2.9 Exercises

1. By Theorem 2.7.9 it follows that for $a < b$, there exists a rational number between a and b. Show there exists an integer k such that $a < \frac{k}{2^m} < b$ for some k, m integers.

2. Show there is no smallest number in $(0, 1)$. Recall $(0, 1)$ means the real numbers which are strictly larger than 0 and smaller than 1.

3. Show there is no smallest number in $\mathbb{Q} \cap (0, 1)$.

4. Show that if $S \subseteq \mathbb{R}$ and S is well ordered with respect to the usual order on $\mathbb{R}$ then S cannot be dense in $\mathbb{R}$.

5. Prove by induction that $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.

6. It is a fine thing to be able to prove a theorem by induction but it is even better to be able to come up with a theorem to prove in the first place. Derive a formula for $\sum_{k=1}^{n} k^4$ in the following way. Look for a formula in the form $An^5 + Bn^4 + Cn^3 + Dn^2 + En + F$. Then try to find the constants A, B, C, D, E, and F such that things work out right. In doing this, show

$$(n+1)^4 =$$

$$\left(A(n+1)^5 + B(n+1)^4 + C(n+1)^3 + D(n+1)^2 + E(n+1) + F\right)$$

$$-An^5 + Bn^4 + Cn^3 + Dn^2 + En + F$$

and so some progress can be made by matching the coefficients. When you get your answer, prove it is valid by induction.

7. Prove by induction that whenever $n \geq 2$, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

8. If $r \neq 0$, show by induction that $\sum_{k=1}^{n} ar^k = a\frac{r^{n+1}}{r-1} - a\frac{r}{r-1}$.

9. Prove by induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

10. Let a and d be real numbers. Find a formula for $\sum_{k=1}^{n} (a + kd)$ and then prove your result by induction.

11. Consider the geometric series, $\sum_{k=1}^{n} ar^{k-1}$. Prove by induction that if $r \neq 1$, then $\sum_{k=1}^{n} ar^{k-1} = \frac{a - ar^n}{1-r}$.

12. This problem is a continuation of Problem 11. You put money in the bank and it accrues interest at the rate of r per payment period. These terms need a little explanation. If the payment period is one month, and you started with \$100 then the amount at the end of one month would equal $100(1+r) = 100 + 100r$. In this the second term is the interest and the first is called the principal. Now you have $100(1+r)$ in the bank. How much will you have at the end of the second month? By analogy to what was just done it would equal

$$100(1+r) + 100(1+r)r = 100(1+r)^2.$$

The amount you would have at the end of n months would be $100(1+r)^n$. (When a bank says they offer 6% compounded monthly, this means r, the rate per payment period equals .06/12.) In general, suppose you start with P and it sits in the bank for n payment periods. Then at the end of the n^{th} payment period, you would have $P(1+r)^n$ in the bank. In an ordinary annuity, you make payments, P at the end of each payment period, the first payment at the end of the first payment