Assignment 1

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Problem Statement

This assignment focuses on Exercise 1.19 from (Driver 2016).

Suppose Harriet has 7 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

Simulation

To get started, let's run a simulation. This was created using `python`. Below, the `harriet_trial` produces one full trial of Harriet's betting sequence above. She starts with 7 dollars and will stop when she either reaches 0 or 50 dollars.

```
import random

from matplotlib import pyplot as plt
import pandas as pd

def harriet_trial(start=7, stop_low=0, stop_high=50):
    state = start
    states = [state]
    rounds = 0
    while state != stop_low and state != stop_high:
        rounds += 1
        if random.random() > 0.5:  # if heads
            state += 1
        else:  # if tails
            state -= 1
        states.append(state)
    return {'final_state': state, 'rounds': rounds, 'states': states}
```

Simulation Constants

Since we want to see what happens for multiple trials, we'll set `N_TRIALS = 10000` and collect the percentage of times Harriet leaves with 50 dollars compared with the percentage of times Harriet leaves with 0 dollars.

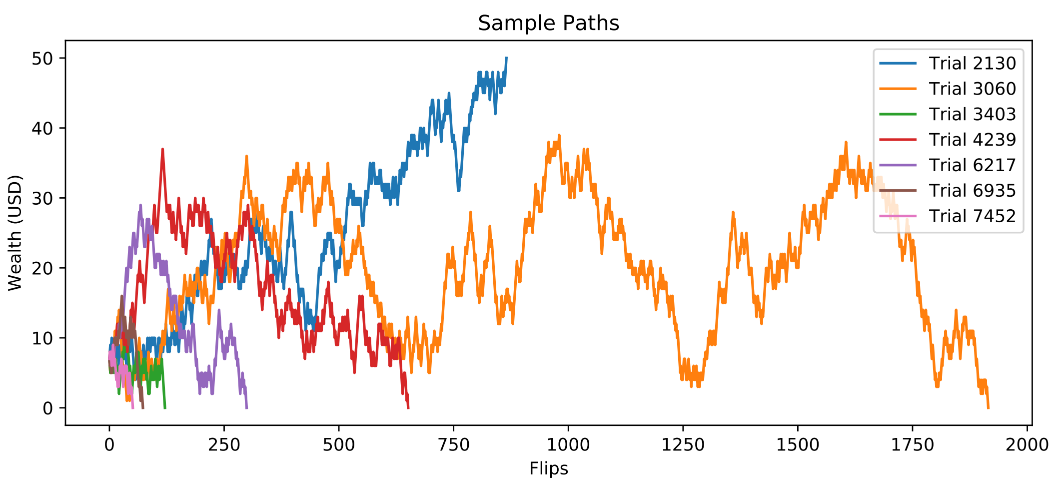
```
CONSISTENT = True
PLOTS = True
N_TRIALS = 10000
```

Run

```
if CONSISTENT:
    random.seed(0)  # keeps the psuedo-random number generator in the same sequence from run to run

if PLOTS:
    plt.figure(figsize=(10, 4))

results = []
for i in range(N_TRIALS):
    results.append(harriet_trial(start=7, stop_low=0, stop_high=50))
    if PLOTS:
      if random.random() < 0.001:
        plt.plot(results[-1]['states'], label='Trial {}'.format(i))
plt.xlabel('Flips')
plt.ylabel('Wealth (USD)')
plt.title('Sample Paths')
plt.legend()
```



Collect Results

```
data = pd.DataFrame(results)
grouped_data = data.groupby(by='final_state').count()[['rounds']]
grouped_data['Percentage'] = grouped_data.rounds / N_TRIALS
```

	rounds	Percentage
0	8622	0.8622
50	1378	0.1378

Simulation Conclusion

Based on our simulation, we expect the probability of Harriet having 50 dollars when she goes home is 13.8%, and the expected value is 6.9 dollars.

Analytical Solution

We will show that we have a discrete martingale, and furthermore, that it is bounded, so that we may apply Doob's Optional Stopping Theorem. Then solving for the probabilities will be a matter of solving a linear system.

Problem Set up

First, let's define our variables. For $i = 0, 1, 2, 3, \ldots$ let S_i be Harriet's wealth after the i-th flip. Since Harriet starts with 7 dollars, $S_0 = 7$.

Let X_i be Harriet's earnings from the i-th flip. For $i = 1, 2, 3, \ldots$, all X_i are independent and identically distributed (i.i.d.) as either 1 with probability 0.5 or -1 with probability 0.5. In otherwords, $P(X_i = 1) = 0.5$ and $P(X_i = -1) = 0.5$ for $i = 1, 2, 3, \ldots$. Then, $E(X_i) = 0$, and for $i \neq j$, $\underbrace{E(X_i|X_j) = E(X_i)}_{\text{by independence}} = 0$.

Discrete Martingale

Also note that, $S_{i+1} = S_i + X_{i+1}$. To show that the sequence $S_0, S_1, S_2 \ldots$ is a discrete martingale, we want to find $E(S_j|S_0, \ldots, S_k)$ for all $j \geq k$.

$$
\begin{aligned}
E(S_j|S_0, \ldots, S_k) &= E(S_k + X_{k+1} + \ldots + X_{j-1} + X_j|S_0, \ldots, S_k) && (1)\\
&= \underbrace{E(S_k|S_0, \ldots, S_k)}_{\text{Since } S_k \text{ is known, } S_k \text{ is a constant and } E(S_k) = S_k} + E(X_{k+1} + \ldots + X_{j-1} + X_j|S_0, \ldots, S_k) && (2)\\
&= S_k + E(X_{k+1} + \ldots + X_{j-1} + X_j|S_0, X_1, \ldots, X_k) && (3)\\
&= S_k + \underbrace{E(X_{k+1}|S_0, X_1, \ldots, X_k) + \ldots + E(X_{j-1}|S_0, X_1, \ldots, X_k) + E(X_j|S_0, X_1, \ldots, X_k)}_{\text{Since } S_0 \text{ is a constant, and all } X_i \text{ are i.i.d.}} && (4)\\
&= S_k + \underbrace{E(X_{k+1}) + \ldots + E(X_{j-1}) + E(X_j)}_{=0} && (5)\\
&= S_k && (6)
\end{aligned}
$$

Therefore, $S_0, S_1, S_2 \ldots$ is a discrete martingale.

Boundedness

Now, let T be the stopping time, that is, the first time such that $S_T = 50$ or $S_T = 0$. Because of the stopping time condition, $S_0, S_1, S_2 \ldots$ is a bounded martingale, since $|S_n| \leq 50$ with probability one for all $n \geq 0$. Now, we can apply Doob's Optional Stopping Theorem for bounded martingales. Thus, $E(S_T) = S_0 = 7$.

Solving for Probabilities

S_T can only be two values. Either, $S_T = 50$ or $S_T = 0$. Thus,

$$
\begin{aligned}
E(S_T) &= 0 * P(S_T = 0) + 50 * P(S_T = 50) && (7)\\
&= 50 * P(S_T = 50) && (8)
\end{aligned}
$$

But, from the previous section,

$$E(S_T) = S_0 = 7 = 50 * P(S_T = 50) \tag{9}$$
$$0.14 = P(S_T = 50) \tag{10}$$
$$0.86 = P(S_T = 0) \tag{11}$$

Conclusion

Thus, the probability that Harriet takes home 50 dollars is 0.14, and our simulation was close at 0.138. The expected amount of money she will take home is 7 dollars, and again, our simulation was close at 6.9 dollars.

Generalization

In a general Harriet trial with a starting wealth of S_0, stopping minimum wealth s_-, and stopping maximum wealth s_+:

$$E(S_T) = S_0 = s_- * P(S_T = s_-) + s_+ * P(S_T = s_+) \tag{12}$$
$$S_0 = s_- * P(S_T = s_-) + s_+ * (1 - P(S_T = s_-)) \tag{13}$$
$$S_0 = s_- * P(S_T = s_-) + s_+ - s_+ * P(S_T = s_-) \tag{14}$$
$$S_0 - s_+ = s_- * P(S_T = s_-) - s_+ * P(S_T = s_-) \tag{15}$$
$$\frac{S_0 - s_+}{s_- - s_+} = P(S_T = s_-) \tag{16}$$
$$1 - \frac{S_0 - s_+}{s_- - s_+} = 1 - P(S_T = s_-) = P(S_T = s_+) \tag{17}$$

Therefore, Harriet's expected amount of money taken home would be S_0. The probability that she takes home s_- dollars is $\frac{S_0 - s_+}{s_- - s_+}$, and the probability that she takes home s_+ dollars is $1 - \frac{S_0 - s_+}{s_- - s_+}$.

References

Driver, Bruce K. 2016. "Math 285 Homework Problem List for S2016." 2016. http://www.math.ucsd.edu/~bdriver/285_S2016/Homeworks/285hm1-7.pdf.