Assignment 11

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Scalar SDE Analytical Solution

This is Exercise 8.1 from (Särkkä and Solin 2019). We are given the following SDE:

$$\mathrm{d}x = -c\,x\mathrm{d}t + g\,x\mathrm{d}\beta, \quad x(0) = x_0, \tag{1}$$

where c, g, and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

We will use the Itô chain rule, where the drift function is $F(x) = -cx$ and the dispersion function $G(x) = gx$. We want to solve for $u(x) = \ln(x)$.

$$\begin{aligned}
u_t &= 0 && (2)\\
u_x &= \frac{1}{x} && (3)\\
u_{xx} &= -\frac{1}{x^2} && (4)\\
\implies \mathrm{d}u &= \left(u_t + u_x F + \frac{1}{2}u_{xx}G^2\right)\mathrm{d}t + u_x\,G\,\mathrm{d}\beta && (5)\\
&= \left(0 - \frac{cx}{x} - \frac{g^2x^2}{2x^2}\right)\mathrm{d}t + \frac{gx}{x}\,\mathrm{d}\beta && (6)\\
&= \left(-c - \frac{g^2}{2}\right)\mathrm{d}t + g\,\mathrm{d}\beta && (7)\\
\implies u &= \left(-c - \frac{g^2}{2}\right)t + g\beta + C && (8)\\
\ln(x) &= \left(-c - \frac{g^2}{2}\right)t + g\beta + C && (9)\\
\implies x &= C\exp\left(\left(-c - \frac{g^2}{2}\right)t + g\beta\right) && (10)
\end{aligned}$$

Applying the initial condition, $$x = x_0\exp\left(\left(-c - \frac{g^2}{2}\right)t + g\beta\right) \tag{11}$$

References

Särkkä, Simo, and Arno Solin. 2019. *Applied Stochastic Differential Equations*. Vol. 10. Cambridge University Press.