27. Obtain a number, $\delta > 0$, such that if $|x-1| < \delta$, then $\left|x^2-1\right| < 1/10$.

28. Obtain a number, $\delta > 0$, such that if $|x-4| < \delta$, then $|\sqrt{x}-2| < 1/10$.

29. Suppose $\varepsilon > 0$ is a given positive number. Obtain a number, $\delta > 0$, such that if $|x-1| < \delta$, then $|\sqrt{x}-1| < \varepsilon$. **Hint:** This δ will depend in some way on ε. You need to tell how.

2.6 The Binomial Theorem

Consider the following problem: You have the integers $S_n = \{1, 2, \cdots, n\}$ and k is an integer no larger than n. How many ways are there to fill k slots with these integers starting from left to right if whenever an integer from S_n has been used, it cannot be re used in any succeeding slot?

$$\overbrace{\_\_\_\_, \_\_\_\_, \_\_\_\_, \_\_\_\_, \cdots, \_\_\_\_}^{k \text{ of these slots}}$$

This number is known as permutations of n things taken k at a time and is denoted by $P(n,k)$. It is easy to figure it out. There are n choices for the first slot. For each choice for the fist slot, there remain $n-1$ choices for the second slot. Thus there are $n(n-1)$ ways to fill the first two slots. Now there remain $n-2$ ways to fill the third. Thus there are $n(n-1)(n-2)$ ways to fill the first three slots. Continuing this way, you see there are

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

ways to do this.

Now define for k a positive integer, $k! \equiv k(k-1)(k-2)\cdots 1$, $0! \equiv 1$. This is called k factorial. Thus $P(k,k) = k!$ and you should verify that $P(n,k) = \frac{n!}{(n-k)!}$. Now consider the number of ways of selecting a set of k different numbers from S_n. For each set of k numbers there are $P(k,k) = k!$ ways of listing these numbers in order. Therefore, denoting by $\left(\begin{array}{c} n \\ k \end{array}\right)$ the number of ways of selecting a set of k numbers from S_n, it must be the case that

$$\left(\begin{array}{c} n \\ k \end{array}\right) k! = P(n,k) = \frac{n!}{(n-k)!}$$

Therefore, $\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$. How many ways are there to select no numbers from S_n? Obviously one way. Note the above formula gives the right answer in this case as well as in all other cases due to the definition which says $0! = 1$.

Now consider the problem of writing a formula for $(x+y)^n$ where n is a positive integer. Imagine writing it like this:

$$\overbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$$

Then you know the result will be sums of terms of the form $a_k x^k y^{n-k}$. What is a_k? In other words, how many ways can you pick x from k of the factors above and y from the other $n-k$. There are n factors so the number of ways to do it is $\left(\begin{array}{c} n \\ k \end{array}\right)$. Therefore, a_k is the above formula and so this proves the following important theorem known as the binomial theorem.

Theorem 2.6.1 *The following formula holds for any n a positive integer.*

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k y^{n-k}.$$