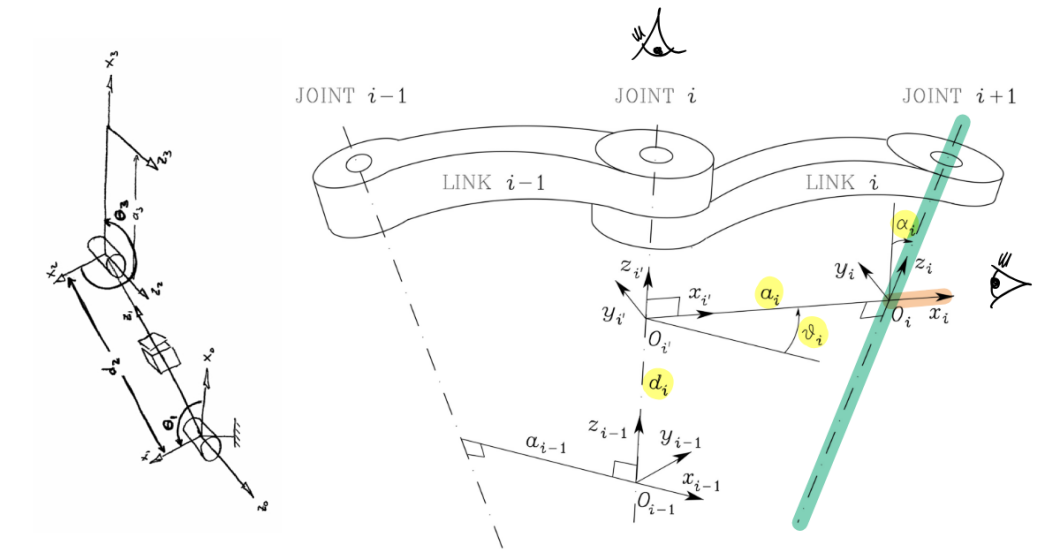


1. Denavit-Hartenberg



Il sistema di riferimento  $\mathcal{R}_i$  solidale con  $LINK_i$  viene definito secondo le seguenti regole:

- ① **Asse  $z_i$  e origine  $O_i$**
- L’asse  $z_i$**  è posto lungo l’asse di movimento di  $g_{i+1}$  (asse di rotazione o di traslazione a seconda del tipo di giunto)
  - L’origine  $O_i$**  è posta all’intersezione di  $z_i$  con la normale comune (*common normal*) fra gli assi  $z_{i-1}$  e  $z_i$ . La normale comune è quella retta perpendicolare ad entrambi gli assi (nota: entrambi angoli retti nella figura)
- Casi particolari:**
    - $\mathcal{R}_0$ :** origine  $O_0$  e  $x_0$  possono essere fissati a piacimento (solo  $z_0$  univocamente definito).
    - $\mathcal{R}_n$ :**  $\nexists g_{n+1} \implies z_n, O_n$  non univocamente definiti. Per consuetudine: origine nel centro della pinza e  $z_n$  coincidente a a  $z_{n-1}$  (visto che tipicamente l’ultimo giunto è rotoidale).
- ② **Asse  $x_i$  e  $y_i$**
- L’asse  $x_i$**  è fissato lungo la normale comune fra gli assi  $z_{i-1}$  e  $z_i$ 
    - Se  $z_{i-1}$  e  $z_i$  si intersecano  $\implies$  direzione di  $x_i$  ( $\perp z_i$ ) è arbitraria
    - se  $z_{i-1}$  e  $z_i$  sono paralleli  $\implies$  origine arbitraria,  $x_i$  nel piano normale a  $z_{i-1}$  e  $z_i$  con direzione e verso arbitrari.
  - L’asse  $y_i$**  completa la terna destrorsa ( $j = k \times i$ )

③ Sistema di riferimento intermedio

$z_{i'}$  diretto lungo  $z_{i-1}$  |  $O_{i'}$  posta all’intersezione di  $z_{i-1}$  con la normale comune fra  $z_{i-1}$  e  $z_i$  |  $x_{i'}$  diretto lungo la normale comune fra  $z_{i-1}$  e  $z_i$  (come  $x_i$ )

- $d_i \rightarrow$  **link offset**: coordinata di  $O_{i'}$  lungo  $z_{i-1}$
- $\theta_i \rightarrow$  **joint angle**: angolo di rotazione da  $x_{i-1}$  a  $x_i$  attorno all’asse  $z_{i'}$  (positivo quando la rotazione è anti-oraria)
- $a_i \rightarrow$  **link length**: distanza (con segno) fra  $O_i$  e  $O_{i'}$
- $\alpha_i \rightarrow$  **link twist**: angolo di rotazione da  $z_{i-1}$  a  $z_i$  attorno all’asse  $x_i$  (positivo quando la rotazione è anti-oraria)

$${}^{i-1}\mathbf{T}_i(q_i) = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Trigonometric inequalities:**  
 $c_{12} + s_{12} = c_{1-2} \qquad c_{12} - s_{12} = c_{1+2} \qquad s_1 c_2 - c_1 s_2 = s_{1-2} \qquad s_1 c_2 + c_1 s_2 = s_{1+2}$

Tips:

$a_i \rightarrow \left\{ \begin{array}{l} z_{i-1} \xleftrightarrow{\text{dist.}} z_i \quad \text{along } x_i \\ \alpha_i \rightarrow \left\{ \begin{array}{l} z_{i-1} \xrightarrow{\quad} z_i \quad \text{around } x_i \\ d_i \rightarrow \left\{ \begin{array}{l} x_{i-1} \xleftrightarrow{\text{dist.}} x_i \quad \text{along } z_{i-1} \\ \theta_i \rightarrow \left\{ \begin{array}{l} x_{i-1} \xrightarrow{\quad} x_i \quad \text{around } z_{i-1} \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right.$

**$d$  e  $a$  sono con segno!**  
Relativ. ad "along":  
  
> 0 se concorde  
< 0 se discorde

2. Differential Kinematics

2.1 Geometric Jacobian

i-th column of  $\mathbf{J}$ :

$$\begin{bmatrix} \mathbf{J}_{p,i} \\ \mathbf{J}_{o,i} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix} & \text{for a *prismatic* joint} \\ \begin{bmatrix} \mathbf{z}_{i-1} \times (\mathbf{p} - \mathbf{p}_{i-1}) \\ \mathbf{z}_{i-1} \end{bmatrix} & \text{for a *revolute* joint} \end{cases}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

E.g. planar RRR  
$$\mathbf{J}(q) = \begin{bmatrix} z_0 \times (p - p_0) & z_1 \times (p - p_1) & z_2 \times (p - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix} \qquad \text{here } c_{12} = c(\theta_1 + \theta_2)$$

$${}^0T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}$$

$${}^0T_2 = {}^0T_1 {}^1T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{p} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix}$$

2.2 Analytical Jacobian

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\boldsymbol{\phi}} \end{bmatrix} = \frac{d\mathbf{x}}{dt} = \underbrace{\frac{\partial \mathbf{x}}{\partial \mathbf{q}}}_{\mathbf{J}_A(\mathbf{q})} \underbrace{\frac{d\mathbf{q}}{dt}}_{\dot{\mathbf{q}}} = \mathbf{J}_A(\mathbf{q}) \dot{\mathbf{q}} \qquad \mathbf{J}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\boldsymbol{\phi}) \end{bmatrix} \mathbf{J}_A(\mathbf{q})$$

2.3 Inverse differential kinematics

$$\begin{cases} \text{minimize} & g(\dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{W} \dot{\mathbf{q}} \\ \text{subject to} & \mathbf{v} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \end{cases} \quad \mathbf{W} \equiv \mathbf{I} \quad \dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v} \quad \mathbf{J}^\dagger \triangleq \mathbf{J}^T (\mathbf{J} \mathbf{J}^T)^{-1}$$

$$\begin{cases} \text{minimize} & g'(\dot{\mathbf{q}}) = \frac{1}{2} (\dot{\mathbf{q}} - \dot{\mathbf{q}}_d)^T (\dot{\mathbf{q}} - \dot{\mathbf{q}}_d) \\ \text{subject to} & \mathbf{v} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \end{cases} \quad \iff \quad \dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v} + \underbrace{(\mathbf{I} - \mathbf{J}^\dagger \mathbf{J})}_{\text{👉 in } \mathcal{N}} \dot{\mathbf{q}}_d$$

**Damped least-square:**  
 $\mathbf{J} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^T, \Sigma_{ii} = \sqrt{eig(\mathbf{J} \mathbf{J}^T)_i} \implies \mathbf{J}^\dagger = \mathbf{V} \boldsymbol{\Sigma}^\dagger \mathbf{U}^T \quad \sigma_i \rightarrow 1 / ( \sigma_i + k^2 ) \quad \mathbf{J}^* = \mathbf{J}^T (\mathbf{J} \mathbf{J}^T + k^2 \mathbf{I})^{-1}$

Secondary objectives:

$$\dot{\mathbf{H}} = \frac{d\mathbf{H}}{dt} = \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad \text{with} \quad \dot{\mathbf{q}}_d = -K \left( \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \right)^T, \quad K > 0$$
$$\dot{\mathbf{H}} = \underbrace{\frac{\partial \mathbf{H}}{\partial \mathbf{q}} \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v}}_{\text{non si sa}} + \underbrace{-K \frac{\partial \mathbf{H}}{\partial \mathbf{q}} (\mathbf{I} - \mathbf{J}^\dagger \mathbf{J}) \left( \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \right)^T}_{< 0}$$

- Max dist. obstacles:**  $H = \min_{p,o} \|p(q) - o\| \rightsquigarrow H \uparrow$
- Max dist. joint limit:**  $H(q) = -\frac{1}{2n} \sum_{i=1}^n \left( \frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2 \rightsquigarrow H \downarrow$
- Max dist. from singularities:**  $H(q) = \sqrt{\det(\mathbf{J}(q) \mathbf{J}^T(q))} \rightsquigarrow H \uparrow$

3. Statics

$$\boldsymbol{\tau}^T \delta \mathbf{q} = \mathbf{F}^T \delta \mathbf{p} \implies \text{🔗} \quad \boldsymbol{\tau} = -\mathbf{J}^T(\mathbf{q}) \mathbf{F}$$

$$\mathcal{N}(\mathbf{J}) \equiv \mathcal{R}^\perp(\mathbf{J}^T) \qquad \mathcal{R}(\mathbf{J}) \equiv \mathcal{N}^\perp(\mathbf{J}^T)$$

**Ellipsoids:**

$$\|\dot{\mathbf{q}}\|^2 = 1 \iff \dot{\mathbf{q}}^T \dot{\mathbf{q}} = 1 \xrightarrow{\dot{\mathbf{q}} = \mathbf{J}^\dagger \mathbf{v}} \mathbf{v}^T (\mathbf{J} \mathbf{J}^T)^{-1} \mathbf{v} = 1 \implies E_v = \{ \mathbf{v} : \mathbf{v}^T (\mathbf{J} \mathbf{J}^T)^{-1} \mathbf{v} = 1 \}$$

$$\|\dot{\boldsymbol{\tau}}\|^2 = 1 \iff \boldsymbol{\tau}^T \boldsymbol{\tau} = 1 \xrightarrow{\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}} E_F = \{ \mathbf{F} : \mathbf{F}^T (\mathbf{J} \mathbf{J}^T) \mathbf{F} = 1 \}$$

**Manipulability measure:**  
 $w(\mathbf{q}) = \sqrt{\det(\mathbf{J} \mathbf{J}^T)} = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\det(\mathbf{J})|$

4. Dynamics

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \quad (= \mathcal{K} - \mathcal{P})$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} = \mathcal{F}_i \qquad i = 1, \dots, n$$

**Kinetic:**

$$\mathcal{T} = \sum_{i=1}^n \mathcal{T}_i + \mathcal{T}_{m_i} \implies \mathcal{T} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_{ij}(\mathbf{q}) \dot{\mathbf{q}}_i \dot{\mathbf{q}}_j = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{B}(\mathbf{q}) \dot{\mathbf{q}}$$
$$\begin{cases} \mathcal{T}_i = \frac{1}{2} m_{l_i} \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i + \frac{1}{2} \boldsymbol{\omega}_i^T {}^b \mathbf{R}_i {}^i \mathbf{I}_{l_i} ({}^b \mathbf{R}_i)^T \boldsymbol{\omega}_i \\ \mathcal{T}_{m_i} = \frac{1}{2} m_{m_i} \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i + \frac{1}{2} \boldsymbol{\omega}_i^T {}^b \mathbf{R}_i {}^i \mathbf{I}_{m_i} ({}^b \mathbf{R}_i)^T \boldsymbol{\omega}_i \end{cases}$$

$$\begin{cases} \mathcal{T}_{l_i} = \frac{1}{2} m_{l_i} \left( \dot{\mathbf{q}}^T \mathbf{J}_p^{(l_i)T} \right) \left( \mathbf{J}_p^{(l_i)} \dot{\mathbf{q}} \right) + \frac{1}{2} \left( \dot{\mathbf{q}}^T \mathbf{J}_o^{(l_i)T} \right) ({}^b \mathbf{R}_i {}^i \mathbf{I}_{l_i} ({}^b \mathbf{R}_i)^T) \left( \mathbf{J}_o^{(l_i)} \dot{\mathbf{q}} \right) \\ \mathcal{T}_{m_i} = \frac{1}{2} m_{m_i} \left( \dot{\mathbf{q}}^T \mathbf{J}_p^{(m_i)T} \right) \left( \mathbf{J}_p^{(m_i)} \dot{\mathbf{q}} \right) + \frac{1}{2} \left( \dot{\mathbf{q}}^T \mathbf{J}_o^{(m_i)T} \right) ({}^b \mathbf{R}_{m_i} {}^{m_i} \mathbf{I}_{m_i} ({}^b \mathbf{R}_{m_i})^T) \left( \mathbf{J}_o^{(m_i)} \dot{\mathbf{q}} \right) \end{cases}$$

**Potential:**

$$\mathcal{U} = \sum_{i=1}^n \mathcal{U}_{l_i} + \mathcal{U}_{m_i} \implies \mathcal{U} = - \sum_{i=1}^n m_{l_i} \mathbf{g}_0^T \mathbf{p}_{l_i} + m_{m_i} \mathbf{g}_0^T \mathbf{p}_{m_i}$$

**Dynamic equations:**

$$\sum_{j=1}^n \mathbf{B}_{ij}(\mathbf{q}) \ddot{\mathbf{q}}_j + \sum_{j=1}^n \sum_{k=1}^n h_{ijk}(\mathbf{q}) \dot{\mathbf{q}}_k \dot{\mathbf{q}}_j + g_i(\mathbf{q}) = \mathcal{F}_i \qquad h_{ijk} = \frac{\partial B_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial B_{jk}}{\partial q_i}$$
$$g_i(\mathbf{q}) = \frac{\partial \mathcal{U}}{\partial q_i} = - \sum_{j=1}^n m_{l_j} \mathbf{g}_0^T \mathbf{J}_{p_i}^{(l_j)}(\mathbf{q}) + m_{m_j} \mathbf{g}_0^T \mathbf{J}_{p_i}^{(m_j)}(\mathbf{q})$$

$$\mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathcal{F}$$
$$\Downarrow$$
$$\mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{F}_{viscous} \dot{\mathbf{q}} + \mathbf{F}_{static} \text{sgn}(\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} - \mathbf{J}^T(\mathbf{q}) \mathbf{h}$$

## 5. Trajectories

### 5.1 PTP

minimize

$\int_0^{t_f} \boldsymbol{\tau}^2(t)dt$

subject to

$\int_0^{t_f} \boldsymbol{\omega}(t)dt = \mathbf{q}_f - \mathbf{q}_i \qquad (\tau = I\dot{\omega})$

$$\begin{cases} \mathbf{q}(t) &= a_3t^3 + a_2t^2 + a_1t + a_0 \\ \dot{\mathbf{q}}(t) &= 3a_3t^2 + 2a_2t + a_1 \\ \ddot{\mathbf{q}}(t) &= 6a_3t + 2a_2 \end{cases}$$

$$\begin{cases} \mathbf{q}(t_i) = a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\ \mathbf{q}(t_f) = a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{\mathbf{q}}(t_i) = 3a_3t_i^2 + 2a_2t_i + a_1 \\ \dot{\mathbf{q}}(t_i) = 3a_3t_f^2 + 2a_2t_f + a_1 \\ \mathbf{q}(t_i) = \mathbf{q}_i \\ \mathbf{q}(t_f) = \mathbf{q}_f \\ \dot{\mathbf{q}}(t_f) = \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}(t_f) = \ddot{\mathbf{q}}_f \end{cases} \xRightarrow{t_f=0} \begin{cases} \mathbf{q}_i = a_0 \\ \mathbf{q}_f = a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{\mathbf{q}}_i = a_1 \\ \dot{\mathbf{q}}_f = 3a_3t_f^2 + 2a_2t_f + a_1 \end{cases}$$

### 5.2 2-1-2

$$\left[\dot{\mathbf{q}}_c = \frac{\mathbf{q}_m - \mathbf{q}_c}{t_m - t_c} = \frac{\text{rise}}{\text{run}} \qquad \ddot{\mathbf{q}}_ct_c = \dot{\mathbf{q}}_c = \frac{\mathbf{q}_m - \mathbf{q}_c}{t_m - t_c} \qquad \ddot{\mathbf{q}}_ct_c^2 - \ddot{\mathbf{q}}_ct_ft_c + \mathbf{q}_f - \mathbf{q}_i = 0\right]$$

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{t_f^2\ddot{\mathbf{q}}_c - 4(\mathbf{q}_f - \mathbf{q}_i)}{\ddot{\mathbf{q}}_c}} \qquad \text{sgn}(\ddot{\mathbf{q}}_c) = \text{sgn}(\mathbf{q}_f - \mathbf{q}_i) \qquad |\ddot{\mathbf{q}}_c| \geq \frac{4|\mathbf{q}_f - \mathbf{q}_i|}{t_f^2}$$

$$t_m = \frac{t_f}{2} \text{ , } \mathbf{q}_m = \frac{\mathbf{q}_f + \mathbf{q}_i}{2} \qquad \mathbf{q}(t) = \begin{cases} \mathbf{q}_i + \frac{1}{2}\ddot{\mathbf{q}}_ct^2 & 0 \leq t \leq t_c \\ \mathbf{q}_i + \ddot{\mathbf{q}}_ct_c(t - t_c/2) & t_c < t \leq t_f - t_c \\ \mathbf{q}_f - \frac{1}{2}\ddot{\mathbf{q}}_c(t_f - t)^2 & t_f - t_c < t \leq t_f \end{cases}$$

Assegnazione di  $\dot{\mathbf{q}}_c$  invece di  $\ddot{\mathbf{q}}_c$

$$\frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} < |\dot{\mathbf{q}}_c| \leq 2\frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} \qquad t_c = \frac{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_ct_f}{\dot{\mathbf{q}}_c} \qquad \ddot{\mathbf{q}}_c = \frac{\dot{\mathbf{q}}_c^2}{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_ct_f}$$

### 5.3 Operational space

$$x_{traj} = \begin{bmatrix} p(t) \\ \phi(t) \end{bmatrix} \qquad \dot{p} = \dot{s}\frac{dp}{ds} = \dot{s}t \qquad ; \qquad \mathbf{t} = \frac{d\mathbf{p}}{ds} \qquad \mathbf{n} = \frac{\frac{d^2\mathbf{p}}{ds^2}}{\|\frac{d^2\mathbf{p}}{ds^2}\|} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

#### 5.3.1 Segment

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \qquad t = \frac{dp}{ds} = \frac{(p_f - p_i)}{\|p_f - p_i\|} \qquad \frac{d^2p}{ds^2} = 0$$

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \qquad \dot{p} = \frac{\dot{s}(p_f - p_i)}{\|p_f - p_i\|} = \dot{s}t \qquad \ddot{p} = \frac{\ddot{s}(p_f - p_i)}{\|p_f - p_i\|} = \ddot{s}t$$

#### 5.3.2 Circonference

$$p'(s) = \begin{bmatrix} \rho \cos(\frac{s}{\rho}) & \rho \sin(\frac{s}{\rho}) & 0 \end{bmatrix} \implies p(s) = c + {}^{\mathcal{O}}R_{\mathcal{O}'}p'(s)$$

$$\frac{dp}{ds} = R \begin{bmatrix} -\sin(s/\rho) & \cos(s/\rho) & 0 \end{bmatrix} \qquad \frac{d^2p}{ds^2} = R \begin{bmatrix} -\cos(s/\rho)/\rho & -\sin(s/\rho)/\rho & 0 \end{bmatrix}$$

$$p(s) = c + R \begin{bmatrix} \rho \cos(s/\rho) \\ \rho \sin(s/\rho) \\ 0 \end{bmatrix} \qquad \dot{p} = R \begin{bmatrix} -\dot{s} \sin(s/\rho) \\ \dot{s} \sin(s/\rho) \\ 0 \end{bmatrix} \qquad \ddot{p} = R \begin{bmatrix} -\dot{s}^2\rho^{-1} \cos(s/\rho) - \ddot{s} \sin(s/\rho) \\ -\dot{s}^2\rho^{-1} \sin(s/\rho) + \ddot{s} \sin(s/\rho) \\ 0 \end{bmatrix}$$

#### 5.3.3 Attitude trajectory

$$\phi(s) = \phi_i + \frac{s(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \qquad \dot{\phi} = \frac{\dot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \qquad \ddot{\phi} = \frac{\ddot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|}$$

## 6. Control

### 6.1 Actuator model

**mech:**

$K_r^{-1}\tau = K_t i_a$

**electr:**

$v_a = R_a i_a + K_v \dot{q}_m \text{ , } v_a = G_v V_c \qquad ; \qquad [K_rq = q_m]$

#### 6.1.1 Velocity generator:

$$\omega_m = \frac{G_v}{k_v} v'_c \qquad F = F_v K_r K_t R_a^{-1} K_v K_r \qquad u = K_r K_t R_a^{-1} G_v v_c$$

$$u = \tau + F\dot{q} \implies \tau = K_r K_t R_a^{-1} (G_v v_c - K_v K_r \dot{q}) \qquad v_c \approx G_v^{-1} K_v K_r \dot{q}$$

#### 6.1.2 Torque generator:

$$c_m \approx \frac{k_t}{k_i} (v'_c - \frac{k_v}{G_v} \omega_m)$$

### 6.2 Decentralized joint control

$$\tau = K_r \tau_m \qquad q = K_r^{-1} q_m \qquad B(q) = \bar{B} + \Delta B(q)$$

$$K_r^{-1} \bar{B} K_r^{-1} \ddot{q}_m + \underbrace{K_r^{-1} \Delta B(q) K_r^{-1} \ddot{q}_m + K_r^{-1} C(q, \dot{q}) K_r^{-1} \dot{q}_m + K_r^{-1} g(q)}_d + \underbrace{K_r^{-1} F_v K_r^{-1}}_{F_m} \dot{q}_m = \tau_m$$

**t.f. motor:**

$M(s) = \frac{k_m}{s(1 + T_ms)} \qquad k_m = \frac{1}{k_v} \text{ , } T_m = \frac{R_a I}{k_t k_v}$

**PI control:**

$C(s) = K_c \frac{1 + sT_c}{s} \qquad [ \text{ } K_c \equiv K_p, T_c \equiv T_p \parallel K_c \equiv K_v, \dots ]$

#### 6.2.1 Position feedback

forward path:

$$G(s) = \frac{k_m K_p (1 + sT_p)}{s^2 (1 + sT_m)} \implies \begin{cases} \times & T_p < T_m \\ \checkmark & T_p > T_m \\ \checkmark \frown & T_p \gg T_m \end{cases}$$

### 6.3 Centralized joint control

$$\text{current controlled} \implies i_a = G_i v_c \implies u = K_r K_t G_i v_c = \tau$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \implies \dot{x} = \begin{bmatrix} \dot{q} \\ B^{-1}(q) [u - C(q, \dot{q})\dot{q} - F\dot{q} - g(q)] \end{bmatrix} \qquad x_{eq} \iff \dot{x} = 0 \iff \begin{cases} \dot{q} = 0 \\ \bar{u} = g(\bar{q}) \end{cases}$$

#### 6.3.1 PD control with gravity compensation

$$V(\dot{q}, e) = \frac{1}{2} \dot{q}^T B(q) \dot{q} + \frac{1}{2} e^T K_P e > 0 \qquad \forall \dot{q}, e \neq 0 \qquad e \triangleq q_d - q$$

$$\dot{V} = \dot{q}^T B(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{B}(q) \dot{q} - \dot{q}^T K_P e \qquad u = g(q) + K_P e - K_d \dot{q}$$

#### 6.3.2 Inverse dynamics (feedback linearization)

$$B(q) \ddot{q} + n(q, \dot{q}) = \tau = u \xRightarrow{F.L.} u \triangleq B(q) y + n(q, \dot{q}) \implies \ddot{q} = y$$

$$\text{PD control:} \qquad y \triangleq -K_P q - K_D \dot{q} + r \quad , \quad r \triangleq \ddot{q}_d + K_P q_d + K_D \dot{q}_d$$

$$\left[ \begin{array}{c} \xRightarrow{\ddot{q}=y} \ddot{q} + K_P q + K_D \dot{q} = r \implies \ddot{e} + K_D \dot{e} + K_P e = 0 \end{array} \right]$$

$$\implies \mathbf{y} = \mathbf{K}_p (\mathbf{q}_d - \mathbf{q}) + \mathbf{K}_d (\dot{\mathbf{q}}_d - \dot{\mathbf{q}}) + \ddot{\mathbf{q}}_d$$

$$\left[ \begin{array}{c} K_P = diag\{\omega_{n1}^2, \dots, \omega_{nn}^2\} \quad K_D = diag\{2\zeta\omega_{n1}, \dots, 2\zeta\omega_{nn}^2\} \end{array} \right]$$

### 6.4 Operational space control

#### 6.4.1 PD with gravity compensation

$$e \triangleq x_d - x \qquad u = g(q) + J_A^T(q) K_P e - J_A^T(q) K_P K_D J_A(q) \dot{q}$$

$$V(\dot{q}, e) = \frac{1}{2} \dot{q}^T B(q) \dot{q} + \frac{1}{2} e^T K_P e > 0 \qquad \forall \dot{q}, e \neq 0$$

$$\text{at equilibrium} \quad \dot{x} = (\dot{q}, \ddot{q}) = 0 \implies -J_A^T(q) K_P e = 0$$

#### 6.4.2 Inverse dynamics (feedback linearization)

$$\dot{x} = J_A(q) \dot{q} \implies \ddot{x} = \dot{J}_A(q) \dot{q} + \ddot{J}_A(q, \dot{q}) \dot{q}$$

$$u \triangleq B(q) y + n(q, \dot{q}) \implies \ddot{q} = y \implies y \triangleq J_A^{-1}(q) (\ddot{x}_d + K_D \dot{e} + K_P e - \dot{J}_A(q, \dot{q}) \dot{q})$$

## 7. Control of the interaction

$$B(q) \ddot{q} + n(q, \dot{q}) = u - \underbrace{J^T(q) h}_{interaction} \implies \text{PD with gravity comp.} \quad J_A^T(q) K_P e = J^T(q) h$$

$$h_A = T_A^T(x) K T_A(x) dx = K_A(x) (x - x_e) \implies e = K_P^{-1} K_A(x) (x - x_e)$$

$$x_\infty = (I - K_P^{-1} K_A(x))^{-1} (x_d + K_P^{-1} K_A(x) x_e) \quad h_{A\infty} = (I + K_A(x) K_P^{-1})^{-1} K_A(x) (x_d - x_e)$$

## 8. Mobile Robotics

holonomic constr.  
(integrable)

$$\iff h_i(\mathbf{q}) = 0, \text{ } i = 1 \dots k < n \implies \begin{array}{c} \text{kin.} \\ \text{constr.} \end{array} : \quad \frac{dh_i(\mathbf{q})}{dt} = \frac{dh_i(\mathbf{q})}{d\mathbf{q}} \dot{\mathbf{q}} = 0$$

$$\text{kin. constr.} \quad a_i(\mathbf{q}, \dot{\mathbf{q}}) = 0 \implies \text{pfaffian} \quad \mathbf{a}_i^T(\mathbf{q}) \dot{\mathbf{q}} = 0 \leftrightarrow \mathbf{A}^T(\mathbf{q}) \dot{\mathbf{q}} = 0$$

$$\dot{\mathbf{q}} \in \mathcal{N}(\mathbf{A}^T(\mathbf{q})) \implies \langle \mathbf{g}_1(\mathbf{q}), \dots, \mathbf{g}_n - k(\mathbf{q}) \rangle \text{ base of } \mathcal{N} \implies \dot{\mathbf{q}} = \mathbf{G}(\mathbf{q}) \mathbf{u}$$

### 8.1 Unicycle

$$\text{pure rolling c.:} \quad \frac{dy}{dx} = \tan \theta \implies \mathbf{q} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \implies \begin{bmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{bmatrix}^T \dot{\mathbf{q}} = 0 \quad , \quad \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix}$$

### 8.2 Differential drive

$$v = \frac{r(\omega_R + \omega_L)}{2} \qquad \omega = \frac{r(\omega_R - \omega_L)}{d}$$

### 8.3 Bike

$$\text{pure rolling (front w., back w.):} \quad \dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0 \quad , \quad \dot{x} \sin \theta - \dot{y} \cos \theta = 0$$

$$x_f = x + L \cos \theta \text{ , } y_f = y + L \sin \theta \xRightarrow{1^{st}c.} \dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - L \dot{\theta} \cos \phi = 0$$

$$\mathbf{A}^T(\mathbf{q}) = \begin{bmatrix} \sin \theta & -\cos \theta & 0 & 0 \\ \sin(\theta + \phi) & -\cos(\theta + \phi) & -L \cos \phi & 0 \end{bmatrix} \qquad \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \cos \theta \cos \phi & 0 \\ \sin \theta \cos \phi & 0 \\ \sin \phi / L & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \mathbf{G}_1(\mathbf{q}) u_1 + \mathbf{G}_2(\mathbf{q}) u_2 \quad (u_2 \equiv \omega) \qquad \text{if front drive: } u_1 = v \text{ , if back d.: } u_1 = \frac{v}{\cos \phi}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \qquad R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$