ELEC 4700

Assignment 2

Finite Difference Method

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Table of Contents

Figures

Part 1: Finite Difference in 1D region

In a one dimensional region that is 150 units long and 100 units wide, where there is a potential on one side and no potential on the other, the potential will tend to try to fill out the region and average itself out. This can be modelled by implementing a nodal network between the two sides and calculating the average across them. One method that can be used to calculate this is the Finite Difference method. It takes the potential of neighbouring nodes and sums them with the voltage at the node in question, and averaging to find the actual potential at that node. This can be done by populating a G matrix that is nx*ny nodes in the x direction and nx*ny nodes in the y direction, where nx and ny are the number of nodes of the nodal network in the x and y direction respectively. Next, a B matrix can be created to contain the boundary conditions set in the problem, where in this case, the left side of the matrix has the potential of V0 and the right side does not. Once these matrices have been created and populated, we can use the equation

$$GX = B$$

to solve for the X matrix, which, after being remapped to an nx by ny matrix, will be the final solution for the potential distribution between the two sides. This final solution is shown in the figure below:

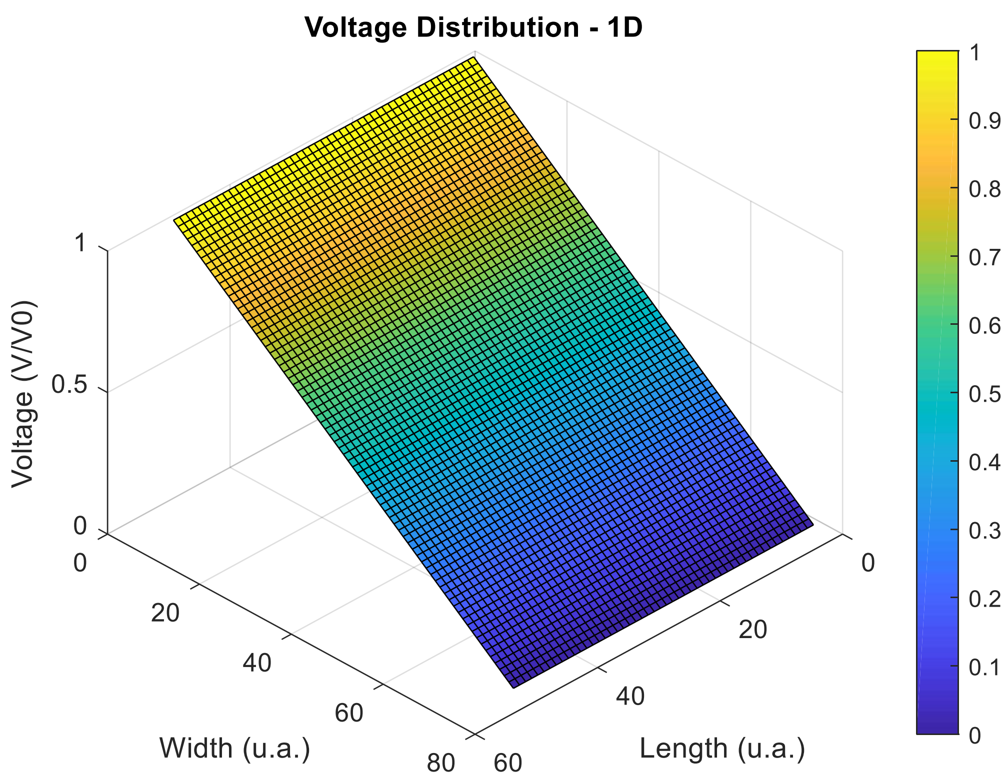


Figure 1: Voltage Distribution – 1D

Part 2: Finite Difference in 2D region

If the solution for the one dimensional region is shown to work as desired, the problem can be expanded into two dimensions. Finite Difference method can also be used to determine the solution for this problem as well. It is implemented almost the same way as in the one dimensional problem, but with a few additions to the populations of the G and B matrices. Instead of adding just the boundary conditions for the left and right side, the top and bottom sides of the boundary conditions have to be implemented as well. For this problem, the left and right sides of the region have a potential of V0 while the top and bottom of the region have no potential. This is implemented while populating the B matrix. Once the G and B matrices have been populated, we can solve for the X matrix and remap it like it was done in the one dimensional problem. The graph below shows the solution for the two dimensional region:

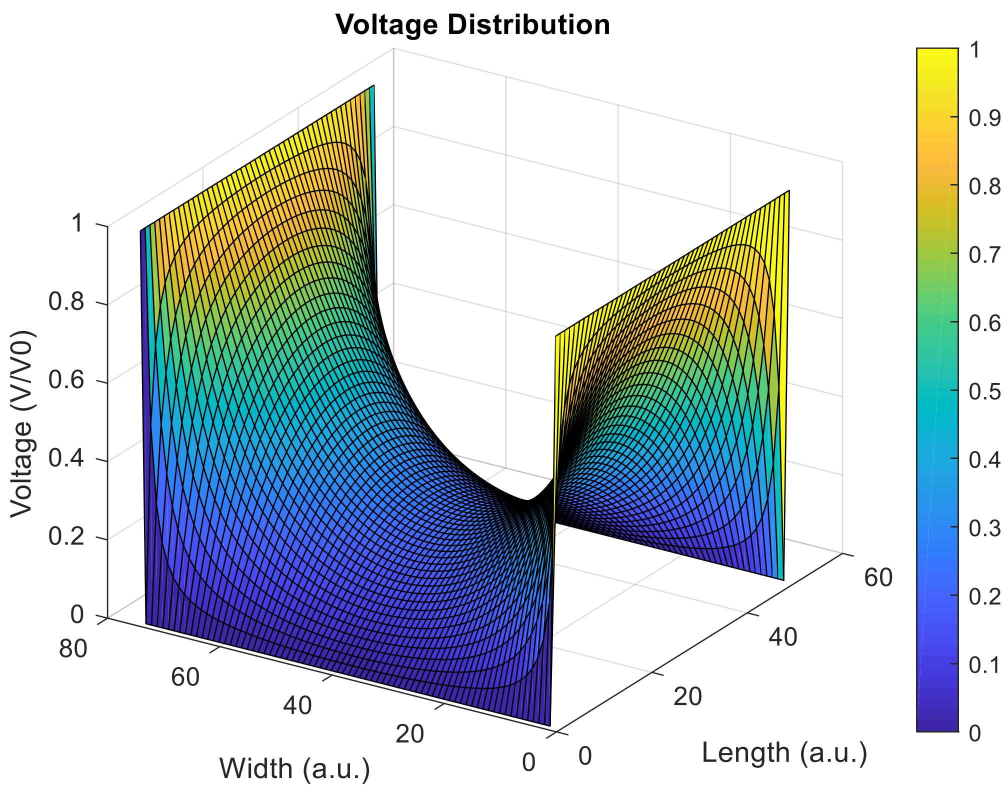


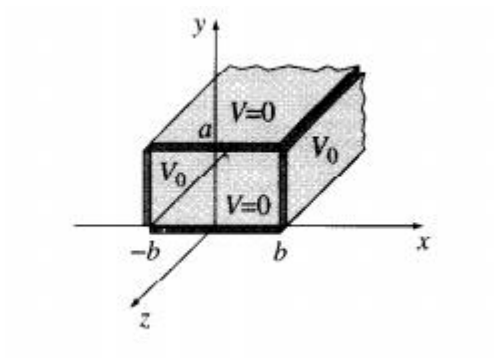
Figure 2: Voltage Distribution – 2D

Part 3: Analytical Series Solution to 2D region with Comparison

Finite Difference is not the only method that can be used to solve this problem. Another method is using an analytical series solution. This calculated the potential at a given node by the number of sums at the given node as well as the node's x and y coordinates within the nodal network. As described in Griffiths "Intro to Electrodynamics 3e", the equation:

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi x}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right)$$

can be used to solve this problem as well. For this equation, x and y are the x and y coordinates of the node in the network, a is the maximum y coordinate from 0, and b is half the maximum x coordinate from 0, as shown in the image below:



To solve this series solution, every x and y coordinate where $-b \le x \le b$ and $0 \le y \le a$ will be calculated for a given number of n sums. If we set b = 75 and a = 100, we can vary the number of sums we wish to do for each node. The more sums that are performed, the more accurate the solution will be. The following graph is produced for n = 151 sums:

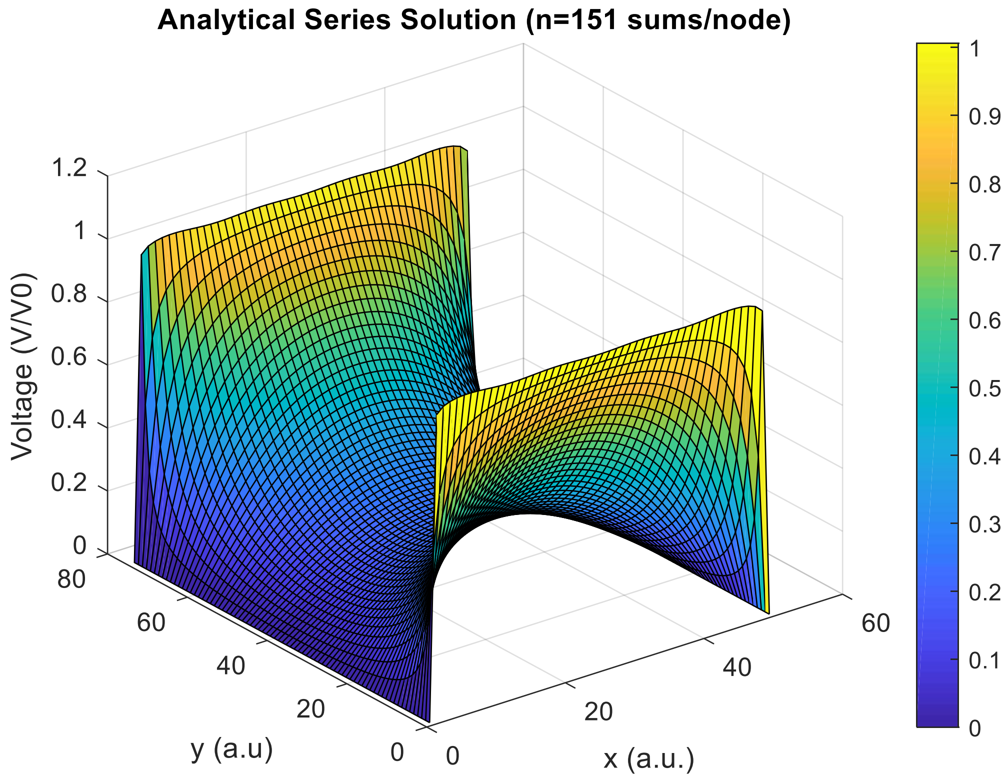


Figure 3:Analytical Series Solution for 151 Sums

When comparing Figure 2 and Figure 3, it can be seen that both solutions are very accurate. The more sums that are done in the analytical solution, the closer the final solution is to the finite difference solution. If we increase the number of nodes in the network, the final solution becomes even closer still.

Part 4: Bottle-Necks Included

Now, two small “boxes” will be added to the potential distribution that aim to add conductivity to the network. Two boxes, located halfway through the x direction of the region and that take up a third of the region around it are placed inside the region. Outside of the boxes, the conductivity of the region remains the same, so current flow doesn’t change. However, inside the boxes, the conductivity of the region is one percent of the conductivity of the region outside the box, so there is not as much current flow. This can be shown in the conductivity matrix below:

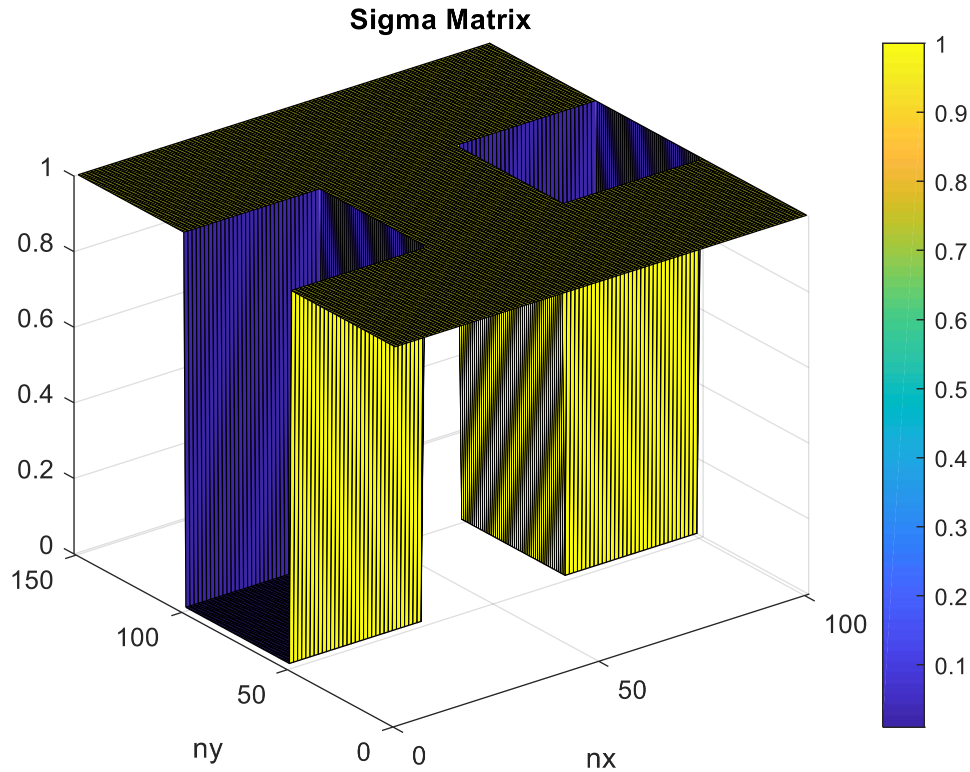


Figure 4: Sigma Matrix

When creating and populating this matrix, the G matrix can be created and populated. If the node's coordinates are within the two box regions, the G matrix is populated based on whether the coordinates are within the boxes or not. Next, the matrix is solved for the X matrix like it was done above to find the potential distribution. This is shown below:

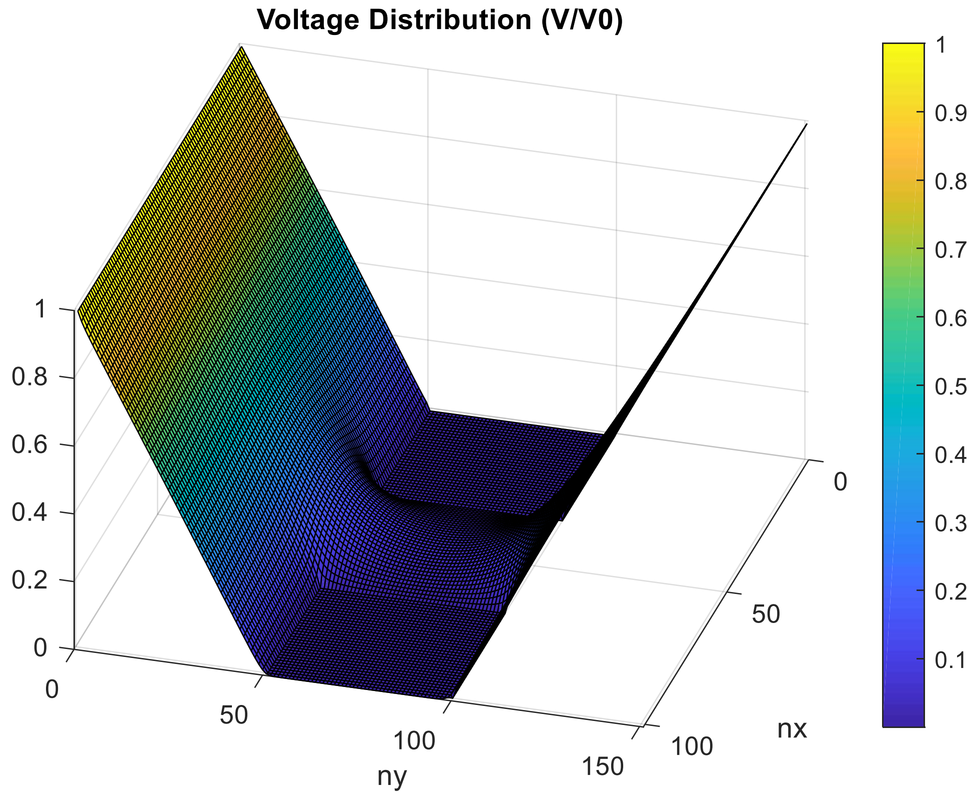


Figure 5: Voltage Distribution

The electric field of any potential is found by taking the gradient of the voltage of that region, so the Ex and Ey vectors of that electric field can be determined from taking the potential distribution gradient, which is shown below:

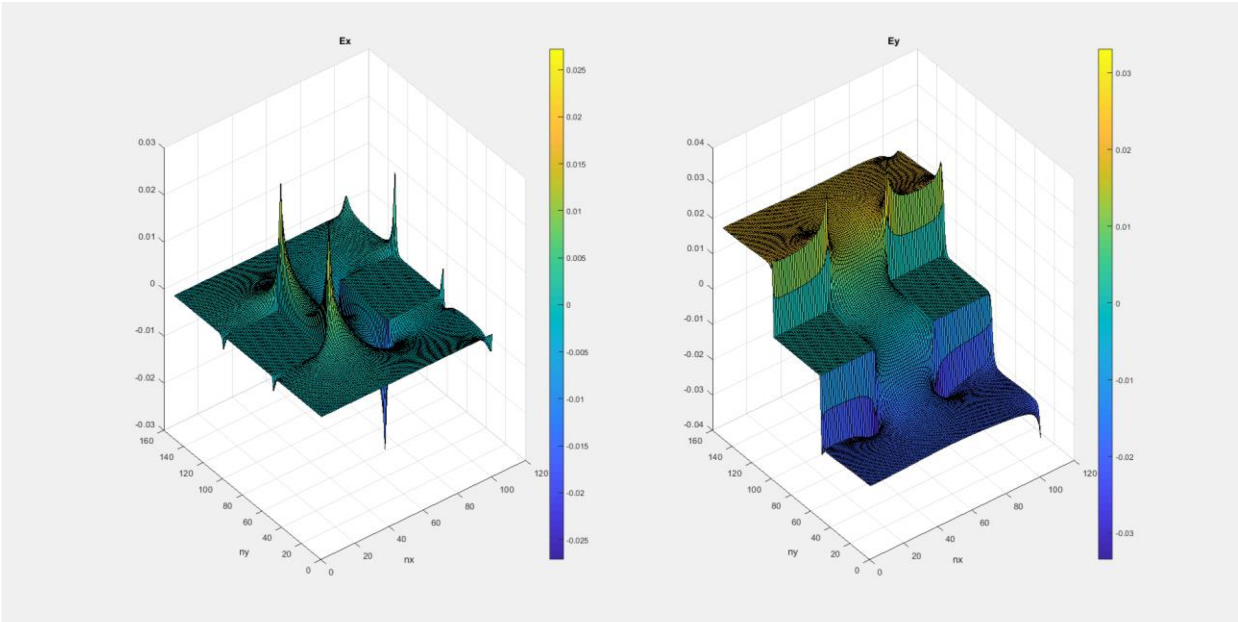


Figure 6: Ex and Ey

Finally, the current density can be calculated by multiplying the conductivity of a specific node by the gradient of the potential at that node. The current density is shown below:

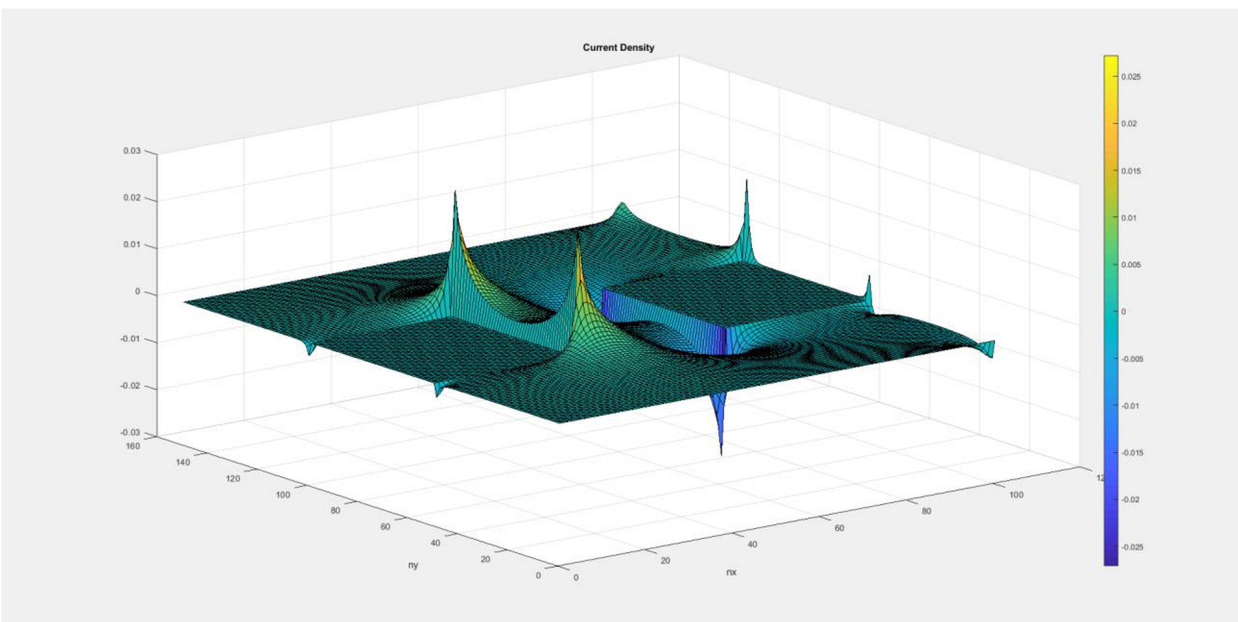


Figure 7: Current Density

As the number of nodes increases, the result becomes more accurate due to more data points being plotted. As we decrease the distance in the y direction between the two boxes, the middle section that current can flow through gets smaller and the voltage within it also gets smaller. If the value for sigma inside the boxes gets bigger, the peaks of the current density get smaller, and if the value for sigma inside the boxes gets smaller, the peaks of the current density get bigger.