19. ↑Suppose you have a rational function $\frac{a(\lambda)}{b(\lambda)}$.

    (a) Show it is of the form $p(\lambda) + \frac{n(\lambda)}{\prod_{i=1}^{m} p_i(\lambda)^{m_i}}$ where $\{p_1(\lambda), \cdots, p_m(\lambda)\}$ are relatively prime and the degree of $n(\lambda)$ is less than the degree of $\prod_{i=1}^{m} p_i(\lambda)^{m_i}$.

    (b) Using Proposition 2.14.6 and the division algorithm for polynomials, show that the original rational function is of the form

    $$\hat{p}(\lambda) + \sum_{i=1}^{m} \sum_{k=1}^{m_i} \frac{n_{ki}(\lambda)}{p_i(\lambda)^k}$$

    where the degree of $n_{ki}(\lambda)$ is less than the degree of $p_i(\lambda)$ and $\hat{p}(\lambda)$ is some polynomial.

    This is the partial fractions expansion of the rational function. Actually carrying out this computation may be impossible, but this shows the existence of such a partial fractions expansion.

20. One can give a fairly simple algorithm to find the *g.c.d.*, greatest common divisor of two polynomials. The coefficients are in some field. For us, this will be either the real, rational, or complex numbers. However, in general, the algorithm for long division would be carried out in whatever field includes the coefficients. Explain the following steps. Let $r_0(\lambda), r_1(\lambda)$ be polynomials with the degree of $r_0(\lambda)$ at least as large as the degree of $r_1(\lambda)$. Then do division.

    $$r_0(\lambda) = r_1(\lambda) f_1(\lambda) + r_2(\lambda)$$

    where $r_2(\lambda)$ has smaller degree than $r_1(\lambda)$ or else is 0. If $r_2(\lambda)$ is 0, then the *g.c.d.* of $r_1(\lambda), r_0(\lambda)$ is $r_1(\lambda)$. Otherwise, $l(\lambda)/r_0(\lambda), r_1(\lambda)$ if and only if $l(\lambda)/r_1(\lambda), r_2(\lambda)$. Do division again

    $$r_1(\lambda) = r_2(\lambda) f_2(\lambda) + r_3(\lambda)$$

    where $\deg(r_3(\lambda)) < \deg(r_2(\lambda))$ or $r_3(\lambda)$ is 0. Then $l(\lambda)/r_2(\lambda), r_3(\lambda)$ if and only if $l(\lambda)/r_1(\lambda), r_2(\lambda)$ if and only if $l(\lambda)/r_0(\lambda), r_1(\lambda)$. If $r_3(\lambda) = 0$, then $r_2(\lambda)/r_2(\lambda), r_1(\lambda)$ so also $r_2(\lambda)/r_0(\lambda), r_1(\lambda)$ and also, if $l(\lambda)/r_0(\lambda), r_1(\lambda)$, then $l(\lambda)/r_1(\lambda), r_2(\lambda)$ and in particular, $l(\lambda)/r_1(\lambda)$ so if this happens, then $r_2(\lambda)$ is the *g.c.d.* of $r_0(\lambda)$ and $r_1(\lambda)$. Continue doing this. Eventually either $r_{m+1}(\lambda) = 0$ or has degree 0. If $r_{m+1}(\lambda) = 0$, then $r_m(\lambda)$ multiplied by a suitable scalar to make the result a monic polynomial is the *g.c.d.* of $r_0(\lambda)$ and $r_1(\lambda)$. If the degree is 0, then the two polynomials $r_0(\lambda), r_1(\lambda)$ must be relatively prime. It is really significant that this can be done because fundamental theorems in linear algebra depend on whether two polynomials are relatively prime having *g.c.d.* equal to 1. In this application, it is typically a question about a polynomial and its derivative.

21. Find the *g.c.d.* for $(x^4 + 3x^2 + 2), (x^2 + 3)$.

22. Find *g.c.d.* of $(x^5 + 3x^3 + x^2 + 3), (x^2 + 3)$.

23. Find the *g.c.d.* of $(x^6 + 2x^5 + x^4 + 3x^3 + 2x^2 + x + 2), (x^4 + 2x^3 + x + 2)$.

24. Find the *g.c.d.* of $(x^4 + 3x^3 + 2x + 1)$, $(4x^3 + 9x^2 + 2)$. If you do this one by hand, it might be made easier to note that the question of interest is resolved if you multiply everything with a nonzero scalar before you do long division.