2.7 Well Ordering and Archimedean Property

Definition 2.7.1 *A set is well ordered if every nonempty subset S, contains a smallest element z having the property that $z \leq x$ for all $x \in S$.*

Axiom 2.7.2 *Any set of integers larger than a given number is well ordered.*

In particular, the natural numbers defined as $\mathbb{N} \equiv \{1, 2, \cdots\}$ is well ordered.

The above axiom implies the principle of mathematical induction.

Theorem 2.7.3 *(Mathematical induction) A set $S \subseteq \mathbb{Z}$, having the property that $a \in S$ and $n+1 \in S$ whenever $n \in S$ contains all integers $x \in \mathbb{Z}$ such that $x \geq a$.*

Proof: Let $T \equiv ([a, \infty) \cap \mathbb{Z}) \setminus S$. Thus T consists of all integers larger than or equal to a which are not in S. The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T, denoted as b. It must be the case that $b > a$ since by definition, $a \notin T$. Then the integer, $b - 1 \geq a$ and $b - 1 \notin S$ because if $b - 1 \in S$, then $b - 1 + 1 = b \in S$ by the assumed property of S. Therefore, $b - 1 \in ([a, \infty) \cap \mathbb{Z}) \setminus S = T$ which contradicts the choice of b as the smallest element of T. ($b - 1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that everything in $[a, \infty) \cap \mathbb{Z}$ is also in S. ■

Mathematical induction is a very useful device for proving theorems about the integers.

Example 2.7.4 *Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.*

By inspection, if $n = 1$ then the formula is true. The sum yields 1 and so does the formula on the right. Suppose this formula is valid for some $n \geq 1$ where n is an integer. Then

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 .$$

The step going from the first to the second equality is based on the assumption that the formula is true for n. This is called the induction hypothesis. Now simplify the expression in the second line,

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 .$$

This equals $(n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right)$ and

$$\frac{n(2n+1)}{6} + (n+1) = \frac{6(n+1) + 2n^2 + n}{6} = \frac{(n+2)(2n+3)}{6}$$

Therefore,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},$$

showing the formula holds for $n+1$ whenever it holds for n. This proves the formula by mathematical induction.

Example 2.7.5 *Show that for all $n \in \mathbb{N}$, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.*