# Assignment 2

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## Simulation

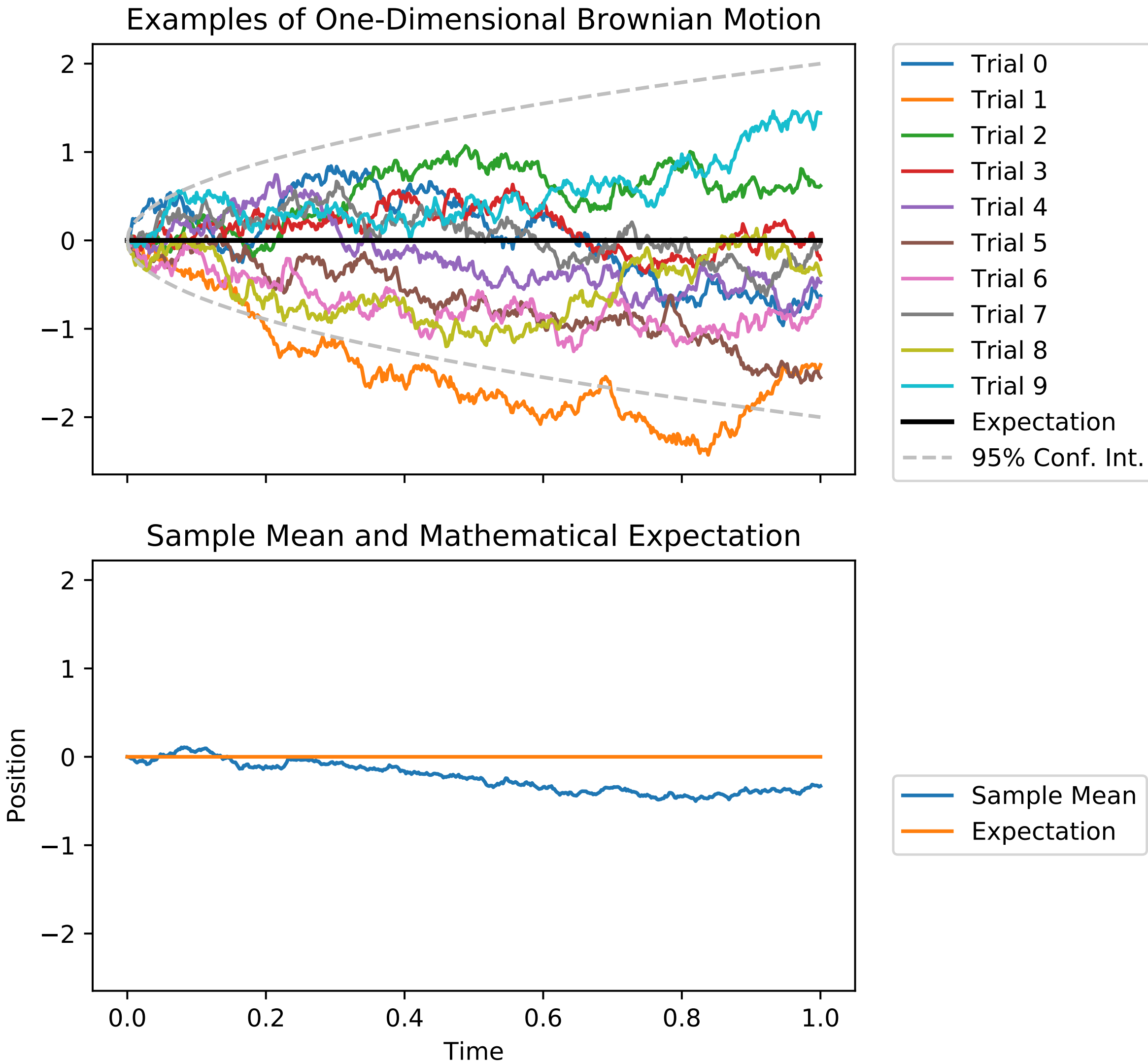
First I will simulate one-dimensional Brownian motion using `python` following (Higham 2001).

```
from matplotlib import pyplot as plt
import numpy as np


def brownian_motion(end_time=1., num_tsteps=500):
    dt = (end_time - 0) / num_tsteps
    dw = np.sqrt(dt) * np.random.normal(size=num_tsteps+1)
    # Brownian motion must start at time 0 with value 0
    dw[0] = 0
    w = dw.cumsum()
    t = np.linspace(0, end_time, num=num_tsteps+1)  # not used in calculations
    assert len(w) == len(t), f'time and position arrays are not the same length. len(t) - len(w) = {len
    return t, w, dw
```

```
END_TIME = 1.
NUM_TSTEPS = 500
N_TRIALS = 10
CONSISTENT = True
PLOTS = True
```

```
if CONSISTENT:
    np.random.seed(0)  # keeps the psuedo-random number
    #generator in the same sequence from run to run

if PLOTS:
    fig, (ax0, ax1) = plt.subplots(nrows=2, sharex=True,
                                   sharey=True, figsize=(7,7))

results = []

for i in range(N_TRIALS):
    time, position, velocity = brownian_motion(end_time=END_TIME, num_tsteps=NUM_TSTEPS)
    results.append((position, velocity))

    if PLOTS:
        ax0.plot(time, position, label=f'Trial {i}')

if PLOTS:
    ax0.plot(time, np.zeros_like(time), color='0.0', label='Expectation', linewidth=2)
    ax0.plot(time, 2 * np.sqrt(time), '--', color='0.75')
    ax0.plot(time, -2 * np.sqrt(time) * np.ones_like(time), '--', color='0.75', label='95% Conf. Int.')
    ax0.set_title('Examples of One-Dimensional Brownian Motion')
```

```
box = ax0.get_position()
ax0.set_position([box.x0, box.y0, box.width * 0.8, box.height])
ax0.legend(bbox_to_anchor=(1.05, 1), loc='upper left', borderaxespad=0.)

mean = np.array([res[0] for res in results]).T.mean(axis=1)

ax1.plot(time, mean, label='Sample Mean')
ax1.plot(time, np.zeros_like(time), label='Expectation')
ax1.set_title('Sample Mean and Mathematical Expectation')
box = ax1.get_position()
ax1.set_position([box.x0, box.y0, box.width * 0.8, box.height])
ax1.legend(bbox_to_anchor=(1.05, 0.5), loc='upper left', borderaxespad=0.)
plt.xlabel('Time')
plt.ylabel('Position')
plt.show()
```


Examples of One-Dimensional Brownian Motion
Trial 0
Trial 1
Trial 2
Trial 3
Trial 4
Trial 5
Trial 6
Trial 7
Trial 8
Trial 9
Expectation
95% Conf. Int.
Sample Mean and Mathematical Expectation
Sample Mean
Expectation
Position
Time
0.0
0.2
0.4
0.6
0.8
1.0
2
1
0
−1
−2

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# Analytical Problem

From (Evans 2012), Exercise (19): Let $W(\cdot)$ be a one-dimensional Brownian motion. Show

$$E(W^{2k}(t)) = \frac{(2k)!t^k}{2^k k!}$$

## My Solution

From the text, we know that $W(t) \sim N(0,t)$, ie. that Brownian motion is normally distributed with variance $t$. Thus, the probability density function of $W(t)$, $f_{W(t)}(w) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{w^2}{2t}}$.

A few strategies will be used in the integration. First, we'll use a change of variables: $x = \frac{w}{\sqrt{2t}}$, to make the integration clearner. Then we'll use the gamma function (denoted by $\Gamma$) in the integration. The integration is as follows:

$$\int_{-\infty}^{\infty} x^{2k}\mathrm{e}^{-x^2}dx = \Gamma(k+\frac{1}{2})$$

and that gamma function evaluates to:

$$\Gamma(k+\frac{1}{2}) = \frac{(2k)!}{4^k k!}\sqrt{\pi}.$$

Now, for the problem at hand:

$$\begin{aligned}
E(W^{2k}(t)) &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} w^{2k}\mathrm{e}^{-\frac{w^2}{2t}}dw && (1)\\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\sqrt{2t}x)^{2k}\mathrm{e}^{-x^2}\sqrt{2t}dx && (2)\\
&= \frac{(2t)^k\sqrt{2t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} x^{2k}\mathrm{e}^{-x^2}dx && (3)\\
&= \frac{(2t)^k}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^{2k}\mathrm{e}^{-x^2}dx && (4)\\
&= \frac{(2t)^k}{\sqrt{\pi}}\Gamma(k+\frac{1}{2}) && (5)\\
&= \frac{(2t)^k}{\sqrt{\pi}}\frac{(2k)!}{4^k k!}\sqrt{\pi} && (6)\\
&= \frac{(2k)!2^k t^k}{(2^k)^2 k!} && (7)\\
&= \frac{(2k)!t^k}{2^k k!} && (8)
\end{aligned}$$

# References


Evans, Lawrence C. 2012. *An Introduction to Stochastic Differential Equations*. Vol. 82. American Mathematical Soc.

Higham, Desmond J. 2001. "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations." *SIAM Review* 43 (3). SIAM: 525–46.