7. Using the preceding problem, show the number of ways of selecting a set of k things from a set of n things is $\binom{n}{k}$.

8. Prove the binomial theorem from Problem 7. **Hint:** When you take $(x+y)^n$, note that the result will be a sum of terms of the form, $a_k x^{n-k} y^k$ and you need to determine what a_k should be. Imagine writing $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ where there are n factors in the product. Now consider what happens when you multiply. Each factor contributes either an x or a y to a typical term.

9. Prove by induction that $n < 2^n$ for all natural numbers, $n \geq 1$.

10. Prove by the binomial theorem and Problem 7 that the number of subsets of a given finite set containing n elements is 2^n.

11. Let n be a natural number and let $k_1 + k_2 + \cdots k_r = n$ where k_i is a non negative integer. The symbol

$$\binom{n}{k_1 k_2 \cdots k_r}$$

denotes the number of ways of selecting r subsets of $\{1, \cdots, n\}$ which contain $k_1, k_2 \cdots k_r$ elements in them. Find a formula for this number.

12. Is it ever the case that $(a+b)^n = a^n + b^n$ for a and b positive real numbers?

13. Is it ever the case that $\sqrt{a^2+b^2} = a+b$ for a and b positive real numbers?

14. Is it ever the case that $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$ for x and y positive real numbers?

15. Derive a formula for the multinomial expansion, $\left(\sum_{k=1}^{p} a_k\right)^n$ which is analogous to the binomial expansion. **Hint:** See Problem 8.

16. Suppose $a > 0$ and that x is a real number which satisfies the quadratic equation,

$$ax^2 + bx + c = 0.$$

Find a formula for x in terms of a and b and square roots of expressions involving these numbers. **Hint:** First divide by a to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Then add and subtract the quantity $b^2/4a^2$. Verify that

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2.$$

Now solve the result for x. The process by which this was accomplished in adding in the term $b^2/4a^2$ is referred to as completing the square. You should obtain the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is called the discriminant. When it is positive there are two different real roots. When it is zero, there is exactly one real root and when it equals a negative number there are no real roots.

17. Find u such that $-\frac{b}{2} + u$ and $-\frac{b}{2} - u$ are roots of $x^2 + bx + c = 0$. Obtain the quadratic formula from this. [2]

[2] The ancient Babylonians knew how to solve these quadratic equations sometime before 1700 B.C.