Axiom 2.4.2 *The product of two positive real numbers is positive.*

Axiom 2.4.3 *For a given real number x one and only one of the following alternatives holds. Either x is positive, $x = 0$, or $-x$ is positive.*

Definition 2.4.4 *$x < y$ exactly when $y + (-x) \equiv y - x \in \mathbb{R}^+$. In the usual way, $x < y$ is the same as $y > x$ and $x \leq y$ means either $x < y$ or $x = y$. The symbol $\geq$ is defined similarly.*

Theorem 2.4.5 *The following hold for the order defined as above.*

1. *If $x < y$ and $y < z$ then $x < z$ (Transitive law).*
2. *If $x < y$ then $x + z < y + z$ (addition to an inequality).*
3. *If $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.*
4. *If $x > 0$ then $x^{-1} > 0$.*
5. *If $x < 0$ then $x^{-1} < 0$.*
6. *If $x < y$ then $xz < yz$ if $z > 0$, (multiplication of an inequality).*
7. *If $x < y$ and $z < 0$, then $xz > zy$ (multiplication of an inequality).*
8. *Each of the above holds with $>$ and $<$ replaced by $\geq$ and $\leq$ respectively except for 4 and 5 in which we must also stipulate that $x \neq 0$.*
9. *For any x and y, exactly one of the following must hold. Either $x = y$, $x < y$, or $x > y$ (trichotomy).*
10. *$xy > 0$ if and only if both x, y are positive or both $-x, -y$ are positive. Thus $xy = 0$ means x, y have the same sign.*

Proof: First consider 1, the transitive law. Suppose $x < y$ and $y < z$. Why is $x < z$? In other words, why is $z - x \in \mathbb{R}^+$? It is because $z - x = (z - y) + (y - x)$ and both $z - y, y - x \in \mathbb{R}^+$. Thus by 2.4.1 above, $z - x \in \mathbb{R}^+$ and so $z > x$.

Next consider 2, addition to an inequality. If $x < y$ why is $x + z < y + z$? it is because

$$
\begin{aligned}
(y + z) + -(x + z) &= (y + z) + (-1)(x + z) \\
&= y + (-1)x + z + (-1)z \\
&= y - x \in \mathbb{R}^+.
\end{aligned}
$$

Next consider 3. If $x \leq 0$ and $y \leq 0$, why is $xy \geq 0$? First note there is nothing to show if either x or y equal 0 so assume this is not the case. By 2.4.3 $-x > 0$ and $-y > 0$. Therefore, by 2.4.2 and what was proved about $-x = (-1)x$,

$$(-x)(-y) = (-1)^2 xy \in \mathbb{R}^+.$$

Is $(-1)^2 = 1$? If so the claim is proved. But $-(-1) = (-1)^2$ and $-(-1) = 1$ because $-1 + 1 = 0$.

Next consider 4. If $x > 0$ why is $x^{-1} > 0$? By 2.4.3 either $x^{-1} = 0$ or $-x^{-1} \in \mathbb{R}^+$. It can't happen that $x^{-1} = 0$ because then you would have to have $1 = 0x$ and as was shown earlier, $0x = 0$. Therefore, consider the possibility that $-x^{-1} \in \mathbb{R}^+$. This can't work either because then you would have

$$(-1)x^{-1}x = (-1)(1) = -1$$