were also able to show $\sqrt{2}$ could not be written as the quotient of two integers. This discovery that the rational numbers could not even account for the results of geometric constructions was very upsetting to the Pythagoreans, especially when it became clear there were an endless supply of such "irrational" numbers.

This shows that if it is desired to consider all points on the number line, it is necessary to abandon the attempt to describe arbitrary real numbers in a purely algebraic manner using only the integers. Some might desire to throw out all the irrational numbers, and considering only the rational numbers, confine their attention to algebra, but this is not the approach to be followed here because it will effectively eliminate every major theorem of calculus and analysis. In this book real numbers will continue to be the points on the number line, a line which has no holes. This lack of holes is more precisely described in the following way.

Definition 2.10.1 *A non empty set, $S \subseteq \mathbb{R}$ is bounded above (below) if there exists $x \in \mathbb{R}$ such that $x \geq (\leq)\, s$ for all $s \in S$. If S is a nonempty set in $\mathbb{R}$ which is bounded above, then a number, l which has the property that l is an upper bound and that every other upper bound is no smaller than l is called a least upper bound, l.u.b. (S) or often $\sup(S)$. If S is a nonempty set bounded below, define the greatest lower bound, g.l.b. (S) or $\inf(S)$ similarly. Thus g is the g.l.b. (S) means g is a lower bound for S and it is the largest of all lower bounds. If S is a nonempty subset of $\mathbb{R}$ which is not bounded above, this information is expressed by saying $\sup(S) = +\infty$ and if S is not bounded below, $\inf(S) = -\infty$.*

In an appendix, there is a proof that the real numers can be obtained as equivalence classes of Cauchy sequences of rational numbers but in this book, we follow the historical development of the subject and accept it as an axiom. In other words, we will believe in the real numbers and this axiom.

The completeness axiom was identified by Bolzano as the reason for the truth of the intermediate value theorem for continuous functions around 1818. However, every existence theorem in calculus depends on some form of the completeness axiom.

Axiom 2.10.2 *(completeness) Every nonempty set of real numbers which is bounded above has a least upper bound and every nonempty set of real numbers which is bounded below has a greatest lower bound.*

It is this axiom which distinguishes Calculus from Algebra. A fundamental result about sup and inf is the following.

Proposition 2.10.3 *Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

Proof: Consider the first claim. If the indicated set equals $\emptyset$, then $\sup(S) - \delta$ is an upper bound for S which is smaller than $\sup(S)$, contrary to the definition of $\sup(S)$ as the least upper bound. In the second claim, if the indicated set equals $\emptyset$, then $\inf(S) + \delta$ would be a lower bound which is larger than $\inf(S)$ contrary to the definition of $\inf(S)$. ■

The wonderful thing about sup is that you can switch the order in which they occur. The same thing holds for inf. It is also convenient to generalize the notion of sup and inf so that we don't have to worry about whether it is a real number.