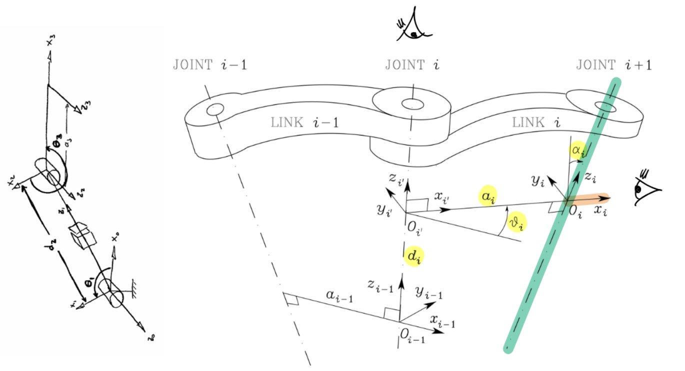
# 1. Denavit-Hartenberg




Il sistema di riferimento $\mathcal{R}_i$ solidale con $LINK_i$ viene definito secondo le seguenti regole:

1 **Asse** $z_i$ **e origine** $O_i$

- **L'asse $z_i$** è posto lungo l'asse di movimento di $g_{i+1}$ (asse di rotazione o di traslazione a seconda del tipo di giunto)
- **L'origine $O_i$** è posta all'intersezione di $z_i$ con la normale comune (*common normal*) fra gli assi $z_{i-1}$ e $z_i$. La normale comune è quella retta perpendicolare ad entrambi gli assi (nota: entrambi angoli retti nella figura)
- **Casi particolari**:
  - $\mathcal{R}_0$: origine $O_0$ e $x_0$ possono essere fissati a piacimento (solo $z_0$ univocamente definito).
  - $\mathcal{R}_n$: $\nexists\ g_{n+1} \implies z_n,\ O_n$ non univocamente definiti. Per consuetudine: origine nel centro della pinza e $z_n$ coincidente a a $z_{n-1}$ (visto che tipicamente l'ultimo giunto è rotoidale).

2 **Asse** $x_i$ **e** $y_i$

- **L'asse $x_i$** è fissato lungo la normale comune fra gli assi $z_{i-1}$ e $z_i$
  - Se $z_{i-1}$ e $z_i$ si intersecano $\implies$ direzione di $x_i$ ($\perp z_i$) è arbitraria
  - se $z_{i-1}$ e $z_i$ sono paralleli $\implies$ origine arbitraria, $x_i$ nel piano normale a $z_{i-1}$ e $z_i$ con direzione e verso arbitrari.
- **L'asse $y_i$** completa la terna destrorsa ($j = k \times i$)

3 **Sistema di riferimento intermedio**

$z_{i'}$ diretto lungo $z_{i-1}$ | $O_{i'}$ posta all'intersezione di $z_{i-1}$ con la normale comune fra $z_{i-1}$ e $z_i$ | $x_{i'}$ diretto lungo la normale comune fra $z_{i-1}$ e $z_i$ (come $x_i$)

- $\boldsymbol{d_i}$ → **link offset**: coordinata di $O_{i'}$ lungo $z_{i-1}$
- $\boldsymbol{\theta_i}$ → **joint angle**: angolo di rotazione da $x_{i-1}$ a $x_i$ attorno all'asse $z_{i'}$ (positivo quando la rotazione è anti-oraria)
- $\boldsymbol{a_i}$ → **link length**: distanza (con segno) fra $O_i$ e $O_{i'}$
- $\boldsymbol{\alpha_i}$ → **link twist**: angolo di rotazione da $z_{i-1}$ a $z_i$ attorno all'asse $x_i$ (positivo quando la rotazione è anti-oraria)

$$
{}^{i-1}\mathbf{T}_i(q_i) = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

**Trigonometric inequalities**:

$$
c_{12} + s_{12} = c_{1-2} \qquad c_{12} - s_{12} = c_{1+2} \qquad s_1 c_2 - c_1 s_2 = s_{1-2} \qquad s_1 c_2 + c_1 s_2 = s_{1+2}
$$

**Tips**:

$$
\begin{aligned}
a &\rightarrow \left\{ z_{i-1} \overset{\text{dist.}}{\longleftrightarrow} z_i \quad \text{along } x_i \right. \\
\alpha_i &\rightarrow \left\{ z_{i-1} \overset{\angle}{\longrightarrow} z_i \quad \text{around } x_i \right. \\
d &\rightarrow \left\{ x_{i-1} \overset{\text{dist.}}{\longleftrightarrow} x_i \quad \text{along } z_{i-1} \right. \\
\theta &\rightarrow \left\{ x_{i-1} \overset{\angle}{\longrightarrow} x_i \quad \text{around } z_{i-1} \right.
\end{aligned}
$$

# 2. Differential Kinematics

## 2.1 Geometric Jacobian

$$
\text{i-th column of } \boldsymbol{J}: \qquad \begin{bmatrix} \boldsymbol{J}_{p,i} \\ \boldsymbol{J}_{o,i} \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix} & \textit{for a } \textbf{\textit{prismatic}} \textit{ joint} \\[2ex] \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix} & \textit{for a } \textbf{\textit{revolute}} \textit{ joint} \end{cases}
$$

$$
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}
$$

*E.g. planar RRR*

$$
J(q) = \begin{bmatrix} z_0 \times (p - p_0) & z_1 \times (p - p_1) & z_2 \times (p - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix} \qquad \textit{here} \quad c_{12} = c(\theta_1 + \theta_2)
$$

$$
{}^0T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}
$$

$$
{}^0T_2 = {}^0T_1\,{}^1T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix}
$$

$$
{}^0T_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix}
$$

## 2.2 Analytical Jacobian

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{\phi}} \end{bmatrix} = \frac{d\boldsymbol{x}}{dt} = \underbrace{\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}}_{\boldsymbol{J}_A(\boldsymbol{q})} \underbrace{\frac{d\boldsymbol{q}}{dt}}_{\dot{\boldsymbol{q}}} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}(\boldsymbol{\phi}) \end{bmatrix} \boldsymbol{J}_A(\boldsymbol{q})
$$

## 2.3 Inverse differential kinematics

$$
\begin{cases} \text{minimize} & g(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}} \\ \text{subject to} & \boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \end{cases} \overset{\boldsymbol{W}=\boldsymbol{I}}{\iff} \dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger(\boldsymbol{q})\boldsymbol{v} \qquad \boldsymbol{J}^\dagger \triangleq \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}
$$

$$
\begin{cases} \text{minimize} & g'(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d) \\ \text{subject to} & \boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \end{cases} \iff \dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger(\boldsymbol{q})\boldsymbol{v} + \underbrace{(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})}_{\text{📽 in } \mathcal{N}} \dot{\boldsymbol{q}}_d
$$

**Damped least-square:**

$$
J = \boldsymbol{U\Sigma V}^T,\ \Sigma_{ii} = \sqrt{eig(JJ^T)_i} \implies J^\dagger = V\Sigma^\dagger U^T \overset{\sigma_i \to 1/(\sigma_i + k^2)}{\implies} \boldsymbol{J}^* = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1}
$$

**Secondary objectives**:

$$
\dot{\boldsymbol{H}} = \frac{d\boldsymbol{H}}{dt} = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}\frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} \quad \text{with} \quad \dot{\boldsymbol{q}}_d = -K\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}\right)^T,\ K > 0
$$

$$
\dot{\boldsymbol{H}} = \underbrace{\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}} \boldsymbol{J}^\dagger(\boldsymbol{q})\boldsymbol{v}}_{\text{non si sa}} + \underbrace{-K\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}(\boldsymbol{I} - \boldsymbol{J}^\dagger\boldsymbol{J})\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}\right)^T}_{<0}
$$

- **Max dist. obstacles**: $H = \min_{p,o} \|p(q) - o\| \qquad \rightsquigarrow H \uparrow$
- **Max dist. joint limit**: $H(q) = -\frac{1}{2n}\sum_{i=1}^n \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}}\right)^2 \qquad \rightsquigarrow H \downarrow$
- **Max dist. from singularities**: $H(q) = \sqrt{\det(J(q)J^T(q))} \qquad \rightsquigarrow H \uparrow$

# 3. Statics

$$
\boldsymbol{\tau}^T \delta \boldsymbol{q} = \boldsymbol{F}^T \delta \boldsymbol{p} \implies \text{⚖️}: \ \boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}
$$

$$
\mathcal{N}(\boldsymbol{J}) \equiv \mathcal{R}^\perp(\boldsymbol{J}^T) \qquad \mathcal{R}(\boldsymbol{J}) \equiv \mathcal{N}^\perp(\boldsymbol{J}^T)
$$

**Ellipsoids:**

$$
\|\dot{\boldsymbol{q}}\|^2 = 1 \iff \dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = 1 \overset{\dot{q} = J^\dagger v}{\implies} \boldsymbol{v}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{v} = 1 \implies E_v = \{\boldsymbol{v} \ : \ \boldsymbol{v}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{v} = 1\}
$$

$$
\|\dot{\boldsymbol{\tau}}\|^2 = 1 \iff \boldsymbol{\tau}^T\boldsymbol{\tau} = 1 \overset{\tau = J^T F}{\implies} E_F = \{\boldsymbol{F} \ : \ \boldsymbol{F}^T(\boldsymbol{J}\boldsymbol{J}^T)\boldsymbol{F} = 1\}
$$

**Manipulability measure:**

$$
w(\boldsymbol{q}) = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^T)} = |\lambda_1\lambda_2\cdots\lambda_n| = |\det(\boldsymbol{J})|
$$

# 4. Dynamics

$$
\mathcal{L} = \mathcal{T} - \mathcal{U} \quad (= \mathcal{K} - \mathcal{P}) \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_i}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_i} = \mathcal{F}_i \qquad i = 1, \dots, n
$$

**Kinetic:**

$$
\mathcal{T} = \sum_{i=1}^n \mathcal{T}_{l_i} + \mathcal{T}_{m_i} \implies \mathcal{T} = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n b_{ij}(\boldsymbol{q})\dot{\boldsymbol{q}}_i\dot{\boldsymbol{q}}_j = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}
$$

$$
\begin{cases} \mathcal{T}_{l_i} = \frac{1}{2}m_{l_i}\dot{\boldsymbol{p}}_i^T\dot{\boldsymbol{p}}_i + \frac{1}{2}\boldsymbol{\omega}_i^T\,{}^b\boldsymbol{R}_i\,{}^i\boldsymbol{I}_{l_i}({}^b\boldsymbol{R}_i)^T\boldsymbol{\omega}_i \\ \mathcal{T}_{m_i} = \frac{1}{2}m_{m_i}\dot{\boldsymbol{p}}_i^T\dot{\boldsymbol{p}}_i + \frac{1}{2}\boldsymbol{\omega}_i^T\,{}^b\boldsymbol{R}_i\,{}^i\boldsymbol{I}_{m_i}({}^b\boldsymbol{R}_i)^T\boldsymbol{\omega}_i \end{cases}
$$

$$
\begin{cases} \mathcal{T}_{l_i} = \frac{1}{2}m_{l_i}\left(\dot{\boldsymbol{q}}^T\boldsymbol{J}_p^{(l_i)T}\right)\left(\boldsymbol{J}_p^{(l_i)}\dot{\boldsymbol{q}}\right) + \frac{1}{2}\left(\dot{\boldsymbol{q}}^T\boldsymbol{J}_o^{(l_i)T}\right)\left({}^b\boldsymbol{R}_i\,{}^i\boldsymbol{I}_{l_i}({}^b\boldsymbol{R}_i)^T\right)\left(\boldsymbol{J}_o^{(l_i)}\dot{\boldsymbol{q}}\right) \\ \mathcal{T}_{m_i} = \frac{1}{2}m_{m_i}\left(\dot{\boldsymbol{q}}^T\boldsymbol{J}_p^{(m_i)T}\right)\left(\boldsymbol{J}_p^{(m_i)}\dot{\boldsymbol{q}}\right) + \frac{1}{2}\left(\dot{\boldsymbol{q}}^T\boldsymbol{J}_o^{(m_i)T}\right)\left({}^b\boldsymbol{R}_{m_i}\,{}^{m_i}\boldsymbol{I}_{m_i}({}^b\boldsymbol{R}_{m_i})^T\right)\left(\boldsymbol{J}_o^{(m_i)}\dot{\boldsymbol{q}}\right) \end{cases}
$$

**Potential:**

$$
\mathcal{U} = \sum_{i=1}^n \mathcal{U}_{l_i} + \mathcal{U}_{m_i} \implies \mathcal{U} = -\sum_{i=1}^n m_{l_i}\boldsymbol{g}_0^T\boldsymbol{p}_{l_i} + m_{m_i}\boldsymbol{g}_0^T\boldsymbol{p}_{m_i}
$$

**Dynamic equations:**

$$
\sum_{j=1}^n \boldsymbol{B}_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^n\sum_{k=1}^n h_{ijk}(\boldsymbol{q})\dot{q}_k\dot{q}_j + g_i(\boldsymbol{q}) = \mathcal{F}_i \qquad h_{ijk} = \frac{\partial B_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial B_{jk}}{\partial q_i}
$$

$$
g_i(\boldsymbol{q}) = \frac{\partial \mathcal{U}}{\partial q_i} = -\sum_{j=1}^n m_{l_j}\boldsymbol{g}_0^T\boldsymbol{J}_{p_i}^{(l_j)}(\boldsymbol{q}) + m_{m_j}\boldsymbol{g}_0^T\boldsymbol{J}_{p_i}^{(m_j)}(\boldsymbol{q})
$$

$$
\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} + \boldsymbol{g}(\boldsymbol{q}) = \mathcal{F}
$$

$$
\Downarrow
$$

$$
\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} + \boldsymbol{F}_{viscous}\dot{\boldsymbol{q}} + \boldsymbol{F}_{static}\mathrm{sgn}(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}
$$

# 5. Trajectories

## 5.1 PTP

$$\text{minimize} \int_0^{t_f} \boldsymbol{\tau}^2(t)dt \qquad \text{subject to} \int_0^{t_f} \boldsymbol{\omega}(t)dt = \boldsymbol{q}_f - \boldsymbol{q}_i \qquad (\tau = I\dot{\omega})$$

$$\begin{cases} \boldsymbol{q}(t) &= a_3t^3 + a_2t^2 + a_1t + a_0 \\ \dot{\boldsymbol{q}}(t) &= 3a_3t^2 + 2a_2t + a_1 \\ \ddot{\boldsymbol{q}}(t) &= 6a_3t + 2a_2 \end{cases}$$

$$\begin{cases} \boldsymbol{q}(t_i) = a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\ \boldsymbol{q}(t_f) = a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{\boldsymbol{q}}(t_i) = 3a_3t_i^2 + 2a_2t_i + a_1 \\ \dot{\boldsymbol{q}}(t_i) = 3a_3t_f^2 + 2a_2t_f + a_1 \\ \boldsymbol{q}(t_i) = \boldsymbol{q}_i \\ \boldsymbol{q}(t_f) = \boldsymbol{q}_f \\ \dot{\boldsymbol{q}}(t_f) = \dot{\boldsymbol{q}}_i \\ \dot{\boldsymbol{q}}(t_f) = \dot{\boldsymbol{q}}_f \end{cases} \overset{t_f=0}{\Longrightarrow} \begin{cases} \boldsymbol{q}_i = a_0 \\ \boldsymbol{q}_f = a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{\boldsymbol{q}}_i = a_1 \\ \dot{\boldsymbol{q}}_f = 3a_3t_f^2 + 2a_2t_f + a_1 \end{cases}$$

## 5.2 2-1-2

$$\left[\dot{\boldsymbol{q}}_c = \frac{\boldsymbol{q}_m - \boldsymbol{q}_c}{t_m - t_c} = \frac{\text{rise}}{\text{run}} \qquad \ddot{\boldsymbol{q}}_c t_c = \dot{\boldsymbol{q}}_c = \frac{\boldsymbol{q}_m - \boldsymbol{q}_c}{t_m - t_c} \qquad \ddot{\boldsymbol{q}}_c t_c^2 - \ddot{\boldsymbol{q}}_c t_f t_c + \boldsymbol{q}_f - \boldsymbol{q}_i = 0\right]$$

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{t_f^2\ddot{\boldsymbol{q}}_c - 4(\boldsymbol{q}_f - \boldsymbol{q}_i)}{\ddot{\boldsymbol{q}}_c}} \qquad \operatorname{sgn}(\ddot{\boldsymbol{q}}_c) = \operatorname{sgn}(\boldsymbol{q}_f - \boldsymbol{q}_i) \qquad |\ddot{\boldsymbol{q}}_c| \geq \frac{4|\boldsymbol{q}_f - \boldsymbol{q}_i|}{t_f^2}$$

$$t_m = \frac{t_f}{2}\ ,\ \boldsymbol{q}_m = \frac{\boldsymbol{q}_f - \boldsymbol{q}_i}{2} \qquad \boldsymbol{q}(t) = \begin{cases} \boldsymbol{q}_i + \frac{1}{2}\ddot{\boldsymbol{q}}_c t^2 & 0 \leq t \leq t_c \\ \boldsymbol{q}_i + \ddot{\boldsymbol{q}}_c t_c(t - t_c/2) & t_c < t \leq t_f - t_c \\ \boldsymbol{q}_f - \frac{1}{2}\ddot{\boldsymbol{q}}_c(t_f - t)^2 & t_f - t_c < t \leq t_f \end{cases}$$

**Assegnazione di $\dot{\boldsymbol{q}}_c$ invece di $\ddot{\boldsymbol{q}}_c$**

$$\frac{|\boldsymbol{q}_f - \boldsymbol{q}_i|}{t_f} < |\dot{\boldsymbol{q}}_c| \leq 2\frac{|\boldsymbol{q}_f - \boldsymbol{q}_i|}{t_f} \qquad t_c = \frac{\boldsymbol{q}_i - \boldsymbol{q}_f + \dot{\boldsymbol{q}}_c t_f}{\dot{\boldsymbol{q}}_c} \qquad \ddot{\boldsymbol{q}}_c = \frac{\dot{\boldsymbol{q}}_c^2}{\boldsymbol{q}_i - \boldsymbol{q}_f + \dot{\boldsymbol{q}} t_f}$$

## 5.3 Operational space

$$x_{traj} = \begin{bmatrix} p(t) \\ \phi(t) \end{bmatrix} \quad \dot{p} = \dot{s}\frac{dp}{ds} = \dot{s}t \quad ; \quad \boldsymbol{t} = \frac{d\boldsymbol{p}}{ds} \quad \boldsymbol{n} = \frac{\frac{d^2\boldsymbol{p}}{ds^2}}{\|\frac{d^2\boldsymbol{p}}{ds^2}\|} \quad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$$

### 5.3.1 Segment

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \qquad t = \frac{dp}{ds} = \frac{(p_f - p_i)}{\|p_f - p_i\|} \qquad \frac{d^2p}{ds^2} = 0$$

$$p(s) = p_i + \frac{s(p_f - p_i)}{\|p_f - p_i\|} \qquad \dot{p} = \frac{\dot{s}(p_f - p_i)}{\|p_f - p_i\|} = \dot{s}t \qquad \ddot{p} = \frac{\ddot{s}(p_f - p_i)}{\|p_f - p_i\|} = \ddot{s}t$$

### 5.3.2 Circonference

$$p'(s) = \left[\rho\cos(\tfrac{s}{\rho}) \quad \rho\sin(\tfrac{s}{\rho}) \quad 0\right] \implies p(s) = c + {}^{\mathcal{O}}R_{\mathcal{O}'}p'(s)$$

$$\frac{dp}{ds} = R\left[-\sin(s/\rho) \quad cos(s/\rho) \quad 0\right] \qquad \frac{d^2p}{ds^2} = R\left[-\cos(s/\rho)/\rho \quad -sin(s/\rho)/\rho \quad 0\right]$$

$$p(s) = c + R\begin{bmatrix} \rho\cos(s/\rho) \\ \rho\sin(s/\rho) \\ 0 \end{bmatrix} \qquad \dot{p} = R\begin{bmatrix} -\dot{s}\sin(s/\rho) \\ \dot{s}\sin(s/\rho) \\ 0 \end{bmatrix} \qquad \ddot{p} = R\begin{bmatrix} -\dot{s}^2\rho^{-1}\cos(s/\rho) - \ddot{s}\sin(s/\rho) \\ -\dot{s}^2\rho^{-1}\sin(s/\rho) + \ddot{s}\sin(s/\rho) \\ 0 \end{bmatrix}$$

### 5.3.3 Attitude trajectory

$$\phi(s) = \phi_i + \frac{s(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \qquad \dot{\phi} = \frac{\dot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|} \qquad \ddot{\phi} = \frac{\ddot{s}(\phi_f - \phi_i)}{\|\phi_f - \phi_i\|}$$

# 6. Control

## 6.1 Actuator model

**mech:** $K_r^{-1}\tau = K_t i_a$ **electr:** $v_a = R_a i_a + K_v\dot{q}_m\ ,\ v_a = G_v V_c \quad ; \quad [K_r q = q_m]$

### 6.1.1 Velocity generator:

$$\omega_m = \frac{G_v}{k_v}v_c' \qquad F = F_v K_r K_t R_a^{-1} K_v K_r \qquad u = K_r K_t R_a^{-1} G_v v_c$$

$$u = \tau + F\dot{q} \implies \tau = K_r K_t R_a^{-1}(G_v v_c - K_v K_r \dot{q}) \qquad v_c \approx G_v^{-1} K_v K_r \dot{q}$$

### 6.1.2 Torque generator:

$$c_m \approx \frac{k_t}{k_i}(v_c' - \frac{k_v}{G_v}\omega_m)$$

## 6.2 Decentralized joint control

$$\tau = K_r\tau_m \qquad q = K_r^{-1}q_m \qquad B(q) = \bar{B} + \Delta B(q)$$

$$K_r^{-1}\bar{B}K_r^{-1}\ddot{q}_m + \underbrace{K_r^{-1}\Delta B(q)K_r^{-1}\ddot{q}_m + K_r^{-1}C(q,\dot{q})K_r^{-1}\dot{q}_m + K_r^{-1}g(q)}_{d} + \underbrace{K_r^{-1}F_v K_r^{-1}}_{F_m}\dot{q}_m = \tau_m$$

**t.f. motor:** $M(s) = \dfrac{k_m}{s(1 + T_m s)} \qquad k_m = \dfrac{1}{k_v}\ ,\ T_m = \dfrac{R_a I}{k_t k_v}$

**PI control:** $C(s) = K_c\dfrac{1 + sT_c}{s} \qquad [\ K_c \equiv K_p, T_c \equiv T_p \parallel K_c \equiv K_v,\ \cdots\ ]$

### 6.2.1 Position feedback

$$\text{forward path:} \quad G(s) = \frac{k_m K_p(1 + sT_p)}{s^2(1 + sT_m)} \implies \begin{cases} \times & T_p < T_m \\ \checkmark & T_p > T_m \\ \checkmark\Uparrow & T_p \gg T_m \end{cases}$$

## 6.3 Centralized joint control

$$\text{current controlled} \implies i_a = G_i v_c \implies u = K_r K_t G_i v_c = \tau$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \implies \dot{x} = \begin{bmatrix} \dot{q} \\ B^{-1}(q)\left[u - C(q,\dot{q})\dot{q} - F\dot{q} - g(q)\right] \end{bmatrix} \qquad x_{eq} \iff \dot{x} = 0 \iff \begin{cases} \dot{q} = 0 \\ \bar{u} = g(\bar{q}) \end{cases}$$

### 6.3.1 PD control with gravity compensation

$$V(\dot{q}, e) = \frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}e^T K_P e > 0 \qquad \forall \dot{q}, e \neq 0 \qquad e \triangleq q_d - q$$

$$\dot{V} = \dot{q}^T B(q)\ddot{q} + \frac{1}{2}\dot{q}^T\dot{B}(q)\dot{q} - \dot{q}^T K_P e \qquad u = g(q) + K_p e - K_d\dot{q}$$

### 6.3.2 Inverse dynamics (feedback linearization)

$$B(q)\ddot{q} + n(q,\dot{q}) = \tau = u \overset{F.L.}{\Longrightarrow} u \triangleq B(q)y + n(q,\dot{q}) \implies \ddot{q} = y$$

$$\text{PD control:} \qquad y \triangleq -K_P q - K_D\dot{q} + r \quad , \quad r \triangleq \ddot{q}_d + K_P q_d + K_D\dot{q}_d$$

$$\left[\ \overset{\ddot{q}=y}{\Longrightarrow} \ddot{q} + K_P q + K_D\dot{q} = r \implies \ddot{e} + K_D\dot{e} + K_P e = 0\ \right]$$

$$\implies \boldsymbol{y} = \boldsymbol{K_p}(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K_d}(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \ddot{\boldsymbol{q}}_d$$

$$\left[\quad K_P = diag\{\omega_{n1}^2,\ \ldots,\ \omega_{nn}^2\} \quad K_D = diag\{2\zeta\omega_{n1},\ \ldots,\ 2\zeta\omega_{nn}^2\}\quad\right]$$

## 6.4 Operational space control

### 6.4.1 PD with gravity compensation

$$e \triangleq x_d - x \qquad u = g(q) + J_A^T(q)K_P e - J_A^T(q)K_P K_D J_A(q)\dot{q}$$

$$V(\dot{q}, e) = \frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}e^T K_P e > 0 \qquad \forall \dot{q}, e \neq 0$$

$$\text{at equilibrium} \quad \dot{x} = (\dot{q}, \ddot{q}) = 0 \implies -J_A^T(q)K_P e = 0$$

### 6.4.2 Inverse dynamics (feedback linearization)

$$\dot{x} = J_A(q)\dot{q} \implies \ddot{x} = \dot{J}_A(q)\ddot{q} + \dot{J}_A(q,\dot{q})\dot{q}$$

$$u \triangleq B(q)y + n(q,\dot{q}) \implies \ddot{q} = y \implies y \triangleq J_A^{-1}(q)\big(\ddot{x}_d + K_D\dot{e} + K_P e - \dot{J}_A(q,\dot{q})\dot{q}\big)$$

# 7. Control of the interaction

$$B(q)\ddot{q} + n(q,\dot{q}) = u - \underbrace{J^T(q)h}_{interaction} \implies \text{PD with gravity comp.} \quad J_A^T(q)K_P e = J^T(q)h$$

$$h_A = T_A^T(x)KT_A(x)dx = K_A(x)(x - x_e) \implies e = K_P^{-1}K_A(x)(x - x_e)$$

$$x_\infty = (I - K_P^{-1}K_A(x))^{-1}(x_d + K_P^{-1}K_A(x)x_e) \qquad h_{A\infty} = (I + K_A(x)K_P^{-1})^{-1}K_A(x)(x_d - x_e)$$

# 8. Mobile Robotics

$$\begin{matrix}\text{holonomic constr.} \\ \text{(integrable)}\end{matrix} \iff h_i(\boldsymbol{q}) = 0,\ i = 1\ldots k < n \implies \begin{matrix}\text{kin.} \\ \text{constr.}\end{matrix} : \quad \frac{dh_i(\boldsymbol{q})}{dt} = \frac{dh_i(\boldsymbol{q})}{d\boldsymbol{q}}\dot{\boldsymbol{q}} = 0$$

$$\text{kin. constr.} \quad a_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \implies \text{pfaffian} \quad \boldsymbol{a}_i^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \leftrightarrow \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$$

$$\dot{\boldsymbol{q}} \in \mathcal{N}(\boldsymbol{A}^T(\boldsymbol{q})) \implies \langle \boldsymbol{g}_1(\boldsymbol{q}), \ldots, \boldsymbol{g}_n - k(\boldsymbol{q})\rangle \text{ base of } \mathcal{N} \implies \dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}$$

## 8.1 Unicycle

$$\text{pure rolling c.:} \quad \frac{dy}{dx} = \tan\theta \implies \boldsymbol{q} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \implies \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}^T \dot{\boldsymbol{q}} = 0 \quad , \quad \boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$$

## 8.2 Differential drive

$$v = \frac{r(\omega_R + \omega_L)}{2} \qquad \omega = \frac{r(\omega_R - \omega_L)}{d}$$

## 8.3 Bike

$$\text{pure rolling (front w., back w.):} \quad \dot{x}_f\sin(\theta + \phi) - \dot{y}_f\cos(\theta + \phi) = 0 \quad , \quad \dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

$$x_f = x + L\cos\theta\ ,\ y_f = y + L\sin\theta \overset{1^{st}c.}{\Longrightarrow} \dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) - L\dot{\theta}\cos\phi = 0$$

$$\boldsymbol{A}^T(\boldsymbol{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta + \phi) & -cos(\theta + \phi) & -L\cos\phi & 0 \end{bmatrix} \qquad \boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ \sin\phi/L & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \boldsymbol{G}_1(\boldsymbol{q})u_1 + \boldsymbol{G}_2(\boldsymbol{q})u_2 \quad (u_2 \equiv \omega) \qquad \text{if front drive: } u_1 = v\ ,\ \text{if back d.: } u_1 = \frac{v}{\cos\phi}$$

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$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$