Assignment 4

Reid Ginoza

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Analytical

This assignment follows Exercise 35 from (Evans 2012):

Use the Itô chain rule to show that $Y(t) := \mathrm{e}^{\frac{t}{2}} \cos(W(t))$ is a martingale.

My Solution

Let $X(t) = W(t)$. Then $dX = dW$. Rewriting X in the general differential form: we have

$$dX = Fdt + GdW$$

And since we know that $dX = dW$, then we know that $F \equiv 0$ and $G \equiv 1$.

Now let $u(x, t) := \mathrm{e}^{\frac{t}{2}} \cos(x)$. Then we know the following:

- $u_t = \frac{1}{2}\mathrm{e}^{\frac{t}{2}} \cos(x)$
- $u_x = -\mathrm{e}^{\frac{t}{2}} \sin(x)$
- $u_{xx} = -\mathrm{e}^{\frac{t}{2}} \cos(x)$

Since we chose $u(x, t)$ so that $Y = u(X, t)$, we can apply the Itô chain rule:

$$\begin{aligned}
dY = du(X, t) &= \left(u_t + u_x F + \frac{1}{2}u_{xx}G^2\right) dt + u_x G dW && (1)\\
&= \left(u_t + \frac{1}{2}u_{xx}\right) dt + u_x dW && (2)\\
&= \left(\frac{1}{2}\mathrm{e}^{\frac{t}{2}} \cos(x) - \frac{1}{2}\mathrm{e}^{\frac{t}{2}} \cos(x)\right) dt - \mathrm{e}^{\frac{t}{2}} \sin(x) dW && (3)\\
dY &= -\mathrm{e}^{\frac{t}{2}} \sin(x) dW && (4)
\end{aligned}$$

Now let s and r be times such that $0 \le s \le r \le T$. Then the stochastic process $Y(\cdot)$ can be written as:

$$Y(r) = Y(s) - \int_s^r \mathrm{e}^{\frac{t}{2}} \sin(x) dW.$$

To see whether $Y(\cdot)$ is a martingale, we want to take the mathematical expectation of $Y(r)$ with the history $\mathcal{U}(s)$.

$$\begin{aligned}
\mathrm{E}(Y(r)|\mathcal{U}(s)) &= \mathrm{E}\left(Y(s) - \int_s^r \mathrm{e}^{\frac{t}{2}} \sin(x) dW |\mathcal{U}(s)\right) && (5)\\
&= \mathrm{E}\left(Y(s)|\mathcal{U}(s)\right) - \mathrm{E}\left(\int_s^r \mathrm{e}^{\frac{t}{2}} \sin(x) dW |\mathcal{U}(s)\right) && (6)\\
&= Y(s) - \mathrm{E}\left(\int_s^r \mathrm{e}^{\frac{t}{2}} \sin(x) dW |\mathcal{U}(s)\right) && (7)\\
\mathrm{E}(Y(r)|\mathcal{U}(s)) &= Y(s) && (8)
\end{aligned}$$

The last line simplified based on the Theorem in Section 4.2.3 of (Evans 2012), which states that:

$$\mathrm{E}\left(\int_0^T G dW\right) = 0$$

and a similar statement can be made of an interval in the support.

Simulation

This will simulate multiple sample paths of the stochastic process $Y(t) := \mathrm{e}^{\frac{t}{2}} \cos(W(t))$.

```
from matplotlib import pyplot as plt
import numpy as np


def plot_on_axis(ax, time, pos, cols, title, with_mean=False):
    ax.plot(time, pos[:, cols])
    ax.set_title(title)
    if with_mean:
        ax.plot(time, pos.mean(axis=1), color='black',
            label='Sample Mean', linewidth=2)
        ax.legend()


def multiple_brownian_motion(end_time=1., num_tsteps=500, n_trials=1000):
    dt = (end_time - 0) / num_tsteps
    dw = np.random.normal(scale=np.sqrt(dt), size=(num_tsteps+1, n_trials))
    # Brownian motion must start at time 0 with value 0
    dw[0] = np.zeros_like(dw[0])
    w = dw.cumsum(axis=0)
    t = np.linspace(0, end_time, num=num_tsteps+1)  # not used in calculations
    assert w.shape[0] == t.shape[0], f'time and position arrays are not the same length. w.shape[0] - t
    assert w.shape == dw.shape, f'position and velocity arrays are not the same shape: w.shape: {w.shap
    return t, w, dw


def y_process(t, w):
    """ The stochastic process defined in the exercise.
    This function assumes that the same time axis is used for all Brownian motion,
    can pass the time array output from multiple_brownian_motion.
    """
    assert t.shape[0] == w.shape[0], f'Time and Brownian and motion need to be the same length: time {t

    return np.exp(t/2).reshape((t.shape[0], 1)) * np.cos(w)
```

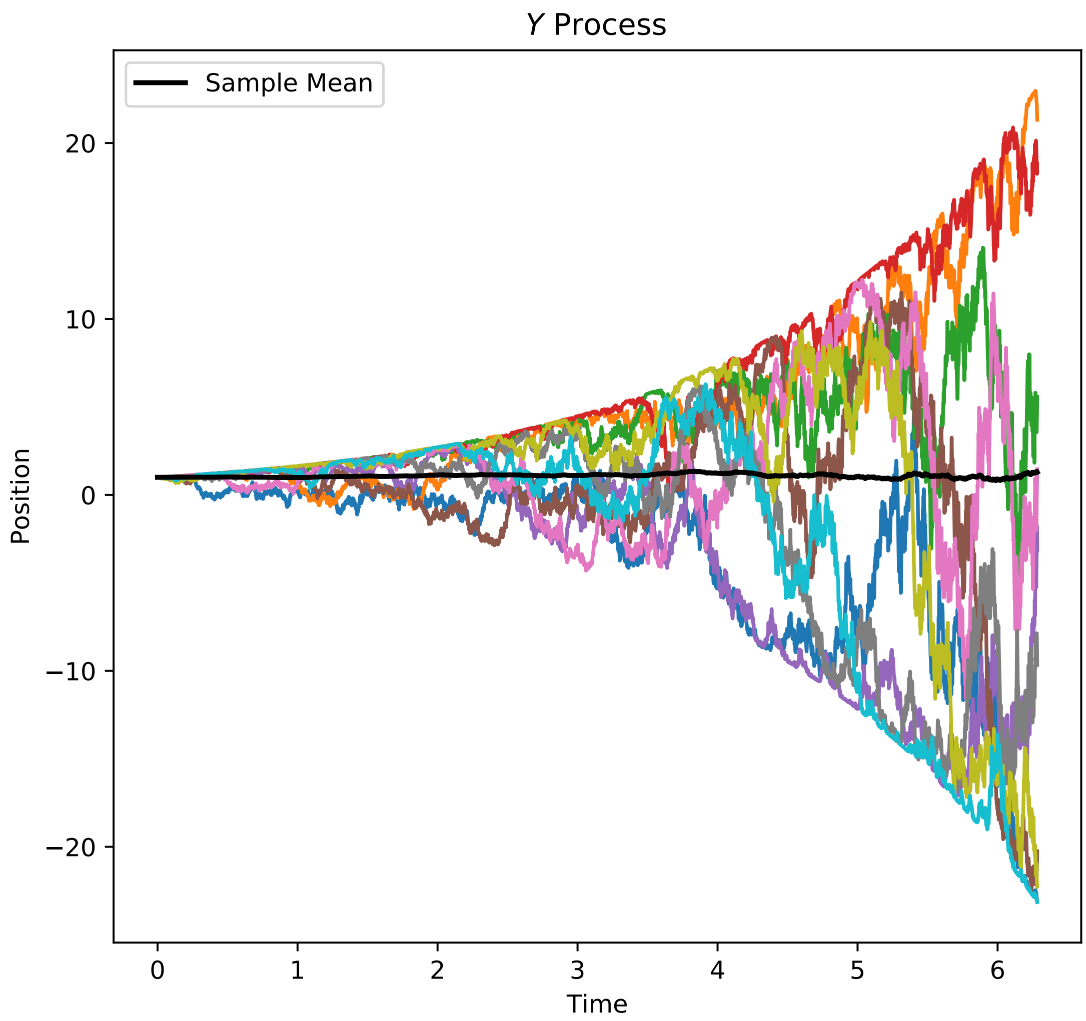
```
END_TIME = 2*np.pi
NUM_TSTEPS = round(END_TIME * 1000)
N_TRIALS = 1000
N_PATHS = 10
```

```
time, w, dw = multiple_brownian_motion(
    END_TIME, NUM_TSTEPS, N_TRIALS)

position = y_process(time, w)

columns = np.arange(position.shape[1])
np.random.shuffle(columns)
plot_columns = columns[:N_PATHS]

fig, ax = plt.subplots(1, figsize=(7, 6.5))
plot_on_axis(ax, time, position, plot_columns, r'$Y$ Process', with_mean=True)
plt.xlabel('Time')
plt.ylabel('Position')
plt.show()
```



References

Evans, Lawrence C. 2012. *An Introduction to Stochastic Differential Equations*. Vol. 82. American Mathematical Soc.