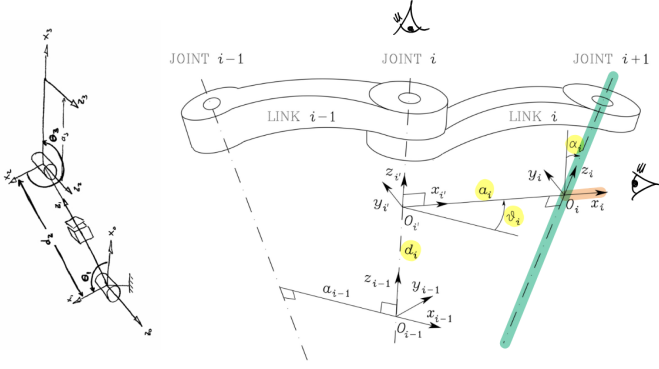


1. Denavit-Hartenberg



Il sistema di riferimento \mathcal{R}_i solidale con $LINK_i$ viene definito secondo le seguenti regole:

① Asse z_i e origine O_i

- **L'asse z_i** è posto lungo l'asse di movimento di g_{i+1} (asse di rotazione o di traslazione a seconda del tipo di giunto)
- **L'origine O_i** è posta all'intersezione di z_i con la normale comune (*common normal*) fra gli assi z_{i-1} e z_i . La normale comune è quella retta perpendicolare ad entrambi gli assi (nota: entrambi angoli retti nella figura)
- **Casi particolari:**
 - \mathcal{R}_0 : origine O_0 e x_0 possono essere fissati a piacimento (solo z_0 univocamente definito).
 - \mathcal{R}_n : $\nexists g_{n+1} \implies z_n, O_n$ non univocamente definiti. Per consuetudine: origine nel centro della pinza e z_n coincidente a z_{n-1} (visto che tipicamente l'ultimo giunto è rotoidale).

② Asse x_i e y_i

- **L'asse x_i** è fissato lungo la normale comune fra gli assi z_{i-1} e z_i
 - Se z_{i-1} e z_i si intersecano \implies direzione di x_i ($\perp z_i$) è arbitraria
 - se z_{i-1} e z_i sono paralleli \implies origine arbitraria, x_i nel piano normale a z_{i-1} e z_i con direzione e verso arbitrari.
- **L'asse y_i** completa la terna destrorsa ($j = k \times i$)

③ Sistema di riferimento intermedio

$z_{i'}$ diretto lungo $z_{i-1} \mid O_{i'}$ posta all'intersezione di z_{i-1} con la normale comune fra z_{i-1} e $z_i \mid x_{i'}$ diretto lungo la normale comune fra z_{i-1} e z_i (come x_i)

- $d_i \rightarrow$ **link offset**: coordinata di $O_{i'}$ lungo z_{i-1}
- $\theta_i \rightarrow$ **joint angle**: angolo di rotazione da x_{i-1} a x_i attorno all'asse $z_{i'}$ (positivo quando la rotazione è anti-oraria)
- $a_i \rightarrow$ **link length**: distanza (con segno) fra O_i e $O_{i'}$
- $\alpha_i \rightarrow$ **link twist**: angolo di rotazione da z_{i-1} a z_i attorno all'asse x_i (positivo quando la rotazione è anti-oraria)

$${}^{i-1}\mathbf{T}_i(q_i) = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trigonometric inequalities:

$$c_{12} + s_{12} = c_{1-2} \quad c_{12} - s_{12} = c_{1+2} \quad s_{12} c_2 - c_{12} s_2 = s_{1-2} \quad s_{12} c_2 + c_{12} s_2 = s_{1+2}$$

Tips:

$$\begin{aligned} a &\rightarrow \begin{cases} z_{i-1} \xleftrightarrow{\text{dist.}} z_i & \text{along } x_i \\ z_{i-1} \xrightarrow{\quad} z_i & \text{around } x_i \end{cases} \\ d &\rightarrow \begin{cases} x_{i-1} \xleftrightarrow{\text{dist.}} x_i & \text{along } z_{i-1} \\ x_{i-1} \xrightarrow{\quad} x_i & \text{around } z_{i-1} \end{cases} \end{aligned}$$

2. Differential Kinematics

2.1 Geometric Jacobian

$$\text{i-th column of } \mathbf{J}: \begin{bmatrix} \mathbf{J}_{p,i} \\ \mathbf{J}_{o,i} \end{bmatrix} = \begin{cases} \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} & \text{for a } \mathbf{prismatic} \text{ joint} \\ \begin{bmatrix} z_{i-1} \times (p - p_{i-1}) \\ z_{i-1} \end{bmatrix} & \text{for a } \mathbf{revolute} \text{ joint} \end{cases}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

E.g. planar RRR

$$\mathbf{J}(q) = \begin{bmatrix} z_0 \times (p - p_0) & z_1 \times (p - p_1) & z_2 \times (p - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix} \quad \text{here } c_{12} = c(\theta_1 + \theta_2)$$

$$\begin{aligned} {}^0T_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ {}^0T_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} \\ {}^0T_2 &= {}^0T_1 {}^1T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix} \\ {}^0T_3 &= \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{p} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix} \end{aligned}$$

2.2 Analytical Jacobian

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\boldsymbol{\phi}} \end{bmatrix} = \frac{d\mathbf{x}}{dt} = \underbrace{\frac{\partial \mathbf{x}}{\partial \mathbf{q}}}_{\mathbf{J}_A(\mathbf{q})} \underbrace{\frac{d\mathbf{q}}{dt}}_{\dot{\mathbf{q}}} = \mathbf{J}_A(\mathbf{q}) \dot{\mathbf{q}} \quad \mathbf{J}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\boldsymbol{\phi}) \end{bmatrix} \mathbf{J}_A(\mathbf{q})$$

2.3 Inverse differential kinematics

$$\begin{cases} \text{minimize} & g(\dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{W} \dot{\mathbf{q}} \\ \text{subject to} & \mathbf{v} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \end{cases} \quad \mathbf{W} \stackrel{W}{=} \mathbf{I} \quad \dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v} \quad \mathbf{J}^\dagger \triangleq \mathbf{J}^T (\mathbf{J} \mathbf{J}^T)^{-1}$$

$$\begin{cases} \text{minimize} & g'(\dot{\mathbf{q}}) = \frac{1}{2} (\dot{\mathbf{q}} - \dot{\mathbf{q}}_d)^T (\dot{\mathbf{q}} - \dot{\mathbf{q}}_d) \\ \text{subject to} & \mathbf{v} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \end{cases} \iff \dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v} + \underbrace{(\mathbf{I} - \mathbf{J}^\dagger \mathbf{J})}_{\text{in } \mathcal{N}} \dot{\mathbf{q}}_d$$

Damped least-square:

$$\mathbf{J} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^T, \quad \Sigma_{ii} = \sqrt{\text{eig}(\mathbf{J} \mathbf{J}^T)} \implies \mathbf{J}^\dagger = \mathbf{V} \boldsymbol{\Sigma}^\dagger \mathbf{U}^T \quad \sigma_i \rightarrow 1/(\sigma_i + k^2) \quad \mathbf{J}^* = \mathbf{J}^T (\mathbf{J} \mathbf{J}^T + k^2 \mathbf{I})^{-1}$$

Secondary objectives:

$$\dot{\mathbf{H}} = \frac{d\mathbf{H}}{dt} = \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad \text{with} \quad \dot{\mathbf{q}}_d = -K \left(\frac{\partial \mathbf{H}}{\partial \mathbf{q}} \right)^T, \quad K > 0$$

$$\dot{\mathbf{H}} = \underbrace{\frac{\partial \mathbf{H}}{\partial \mathbf{q}} \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v}}_{\text{non si sa}} + \underbrace{-K \frac{\partial \mathbf{H}}{\partial \mathbf{q}} (\mathbf{I} - \mathbf{J}^\dagger \mathbf{J}) \left(\frac{\partial \mathbf{H}}{\partial \mathbf{q}} \right)^T}_{< 0}$$

- **Max dist. obstacles:** $H = \min_{p,o} \|p(q) - o\| \rightsquigarrow H \uparrow$

- **Max dist. joint limit:** $H(q) = -\frac{1}{2n} \sum_{i=1}^n \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{iM}} \right)^2 \rightsquigarrow H \downarrow$

- **Max dist. from singularities:** $H(q) = \sqrt{\det(\mathbf{J}(q) \mathbf{J}^T(q))} \rightsquigarrow H \uparrow$

3. Statics

$$\boldsymbol{\tau}^T \delta \mathbf{q} = \mathbf{F}^T \delta \mathbf{p} \implies \mathbf{J}^\dagger: \quad \boldsymbol{\tau} = -\mathbf{J}^T(\mathbf{q}) \mathbf{F}$$

$$\mathcal{N}(\mathbf{J}) \equiv \mathcal{R}^\perp(\mathbf{J}^T) \quad \mathcal{R}(\mathbf{J}) \equiv \mathcal{N}^\perp(\mathbf{J}^T)$$

Ellipsoids:

$$\|\dot{\mathbf{q}}\|^2 = 1 \iff \dot{\mathbf{q}}^T \dot{\mathbf{q}} = 1 \xrightarrow{\dot{\mathbf{q}} = \mathbf{J}^\dagger \mathbf{v}} \mathbf{v}^T (\mathbf{J} \mathbf{J}^T)^{-1} \mathbf{v} = 1 \implies E_v = \{ \mathbf{v} : \mathbf{v}^T (\mathbf{J} \mathbf{J}^T)^{-1} \mathbf{v} = 1 \}$$

$$\|\dot{\boldsymbol{\tau}}\|^2 = 1 \iff \boldsymbol{\tau}^T \boldsymbol{\tau} = 1 \xrightarrow{\boldsymbol{\tau} = -\mathbf{J}^T \mathbf{F}} E_F = \{ \mathbf{F} : \mathbf{F}^T (\mathbf{J} \mathbf{J}^T) \mathbf{F} = 1 \}$$

Manipulability measure:

$$w(\mathbf{q}) = \sqrt{\det(\mathbf{J} \mathbf{J}^T)} = |\lambda_1 \lambda_2 \dots \lambda_n| = |\det(\mathbf{J})|$$

4. Dynamics

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \quad (\mathcal{K} - \mathcal{P}) \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} = \mathcal{F}_i \quad i = 1, \dots, n$$

Kinetic:

$$\mathcal{T} = \sum_{i=1}^n \mathcal{T}_i + \mathcal{T}_{m_i} \implies \mathcal{T} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_{ij}(\mathbf{q}) \dot{\mathbf{q}}_i \dot{\mathbf{q}}_j = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{B}(\mathbf{q}) \dot{\mathbf{q}}$$

$$\begin{cases} \mathcal{T}_i = \frac{1}{2} m_i \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i + \frac{1}{2} \boldsymbol{\omega}_i^T \mathbf{R}_i^i \mathbf{I}_i \boldsymbol{\omega}_i \\ \mathcal{T}_{m_i} = \frac{1}{2} m_{m_i} \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i + \frac{1}{2} \boldsymbol{\omega}_i^T \mathbf{R}_i^i \mathbf{I}_{m_i} \boldsymbol{\omega}_i \end{cases}$$

$$\begin{cases} \mathcal{T}_i = \frac{1}{2} m_i \left(\dot{\mathbf{q}}^T \mathbf{J}_p^{(i)} \right) \left(\mathbf{J}_p^{(i)} \dot{\mathbf{q}} \right) + \frac{1}{2} \left(\dot{\mathbf{q}}^T \mathbf{J}_o^{(i)} \right) \left(\mathbf{R}_i^i \mathbf{I}_i \mathbf{R}_i^T \right) \left(\mathbf{J}_o^{(i)} \dot{\mathbf{q}} \right) \\ \mathcal{T}_{m_i} = \frac{1}{2} m_{m_i} \left(\dot{\mathbf{q}}^T \mathbf{J}_p^{(m_i)} \right) \left(\mathbf{J}_p^{(m_i)} \dot{\mathbf{q}} \right) + \frac{1}{2} \left(\dot{\mathbf{q}}^T \mathbf{J}_o^{(m_i)} \right) \left(\mathbf{R}_{m_i}^{m_i} \mathbf{I}_{m_i} \mathbf{R}_{m_i}^T \right) \left(\mathbf{J}_o^{(m_i)} \dot{\mathbf{q}} \right) \end{cases}$$

Potential:

$$\mathcal{U} = \sum_{i=1}^n \mathcal{U}_{l_i} + \mathcal{U}_{m_i} \implies \mathcal{U} = - \sum_{i=1}^n m_i g_0^T \mathbf{p}_{l_i} + m_{m_i} g_0^T \mathbf{p}_{m_i}$$

Dynamic equations:

$$\sum_{j=1}^n \mathbf{B}_{ij}(\mathbf{q}) \ddot{\mathbf{q}}_j + \sum_{j=1}^n \sum_{k=1}^n h_{ijk}(\mathbf{q}) \dot{\mathbf{q}}_k \dot{\mathbf{q}}_j + g_i(\mathbf{q}) = \mathcal{F}_i \quad h_{ijk} = \frac{\partial B_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial B_{jk}}{\partial q_i}$$

$$g_i(\mathbf{q}) = \frac{\partial \mathcal{U}}{\partial q_i} = - \sum_{j=1}^n m_{l_j} g_0^T \mathbf{J}_{p_i}^{(l_j)}(\mathbf{q}) + m_{m_j} g_0^T \mathbf{J}_{p_i}^{(m_j)}(\mathbf{q})$$

$$\mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathcal{F}$$

\Downarrow

$$\mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{F}_{viscous} \dot{\mathbf{q}} + \mathbf{F}_{static} \text{sgn}(\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} - \mathbf{J}^T(\mathbf{q}) \mathbf{h}$$

5. Trajectories

5.1 PTP

$$\text{minimize} \int_0^{t_f} \tau^2(t) dt \quad \text{subject to} \quad \int_0^{t_f} \omega(t) dt = \mathbf{q}_f - \mathbf{q}_i \quad (\tau = I\dot{\omega})$$

$$\begin{cases} \mathbf{q}(t) &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\ \dot{\mathbf{q}}(t) &= 3a_3 t^2 + 2a_2 t + a_1 \\ \ddot{\mathbf{q}}(t) &= 6a_3 t + 2a_2 \end{cases}$$

$$\begin{cases} \mathbf{q}(t_i) = a_3 t_i^3 + a_2 t_i^2 + a_1 t_i + a_0 \\ \mathbf{q}(t_f) = a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{\mathbf{q}}(t_i) = 3a_3 t_i^2 + 2a_2 t_i + a_1 \\ \dot{\mathbf{q}}(t_f) = 3a_3 t_f^2 + 2a_2 t_f + a_1 \\ \mathbf{q}(t_i) = \mathbf{q}_i \\ \mathbf{q}(t_f) = \mathbf{q}_f \\ \dot{\mathbf{q}}(t_f) = \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}(t_f) = \ddot{\mathbf{q}}_f \end{cases} \xrightarrow{t_f=0} \begin{cases} \mathbf{q}_i = a_0 \\ \mathbf{q}_f = a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{\mathbf{q}}_i = a_1 \\ \dot{\mathbf{q}}_f = 3a_3 t_f^2 + 2a_2 t_f + a_1 \end{cases}$$

5.2 2-1-2

$$\left[\dot{\mathbf{q}}_c = \frac{\mathbf{q}_m - \mathbf{q}_c}{t_m - t_c} = \frac{\text{rise}}{\text{run}} \quad \ddot{\mathbf{q}} t_c = \dot{\mathbf{q}} = \frac{\mathbf{q}_m - \mathbf{q}_c}{t_m - t_c} \quad \ddot{\mathbf{q}}_c t_c^2 + \ddot{\mathbf{q}}_c t_f t_c + \mathbf{q}_f - \mathbf{q}_i = 0 \right]$$

$$t_c = \frac{t_f}{2} - \frac{1}{2} \sqrt{\frac{t_f^2 \ddot{\mathbf{q}} - 4(\mathbf{q}_f - \mathbf{q}_i)}{\ddot{\mathbf{q}}_c}} \quad \text{sgn}(\ddot{\mathbf{q}}) = \text{sgn}(\mathbf{q}_f - \mathbf{q}_i) \quad |\ddot{\mathbf{q}}| \geq \frac{4|\mathbf{q}_f - \mathbf{q}_i|}{t_f^2}$$

$$\mathbf{q}(t) = \begin{cases} \mathbf{q}_i + \frac{1}{2} \ddot{\mathbf{q}}_c t^2 & 0 \leq t \leq t_c \\ \mathbf{q}_i + \ddot{\mathbf{q}}_c t_c (t - t_c/2) & t_c < t \leq t_f - t_c \\ \mathbf{q}_f - \frac{1}{2} \ddot{\mathbf{q}}_c (t_f - t)^2 & t_f - t_c < t \leq t_f \end{cases}$$

Assegnazione di $\dot{\mathbf{q}}_c$ invece di $\ddot{\mathbf{q}}_c$

$$\frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} < |\dot{\mathbf{q}}_c| \leq 2 \frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} \quad t_c = \frac{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_c t_f}{\dot{\mathbf{q}}_c} \quad \ddot{\mathbf{q}} = \frac{\dot{\mathbf{q}}_c^2}{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_c t_f}$$

5.3 Operational space

$$x_{traj} = \begin{bmatrix} p(t) \\ \phi(t) \end{bmatrix} \quad \dot{p} = \dot{s} \frac{dp}{ds} = \dot{s} t \quad ; \quad t = \frac{dp}{ds} \quad \mathbf{n} = \frac{\frac{d^2 \mathbf{p}}{ds^2}}{\|\frac{d^2 \mathbf{p}}{ds^2}\|} \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

5.3.1 Segment

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \quad t = \frac{dp}{ds} = \frac{(p_f - p_i)}{\|p_f - p_i\|} \quad \frac{d^2 p}{ds^2} = 0$$

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \quad \dot{p} = \frac{\dot{s}(p_f - p_i)}{\|p_f - p_i\|} = \dot{s} t \quad \ddot{p} = \frac{\ddot{s}(p_f - p_i)}{\|p_f - p_i\|} = \ddot{s} t$$

5.3.2 Circonference

$$p'(s) = [\rho \cos(\frac{s}{\rho}) \quad \rho \sin(\frac{s}{\rho}) \quad 0] \implies p(s) = c + {}^O R_O p'(s)$$

$$\frac{dp}{ds} = R \begin{bmatrix} -\sin(s/\rho) & \cos(s/\rho) & 0 \end{bmatrix} \quad \frac{d^2 p}{ds^2} = R \begin{bmatrix} -\cos(s/\rho)/\rho & -\sin(s/\rho)/\rho & 0 \end{bmatrix}$$

$$p(s) = c + R \begin{bmatrix} \rho \cos(s/\rho) \\ \rho \sin(s/\rho) \\ 0 \end{bmatrix} \quad \dot{p} = R \begin{bmatrix} -\dot{s} \sin(s/\rho) \\ \dot{s} \cos(s/\rho) \\ 0 \end{bmatrix} \quad \ddot{p} = R \begin{bmatrix} -\ddot{s} \rho^{-1} \cos(s/\rho) - \dot{s} \sin(s/\rho) \\ -\ddot{s} \rho^{-1} \sin(s/\rho) + \dot{s} \cos(s/\rho) \\ 0 \end{bmatrix}$$

5.3.3 Attitude trajectory

$$\phi(s) = \phi_i + \frac{s(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \quad \dot{\phi} = \frac{\dot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \quad \ddot{\phi} = \frac{\ddot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|}$$

6. Control

6.1 Actuator model

$$\text{mech: } K_r^{-1} \tau = K_t \dot{i}_a \quad \text{electr: } v_a = R_a i_a + K_v \dot{q}_m, \quad v_a = G_v V_c \quad ; \quad [K_r q = q_m]$$

6.1.1 Velocity generator:

$$\omega_m = \frac{G_v}{k_v} v'_c \quad F = F_v K_r K_t R_a^{-1} K_v K_r \quad u = K_r K_t R_a^{-1} G_v v_c$$

$$u = \tau + F \dot{q} \implies \tau = K_r K_t R_a^{-1} (G_v v_c - K_v K_r \dot{q}) \quad v_c \approx G_v^{-1} K_v K_r \dot{q}$$

6.1.2 Torque generator:

$$c_m \approx \frac{k_t}{k_i} (v'_c - \frac{k_v}{G_v} \omega_m)$$

6.2 Decentralized joint control

$$\tau = K_r \tau_m \quad q = K_r^{-1} q_m \quad B(q) = \bar{B} + \Delta B(q)$$

$$K_r^{-1} \bar{B} K_r^{-1} \ddot{q}_m + \underbrace{K_r^{-1} \Delta B(q) K_r^{-1} \ddot{q}_m + K_r^{-1} C(q, \dot{q}) K_r^{-1} \dot{q}_m + K_r^{-1} g(q)}_{\hat{d}} + \underbrace{K_r^{-1} F_v K_r^{-1} \dot{q}_m}_{\hat{F}_m} = \tau_m$$

$$\text{t.f. motor: } M(s) = \frac{k_m}{s(1 + T_m s)} \quad k_m = \frac{1}{k_v}, \quad T_m = \frac{R_a I}{k_t k_v}$$

$$\text{PI control: } C(s) = K_c \frac{1 + s T_c}{s} \quad [K_c \equiv K_p, T_c \equiv T_p \parallel K_c \equiv K_v, \dots]$$

6.2.1 Position feedback

$$\text{forward path: } G(s) = \frac{k_m K_p (1 + s T_p)}{s^2 (1 + s T_m)} \implies \begin{cases} \times & T_p < T_m \\ \checkmark & T_p > T_m \\ \checkmark \nless & T_p \gg T_m \end{cases}$$

6.3 Centralized joint control

$$\text{current controlled} \implies i_a = G_i v_c \implies u = K_r K_t G_i v_c = \tau$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \implies \dot{x} = \begin{bmatrix} B^{-1}(q) [u - C(q, \dot{q}) \dot{q} - F \dot{q} - g(q)] \\ x_{eq} \iff \dot{x} = 0 \iff \begin{cases} \dot{q} = 0 \\ \ddot{u} = g(\ddot{q}) \end{cases} \end{bmatrix}$$

6.3.1 PD control with gravity compensation

$$V(\dot{q}, e) = \frac{1}{2} \dot{q}^T B(q) \dot{q} + \frac{1}{2} e^T K_P e > 0 \quad \forall \dot{q}, e \neq 0 \quad e \triangleq q_d - q$$

$$\dot{V} = \dot{q}^T B(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{B}(q) \dot{q} - \dot{q}^T K_P e \quad u = g(q) + K_P e - K_d \dot{q}$$

6.3.2 Inverse dynamics (feedback linearization)

$$B(q) \ddot{q} + n(q, \dot{q}) = \tau = u \quad \xrightarrow{F.L.} \quad u \triangleq B(q) y + n(q, \dot{q}) \implies \ddot{q} = y$$

$$\text{PD control: } y \triangleq -K_P q - K_D \dot{q} + r \quad , \quad r \triangleq \ddot{q}_d + K_P q_d + K_D \dot{q}_d$$

$$\left[\begin{array}{c} \ddot{q} = y \\ \ddot{q} + K_P q + K_D \dot{q} = r \end{array} \implies \ddot{e} + K_D \dot{e} + K_P e = 0 \right]$$

$$\implies \mathbf{y} = \mathbf{K}_p (\mathbf{q}_d - \mathbf{q}) + \mathbf{K}_d (\dot{\mathbf{q}}_d - \dot{\mathbf{q}}) + \ddot{\mathbf{q}}_d$$

$$\left[\begin{array}{c} K_P = \text{diag}\{\omega_{n1}^2, \dots, \omega_{nn}^2\} \quad K_D = \text{diag}\{2\zeta\omega_{n1}, \dots, 2\zeta\omega_{nn}\} \end{array} \right]$$

6.4 Operational space control

6.4.1 PD with gravity compensation

$$e \triangleq x_d - x \quad u = g(q) + J_A^T(q) K_P e - J_A^T(q) K_P K_D J_A(q) \dot{q}$$

$$V(\dot{q}, e) = \frac{1}{2} \dot{q}^T B(q) \dot{q} + \frac{1}{2} e^T K_P e > 0 \quad \forall \dot{q}, e \neq 0$$

$$\text{at equilibrium} \quad \dot{x} = (\dot{q}, \ddot{q}) = 0 \implies -J_A^T(q) K_P e = 0$$

6.4.2 Inverse dynamics (feedback linearization)

$$\dot{x} = J_A(q) \dot{q} \implies \ddot{x} = \dot{J}_A(q) \dot{q} + \ddot{J}_A(q, \dot{q}) \dot{q}$$

$$u \triangleq B(q) y + n(q, \dot{q}) \implies \ddot{q} = y \implies y \triangleq J_A^{-1}(q) (\ddot{x}_d + K_D \dot{e} + K_P e - \dot{J}_A(q, \dot{q}) \dot{q})$$

7. Control of the interaction

$$B(q) \ddot{q} + n(q, \dot{q}) = u - \underbrace{J^T(q) h}_{\text{interaction}} \implies \text{PD with gravity comp.} \quad J_A^T(q) K_P e = J^T(q) h$$

$$h_A = T_A^T(x) K T_A(x) dx = K_A(x) (x - x_e) \implies e = K_P^{-1} K_A(x) (x - x_e)$$

$$x_\infty = (I - K_P^{-1} K_A(x))^{-1} (x_d + K_P^{-1} K_A(x) x_e) \quad h_{A\infty} = (I + K_A(x) K_P^{-1})^{-1} K_A(x) (x_d - x_e)$$

8. Mobile Robotics

$$\text{holonomic constr. (integrable)} \iff h_i(\mathbf{q}) = 0, \quad i = 1 \dots k < n \implies \text{kin. constr. : } \frac{dh_i(\mathbf{q})}{dt} = \frac{dh_i(\mathbf{q})}{d\mathbf{q}} \dot{\mathbf{q}} = 0$$

$$\text{kin. constr. } a_i(\mathbf{q}, \dot{\mathbf{q}}) = 0 \implies \text{pfaffian } \mathbf{a}_i^T(\mathbf{q}) \dot{\mathbf{q}} = 0 \leftrightarrow \mathbf{A}^T(\mathbf{q}) \dot{\mathbf{q}} = 0$$

$$\dot{\mathbf{q}} \in \mathcal{N}(\mathbf{A}^T(\mathbf{q})) \implies \langle \mathbf{g}_1(\mathbf{q}), \dots, \mathbf{g}_n - k(\mathbf{q}) \rangle \text{ base of } \mathcal{N} \implies \dot{\mathbf{q}} = \mathbf{G}(\mathbf{q}) \mathbf{u}$$

8.1 Unicycle

$$\text{pure rolling c.: } \frac{dy}{dx} = \tan \theta \implies \mathbf{q} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \implies \begin{bmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{bmatrix}^T \dot{\mathbf{q}} = 0 \quad , \quad \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix}$$

8.2 Differential drive

$$v = \frac{r(\omega_R + \omega_L)}{2} \quad \omega = \frac{r(\omega_R - \omega_L)}{d}$$

8.3 Bike

$$\text{pure rolling (front w., back w.): } \dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0 \quad , \quad \dot{x} \sin \theta - \dot{y} \cos \theta = 0$$

$$x_f = x + L \cos \theta, \quad y_f = y + L \sin \theta \xrightarrow{1^{st} c.} \dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - L \dot{\theta} \cos \phi = 0$$

$$\mathbf{A}^T(\mathbf{q}) = \begin{bmatrix} \sin \theta & -\cos \theta & 0 & 0 \\ \sin(\theta + \phi) & -\cos(\theta + \phi) & -L \cos \phi & 0 \end{bmatrix} \quad \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \cos \theta \cos \phi & 0 \\ \sin \theta \cos \phi & 0 \\ \sin \phi / L & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \mathbf{G}_1(\mathbf{q}) u_1 + \mathbf{G}_2(\mathbf{q}) u_2 \quad (u_2 \equiv \omega) \quad \text{if front drive: } u_1 = v, \quad \text{if back d.: } u_1 = \frac{v}{\cos \phi}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$