Assignment 8

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Analytical

This is Exercise 5.11 from (Särkkä and Solin 2019), which asks for the mean and variance of the Black-Scholes function, defined as:

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}\beta, \tag{1}$$

where β is standard Brownian motion.

The author provides the following system of equations:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \mu m \tag{2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2\mu P + \sigma^2 P + \sigma^2 m^2, \tag{3}$$

where m is the mean, P is the variance, and μ and σ are parameters of the Black-Scholes SDE. We are also given the intial conditions:

$$m\left(0\right) = m_0 \tag{4}$$

$$P\left(0\right) = P_0 \tag{5}$$

My solution

Mean

First we'll solve the mean equation by separation of variables.

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \mu m \tag{6}$$

$$\frac{1}{m}\mathrm{d}m = \mu \mathrm{d}t \tag{7}$$

$$\int \frac{1}{m}\mathrm{d}m = \int \mu \mathrm{d}t \tag{8}$$

$$\ln m = \mu t + c_1 \tag{9}$$

$$m = c\,\mathrm{e}^{(\mu t)} \tag{10}$$

and with the initial condition $m\left(0\right) = m_0$:

$$m\left(0\right) = m_0 = c\mathrm{e}^{\mu(0)} \tag{11}$$

$$m_0 = c \tag{12}$$

$$\therefore m\left(t\right) = m_0\,\mathrm{e}^{\mu t} \tag{13}$$

Variance

We will solve the Variance equation by substituting in the mean equation and then using an integrating factor, I. Note that we assume that $\sigma > 0$.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2\mu P + \sigma^2 P + \sigma^2 m^2 \tag{14}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2\mu P + \sigma^2 P + \sigma^2 m_0^2 \,\mathrm{e}^{2\mu t} \tag{15}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} - \left(2\mu + \sigma^2\right) P = \sigma^2 m_0^2 \,\mathrm{e}^{2\mu t} \tag{16}$$

$$\text{Let } I := \mathrm{e}^{\int -\left(2\mu+\sigma^2\right)\mathrm{d}t} = \mathrm{e}^{-\left(2\mu+\sigma^2\right)t} \tag{17}$$

$$\text{Then,} \quad \mathrm{e}^{-\left(2\mu+\sigma^2\right)t}\frac{\mathrm{d}P}{\mathrm{d}t} - \left(2\mu + \sigma^2\right) P\mathrm{e}^{-\left(2\mu+\sigma^2\right)t} = \sigma^2 m_0^2 \,\mathrm{e}^{2\mu t}\mathrm{e}^{-\left(2\mu+\sigma^2\right)t} \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\left(2\mu+\sigma^2\right)t}P\right) = \sigma^2 m_0^2 \,\mathrm{e}^{-\sigma^2 t} \tag{19}$$

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\left(2\mu+\sigma^2\right)t}P\right)\mathrm{d}t = \sigma^2 m_0^2 \int \mathrm{e}^{-\sigma^2 t}\mathrm{d}t \tag{20}$$

$$\mathrm{e}^{-\left(2\mu+\sigma^2\right)t}P = -\sigma^2 \frac{1}{\sigma^2} m_0^2 \,\mathrm{e}^{-\sigma^2 t} + c_2 \tag{21}$$

$$P = -m_0^2 \,\mathrm{e}^{-\sigma^2 t}\mathrm{e}^{\left(2\mu+\sigma^2\right)t} + c_2 \,\mathrm{e}^{\left(2\mu+\sigma^2\right)t} \tag{22}$$

$$P = -m_0^2 \,\mathrm{e}^{2\mu t} + c_2 \,\mathrm{e}^{\left(2\mu+\sigma^2\right)t} \tag{23}$$

with the initial condition $P\left(0\right) = P_0$,

$$P\left(0\right) = P_0 = -m_0^2 \,\mathrm{e}^{2\mu(0)} + c_2 \,\mathrm{e}^{\left(2\mu+\sigma^2\right)(0)} \tag{24}$$

$$P_0 = -m_0^2 + c_2 \tag{25}$$

$$\implies c_2 = P_0 + m_0^2 \tag{26}$$

$$\therefore P\left(t\right) = -m_0^2 \,\mathrm{e}^{2\mu t} + \left(P_0 + m_0^2\right) \,\mathrm{e}^{\left(2\mu+\sigma^2\right)t} \tag{27}$$

Numerical Problem

We will

```
from matplotlib.colors import Normalize
from matplotlib import pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import numpy as np
import pandas as pd
from scipy.stats import kde
import seaborn as sns
```

References

Särkkä, Simo, and Arno Solin. 2019. *Applied Stochastic Differential Equations*. Vol. 10. Cambridge University Press.