Assignment 3

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Simulation

First I will simulate one-dimensional Brownian motion using `python` following (Higham 2001).

```
from matplotlib import pyplot as plt
import numpy as np


def brownian_motion(end_time=1., num_tsteps=500):
    dt = (end_time - 0) / num_tsteps
    dw = np.sqrt(dt) * np.random.normal(size=num_tsteps+1)
    # Brownian motion must start at time 0 with value 0
    dw[0] = 0
    w = dw.cumsum()
    t = np.linspace(0, end_time, num=num_tsteps+1)  # not used in calculations
    assert len(w) == len(t), f'time and position arrays are not the same length. len(t) - len(w) = {len
    return t, w, dw
```

```
END_TIME = 1.
NUM_TSTEPS = 500
N_TRIALS = 10
CONSISTENT = True
PLOTS = True
```

```
# space for simulation
```

Analytical Problem

From (Evans 2012), Exercise (33.): Show directly that $I(t) := W^2(t) - t$ is a martingale.

(Hint: $W^2(t) = (W(t) - W(s))^2 - W^2(s) + 2W(t)W(s)$. Take the conditional expectation with respect to $\mathcal{W}(s)$, the history of $W(\cdot)$, and then condition with respect to the history of $I(\cdot)$.)

My Solution

To show that $I(t)$ is a martingale, we want to show that $I(s) = \mathrm{E}(I(t)|\mathcal{U}(s))$.

Following the hint about W^2:

$$I(t) := W^2(t) - t = (W(t) - W(s))^2 - W^2(s) + 2W(t)W(s) - t$$

Now, let $\mathcal{U}(s)$ be the history of the $I(\cdot)$ until time s, and $\mathcal{W}(s)$ be the history of $W(\cdot)$ until time s.

$$\begin{aligned}
\mathrm{E}(I(t)|\mathcal{W}(s)) &= \mathrm{E}((W(t)-W(s))^2 - W^2(s) + 2W(t)W(s) - t|\mathcal{W}(s)) && (1)\\
&= \mathrm{E}((W(t)-W(s))^2|\mathcal{W}(s)) - \mathrm{E}(W^2(s)|\mathcal{W}(s)) + \mathrm{E}(2W(t)W(s)|\mathcal{W}(s)) - \mathrm{E}(t|\mathcal{W}(s)) && (2)\\
&= \mathrm{Var}(W(t)-W(s)|\mathcal{W}(s)) + |\mathrm{E}(W(t)-W(s))|^2 - W^2(s) + 2W(s)\mathrm{E}(W(t)|\mathcal{W}(s)) - t && (3)\\
&= \mathrm{Var}(W(t)|\mathcal{W}(s)) - W^2(s) + 2W^2(s) - t && (4)\\
&= t + W^2(s) - t && (5)\\
&= W^2(s) && (6)
\end{aligned}$$

References

Evans, Lawrence C. 2012. *An Introduction to Stochastic Differential Equations*. Vol. 82. American Mathematical Soc.

Higham, Desmond J. 2001. "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations." *SIAM Review* 43 (3). SIAM: 525–46.