CS4243 Computer Vision and Pattern Recognition
Lab 1

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I. Introduction

There are two objectives for this lab:

- *To review some important results in linear algebra that we had learned in class*

- *To learn Matlab.*
  Matlab is a software package with a bunch of built-in mathematical functions. It is good in computation of vectors and matrices. You may look at the sample program sample_Mfile.m to get a feel of how matlab programs are written. If you have time and want a more detailed introduction to Matlab, you may want to browse through the following web site:
  http://www.math.utah.edu/lab/ms/matlab/matlab.html
  If you don't have time to browse through the above web site, it is not the end of the world, but make sure you put in a lot of, yes, a lot of effort in understanding the math results in Sections II and III.

Sections II and III gives you a quick run through of some basic linear algebra that we had covered in the lecture.

Section IV is the problem set that you need to do for this lab session.

II. Vector Algebra

A quantity which is characterized by magnitude and direction is called a vector. We represent a vector in *N*-dimensional space by an *N*x1 tuple:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix}$$

Transpose of a Vector

The transpose of a vector changes it from Nx1 to 1xN, or from 1xN to Nx1.

Example:

$$a^T = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \end{bmatrix}$$

Magnitude of Vectors

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_N^2}$$

Multiplication of Vectors

Scalar Multiplication

$$b = \alpha\ a$$

$$= \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \\ \vdots \\ \alpha a_N \end{bmatrix}$$

where α is a scalar

Vector Multiplication

dot-product:

$$
\begin{aligned}
c &= a \cdot b \\
&= a^T b \\
&= \sum_{i=1}^{N} a_i b_i
\end{aligned}
$$

note that c is a scalar and it is also given by

$$c = \|a\| \, \|b\| \cos\theta$$

cross-product:

let $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

the vector cross product is given by

$$
\begin{aligned}
c &= a \wedge b \\
&= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= (a_2 b_3 - a_3 b_2)\, i \quad - \quad (a_1 b_3 - a_3 b_1)\, j \quad + \quad (a_1 b_2 - a_2 b_1)\, k
\end{aligned}
$$

The magnitude of c is given by

$$\|c\| = \|a\| \, \|b\| \sin\theta$$

The direction of c is perpendicular to both a and b.

Properties of Vector Products

If the unit vectors i¸ j, k are the three orthogonal axis of a right-handed coordinate system, then

- $i \cdot i = j \cdot j = k \cdot k = 1$

- $i \cdot j = j \cdot k = k \cdot i = 0$
- $i \wedge i = j \wedge j = k \wedge k = 0$
- $i \wedge j = -j \wedge i = k$
- $j \wedge k = -k \wedge j = i$
- $k \wedge i = -i \wedge k = j$

We also have the following results:

- $a \cdot b \wedge c = b \cdot c \wedge a = c \cdot a \wedge b$
- $a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$

Linear Dependence and Independence

A set of vectors $\{x_1, x_2, \cdots, x_m\}$ is linearly dependent if there exist a set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i x_i = 0$$

If the only way to satisfy the equation is to have
$\alpha_i = 0 \quad \forall i$, then we say that the set of vectors $\{x_1, x_2, \cdots, x_m\}$ are linearly independent.

Basis Set

The set of vectors that can be used to represent all Nx1 vectors is called a basis vector set. The basis vector set is said to "span" the Nx1 vector space.

For example, if $\{v_i\}_{1 \le i \le N}$ is a basis set, then any Nx1 vector *x* can be written as

$$x = \sum_{i=1}^{N} c_i v_i$$

where c_i is a scalar.

Vector Space

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers.   The addition and multiplication must produce vectors that are within the space. In other words, if we add any vectors in the space, their sum is in the space. If we scale any vector, it still remains in the space.

III.   Matrix Algebra

A matrix of dimensions M by N is a rectangular block of numbers (real or complex) with M rows and N columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}$$

Transpose

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{M1} \\ a_{12} & a_{22} & \cdots & a_{M2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1N} & a_{2N} & \cdots & a_{MN} \end{pmatrix}$$

Diagonal Matrix

A diagonal matrix is a matrix with all off-diagonal entries equal to zero.

Identity Matrix

An identity Matrix, written as *I*, is a diagonal matrix will all its diagonal entries equal to one.

Symmetric Matrix

A symmetric matrix is a square matrix whose transpose is equal to itself, i.e.

$$A = A^T$$

Determinants

For a 2x2 matrix A, its determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

For a 3x3 matrix A, its determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{31}a_{22}a_{13} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23}$$

The determinant of a matrix A equals the volume of a parallelepipe. The edges of the parallelepipe come from the rows of A.

Properties of Determinants

- $|A| = \dfrac{1}{|A^{-1}|}$
- $|A| = |A^T|$
- If any two rows of A are interchanged, the sign of its determinant is changed.
- $|AB| = |A|\,|B|$

Rank of a Matrix

The rank of a matrix indicates the number of linearly independent rows (or columns) in the matrix.

Let $r(A)$ represents the rank of a matrix A, then

$$r(AB) \le r(A)$$
$$r(AB) \le r(B)$$

which also means that

$$r(AB) \le \min(r(A), r(B))$$

Some General Properties of Matrices

- $IA = AI = A$
- $A^{-1}A = I$ and $AA^{-1} = I$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(B + C)D = BD + CD$
- $AB \ne BA$ in general

Null Space

The nullspace of a matrix A consists of all vectors x such that $Ax = 0$.

If A is rank-deficient (i.e. not full rank), then there is a non-zero solution for x.

Solution of a Linear System

Given the matrix A and vector b,

the problem $Ax = b$ has a solution if and only if the vector b can be expressed as a linear combination of the columns of A.

If A is a square matrix and invertible, then the solution is

$$x = A^{-1}b$$

<u>Overdetermined system:</u>
If A is a rectangular matrix of size M by N, where $M>N$, and if $A^T A$ is invertible (i.e. square and full rank), then the linear least squares solution to x is given by

$$x = (A^T A)^{-1} A^T b$$

Eigenvalues and Eigenvectors

If x is an eigenvector of A and λ is the eigenvalue,
then $Ax = \lambda x$,
where A is a matrix, x a vector, and λ a scalar,

Properties of Eigenvalues and Eigenvectors

The sum of all the eigenvalues of the matrix A equals the sum of the diagonal entries of A (note: sum of diagonal entries is also known as trace), i.e.

$$\sum_{i=1}^{n} \lambda_i = trace(A)$$

The product of all the eigenvalues of the matrix A equals the determinant of A.

Eigenvectors corresponding to different eigenvalues are linearly independent.

Eigenvectors of a real symmetric matrix are orthogonal.

Eigenvalues of a real symmetric matrix are also real.

Singular Value Decomposition

Any m by n matrix A can be decomposed into

$$A = U \, \Sigma \, V^T$$

The columns of U are eigenvectors of AA^T. U is an m by m matrix.

The columns of V are eigenvectors of $A^T A$. V is an n by n matrix.

The entries on the diagonal of Σ are known as the singular values. They are the square roots of the eigenvalues of both AA^T and $A^T A$.

The number of non-zero singular values equals the rank of matrix A.

IV. Problem Set

The file 'sample_mfiles.m' is a sample Matlab program to show you how some basic commands in Matlab are used. Read through the program and run it to see the results.

Note:

- To activate matlab, type "matlab" at the command prompt, then type "sample_mfiles" to run sample_mfiles.m
- In Matlab, the command for the transpose of a vector or matrix A is **A.'** or **transpose(A)**
- For all the matlab programs in our lab series, start and end the program with the statements **diary on** and **diary off** respectively. This will create/append a file called diary that will keep the output of the program.

1. Write a matlab program to do the following:

<u>Q1.1</u>

At the beginning of your program, use the following commands to close all figures and clear all variables.

close all
clear all

Define the following vectors and matrices:

$$A = \begin{pmatrix} 1 & 7 & 3 \\ 2 & 4 & 1 \\ 4 & 8 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0.25 \\ 2 & 4 & 1 \\ 4 & 8 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

program statements:

A = [1 7 3; 2 4 1; 4 8 6];

B = [1 1 0.25; 2 4 1; 4 8 2];

C = [0; 0; 0];

Run the codes and print out the matrices A and B, and vector C.

Q1.2

Continuing from the codes written so far, perform singular value decomposition on the matrices A and B respectively.

program statements:

[Ua, Sa, Va] = svd(A)

[Ub, Sb, Vb] = svd(B)

Type "help svd" at the matlab command prompt to understand what you are doing.

Verify that U is an orthonormal matrix (i.e. all columns are orthonormal).

Verify that V is an orthonormal matrix.

Examine the singular values of A. What is the minimum singular value? Why is it so? What is the rank of A?

Examine the singular values of B. What is the minimum singular value? Explain what it means in terms of the rank of matrix B.

Q1.3

Using the results obtained so far, write down the solution (for vector x) for each of the equations below, using the SVD output:

Solve the equation Ax=C

Solve the equation $Bx=C$

Where x can be non-zero, you must provide the non-zero solution.

Q1.4

Form a 4 by 3 matrix F :

$$F = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 3 & 2 \\ 4 & 2 & 3 \\ -1 & 7 & 2 \end{pmatrix}$$

Can the matrix F be of rank 4? Why?

Solve the following equation using linear least squares:

$$F\ x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Let $E = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 3 & 2 \\ 4 & 3 & 3 \\ -1 & 7 & 2 \end{pmatrix}$

Solve the following equation using linear least squares:

$$E\ x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

If $\hat{x}$ is your solution, compute $E\hat{x}$ and comment on your result.

2. This question is on eigenvalues and eigenvectors.

   Use the random number generator to generate a 3x3 matrix A with all elements in the range 0 to 100.

   Find the eigenvectors and eigenvalues of the matrix A.

   Display all the eigenvalue-eigenvector pairs.

   For each eigenvalue-eigenvector pair, verify that $Ax = \lambda x$

   Hint: Type "help eig" to see how the eigenvector/eigenvalue function can be used.

Submission Instructions

Submit the softcopy of your Matlab script to IVLE by 1st Sep (Friday) 2359hrs. Please put your Matlab script in a folder and submit the folder. Use the following convention to name your folder:

> MatriculationNumber_yourName_Lab#.
>
> For example, if your matriculation number is A1234567B, and your name is Chow Yuen Fatt, for this lab, your file name should be A1234567B_ChowYuenFatt_Lab1.