Assignment 5

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Analytical

Exercise

This is a problem from (Douglas and Peter 2006): Verify that

$$N = N_0 \mathrm{e}^{\left(r - \frac{u^2}{2}\right)t + uB_t}$$

is the solution to

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN + uW(t)N$$

where r and u are constants.

This is a stochastic form of the population growth model, which models the population growth as propotional to the current population. This stochastic form models the stochastic term as also propotional to the current population.

My Solution

First, we write the differential equation in the form used in (Evans 2012).

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN + uWN \tag{1}$$

$$\mathrm{d}N = F\mathrm{d}t + G\mathrm{d}B_t, \tag{2}$$

where

$$F = rN \tag{3}$$

$$G = uN \tag{4}$$

$$dB_t = W\mathrm{d}t \tag{5}$$

From this form, we can apply Itô chain rule by taking the logarithm of both sides. In otherwords, we first study a function of the random variable N, $\Phi(N) = \ln(N)$. Then for our new function Φ we have the following derivatives:

$$\Phi_t = 0; \tag{6}$$

$$\Phi_N = \frac{1}{N} \tag{7}$$

$$\Phi_{NN} = -\frac{1}{N^2} \tag{8}$$

And we can proceed

$$\mathrm{d}\Phi = \mathrm{d}\ln(N) = \left(\Phi_t + \Phi_N F + \frac{1}{2}\Phi_{NN} G^2\right)\mathrm{d}t + \Phi_N G \mathrm{d}B_t \tag{9}$$

$$= \left(0 + \left(\frac{1}{N}\right)(rN) - \frac{1}{2}\left(\frac{1}{N^2}\right)(uN)^2\right)\mathrm{d}t + \left(\frac{1}{N}\right)(uN)\,\mathrm{d}B_t \tag{10}$$

$$\mathrm{d}\Phi = \left(r - \frac{u^2}{2}\right)\mathrm{d}t + u\mathrm{d}B_t \tag{11}$$

In the last line above, r and u are constants from the problem definition. By the definition of the stochastic differential, we know the solution Φ, and we can solve for Φ by integration.

$$\Phi = \ln(N) = \int_0^t \left(r - \frac{u^2}{2}\right)\mathrm{d}t + \int_0^t u\mathrm{d}B_t \tag{12}$$

$$\ln(N) = \left(r - \frac{u^2}{2}\right)t + uB_t + C \tag{13}$$

$$N = C\mathrm{e}^{\left(r - \frac{u^2}{2}\right)t + uB_t} \tag{14}$$

Lastly, when $t = 0$, $B_t = 0$ by virtue of Brownian motion. Let $N_0 = N(0)$, our initial condition, and we have:

$$N = N_0\mathrm{e}^{\left(r - \frac{u^2}{2}\right)t + uB_t}.$$

Simulation

This code is calculating the Itô integral and the Stratonovich integral for integrals with known solutions. This follows (Higham 2001), but adds in multiple trials in a vectorized manner.

References

Douglas, Henderson, and Plaschko Peter. 2006. *Stochastic Differential Equations in Science and Engineering (with Cd-Rom)*. World Scientific.

Evans, Lawrence C. 2012. *An Introduction to Stochastic Differential Equations*. Vol. 82. American Mathematical Soc.

Higham, Desmond J. 2001. "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations." *SIAM Review* 43 (3). SIAM: 525–46.