Math 172 Notes: Partial fractions decomposition of a rational function

May 31, 2018

Rational functions and their arithmetic

A *rational function* is an expression p/q where p and q are polynomials.* They could be polynomials in any number of unknowns, but for present purposes we are interested in the case of a single unknown (which we'll call x).

Arithmetic of polynomials is closely analogous to arithmetic of integers, and arithmetic of rational functions then corresponds, extending the analogy, to arithmetic of rational numbers (fractions).

In that analogy, the *size* of an integer is replaced by the *degree* of a polynomial. We say that a fraction is *proper* if the size of the numerator is less than the size of the denominator. That concept carries over to rational functions as: a rational function is proper if the *degree* of the numerator is less than the *degree* of the denominator. The concept of "lowest terms" — no integer factors common to the numerator and denominator — carries over as well, as "no *polynomial* factors common to the numerator and denominator". The "mixed number" idea, the sum of an integer plus a proper fraction, carries over to "polynomial plus proper rational function" (I am not aware of a name for this form). To put a fraction or rational function in this form just requires division with remainder.

Partial fractions decomposition

There is a useful way to express a rational function that is often not covered in algebra or precalculus classes. If the denominator q can be factored as $q_1 q_2$ where q_1 and q_2 have no common factors, then this expression is available.

It is known as the "partial fractions decomposition" of p/q relative to the given factorization of q, and it expresses the rational function p/q as a sum of a polynomial part plus *proper* rational functions p_1/q_1 and p_2/q_2.

*The term "rational function" for these expressions goes back to the days of yore, when the function concept was rather vague and the term "function" used rather indiscriminately.

As we've defined it, a rational function is *not* something of type "function", but rather of type "expression". However, as with other expressions we can often *associate* a function with one of these (in many ways). The resulting functions are also called "rational functions".

Example 1. For the rational function $1/(x^3-1)$, we can factor the denominator x^3-1 as $(x-1)(x^2+x+1)$. These factors have no factors in common, since $x-1$ is not a factor of x^2+x+1 (check by division, or just note that 1 is not a root of x^2+x+1).

Then the partial fractions decomposition looks like

$$\frac{1}{x^3-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1} \tag{1}$$

How do we find the coefficients a, b, c? The "method of comparing coefficients" will do this.

The idea is to get a common denominator on the right, express the resulting numerator as a polynomial in x, and then write down the equations which say that the numerator is 1, using the fact that *two polynomials are equal if and only if their coefficient sequences are the same.*

Carrying out that plan, we get

$$\frac{a}{x-1} + \frac{bx+c}{x^2+x+1} = \frac{a(x^2+x+1)+(bx+c)(x-1)}{(x-1)(x^2+x+1)}$$

so the numerator as a polynomial in x is $(a+b)x^2+(a-b+c)x+a-c$. For this to be 1, we need the coefficients of x and x^2 to be 0, while the constant term needs to be 1. That translates into the following equations

$$\begin{aligned} a+b &= 0 \\ a-b+c &= 0 \\ a-c &= 1 \end{aligned}$$

Solve this system of linear equations to find the coefficients a, b, c which make (1) work. *The method of substitution* is not too messy here: $b=-a$ from the first equation, substitute into the second to get $a-(-a)+c=0$, that is, $2a+c=0$, so $c=-2a$. Substitute that into the last equation to get $3a=1$. Solve to get $a=1/3$, then substitute back into the previous equations to get $b=-a=-1/3$, $c=-2/3$.

So we have

$$\frac{1}{x^3-1} = \frac{1/3}{x-1} + \frac{-1/3x+-2/3}{x^2+x+1} = \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$$

There are more examples of the method of comparing coefficients in my Math 118 notes on polynomial division, and related facts about polynomials and rational functions.

There are other ways to determine the partial fractions decomposition for a given factorization of the denominator. I'll explain one for the factorizations we're likely to want for purposes of integration. It exploits a useful connection between algebra and calculus.

First let's see how the partial fractions decomposition gets used in integration. Then we'll come back to another way to find it, more tailored for our present uses.

Using the partial fractions decomposition in integration

The method just illustrated is completely general-purpose. For purposes of integration, however, we would usually want to choose the factorization of the denominator into real irreducible factors. These are either *linear*, corresponding to real roots, or *quadratic with negative discriminant*, corresponding to a conjugate pair of complex (non-real) roots.*

If we do that, then the partial fractions decomposition is a sum of proper rational functions like

$$\frac{r(x)}{(x-a)^m}, \qquad \frac{s(x)}{(x^2+bx+c)^m}$$

In the first case, we can express $r(x)$ as a polynomial in $x-a$ (of degree at most $m-1$). Then we can cancel factors of $x-a$ in each term:

$$\frac{r_0 + r_1(x-a) + r_2(x-a)^2 + \cdots + r_{m-1}(x-a)^{m-1}}{(x-a)^m} = r_0(x-a)^{-m} + \cdots + r_{m-1}(x-a)^{-1}$$

so an antiderivative is easy to find: only the degree -1 term needs anything exotic (log).

In the quadratic case, we can do something similar. It's more cumbersome, since we're not using complex numbers. We'll do it for the denominator x^2+1, corresponding to the two roots i,$-i$. In integration you can *reduce* to this case by substitution, as long as the discriminant is negative.

We express the numerator $s(x)$ in the form

$$s(x) = p_e(x^2+1) + xp_o(x^2+1)$$

where p_e and p_o are polynomials of degree at most $m-1$. It's not hard to see that this is possible. Let $s_e(x)$ be the sum of the even-degree terms of $s(x)$. This is a polynomial in x^2, say k, so $k(x^2) = s_e(x)$. Now express $k(u)$ as a polynomial in $u+1$. This is $p_e(u+1)$. Now $p_e(x^2+1) = s_e(x)$.

Now $s(x) - s_e(x)$ has only odd-degree terms. So it has a factor of x. Then $s(x) - s_e(x) = xg(x^2)$ for some polynomial g. Express $g(x^2)$ as a polynomial in x^2+1 as done for $s_e(x)$. This is $p_o(x^2+1)$.

As I said, this is cumbersome to explain, but not very hard to do.

First, a warm-up:

Exercise 1. Write the polynomial u^3+3u^2-2u+4 as a polynomial in $u+1$. Check to see that you've actually done what you intended to do.

Now

Exercise 2. Now write $x^6+3x^4-2x^2+4$ as a polynomial in x^2+1. Why is there almost nothing extra to do here after having done the previous exercise?

Then do the whole thing.

*If we have a repeated root, we'd use the appropriate power of the corresponding factor, since we need the factors to not have any factor in common. The "appropriate power" is the multiplicity of the root.

Exercise 3. Express the even degree part of $x^4 + 3x^3 + 2x^2 - 4x + 1$ as a polynomial in x^2. Now express it as a polynomial in $x^2 + 1$ and *check to see that it works*. Express the odd degree part of $x^4 + 3x^3 + 2x^2 - 4x + 1$ as x times a polynomial in x^2 of even degree. Write this latter polynomial as a polynomial in $x^2 + 1$ and finally give the expression above: $p_e(x^2 + 1) + xp_o(x^2 + 1)$

Having gotten our expression for the numerator in the form $p_e(x^2 + 1) + xp_o(x^2 + 1)$, we can proceed as before:

$$\frac{s(x)}{(x^2+1)^m} = \frac{p_e(x^2+1) + xp_o(x^2+1)}{(x^2+1)^m}$$

and expanding and cancelling common factors of $x^2 + 1$ gives a sum of terms of the following two types:

1. $c_e \frac{1}{(x^2+1)^k}$ and
2. $c_o \frac{x}{(x^2+1)^k}$

for $k = 0, 1, \ldots, m - 1$ (where c_e and c_o are constants).

Finding an antiderivative for terms of the *second* type isn't hard, since $2x$ is the derivative of $x^2 + 1$, so $u = x^2 + 1$ is a good substitution. Antiderivatives will be either a constant times $(x^2 + 1)^{-k+1}$ or, when $k = 1$, $\log(x^2 + 1)$.

Finding an explicit antiderivative for terms of the *first* type (proportional to $(x^2 + 1)^{-k}$) when $k > 1$ can be done using integration by parts to get a "reduction formula" which can be used recursively. See the "techniques of integration" notes.

When $k = 1$, the arctangent function gives an antiderivative.

So the reason we care about the partial fractions decomposition for purposes of integration is that *it breaks a rational function down into a sum of terms, each of which we can find an explicit antiderivative for.*

Exercise 4. Find an antiderivative for $1/(x^3 - 1)$ using the partial fractions decomposition in Example 1. Check that it works. Can you think of another way to find an antiderivative?

How to carry out these steps efficiently

Finding the partial fractions decomposition can often be done very efficiently. The easiest case is when the roots of the denominator are all real and distinct, in which case the partial fractions decomposition looks like

$$\frac{p(x)}{q(x)} = \frac{a_1}{x - r_1} + \frac{a_2}{x - r_2} + \cdots + \frac{a_k}{x - r_k}$$

and we need to find the coefficients $a_1, a_2, \ldots, a_k$. If you clear denominators, you get

$$p(x) = a_1q_1(x) + a_2q_2(x) + \cdots + a_kq_k(x)$$

where $q_j(x)$ is the product of all the factors $x - r_i$ *except* $x - r_j$. That is, $q_j(x) = q(x)/(x - r_j)$ (which is a polynomial, since r_j is a root).

Now the idea is that you can plug in any number you want for x and *you must still have a true statement**. The clever numbers to choose are the *roots* $r_1, r_2, \ldots, r_k$. That's because the *only* polynomial $q_j(x)$ that isn't zero when you plug in a root is the one corresponding to that root! That is, $q_j(x)$ has all the roots *except* r_j as a root. So when you plug in r_1, for example, you get

$$\begin{aligned} p(r_1) &= a_1 q_1(r_1) + a_2 q_2(r_1) + \cdots + a_k q_k(r_1) \\ &= a_1 q_1(r_1) + a_2 \cdot 0 + \cdots + a_k \cdot 0 \\ &= a_1 q_1(r_1) \end{aligned}$$

So we've "isolated" the coefficient a_1 without doing anything! In general, we have

$$a_j = p(r_j)/q_j(r_j)$$

Now we can actually streamline this further. You perhaps think of plugging numbers into polynomials as easy. But it's tedious, error-prone, and inefficient for polynomials of degree higher than about 2, unless you use Horner's rule (aka "synthetic division"). Read the Math 118 notes on polynomial division for this. The idea is that division by $x - r_j$ leaves a remainder that's equal to the value of the polynomial at r_j (again, see polynomial division notes for details). So to calculate $p(r_j)$ efficiently, just do "synthetic division" of $p(x)$ by $x - r_j$.

You could do the same for $q_j(r_j)$, but you'd have to first get $q_j(x)$ in polynomial form, which requires another division. But notice that by the product rule,

$$q'(x) = [(x - r_j)q_j(x)]' = q_j(x) + (x - r_j)q_j'(x)$$

and if you plug in r_j, you get $q'(r_j) = q_j(r_j)$.

So calculate the derivative $q'(x)$ and then do synthetic division of $q'(x)$ by $x - r_j$ to get $q'(r_j)$ $(= q_j(r_j))$. Do the division for each root to get $p(r_j)$ and $q'(r_j)$ and then you've got all the coefficients.

What about complex roots? OK, that's pretty streamlined, but it's only for the easiest case. What if, for example, the denominator $q(x)$ has *complex* (non-real) roots? If they're distinct, you could actually do the same thing. You'd just have to be willing to do the arithmetic with complex numbers. Then when you've found the coefficients, put things together into its "real form". Since the complex roots of a real polynomial come in complex conjugate pairs, you'll get terms in pairs.

*This is an important and basic idea you should be, or quickly *become*, very familiar with.

Corresponding to the pair of roots $r, \bar{r}$ you've got two terms,

$$\frac{a}{x-r} + \frac{b}{x-\bar{r}}$$

You'll find that a and b are conjugate. When you get a common denominator, you'll have

$$\frac{a(x-\bar{r}) + b(x-r)}{x^2 - tx + n}$$

where $t = r + \bar{r}$ is twice the real part of r and $n = r\bar{r}$ is the squared modulus of r. (If we write $r = c + id$ for real numbers c, d then $t = 2c$ and $n = c^2 + d^2$.) Then the numerator is

$$a(x-r) + b(x-\bar{r}) = (a+b)x - (ar + b\bar{r})$$

Now since $b = \bar{a}$, the coefficients are in fact real numbers: $a + b$ is twice the real part of a, and $ar + b\bar{r} = ar + \bar{a}\bar{r} = ar + \overline{ar}$ is twice the real part of ar.

Exercise 5. Redo example 1 using this approach.

So complex roots aren't much more trouble. The only reason they're *any* trouble is that we don't have a way to say what an antiderivative of $1/(x-r)$ is if r isn't real. That is, our logarithm function doesn't accept complex number inputs.

What if you have repeated roots? I'm going to illustrate by an example, since the notation to express things in a general form is somewhat cumbersome. Take the rational function

$$\frac{x^2+x+1}{x(x-1)^2(x-2)^3}$$

with $p(x) = x^2 + x + 1$ and $q(x) = x(x-1)^2(x-2)^3$ (in factored form; determining this is the only "hard" part of the whole procedure). The partial fractions decomposition we're after looks like

$$\frac{x^2+x+1}{x(x-1)^2(x-2)^3} = \frac{p_0(x)}{x} + \frac{p_1(x)}{(x-1)^2} + \frac{p_2(x)}{(x-2)^3}$$

where $p_0(x)$ has degree 0, $p_1(x)$ has degree ≤ 1 and $p_2(x)$ has degree ≤ 2.

Now looking ahead, we're going to want the numerators expressed as polynomials in $x - r$ where r is the root of the corresponding denominator. So

$$\begin{aligned} p_0(x) &= a_0 \\ p_1(x) &= b_0 + b_1(x-1) \\ p_2(x) &= c_0 + c_1(x-2) + c_2(x-2)^2 \end{aligned}$$

and we need to find the coefficients.

Check that the standard "clearing denominators and plugging in roots" gives the coefficients a_0, b_0 and c_0.

How do we get the remaining coefficients? They're the ones forced on us by multiplicity, and our problem is that we have 6 coefficients to find but only 3 roots to plug in. We need 3 more equations. You could get them in many ways: plug some non-roots in, for example. But we're going to use a connection between the derivative and multiplicity we've already used.

Theorem 1. *Suppose $s(x)$ is a polynomial having a number r as a root with multiplicity m. That is, suppose $s(x) = (x-r)^m g(x)$ where the polynomial $g(x)$ has no factor $x - r$. Then*

$$s^{(m)}(r) = m!g(r)$$

while $s^{(k)}(r) = 0$ for $k < m$.

Proof. Your turn; use the product rule repeatedly. □

Exercise 6. Prove (that is: *give the reasoning for*) the theorem for $m = 2$.

Notes on this exercise:

1. We've already proved the theorem for $m = 1$ above. Look back at that argument and understand it.

2. Perhaps it goes without saying, but I'm going to say it: *write down what the theorem says* in the specific case $m = 2$ *before you do anything else,* and *think about what it means.* Now think about how you might prove it. Then prove it.

Exercise 7. What would it take to prove for $m = 3$? Do that. Now can you see how to express your reasoning in a way that applies for any m?

How does this theorem figure in to our strategy? Notice we've used it already to avoid having to explicitly compute q_j. We'll use it in a similar way here.

We've got the polynomial identity

$$p(x) = p_0(x)q_0(x) + p_1(x)q_1(x) + p_2(x)q_2(x) \tag{2}$$

where

$$\begin{aligned} q_0(x) &= (x-1)^2(x-2)^3 \\ q_1(x) &= x(x-2)^3 \\ q_2(x) &= x(x-1)^2 \end{aligned}$$

We've gotten three equations from it by plugging in the roots $0, 1, 2$. That "isolates" $p_0(0) = a_0$, $p_1(1) = b_0$, and $p_2(2) = c_0$.

Exercise 8. Find a_0, b_0, c_0.

Then *take derivatives* of the basic identity (2):

$$p'(x) = p_0'(x)q_0(x) + p_0(x)q_0'(x) + \ldots + p_2(x)q_2'(x)$$

Plugging in the root 1 makes the terms with a factor of $q_0(x)$ or $q_2(x)$ go away. But also, it makes the terms with a factor of $q_0'(x)$ or $q_2'(x)$ go away, since 1 is a root of q_0 and q_2 of multiplicity 2!

So we get

$$p'(1) = p_1'(1)q_1(1) + p_1(1)q_1'(1)$$

and by the theorem, $q_1(1) = q^{(2)}(1)/2$ while $p_1'(1) = b_1$ and $p_1(1) = b_0$. Finally, $q_1'(1)$ can be determined using the same ideas: $q_1'(1) = q^{(3)}(1)/3!$.

So we get

$$b_1 = \frac{p'(1) - b_0 q^{(3)}(1)/3!}{q^{(2)}(1)/2!}$$

Exercise 9. Verify that if $q(x) = (x-1)^2 q_1(x)$, then $q^{(3)}(1) = 3! q_1'(1)$.

Applying this idea to the root 2, you can solve for c_1. Then you can differentiate (2) one more time, plug in 2, and solve for c_2.

Exercise 10. Solve for b_1, c_1, c_2 and thereby obtain the partial fraction expansion of

$$\frac{x^2+x+1}{x(x-1)^2(x-2)^3}$$

Give an antiderivative for this rational function.

Note. In working with the product rule used repeatedly, you may have noticed something suspicious. It looks as if

$$(fg)^{(3)} = f^{(3)}g + 3f^{(2)}g' + 3f'g^{(2)} + g^{(3)}$$

for example, which suggests $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. What is the "expanded" form of $(fg)^{(4)}$ and does it fit the apparent pattern? Is there a reason? How far does it go?

Repeated complex roots A natural approach to dealing with repeated complex roots, after what we've done already, is to see if going through the above with complex roots can yield an expansion that can be put into "real form" as we did for distinct complex roots.

Another approach to dealing with repeated roots is to consider the case of distinct roots, then find a way to "take limits" as one root approaches another. The trick is to get something useful. For example,

$$\frac{1}{(x-r)(x-s)} = \frac{c}{x-r} - \frac{c}{x-s}$$

where $c = 1/(r - s)$. If we let $r \to s$ we appear to get garbage, as $c \to \infty$. But what if we first find an antiderivative? That would be

$$\frac{\log(x - r) - \log(x - s)}{r - s}$$

and the limit of this as $r \to s$ is (by definition!) the derivative of $f(t) = \log(x - t)$ at $t = s$, namely

$$\frac{-1}{x - s}$$

which is a correct antiderivative of

$$\frac{1}{(x - s)^2}$$

(the case where $r = s$).

The point is just that it's sometimes possible to weasel out of dealing explicitly with repeated roots, treating the situation as a limit of distinct roots. This is sometimes an illuminating perspective.

Additional exercises

I'll post them separately.