Math 172: Series

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Abstract

These notes cover the basics of series. They refer back to notes on sequences and integration. Power series and other series of functions are covered in another set of notes.

An expression like

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{1}$$

which goes on forever is called a *series*. Sometimes, especially in older sources, the term *infinite series* is used. However, you're unlikely to hear any mention of a "finite series", since the term "sum" is already available for that idea.

To give an actual definition, we use the concept of sequence: a series is just a sequence with "+" symbols between consecutive terms*. The terms of the sequence are also called the *terms* of the series.

While a series looks like a sum, adding infinitely many terms is very different from adding finitely many terms. The latter is determined as soon as we say how to add two terms, by the associative property of addition†. But the associative property doesn't necessarily apply‡ to sums of infinitely many terms: for example,

$$1 + -1 + 1 + -1 + \cdots$$

*This definition is meant to emphasize as strongly as possible that a series is an *expression*. To see the significance of this distinction, it's crucial to keep in mind the fact that *not all expressions can be evaluated:* for example, $3 \div 0$ is a perfectly good expression, but cannot be evaluated. A sequence, however, is *not* an expression.

†and the Principle of Mathematical Induction.

‡To be more specific and accurate, the problem is that the Principle of Mathematical Induction allows us to say that any *finite* sum has a value independent of the way we group, but says nothing about *infinite* sums. Moreover, there is no actual value determined by an infinite sum until we decide what that should mean. As for the associative property, what is meant is that attempting to extend it to work for infinite sums gives what appear to be contradictions — if infinite sums work like finite sums. Conclusion: they don't!

could be grouped as

$$1 + (-1 + 1) + (-1 + 1) + \cdots$$

or as

$$(1 + -1) + (1 + -1) + \cdots$$

and the first would appear to be 1 as it is 1 plus a sum of zeros, while the latter looks like a sum of zeros, and therefore zero.

Notation It is common to call a series an "infinite sum", even when we also use the phrase "sum of a series" to mean the value of the series. This is the same overloading of the word "sum" as when we say that the expression $4 + 2$ is a sum, and that its sum is 6. In other words, "sum" is both a *type of expression*, and the *result of evaluating an expression of that type*. So it shouldn't cause too much consternation, and careful attention to context should be sufficient to keep things straight.

1 Definition: Sum of a series

We can "evaluate" infinite sums (or really, we *define* their value) as *limits*: specifically, as the *limit of a sequence of finite sums.* If $a_1 + a_2 + a_3 + \cdots$ is a series — that is, (a_n) is its sequence of terms — then we define *another* sequence from it, the *sequence of partial sums*, as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n \\ &\vdots \end{aligned}$$

so s_n is the sum of the first n terms of the series, for any integer $n \geq 1$. Using sum notation, we can write the definition as

$$s_n = \sum_{i=1}^{n} a_i$$

Make sure you are clear that each s_n is a *finite* sum (with n terms).

If the sequence (s_n) of partial sums has a real number L as a limit*, then we will say

*That is, not $+\infty$ or $-\infty$. If we allow the sequence to consist of complex numbers, very little changes.

that the *sum of the series* is L, or that the series *converges* to L. If the sequence of partial sums has limit $+\infty$ (or $-\infty$) we will say that the series *diverges* to $+\infty$ (or $-\infty$). Finally, if the sequence of partial sums has *no* limit, not even $\pm\infty$, then we just say that the series *diverges.*

Exercise 1. Find s_1, s_2, s_3, s_4 for the series (1).

Note. One convention I use here is numbering of "formulas" or other items (numbered on the right) so as to make cross-reference convenient. A cross-reference in the text looks like (2) and refers to whatever "displayed item" (item on a line by itself) is labelled 2 (in parentheses on the right). The item could be on an earlier page.

2 Basic examples

The geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

(where the sequence of terms is $a_n = 2^{-n}$) converges. In this case we can give an *explicit formula for the partial sums.* In fact, think of the partial sums as the distance travelled by a runner who covers 1 unit in the first minute, half a unit in the second minute, and each succeeding minute covers half the previous minute's distance. Then we have essentially the bisection method starting at 1 and 2 and always taking the right half-interval. You can verify that $s_n = 1 + \frac{2^n - 1}{2^n} = 2 - \frac{1}{2^n}$. This sequence has limit 2, so the series converges and its sum is 2.

The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. We don't have an explicit formula* for the partial sums (they're called the "harmonic numbers" and often written H_n), but we can still show that the limit is $+\infty$. There are several ways to do this, each illustrating an important, general-purpose technique.

Way #1 One way is to consider a subsequence of the sequence of partial sums, namely the partial sums of the first $1, 2, 4, 8, 16$ etc terms. We can show that, for example, $H_{16} > H_8 + \frac{1}{2}$

*Of course, the defining expression is a perfectly explicit formula. It's difficult to say exactly what we're after here, but the idea is that we'd like a formula which doesn't get longer the larger n gets. A prototypical example would be $1+2+3+\cdots+n = n(n+1)/2$. The formula on the right is often described as "closed-form", but what that means exactly is hard to say.

by a crude inequality:

$$
\begin{aligned}
H_{16} &= \underbrace{1+\frac{1}{2}+\cdots+\frac{1}{8}}+\frac{1}{9}+\cdots+\frac{1}{16} \\
&= \qquad H_8 \qquad\quad +\frac{1}{9}+\cdots+\frac{1}{16} \\
&> \qquad H_8 \qquad\quad +\frac{1}{16}+\cdots+\frac{1}{16} \\
&= \qquad H_8 \qquad +8\cdot\frac{1}{16} \\
&= H_8+\frac{1}{2}
\end{aligned}
$$

where the $>$ symbol comes in because the terms from $\frac{1}{9}$ to $\frac{1}{15}$ are all bigger than $\frac{1}{16}$. So we replaced the sum of the reciprocals from $\frac{1}{9}$ to $\frac{1}{16}$ (that's 8 numbers) with the smaller sum $\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}$ (also 8 terms) and the latter is $\frac{1}{2}$.

We can play the same trick with H_{32} and H_{16} to get $H_{32} > H_{16}+\frac{1}{2}$ and so on (try it yourself). In general you can see that $H_{2^n} > H_{2^{n-1}}+\frac{1}{2}$ for all $n>1$ (why not for $n=1$?). This shows that the subsequence H_{2^n} is increasing without bound, in fact that $H_{2^n} > \frac{1}{2}n$. So this subsequence has limit $+\infty$. Therefore the sequence H_n cannot have a finite limit (and since it's increasing, it must have limit $+\infty$ also). (See the notes on sequences if this isn't apparent.)

The previous argument illustrated a useful technique called a "condensation criterion". It is loosely analogous to substitution in integration. We'll see it again.

Way #2 A different way to show that the limit is $+\infty$ interprets the partial sums H_n as *upper sums for an integral.* Choosing the partition $(1,2,3,4,5,\ldots,n+1)$, note that the upper sum for the integral

$$\int_1^{n+1}\frac{1}{x}\,dx \tag{2}$$

is exactly H_n. The value of the integral is $\log(n+1)$ and since H_n is an upper sum, we have $H_n \geq \log(n+1)$. As the sequence $\log(n+1)$ has limit $+\infty$, so does H_n.

Wait, how do we know the limit of $\log(n+1)$ is $+\infty$? *This is a basic and important property of the logarithm function,* but if you didn't know it, you do now. That's because a *lower* sum for this integral, for the same partition, is $H_{n+1}-1$ (draw a picture for $n=4$ for example.) Our first argument shows that H_{n+1} has limit $+\infty$, and $\log(n+1) \geq H_{n+1}-1$ since $H_{n+1}-1$ is a lower sum, so $\log(n+1)$ must have limit $+\infty$ also!

Just as for the geometric series above, this is also an illustration of a generally useful technique, this time *comparison with an integral.* We'll see it again, too.

Exercise 2. Write out the lower sum for the integral (2) for the partition $(1, 2, 3, \ldots, n+1)$ and verify that it is $H_{n+1} - 1$.

Exercise 3. (This is only tangentially related to series, but it may set you on the road to great fame and fabulous wealth.) Show that $H_n - \log(n)$ has a limit. (Hint: show first that $H_n - \log(n)$ is decreasing:

$$H_n - \log(n) < H_{n-1} - \log(n-1)$$

This is the same as saying that

$$\frac{1}{n} = H_n - H_{n-1} < \log(n) - \log(n-1) = \int_{n-1}^{n} \frac{1}{x}\, dx$$

which you can verify easily, using a basic property of integrals or a picture. Second, show that $H_n > \log(n)$. So $H_n - \log(n)$ is bounded below by 0. Third, do a victory lap. You're done. Why?)

Euler's constant γ The limit of this sequence $H_n - \log(n)$ is called *Euler's constant* and written γ (lower case Greek letter gamma). Unlike its more well-known number-with-letter-name cousins π and e, it is not known whether γ is rational or irrational. If you settle this question, please cut me in on the prize money. I take only a 3% commission. You can keep the fame.

The number γ arises in a surprising number of places in mathematics.

Exercise 4. Let $a_n = 1/(2n-1)$. Write out the first six partial sums $s_1, s_2, \ldots, s_6$ for the series $\sum a_n$. Show that $s_n > \frac{1}{2}H_n$ for all n (here s_n is the nth partial sum of the series $\sum a_n$). Conclude that $\sum a_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges.

This next exercise uses this and will be referred to later.

Exercise 5. Let $a_n = 2/n$ if n is odd and $-1/(n-1)$ if n is even. Write out the first six partial sums of the series $\sum a_n$. Show that $s_{2n} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. Conclude that $\sum a_n$ diverges.

2.1 Two important families of series

The two series we gave as basic examples are specific examples of two very important *families* of series: the *geometric series* (plural) and the so-called "*p-series*" (one for each value of p).

2.1.1 Geometric series

For any real number r we can consider the series

$$1 + r + r^2 + r^3 + \cdots$$

whose terms are r^n (indexing starting at 0). This is called the geometric series with *ratio* r.

Theorem 1. *The geometric series with ratio r converges exactly when $|r| < 1$. In that case, its limit is $\frac{1}{1-r}$.*

Proof. Check that the partial sums $s_n = 1 + r + r^2 + \cdots + r^n$ (note this is a sum of $n+1$ terms) satisfy $s_n(1-r) = 1 - r^{n+1}$. Now if r^{n+1} has a limit, so does s_n (as long as $r \neq 1$). So we are finished if we prove the following: □

Fact 1. *The sequence $1, r, r^2, r^3, \ldots$ has limit 0 if $|r| < 1$.*

Exercise 6. Write out the proof of the theorem in full, dealing with all cases. What happens when $r = 1$?

2.1.2 p-series

For any real number p, the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is called (in calculus textbooks) a "p-series" with exponent p. (As a *function* of p, it defines the famous Riemann zeta function.)

Theorem 2. *The p-series converges if and only if $p > 1$.*

Proof. The partial sum s_n can be viewed as (almost) a lower sum for the partition $(1, 2, 3, \ldots n)$ for the integral $\int_1^n x^{-p} dx$. In fact, $s_n - 1$ is *exactly* this lower sum. Since lower sums are less than or equal the integral,

$$s_n - 1 \leq \int_1^n x^{-p} dx = \frac{1}{-p+1} x^{-p+1}|_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

for $p \neq 1$. The partial sum can also be seen as an *upper* sum: this time for the integral $\int_1^{n-1} x^{-p} dx$. So we get

$$\frac{(n-1)^{1-p}}{1-p} - \frac{1}{1-p} = \int_1^{n-1} x^{-p} dx \leq s_n$$

So the statement will be established if we prove the following: □

Fact 2. *The sequence $1^k, 2^k, 3^k, \ldots$ has limit 0 for $k < 0$ and limit $+\infty$ for $k > 0$.*

Exercise 7. Prove this fact and use it to give a full proof of the theorem. What about when $k = 0$? Why don't we need that for the theorem? How does the inequality with the integral work when $p = 1$?

Remark. A hint for the exercise above: you will need to note that the sequence of partial sums is *increasing*, and the *bounded increasing sequence property* will come in handy.

Unlike the geometric series, it is not at all easy to determine limits (that is, their sum, or value) for the p-series. Euler determined the limits for p an *even positive integer* (the famous "Basel problem" was the case $p = 2$ — Euler went way beyond the call of duty here!). The limit is a rational multiple of π^p; the rational number involved is the pth Bernoulli number divided by $2p!$ (not $(2p)!$). (I give some explanation of the Bernoulli numbers in the notes on power sums.) That is, Euler proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k} B_{2k}}{2(2k)!} \pi^{2k}$$

for any positive integer k.

By contrast, before the 1980s it was unknown even whether the sum for $p = 3$ was *irrational* or not. Apéry in 1979 proved that it is irrational. It seems fair to say that little else is known about the sums of p series for integer exponents.

Series of functions If we view the geometric series as a function of r, or rather as a *series* of functions, then we have shown that it converges to the *function* $1/(1 - r)$. This is the *power series expansion* for the *rational function* $1/(1 - r)$. Similarly, the p-series as a function of p can be thought of as a series of functions. This is a so-called *Dirichlet series*. Its limit is the Riemann zeta function, which is most definitely *not* a rational function. We will study power series in some detail later; Dirichlet series require more sophisticated methods (complex analysis, Fourier analysis) and we will not do much with them. We will at least find the sum

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

that Euler evaluated to solve the Basel problem, however.

3 Determining convergence: series with nonnegative terms

The first and most useful idea in determining convergence of series is *comparison*. Its usefulness derives from the fact that a *bounded increasing sequence has a limit* (completeness of $\mathbf{R}$ — we need the *real* numbers here!).

But it's useful to state a version specifically for comparison of series, and a version for comparison of series with integrals.

We will also start with series all of whose terms are nonnegative. This simplifies some things (as in the case of sequences) and allows us to get started without extra distracting conditions. It's helpful to keep in mind the following

Fact 3. *If $\sum a_n$ has nonnegative terms then the sequence of partial sums is (weakly) increasing.*

Proof. Exercise. If s_n is the sequence of partial sums, you want to show $s_{n+1} \geq s_n$ (for all n). That's the same as showing that $s_{n+1} - s_n \geq 0$. What is $s_{n+1} - s_n$? Write out what the partial sums are sums of. □

Recall that increasing sequences (and increasing functions) are substantially simpler than sequences (functions) in general. Here, the main thing we get is a simple criterion for limits. Recall once again that *an increasing sequence which is bounded above has a limit.* That's one expression of the *completeness property* of the real numbers. Now we see how this translates to series, through the sequence of partial sums:

Fact 4. *A series whose terms are nonnegative and whose partial sums are bounded above converges.*

Proof. Exercise. □

It's not often easy to recognize directly that a sequence of partial sums is bounded above, but we can often use *comparison* (inequalities) to establish this.

3.1 Comparison tests

Theorem 3. *Suppose $0 \leq a_n \leq b_n$ for all indices n. Then:*

1. *If $\sum b_n$ converges then $\sum a_n$ converges, and*
2. *If $\sum a_n$ diverges then $\sum b_n$ diverges*

Proof. Recalling that convergence/divergence is defined in terms of the sequence of partial sums, we need to determine what the condition given says about the sequences of partial sums. To get started, let's give them names: say the partial sums for $\sum a_n$ are s_n and the partial sums for $\sum b_n$ are called t_n.

Claim: $s_n \leq t_n$ for all indices n.

Given this, let's see how it's used. If $\sum b_n$ converges, then its sequence of partial sums t_n converges to a limit L. As t_n is increasing, $t_n \leq L$ for all indices n. Then by the claim, $s_n \leq L$ for all indices n, so s_n is bounded above. As s_n is also increasing, it must have a limit, so $\sum a_n$ converges. □

Exercise 8. Prove the claim in the proof above and prove the second statement of the theorem.

Exercise 9. In the theorem above, if $L = \sum b_n$ and $M = \sum a_n$ show that $M \leq L$.

Comparison with one of our two types of series is used so often that they are usually stated as separate tests:

The ratio test (comparison with geometric series):

Theorem 4. *If $\sum a_n$ has positive terms and there is a number $r < 1$ so that the ratios $a_{n+1}/a_n \leq r$ for all indices n then $\sum a_n$ converges. If the ratios $a_{n+1}/a_n \geq 1$ for all n then $\sum a_n$ diverges.*

Proof. $a_{n+1}/a_n \leq r$ means that $a_{n+1} \leq ra_n$. So for $n = 1$ this means $a_2 \leq ra_1$. For $n = 2$ this means $a_3 \leq ra_2$. Putting these together gives $a_3 \leq r^2 a_1$. Continuing up the indices, we can see that $a_{n+1} \leq r^n a_1$ for all indices n. (This is the "principle of mathematical induction", named so as to insure confusion.) Now the geometric series with positive ratio r converges if $r < 1$, and therefore the series $\sum a_1 r^n$ converges as well (exercise: why?). Now use comparison. □

A subtle aspect of this convergence criterion is that *it is not the same to check that all the ratios are less than* 1. The harmonic series is a counterexample.

Make sure you understand the difference between

Condition 1. For all n, $a_{n+1}/a_n < 1$.

and

Condition 2. There is a number $r < 1$ so that for all n, $a_{n+1}/a_n < r$.

and that the harmonic series satisfies condition 1 but not condition 2, and that condition 2 is what the ratio test involves.

Note. It might help to point out that the "subtle distinction" here arises in many forms in mathematics when inequalities or other conditions are asserted for infinitely many cases. A term used in some other contexts to make this type of distinction is "uniform". To use it here, we could say that condition 2 says that the ratios need to be *uniformly* less than 1, that is, they are all *at least some fixed amount* less than 1.

As an exercise you will prove limit versions of these tests. These are often easier to apply, but have stronger requirements.

Exercise 10. Show how the harmonic series is a counterexample to the claim that a series with positive terms converges if its ratios of successive terms are all less than 1. Why does this not contradict the theorem? (Yes, I've attempted to explain that already. Your turn!)

To prove *divergence* it is enough to check that the ratios are all *at least* 1:

Exercise 11. If $a_{n+1}/a_n \geq 1$ for all n, compare $\sum a_n$ with the series $\sum a_1 = a_1 + a_1 + a_1 + \cdots$ to prove divergence.

It is a good exercise to try to make up some series where the ratios of successive terms do not satisfy either condition and which converge or diverge. For example, "interleave" the terms of two geometric series, like $a + b + a^2 + b^2 + \cdots$. What kinds of behavior can you get?

The root test (comparison with geometric series) A slightly different way to compare to a geometric series gives the so-called "root test". It's more powerful than the ratio test, but in many common examples encountered in calculus often more difficult to apply.

Theorem 5. *If $\sum a_n$ has nonnegative terms and there is some real number $r < 1$ so that $a_n^{1/n} \leq r$ for all indices n, then $\sum a_n$ converges.*

Proof. The condition says that $a_n \leq r^n$ so we compare with the geometric series with ratio r. $\square$

Exercise 12. Write out the proof carefully.

Remark 1. All of the above tests work if we only ask that the test is passed for all n past a certain point.

Remark 2. The ratio test is actually a consequence of the root test. If all ratios are $\leq r$, and $r < 1$, then $a_n \leq r^n(a_1/r)$ so $a_n^{1/n} \leq (a_1/r)^{1/n} r$ and, since $(a_1/r)^{1/n} \to 1$ (exercise), for large enough n we will have $(a_1/r)^{1/n} < 1 + \frac{\varepsilon}{2} < 1/r$ where $\varepsilon = \frac{1}{r} - 1 > 0$. Then

$$a_n^{1/n} \leq (a_1/r)^{1/n} r < (1 + \frac{\varepsilon}{2}) r$$

for large n, and $(1 + \frac{\varepsilon}{2}) r < 1$. So by the root test, the series converges.

Exercise 13. Prove that if $a > 0$ then $a^{1/n} \to 1$. (Hint: one approach is to use the logarithm and continuity. Another is to use the factorization that gave the formula for the partial sums of the geometric series.)

Exercise 14. Show that the sequence $n^{1/n}$ has limit 1. Explain why this shows that the root test is ineffective for p-series. (Hint: take logarithm and use L'Hôpital's rule, or show that $f(x) = \log(x)/x$ is decreasing for $x > e$, and bounded below, so $\log(n)/n$ must have a limit by completeness. Therefore $n^{1/n}$ must have a limit, say L. To evaluate it, consider the limit of the sequence $2^{1/n}n^{1/n} = (2n)^{1/n}$. On the one hand, the 1/2 power of this sequence is the even-index subsequence of $n^{1/n}$ so must have the same limit L. Therefore $(2n)^{1/n}$ has limit L^2. On the other hand, $2^{1/n}$ has limit 1 so $2^{1/n}n^{1/n}$ has limit $1 \cdot L = L$. So $L = L^2$ and therefore $L = 0$ or 1. But $L = 0$ is impossible as L is the exponential of a real number. QED.)

Here is a comparison-with-p-series test.

Theorem 6. *If there is a real number $p > 1$ so that $n^p a_n$ is bounded above then $\sum a_n$ converges. If there is a real number $p \leq 1$ so that $n^p a_n$ is bounded below by a positive number, then $\sum a_n$ diverges.*

The conditions in the theorem are usually expressed as "a_n is $O(n^{-p})$" and "a_n is $\Omega(n^{-p})$" (the latter seen mostly in computer science). We say that a sequence a_n is $O(b_n)$ if there is a number $M > 0$ so that $a_n \leq Mb_n$. We say a_n is $\Omega(b_n)$ if there is an $M > 0$ so that $a_n \geq Mb_n$.

Exercise 15. Show that this test is ineffective for the series $\sum \frac{1}{n\log(n)}$ and $\sum \frac{1}{n\log(n)^2}$. Use comparisons with integrals to determine convergence/divergence.

In comparing a series with lower/upper sums for an integral, it helps to use decreasing functions for the integrand so that you can use the endpoints of the partition. Here is a statement set up this way.

Integral comparison test

Theorem 7. *Suppose f is a decreasing continuous nonnegative function on $[0, \infty)$. Then the series $\sum f(n)$ converges if the improper integral $\int_0^\infty f$ exists, that is, if $\int_0^N f$ has a finite limit as $N \to \infty$.*

Proof. As in the example of the harmonic series and the exercise above, relate the partial sums to upper and lower sums of the integral for the partition $(1, 2, 3, \ldots, N)$ □

It is possible to reformulate this in a number of ways.

4 Improper integrals (while we're on the topic)

When integration is introduced, there are usually two conditions imposed:

1. The integral is over a finite interval $[a, b]$, and

2. The function to be integrated is defined and bounded on $[a, b]$.

It is possible (and very useful) to remove these restrictions. You do that with limits. For example, we define

$$\int_a^\infty f = \lim_{b\to\infty} \int_a^b f \tag{3}$$

which makes sense if f is defined on $[a, \infty)$ and bounded on all finite intervals $[a, b]$ with $b > a$. That is,

$$F(x) = \int_a^x f$$

defines a function F on $[a, \infty)$ and (3) just defines the integral over $[a, \infty)$ as the limit of F at ∞, which may or may not exist.

Example 1. $\int_1^\infty x^{-2}dx = \lim_{b\to\infty} -x^{-1}|_1^b = \lim_{b\to\infty}(-b^{-1} + 1) = 1$

Example 2. $\int_1^\infty x^{-1}dx = \lim_{b\to\infty} \log(x)|_1^b = \lim_{b\to\infty} \log(b) = \infty$ so this integral diverges (to ∞) or "does not exist".

Integrals not meeting the two conditions above and defined as limits of ones that do are called "improper integrals". The two conditions make for two ways of being "improper". The first has to do with the interval. As we've seen, that's straightforward to deal with.

The other way of being "improper" is for the function being integrated (the *integrand*) to be unbounded. We deal with that using limits as well. Here is the basic idea. Suppose first f is bounded on $[a, c]$ for any $c < b$ but not on $[a, b]$. For example, $f(x) = (1 - x)^{-1/2}$ is bounded on $[0, c]$ for any c less than 1, but not on $[0, 1]$. We define

$$\int_a^b f = \lim_{c\to b-} \int_a^c f$$

so for our example, we have, by definition,

$$\int_0^1 (1 - x)^{-1/2}dx = \lim_{c\to 1-} \int_0^c (1 - x)^{-1/2}dx = \lim_{c\to 1-} -2(1 - x)^{1/2}|_0^c = 2$$

If a function has a point in the middle of the interval of integration like this, we break the integral into two and use the same idea on both:

$$\int_a^b f = \lim_{c_1 \to c-} \int_a^{c_1} f + \lim_{c_2 \to c+} \int_{c_2}^{b} f$$

Example 3. $\int_{-1}^{1} x^{-2/3} dx$. The integrand is unbounded on $[-1, 1]$ but bounded on $[-1, c_1]$ and $[c_2, 1]$ for any $c_1 < 0$, $c_2 > 0$. In other words, the only "problem point" is 0. So by definition, the integral is the sum of limits:

$$\lim_{c_1 \to 0-} \int_{-1}^{c_1} x^{-2/3} dx + \lim_{c_2 \to 0+} \int_{c_2}^{1} x^{-2/3} dx = \lim_{c_1 \to 0-} (3c_1^{1/3} - 3(-1)^{1/3}) + \lim_{c_2 \to 0+} (3 - 3c_2^{1/3}) = 6$$

It is a subtle point that the limits are to be taken separately. See the difference with $\int_{-1}^{1} x^{-2} dx$!

There are many more complicated ways for a function to be unbounded; a more sophisticated approach to integration is the easiest way to deal with them.

Finally, we can deal with integrals that are improper for both reasons. For example,

$$\int_0^\infty \frac{\sin(x)}{x} dx = \lim_{a \to 0+} \int_a^1 \frac{\sin(x)}{x} dx + \lim_{b \to \infty} \int_1^b \frac{\sin(x)}{x} dx$$

We'll see the value of this integral later, after we've discussed power series.

Improper integrals may seem like an extravagant extension of the integral idea, thought up just to provide hard homework problems. But have a look at any scientific use of calculus: improper integrals are all over the place. "Integration over all space" in electricity and magnetism, integration of unbounded functions, various transforms, integration over the sample space in probability and statistics, etc. They are actually the norm, while the integrals we have defined are a simplified version.

[Gamma and beta integrals? Fresnel integral?]

5 Series with general terms

We now drop the condition that the terms of a series be nonnegative.

There are two facts we should point out somewhere. The first is

Fact 5. *If $\sum a_n$ converges, then $a_n \to 0$.*

Proof. Exercise (hint: if the series converges, then the sequence s_n of partial sums must have a limit, say L. What is the limit of the sequence $s_n - s_{n-1}$? Why is this relevant?) $\square$

You can use this fact to show that certain series don't converge. You *can't* use it to show that a series *does* converge, though, as the example of the harmonic series shows.

Example 4. If r is a real number ≥ 1, the series $\sum r^n$ does not converge since r^n does not converge to 0.

Example 5. The series

$$\sum \frac{n^{0.0000001}}{\log(n)^{1000000}}$$

doesn't converge, since $n^\varepsilon / \log(n)^k$ has limit $+\infty$ for any real number $\varepsilon > 0$ and any k.

The second fact that's sometimes helpful concerns the so-called tail of a series.

Tail of a series If $\sum a_n$ is a series and s_n is the nth partial sum, then we define the *tail* t_n to be the sum of the terms from $n+1$ on. (We regard this merely as an expression until we know the series converges.) Here is an admittedly sloppy statement of a fact that gets used frequently.

Fact 6. *$\sum a_n$ converges if and only if $t_n \to 0$.*

A less sloppy way to put this is:

Criterion. *$\sum a_n$ converges if and only if, for any $\varepsilon > 0$ there is an N so that $\left|\sum_{n=m}^{k} a_n\right| < \varepsilon$ for all $k \geq m > N$.*

Proof. This (the less-sloppy version) is actually the Cauchy Convergence Criterion translated into series language. See the notes on sequences. It says that a sequence s_n converges if and only if, given any $\varepsilon > 0$ there is an N so that $|s_n - s_m| < \varepsilon$ for all $n, m > N$. Apply this to the sequence of partial sums. (What is $s_n - s_m$ then?) $\square$

If the sequence of partial sums does converge, call the limit L. Then the series defining the term t_n converges, and $t_n = L - s_n$. The sequence (t_n) then has limit $L - L = 0$. On the other hand, if the series defining any one of the terms t_n converges, to any finite limit at all, then so does $\sum a_n$, to the limit $s_n + t_n$.

5.1 Alternating series

An alternating series is one whose sequence of terms is $(-1)^n b_n$ where b_n is a *positive decreasing* sequence. The partial sums of an alternating series zigzag back and forth, taking smaller steps each time. As a consequence, they must converge... well, almost.

What about $\sum(-1)^n(1+\frac{1}{n})$? Its partial sums zigzag back and forth, but do *not* become arbitrarily close to each other, so the series does not converge.

Theorem 8. *If b_n is a decreasing sequence of positive numbers with limit 0, then the alternating series $\sum(-1)^n b_n$ converges to a limit between any two consecutive partial sums.*

With alternating sequences of this type, it is easy to estimate the limit and have a quantitative bound for its accuracy: Calculate two consecutive partial sums, and estimate the limit by the average. You are then guaranteed to be accurate to within half of the last term.

Example 6. $\sum(-1)^{n-1}\frac{1}{n}$ has limit approximately $\frac{41}{60}$ since $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}$ and $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}=\frac{47}{60}$. The actual limit must be between $\frac{7}{12}$ and $\frac{47}{60}$ so $\frac{41}{60}$ is off by at most $\frac{1}{10}=0.1$. (The actual limit is $\log(2)$, between $\frac{41}{60}$ and $\frac{42}{60}$!)

Example 7. We'll see that $\frac{1}{e}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots$. We can say that $\frac{1}{e}$ is between $\frac{1}{2}$ and $\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$ (in case you didn't know already! We'll see better estimates of the error when we talk about approximation and Taylor polynomials.)

The two conditions on the sequence (b_n) are both indispensible. Exercise 5 gives an example of a series $\sum(-1)^n b_n$ where (b_n) is positive and has limit 0, yet the series diverges.

Exercise 16. Go back to Exercise 5 and determine why the alternating series criterion does not apply to the series there.

5.2 Absolute convergence

A series $\sum a_n$ is said to *converge absolutely* if the series $\sum |a_n|$ of absolute values of terms converges. From this definition, it is not clear that absolute convergence of $\sum a_n$ really has anything to do with $\sum a_n$. In fact, it turns out to be a *stronger* requirement than convergence, strong enough that a number of things we'd like to be true become true.

Absolute convergence is defined in terms of convergence of a series with nonnegative terms, so we can use all our work with such series. In particular, we can use it to prove that absolute convergence of a series really says something about that series.

Theorem 9. *If $\sum |a_n|$ converges, then $\sum a_n$ converges.*

Proof. We define $a_n^+ = a_n$ if $a_n > 0$ and $a_n^+ = 0$ otherwise. Likewise we define $a_n^- = -a_n$ if $a_n < 0$ and $a_n^- = 0$ otherwise. Then $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$ and $a_n^+ \leq |a_n|$ and the same for a_n^-. Then $\sum |a_n|$, $\sum a_n^+$ and $\sum a_n^-$ are all series with nonnegative terms.

Now by comparison, $\sum a_n^+$ and $\sum a_n^-$ converge if $\sum |a_n|$ does. We claim that the difference of their partial sums gives the partial sums of $\sum a_n$. Exercise: prove this. So by the rules for limits, the sequence of partial sums for $\sum a_n$ converges, which is to say that $\sum a_n$ converges. □

Exercise 17. Complete the proof by showing that the sequence of partial sums for $\sum a_n^+$ minus the sequence of partial sums for $\sum a_n^-$ is the sequence of partial sums for $\sum a_n$.

Exercise 18. Show that

$$a_n^+ = \frac{|a_n| + a_n}{2} \qquad a_n^- = \frac{|a_n| - a_n}{2}$$

Note that the theorem says nothing about the relation of the sums of $\sum a_n$ and $\sum |a_n|$. We can see from the proof, however, that the partial sums for $\sum a_n$ are less than or equal to the partial sums for $\sum |a_n|$. In fact, you can prove

Theorem 10. *If $\sum a_n$ converges absolutely, then*

$$\left|\sum a_n\right| \leq \sum |a_n|$$

Exercise 19. Prove this. (Note that to prove $|a| < b$ you can show that both $a < b$ and $-a < b$.)

Remark 3. Recall that there is a similar fact for integrals, and for finite sums. It is generically called the Triangle Inequality. The reason is explained in the notes on sequences.

5.3 Rearrangement

Series which converge but aren't absolutely convergent are the ones we need to be careful with. (These are often called *conditionally convergent.*) Riemann proved that any such series can be rearranged so as to converge to *any value whatsoever*, or to diverge to $+\infty$ or $-\infty$, or to diverge without any limiting value. You can see this with the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

It converges as we've seen (to some number between $\frac{1}{2}$ and 1; we'll see what number later), but not absolutely since the harmonic series diverges. Suppose you rearrange the terms to the pattern (odd + odd + odd) - even: that is,

$$(1+\frac{1}{3}+\frac{1}{5})-\frac{1}{2}+(\frac{1}{7}+\frac{1}{9}+\frac{1}{11})-\frac{1}{4}+\cdots$$

If we add the terms in parentheses, we get another convergent alternating series (we should check that the terms are decreasing in absolute value and have limit 0) and this one has limit between $1+\frac{1}{3}+\frac{1}{5}=1\frac{8}{15}$ and $1\frac{8}{15}-\frac{1}{2}>1$. So this has a different limit.

Exercise 20. Let the rearranged series after grouping above be $\sum(-1)^{n-1}b_n$. So $b_1=1\frac{8}{15}$, $b_2=\frac{1}{2}$, $b_3=\frac{1}{7}+\frac{1}{9}+\frac{1}{11}=\frac{239}{693}$, etc. Give a formula for b_{2n-1}, the sum of the three terms in the nth set of parentheses in the sum above. Use it to show that $b_{2n}>b_{2n+1}$ and that the sequence b_n has limit 0.

Remark 4. One issue may be slightly confusing here. We not only rearranged, but regrouped a series. The point is that the grouping actually only gives a subsequence of the sequence of partial sums. Check, for example, that the partial sums of $\sum(-1)^{n-1}b_n$ are the subsequence $s_3, s_4, s_7, s_8, \ldots$ of the sequence of partial sums of the rearranged series. In an appendix we look at grouping and show that *if* a series converges, then grouping won't change convergence or the limit.

Remark 5. You can see what's going on here if you think about making the series $\sum a_n^+$ and $\sum a_n^-$ as in Theorem 9. Both of these series *must diverge* for a conditionally convergent series. That's because $\sum a_n$ converges and $\sum |a_n|$ diverges. If both $\sum a_n^+$ and $\sum a_n^-$ converge, then so would $\sum |a_n|$ (and it doesn't...) and if one converges and the other diverges then $\sum a_n$ would diverge. So the reason $\sum a_n$ converges is cancellation, and this depends on ordering of terms.

Exercise 21. (a) If $u_n \to U$ and $v_n \to V$ show that *any* subsequence u_{n_k} of u_n minus *any* subsequence v_{m_k} of v_n has limit $U-V$. (b) On the other hand, if $u_n=3n$ and $v_n=6n$ whether $u_{n_k}-v_{m_k}$ has a limit (and if so, what the limit is) depends on the particular subsequences. Give an example of subsequences so the difference has limit 3.

5.3.1 Rearrangements of absolutely convergent series

Absolute convergence is what we need to prevent rearrangements from changing the limit or convergence status. Riemann also proved that if a series converges absolutely, then so does any rearrangement of that series, and the sum is unchanged.

To prove this, we'd have to make "rearrangement" precise. We do that through a *bijection* $f:\mathbf{N}\to\mathbf{N}$. Then the series rearranged by f has terms $a_{f(n)}$. So instead of a_1 being the first term, $a_{f(1)}$ is.

Exercise 22. Write out a formula for the rearrangement f we did in the example above.

If you found that exercise a bit challenging, you're not alone. It might be nice to avoid explicitly dealing with rearrangements and the excruciating detail that seems to entail. Another approach is to *neglect ordering altogether.* If we're going to do that, we need to consider *all* finite sums of terms, not just the partial sums.

It's possible to prove that absolute convergence is the same as:

Condition 3. Given any $\varepsilon > 0$ there is a finite subset S of indices so that for any finite set $T \supset S$ we have

$$|\sum_{n\in T} a_n - L| < \varepsilon \tag{4}$$

This says that there is a finite set of terms (the terms a_n with $n \in S$) whose sum is within ε of L, and adding any more terms still gives a sum within ε of L. So the terms whose indices are not in S make, in total, a "negligible" contribution to the sum.

The condition (4) can be replaced* by:

$$|\sum_{n\in T\setminus S} a_n| < \varepsilon \tag{5}$$

This says essentially that any finite part of the tail should be negligible, so it's an unordered analog of the criterion (5).

The advantage of this notion is that it does not include any order of terms. So a simple consequence is: if a series converges absolutely, any rearrangement of that series also converges to the same sum.

Let's prove this.

Theorem 11. *A series $\sum a_n$ converges absolutely if and only if for any $\varepsilon > 0$ there is a finite set S of indices so that, for any finite set T of indices containing S, the sum of the terms a_n with $n \in T \setminus S$ is less than ε in absolute value.*

Proof. If $\sum |a_n|$ converges, then given $\varepsilon > 0$ we need to find S to make the condition (5) true. We do that by taking the N we are guaranteed by absolute convergence, so that $\sum_{n=m}^{\infty} |a_n| < \varepsilon$ for all $m \geq N$. Let $S = \{1, 2, 3, \ldots, N-1\}$. Now if T is any finite subset containing S, then by the triangle inequality, $|\sum_{n\in T\setminus S} a_n| \leq \sum_{n\in T\setminus S} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \varepsilon$.

Now to prove the converse: if there *is* such a set S of indices, we have to find N. We will show that $\sum a_n^+$ and $\sum a_n^-$ converge. Since the details are similar for both, we will work with $\sum a_n^+$ and leave the other as an exercise. Let T_m be the set of indices of positive terms

*What is meant here is that if you replace the inequality you get an equivalent condition, not that the inequalities are the same in any sense.

up to index m, together with S. That is, $T_m = \{n : n \leq m,\ a_n > 0\} \cup S$. Choose N to be the largest index in S; then

$$\sum_{n=N+1}^{m} a_n^+ \leq \sum_{n \in T_m \setminus S} a_n < \varepsilon$$

as long as $m > N$. So by the criterion (5) the series $\sum a_n^+$ converges. $\square$

Exercise 23. Prove that if a set S of indices exists as in the statement of the theorem, then

$$\sum_{n=N+1}^{m} a_n^- < \varepsilon$$

for all $m > N = \max(S)$. Explain why this proves that $\sum a_n^-$ converges and finish the proof of the theorem.

6 Additional exercises

Exercise 24. (Limit versions of ratio, root and comparison tests.) Prove the following versions of the ratio, root and comparison tests for nonnegative series.

1. Suppose $L = \lim\ a_{n+1}/a_n$ exists. If $L < 1$ then $\sum a_n$ converges; if $L > 1$ then $\sum a_n$ diverges.

2. Suppose $L = \lim\ a_n^{1/n}$ exists. If $L < 1$ then $\sum a_n$ converges; if $L > 1$ then $\sum a_n$ diverges.

3. Suppose $L = \lim\ a_n/b_n$ exists. If $L > 0$ then $\sum a_n$ converges if and only if $\sum b_n$ converges. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

4. If $\frac{a_n}{b_n}$ has limit $+\infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.

Exercise 25. Consider the series whose terms are defined by

$$a_n = 2^{-\lfloor \frac{n}{2} \rfloor} 3^{-\lfloor \frac{n+1}{2} \rfloor}$$

Here the symbol $\lfloor\rfloor$ means the "floor" function, which gives the largest integer less than or equal to its input.

1. Check that when n is even, $a_n = 6^{-n/2}$.

2. Check that when n is odd, $a_n = \frac{1}{3} 6^{-(n-1)/2}$

3. What is the ratio a_{n+1}/a_n when n is even? What is it when n is odd?

4. Does the series $\sum a_n$ converge? Explain why the ratio test works here but the limit version doesn't. Does the root test work?

Exercise 26. Does the series $\sum \sin(n)2^{-n}$ converge? Explain why the root test works here but the ratio test would be difficult to use or inconclusive, and the limit versions of these tests not applicable. How about simply comparison with $\sum 2^{-n}$?

Exercise 27. Show that the series $\sum 1/\sqrt{n!}$ converges.

Exercise 28. (A "condensation criterion") Suppose $\sum a_n$ is a series with nonnegative terms, and suppose the sequence a_n of terms is (weakly) decreasing. Prove that if $\sum 2^n a_{2^n}$ converges, then so does $\sum a_n$.

Hint: how does the partial sum s_{2^n} of the series $\sum a_n$ compare to the partial sum t_n of $\sum 2^n a_{2^n}$?

Exercise 29. (a) Use the "condensation criterion" above to show that $\sum \frac{1}{n \log(n)^p}$ converges for $p > 1$. (b) Use comparison with an integral to establish the same result. (c) How did you evaluate the integral? Discuss the similarities between the condensation criterion and substitution.

7 Appendix

Claim. If a series $\sum a_n$ converges, then any way of grouping terms gives the same sum.

How do we interpret a "way of grouping" so as to be able to prove or disprove this statement? Let's say that a "way of grouping" is determined by an increasing sequence of indices n_k, and the "grouped" series has terms

$$g_k = \sum_{n=n_{k-1}+1}^{n_k} a_n$$

(where we set n_0 to be one less than the starting index of the sequence of terms). Then the claim is that $\sum g_k$ converges and has the same limit as $\sum a_n$.

Example 8. $a_n = (-1)^n / \log(n+2)$

The series $\sum a_n$ converges by the alternating series criterion. What if we group as:

$$\left(\frac{1}{\log(2)} - \frac{1}{\log(3)} + \frac{1}{\log(4)}\right) + \frac{-1}{\log(5)} + \left(\frac{1}{\log(6)} + \frac{-1}{\log(7)}\right) + \cdots ?$$

We specify this grouping by $n_0 = -1$, $n_1 = 2$, $n_2 = 3$, $n_3 = 5$, etc. Then we have

$$\begin{aligned} g_1 &= a_0 + a_1 + a_2 \\ g_2 &= a_3 \\ g_3 &= a_4 + a_5 \\ &\vdots \end{aligned}$$

which gives exactly the sums of the grouped terms.

Proof. (of claim) This is easy now that we've been able to capture the notion of grouping this way: the partial sums of the series $\sum g_k$ are simply a subsequence of the partial sums of $\sum a_n$. Namely,

$$\sum_{k=1}^{m} g_k = \sum_{n=1}^{n_m} a_n$$

Since $\sum a_n$ converges, its partial sums converge, and any subsequence then converges to the same limit. $\square$

On the other hand, you can see that a "grouped" series can converge when the original doesn't. This is actually a topic of study in mathematics, called "summability methods".

8 Answers to some exercises

16 (b) $n_k = 2k + 1$, $m_k = k$ so $u_{n_k} = 3(2k + 1) = 6k + 3$ and $v_{m_k} = v_k = 6k$. The difference is the constant sequence 3.

17 You need f to start

$$\begin{array}{ccc} 1 & \to & 1 \\ 2 & \to & 3 \\ 3 & \to & 5 \\ 4 & \to & 2 \\ 5 & \to & 7 \\ 6 & \to & 9 \\ 7 & \to & 11 \\ 8 & \to & 4 \\ \vdots & \vdots & \vdots \end{array}$$

a group of 3 odds, then an even, so we have one even number every fourth index, so $f(4n) = 2n$. Then $f(4n + k) = 6n + 2k - 1$ for $k = 1, 2, 3$ and $n \geq 0$.