Definition 2.10.4 *Let $f(a,b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are nonempty sets which means that $f(a,b)$ is either a number, ∞, or $-\infty$. The symbol, $+\infty$ is interpreted as a point out at the end of the number line which is larger than every real number. Of course there is no such number. That is why it is called ∞. The symbol, $-\infty$ is interpreted similarly. Then $\sup_{a\in A} f(a,b)$ means $\sup(S_b)$ where $S_b \equiv \{f(a,b) : a \in A\}$. A similar convention holds for* inf .

Unlike limits, you can take the sup in different orders, same for inf .

Lemma 2.10.5 *Let $f(a,b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are sets. Then*

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Also, you can replace sup *with* inf .

Proof: Note that for all a, b, $f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b)$ and therefore, for all a, $\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b)$. Therefore,

$$\sup_{a\in A}\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Repeat the same argument interchanging a and b, to get the conclusion of the lemma. Similar considerations give the same result for inf . ■

2.11 Existence of Roots

What is $\sqrt[5]{7}$ and does it even exist? You can ask for it on your calculator and the calculator will give you a number which multiplied by itself 5 times will yield a number which is close to 7 but it isn't exactly right. Why should there exist a number which works exactly? Every one you find, appears to be some sort of approximation at best. If you can't produce one, why should you believe it is even there? The following is an argument that roots exist. You fill in the details of the argument. Basically, roots exist in analysis because of completeness of the real line. Here is a lemma.

Lemma 2.11.1 *Suppose $n \in \mathbb{N}$ and $a > 0$. Then if $x^n - a \neq 0$, there exists $\delta > 0$ such that whenever $y \in (x-\delta, x+\delta)$, it follows $y^n - a \neq 0$ and has the same sign as $x^n - a$.*

Proof: From the binomial theorem and the triangle inequality, assuming always that $|y-x| < 1$, and $(y^n - a) = ((y-a+a)^n - a)$,

$$\begin{aligned}
(x^n - a)(y^n - a) &= (x^n - a)\left(\sum_{k=0}^{n-1}\binom{n}{k}(y-x)^{n-k}x^k + (x^n - a)\right) \\
&= (x^n - a)^2 + (x^n - a)\sum_{k=0}^{n-1}\binom{n}{k}(y-x)^{n-k}x^k \\
&\geq |x^n - a|^2 - |x^n - a|\,|y-x|\sum_{k=0}^{n-1}\binom{n}{k}|y-x|^{n-(k+1)}|x|^k \\
&\geq |x^n - a|^2 - |x^n - a|\,|y-x|\sum_{k=0}^{n-1}\binom{n}{k}|x|^k
\end{aligned}$$