

the same thing. Thus the rest of the proof goes just as before with real numbers. Using the results of Problem 6 on Page 36, the following holds.

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
 &= |z|^2 + |w|^2 + z\bar{w} + \overline{wz} \\
 &= |z|^2 + |w|^2 + 2\operatorname{Re} z\bar{w} \\
 &\leq |z|^2 + |w|^2 + 2|z\bar{w}| = |z|^2 + |w|^2 + 2|z||w| \\
 &= (|z| + |w|)^2
 \end{aligned}$$

and so  $|z + w| \leq |z| + |w|$  as claimed. The other inequality follows as before.

$$|z| \leq |z - w| + |w|$$

and so  $|z| - |w| \leq |z - w| = |w - z|$ . Now do the same argument switching the roles of  $z$  and  $w$  to conclude

$$|z| - |w| \leq |z - w|, \quad |w| - |z| \leq |z - w|$$

which implies the desired inequality. ■

Since  $\mathbb{R} \subseteq \mathbb{C}$  and the absolute value is consistently defined, the inequality holds also on  $\mathbb{R}$ .

## 2.14 Dividing Polynomials

It will be very important to be able to work with polynomials, especially in subjects like linear algebra and with the technique of partial fractions. It is surprising how useful this junior high material will be. In this section, a polynomial is an expression. Later, the expression will be used to define a function. These two ways of looking at a polynomial are very different.

**Definition 2.14.1** A polynomial is an expression of the form  $p(\lambda) =$

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0,$$

$a_n \neq 0$  where the  $a_i$  are (real or complex) numbers. Two polynomials are equal means that **the coefficients match for each power of  $\lambda$** . The degree of a polynomial is the largest power of  $\lambda$ . Thus the degree of the above polynomial is  $n$ . Addition of polynomials is defined in the usual way as is multiplication of two polynomials. The leading term in the above polynomial is  $a_n \lambda^n$ . The coefficient of the leading term is called the leading coefficient. It is called a monic polynomial if  $a_n = 1$ . A **root** of a polynomial  $p(\lambda)$  is  $\mu$  such that  $p(\mu) = 0$ . This is also called a zero.

Note that the degree of the zero polynomial is not defined in the above. The following is called the division algorithm. First is an important property of multiplication.

**Lemma 2.14.2** If  $f(\lambda)g(\lambda) = 0$ , then either  $f(\lambda) = 0$  or  $g(\lambda) = 0$ . That is, there are no nonzero divisors of 0.

**Proof:** Let  $f(\lambda)$  have degree  $n$  and  $g(\lambda)$  degree  $m$ . If  $m + n = 0$ , it is easy to see that the conclusion holds. Suppose the conclusion holds for  $m + n \leq M$  and suppose  $m + n = M + 1$ . Then  $f(\lambda)g(\lambda) =$

$$\begin{aligned}
 &(a_0 + a_1 \lambda + \cdots + a_{n-1} \lambda^{n-1} + a_n \lambda^n)(b_0 + b_1 \lambda + \cdots + b_{m-1} \lambda^{m-1} + b_m \lambda^m) \\
 &= (a(\lambda) + a_n \lambda^n)(b(\lambda) + b_m \lambda^m) \\
 &= a(\lambda)b(\lambda) + b_m \lambda^m a(\lambda) + a_n \lambda^n b(\lambda) + a_n b_m \lambda^{n+m}
 \end{aligned}$$