Naive Set Theory Refresher

Lecture 2, No Associated Chapter

John Connor

Why Sets?

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This directory can be seen as containing a **set** of files, and the set is displayed by the `ls` command.

The `rm` command can be seen an operation which removes elements from the set. In the example, we removed all of the files that begin with the letter "b".

Definition: Set

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But what's a collection and what is an element?

Set: Definition by Example

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1. The empty set $\emptyset$ is a collection of no elements.
2. The set containing only the elements 1, 2, and 3 can be written as $\{1, 2, 3\}$.
3. The order of elements in a set does not matter; i.e. $\{1, 2, \{3, 4\}\} = \{2, 1, \{3, 4\}\} = \{2, 1, \{4, 3\}\}$ but $\{1, 2, \{3, 4\}\} \neq \{1, 3, \{2, 4\}\}$.

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Note: The emptyset is a subset of *every* set!

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This can also be stated using subsets:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

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$$|\mathcal{P}(A)| = 2^{|A|}$$

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Making Sense of Set Difference

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rm: cannot remove "no-such-file.txt":
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We can give `rm` a set of files to remove, and it will remove all of the files in the set that actually appear in the directory.

This is like:

$$\{\text{foo.txt}, \text{bar.txt}\} \setminus \{\text{foo.txt}, \text{no-such-file.txt}\} = \{\text{bar.txt}\}$$

Definition: Tuple

Like a set, a tuple is a collection of elements, but unlike a set a tuple

1. is ordered: $(0, 1, 2, 3) \neq (1, 2, 3, 0)$.
2. can have "duplicate" elements: $(0, 0, 1, 2, 3) \neq (0, 1, 2, 3)$.

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Note: A tuple with exactly two elements is called a **pair**.

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$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$

Beyond Sets: Multisets and Lists

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