12. Does there exist a subset of $\mathbb{C}$, $\mathbb{C}^+$ which satisfies 2.4.1 - 2.4.3? **Hint:** You might review the theorem about order. Show -1 cannot be in $\mathbb{C}^+$. Now ask questions about $-i$ and i. In mathematics, you can sometimes show certain things do not exist. It is very seldom you can do this outside of mathematics. For example, does the Loch Ness monster exist? Can you prove it does not?

13. Show that if a/b is irrational, then $\{ma+nb\}_{m,n\in\mathbb{Z}}$ is dense in $\mathbb{R}$, each an irrational number. If a/b is rational, show that $\{ma+nb\}_{m,n\in\mathbb{Z}}$ is not dense. **Hint:** From Theorem 2.16.1 there exist integers, m_l, n_l such that $|m_l a + n_l b| < 2^{-l}$. Let $P_l \equiv \cup_{k\in\mathbb{Z}} \{k(m_l a + n_l b)\}$. Thus this is a collection of numbers which has successive numbers 2^{-l} apart. Then consider $\cup_{l\in\mathbb{N}} P_l$. In case the ratio is rational and $\{ma+nb\}_{m,n\in\mathbb{Z}}$ is dense, explain why there are relatively prime integers p, q such that $p/q = a/b$ is rational and $\{mp+nq\}_{m,n\in\mathbb{Z}}$ would be dense. Isn't this last a collection of integers?

14. This problem will show, as a special case, that the rational numbers are dense in $\mathbb{R}$. Referring to the proof of Theorem 2.16.1.

    (a) Suppose $\alpha \in (0,1)$ and is irrational. Show that if N is a positive integer, then there are integers m, n such that $0 < n \leq N$ and $|n\alpha - m| < \frac{1}{N}\frac{1}{2}(1+\alpha) < \frac{1}{N}$. Thus $\left|\alpha - \frac{m}{n}\right| < \frac{1}{nN} \leq \frac{1}{n^2}$.

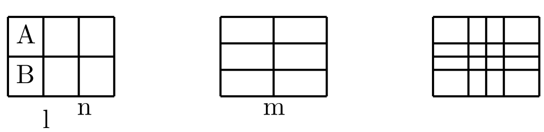
    (b) Show that if β is any nonzero irrational number, and N is a positive integer, there exists $0 < n \leq N$ and an integer m such that $\left|\beta - \frac{m}{n}\right| < \frac{1}{nN} \leq \frac{1}{n^2}$. **Hint:** You might consider $\beta - [\beta] \equiv \alpha$ where $[\beta]$ is the integer no larger than β which is as large as possible.

    (c) Next notice that from the proof, the same will hold for any β a positive number. **Hint:** In the proof, if there is a repeat in the list of numbers, then you would have an exact approximation. Otherwise, the pigeon hole principle applies as before. Now explain why nothing changes if you only assume β is a nonzero real number.

15. This problem outlines another way to see that rational numbers are dense in $\mathbb{R}$. Pick $x \in \mathbb{R}$. Explain why there exists m_l, the smallest integer such that $2^{-l} m_l \geq x$ so $x \in (2^{-l}(m_l - 1), 2^{-l} m_l]$. Now note that $2^{-l} m_l$ is rational and closer to x than 2^{-l}.

16. You have a rectangle R having length 4 and height 3. There are six points in R. One is at the center. Show that two of them are as close as $\sqrt{5}$. You might use pigeon hole principle.

17. Do the same problem without assuming one point is at the center. **Hint:** Consider the pictures. If not, then by pigeon hole principle, there is exactly one point in each of the six rectangles in first two pictures.



18. Suppose $r(\lambda) = \frac{a(\lambda)}{p(\lambda)^m}$ where $a(\lambda)$ is a polynomial and $p(\lambda)$ is an irreducible polynomial meaning that the only polynomials dividing $p(\lambda)$ are numbers and scalar multiples of $p(\lambda)$. That is, you can't factor it any further. Show that $r(\lambda)$ is of the form

$$r(\lambda) = q(\lambda) + \sum_{k=1}^{m} \frac{b_k(\lambda)}{p(\lambda)^k}, \text{ where degree of } b_k(\lambda) < \text{ degree of } p(\lambda)$$