \frac{V_0 \sin\beta dV}{V\sqrt{V^2 - V_0^2 \sin^2\beta_0}}$$

$$i = \frac{2}{\pi} \sin^{-1} \frac{V_0 \sin\beta_0}{V} - \frac{2\beta_0}{\pi}$$

. . . . [43]

Eqs. 42 and 43 can be combined to give:

(a) For $\frac{\Delta V}{V_0} \leq \cos\beta_0$:

$$i = \frac{2}{\pi} \sin^{-1} \frac{V_0 \sin\beta_0}{\sqrt{V_0^2 - 2V_0\Delta V \cos\beta_0 + \Delta V^2}} - \frac{2\beta_0}{\pi} \qquad [44a]$$

(b) For $\frac{\Delta V}{V_0} \geq \cos\beta_0$:

$$i = 114.6° - \frac{2}{\pi} \sin^{-1} \frac{V_0 \sin\beta_0}{\sqrt{V_0^2 - 2V_0\Delta V \cos\beta_0 + \Delta V^2}} - \frac{2\beta_0}{\pi} \qquad [44b]$$

Eqs. 42 and 44 represent the changes in velocity and inclination that can be obtained with a given characteristic velocity as the initial steering angle is changed. These equations may be solved simultaneously to determine the characteristic velocity requirements directly in terms of the orbital velocities and the inclination

$$\Delta V = \sqrt{V_0^2 - 2VV_0 \cos\frac{\pi}{2} i + V^2} \qquad [45]$$

Nomenclature

a	=	semimajor axis of orbit
e	=	eccentricity of orbit
E	=	energy per unit mass, complete elliptic integral of the second kind
F	=	thrust force
i	=	inclination of orbit to a reference plane
k, k'	=	constants which represent the moduli of elliptic integrals
K	=	complete elliptic integral of the first kind
L	=	angular momentum per unit mass
m	=	mass
n	=	number of revolutions
P	=	period of orbit

R = radius
t = time
V = velocity
ΔV = characteristic velocity
Ω = position angle of the line of nodes (see Fig. 1)
α = angle between the velocity vector and the component of the thrust vector in the plane of the orbit
β = angle between the thrust line and the plane of the orbit
θ = position angle in the orbit measured from perigee
θ' = position angle in the orbit measured from a fixed reference
λ = Lagrange multiplier
ϕ = flight path angle with the horizontal
ω = angle between the line of nodes and the line of apsides

Subscripts

a = apogee
N = normal component
opt = optimum
p = perigee
T = tangential component
z = orthogonal component
0 = initial; circular
1 = intermediate
2 = final

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Effects of Plane Librations on the Orbital Motion of a Dumbbell Satellite

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The equations of motion of a dumbbell satellite oscillating or tumbling in the plane of its orbit are treated by a perturbation technique which assumes the satellite's length to be small with respect to its orbital radius. The unperturbed orbital motion is that of a point mass at the mass center of the satellite, while the equations for the perturbed motion are essentially decoupled. Analytic solutions are obtained under initial conditions which would have yielded a circular orbit were the satellite a point mass. Although the disturbances induced in the orbital motion by the librations are usually quite small, for certain frequencies of the librations a resonance phenomenon occurs; i.e., the perturbation quantities contain secular terms, so that they grow indefinitely with increasing time.

The MOTION of a satellite about its mass center is a problem of prime importance in connection with many satellite missions, the more exciting of which necessitate the directional stabilization of the vehicle. An important factor in this motion is the variation of the gravitational force on a mass particle in the satellite with its distance from the body about which the vehicle is orbiting. As a result, the point at which the net force acts—the center of gravity of the satellite—does not, in general, coincide with its center of mass. It is then possible for the line between the gravitational and mass centers to be noncoincident with the line of action of the gravitational force, so that a torque is exerted about the mass center of the satellite.

The existence of a gravitational torque was first noted over 150 years ago in connection with the longitudinal oscillations

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