and it would follow from 2.4.2 that $-1 \in \mathbb{R}^+$. But this is impossible because if $x \in \mathbb{R}^+$, then if $-1 \in \mathbb{R}$, $(-1)\,x = -x \in \mathbb{R}^+$ and contradicts 2.4.3 which states that either $-x$ or x is in $\mathbb{R}^+$ but not both.

Next consider 5. If $x < 0$, why is $x^{-1} < 0$? As before, $x^{-1} \neq 0$. If $x^{-1} > 0$, then as before,

$$-x\left(x^{-1}\right) = -1 \in \mathbb{R}^+$$

which was just shown not to occur.

Next consider 6. If $x < y$ why is $xz < yz$ if $z > 0$? This follows because

$$yz - xz = z\,(y - x) \in \mathbb{R}^+$$

since both z and $y - x \in \mathbb{R}^+$.

Next consider 7. If $x < y$ and $z < 0$, why is $xz > zy$? This follows because

$$zx - zy = z\,(x - y) \in \mathbb{R}^+$$

by what was proved in 3.

The next two claims are obvious and left for you.

Now suppose $xy > 0$. If $-x > 0$ and $y > 0$, then $-xy > 0$ contrary to $xy > 0$. It is similar if $x > 0$. Thus if $xy > 0$ either both x, y are positive or both $-x, -y$ are positive. In the second case, we say both x, y are negative. If both x, y are positive, then $xy > 0$ by the order axioms. If $-x, -y$ both positive, then $xy = (-1)^2\, xy = (-x)\,(-y) > 0$. ■

Note that trichotomy could be stated by saying $x \leq y$ or $y \leq x$.

Definition 2.4.6 $|x| \equiv \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$

Note that $|x|$ can be thought of as the distance between x and 0.

Theorem 2.4.7 $|xy| = |x|\,|y|$.

Proof: You can verify this by checking all available cases. Do so. ■

Theorem 2.4.8 *The following inequalities hold.*

$$|x + y| \leq |x| + |y|\,, \quad ||x| - |y|| \leq |x - y|\,.$$

Either of these inequalities may be called the triangle inequality.

Proof: First note that if $a, b \in \mathbb{R}^+ \cup \{0\}$ then $a \leq b$ if and only if $a^2 \leq b^2$. Here is why. Suppose $a \leq b$. Then by the properties of order proved above, $a^2 \leq ab \leq b^2$ because $b^2 - ab = b\,(b - a) \in \mathbb{R}^+ \cup \{0\}$. Next suppose $a^2 \leq b^2$. If both $a, b = 0$ there is nothing to show. Assume then they are not both 0. Then

$$b^2 - a^2 = (b + a)\,(b - a) \in \mathbb{R}^+.$$

By the above theorem on order, $(a + b)^{-1} \in \mathbb{R}^+$ and so using the associative law,

$$(a + b)^{-1}\,((b + a)\,(b - a)) = (b - a) \in \mathbb{R}^+$$

Now

$$\begin{aligned} |x + y|^2 &= (x + y)^2 = x^2 + 2xy + y^2 \\ &\leq |x|^2 + |y|^2 + 2\,|x|\,|y| = (|x| + |y|)^2 \end{aligned}$$

and so the first of the inequalities follows. Note I used $xy \leq |xy| = |x|\,|y|$ which follows from the definition.

To verify the other form of the triangle inequality, $x = x - y + y$ so $|x| \leq |x - y| + |y|$ and so $|x| - |y| \leq |x - y| = |y - x|$. Now repeat the argument replacing the roles of x and y to conclude $|y| - |x| \leq |y - x|$. Therefore, $||y| - |x|| \leq |y - x|$. ■