Linear and Angular Momentum

Linear Momentum and Force

- Linear motion: apply force to a mass
- The force causes the linear momentum to change
- The net force acting on a body is the time rate of change of its linear momentum

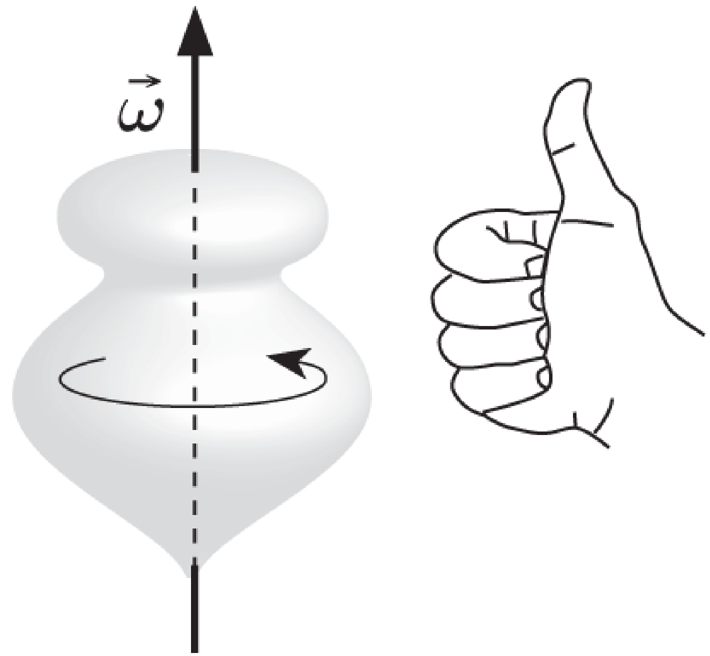
$$\mathbf{F}_{\text{net}} = \Sigma\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{p} = m\mathbf{v}$$

Angular Momentum

- Same basic techniques that were used in linear motion can be applied to rotational motion.
  - $a = r\alpha$
  - $v = r\omega$
  - $x = r\theta$
- Linear momentum defined as $\mathbf{p} = m\mathbf{v}$
- Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})$

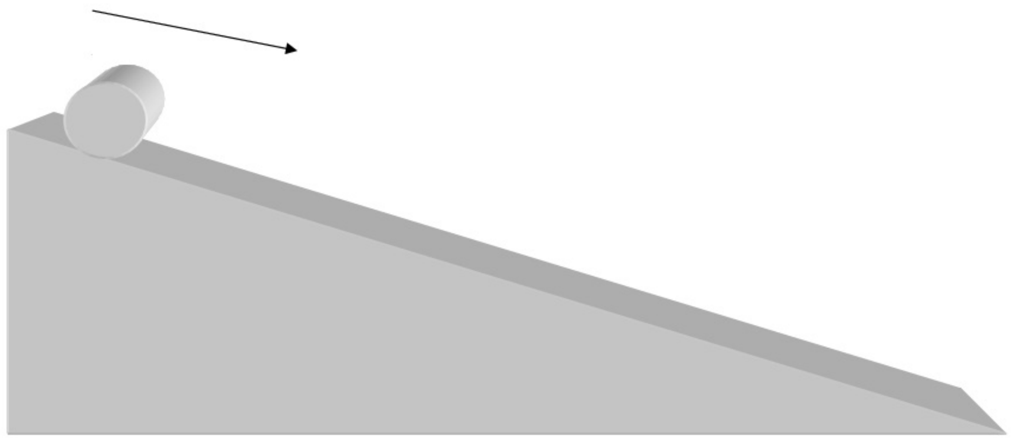
Angular Velocity



$$\text{magnitude: } |\vec{\omega}| \equiv \left|\frac{d\theta}{dt}\right| \equiv \lim_{\Delta t \to 0} \left|\frac{\Delta\theta}{\Delta t}\right| = |\text{rate of change of } \theta|$$

$$\text{direction: } \begin{pmatrix} \text{along the axis of rotation in the direction indicated} \\ \text{by one's right thumb if one's right fingers curl in} \\ \text{the direction that particles in the object move} \end{pmatrix}$$

Clicker Question



C6T.1 A cylinder rolls, without slipping, down an incline directly toward you. The cylinder's angular velocity $\vec{\omega}$ points in which direction?

A. Toward you

B. Away from you

C. To your right

D. To your left

E. Some other direction (specify)

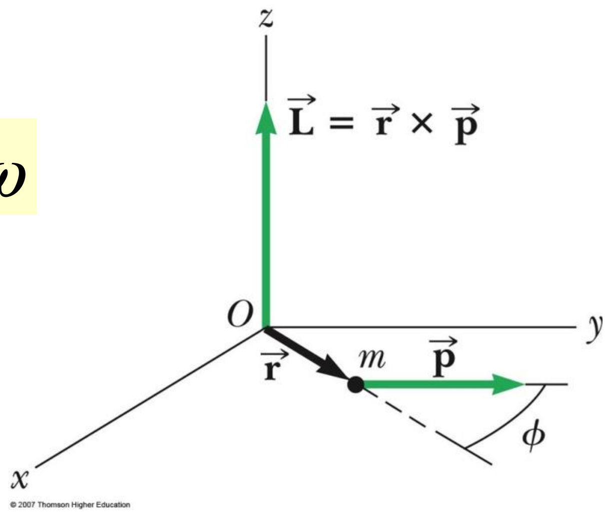
C. To your right (use right hand rule).

Angular Momentum II

- Angular momentum of a particle

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})$$

$$L = mr^2\omega$$



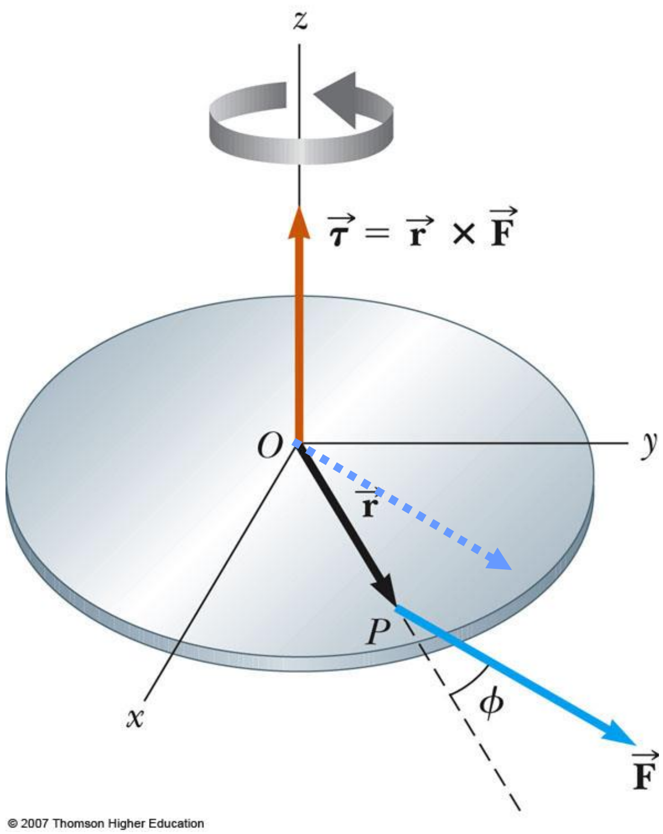
  - **r** is the particle’s instantaneous position vector
  - **p** is its instantaneous linear momentum

Angular Momentum and Torque

- The torque causes the angular momentum to change
- The net torque acting on a body is the time rate of change of its angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \quad \Rightarrow \quad \boxed{\boldsymbol{\tau}_{\text{net}} = \Sigma \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

Torque as a Cross Product



- The torque is the cross product of a force vector with the position vector to its point of application

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\tau| = rF\sin\theta = r_{\perp}F = r\,F_{\perp}$$

- The torque vector is perpendicular to the plane formed by the position vector and the force vector
- Right Hand Rule: curl fingers from r to F, thumb points along torque.

Demonstration

$$\vec{F}_{net} = \Sigma\vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \vec{\tau}_{net} = \Sigma\vec{\tau} = \frac{d\vec{L}}{dt}$$

- Start from $\dfrac{d\vec{L}}{dt} = \dfrac{d}{dt}(\vec{r}\times\vec{p}) = m\dfrac{d}{dt}(\vec{r}\times\vec{v})$

- Expand using derivative chain rule

$$\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{r}\times\vec{v}) = m\left[\frac{d\vec{r}}{dt}\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt}\right] = m\left[\cancelto{\text{Zero}}{\vec{v}\times\vec{v}} + \vec{r}\times\vec{a}\right]$$

$$\frac{d\vec{L}}{dt} = m\left[\vec{v}\times\vec{v} + \vec{r}\times\vec{a}\right] = m\vec{r}\times\vec{a} = \vec{r}\times(m\vec{a}) = \vec{r}\times\vec{F}_{net} = \vec{\tau}_{net}$$

Same basic techniques that were used in linear motion can be applied to rotational motion.

- F becomes τ
- a becomes α
- v becomes ω
- x becomes θ

Linear momentum defined as $\mathbf{p} = m\mathbf{v}$

Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})$

Conservation Laws

A **symmetry principle** implies an associated **conservation law**.

Symmetry	Conservation of
Translation in time	Energy
Translation in space	Momentum
Rotation in space	Angular Momentum

Position

Orientation

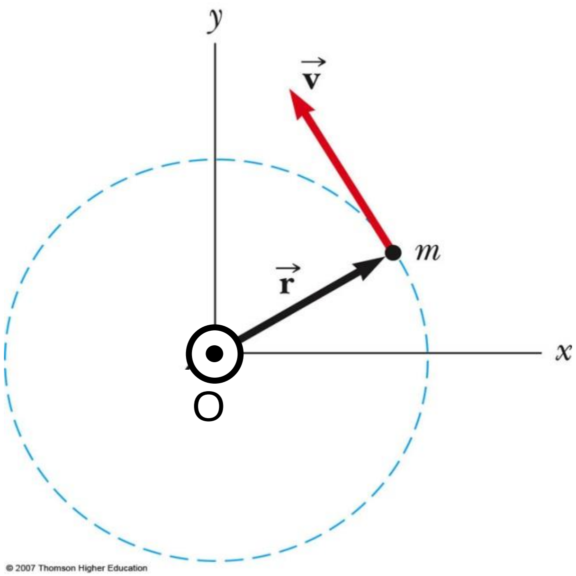
Angular Momentum of a Particle in Uniform Circular Motion

Example: A particle moves in the xy plane in a circular path of radius r. Find the magnitude and direction of its angular momentum relative to an axis through O when its velocity is v.

- The angular momentum vector points out of the diagram
- The magnitude is

$$L = rp\sin\theta = mvr\sin(90^{o}) = mvr$$

- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path



Centripetal Force

Thus, in uniform circular motion there must be a net force to produce the centripetal acceleration (a_c).

The centripetal force is the name given to the net force required to keep an object moving on a circular path.

This force could be Normal, Friction, Tension, or both

The direction of the centripetal force always points toward the center of the circle and continually changes direction as the object moves.

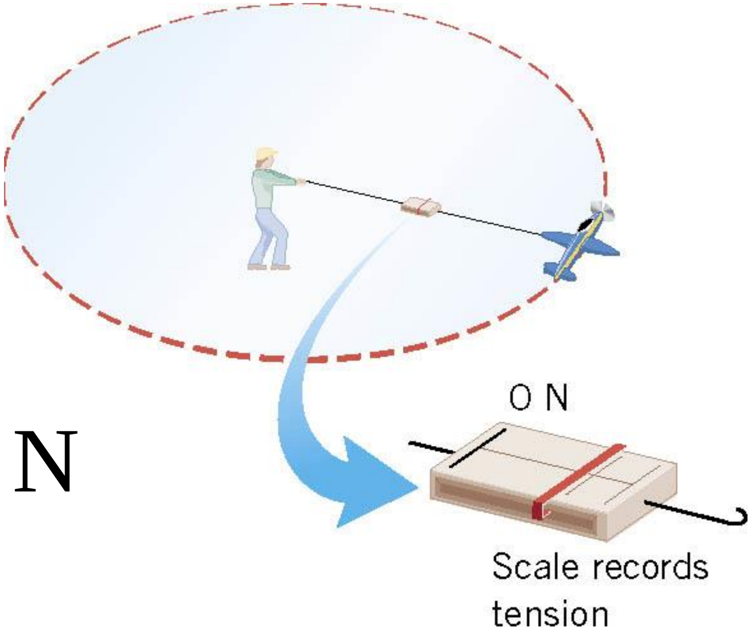
$$F_c = ma_c = m\frac{v^2}{r}$$

Centripetal Force

Example 1: The Effect of Speed on Centripetal Force
The model airplane has a mass of 0.90 kg and moves at constant speed on a circle that is parallel to the ground. Find the tension in the 17 m guideline for a speed of 19 m/s.

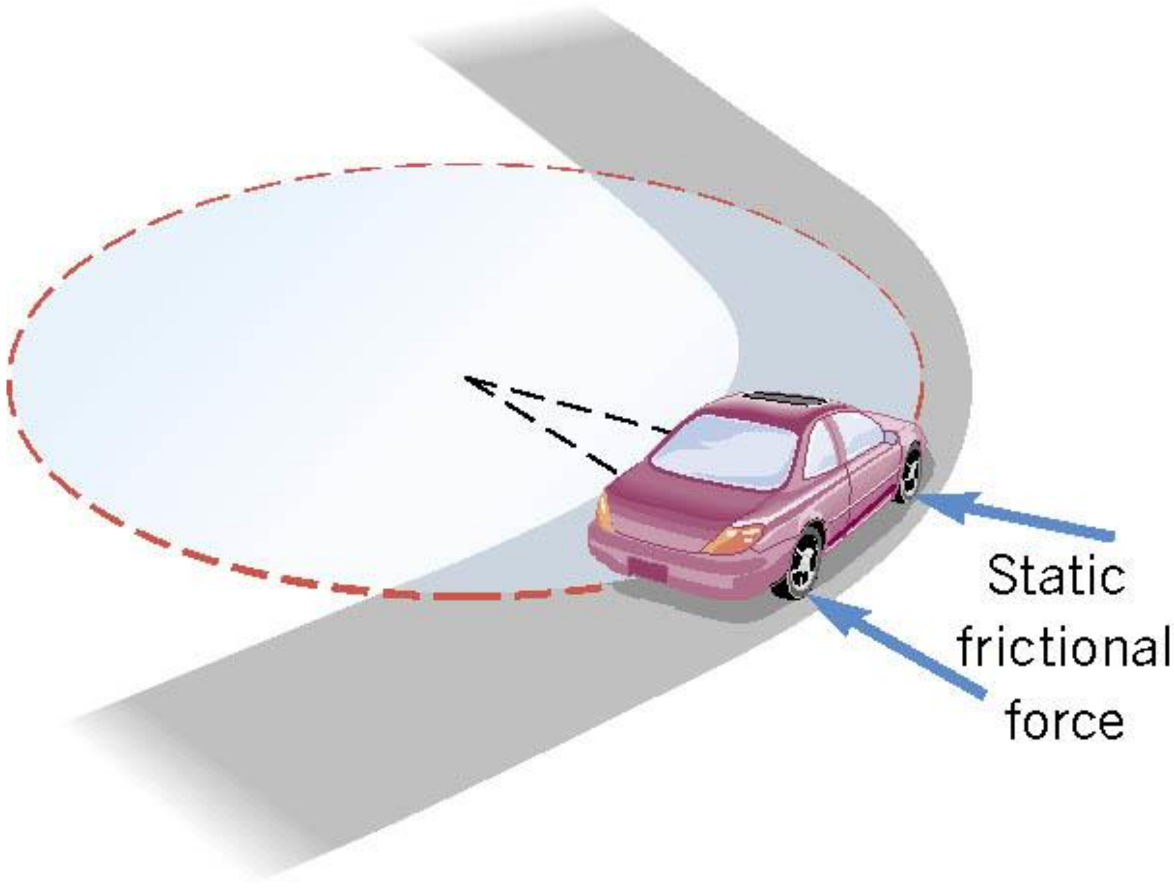
$$F_c = T = m\frac{v^2}{r}$$

$$T = (0.90\,\text{kg})\frac{(19\,\text{m/s})^2}{17\,\text{m}} = 19\,\text{N}$$



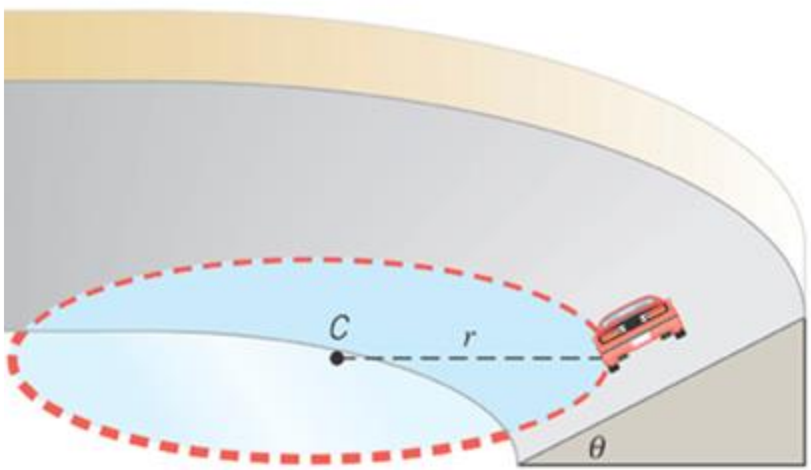
Banked Curves

On an unbanked curve, the static frictional force provides the centripetal force.

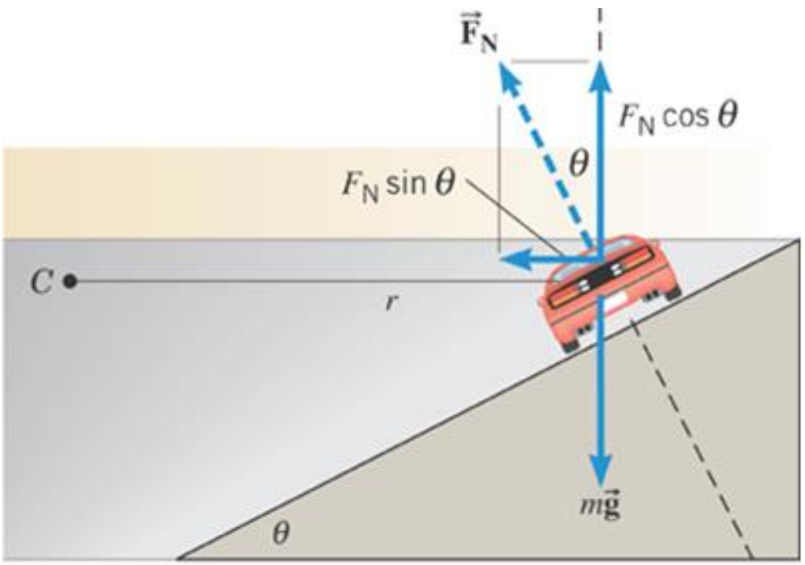


Banked Curves

Example 2: On a frictionless banked curve, the centripetal force is the horizontal component of the normal force. The vertical component of the normal force balances the car's weight.

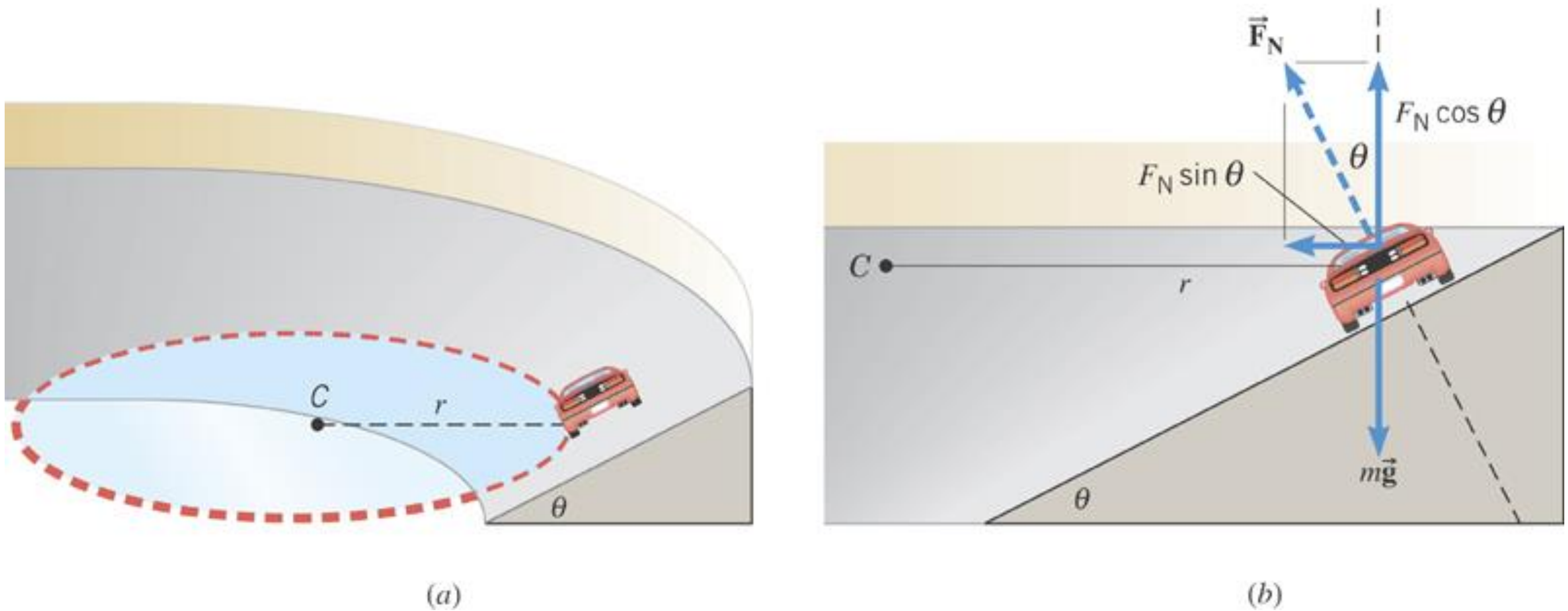


(a)



(b)

Banked Curves



$$F_c = F_N \sin\theta = m\frac{v^2}{r}$$

$$F_N \cos\theta = mg$$

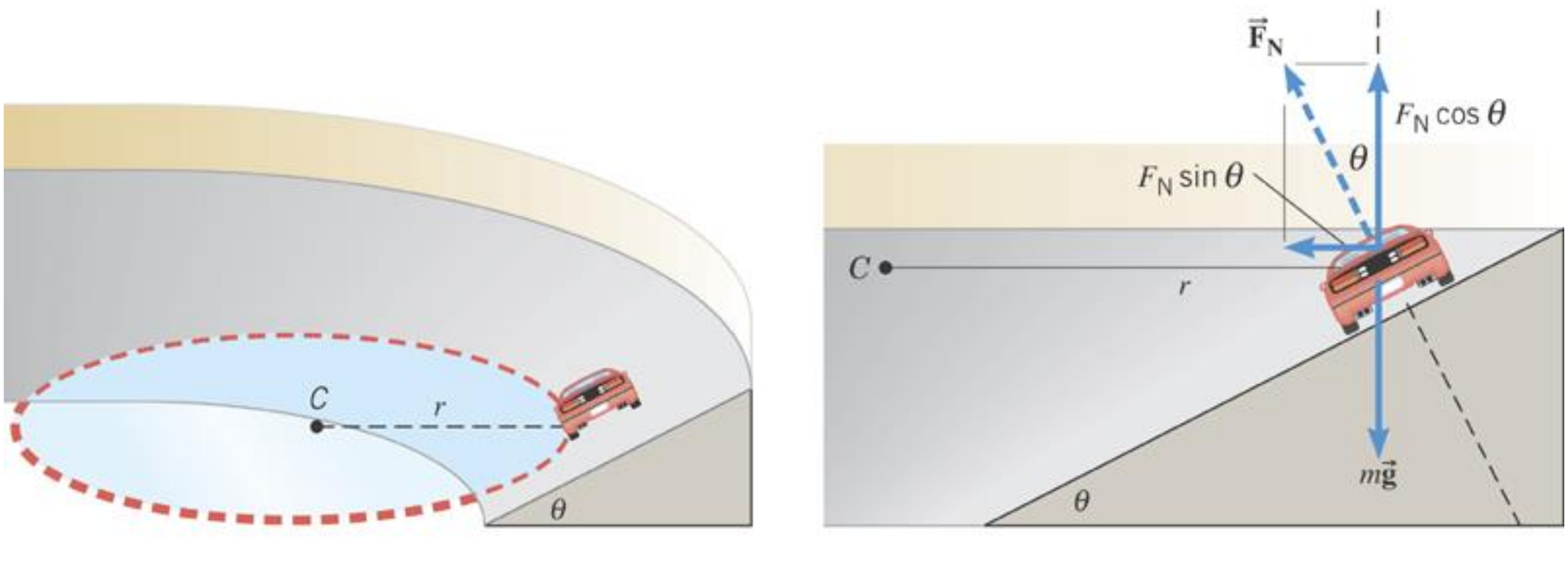
Banked Curves

$$F_N \sin\theta = m\frac{v^2}{r}$$

$$F_N \cos\theta = mg$$

$$\Rightarrow$$

$$\tan\theta = \frac{v^2}{rg}$$



(a)

(b)

Ferris wheel- park fun

Ferris wheel, which has seats on the rim of a circle with a radius of 25 m. It rotates at a constant speed and makes one complete revolution every 20 seconds.

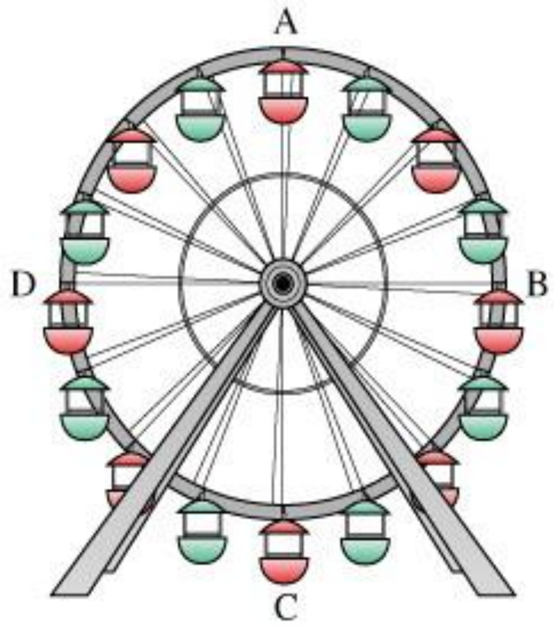
1- What is the acceleration direction at each position?

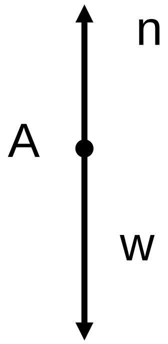
2- What must the forces (normal and weight) look like at each position?

3- what would the normal force be on a rider whose weight is 600 N at:

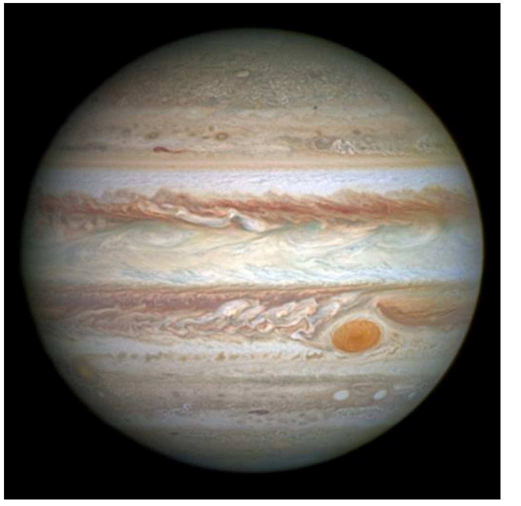
A) the highest point?

B) the lowest point?





Practice Problem



Jupiter spins once every 9.92 h and has an equatorial radius of 71,500 km. What is the ordinary speed (in m/s) of a point on Jupiter's equator due to Jupiter's rotation?

Jupiter's radius is $r = 71{,}500$ km and its rate of rotation is $|\vec{\omega}| = 1$ revolution (2π rad) per 9.92 h. the speed of a point on the Jupiter's equator is

$$|\vec{v}| = r|\vec{\omega}| = (71{,}500\ \cancel{\text{km}})\left(\frac{2\pi\ \text{rad}}{9.92\ \cancel{\text{h}}}\right)\left(\frac{1\ \cancel{\text{h}}}{3600\ \text{s}}\right)\left(\frac{1000\ \text{m}}{1\ \cancel{\text{km}}}\right) = 12{,}600\ \frac{\text{m}}{\text{s}}.$$