20. Using the binomial theorem prove that for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}$.
Hint: Show first that $\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$. By the binomial theorem,
$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{\overbrace{n\cdot(n-1)\cdots(n-k+1)}^{k\text{ factors}}}{k!n^k}.$$
Now consider the term $\frac{n\cdot(n-1)\cdots(n-k+1)}{k!n^k}$ and note that a similar term occurs in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$ except that n is replaced with $n+1$ whereever this occurs. Argue the term got bigger and then note that in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$, there are more terms.

21. Prove by induction that for all $k \geq 4$, $2^k \leq k!$

22. Use the Problems 21 and 20 to verify for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n \leq 3$.

23. Prove by induction that $1 + \sum_{i=1}^{n} i\,(i!) = (n+1)!$.

24. I can jump off the top of the Empire State Building without suffering any ill effects. Here is the proof by induction. If I jump from a height of one inch, I am unharmed. Furthermore, if I am unharmed from jumping from a height of n inches, then jumping from a height of $n+1$ inches will also not harm me. This is self evident and provides the induction step. Therefore, I can jump from a height of n inches for any n. What is the matter with this reasoning?

25. All horses are the same color. Here is the proof by induction. A single horse is the same color as himself. Now suppose the theorem that all horses are the same color is true for n horses and consider $n+1$ horses. Remove one of the horses and use the induction hypothesis to conclude the remaining n horses are all the same color. Put the horse which was removed back in and take out another horse. The remaining n horses are the same color by the induction hypothesis. Therefore, all $n+1$ horses are the same color as the $n-1$ horses which didn't get moved. This proves the theorem. Is there something wrong with this argument?

26. Let $\binom{n}{k_1, k_2, k_3}$ denote the number of ways of selecting a set of k_1 things, a set of k_2 things, and a set of k_3 things from a set of n things such that $\sum_{i=1}^{3} k_i = n$. Find a formula for $\binom{n}{k_1, k_2, k_3}$. Now give a formula for a trinomial theorem, one which expands $(x+y+z)^n$. Could you continue this way and get a multinomial formula?

2.10 Completeness of $\mathbb{R}$

By Theorem 2.7.9, between any two real numbers, points on the number line, there exists a rational number. This suggests there are a lot of rational numbers, but it is not clear from this Theorem whether the entire real line consists of only rational numbers. Some people might wish this were the case because then each real number could be described, not just as a point on a line but also algebraically, as the quotient of integers. Before 500 B.C., a group of mathematicians, led by Pythagoras believed in this, but they discovered their beliefs were false. It happened roughly like this. They knew they could construct the square root of two as the diagonal of a right triangle in which the two sides have unit length; thus they could regard $\sqrt{2}$ as a number. Unfortunately, they