Rigid Body

ROTATION

The Motion of Rigid Bodies

Rigid bodies

A rigid body is defined as a body of which the distance between any two particles is not changed under force or torque.

The classification of the motion of rigid bodies

(i) Translational motion: The motion in which any straight line in a rigid body is always parallel;

(ii) Rotation about a fixed axis: The motion in which any particle in a body is in a circular motion of a radius about an identical straight line keeping constant position;

Kinetic Energy of Rotation

Kinetic energy of rotation

The kinetic energy of a rigid body is

$$K = \sum \frac{1}{2} m_i v_i^{\,2} \qquad = \sum \frac{1}{2} m_i (r_i^{\,2} \omega^2)$$

$$= \frac{1}{2}\left(\sum m_i r_i^{\,2}\right)\omega^2 \qquad = \frac{1}{2} I \omega^2$$

Rotational inertia:

$$\boxed{I = \sum m_i r_i^{\,2}}$$

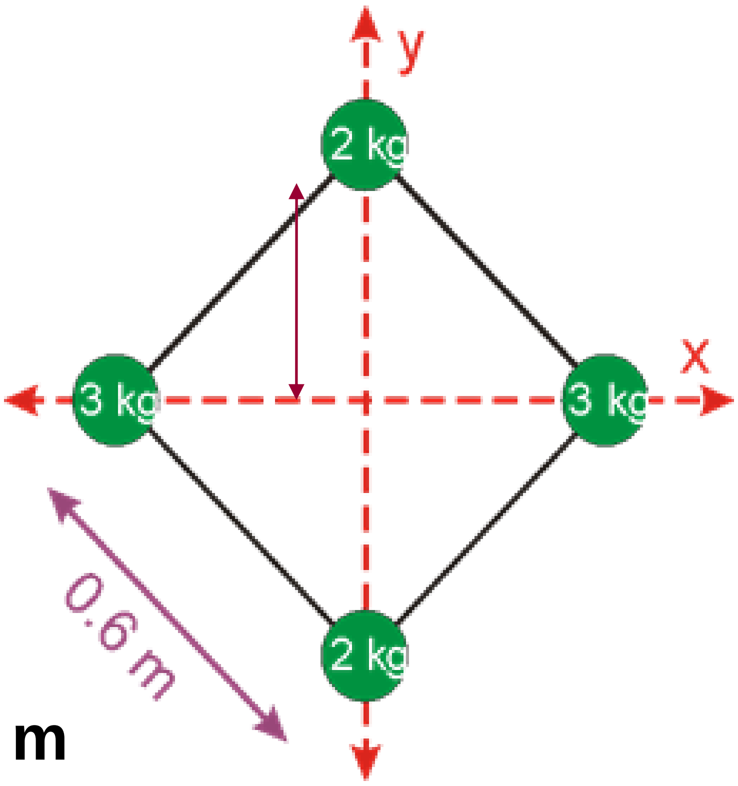
Example

- What is the moment of inertia of the following point masses arranged in a square?

a) about the x-axis?

b) about the y-axis?

c) about the z-axis?



Given: M2=2, M3=3 kg, L=0.6 m

a) Find I about the x-axis?

Basic formula

$$I = \sum m_i r_i^2$$

First, find distance to 2-kg masses $r = 0.6 \cdot \sin(45^\circ)$

$$I = M_2 r^2 + M_2 r^2 \quad = 0.72\ \mathrm{kg \cdot m^2}$$

b) Find I about the y-axis?

Same as before, except you use the 3-kg masses

$$I = M_3 r^2 + M_3 r^2 \quad = 1.08\ \mathrm{kg \cdot m^2}$$

c) Find I about the z-axis?

Use all the masses

$$I = M_2 r^2 + M_2 r^2 + M_3 r^2 + M_3 r^2 \quad = 1.8\ \mathrm{kg \cdot m^2}$$

Moment of Inertia

- What is the moment of inertia of an extended object (Many particles)

$$I = \sum m_i r_i^{\,2}$$

  - Break it up into little pieces (***dm*** Or ***dr***)

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^{\,2} \Delta m_i$$

$$= \int_V r^2 dm$$

$$= \int_V r^2 \rho(r) dV$$

Calculating the rotational inertia

For a collection of point masses :

$$I = \sum m_i r_i^2$$

For continuously distributed mass:

$$I = \int \rho r^2 dV$$

It is related to the mass distribution about an axis.

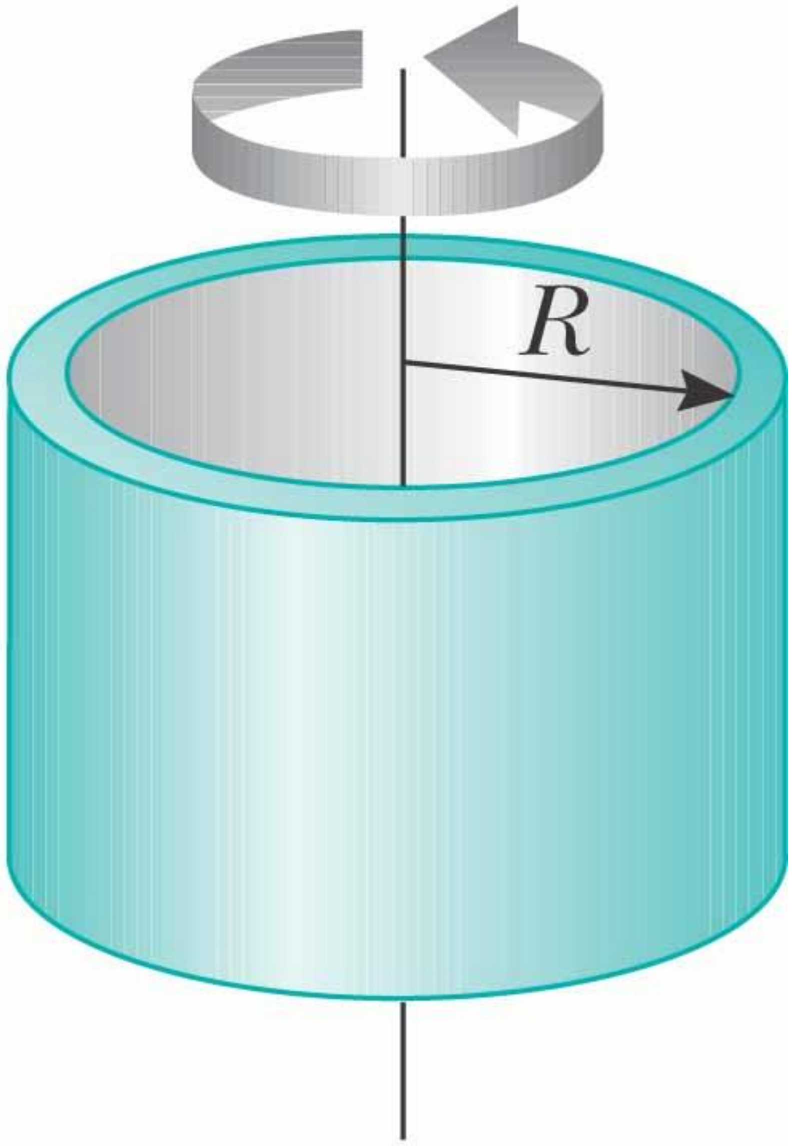
Moments of Inertia

- Moment of inertia of a hoop or a thin cylinder
  - All the mass is at the same R!

$$I = \int_V r^2 dm$$

$$= R^2 \int_V dm$$

$$= MR^2$$



Moments of Inertia

The volume element here is dV = 2πLr dr

And the mass element dm = ρ dV = 2πρLr dr

- Moment of inertia of a disk or solid cylinder
  - Consider ring at *r*, with volume *2πrLdr*

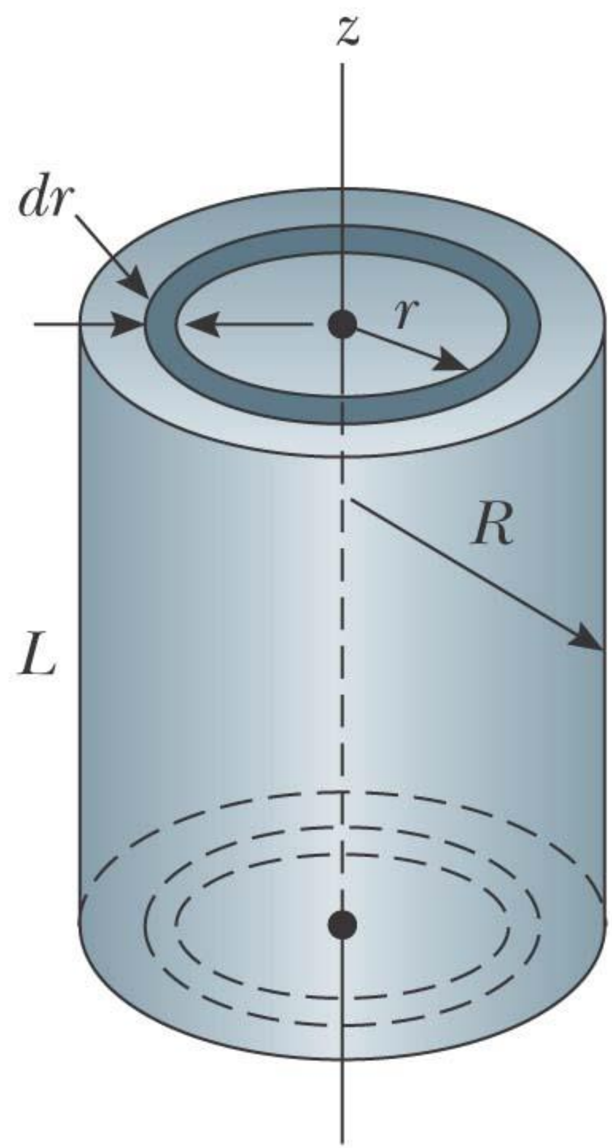
$$I = \int_V r^2 dm = \int_V r^2 \rho dV$$

$$= \rho \int_0^R r^2 \left(2\pi r L dr\right) = 2\pi\rho L \int_0^R r^3 dr$$

$$= 2\pi\rho L \tfrac{1}{4} r^4 \Big|_0^R = \tfrac{1}{2}\pi\rho L R^4$$

$$= \tfrac{1}{2}\left(\rho\pi R^2 L\right) R^2$$

$$= \tfrac{1}{2} M R^2$$



Moments of Inertia

The linear density, (λ), indicates the amount of a quantity, indicated by m, per unit length along a single dimension.

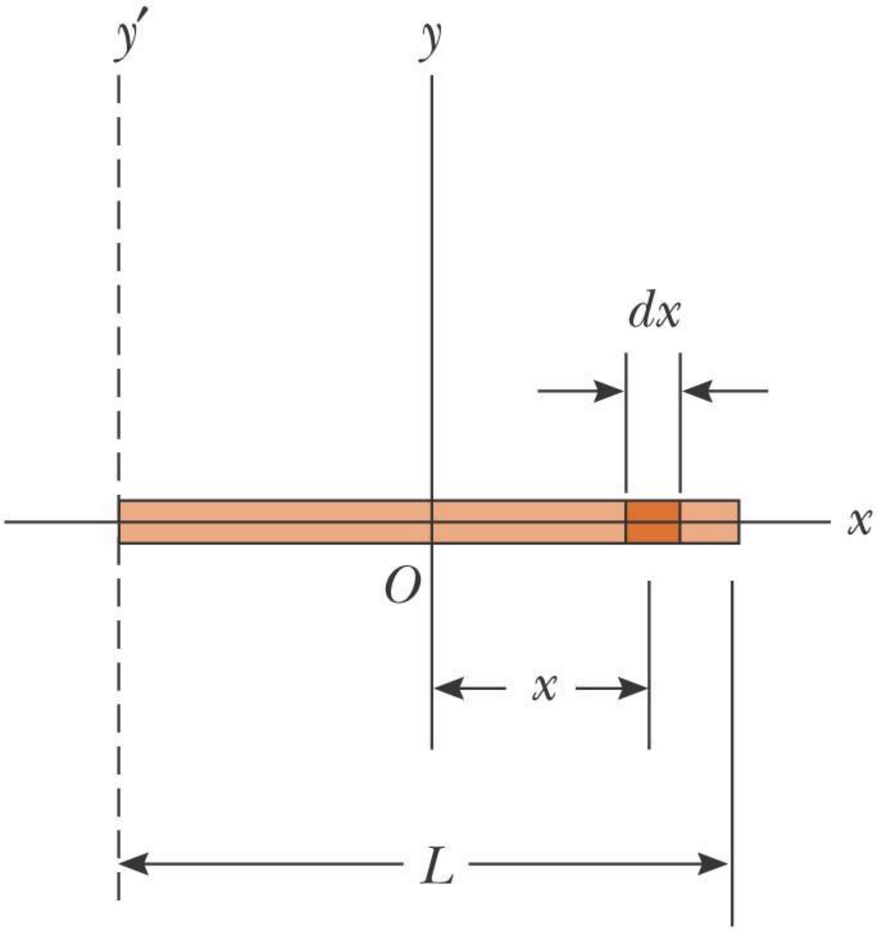
- Moment of inertia of a thin rod about CM

$$I = \int_V r^2 dm = \lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$$

$$= \lambda \tfrac{1}{3} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \tfrac{\lambda}{3}\left[\left(\tfrac{L}{2}\right)^3 - \left(\tfrac{-L}{2}\right)^3\right]$$

$$= \tfrac{\lambda}{12} L^3 = \tfrac{1}{12}(\lambda L)L^2$$

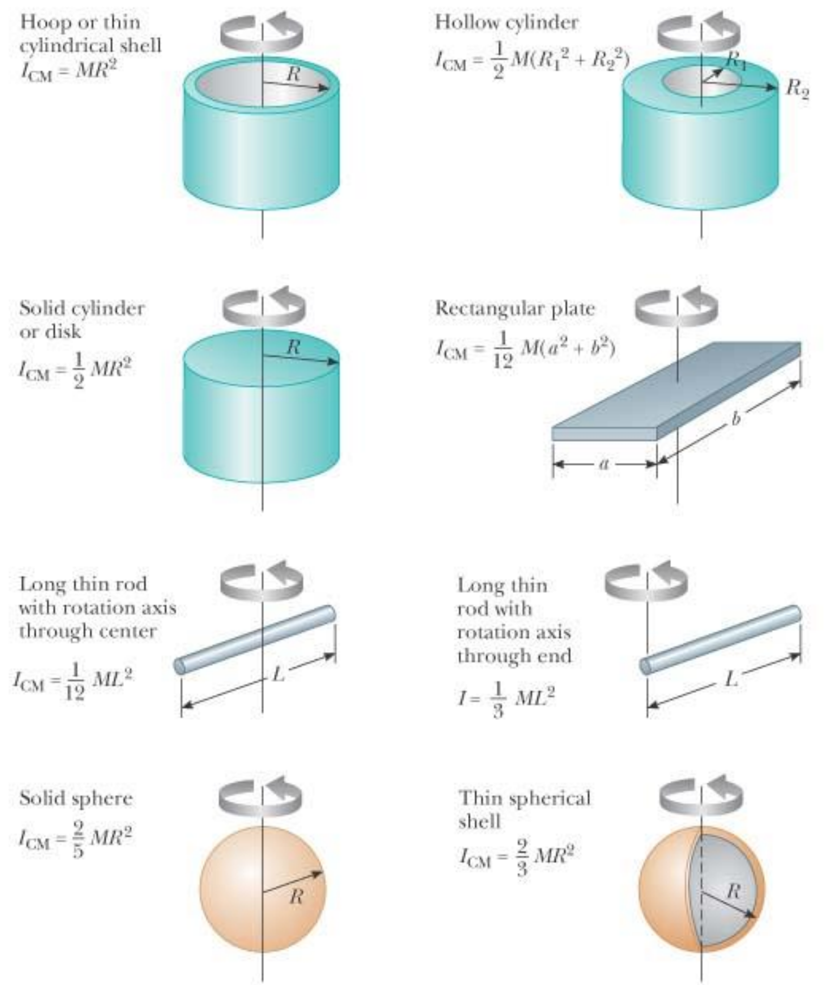
$$= \tfrac{1}{12} ML^2$$



Moments of Inertia

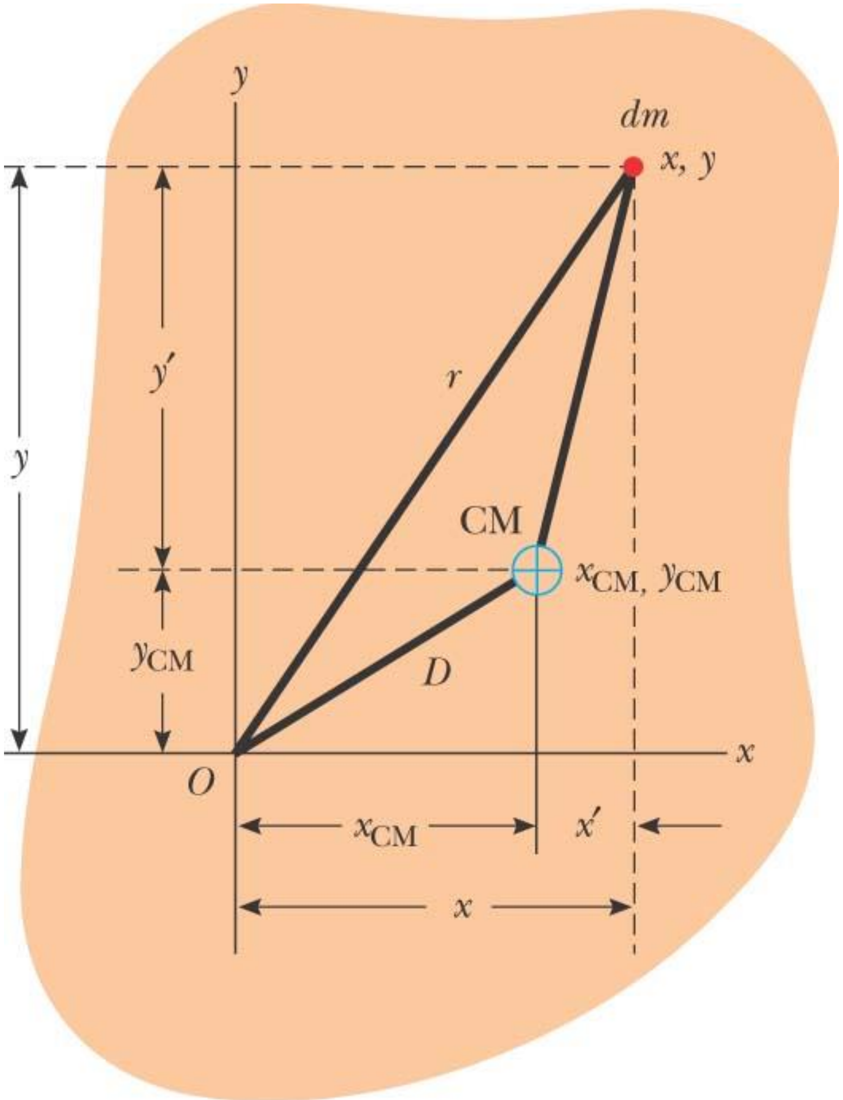
Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries



Moments of Inertia

- What about an arbitrary axis?
  – Use “Parallel Axis Theorem”
    $I = I_{CM} + MD^2$
  – Moment of inertia about any axis is just moment of inertia about center of mass plus moment of inertia of “CM” about the axis



(a)

Parallel Axis Theorem

$$I = I_{CM} + mx^2$$

Where;

I = moment of inertia about a parallel axis,

I_{CM} = moment of inertia about the center of mass,

m = mass of object (or segment), and

x = distance from the center of mass and the center of rotation

Parallel-axis theorem:

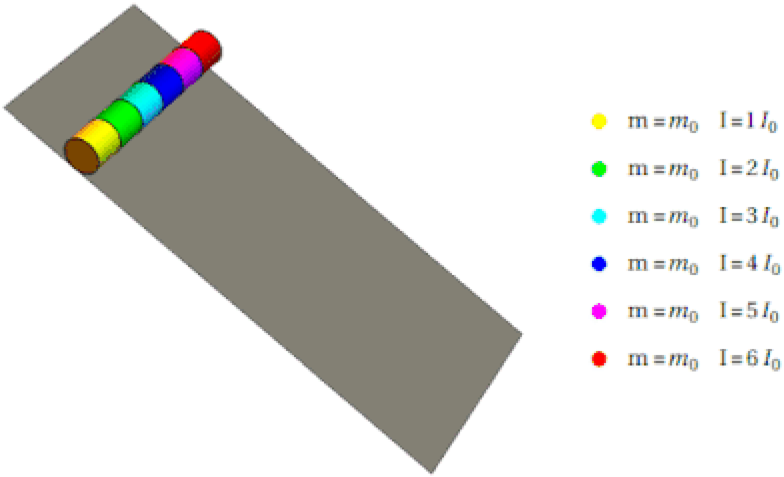
$$I = I_{\mathrm{CM}} + Md^2$$

Normal-axis theorem:

$$I_z = I_x + I_y$$

What are the Factors on which Moment of Inertia Depends?

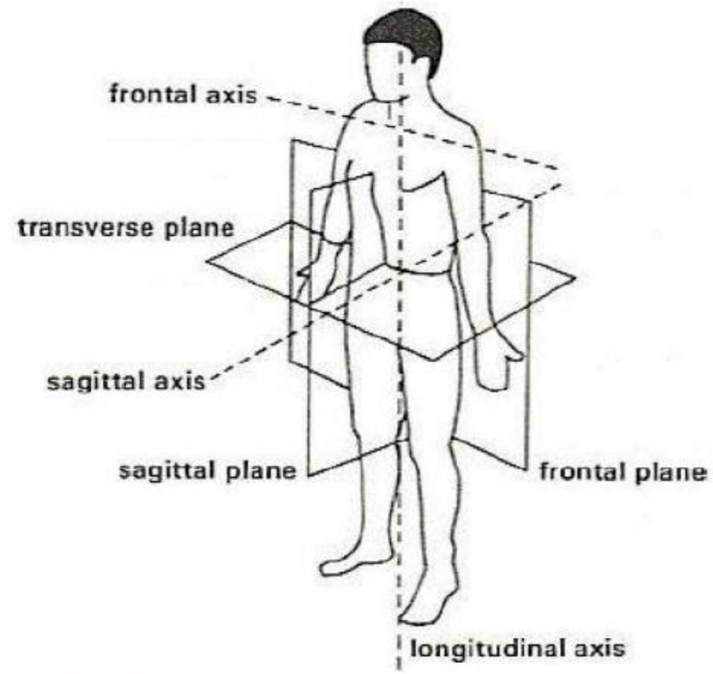
- The density of the material
- Shape and size of the body
- Axis of rotation (distribution of mass relative to the axis)

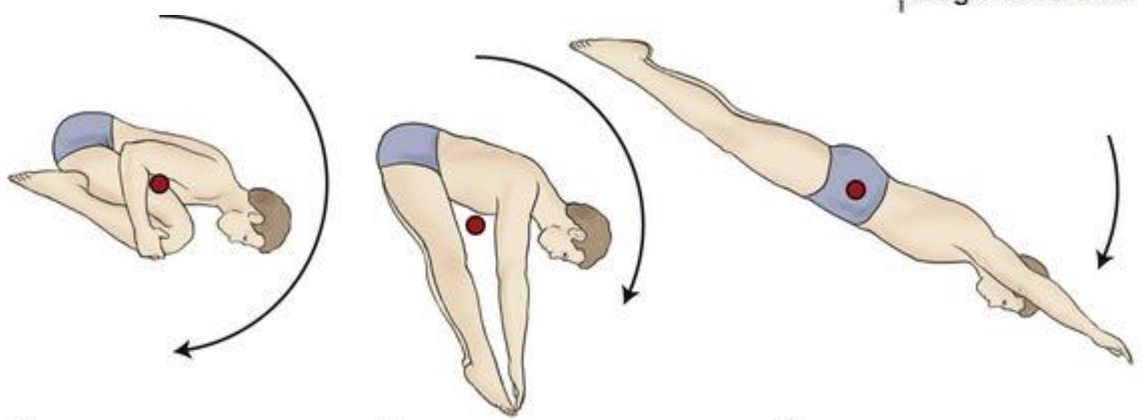


- The cylinders with higher moment of inertia roll down a slope with a smaller acceleration, as more of their potential energy needs to be converted into the rotational kinetic energy.

Human body

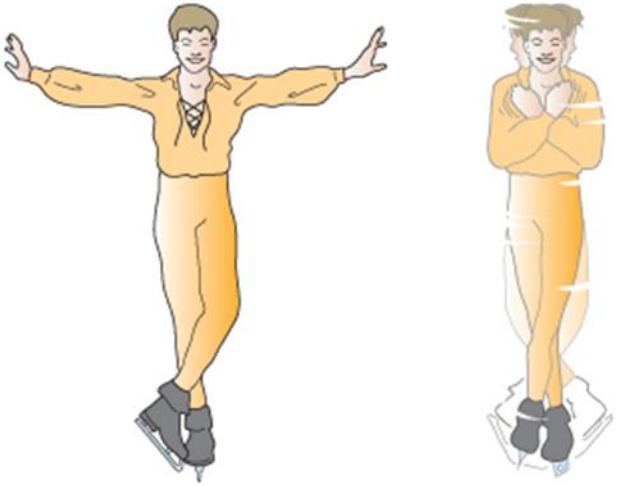
- Human body planes and axes





Conceptual Example 14 A Spinning Skater

An ice skater is spinning with both arms and a leg outstretched. He pulls his arms and leg inward and his spinning motion changes dramatically.



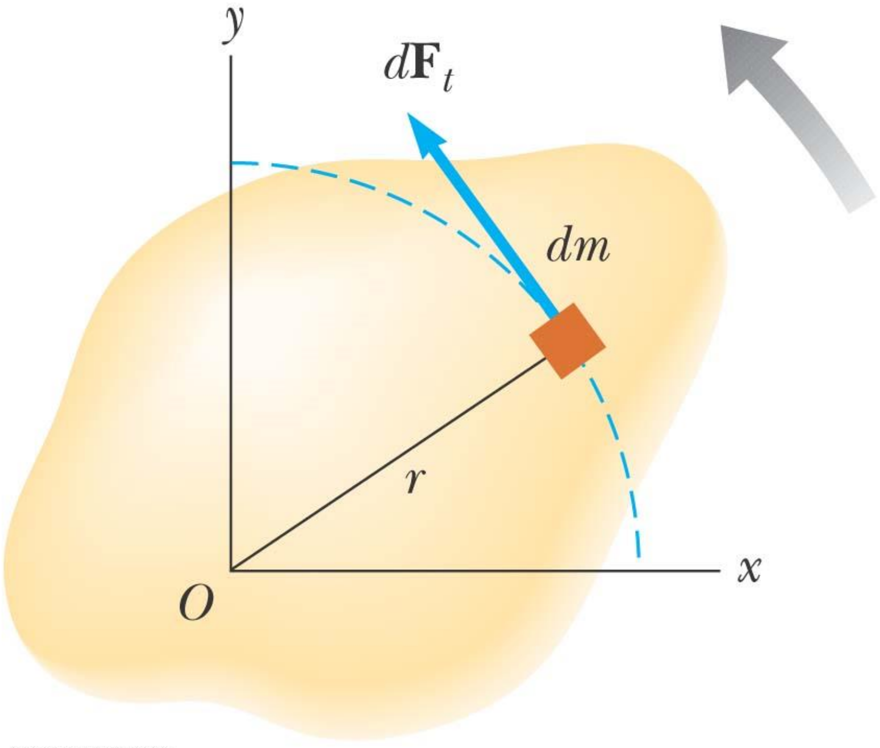
Use the principle of conservation of angular momentum to explain how and why his spinning motion changes.

Torque and Angular Acceleration

- What about for extended object?

$$\sum_i \tau_i = \sum_i m r_i^2 \alpha_i$$
$$= \alpha \sum_i m r_i^2$$
$$= I\alpha$$

  - *Net* torque gives rise to angular acceleration



Work and Power

- Work and power for a rotating object

$$dW = Fdx = Fds$$
$$= F\sin\phi \cdot rd\theta$$
$$= \tau d\theta$$

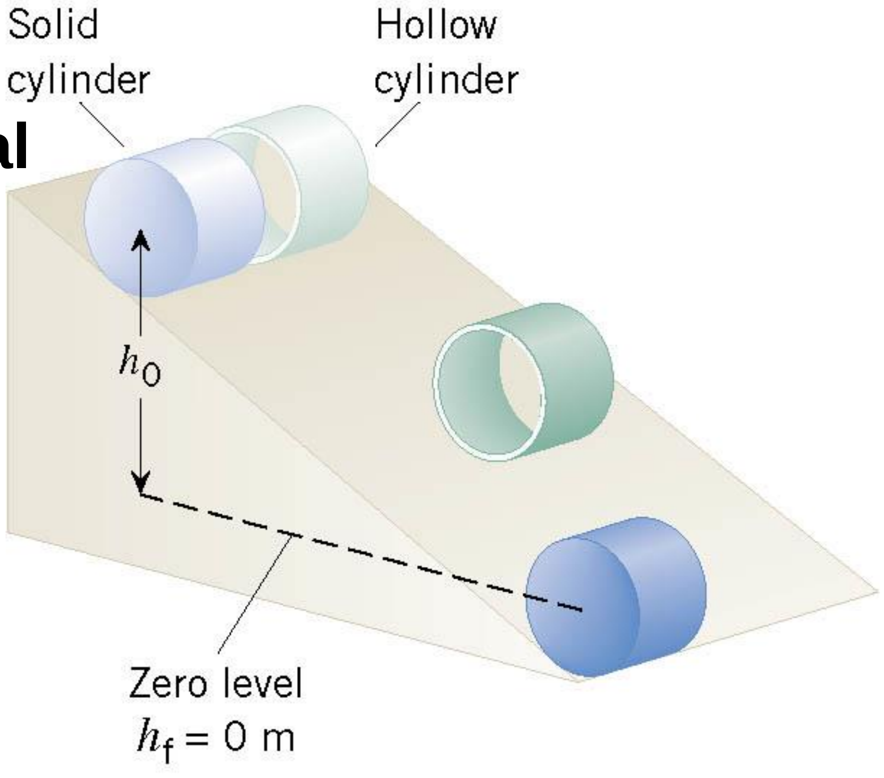
$$P = \frac{dW}{dt} = \tau\omega$$

$$W = \int_a^b Pdt = \int_a^b \tau\omega dt = \int_a^b I\alpha\omega dt$$
$$= I\int_a^b \omega \tfrac{d\omega}{dt} dt = \tfrac{1}{2} I \int_a^b \tfrac{d\omega^2}{dt} dt$$
$$= \tfrac{1}{2} I\left(\omega_b^2 - \omega_a^2\right)$$

Example 13 Rolling Cylinders

A thin-walled hollow cylinder (mass = m_h, radius = r_h) and a solid cylinder (mass = m_s, radius = r_s) start from rest at the top of an incline.

Determine which cylinder has the greatest translational speed upon reaching the bottom.



Rotational Work and Energy

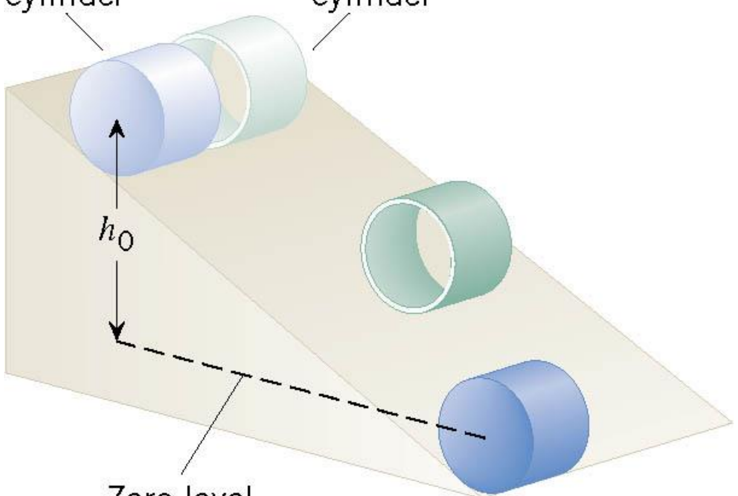
$$E = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2 + mgh$$

ENERGY CONSERVATION

$$\tfrac{1}{2} m v_f^2 + \tfrac{1}{2} I \omega_f^2 + mgh_f = \tfrac{1}{2} m v_i^2 + \tfrac{1}{2} I \omega_i^2 + mgh_o$$

$$\tfrac{1}{2} m v_f^2 + \tfrac{1}{2} I \omega_f^2 = mgh_o$$

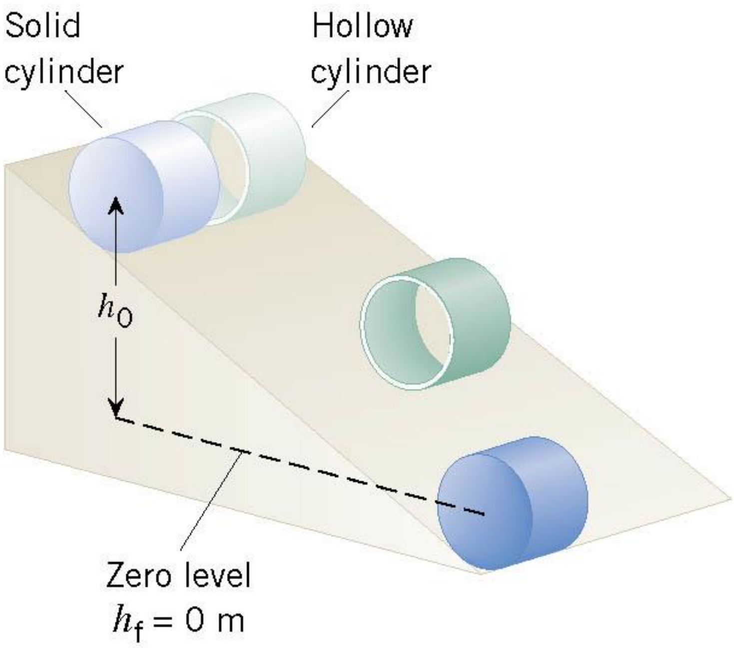
$$\omega_f = v_f / r$$



Rotational Work and Energy

$$\tfrac{1}{2} m v_f^2 + \tfrac{1}{2} I\, v_f^2 \big/ r^2 = mgh_o$$

$$v_f = \sqrt{\frac{2mgh_o}{m + I\big/ r^2}}$$



The cylinder with the smaller moment of inertia will have a greater final translational speed.