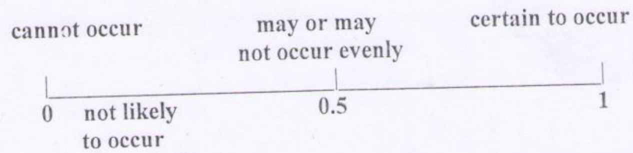
Introduction to Probability

Probabilities are associated with experiments where the outcome is not known in advance or cannot be predicted. For example, if you toss a coin, will you obtain a head or tail? If you roll a die will obtain 1, 2, 3, 4, 5 or 6? Probability measures and quantifies "how likely" an event, related to these types of experiment, will happen. The value of a probability is a number between 0 and 1 inclusive. An event that cannot occur has a probability (of happening) equal to 0 and the probability of an event that is certain to occur has a probability equal to 1.(see probability scale below).



In order to quantify probabilities, we need to define the **sample space** of an experiment and the **events** that may be associated with that experiment.

1. Sample Space and Events

The sample space is the set of all possible outcomes in an experiment.

Example 1: If a die is rolled, the sample space S is given by
S = {1, 2, 3, 4, 5, 6}

Example 2: If two coins are tossed, the sample space S is given by
S = {HH, HT, TH, TT}, where H = head and T = tail.

Example 3: Toss coins three times and observes the sequence of head (H) and tail (T) that appears in the sample space S is given by
S = {HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

Example 4: If two dice are rolled, the sample space S is given by

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

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*** Event: -** Is a subset of the sample space that is a some specific outcome of an experiment. S= {H, T}, E_1= H, E_2= T

Example 1: A die is rolled (see example 1 above for the sample space). Let us define event E as the set of possible outcomes where the number on the face of the die is even. Event E is given by E = {2, 4, 6}

Example 2: Two coins are tossed (see example 2 above for the sample space). Let us define event E as the set of possible outcomes where the number of head obtained is equal to two. Event E is given by E = {(HT), (TH)}

Example 3: Two dice are rolled (see example 3 above for the sample space). Let us define event E as the set of possible outcomes where the sum of the numbers on the faces of the two dice is equal to four. Event E is given by E = {(1, 3), (2, 2), (3, 1)}

* Axioms of probability

1- For any event A we have $p(A) \geq 0$

2- For any certain event S we have P(S) = 1

3- For any two disjoint event A and B we have $p(A \cup B) = p(A) + p(B)$

4- For any infinite sequence of mutually disjoint A_1, A_2... A_n we have

$$p(A_1 \cup A_1 \cup ... An) = p(A_1) + p(A_2) + ... + p(A_n)$$

* Theorem on probability space

1- The impossible event or the other word in set is empty ϕ has probability zero that is

$$P(\phi) = 0.$$

Proof: -

for any event A, we have $A \cup \phi = A$

$$p(A \cup \phi) = p(A) + p(\phi)$$

disjoint

$$p(A \cup \phi) = p(A)$$

$$p(A) = p(A) + p(\phi)$$

$$p(A) - p(A) = p(\phi) = 0 = p(\phi)$$

we subsitutite $p(A \cup \phi) = p(A)$

2- for any event A, we have $p(A') = 1 - p(A)$

Proof:-

$$S = A \cup A'$$

$$p(S) = p(A \cup A')$$

$$= p(A) + p(A')$$

$$p(S) = 1 ....from\ Axioms\ 2$$

$$1 = p(A) + p(A')$$

$$p(A') = 1 - p(A)$$

3- for any event A, we have $0 \le p(A) \le 1$

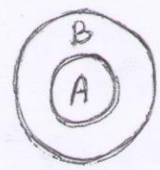
Proof: - from Axiom 1, $0 \le p(A)$, $p(A) \ge 0$ then we need only show that $p(A) = S$

* Since $S = A \cup A'$, when $A\, and\, A'$ are disjoint we get

$1 = p(A \cup A') = p(A) + p(A')$

$\because p(A') = 1 - p(A)$ and since $p(A') \ge 0$

$\because p(A) \le 1,\ 0 \le p(A) \le 1$



4- If $A \subseteq B, then\ p(A) \le p(B)$

Proof:-

Let we have event A and B\A

Where A and B\A disjoint

$$B = A \cup (B \setminus A)$$
$$p(B) = p(A \cup B \setminus A)$$
$$= p(A) + p(B \setminus A)$$

From Axioms 1 we have $p(B \setminus A) \ge 0$, Hence we have $p(A) \le p(B)$

5- For any event A and B we have $p(A \setminus B) = p(A) - p(A \cap B)$

Proof: -

Let we have event $p(A \setminus B)$ and $A \cap B$ where $(A \setminus B)\, and\, A \cap B\, is\, disjiont$

$$A = (A \setminus B) \cup (A \cap B)$$
$$p(A) = p[(A \setminus B) \cup (A \cap B)]$$
$$p(A) = p[(A \setminus B) + (A \cap B)]$$
$$p(A \setminus B) = p(A) - p(A \cap B)$$

Theorem (Addition Rule)

For any two event A and B, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Proof:-

Let we have two event B and $(A \setminus B)$

Hence
$$(A \cup B) = (A \setminus B) \cup B$$
$$p(A \cup B) = p(A \setminus B) + p(B)$$
$$p(A \setminus B) = p(A) - p(A \cap B) \text{ from theorem 5}$$
$$p(A \cup B) = p(A) - p(A \cap B) + p(B)$$
$$= p(A) + p(B) - p(A \cap B)$$

Corollary: - From any events A, B, C we have

$p(A \cup B \cup C) = p(A) + p(B) + p(C) - p(A \cap B) - p(A \cap C) - p(A \cap B \cap C)$

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Example1:- Let $p(A)=0.5, p(B)=0.5, p(A\cap B)=0.25, \textit{find } p(A\cup B)$

$p(A\cup B)=p(A)-p(A\cap B)+p(B)=0.5+0.5-0.25=0.75$

Example2:- Let the sample space $S=A\cup B$ and let

$p(A)=0.6 \textit{ and } p(B)=0.7 \textit{ find } p(A\cap B)$

$\because S=A\cup B$

$p(S)=p(A\cup B)$

$1=p(A\cup B)$

$p(A\cup B)=p(A)+p(B)-p(A\cap B)$

$1=0.6+0.7-p(A\cap B)$

$p(A\cap B)=1.3-1=0.3$

How to Calculate Probabilities?

1 - Classical Probability Formula: It is based on the fact that all outcomes are equally likely.

$$P(E)=\frac{\textit{Total number of outcomes in } E}{\textit{Total number of outcomes in the sample space}}$$

Example 1: A die is rolled; find the probability of getting a 3.
The event of interest is "getting a 3". So E = {3}.
The sample space S is given by S = {1, 2, 3, 4, 5, 6}.
The number of possible outcomes in E is 1 and the number of possible outcomes in S is 6. Hence the probability of getting a 3 is P ("3") = 1 / 6.

Example 2: A die is rolled; find the probability of getting an even number.
The event of interest is "getting an even number". So E = {2, 4, 6}, the even numbers on a die.
The sample space S is given by S = {1, 2, 3, 4, 5, 6}.
The number of possible outcomes in E is 3 and the number of possible outcomes in S is 6. Hence the probability of getting a 3 is P ("3") = 3 / 6 = 1 / 2.

Example 3: A card is selected at random from 52 playing card A= {heart}, B= {face card}, find p(A), p(B), $p(A\cap B)$ and $p(A\cup B)$

Sol.

P (A) = 13/52, p (B)=12/52, $p(A\cap B)$=p(A).p(B)=13/52*12/52=3/52,

$p(A\cup B)$= p (A) + p (B) - $p(A\cap B)$=13/52+12/52-3/52=11/26

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2 - Empirical Probability Formula: It uses real data on present situations to determine how likely outcomes will occur in the future. Let us clarify this using an example 30 people were asked about the colors they like and here are the results:

Color	frequency
Red	10
Blue	15
Green	5

If a person is selected at random from the above group of 30, what is the probability that this person likes the red color? Let event E be "likes the red color". Hence

$$P(E) = \frac{Frequency\ for\ red\ color}{Total\ frequency\ in\ the\ above\ table} = 10 / 30 = 1 / 3$$

Example1: The table below shows students distribution per grade in a school.

Grade	Frequency
1	50
2	30
3	40
4	42
5	38
6	50

If a student is selected at random from this school, what is the probability that this student is in grade 3? Let event E be "student from grade 3". Hence

$$P(E) = \frac{Frequency\ for\ grade\ 3}{Total\ frequencies} = \frac{40}{250} = 0.6$$

of the experiment in question and E is the event of interest. n (S) is the number of elements in the sample space S and n (E) is the number of elements in the event E.

Example2:- Let three coins toss one time and observes the number of heads observed the sample space of S is (0, 1, 2, 3}. Find the probability of appeared the head.

S = {HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

Sol:-

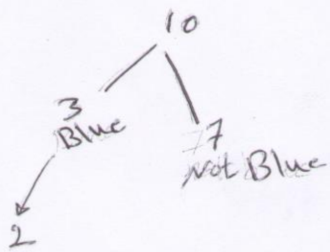
P (0) =1\8, P (1)=3\8, P (2)=3\8, P (0)=1\8

$$\sum_{i=1}^{n} pi = 1$$ i.e. 1\8+3\8+3\8+1\8=1

Example3:- of 10 girls is in a class 3 have blue eyes, two of girls chosen at random. Find: - **a-** both have blue eyes **b-** no one has blue eyes **c-** at least one has blue eyes **d-** Exactly one is blue eyes.

Sol:-

$$C_2^{10} = \frac{10!}{2!(10-2)!} = \frac{10*9*8!}{2!*8!}$$



$$a- \frac{C_2^3 * C_0^7}{C_2^{10}} = \frac{\frac{3!}{2!(3-2)!} * \frac{7!}{0!(7-0)!}}{\frac{10!}{2!(10-2)!}!} = \frac{2}{45}$$

$$b- \frac{C_0^3 * C_2^7}{C_2^{10}} = \frac{1 * \frac{7!}{2!(7-2)!}}{\frac{10!}{2!(10-2)!}!} = \frac{21}{45}$$

$$c- \frac{C_1^3 * C_1^7}{C_2^{10}} + \frac{C_2^3 * C_0^7}{C_2^{10}} = ?$$

$d- C_1^3 . C_1^7 / C_2^{10} = ?$

Conditional Prob. and independent:-

$$p(A \cap B) = p(A) * p(B) \rightarrow indep.$$

$$p(A \setminus B) = \frac{p(A \cap B)}{p(B)}; p(B) > 0$$

$$p(B \setminus A) = \frac{p(A \cap B)}{p(A)}; p(A) > 0$$

$$p(A \cap B) = p(A \setminus B) * p(B)$$

$$p(A \cap B) = p(B \setminus A) * p(A)$$

Example 1:- Let a pair of dice be tossed if the sum is (6). Find the prob. that dice is 6.

If E= {(2, 4), (4, 2), (1, 5), (5, 1), (3, 3)} If the sum of pairs are even no.

A= {(2, 4), (4, 2)}

E $\cap$ A=2

$p(A)$=2

$p(E) = 5 \Rightarrow p(A \cap E) = \frac{2}{5}$

Example 2:- If a certain college 25% of the students failed in Math. ; 15% of the students failed in Chem. and 10% of the students failed both Math. and Chem.

A student is selected at random.

1- If they failed Chem. what is that failed in Math.?

2- If they failed Math. What is that failed in Chem.?
3- What is the prob. that failed Chem. and Math.?
Let M= {student who failed Math.}
C= {student who failed Chem.}

25% M   15% Ch   10% M&Ch

Sol.

$$P(M) = \frac{25}{100} = 0.25$$

$$P(C) = \frac{15}{100} = 0.15$$

$$P(M \cap C) = \frac{10}{100} = 0.10$$

1- $p(C \setminus M) = \frac{p(C \cap M)}{p(M)} = \frac{0.10}{0.25}$

2- $p(M \setminus C) = \frac{p(M \cap C)}{p(C)} = \frac{0.10}{0.15}$

3- $p(M \cup C) = P(M) + P(C) - p(M \cap C)$

Three conditional events or more:-

$$p(A_1, A_2, A_3, \ldots A_n) = p(A_1).p(A_2 \setminus A_1).(A_3 \setminus A_2 \cap A_1)......(A_n \setminus A_1 \cap A_2........A_{n-1})$$

Example 3:- Contain (12) items of which (4) are defect. Three items are draw at random from the box one after other. Find the prob. that all three non- defective.
Sol:-

$$P(A_1) = \frac{8}{12}$$

$$P(A_2) = \frac{7}{11}$$

$$P(A_3) = \frac{6}{10}$$

$$p = P(A_1).P(A_2).P(A_3) = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10}$$

12
4 D   8 ND
3

Example 4:- Four cards are to be drawn successively at a random and without replacement, what is the prob. of receiving a spade; heart; diamond and club.
Sol:-

$$p(ABCD) = P(A)..P(B \setminus A)..P(C \setminus A \cap B).P(D \setminus A \cap B \cap C)$$

$$= \frac{13}{52} * \frac{13}{51} * \frac{13}{50} * \frac{13}{49}$$

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Example 5:- A class has (12) boys and (4) girls if three student are selected at random from a class. What is the prob. that they are all boys?

Sol:-

$$p = \frac{12}{16} * \frac{11}{15} * \frac{10}{14}$$

12 4
B G
3

Example 6:- Consider two urns, Urn 1 contain (5) white and (7) red balls. Urn2 contain (6) white and (4) red balls. One of the Urns is selected at random and a ball is drawn from it. Find the prob. that the ball drawn be white.

Let A1= Urn 1 is chosen

A2= Urn 2 is chosen

B= white ball is drawn

Sol:-

$$P(B) = P(B \setminus A_1)P(A_1) + .P(B \setminus A_2).P(A_2)$$

$$P(A_1) = P(A_2) = \frac{1}{2}$$

$$P(W) = \frac{5}{12}.\frac{1}{2} + \frac{6}{10}.\frac{1}{2} = \frac{61}{120} = 0.51$$

$\boxed{5W \; 7R}$ $\boxed{6W \; 4R}$

Urn I Urn II

white

$\frac{5}{12}.\frac{1}{2} + \frac{\cancel{6}^{\,3}}{10}.\frac{1}{2}$

More question with solutions for prob.

<u>**Question 1:**</u> A die is rolled, find the probability that an even number is obtained.

<u>**Solution to Question 1:**</u>

- Let us first write the <u>sample space</u> S of the experiment.
  S = {1,2,3,4,5,6}
- Let E be the <u>event</u> "an even number is obtained" and write it down.
  E = {2,4,6}
- We now use the formula of the <u>classical</u> probability.
  P(E) = n(E) / n(S) = 3 / 6 = 1 / 2

<u>**Question 2:**</u> Two coins are tossed, find the probability that two heads are obtained.

Note: Each coin has two possible outcomes H (heads) and T (Tails).

<u>**Solution to Question 2:**</u>

- The sample space S is given by.
  S = {(H,T),(H,H),(T,H),(T,T)}
- Let E be the event "two heads are obtained".
  E = {(H,H)}
- We use the formula of the classical probability.
  P(E) = n(E) / n(S) = 1 / 4

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Question 3: Which of these numbers cannot be a probability?

a) -0.00001
b) 0.5
c) 1.001
d) 0
e) 1
f) 20%

Solution to Question 3:
A probability is always greater than or equal to 0 and less than or equal to 1, hence only **a)** and **c)** above cannot represent probabilities: -0.00010 is less than 0 and 1.001 is greater than 1.

Question 4: Two dice are rolled, find the probability that the sum is

a) equal to 1
b) equal to 4
c) less than 13

Solution to Question 4:

- a) The sample space S of two dice is shown below.

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

Let E be the event "sum equal to 1". There are no outcomes which correspond to a sum equal to 1, hence

P(E) = n(E) / n(S) = 0 / 36 = 0

b) Three possible outcomes give a sum equal to 4: E = {(1,3),(2,2),(3,1)}, hence.

P (E) = n(E) / n(S) = 3 / 36 = 1 / 12

c) All possible outcomes, E = S, give a sum less than 13, hence.

P(E) = n(E) / n(S) = 36 / 36 = 1

Question 5: A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution to Question 5:

- The sample space S of the experiment described in question 5 is as follows

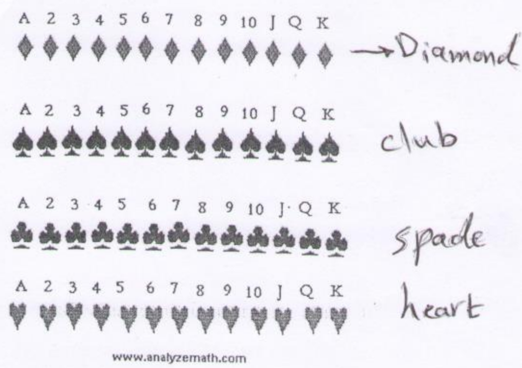
S = { (1,H),(2,H),(3,H),(4,H),(5,H),(6,H)
(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)}

Let E be the event "the die shows an odd number and the coin shows a head".
Event E may be described as follows E={(1,H),(3,H),(5,H)}

- The probability P(E) is given by P(E) = n(E) / n(S) = 3 / 12 = 1 / 4

<u>Question 6:</u> A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

<u>Solution to Question 6:</u> The sample space S of the experiment in question 6 is shown below



$\frac{1}{13} \cdot \frac{13}{52} = \frac{1}{52}$

- Let E be the event "getting the 3 of diamond". An examination of the sample space shows that there is one "3 of diamond" so that n (E) = 1 and n(S) = 52. Hence the probability of event E occurring is given by
  P(E) = 1 / 52

<u>Question 7:</u> A card is drawn at random from a deck of cards. Find the probability of getting a queen.

<u>Solution to Question 7:</u>

- The sample space S of the experiment in question 7 is shown above (see question 6)
- Let E be the event "getting a Queen". An examination of the sample space shows that there are 4 "Queens" so that n (E) = 4 and n(S) = 52. Hence the probability of event E occurring is given by
  P(E) = 4 / 52 = 1 / 13

<u>Question 8:</u> A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

<u>Solution to Question 8:</u>

- We first construct a table of frequencies that gives the marbles color distributions as follows

Color	Frequency
red	3
green	7
white	10

- We now use the <u>empirical</u> formula of the probability

$$P(E) = \frac{Frequency\ for\ white\ color}{Total\ frequencies\ in\ the\ above\ table} = 10 / 20 = 1 / 2$$

Question 9: The blood groups of 200 people are distributed as follows: 50 have type **A** blood, 65 have **B** blood type, 70 have **O** blood type and 15 have type **AB** blood. If a person from this group is selected at random, what is the probability that this person has O blood type?

Solution to Question 9:

- We construct a table of frequencies for the blood groups as follows

group	frequency
A	50
B	65
O	70
AB	15

- We use the empirical formula of the probability

$$p(E) = \frac{Frequency\ for\ O\ blood}{Total\ for\ frequencies} = 70 / 200 = 0.35$$

Exercises:

a) A die is rolled; find the probability that the number obtained is greater than 4.
b) Two coins are tossed, find the probability that one head only is obtained.
c) Two dice are rolled; find the probability that the sum is equal to 5.
d) A card is drawn at random from a deck of cards. Find the probability of getting the King of heart.

Answers to above exercises:

a) 2 / 6 = 1 / 3
b) 2 / 4 = 1 / 2
c) 4 / 36 = 1 / 9
d) 1 / 52