Therefore, if also, $|y-x| < \frac{|x^n-a|}{2}\left(\sum_{k=0}^{n-1}\binom{n}{k}|x|^k\right)^{-1}$, the above is as large as $|x^n-a|^2/2>0$ and so x^n-a and y^n-a have the same sign when

$$0<\delta<\min\left(1,\frac{|x^n-a|}{2}\left(\sum_{k=0}^{n-1}\binom{n}{k}|x|^k\right)^{-1}\right) \blacksquare$$

Theorem 2.11.2 *Let $a>0$ and let $n>1$. Then there exists a unique $x>0$ such that $x^n=a$.*

Proof: Let S denote those numbers $y\geq 0$ such that $t^n-a<0$ for all $t\in[0,y]$. Now note that from the binomial theorem,

$$(1+a)^n-a=\sum_{k=0}^{n}\binom{n}{k}a^k1^{n-k}-a\geq 1+a-a=1>0$$

Thus S is bounded above and $0\in S$. Let $x\equiv\sup(S)$. Then by definition of sup, for every $\delta>0$, there exists $t\in S$ with $|x-t|<\delta$.

If $x^n-a>0$, then by the above lemma, for $t\in S$ sufficiently close to x, $(t^n-a)(x^n-a)>0$ which is a contradiction because the first factor is negative and the second is positive. Hence $x^n-a\leq 0$. If $x^n-a<0$, then from the above lemma, there is a $\delta>0$ such that if $t\in(x-\delta,x+\delta)$, x^n-a and t^n-a have the same sign. This is also a contradiction because then $x\neq\sup(S)$. It follows $x^n=a$. $\blacksquare$

From now on, we will use this fact that n^{th} roots exist whenever it is convenient to do so.

2.12 Exercises

1. Let $S=[2,5]$. Find $\sup S$. Now let $S=[2,5)$. Find $\sup S$. Is $\sup S$ always a number in S? Give conditions under which $\sup S\in S$ and then give conditions under which $\inf S\in S$.

2. Show that if $S\neq\emptyset$ and is bounded above (below) then $\sup S$ ($\inf S$) is unique. That is, there is only one least upper bound and only one greatest lower bound. If $S=\emptyset$ can you conclude that 7 is an upper bound? Can you conclude 7 is a lower bound? What about 13.5? What about any other number?

3. Let S be a set which is bounded above and let $-S$ denote the set $\{-x:x\in S\}$. How are $\inf(-S)$ and $\sup(S)$ related? **Hint:** Draw some pictures on a number line. What about $\sup(-S)$ and $\inf S$ where S is a set which is bounded below?

4. Which of the field axioms is being abused in the following argument that $0=2$? Let $x=y=1$. Then
$$0=x^2-y^2=(x-y)(x+y)$$
and so $0=(x-y)(x+y)$. Now divide both sides by $x-y$ to obtain $0=x+y=1+1=2$.

5. Give conditions under which equality holds in the triangle inequality.

6. Let $k\leq n$ where k and n are natural numbers. $P(n,k)$, permutations of n things taken k at a time, is defined to be the number of different ways to form an ordered list of k of the numbers, $\{1,2,\cdots,n\}$. Show
$$P(n,k)=n\cdot(n-1)\cdots(n-k+1)=\frac{n!}{(n-k)!}.$$