Therefore, some interval contains two points of P_N.[3] But each interval has length no more than C/N and so there exist $k, \hat{k}, l, \hat{l}$ integers such that

$$\left|ka + lb - \left(\hat{k}a + \hat{l}b\right)\right| \equiv |ma + nb| < \frac{C}{N}$$

Now let $\varepsilon > 0$ be given. Choose N large enough that $C/N < \varepsilon$. Then the above inequality holds for some integers m, n. ■

2.17 Exercises

1. Let $z = 5 + i9$. Find z^{-1}.

2. Let $z = 2 + i7$ and let $w = 3 - i8$. Find $zw, z + w, z^2$, and w/z.

3. If z is a complex number, show there exists ω a complex number with $|\omega| = 1$ and $\omega z = |z|$.

4. For those who know about the trigonometric functions from calculus or trigonometry[4], De Moivre's theorem says $[r(\cos t + i \sin t)]^n = r^n (\cos nt + i \sin nt)$ for n a positive integer. Prove this formula by induction. Does this formula continue to hold for all integers n, even negative integers? Explain.

5. Using De Moivre's theorem from Problem 4, derive a formula for $\sin(5x)$ and one for $\cos(5x)$. **Hint:** Use Problem 18 on Page 24 and if you like, you might use Pascal's triangle to construct the binomial coefficients.

6. If z, w are complex numbers prove $\overline{zw} = \overline{z}\overline{w}$ and then show by induction that $\overline{z_1 \cdots z_m} = \overline{z_1} \cdots \overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.

7. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where all the a_k are real numbers. Suppose also that $p(z) = 0$ for some $z \in \mathbb{C}$. Show it follows that $p(\overline{z}) = 0$ also.

8. I claim that $1 = -1$. Here is why. $-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$. This is clearly a remarkable result but is there something wrong with it? If so, what is wrong? **Hint:** When we push symbols without consideration of their meaning, we can accomplish many strange and wonderful but false things.

9. De Moivre's theorem of Problem 4 is really a grand thing. I plan to use it now for rational exponents, not just integers. $1 = 1^{(1/4)} = (\cos 2\pi + i \sin 2\pi)^{1/4} = \cos(\pi/2) + i \sin(\pi/2) = i$. Therefore, squaring both sides it follows $1 = -1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?

10. Review Problem 4 at this point. Now here is another question: If n is an integer, is it always true that $(\cos\theta - i \sin\theta)^n = \cos(n\theta) - i \sin(n\theta)$? Explain.

11. Suppose you have any polynomial in $\cos\theta$ and $\sin\theta$. By this I mean an expression of the form $\sum_{\alpha=0}^m \sum_{\beta=0}^n a_{\alpha\beta} \cos^\alpha \theta \sin^\beta \theta$ where $a_{\alpha\beta} \in \mathbb{C}$. Can this always be written in the form $\sum_{\gamma=-(n+m)}^{m+n} b_\gamma \cos\gamma\theta + \sum_{\tau=-(n+m)}^{n+m} c_\tau \sin\tau\theta$? Explain.

[3] This is called the pigeon hole principle. It was used by Jacobi and Dirichlet. Later, Besicovitch used it in his amazing covering theorem. In terms of pigeons, it says that if you have more pigeons than holes and they each need to go in a hole, then some hole must have more than one pigeon.

[4] I will present a treatment of the trig functions which is independent of plane geometry a little later.