# 1 | WEAK FORM OF THE MACRO-SCALE BALANCE EQUATIONS

The balance equations in a weak form postulate that the external virtual power exerted by the set of external agencies must be equal to the internal virtual power of the stress-like variables, for all admissible variations in primal descriptors. In view of (**??**), we can write

$$\int_{\Omega} \left( \boldsymbol{\sigma} : \delta\dot{\boldsymbol{\varepsilon}} + \dot{\chi}\,\delta p - \mathcal{V}\cdot\delta\boldsymbol{\varphi} \right) d\Omega = \int_{\Omega} \mathbf{f}\cdot\delta\dot{\mathbf{u}}\,d\Omega + \int_{\Gamma_{\mathrm{N}}^{u}} \mathbf{t}\cdot\delta\dot{\mathbf{u}}\,d\Gamma - \int_{\Gamma_{\mathrm{N}}^{p}} \frac{q}{\rho^{f}}\,\delta p\,d\Gamma, \quad \forall\,\delta\dot{\mathbf{u}} \text{ and } \delta p \text{ admissible.} \tag{1}$$

The underlying admissibility requirements invoked in (1), for virtual variations $\delta\dot{\mathbf{u}}$ and $\delta p$, take into account proper regularity demands such that all the integral terms can be formally evaluated as well as homogeneous prescribed values for both continuous fields on $\Gamma_{\mathrm{N}}^{u}$ and $\Gamma_{\mathrm{N}}^{p}$ (i.e. where Dirichlet boundary conditions are specified for $\mathbf{u}$ and $p$, see (**??**) and (**??**)), respectively.

Since $\delta\dot{\mathbf{u}}$ and $\delta p$ are independent of each other, the variational form of equilibrium is finally described as a system of two coupled scalar equations[?,?,?]

$$\begin{aligned} G &\equiv \int_{\Omega} \boldsymbol{\sigma}:\delta\dot{\boldsymbol{\varepsilon}}\,d\Omega - \int_{\Omega} \mathbf{f}\cdot\delta\dot{\mathbf{u}}\,d\Omega - \int_{\Gamma_{\mathrm{N}}^{u}} \mathbf{t}\cdot\delta\dot{\mathbf{u}}\,d\Gamma = 0, \quad \forall\,\delta\dot{\mathbf{u}} \text{ admissible,} \\ H &\equiv \int_{\Omega} \left(\dot{\chi}\,\delta p - \mathcal{V}\cdot\delta\boldsymbol{\varphi}\right) d\Omega + \int_{\Gamma_{\mathrm{N}}^{p}} \frac{q}{\rho^{f}}\,\delta p\,d\Gamma = 0, \quad \forall\,\delta p \text{ admissible,} \end{aligned} \tag{2}$$

which is valid $\forall\,t$.

# 2 | SOLUTION OF THE VARIATIONAL EQUATIONS OF THE MACRO-SCALE

This section briefly describes the properties of the numerical model to be implemented at the macro-scale. Considering a finite element discretization h of the $\Omega$ domain, it is possible to build the interpolation functions of the displacement, $\mathbf{N}^{u}$, and pore pressure, $\mathbf{N}^{p}$, fields as follows

$$\begin{aligned} \mathbf{u} &= \mathbf{N}^{u}\,\bar{\mathbf{u}}, \\ p &= \mathbf{N}^{p}\,\bar{\mathbf{p}}, \end{aligned} \tag{3}$$

where $\bar{\mathbf{u}}$ and $\bar{\mathbf{p}}$ are the vectors of the nodal displacement and pore pressure (unknowns), respectively. In order to satisfy the Babuska-Brezzi convergence conditions[?], it is necessary to adopt shape functions of a different order. For the present isoparametric bi-quadratic, $\mathbf{N}^{u}$, and bi-linear, $\mathbf{N}^{p}$, elements are proposed, respectively. In turn, it can be written for virtual variations

$$\begin{aligned} \delta\dot{\mathbf{u}} &= \mathbf{N}^{u}\,\delta\dot{\bar{\mathbf{u}}}, \\ \delta p &= \mathbf{N}^{p}\,\delta\bar{\mathbf{p}}, \end{aligned} \tag{4}$$

$\delta\dot{\bar{\mathbf{u}}}$ and $\delta\bar{\mathbf{p}}$ being the vectors of the rate of the nodal virtual displacement and virtual pore pressure, respectively. Thus, replace the virtual variations with the interpolation functions, and considering that $\delta\dot{\bar{\mathbf{u}}}$ and $\delta\bar{\mathbf{p}}$ are arbitrary, the macro-scale integral variational equations Eqs. (2) can be rewritten as

$$\begin{aligned} G_h &\equiv \int_{\Omega_h} \mathbf{B}^{uT}\,\boldsymbol{\sigma}\,d\Omega - \int_{\Omega_h} \mathbf{N}^{uT}\,\mathbf{f}\,d\Omega - \int_{\Gamma_{\mathrm{N},h}^{u}} \mathbf{N}^{uT}\,\mathbf{t}\,d\Gamma = \mathbf{0}, \ \forall\,t, \\ H_h &\equiv \int_{\Omega_h} \mathbf{N}^{pT}\,\dot{\chi}\,d\Omega - \int_{\Omega_h} \mathbf{B}^{pT}\,\mathcal{V}\,d\Omega + \int_{\Gamma_{\mathrm{N},h}^{p}} \mathbf{N}^{pT}\,\frac{q}{\rho^{f}}\,d\Gamma = \mathbf{0}, \ \forall\,t, \end{aligned} \tag{5}$$

where $\Omega_h$ and $\Gamma_{\mathrm{N},h}^{(\bullet)}$ denote the domain and boundary of the discretized macro-scale, $\mathbf{B}^{u} = \boldsymbol{\nabla}^{sym}\,\mathbf{N}^{u}$ is the deformation-displacement matrix, and $\mathbf{B}^{p} = \boldsymbol{\nabla}\,\mathbf{N}^{p}$ is the matrix that relates pore pressures to their gradient. Meanwhile, $(\cdot)^{T}$ indicates the transpose of $(\cdot)$.

Transformando la Eq. ((5)) derivandola en el tiempo

$$\begin{aligned}
\dot{G}_h &\equiv \int_{\Omega_h} \mathbf{B}^{uT}\, \dot{\boldsymbol{\sigma}}\, d\Omega - \int_{\Omega_h} \mathbf{N}^{uT}\, \dot{\mathbf{f}}\, d\Omega - \int_{\Gamma^u_{\mathrm{N},h}} \mathbf{N}^{uT}\, \dot{\mathbf{t}}\, d\Gamma = \mathbf{0},\ \forall t, \\
H_h &\equiv \int_{\Omega_h} \mathbf{N}^{pT}\, \dot{\chi}\, d\Omega - \int_{\Omega_h} \mathbf{B}^{pT}\, \mathcal{V}\, d\Omega + \int_{\Gamma^p_{\mathrm{N},h}} \mathbf{N}^{pT}\, \frac{q}{\rho^f}\, d\Gamma = \mathbf{0},\ \forall t,
\end{aligned} \tag{6}$$

These equations (5) are solved through the Newton-Raphson procedure in an incremental iterative algorithm remaining for the time increment ${}^{n+1}t$ in the iteration $(k)$??? , as follows (subscript $h$ is omitted in the following):

$$\begin{bmatrix} {}^{n+1}\dot{G}_h^{(k)} \\ {}^{n+1}H_h^{(k)} \end{bmatrix} + {}^{n+1}\mathbf{J}^{(k)} \begin{bmatrix} \Delta\bar{\mathbf{u}} \\ \Delta\bar{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{7}$$

Donde ${}^{n+1}\dot{G}_h^{(k)}$ y ${}^{n+1}H_h^{(k)}$ son los residuos en el tiempo $t = n+1$ iteracion $k$:

$$\begin{aligned}
{}^{n+1}\dot{G}_h^{(k)} &\equiv \int_{\Omega_h} \mathbf{B}^{uT}\, {}^{n+1}\dot{\boldsymbol{\sigma}}^{(k)}\, d\Omega - \int_{\Omega_h} \mathbf{N}^{uT}\, {}^{n+1}\dot{\mathbf{f}}^{(k)}\, d\Omega - \int_{\Gamma^u_{\mathrm{N},h}} \mathbf{N}^{uT}\, {}^{n+1}\dot{\mathbf{t}}^{(k)}\, d\Gamma = \mathbf{0},\ t = n+1, \\
{}^{n+1}H_h^{(k)} &\equiv \int_{\Omega_h} \mathbf{N}^{pT}\, {}^{n+1}\dot{\chi}^{(k)}\, d\Omega - \int_{\Omega_h} \mathbf{B}^{pT}\, {}^{n+1}\mathcal{V}^{(k)}\, d\Omega + \int_{\Gamma^p_{\mathrm{N},h}} \mathbf{N}^{pT}\, \frac{{}^{n+1}q^{(k)}}{\rho^f}\, d\Gamma = \mathbf{0},\ t = n+1,
\end{aligned} \tag{8}$$

The expressions obtained have the macroscopic nodal increment displacement $\Delta\bar{\mathbf{u}}$ and macroscopic nodal increment pore pressure $\Delta\bar{\mathbf{p}}$ as unknowns. The *macro-scale Jacobian* matrix is composed of (superscripts left and right are omitted):

$$\mathbf{J} = \begin{bmatrix} \frac{\partial^{n+1}\dot{G}}{\partial^{n+1}\bar{\mathbf{u}}} & \frac{\partial^{n+1}\dot{G}}{\partial^{n+1}\bar{\mathbf{p}}} \\ \frac{\partial^{n+1}H}{\partial^{n+1}\bar{\mathbf{u}}} & \frac{\partial^{n+1}H}{\partial^{n+1}\bar{\mathbf{p}}} \end{bmatrix} \tag{9}$$

The components of the *macro-scale Jacobian* matrix?? are

$$\begin{aligned}
\frac{\partial \dot{G}}{\partial \bar{\mathbf{u}}} &= \int_{\Omega} \mathbf{B}^{uT} \frac{\partial \dot{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}} \mathbf{B}^u\, d\Omega \\
\frac{\partial \dot{G}}{\partial \bar{\mathbf{p}}} &= \int_{\Omega} \mathbf{B}^{uT} \frac{\partial \dot{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varphi}} \mathbf{B}^p\, d\Omega + \int_{\Omega} \mathbf{B}^{uT} \frac{\partial \dot{\boldsymbol{\sigma}}}{\partial p} \mathbf{N}^p\, d\Omega
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\partial H}{\partial \bar{\mathbf{u}}} &= -\int_{\Omega} \mathbf{B}^{pT} \frac{\partial \mathcal{V}}{\partial \boldsymbol{\varepsilon}} \mathbf{B}^u\, d\Omega + \int_{\Omega} \mathbf{N}^{pT} \frac{\partial \dot{\chi}}{\partial \boldsymbol{\varepsilon}} \mathbf{B}^u\, d\Omega \\
\frac{\partial H}{\partial \bar{\mathbf{p}}} &= -\int_{\Omega} \mathbf{B}^{pT} \frac{\partial \mathcal{V}}{\partial \boldsymbol{\varphi}} \mathbf{B}^p\, d\Omega - \int_{\Omega} \mathbf{B}^{pT} \frac{\partial \mathcal{V}}{\partial p} \mathbf{N}^p\, d\Omega + \int_{\Omega} \mathbf{N}^{pT} \frac{\partial \dot{\chi}}{\partial \boldsymbol{\varphi}} \mathbf{B}^p\, d\Omega + \int_{\Omega} \mathbf{N}^{pT} \frac{\partial \dot{\chi}}{\partial p} \mathbf{N}^p\, d\Omega
\end{aligned} \tag{11}$$

En donde introduciendo los operadores tangentes homogeneizados quedaria:

$$\frac{\partial \dot{G}}{\partial \bar{\mathbf{u}}} = \int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \left( \frac{\partial \dot{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varepsilon}} \right) - \mathbf{b}_\mu \otimes \beta\, \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varepsilon}} \right] d\Omega_\mu \right] \mathbf{B}^u\, d\Omega \tag{12}$$

$$\begin{aligned}
\frac{\partial \dot{G}}{\partial \bar{\mathbf{p}}} = &\int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varphi}} - \mathbf{b}_\mu \otimes \beta \left( \frac{\partial \dot{\boldsymbol{\varphi}}}{\partial \boldsymbol{\varphi}} \cdot \mathbf{y} + \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varphi}} \right) \right] d\Omega_\mu \right] \mathbf{B}^p\, d\Omega \\
&+ \int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial p} - \mathbf{b}_\mu \left( \frac{\partial \dot{p}}{\partial p} + \beta\, \frac{\partial \dot{\tilde{p}}_\mu}{\partial p} \right) \right] d\Omega_\mu \right] \mathbf{N}^p\, d\Omega
\end{aligned} \tag{13}$$

$$
\begin{aligned}
\frac{\partial H}{\partial \bar{\mathbf{u}}} = &- \int_{\Omega} \mathbf{B}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ -\mathbf{k}_\mu \cdot \frac{\partial \tilde{\boldsymbol{\varphi}}_\mu}{\partial \boldsymbol{\varepsilon}} - \mathbf{b}_\mu : \left( \frac{\partial \dot{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} - \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varepsilon}} \right) \otimes \mathbf{y} + \frac{1}{M_\mu} \beta \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varepsilon}} \otimes \mathbf{y} \right] d\Omega_\mu \right] \mathbf{B}^u \, d\Omega \\
&+ \int_{\Omega} \mathbf{N}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{b}_\mu : \left( \frac{\partial \dot{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varepsilon}} \right) + \frac{1}{M_\mu} \beta \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varepsilon}} \right] d\Omega_\mu \right] \mathbf{B}^u \, d\Omega
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial \bar{\mathbf{p}}} = &- \int_{\Omega} \mathbf{B}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ -\mathbf{k}_\mu \cdot \left( \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\varphi}} + \frac{\partial \tilde{\boldsymbol{\varphi}}_\mu}{\partial \boldsymbol{\varphi}} \right) - \mathbf{b}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varphi}} \otimes \mathbf{y} - \frac{1}{M_\mu} \beta \left( \frac{\partial \dot{\boldsymbol{\varphi}}}{\partial \boldsymbol{\varphi}} \cdot \mathbf{y} + \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varphi}} \right) \otimes \mathbf{y} \right] d\Omega_\mu \right] \mathbf{B}^p \, d\Omega \\
&- \int_{\Omega} \mathbf{B}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ -\mathbf{k}_\mu \cdot \frac{\partial \tilde{\boldsymbol{\varphi}}_\mu}{\partial p} - \left( \mathbf{b}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial p} \right) \mathbf{y} - \frac{1}{M_\mu} \left( \frac{\partial \dot{p}}{\partial p} + \beta \frac{\partial \dot{\tilde{p}}_\mu}{\partial p} \right) \mathbf{y} \right] d\Omega_\mu \right] \mathbf{N}^p \, d\Omega \\
&+ \int_{\Omega} \mathbf{N}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{b}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial \boldsymbol{\varphi}} + \frac{1}{M_\mu} \beta \left( \frac{\partial \dot{\boldsymbol{\varphi}}}{\partial \boldsymbol{\varphi}} \cdot \mathbf{y} + \frac{\partial \dot{\tilde{p}}_\mu}{\partial \boldsymbol{\varphi}} \right) \right] d\Omega_\mu \right] \mathbf{B}^p \, d\Omega \\
&+ \int_{\Omega} \mathbf{N}^{pT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{b}_\mu : \frac{\partial \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu}{\partial p} + \frac{1}{M_\mu} \left( \frac{\partial \dot{p}}{\partial p} + \beta \frac{\partial \dot{\tilde{p}}_\mu}{\partial p} \right) \right] d\Omega_\mu \right] \mathbf{N}^p \, d\Omega
\end{aligned} \tag{15}
$$

Mediante el metodo de integracion en el tiempo se tiene que:

$$
{}^{n+\alpha}\dot{\boldsymbol{\varphi}} = \frac{{}^{n+1}\boldsymbol{\varphi} - {}^{n}\boldsymbol{\varphi}}{\Delta t} \tag{16}
$$

$$
{}^{n+\alpha}\boldsymbol{\varphi} = (1-\alpha)\,{}^{n}\boldsymbol{\varphi} + \alpha\,{}^{n+1}\boldsymbol{\varphi} = \alpha \left( {}^{n+1}\boldsymbol{\varphi} - {}^{n}\boldsymbol{\varphi} \right) + {}^{n}\boldsymbol{\varphi} \tag{17}
$$

si “derivo” respecto de $\boldsymbol{\varphi}$ quedaria

$$
\frac{\partial^{n+\alpha}\dot{\boldsymbol{\varphi}}}{\partial^{n+1}\boldsymbol{\varphi}} = \frac{1}{\Delta t}\mathbf{I} \tag{18}
$$

$$
\frac{\partial^{n+\alpha}\boldsymbol{\varphi}}{\partial^{n+1}\boldsymbol{\varphi}} = \frac{1}{\partial^{n+1}\boldsymbol{\varphi}}\alpha \left( {}^{n+1}\boldsymbol{\varphi} - {}^{n}\boldsymbol{\varphi} \right) + \frac{1}{\partial^{n+1}\boldsymbol{\varphi}}{}^{n}\boldsymbol{\varphi} = \alpha \mathbf{I} \tag{19}
$$

siendo ${}^{n}\boldsymbol{\varphi}$ el paso previo convergido.
Se opera de manera semejante con las demas variables. Introduciendo ello en las cuatro ecuaciones previas se llega a:

$$
\frac{\partial \dot{G}}{\partial \bar{\mathbf{u}}} = \int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \left( \frac{1}{\Delta t}\mathbb{I} + \frac{1}{\Delta t}\frac{\partial \tilde{\boldsymbol{\varepsilon}}_\mu}{\partial \boldsymbol{\varepsilon}} \right) - \mathbf{b}_\mu \otimes \beta \frac{1}{\Delta t}\frac{\partial \tilde{p}_\mu}{\partial \boldsymbol{\varepsilon}} \right] d\Omega_\mu \right] \mathbf{B}^u \, d\Omega \tag{20}
$$

$$
\begin{aligned}
\frac{\partial \dot{G}}{\partial \bar{\mathbf{p}}} = &\int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \frac{1}{\Delta t}\frac{\partial \tilde{\boldsymbol{\varepsilon}}_\mu}{\partial \boldsymbol{\varphi}} - \mathbf{b}_\mu \otimes \beta \left( \frac{1}{\Delta t}\mathbf{I} \cdot \mathbf{y} + \frac{1}{\Delta t}\frac{\partial \tilde{p}_\mu}{\partial \boldsymbol{\varphi}} \right) \right] d\Omega_\mu \right] \mathbf{B}^p \, d\Omega \\
&+ \int_{\Omega} \mathbf{B}^{uT} \left[ \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left[ \mathbf{C}_\mu : \frac{1}{\Delta t}\frac{\partial \tilde{\boldsymbol{\varepsilon}}_\mu}{\partial p} - \mathbf{b}_\mu \left( \frac{1}{\Delta t}1 + \beta \frac{1}{\Delta t}\frac{\partial \tilde{p}_\mu}{\partial p} \right) \right] d\Omega_\mu \right] \mathbf{N}^p \, d\Omega
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial \bar{\mathbf{u}}} = &-\int_{\Omega} \mathbf{B}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[-\mathbf{k}_\mu\cdot\alpha\,\frac{\partial\tilde{\boldsymbol{\varphi}}_\mu}{\partial\boldsymbol{\varepsilon}} - \mathbf{b}_\mu : \left(\frac{1}{\Delta t}\mathbb{I} + \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial\boldsymbol{\varepsilon}}\right)\otimes\mathbf{y} - \frac{1}{M_\mu}\,\beta\,\frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial\boldsymbol{\varepsilon}}\otimes\,\mathbf{y}\right] d\Omega_\mu\right]\mathbf{B}^u\, d\Omega \\
&+\int_{\Omega} \mathbf{N}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[\mathbf{b}_\mu : \left(\frac{1}{\Delta t}\mathbb{I} + \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial\boldsymbol{\varepsilon}}\right) + \frac{1}{M_\mu}\,\beta\,\frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial\boldsymbol{\varepsilon}}\right] d\Omega_\mu\right]\mathbf{B}^u\, d\Omega
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial \bar{\mathbf{p}}} = &-\int_{\Omega} \mathbf{B}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[-\mathbf{k}_\mu\cdot\left(\alpha\mathbf{I} + \alpha\frac{\partial\tilde{\boldsymbol{\varphi}}_\mu}{\partial\boldsymbol{\varphi}}\right) - \mathbf{b}_\mu : \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial\boldsymbol{\varphi}}\otimes\mathbf{y} - \frac{1}{M_\mu}\,\beta\,\left(\frac{1}{\Delta t}\mathbf{I}\cdot\mathbf{y} + \frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial\,\boldsymbol{\varphi}}\right)\otimes\mathbf{y}\right] d\Omega_\mu\right]\mathbf{B}^p\, d\Omega \\
&-\int_{\Omega} \mathbf{B}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[-\mathbf{k}_\mu\cdot\alpha\frac{\partial\tilde{\boldsymbol{\varphi}}_\mu}{\partial p} - \left(\mathbf{b}_\mu : \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial p}\right)\mathbf{y} - \frac{1}{M_\mu}\left(\frac{1}{\Delta t}\,1 + \beta\,\frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial p}\right)\mathbf{y}\right] d\Omega_\mu\right]\mathbf{N}^p\, d\Omega \\
&+\int_{\Omega} \mathbf{N}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[\mathbf{b}_\mu : \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial\boldsymbol{\varphi}} + \frac{1}{M_\mu}\,\beta\,\left(\frac{1}{\Delta t}\mathbf{I}\cdot\mathbf{y} + \frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial\boldsymbol{\varphi}}\right)\right] d\Omega_\mu\right]\mathbf{B}^p\, d\Omega \\
&+\int_{\Omega} \mathbf{N}^{pT}\left[\frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left[\mathbf{b}_\mu : \frac{1}{\Delta t}\frac{\partial\tilde{\boldsymbol{\varepsilon}}_\mu}{\partial p} + \frac{1}{M_\mu}\left(\frac{1}{\Delta t}\,1 + \beta\,\frac{1}{\Delta t}\frac{\partial\tilde{p}_\mu}{\partial p}\right)\right] d\Omega_\mu\right]\mathbf{N}^p\, d\Omega
\end{aligned}
\tag{23}
$$

In line with the approaches given in sections **??** and **??**, two models describing behavior at the micro-scale were assumed in the paper. For one of them, all arguments of the constitutive functionals were assumed to be fully expanded (FOE constitutive arguments) while the other one combines variables of this type with another one for which a zero-order expansion was adopted (COE constitutive arguments). In particular, these multiscale models differ only in the field of micro-pore pressures. In order to write the functionals including both models in a compact form hereafter, a binary factor is proposed, which is denoted $\beta$ and satisfies the following condition

$$
\beta = \begin{cases} 1 & \text{in case of FOE primal descriptors,} \\ 0 & \text{in case of ZOE primal descriptors.} \end{cases}
\tag{24}
$$

Hence, homogenized relationships can be rewritten

$$
\boldsymbol{\sigma} = \hat{\sigma}\left(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}_\mu, p, \beta\,\boldsymbol{\varphi}, \beta\,\tilde{p}_\mu\right) = \frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left(\boldsymbol{\sigma}_\mu - \mathbf{f}\otimes\mathbf{y}\right) d\Omega_\mu,\ \forall t,
\tag{25}
$$

$$
\dot{\chi} = \hat{\chi}\left(\dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\mu\right) = \frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\dot{\chi}_\mu\, d\Omega_\mu,\ \forall t,
\tag{26}
$$

$$
\mathcal{V} = \hat{\mathcal{V}}\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mu, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\mu\right) = \frac{1}{|\Omega_\mu|}\int_{\Omega_\mu}\left(\mathcal{V}_\mu - \dot{\chi}_\mu\,\mathbf{y}\right) d\Omega_\mu,\ \forall t,
\tag{27}
$$

where $\hat{(\bullet)}$ denotes the homogenized macroscopic constitutive response functional.

From Eqs. (25)-(27) the homogenized tangent operators[?,?,?,?,?,?] are calculated in a straightforward way, which in turn allow the components of the Jacobian matrix Eq. (7) to be determined. The determination of the operators is presented in Appendix **??**.

# 3 | PMVP CON RESTRICCIONES

## 3.1 | Principle of Multiscale Virtual Power with constraints

The Principle of Multiscale Virtual Power, see Blanco et al**?**, states that the total virtual power per unit volume, at a point **x** of the macro-scale, must be equal to the volumetric average of the total micro-scale virtual power (per unit volume) at the corresponding RVE, for all admissible virtual actions.

$$\int_{\Omega_\mu} \tilde{\mathbf{u}}_\mu \, d\Omega_\mu = 0, \tag{28}$$

$$\int_{\Gamma_\mu} \tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu \, d\Gamma_\mu = \mathbf{0}, \tag{29}$$

$$\int_{\Omega_\mu} \tilde{p}_\mu \, d\Omega_\mu = 0, \tag{30}$$

$$\int_{\Gamma_\mu} \tilde{p}_\mu \, \mathbf{n}_\mu \, d\Gamma_\mu = \mathbf{0}, \tag{31}$$

It is necesary to introduce the constraint (28)-(31) trough the method of Lagrange multiplier in the variational way on the PMVP equation. For this purpose, se introducen los multiplicadores de Lagrange para relajar la ecuacion del PMVP e incluir las restricciones que permitiria obtener el modelo minimamente restringido

$$\int_{\Omega_\mu} \boldsymbol{\theta}_u \cdot \tilde{\mathbf{u}}_\mu \, d\Omega_\mu = 0 \;\Rightarrow\; \int_{\Omega_\mu} \boldsymbol{\theta}_u \cdot \delta\tilde{\mathbf{u}}_\mu \, d\Omega_\mu + \int_{\Omega_\mu} \delta\boldsymbol{\theta}_u \cdot \tilde{\mathbf{u}}_\mu \, d\Omega_\mu = 0, \tag{32}$$

$$\int_{\Gamma_\mu} \boldsymbol{\lambda}_u : \left(\tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu\right) d\Gamma_\mu = \mathbf{0} \;\Rightarrow\; \int_{\Gamma_\mu} \boldsymbol{\lambda}_u : \left(\delta\tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu\right) d\Gamma_\mu + \int_{\Gamma_\mu} \delta\boldsymbol{\lambda}_u : \left(\tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu\right) d\Gamma_\mu = \mathbf{0} \tag{33}$$

$$\int_{\Omega_\mu} \theta_p \, \tilde{p}_\mu \, d\Omega_\mu = 0 \;\Rightarrow\; \int_{\Omega_\mu} \theta_p \, \delta\tilde{p}_\mu \, d\Omega_\mu + \int_{\Omega_\mu} \delta\theta_p \, \tilde{p}_\mu \, d\Omega_\mu = 0, \tag{34}$$

$$\int_{\Gamma_\mu} \boldsymbol{\lambda}_p \cdot \left(\tilde{p}_\mu \, \mathbf{n}_\mu\right) d\Gamma_\mu = 0 \;\Rightarrow\; \int_{\Gamma_\mu} \boldsymbol{\lambda}_p \cdot \left(\delta\tilde{p}_\mu \, \mathbf{n}_\mu\right) d\Gamma_\mu + \int_{\Gamma_\mu} \boldsymbol{\lambda}_p \cdot \left(\tilde{p}_\mu \, \mathbf{n}_\mu\right) d\Gamma_\mu = \mathbf{0}, \tag{35}$$

Thus, recalling the definition of macro-scale total virtual power per unit volume (given by the first integrand term of the r.h.s. in Eq. (**??**)), assuming the same mathematical structure for its micro-scale counterpart and including the previously established restriction (**??**), the PMVP gives us the following variational sentence with constraints

$$\begin{aligned}
\boldsymbol{\sigma} : \delta\dot{\boldsymbol{\varepsilon}} + \dot{\chi}\,\delta p - \mathcal{V}\cdot\delta\boldsymbol{\varphi} - \mathbf{f}\cdot\delta\dot{\mathbf{u}} = & \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left(\boldsymbol{\sigma}_\mu : \delta\dot{\boldsymbol{\varepsilon}}_\mu + \dot{\chi}_\mu\,\delta p_\mu - \mathcal{V}_\mu\cdot\delta\boldsymbol{\varphi}_\mu - \mathbf{f}_\mu\cdot\delta\dot{\mathbf{u}}_\mu\right) d\Omega_\mu \\
& + \int_{\Omega_\mu} \boldsymbol{\theta}_u \cdot \delta\tilde{\mathbf{u}}_\mu \, d\Omega_\mu + \int_{\Omega_\mu} \delta\boldsymbol{\theta}_u \cdot \tilde{\mathbf{u}}_\mu \, d\Omega_\mu + \int_{\Gamma_\mu} \boldsymbol{\lambda}_u : \left(\delta\tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu\right) d\Gamma_\mu + \int_{\Gamma_\mu} \delta\boldsymbol{\lambda}_u : \left(\tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu\right) d\Gamma_\mu \\
& + \int_{\Omega_\mu} \theta_p \, \delta\tilde{p}_\mu \, d\Omega_\mu + \int_{\Omega_\mu} \delta\theta_p \, \tilde{p}_\mu \, d\Omega_\mu + \int_{\Gamma_\mu} \boldsymbol{\lambda}_p \cdot \left(\delta\tilde{p}_\mu \, \mathbf{n}_\mu\right) d\Gamma_\mu + \int_{\Gamma_\mu} \boldsymbol{\lambda}_p \cdot \left(\tilde{p}_\mu \, \mathbf{n}_\mu\right) d\Gamma_\mu, \\
& \forall\, \delta\dot{\boldsymbol{\varepsilon}} \in \mathbb{R}^6\left(\Omega_\mu\right),\; \forall\, \delta\dot{\mathbf{u}} \in \mathbb{R}^3\left(\Omega_\mu\right),\; \forall\, \delta\boldsymbol{\varphi} \in \mathbb{R}^3\left(\Omega_\mu\right),\; \forall\, \delta p \in \mathbb{R}^1\left(\Omega_\mu\right), \\
& \forall\, \delta\tilde{\mathbf{u}}_\mu \in H^1\left(\Omega_\mu\right),\; \forall \delta\tilde{p}_\mu \in H^1\left(\Omega_\mu\right), \\
& \forall\, \delta\boldsymbol{\theta}_u \in \mathbb{R}^3\left(\Omega_\mu\right),\; \forall\, \delta\boldsymbol{\lambda}_u \in \mathbb{R}^9\left(\Omega_\mu\right),\; \forall\, \delta\theta_p \in \mathbb{R}\left(\Omega_\mu\right),\; \forall\, \delta\boldsymbol{\lambda}_p \in \mathbb{R}^3\left(\Omega_\mu\right)
\end{aligned} \tag{36}$$

where the compact notation $\dot{\chi}_\mu = \frac{\dot{m}^f_\mu}{\rho^f_\mu}$ and $\mathbf{f}_\mu = \rho_\mu\,\mathbf{g}$ has been considered; $\mathbf{f}_\mu$ being the saturated weight per unit volume of the porous medium which can be obtained from the saturated density $\rho_\mu = \rho^f n_\mu + \rho^s (1 - n_\mu)$, with $\rho^s$ denoting the density on the solid phase. Besides $\mathbb{R}^6$, $\mathbb{R}^3$, and $\mathbb{R}^1$ identify spaces of second-order symmetric tensors, three-dimensional and one-dimensional Euclidean space, respectively. Expression (36) can be viewed as a particular instance of the PMVP for the case of saturated porous media, at both scales of analysis.

## 3.2 Homogenized variables and variational forms of balance at the micro-scale with constraints

The variational identity (36) contains all the ingredients to deduce[??]

(I) the homogenization formulae for the macro-scale stress-like entities $\{\boldsymbol{\sigma}; \dot{\chi}; \mathcal{V}\}$ and body force $\mathbf{f}$,

(II) the variational equilibrium problem at micro-scale.

To attain this goal, definitions (**??**)-(**??**) must be replaced in expression (36) and then, by resorting to simple variational manipulations, the consequences (I)-(II) mentioned above can be easily derived, as we show next.

### 3.2.1 Homogenized variables

(I-a) Homogenized stress tensor:

$$\boldsymbol{\sigma} = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left( \boldsymbol{\sigma}_\mu - \mathbf{f}_\mu \otimes \mathbf{y} \right) d\Omega_\mu, \ \forall t. \tag{37}$$

Which is obtained from (36), taking $\delta \dot{\tilde{\mathbf{u}}}_\mu = \mathbf{0}$, $\delta p = 0$, $\delta \tilde{p}_\mu = 0$, $\delta \boldsymbol{\varphi} = \mathbf{0}$, $\delta \dot{\mathbf{u}} = \mathbf{0}$, $\delta \lambda = 0$ and allowing arbitrary variations of $\delta \dot{\boldsymbol{\varepsilon}}$.

(I-b) Homogenized mass content rate of fluid (per unit fluid density):

$$\dot{\chi} = \frac{\dot{m}^f}{\rho^f} = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \dot{\chi}_\mu \, d\Omega_\mu, \ \forall t. \tag{38}$$

Deduced by admitting, in (36), arbitrary variations of $\delta p$ and adopting $\delta \dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\delta \dot{\tilde{\mathbf{u}}}_\mu = \mathbf{0}$, $\delta \tilde{p}_\mu = 0$, $\delta \dot{\mathbf{u}} = \mathbf{0}$, $\delta \boldsymbol{\varphi} = \mathbf{0}$ and $\delta \lambda = 0$. If the fluid mass density $\rho^f$ is assumed to be uniform throughout the RVE domain, the homogenized formula for the mass content rate, $\dot{m}^f$, can be retrieved after simple manipulations on (38)

$$\dot{m}^f = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \dot{m}^f_\mu \, d\Omega_\mu, \ \forall t. \tag{39}$$

(I-c) Homogenized flux velocity vector:

$$\mathcal{V} = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \left( \mathcal{V}_\mu - \dot{\chi}_\mu \, \mathbf{y} \right) d\Omega_\mu, \ \forall t. \tag{40}$$

It is achieved from Eq. (36), adopting $\delta \dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\delta \dot{\tilde{\mathbf{u}}}_\mu = \mathbf{0}$, $\delta p = 0$, $\delta \tilde{p}_\mu = 0$, $\delta \dot{\mathbf{u}} = \mathbf{0}$ and $\delta \lambda = 0$ with arbitrary variations of $\delta \boldsymbol{\varphi}$.

In view of (40), it is possible to decompose the homogenized flux velocity vector into a stationary part ($\mathcal{V}_{sta}$) and a dynamic or transient part ($\mathcal{V}_{tra}$) as[????]:

$$\mathcal{V} = \mathcal{V}_{sta} + \mathcal{V}_{tra} = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \mathcal{V}_\mu \, d\Omega_\mu - \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \dot{\chi}_\mu \, \mathbf{y} \, d\Omega_\mu, \ \forall t. \tag{41}$$

Decomposition (41) is used latter (see section **??**) to discuss the size effect of RVE.

(I-d) Homogenized body force field:

$$\mathbf{f} = \frac{1}{|\Omega_\mu|} \int_{\Omega_\mu} \mathbf{f}_\mu \, d\Omega_\mu = \frac{\mathbf{g}}{|\Omega_\mu|} \int_{\Omega_\mu} \rho_\mu \, d\Omega_\mu, \ \forall t. \tag{42}$$

It is obtained from Eq. (36), with $\delta \dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\delta \dot{\tilde{\mathbf{u}}}_\mu = \mathbf{0}$, $\delta p = 0$, $\delta \tilde{p}_\mu = 0$, $\delta \boldsymbol{\varphi} = \mathbf{0}$, $\delta \lambda = 0$ and arbitrary variations of $\delta \dot{\mathbf{u}}$.

### 3.2.2 Variational forms of balance in the RVE

(II-a) Integral equation of momentum balance:

$$\int_{\Omega_\mu} \left( \boldsymbol{\sigma}_\mu : \nabla^{sym}_y \delta \dot{\tilde{\mathbf{u}}}_\mu - \mathbf{f}_\mu \cdot \delta \dot{\tilde{\mathbf{u}}}_\mu \right) d\Omega_\mu + \int_{\Omega_\mu} \boldsymbol{\theta}_u \cdot \delta \tilde{\mathbf{u}}_\mu \, d\Omega_\mu + \int_{\Gamma_\mu} \boldsymbol{\lambda}_u : \left( \delta \tilde{\mathbf{u}}_\mu \otimes \mathbf{n}_\mu \right) d\Gamma_\mu = 0, \ \forall \, \delta \dot{\tilde{\mathbf{u}}}_\mu \in \tilde{\mathcal{U}}_\mu, \ \forall t. \tag{43}$$

Eq. (43) derives from kinematically permissible variations of $\delta\dot{\tilde{\mathbf{u}}}_\mu$ in Eq. (36) where, in turn, $\delta\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\delta p = 0$, $\delta\tilde{p}_\mu = 0$, $\delta\boldsymbol{\varphi} = \mathbf{0}$ $\delta\dot{\mathbf{u}} = \mathbf{0}$ and $\delta\lambda = 0$.

(II-b) Integral mass balance equation with micro-pore pressure fluctuation constraint:

$$\int_{\Omega_\mu} \left(\dot{\chi}_\mu\,\delta\tilde{p}_\mu - \mathcal{V}_\mu\cdot\delta\tilde{\boldsymbol{\varphi}}_\mu\right)\,d\Omega_\mu + \int_{\Omega_\mu} \lambda\,\delta\tilde{p}_\mu\,d\Omega_\mu = \int_{\Omega_\mu}\left(\dot{\chi}_\mu\,\delta\tilde{p}_\mu - \mathcal{V}_\mu\cdot\nabla_y\,\delta\tilde{p}_\mu + \lambda\,\delta\tilde{p}_\mu\right)\,d\Omega_\mu = 0,\ \ \forall\,\delta\tilde{p}_\mu\in\tilde{\mathcal{P}}_\mu,\ \forall t. \tag{44}$$

It follows from (36) by allowing for admissible variations of $\delta\tilde{p}_\mu$ with $\delta\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\delta\dot{\tilde{\mathbf{u}}}_\mu = \mathbf{0}$, $\delta p = 0$, $\delta\dot{\mathbf{u}} = \mathbf{0}$, $\delta\boldsymbol{\varphi} = \mathbf{0}$ and $\delta\lambda = 0$.

(II-c) Constraint equation:

$$\int_{\Omega_\mu} \tilde{p}_\mu\,\delta\lambda\,d\Omega_\mu = 0,\ \ \forall\,\delta\lambda\in ?,\ \forall t. \tag{45}$$

It is achieved from Eq. (36) adopting only arbitrary the multiplier variable $\delta\lambda$ and the remaining ones being null

## 3.3 | Solution of the variational equations of the RVE with constraints

Similar to the homogenized variables, the variational forms of balance in the RVE Eqs. (43)-(45) can be rewritten in a functional format, considering in turn the constitutive models introduced above distinguished by the binary factor $\beta$

$$\hat{G}_\mu\left(\boldsymbol{\varepsilon},\tilde{\boldsymbol{\varepsilon}}_\mu,p,\beta\,\boldsymbol{\varphi},\beta\,\tilde{p}_\mu,\delta\dot{\tilde{\mathbf{u}}}_\mu\right) = \int_{\Omega_\mu}\left(\boldsymbol{\sigma}_\mu : \nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\ -\ \mathbf{f}_\mu\cdot\delta\dot{\tilde{\mathbf{u}}}_\mu\right)d\Omega_\mu = 0,\ \ \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu\in\ \tilde{\mathcal{U}}_\mu,\ \forall t, \tag{46}$$

$$H_\mu\left(\boldsymbol{\varphi},\boldsymbol{\varphi}_\mu,\dot{\boldsymbol{\varepsilon}},\dot{\tilde{\boldsymbol{\varepsilon}}}_\mu,\dot{p},\beta\,\dot{\boldsymbol{\varphi}},\beta\,\dot{\tilde{p}}_\mu,\lambda,\delta\tilde{p}_\mu\right) = \int_{\Omega_\mu}\left(\dot{\chi}_\mu\,\delta\tilde{p}_\mu - \mathcal{V}_\mu\cdot\nabla_y\,\delta\tilde{p}_\mu + \lambda\,\delta\tilde{p}_\mu\right)\,d\Omega_\mu = 0,\ \ \forall\,\delta\tilde{p}_\mu\in\ \tilde{\mathcal{P}}_\mu,\ \forall t. \tag{47}$$

$$\hat{J}_\mu\left(\tilde{p}_\mu,\delta\lambda\right) = \int_{\Omega_\mu}\tilde{p}_\mu\,\delta\lambda\,d\Omega_\mu = 0,\ \ \forall\,\delta\lambda\in ?,\ \forall t. \tag{48}$$

Subsequently, in Appendix **??**, the tangent relationships between fluctuations and macro-scale variables that constitute the tangent operators will be determined straightforwardly from these equations.

Furthermore, analogous the macro-scale, it is possible to implement a discretization $h_\mu$ of the domain $\Omega_\mu$ at the micro-scale. When introducing the shape functions, the variables of interest can be related to their nodal counterpart as follows

$$\begin{aligned}\tilde{\mathbf{u}}_\mu &= \mathbf{N}_\mu^u\,\tilde{\bar{\mathbf{u}}}_\mu,\\ \tilde{p}_\mu &= \mathbf{N}_\mu^p\,\tilde{\bar{\mathbf{p}}}_\mu,\\ \lambda &= cte,\end{aligned} \tag{49}$$

where $\tilde{\bar{\mathbf{u}}}_\mu$and $\tilde{\bar{\mathbf{p}}}_\mu$ are the vectors of the nodal micro-displacement fluctuations and nodal micro-pore pressure fluctuations,respectively. $\lambda$ is the constant Lagrange multiplier. In this case, the interpolation functions are specified in the micro-scale space. Following the same procedure as at the macro-scale, virtual variations are assumed for the virtual variations

$$\begin{aligned}\delta\dot{\tilde{\mathbf{u}}}_\mu &= \mathbf{N}_\mu^u\,\delta\dot{\tilde{\bar{\mathbf{u}}}}_\mu,\\ \delta\tilde{p}_\mu &= \mathbf{N}_\mu^p\,\delta\tilde{\bar{\mathbf{p}}}_\mu,\\ \delta\lambda &= \delta\lambda,\end{aligned} \tag{50}$$

$\delta\tilde{\bar{\mathbf{u}}}$ and $\delta\tilde{\bar{\mathbf{p}}}_\mu$ being the vectors of the nodal virtual micro-displacement fluctuations and nodal virtual micro-pore pressure fluctuations, respectively. $\delta\lambda$ is the virtual Lagrange multiplier. Hence, the finite element approximation of system Eqs. (46)-(48) for a given spatial discretization $h$, consists in

$$\hat{G}_\mu^h\left(\boldsymbol{\varepsilon},\tilde{\boldsymbol{\varepsilon}}_\mu,p,\beta\,\boldsymbol{\varphi},\beta\,\tilde{p}_\mu,\delta\dot{\tilde{\mathbf{u}}}_\mu\right) = \left[\int_{\Omega_\mu^h}\left(\mathbf{B}_\mu^{u\,T}\,\boldsymbol{\sigma}_\mu - \mathbf{N}_\mu^{u\,T}\,\mathbf{f}_\mu\otimes\mathbf{y}\right)d\Omega_\mu\right]\cdot\delta\dot{\tilde{\bar{\mathbf{u}}}}_\mu = 0;\quad \forall\,\delta\dot{\tilde{\bar{\mathbf{u}}}}_\mu\in\mathcal{U}_\mu^h,\ \forall t, \tag{51}$$

$$H^h_\mu\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mu, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\mu, \lambda, \delta\tilde{p}_\mu\right) = \left[\int\limits_{\Omega^h_\mu} \left(\mathbf{N}^{p\,T}_\mu\,\dot{\chi}_\mu - \mathbf{B}^{p\,T}_\mu\,\mathcal{V}_\mu + \mathbf{N}^{p\,T}_\mu\,\lambda\right) d\Omega_\mu\right]\cdot\delta\tilde{\mathbf{p}}_\mu = 0; \quad \forall\,\delta\tilde{\mathbf{p}}_\mu \in \mathcal{P}^h_\mu, \ \forall\, t, \tag{52}$$

$$\hat{J}_\mu\left(\tilde{p}_\mu, \delta\lambda\right) = \left[\int\limits_{\Omega^h_\mu} \tilde{p}_\mu\, d\Omega_\mu\right]\cdot\delta\lambda = 0; \quad \forall\,\delta\lambda \in ?, \ \forall\, t, \tag{53}$$

where $\Omega^h_\mu$ denotes the discretized RVE domain, $\mathbf{B}^u_\mu = \nabla^{sym}_y \mathbf{N}^u_\mu$ and $\mathbf{B}^p_\mu = \nabla_y \mathbf{N}^p_\mu$, while $\mathcal{U}^h_\mu$ and $\mathcal{P}^h_\mu$ are the finite-dimensional spaces of the vectors of nodal virtual displacement and pore pressure associated with the finite element discretization $h_\mu$ of the domain $\Omega_\mu$.

Thus, either adopting FOE or ZOE primal descriptors, through a binary factor $\beta$, into Eqs. (51)-(53) yields the following equations system

$$\begin{aligned}
&\left[\overrightarrow{F}_\mu + \mathbf{K}_\mu\,\tilde{\mathbf{u}}_\mu - \mathbf{Q}_\mu\,\beta\,\tilde{\mathbf{p}}_\mu\right]\cdot\delta\dot{\tilde{\mathbf{u}}}_\mu = 0 \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}^h_\mu, \ \forall\, t,\\
&\left[\overline{\mathbf{F}}_\mu + \mathbf{Q}^T_\mu\,\dot{\tilde{\mathbf{u}}}_\mu + \mathbf{S}_\mu\,\beta\,\dot{\tilde{\mathbf{p}}}_\mu + \mathbf{H}_\mu\,\tilde{\mathbf{p}}_\mu + \mathbf{L}_\lambda\,\lambda\right]\cdot\delta\tilde{\mathbf{p}}_\mu = 0 \quad \forall\,\delta\tilde{\mathbf{p}}_\mu \in \mathcal{P}^h_\mu, \ \forall\, t,\\
&\left[\mathbf{L}^T_\lambda\,\tilde{\mathbf{p}}_\mu\right]\cdot\delta\lambda = 0 \quad \forall\,\delta\lambda \in ?, \ \forall\, t,
\end{aligned} \tag{54}$$

where the matrices of the system of Eqs. 54 are defined as

In the first Eq. (54) we define the load vector $\overrightarrow{F}_\mu$, the stiffness matrix $\mathbf{K}_\mu$ and the coupling matrices between the solid and fluid $\mathbf{Q}_\mu$:

$$\begin{aligned}
\overrightarrow{F}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{B}^{u\,T}_\mu\left[\mathbf{C}_\mu : \dot{\boldsymbol{\varepsilon}} - \mathbf{b}_\mu\,\dot{p}\right]\, d\Omega_\mu,\\
\mathbf{K}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{B}^{u\,T}_\mu\,\mathbf{C}_\mu\,\mathbf{B}^u_\mu d\Omega_\mu,\\
\mathbf{Q}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{B}^{u\,T}_\mu\,\mathbf{b}_\mu\,\mathbf{N}^p_\mu\, d\Omega_\mu.
\end{aligned} \tag{55}$$

For the second equation given in Eq. (54), there is the vector $\overline{\mathbf{F}}_\mu$, the compressibility matrix $\mathbf{S}_\mu$ and the water permeability matrix $\mathbf{H}_\mu$:

$$\begin{aligned}
\overline{\mathbf{F}}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{N}^{p\,T}_\mu\left[\mathbf{b}_\mu : \dot{\boldsymbol{\varepsilon}} + \frac{1}{M_\mu}\dot{p}\right] d\Omega_\mu - \int\limits_{\Omega^h_\mu} \mathbf{B}^{p\,T}_\mu\left[\mathbf{k}_\mu\cdot\boldsymbol{\varphi}\right]\, d\Omega_\mu,\\
\mathbf{S}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{N}^{p\,T}_\mu\,\frac{1}{M_\mu}\,\mathbf{N}^p_\mu\, d\Omega_\mu,\\
\mathbf{H}_\mu &= \int\limits_{\Omega^h_\mu} \mathbf{B}^{p\,T}_\mu\,\mathbf{k}_\mu\,\mathbf{B}^p_\mu\, d\Omega_\mu.
\end{aligned} \tag{56}$$

Finally, for the third equation in (54) there is the vector

$$\mathbf{L}_\lambda = \int\limits_{\Omega^h_\mu} \mathbf{N}^{p\,T}_\mu\, d\Omega_\mu, \tag{57}$$

The system of Eqs. (53) must be solved by some time integration method, such as the implicit backward Euler method or the implicit trapezoidal rule[? ? ?], to determine nodal fluctuation variables of the RVE. Proceeding by means of the $\alpha$-method for time integration yields

$$\left[\begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_\mu & \mathbf{L}_\lambda\\ \mathbf{0} & \mathbf{L}^T_\lambda & \mathbf{0}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{u}}_\mu\\ \tilde{\mathbf{p}}_\mu\\ \lambda\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{Q}^T & \mathbf{S}_\mu & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\tilde{\mathbf{u}}}_\mu\\ \dot{\tilde{\mathbf{p}}}_\mu\\ \dot{\lambda}\end{bmatrix} - \begin{bmatrix}\overrightarrow{F}_\mu\\ \overline{\mathbf{F}}_\mu\\ \mathbf{0}\end{bmatrix}\right]\cdot\begin{bmatrix}\delta\dot{\tilde{\mathbf{u}}}_\mu\\ \delta\tilde{\mathbf{p}}_\mu\\ \delta\lambda\end{bmatrix} = 0 \tag{58}$$

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_\mu & \mathbf{L}_\lambda \\ \mathbf{0} & \mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}}_\mu \\ \tilde{\mathbf{p}}_\mu \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ \mathbf{Q}^T & \mathbf{S}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{u}}}_\mu \\ \dot{\tilde{\mathbf{p}}}_\mu \\ \dot{\lambda} \end{bmatrix} - \begin{bmatrix} \vec{F}_\mu \\ \overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{59}$$

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_\mu & \mathbf{L}_\lambda \\ \mathbf{0} & \mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \left[(1-\alpha)\begin{bmatrix} {}^n\tilde{\mathbf{u}}_\mu \\ {}^n\tilde{\mathbf{p}}_\mu \\ {}^n\lambda \end{bmatrix} + \alpha \begin{bmatrix} {}^{n+1}\tilde{\mathbf{u}}_\mu \\ {}^{n+1}\tilde{\mathbf{p}}_\mu \\ {}^{n+1}\lambda \end{bmatrix}\right] + \begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ \mathbf{Q}^T & \mathbf{S}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{{}^{n+1}\tilde{\mathbf{u}}_\mu - {}^n\tilde{\mathbf{u}}_\mu}{\Delta t} \\ \frac{{}^{n+1}\tilde{\mathbf{p}}_\mu - {}^n\tilde{\mathbf{p}}_\mu}{\Delta t} \\ \frac{{}^{n+1}\lambda - {}^n\lambda}{\Delta t} \end{bmatrix} - \begin{bmatrix} \dot{\vec{F}}_\mu \\ \overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{60}$$

$$\left[\begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ -\mathbf{Q}_\mu^T & -\left(\mathbf{S} + \Delta t\,\alpha\,\mathbf{H}_\mu\right) & -\Delta t\,\alpha\,\mathbf{L}_\lambda \\ \mathbf{0} & -\Delta t\,\alpha\,\mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{u}}_\mu \\ \Delta\tilde{\mathbf{p}}_\mu \\ \Delta\lambda \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta t\,\mathbf{H}_\mu & \Delta t\,\mathbf{L}_\lambda \\ \mathbf{0} & \Delta t\,\mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^n\tilde{\mathbf{u}}_\mu \\ {}^n\tilde{\mathbf{p}}_\mu \\ {}^n\lambda \end{bmatrix} - \begin{bmatrix} \dot{\vec{F}}_\mu \\ -\overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{61}$$

Resolviendo este ultimo sistema de ecuaciones NO CONVERGE en la micro-escala. Con $\Delta t$ elevados en la simulacion con paso de tiempo logaritmico el termino de fuerza adicional $\Delta t\ \mathbf{L}_\lambda\ {}^n\tilde{\mathbf{p}}_\mu$ 'crece mucho'.

CONVERGE si se opera de la siguiente manera

$$\hat{G}^h_\mu\left(\dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\mu, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = \left[\int_{\Omega^h_\mu} \left(\mathbf{B}^{u\,T}_\mu\,\dot{\boldsymbol{\sigma}}_\mu - \mathbf{N}^{u\,T}_\mu\,\mathbf{f}_\mu \otimes \mathbf{y}\right) d\Omega_\mu\right] \cdot \delta\dot{\tilde{\mathbf{u}}}_\mu = 0; \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}^h_\mu,\ \forall\,t, \tag{62}$$

$$H^h_\mu\left(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mu, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\mu, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\mu, \lambda, \delta\tilde{\mathbf{p}}_\mu\right) = \left[\int_{\Omega^h_\mu} \left(\mathbf{N}^{p\,T}_\mu\,\dot{\chi}_\mu - \mathbf{B}^{p\,T}_\mu\,\mathcal{V}_\mu + \mathbf{N}^{p\,T}_\mu\,\dot{\lambda}\right) d\Omega_\mu\right] \cdot \delta\tilde{\mathbf{p}}_\mu = 0; \quad \forall\,\delta\tilde{\mathbf{p}}_\mu \in \mathcal{P}^h_\mu,\ \forall\,t, \tag{63}$$

$$\hat{J}_\mu\left(\dot{\tilde{p}}_\mu, \delta\lambda\right) = \left[\int_{\Omega^h_\mu} \dot{\tilde{p}}_\mu\, d\Omega_\mu\right] \cdot \delta\lambda = 0; \quad \forall\,\delta\lambda \in ?,\ \forall\,t, \tag{64}$$

Quedando entonces,

$$\begin{aligned} &\left[\vec{F}_\mu + \mathbf{K}_\mu\,\dot{\tilde{\mathbf{u}}}_\mu - \mathbf{Q}_\mu\,\beta\,\dot{\tilde{\mathbf{p}}}_\mu\right] \cdot \delta\dot{\tilde{\mathbf{u}}}_\mu = 0 \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}^h_\mu,\ \forall\,t, \\ &\left[\overline{\mathbf{F}}_\mu + \mathbf{Q}_\mu^T\,\dot{\tilde{\mathbf{u}}}_\mu + \mathbf{S}_\mu\,\beta\,\dot{\tilde{\mathbf{p}}}_\mu + \mathbf{H}_\mu\,\tilde{\mathbf{p}}_\mu + \mathbf{L}_\alpha\,\dot{\lambda}\right] \cdot \delta\tilde{\mathbf{p}}_\mu = 0 \quad \forall\,\delta\tilde{\mathbf{p}}_\mu \in \mathcal{P}^h_\mu,\ \forall\,t, \\ &\left[\mathbf{L}_\lambda^T\,\dot{\tilde{\mathbf{p}}}_\mu\right] \cdot \delta\lambda = 0 \quad \forall\,\delta\lambda \in ?,\ \forall\,t, \end{aligned} \tag{65}$$

Que al integrar en el tiempo aplicando el metodo $\alpha$ queda

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}}_\mu \\ \tilde{\mathbf{p}}_\mu \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ \mathbf{Q}^T & \mathbf{S}_\mu & \mathbf{L}_\lambda \\ \mathbf{0} & \mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{u}}}_\mu \\ \dot{\tilde{\mathbf{p}}}_\mu \\ \dot{\lambda} \end{bmatrix} - \begin{bmatrix} \dot{\vec{F}}_\mu \\ \overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{66}$$

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \left[(1-\alpha)\begin{bmatrix} {}^n\tilde{\mathbf{u}}_\mu \\ {}^n\tilde{\mathbf{p}}_\mu \\ {}^n\lambda \end{bmatrix} + \alpha \begin{bmatrix} {}^{n+1}\tilde{\mathbf{u}}_\mu \\ {}^{n+1}\tilde{\mathbf{p}}_\mu \\ {}^{n+1}\lambda \end{bmatrix}\right] + \begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ \mathbf{Q}^T & \mathbf{S}_\mu & \mathbf{L}_\lambda \\ \mathbf{0} & \mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{{}^{n+1}\tilde{\mathbf{u}}_\mu - {}^n\tilde{\mathbf{u}}_\mu}{\Delta t} \\ \frac{{}^{n+1}\tilde{\mathbf{p}}_\mu - {}^n\tilde{\mathbf{p}}_\mu}{\Delta t} \\ \frac{{}^{n+1}\lambda - {}^n\lambda}{\Delta t} \end{bmatrix} - \begin{bmatrix} \dot{\vec{F}}_\mu \\ \overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{67}$$

$$\left[\begin{bmatrix} \mathbf{K}_\mu & -\mathbf{Q}_\mu & \mathbf{0} \\ -\mathbf{Q}_\mu^T & -\left(\mathbf{S} + \Delta t\,\alpha\,\mathbf{H}_\mu\right) & -\mathbf{L}_\lambda \\ \mathbf{0} & -\mathbf{L}_\lambda^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{u}}_\mu \\ \Delta\tilde{\mathbf{p}}_\mu \\ \Delta\lambda \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta t\,\mathbf{H}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^n\tilde{\mathbf{u}}_\mu \\ {}^n\tilde{\mathbf{p}}_\mu \\ {}^n\lambda \end{bmatrix} - \begin{bmatrix} \dot{\vec{F}}_\mu \\ -\overline{\mathbf{F}}_\mu \\ \mathbf{0} \end{bmatrix}\right] \cdot \begin{bmatrix} \delta\dot{\tilde{\mathbf{u}}}_\mu \\ \delta\tilde{\mathbf{p}}_\mu \\ \delta\lambda \end{bmatrix} = 0 \tag{68}$$

Este ultimo sistema de ecuaciones es el que esta implementado.

REVISAR el termino de fuerzas adicionales $\Delta t\mathbf{H}_\mu{}^n\tilde{\mathbf{p}}$. Si no considero el caso homogeno presenta micro poro presiones fluctuantes nulas y el caso heterogeneo CONVERGE!

Thus, through the Newton-Rahpson scheme it is possible to determine the unknown nodal vectors of displacements fluctuation and pore pressures fluctuation $\tilde{\bar{\mathbf{u}}}_\mu \in \mathcal{U}^h_\mu$ and $\tilde{\bar{\mathbf{p}}}_\mu \in \mathcal{P}^h_\mu$, respectively.

## 4 | COMPONENTS OF HOMOGENIZED TANGENT OPERATORS WITH CONSTAINT

The expressions of the homogenized tangent constitutive operators can be obtained considering the perturbed displacement fluctuation $\tilde{\mathbf{u}}_\epsilon = \tilde{\mathbf{u}}_\mu + \epsilon\, \tilde{\mathbf{u}}_{(\cdot)}$, the perturbed pore pressure fluctuation $\tilde{p}_\epsilon = \tilde{p}_\mu + \epsilon\, \tilde{p}_{(\cdot)}$ and the perturbed Lagrange multiplier $\lambda_\epsilon = \lambda + \epsilon\, \lambda_{(\cdot)}$ ? ?. Where $\tilde{\mathbf{u}}_{(\cdot)}$ is the incremental micro-displacement fluctuations, $\tilde{p}_{(\cdot)}$ is the incremental micro-pore pressure fluctuations and $\lambda_{(\cdot)}$ is the incremental Lagrange multiplier, all referred with respect to the incremental macro-scale variable $(\cdot)$.

At first, it is shown how it is determined the relationships between the fluctuations variables and the macro-strain tensor. These relations are simplified in determining the tangential relations between the incremental micro-variables ($\tilde{\mathbf{u}}_\varepsilon$, $\tilde{p}_\varepsilon$ and $\lambda_\epsilon$) and the perturbed macro-strains, $\boldsymbol{\varepsilon}_\epsilon = \boldsymbol{\varepsilon} + \epsilon\, \Delta\boldsymbol{\varepsilon}$. These are obtained by linearizing the balance equations defined by Eqs. (46)-(48)

$$\mathcal{L}\hat{G}_\mu\left(\dot{\boldsymbol{\varepsilon}}_\epsilon, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \hat{G}_\mu\left(\dot{\boldsymbol{\varepsilon}}_\epsilon, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0, \quad \forall\, \delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}_\mu, \forall\, t, \tag{69}$$

$$\mathcal{L}H_\mu\left(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}_\epsilon, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \dot{\lambda}_\epsilon \delta\tilde{p}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} H_\mu\left(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}_\epsilon, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \dot{\lambda}_\epsilon \delta\tilde{p}_\mu\right) = 0, \quad \forall\, \delta\tilde{p}_\mu \in \mathcal{P}_\mu, \forall\, t, \tag{70}$$

$$\mathcal{L}\hat{J}_\mu\left(\dot{\tilde{p}}_\epsilon, \delta\lambda\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \hat{J}_\mu\left(\tilde{p}_\epsilon, \delta\lambda\right) = 0, \quad \forall\, \delta\lambda \in ?, \forall\, t. \tag{71}$$

where $\tilde{\boldsymbol{\varepsilon}}_\epsilon = \boldsymbol{\nabla}^{sym}_y\, \tilde{\mathbf{u}}_\epsilon$ and $\tilde{\boldsymbol{\varphi}}_\epsilon = \boldsymbol{\nabla}_y\, \tilde{p}_\epsilon$. In the Eqs. (69-71), the space is discretized and it is obtained,

$$\begin{aligned}\int_{\Omega^h_\mu} \left(\boldsymbol{\nabla}^{sym}_y\, \delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{C}_\mu : \boldsymbol{\nabla}^{sym}_y\, \dot{\tilde{\mathbf{u}}}_\varepsilon\, d\Omega_\mu - \int_{\Omega^h_\mu} \left(\boldsymbol{\nabla}^{sym}_y\, \delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{b}_\mu\, \beta\, \dot{\tilde{p}}_\varepsilon\, d\Omega_\mu &= \\ = -\left[\int_{\Omega^h_\mu} \left(\boldsymbol{\nabla}^{sym}_y\, \delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{C}_\mu\, d\Omega_\mu\right] : \Delta\dot{\boldsymbol{\varepsilon}}, \quad \forall\, \delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}^h_\mu, \forall\, t,\end{aligned} \tag{72}$$

$$\begin{aligned}\int_{\Omega^h_\mu} \left(\delta\tilde{p}_\mu\right)^T \mathbf{b}_\mu : \boldsymbol{\nabla}^{sym}_y\, \dot{\tilde{\mathbf{u}}}_\varepsilon\, d\Omega_\mu + \int_{\Omega^h_\mu} \left(\delta\tilde{p}_\mu\right)^T \frac{1}{M_\mu}\, \beta\, \dot{\tilde{p}}_\varepsilon\, d\Omega_\mu + \int_{\Omega^h_\mu} \left(\boldsymbol{\nabla}_y\, \delta\tilde{p}_\mu\right)^T \cdot \mathbf{k}_\mu \cdot \boldsymbol{\nabla}_y\, \tilde{p}_\varepsilon\, d\Omega_\mu \\ + \int_{\Omega^h_\mu} \left(\delta\tilde{p}_\mu\right)^T d\Omega_\mu\, \lambda_\epsilon = -\left[\int_{\Omega^h_\mu} \left(\delta\tilde{p}_\mu\right)^T \mathbf{b}_\mu\, d\Omega_\mu\right] : \Delta\dot{\boldsymbol{\varepsilon}}, \quad \forall\, \delta\tilde{p}_\mu \in \mathcal{P}^h_\mu, \forall\, t,\end{aligned} \tag{73}$$

$$\delta\lambda \int_{\Omega_\mu} \dot{\tilde{p}}_\epsilon\, d\Omega_\mu = 0, \quad \forall\, \delta\lambda \in ?, \forall\, t. \tag{74}$$

Thus, given a $\Delta\boldsymbol{\varepsilon}$, it is necessary to find the fields $\tilde{\mathbf{u}}_\varepsilon = \mathbf{N}^u\, \tilde{\bar{\mathbf{u}}}_\varepsilon \in \mathcal{U}_\mu$, $\tilde{p}_\varepsilon = \mathbf{N}^p\, \tilde{\bar{\mathbf{p}}}_\epsilon \in \mathcal{P}_\mu$ and $\lambda_\epsilon = cte$ that solve the linear variational Eqs. (72-74) which must be solved by the implicit Euler backward method in the same way as the system of equations of the problem in the RVE and that can be rewritten as follows

$$\begin{aligned}&\mathbf{K}_\mu\, \dot{\tilde{\bar{\mathbf{u}}}}_\varepsilon - \mathbf{Q}_\mu\, \beta\, \dot{\tilde{\bar{\mathbf{p}}}}_\varepsilon = -\dot{\mathbf{f}}^u_\varepsilon, \ \forall\, t, \\ &\mathbf{Q}^T_\mu\, \dot{\tilde{\bar{\mathbf{u}}}}_\varepsilon + \mathbf{S}_\mu\, \beta\, \dot{\tilde{\bar{\mathbf{p}}}}_\varepsilon + \mathbf{H}_\mu\, \tilde{\bar{\mathbf{p}}}_\varepsilon + \mathbf{L}_\lambda\, \dot{\lambda}_\epsilon = -\mathbf{f}^p_\varepsilon, \ \forall\, t, \\ &\mathbf{L}^T_\lambda\, \dot{\tilde{\bar{p}}}_\epsilon = 0, \ \forall\, t.\end{aligned} \tag{75}$$

Este sistema se reuelve igual que 68.

Desde aca propongo los dos sistemas de ecuaciones restantes para obtener las relaciones tangenciales entre variables flcutuantes y las macro poro presiones y macro gradientes. Solamante que puede contener errores, pero la idea es como lo hecho con las macro deformaciones.

In addition, to determine the remaining tangent operators it is required to find the relationship between the fluctuation variables ($\tilde{\mathbf{u}}_\mu$, $\tilde{p}_\mu$, $\lambda$) and the macro-scale variables ($p$, $\boldsymbol{\varphi}$), as shown above for the strains. For this purpose, the balance equations, Eqs. (46)-(48), must be linearized. First, with respect to the perturbed macroscopic pore pressure, $p_\epsilon = p + \epsilon\,\Delta p$,

$$\mathcal{L}\hat{G}_\mu\left(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}_\epsilon, p_\epsilon, \beta\,\boldsymbol{\varphi}, \beta\,\tilde{p}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \hat{G}_\mu\left(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}_\epsilon, p_\epsilon, \beta\,\boldsymbol{\varphi}, \beta\,\tilde{p}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0, \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}_\mu, \forall\,t, \tag{76}$$

$$\mathcal{L}H_\mu\left(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}_\epsilon, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \lambda_\epsilon, \delta\tilde{p}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} H_\mu\left(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}_\epsilon, \beta\,\dot{\boldsymbol{\varphi}}, \beta\,\dot{\tilde{p}}_\epsilon, \lambda_\epsilon, \delta\tilde{p}_\mu\right) = 0, \quad \forall\,\delta\tilde{p}_\mu \in \mathcal{P}_\mu, \forall\,t, \tag{77}$$

$$\mathcal{L}\hat{J}_\mu\left(\tilde{p}_\epsilon, \delta\lambda\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \hat{J}_\mu\left(\tilde{p}_\epsilon, \delta\lambda\right) = 0, \quad \forall\,\delta\lambda \in ?, \forall\,t. \tag{78}$$

In the Eqs. (76-77), the space is discretized and it is obtained,

$$\begin{aligned}\int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{C}_\mu : \nabla_y^{sym}\,\tilde{\mathbf{u}}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu - \int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{b}_\mu\,\beta\,\tilde{p}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu = \\ = \left[\int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{b}_\mu\,d\Omega_\mu\right]\Delta p, \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}_\mu^h, \forall\,t,\end{aligned} \tag{79}$$

$$\begin{aligned}\int_{\Omega_\mu^h} \left(\delta\tilde{p}_\mu\right)^T\,\mathbf{b}_\mu : \nabla_y^{sym}\,\dot{\tilde{\mathbf{u}}}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu + \int_{\Omega_\mu^h} \left(\delta\tilde{p}_\mu\right)^T\,\frac{1}{M_\mu}\,\beta\,\dot{\tilde{p}}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu + \int_{\Omega_\mu^h} \left(\nabla_y\,\delta\tilde{p}_\mu\right)^T \cdot \mathbf{k}_\mu \cdot \nabla_y\,\tilde{p}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu = \\ + \int_{\Omega_\mu^h} \left(\delta\tilde{p}_\mu\right)^T \lambda_\epsilon\,d\Omega_\mu = -\left[\int_{\Omega_\mu} \left(\delta\tilde{p}_\mu\right)^T\,\frac{1}{M_\mu}\,d\Omega_\mu\right]\Delta\dot{p}, \quad \forall\,\delta\tilde{p}_\mu \in \mathcal{P}_\mu^h, \forall\,t.\end{aligned} \tag{80}$$

$$\int_{\Omega_\mu^h} (\delta\lambda)^T\,\tilde{p}_\epsilon\,d\Omega_\mu = 0, \quad \forall\,\delta\lambda \in ?, \forall\,t. \tag{81}$$

Therefore, for $\Delta\,p$, find the fields $\tilde{\mathbf{u}}_p = \mathbf{N}^u\,\tilde{\bar{\mathbf{u}}}_p \in \mathcal{U}_\mu$, $\tilde{p}_p = \mathbf{N}^p\,\tilde{\bar{\mathbf{p}}}_p \in \mathcal{P}_\mu$ and $\lambda_p = \mathbf{N}^\lambda\,\lambda_p$ solution to the linearized equations systems (see Eq. 82).

$$\begin{aligned}&\mathbf{K}_\mu\,\tilde{\bar{\mathbf{u}}}_p - \mathbf{Q}_\mu\,\beta\,\tilde{\bar{\mathbf{p}}}_p = \mathbf{f}_p^u, \forall\,t,\\ &\mathbf{Q}_\mu^T\,\dot{\tilde{\bar{\mathbf{u}}}}_p + \mathbf{S}_\mu\,\beta\,\dot{\tilde{\bar{\mathbf{p}}}}_p + \mathbf{H}_\mu\,\tilde{\bar{\mathbf{p}}}_p + \mathbf{L}_\lambda\,\lambda_p = -\mathbf{f}_p^p, \forall\,t,\\ &\mathbf{L}_\lambda\,\tilde{\bar{\mathbf{p}}}_\mu = 0, \forall\,t.\end{aligned} \tag{82}$$

Then concerning the perturbed macro-pore pressure gradient, $\boldsymbol{\varphi}_\epsilon = \boldsymbol{\varphi} + \epsilon\,\Delta\boldsymbol{\varphi}$.

$$\mathcal{L}\hat{G}_\mu\left(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}_\epsilon, p, \beta\,\boldsymbol{\varphi}_\epsilon, \beta\,\tilde{p}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \hat{G}_\mu\left(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}}_\epsilon, p, \beta\,\boldsymbol{\varphi}_\epsilon, \beta\,\tilde{p}_\epsilon, \delta\dot{\tilde{\mathbf{u}}}_\mu\right) = 0, \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}_\mu, \forall\,t, \tag{83}$$

$$\mathcal{L}H_\mu\left(\boldsymbol{\varphi}_\epsilon, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}_\epsilon, \beta\,\dot{\tilde{p}}_\epsilon, \delta\tilde{p}_\mu\right) = 0 \Rightarrow \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} H_\mu\left(\boldsymbol{\varphi}_\epsilon, \tilde{\boldsymbol{\varphi}}_\epsilon, \dot{\boldsymbol{\varepsilon}}, \dot{\tilde{\boldsymbol{\varepsilon}}}_\epsilon, \dot{p}, \beta\,\dot{\boldsymbol{\varphi}}_\epsilon, \beta\,\dot{\tilde{p}}_\epsilon, \delta\tilde{p}_\mu\right) = 0, \quad \forall\,\delta\tilde{p}_\mu \in \mathcal{P}_\mu, \forall\,t. \tag{84}$$

The equations systems to be solved are presented below.

$$\begin{aligned}\int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{C}_\mu : \nabla_y^{sym}\,\tilde{\mathbf{u}}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu - \int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{b}_\mu\,\beta\,\tilde{p}_{\boldsymbol{\varepsilon}}\,d\Omega_\mu = \\ = \beta\left[\int_{\Omega_\mu^h} \left(\nabla_y^{sym}\,\delta\dot{\tilde{\mathbf{u}}}_\mu\right)^T : \mathbf{b}_\mu \otimes \mathbf{y}\,d\Omega_\mu\right]\Delta p, \quad \forall\,\delta\dot{\tilde{\mathbf{u}}}_\mu \in \mathcal{U}_\mu^h, \forall\,t,\end{aligned} \tag{85}$$

$$\int_{\Omega_\mu^h} \left(\delta\tilde{p}_\mu\right)^T \mathbf{b}_\mu : \boldsymbol{\nabla}_y^{sym}\dot{\tilde{\mathbf{u}}}_{\boldsymbol{\varepsilon}}\, d\Omega_\mu + \int_{\Omega_\mu^h} \left(\delta\tilde{p}_\mu\right)^T \frac{1}{M_\mu}\beta\, \dot{\tilde{p}}_{\boldsymbol{\varepsilon}}\, d\Omega_\mu + \int_{\Omega_\mu^h} \left(\boldsymbol{\nabla}_y\,\delta\tilde{p}_\mu\right)^T \cdot \mathbf{k}_\mu \cdot \boldsymbol{\nabla}_y\,\tilde{p}_{\boldsymbol{\varepsilon}}\, d\Omega_\mu =$$
$$= -\beta\left[\int_{\Omega_\mu} \left(\delta\tilde{p}_\mu\right)^T \frac{1}{M_\mu}\mathbf{y}\, d\Omega_\mu\right]\Delta\dot{p} - \left[\int_{\Omega_\mu} \left(\boldsymbol{\nabla}_y\,\delta\tilde{p}_\mu\right)^T \cdot \mathbf{k}_\mu\, d\Omega_\mu\right]\boldsymbol{\Delta\varphi}, \quad \forall\,\delta\tilde{p}_\mu \in \mathcal{P}_\mu^h,\ \forall\, t. \tag{86}$$

Hence, for $\boldsymbol{\Delta\varphi}$, it is necessary to find the fields $\tilde{\mathbf{u}}_{\boldsymbol{\varphi}} = \mathbf{N}^u\,\tilde{\bar{\mathbf{u}}}_{\boldsymbol{\varphi}} \in \mathcal{U}_\mu$ and $\tilde{p}_{\boldsymbol{\varphi}} = \mathbf{N}^p\,\tilde{\bar{\mathbf{p}}}_{\boldsymbol{\varphi}} \in \mathcal{P}_\mu$ that solve the linear variational Eq. (87).

$$\begin{aligned} &\mathbf{K}_\mu\,\tilde{\bar{\mathbf{u}}}_{\boldsymbol{\varphi}} - \mathbf{Q}_\mu\,\beta\,\tilde{\bar{\mathbf{p}}}_{\boldsymbol{\varphi}} = \beta\,\mathbf{f}_{\boldsymbol{\varphi}}^u,\ \forall\, t,\\ &\mathbf{Q}_\mu^T\,\dot{\tilde{\bar{\mathbf{u}}}}_{\boldsymbol{\varphi}} + \mathbf{S}_\mu\,\beta\,\dot{\tilde{\bar{\mathbf{p}}}}_{\boldsymbol{\varphi}} + \mathbf{H}_\mu\,\tilde{\bar{\mathbf{p}}}_{\boldsymbol{\varphi}} = -\beta\,\mathbf{f}_{\boldsymbol{\varphi}}^{p1} - \mathbf{f}_{\boldsymbol{\varphi}}^{p2},\ \forall\, t. \end{aligned} \tag{87}$$

Once all the equations systems have been solved, it is possible to determine each of the homogenized tangent operators that allow to solve the macro-scale problem.