18. Suppose $f(x) = 3x^2 + 7x - 17$. Find the value of x at which $f(x)$ is smallest by completing the square. Also determine $f(\mathbb{R})$ and sketch the graph of f. **Hint:**
$$\begin{aligned} f(x) &= 3\left(x^2 + \frac{7}{3}x - \frac{17}{3}\right) = 3\left(x^2 + \frac{7}{3}x + \frac{49}{36} - \frac{49}{36} - \frac{17}{3}\right) \\ &= 3\left(\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} - \frac{17}{3}\right). \end{aligned}$$

19. Suppose $f(x) = -5x^2 + 8x - 7$. Find $f(\mathbb{R})$. In particular, find the largest value of $f(x)$ and the value of x at which it occurs. Can you conjecture and prove a result about $y = ax^2 + bx + c$ in terms of the sign of a based on these last two problems?

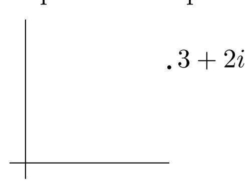
20. Show that if it is assumed $\mathbb{R}$ is complete, then the Archimedean property can be proved. **Hint:** Suppose completeness and let $a > 0$. If there exists $x \in \mathbb{R}$ such that $na \leq x$ for all $n \in \mathbb{N}$, then x/a is an upper bound for $\mathbb{N}$. Let l be the least upper bound and argue there exists $n \in \mathbb{N} \cap [l - 1/4, l]$. Now what about $n + 1$?

21. Suppose you numbers a_k for each k a positive integer and that $a_1 \leq a_2 \leq \cdots$. Let A be the set of these numbers just described. Also suppose there exists an upper bound L such that each $a_k \leq L$. Then there exists N such that if $n \geq N$, then $(\sup A - \varepsilon < a_n \leq \sup A]$.

22. If $A \subseteq B$ for $A \neq \emptyset$ and A, B are sets of real numbers, show that $\inf(A) \geq \inf(B)$ and $\sup(A) \leq \sup(B)$.

2.13 The Complex Numbers

Just as a real number should be considered as a point on the line, a complex number is considered a point in the plane which can be identified in the usual way using the Cartesian coordinates of the point. Thus (a, b) identifies a point whose x coordinate is a and whose y coordinate is b. In dealing with complex numbers, such a point is written as $a + ib$. For example, in the following picture, I have graphed the point $3 + 2i$. You see it corresponds to the point in the plane whose coordinates are $(3, 2)$.



Multiplication and addition are defined in the most obvious way subject to the convention that $i^2 = -1$. Thus,
$$(a + ib) + (c + id) = (a + c) + i(b + d)$$
and
$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd = (ac - bd) + i(bc + ad).$$
Every non zero complex number, $a + ib$, with $a^2 + b^2 \neq 0$, has a unique multiplicative inverse.
$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$
You should prove the following theorem.

Theorem 2.13.1 *The complex numbers with multiplication and addition defined as above form a field satisfying all the field axioms listed on Page 9.*