Math 172: Curves

June 15, 2017

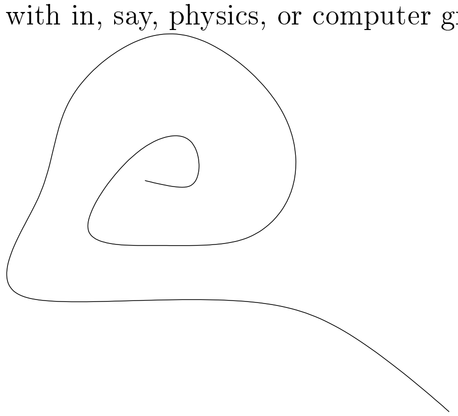
[part 1 only]

These notes apply what we've done so far to study curves. Understanding curves (and especially families of curves) was one of the motivating problems for calculus in the 1600s, as well as one of the first applications of calculus.

One place the kind of curve we're going to study arises is in physics, as the trajectory of a moving particle. We'll consider aspects of motion in two dimensions, as opposed to motion along a line (which you probably dealt with in Calculus I, but if you didn't, it's a special case of what we do here).

Why study curves? Isn't a curve just the graph of a function?

Graphs of functions are particular examples of curves, but in general are just a *makeshift substitute* for curves. They're simple to work with, but not particularly natural if your interest is curves in general. We'll study a notion of curve that's better suited to the curves we might want to work with in, say, physics, or computer graphics. Curves like this, for example:



1 The notion of "parametric curve"

You have undoubtedly seen the description of the unit circle as the solution set of the equation $x^2 + y^2 = 1$. I think of this type of description as "external", since it provides a *test* for points (x, y) to be on the unit circle or not — if the squares of the coordinates add up to 1, then it's on the unit circle, otherwise not. So it "sifts out" the unit circle from the rest of the plane.

The idea here is the punch line of an esteemed joke. How do you sculpt a horse? Get a horse-sized block of marble and chip away anything that doesn't look like a horse!

You may also have seen the description of the unit circle as the set of all points $(\cos(t), \sin(t))$ for all real numbers t (or all t in $[0, 2\pi)$ or similar). This is often called a "parametric description" of the unit circle. This kind of description I think of as "internal", in that whatever you plug in, it spits out a point on the unit circle. We'll see that this type of description is very useful.

Regrettably, I have no joke here.

The "it" that "spits out a point" when you "plug something in" is actually a *function.* So far in Calculus 2 we've dealt entirely with functions whose values are *real numbers*, that is, *points on the number line*, and that was probably true in your Calculus I and precalculus classes as well. However, much of what you've done actually applies more widely, in particular to functions f of the type we've just encountered: from an interval $I = [a, b]$ on the real number line to the *plane**. That is, functions for which $f(x)$ is a *point in the plane*, rather than a single real number, for each real number x in I. When $f(x)$ is a point on a curve C for every x in $[a, b]$ — that is, when f gives a parametric description of C — we also say f *parametrizes* the curve.

In general, a function of this type, at least if it's continuous, can always be thought of as parametrizing a curve in the plane. When this perspective is intended the function itself is called a *parametric curve.*

Definition 1. A *parametric curve* in the plane is a continuous function from an interval $[a, b]$ to the set of points in the plane.

Recall that the *image* of a function is the set of all *outputs* of that function. So the image of a parametric curve is a set of points in the plane. If the function is continuous, these points make a curve in the usual sense of the word. (The image of a parametric curve is sometimes called its *trace.*)

Note that *we haven't given a mathematical definition of "curve".* It is somewhat complicated to give a reasonable one. We will only use the term informally, or as a short form of "parametric curve".

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So you could say that a parametric curve gives a parametric description of its image.

Frequently we think of the interval (the domain of the function) as a time interval, and its image as the *path of a particle moving in the plane*; and we think of the function as giving the *location* of the particle at any specified *time.* That is, you plug *in* the time, you get the *location of the particle in the plane at that time* out.

Nothing mathematical actually depends on this "moving particle" picture, but since you were probably introduced to the derivative of a function in Calc I as something having to do with motion, it's traditional to continue this paradigm. In fact the concept of parametric curve gives us a much

*We use "the plane" and "the set of points in the plane" to mean the same thing more flexible way to model motion. *

Graphical representations In calculus I (or physics) you'd consider a situation like a ball thrown into the air at an angle, and represent it by the function giving height as a function of time. The *graph* of that function is a parabola. Students frequently mistake that graph for the picture you'd see if the ball left a trail. But *that* picture is a graph of height versus *horizontal position*.

Here there are *three* measurements — or really, four, but let's assume the motion happens in a plane. Then the three measurements are *height* (vertical position), *horizontal position*, and *time*. We can only represent *two* of them as coordinates of a point in the plane. The usual Calc I graph leaves out *horizontal position*. By contrast, the image of a parametric curve leaves out *time*.

This particular confusion might appear to have no serious consequences, since horizontal position is closely related to time when the only force acting is vertical. In that situation, according to Galileo and later Newton, the horizontal speed is *constant*. So if that's say 10 feet per second, then when time increases by 3 seconds, horizontal position increases by 30 feet.

But if there are forces acting that aren't vertical, then this is no longer true and mistaking one graph for the other will cause a serious disconnect.

"Parametric equations for a curve" The concept of parametric curve also does business as "parametric equations for a curve". That terminology calls for some explanation. In the older style of expressing the function concept, you'd describe how the x and y coordinates of the output depend on the input, expressed as another "variable" often called t and known as a "parameter"†. You'd do that by giving an equation expressing x in terms of t and another equation expressing y in terms of t.

So instead of the describing the function f by something like

$$f(x) = (x + 2, 3x - 1)$$

(the choice of the letter x being completely irrelevant) or simply $(x+2, 3x-1)$ (where now x means the usual function x) you'd write

$$\begin{aligned} x &= t + 2 \\ y &= 3t - 1 \end{aligned}$$

and a "graph" would be the set of all points in the plane (x, y) that arise from plugging in real

*It's useful for other things as well. For example, the curves that computer drawing software draws for you when you click a few points are dealt with internally as parametric curves.

†I think the distinction between *parameter* and *variable* is that you are free to vary a parameter, but a variable just varies; it's out of your control. But that's also "independent" vs "dependent", right? Hmmm... It's important to realize that these terms do not really have a defined mathematical meaning outside the phrases they occur in, that is, "parametric curve" is a single concept, just a kind of function. We don't have an actual definition for "curve".

numbers for t. The value of t does *not* appear on the graph*, but you can think of it as a *label* for the point. For example, if you plug in $t = 1$ you get the point $(3, 2)$. You could imagine the point $(3, 2)$ in the plane labelled with "$t = 1$".

This notion of "graph of a set of parametric equations" is actually nothing new, if you keep the function concept in mind. This notion of "graph" is simply the *image* of the function, and the function associates *points of the interval* with *points on the curve*. On the other hand, if you apply the usual idea of "graph of a function", you don't even get points in the plane; you get points in three-dimensional space: (t, x, y) where $(x, y) = f(t)$.

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Example 1. Let $f(x) = (x, 2x)$ for x in some interval $[a, b]$. Then the image of f is the *line segment* from $(a, 2a)$ to $(b, 2b)$.

You can see it's the part of the solution set of the equation $y = 2x$ for $a \leq x \leq b$.

Example 2. If $f(x) = (\cos(x), \sin(x))$ then the image of f is an *arc of the unit circle*, from a radians (that is, the point $(\cos(a), \sin(a))$) to b radians (counterclockwise, going a total distance of $b - a$.)

[picture]

2 Describing motion; geometry

Derivatives of such functions can be defined just like in Calc I, namely,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

This definition takes the limit of the *difference quotient* of f at x. It looks exactly like Calc I, but what does it represent?

Remember that $f(x+h)$ and $f(x)$ are *points in the plane* now. The subtraction is just as you'd expect: for points in the plane, $(u, v) - (a, b)$ is *defined* to be $(u - a, v - b)$. The result is often referred to as a *vector* and ideally represents a *displacement* (as here). It is often thought of as an arrow *from* the starting point (a, b) *to* the endpoint (u, v), but which can be "parallel translated"† anywhere and it's still considered the same vector. In particular, if you parallel translate it to start at $(0, 0)$, then it ends at $(u - a, v - b)$. (This is the "standard representation" for a displacement.)

Note. Another, more explicit, way to get the same effect is to treat a vector as a displacement *together with a point of origin.* So the arrow from (a, b) to (u, v) would be represented as having (a, b) as its point of origin, and with displacement $(u - a, v - b)$. Then parallel translation just

*In general, it is not even *measurable* from the graph, so "graph" is perhaps a poor word for this! Using function concepts clears things up greatly.

†"Parallel translate" just means "move without changing direction", that is, if you move the start over 3 and up 5 then you also move the end over 3 and up 5.

swaps out the point of origin, leaving the displacement the same. Keeping explicit track of a point of origin can help sort things out, but it's almost never necessary to actually indicate the point of origin, so most discussions of vectors omit it entirely.

Now the difference quotient is a displacement, that is, a *vector*, divided by h, a *nonzero real number.* (The point of origin is just $f(x)$.)

What does it mean to divide a vector by a nonzero real number?

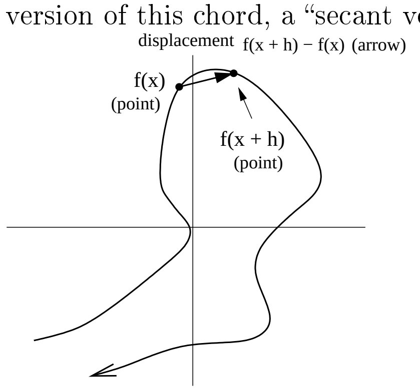
Again, just as you'd expect: we divide *each coordinate* by that real number. That is,

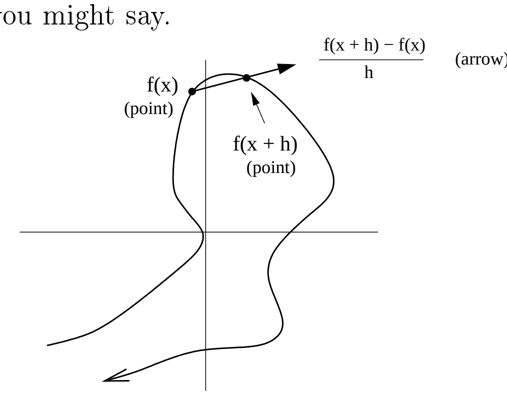
$$\frac{(u,v)}{h} = (\frac{u}{h}, \frac{v}{h}).$$

This says that the "vector difference quotient" above just amounts to a *pair* of *ordinary* difference quotients. But there is another way to look at it that's helpful. This relies on noting that *division by a real number does not change the angle of a vector*, only the length:

Exercise 1. Thinking of a vector as an arrow from the origin, explain why the tangent of the angle the vector (a, b) makes with the positive x-axis is b/a. Then explain why (a, b) and $(a, b)/h$ point along the same line through the origin. (If $h < 0$ they point in opposite directions along this line).

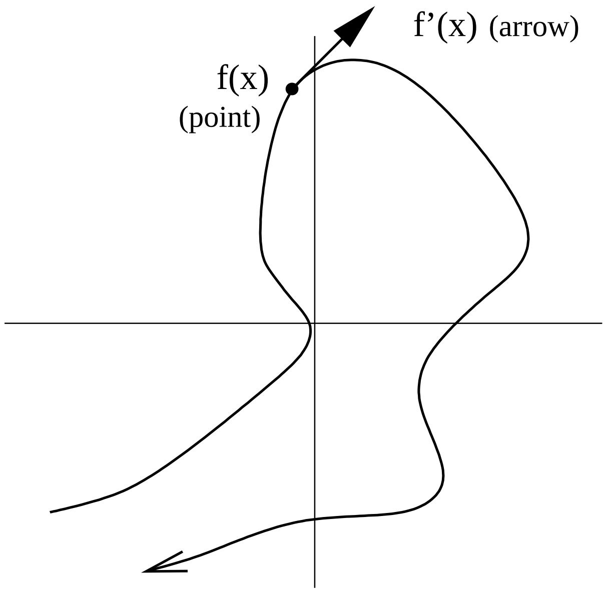
The displacement vector in the numerator of the quotient in (1) is a "chord", that is, a segment of a *secant line* to a curve, from the *point* $f(x)$ to the *point* $f(x+h)$. Division by h gives a stretched version of this chord, a "secant vector" you might say.





So as $h \to 0$ this "secant vector" should approach a "tangent vector" to the curve for the same reason the secant *line* to a graph of a function approaches the tangent *line.* The division by h *rescales the length* so it doesn't just disappear. So if the limit exists, it is a vector *tangent* to the curve*.

*The same sleight-of-hand is present here as in Calc I. Namely, there is no purely geometric definition of "tangent" anything (vector, line, what have you). The limit of the difference quotient IS the definition. So the Calc I statement "$f'(x)$ is the slope of the tangent line to the graph of f at the point x" is true *not* because the derivative is defined to be the slope of the tangent line, but because the tangent line is defined to be "that line through $(x, f(x))$ having slope $f'(x)$".



Brief digression on vectors

At this point you may be feeling that vectors are basically just points. They both have two real-number coordinates, after all. A purist would disagree. A point is a *location*, a vector is a *displacement*. But the same distinction is routinely blurred with real numbers from Day 1. Recall when you learned to add on the number line. To add $4+6$ you *start* at 4 (so the number 4 plays the role of a position or location) and *move* 6 spaces to the right (so 6 is indicating a *displacement*, *not* a position).

[picture of number line]

Purists think of space as being inhabited (or overlaid) by two kinds of things, then: points and arrows (vectors, displacements)*. You can add a *displacement* to a point (like on the number line) but *not* a *point* to a point. You can *subtract* points; that gives you a displacement (as we've seen). And so on — I'll let you work out the rest. You can add or subtract displacements (displacement $\pm$ displacement), getting another displacement as a result.

You have probably determined that I'm not a purist in these matters. I think being *aware* of the distinction is important, but it's rather difficult to maintain. Indeed, you will find that quite a few people, including many who insist on observing this distinction, routinely blur it without giving any indication of being aware of it. That can be confusing.

[better example, below is a more specific issue]

[For example, in most physics textbooks and classes, "force" is given as a major example of a vector. But in what sense is a force a *displacement*? Below we will consider that a force, *acting*

*Or perhaps space consists of points while vectors *act* on those points. That would be identifiying the vector (a, b) with the *transformation* $(x, y) \mapsto (x + a, y + b)$ — that is, "move horizontally by amount a and vertically by amount b", called a *translation*.

through a displacement, gives a quantity physicists call *work*. So a force looks like a real-valued *function* which takes displacements (*bona fide* vectors) as *input*.]

The issue of what mathematical concepts are used to represent physical "quantities" (for lack of a better word) is one that, in my opinion, deserves more attention than I've seen it get. One extreme is to use as few mathematical concepts as possible. Even the concept of vector is unnecessary, from this point of view. That point of view lost favor in the late 1800s, because it's just too cumbersome to write and think about. Maxwell quote

kelvin quote

heaviside quote

During that time, the subject of *linear algebra* was developing. Many concepts of physics turn out to be well-represented by linear algebra concepts. If you've heard the term "tensor", that's what's going on, although often in a form that minimizes linear algebraic overhead.

[Actually, the physics version of these concepts developed in parallel with linear algebra, but apparently with little contact with linear algebra, so it's somewhat difficult to make the correspondence between the mathematical version and the physics version, and some confusion results.]

Derivatives

Back to the story.

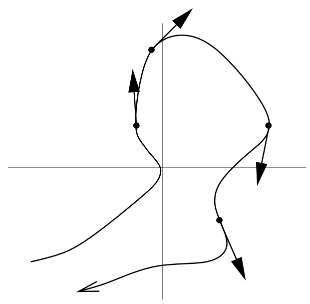
So we've seen that a function f from an interval I on the real number line to the plane has a derivative f' defined in exactly the same way as Calc I.*

Amazingly, you can re-use *all* of Calc I's "rules for derivatives" since $f(x) = (g(x), h(x))$ for a pair of functions g, h from I to the real numbers. We saw that the difference quotient for f is just a pair of Calc I difference quotients, one for g, one for h. So $f'(x) = (g'(x), h'(x))$. (The fact that limits in $\mathbf{R}^2$ are taken *coordinatewise* is assumed here. We'll discuss it later.).

The only difference with the Calc I situation is that $f'(x)$ is a not a real number any more. But it is still a *vector*. It's called the *velocity vector* of the curve f (at the point $f(x)$), inspired by the interpretation of f as the trajectory of a particle moving in the plane.

The velocity vector $f'(x)$ can be pictured as a vector with origin $f(x)$ and tangent to the curve at the point $f(x)$. You don't need the "moving particicle" picture for this. You can picture this for each value of x, and then you get a curve with tangent vectors at each point. (This is a type of "vector field along the curve". The picture necessarily only shows a representative selection of tangent vectors.)

*If the limit of the difference quotient exists, we call f differentiable at that point, just as in Calc I. We're doing calculus, so will want to take derivatives! Therefore we will mostly consider only differentiable parametric curves, and often will require more, such as continuity of the derivative. However, in the Calc I spirit, we'll just let that be an implicit understanding.



The terminology "velocity vector" and "velocity" are often used for $f'(x)$ even when suggestion of motion is not intended. In that case there is no "particle", so saying "velocity of the particle" is a little discordant. We just say "velocity of the curve", even though the curve itself is not moving, to refer to the vectors $f'(x)$ for various values of x (that is, to the "vector-valued function" f').

Notation. It's fairly common to use a distinctive notation for parametric curves. Lower case Greek letters are common for the function name, for no particular reason. (Using a different alphabet for certain kinds of things is a common way to organize your notation.) We all like to use x and y for first and second coordinates of points, so we do that here. But then we can't *also* use x for an input to γ, so we use t (and often call it "time", a reference to the "moving particle" picture.). So a typical name for a parametric curve might be γ (lower case Greek gamma) or α or β, and if there's no confusion we'd write $\gamma(t) = (x(t), y(t))$, that is, x and y are now the *functions* we called g and h above. They are called *coordinate functions* of the curve. (This puts us notationally very close to the old-style "parametric equations" perspective.)

Finally, a notation for the derivative in this context is $\dot{\gamma}(t)$ (an overdot rather than a prime). This is Newton's notation (you were probably wondering when we were going to get to that, right?). It is only used for the "derivative with respect to time".

Velocity and speed In the moving particle picture, the velocity vector really does give *velocity* in the physics meaning: speed *and* direction. It's an arrow, after all, so the "direction" part is there. The speed is the *length* of the arrow.

Wait, why is that the *speed*? And how do we determine the length of a vector?

Length. Pythagoras to the rescue! The vector (a, b), thought of as an arrow from $(0, 0)$ to the point (a, b), is the *hypotenuse of a right triangle* with vertical and horizontal legs. The lengths of the legs are just a and b. So the hypotenuse (the vector) has length $\sqrt{a^2 + b^2}$.

[picture]

We write $|u|$ or $||u||$ for the length of the vector u. (This is often called the *norm* of u.) So $|(a, b)| = \sqrt{a^2 + b^2}$.

Speed. If you write the difference quotient for γ at a point t_0 you can see the speed interpretation easily:

$$\frac{|\gamma(t_0 + h) - \gamma(t_0)|}{|h|} = \frac{\text{distance}}{\text{time}}$$

since the length of the displacement vector $\gamma(t_0+h)-\gamma(t_0)$ is simply the length of the arrow from $\gamma(t_0)$ to $\gamma(t_0+h)$, that is, the *distance*, and that (net) distance is travelled over *time* h.

Taking the limit of this gives the *instantaneous* speed at t_0, $|\gamma'(t_0)|$.

Question. Why can we take the limit inside the norm or absolute value?

Answer: because the norm is a *continuous function*. However, it's a function from points (or vectors) to real numbers, so we don't really have a definition of continuity for it. Rather than give one, we can just note that the function we're taking a limit of is the square root of the sum of squares of two difference quotients. So we just need Calc I rules for limits.

You might wonder whether there's a slight problem in the explanation above. The length of the displacement vector is not actually the distance travelled, but rather a lower bound for it. It turns out that "in the limit" it's true. That is, very short sections of the curve are nearly linear, so as $h \to 0$ the length of the displacement and actual distance along the curve from $\gamma(t)$ to $\gamma(t+h)$ differ only negligibly. (To fully explain this, we need a definition of length along the curve! You'll need to wait for Math 271 for full details here.)

Whether that bothered you or not, it's probably a good idea to have an example to see that this all works correctly.

Example 3. Take $\gamma(t) = (4\cos(t), 4\sin(t))$. Then for each value of t, the point $\gamma(t)$ is a point on the circle of radius 4 centered at $(0,0)$. In other words, γ is the trajectory of a particle moving around the circle. You can even see how it moves: at time t it's at t radians (radian = radius units). So its "angular velocity" is constant. But at what *speed* is it moving *along* the circle? Well, it goes $4t$ units (since the radius is 4, t radians = $4t$ units) in time t, so that's a speed of 4 (distance units per time unit).

That agrees with the derivative: $\gamma'(t) = (-4\sin(t), 4\cos(t))$ so

$$\begin{aligned} |\gamma'(t)| &= \sqrt{(-4\sin(t))^2 + (4\cos(t))^2} \\ &= \sqrt{16\sin^2(t) + 16\cos^2(t)} \\ &= \sqrt{16} = 4 \end{aligned}$$

Note that the derivative is a vector *perpendicular* to the segment from $(0,0)$ to $\gamma(t)$. (That is, perpendicular to $\gamma(t)$ *thought of as an arrow from the origin.* Physicsts call that the *radius vector* or *position vector* to the point and often call it $\vec{r}$ or $\vec{\mathbf{r}}$ or $\mathbf{r}$ depending on how flamboyant they wish to be*.)

[picture of velocity vector]

How do you *tell* that the velocity vector here is perpendicular to the radius vector? Look at slopes! (Linear algebra will give us another way to check that — see the Math 162 notes. Actually,

*Fun aside, the physicist's notation is an efficient way to keep track of things. In this case, the bold and/or decorated r represents a vector while the regular r represents the *length* of that vector.

we'll need this so feel free to look ahead.) This says that the velocity vector $\gamma'(t)$ really is tangent to the circle at the point $\gamma(t)$.

Take the *second* derivative of γ to get the *acceleration vector*. Why do you think it's not zero? Which way is it pointing? (It may help to parallel translate it in your imagination to start at $\gamma(t)$, so it would end at $\gamma(t) + \gamma''(t)$).

Explanation: The acceleration is *not* zero because the *velocity* IS changing (even though the *speed* isn't). The acceleration vector points inward, to the center of the circle. What *should* be zero is the acceleration *in the direction of motion*, that is, the *tangential* acceleration.

This shows another role for linear algebra: *express a vector as a sum of vectors in two given directions*. This is known as "resolving into components" — again, for more information, see the Math 162 notes.

Reparametrizations Two parametric curves can have the same image, yet *not* be the same parametric curve. For example, the function $\gamma(t) = (t^2, 2t^2)$ with domain $[0, 1]$ parametrizes the same curve as the function $\alpha(t) = (t, 2t)$ with the same domain. The difference is in how the curve is "described", that is, how the particle moves along the curve. We have α' constant, so α "describes" uniform motion along the line segment from $(0, 0)$ to $(1, 2)$, while $\gamma'(t) = (2t, 4t)$ so γ "describes" a particle picking up speed as it moves along the same line segment. We have $\gamma''(t) = (2, 4)$ so the *acceleration* is constant.

It is probably clear that $\gamma(t) = \alpha(t^2)$, that is, γ is a *composite function*. In general, if we have a continuous function $\phi : [c, d] \to [a, b]$ we can consider composition of a parametric curve with ϕ to give another parametric curve. We call this process *reparametrization*, and call $\gamma = \alpha \circ \phi$ a *reparametrization* of α. It can be thought of as a kind of "time change". The curve $\alpha \circ \phi$ is at the same point at time t that the curve α is at time $\phi(t)$.

It's natural to wonder if two parametric curves with the same image are necessarily related this way. The answer is: almost. We'll see the details next.

3 Length of a curve ("arc length")

In Calculus I, you saw that if a function f represented a particle's *position* with respect to time (that is, $f(t)$ is the location at time t) then $f'(t)$ gives its *velocity* at time t.

On the other hand, if $v(t)$ is the *velocity* of a particle at time t, then

$$x_0 + \int_a^t v(u)\, du \tag{2}$$

represents the particle's *position* at time t relative to a starting position x_0 at time $t = a$.

Let's pause to see why that's true. We could call on the FTOC, but let's look at what the integral is really defined as: a weighted sum of values of v. It's a limit of sums like

$$v(c_1)(t_1 - t_0) + v(c_2)(t_2 - t_1) + \cdots + v(c_n)(t_n - t_{n-1})$$

of which the ith term is a *velocity* at some point c_i in the time interval $[t_i, t_{i-1}]$, times the *length of time elapsed.* Now for *constant* velocity,

$$\text{velocity} = \frac{\text{displacement}}{\text{time}}$$

so if you rearrange, *velocity* times *time elasped* = *displacement.*

So we're getting the *displacement* over that time interval — approximately, since velocity isn't necessarily constant. Add those displacements up over all the time intervals. That gives, again approximately, the displacement from time $t_0 = a$ to $t_n = t$. If we started at x_0 at time $t = a$ then the expression (2) above gives exactly the ending position at time $t = b$.

OK, so we evidently get f back again by integration of the velocity.

There are a few things to note here.

First, the integral by itself naturally gives us a *displacement.* It doesn't supply the starting point.

Second, *speed* is the absolute value of velocity. If we integrate that, we're getting something that at first glance seems similar. But there is a difference. The integral of speed does not add up *displacements*, but rather *distances.* One is a signed quantity, the other is not. The basic idea is: if you step forward one step, then back one step, you travelled two steps, but the *displacement* is zero.

So we have two different integrals:

$$\begin{aligned} \int_a^t v(t)\,dt &= f(t) - f(a) \qquad \text{displacement over } [a, t] \\ \int_a^t |v(t)|\,dt &= \text{total distance travelled over } [a, t] \end{aligned}$$

For example, if $f(t) = -t^2 + 2t$ represents the height of a ball thrown straight up into the "air" on some planet, the velocity is $v(t) = f'(t) = -2t + 2$. The displacement over $[0, 2]$ is zero: it starts at $t = 0$ at height 0 and returns to height 0 at $t = 2$. But it travels

$$\int_0^2 |-2t + 2|\,dt = \int_0^1 (-2t + 2)\,dt + \int_1^2 (2 - 2t)\,dt = 2 \tag{3}$$

units over the time it's in the air.

Exercise 2. Give at least two explanations for why the evaluation (3) is correct. Give a general "technique" for evaluating an integral like $\int |v|$.

Exercise 3. You can also just check that it peaks at $t = 1$. What height does it have at $t = 1$? How does this give you the total distance travelled? How is this related to the trick for evaluation of the integral with an absolute value used above?

Now with parametric curves, we have the same two integrals, giving us the same two things. The only difference is that now *displacements* are in two dimensions, while *distance* is still a nonnegative real number.

$$\begin{aligned} \text{displacement} = \gamma(b) - \gamma(a) &= \int_a^b \gamma'(t)\, dt \\ \text{length of } \gamma = \ell(\gamma) &= \int_a^b ||\gamma'(t)||\, dt \end{aligned}$$

(This is one of many places where Calc I concepts which are somewhat subtly different become markedly different when extended to more than one dimension.)

Note. Different curves starting and ending at the same point have the same *displacement*, but need not have the same *length*. In other words, the first integral above *does not depend on the curve* γ, only on where it starts and ends.

Reparametrization, the chain rule We can now see in what sense two parametric curves with the same image must be reparametrizations of each other. We'll need the chain rule, which works the same way (for the same reason) as in Calc I:

$$[\gamma(\phi(t))]' = \gamma'(\phi(t))\phi'(t)$$

Here γ' gives vector outputs, so $\gamma'(\phi(t))$ is the velocity vector of the original curve γ at time $\phi(t)$, which is still a tangent vector to the reparametrized curve at the point $\gamma(\phi(t))$, but not necessarily the velocity vector for $\gamma \circ \phi$. The chain rule says that to get the velocity vector for $\gamma \circ \phi$, we multiply by the factor $\phi'(t)$. This is just a real number, so stretches or shrinks the vector $\gamma'(\phi(t))$ it multiplies.

Note that one consequence of this is that if we reparametrize, we don't change the *direction* of the velocity vector at any point, only its *magnitude* (by the factor $\phi'(t)$).

If we write x and y for the coordinate functions of γ, the chain rule above just says

$$(x(\phi(t)), y(\phi(t)))' = (x'(\phi(t)), y'(\phi(t)))\phi'(t)$$

which is just the ordinary Calc I chain rule for each function $x \circ \phi$ and $y \circ \phi$.

Arc length parametrization aka unit speed parametrization There is an especially natural reparametrization for curves whose velocity vector is continuous and never 0. This results in a curve with the same image, but whose velocity vector always has length 1, that is, whose *speed* is always 1.

We define the arclength function for a parametric curve γ by

$$\ell(t) = \int_a^t ||\gamma'(u)||\, du.$$

(assuming γ has a continuous velocity vector). This measures the length along the curve γ from the point $\gamma(a)$ to the point $\gamma(t)$, as a function of t.

Note that, by the FTOC, the function ℓ is automatically differentiable and $\ell'(t)$ is the *speed* at time t, $\ell'(t) = ||\gamma'(t)||$. If the speed is never zero, then $\ell'(t) > 0$ for all t in $[a, b]$ and therefore ℓ is an increasing function. That means ℓ has an inverse. Let $\phi = \ell^{-1}$.

Recall the Inverse Function Theorem:

Theorem 1. *If f is a differentiable function with $f'(x) > 0$ for all x in an interval (a, b) and continuous on $[a, b]$ then the image of f is an interval $[f(a), f(b)]$, f has a differentiable inverse f^{-1} , and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. This is proven in the "rules for derivatives" notes. It uses the same ingredients as the proof of the chain rule, plus the intermediate value theorem. Given the rest, the formula for the derivative of f^{-1} follows from the chain rule applied to the composite $f(f^{-1}(x)) = x$. Just differentiate both sides and solve for $(f^{-1})'(x)$:

$$\begin{aligned} [f(f^{-1}(x))]' &= x' \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

□

We use this to get ϕ', so we can see that the reparametrized curve, $\gamma \circ \phi$, has unit speed (that is, the speed is always 1):

$$[\gamma(\phi(t))]' = \gamma'(\phi(t))\phi'(t) = \gamma'(\phi(t))\frac{1}{\ell'(\phi(t))} = \frac{\gamma'(\phi(t))}{||\gamma'(\phi(t))||}$$

which shows that the velocity vector is a unit vector.

Now if γ and α are two one-to-one* parametric curves with the same image, and with never-zero

*One-to-one is meant as a function. This means the curve does not pass through the same point more than once.

velocity, then there is a differentiable function ϕ with differentiable inverse so that $\gamma = \alpha \circ \phi$. Or to put it another way, if α and γ start at the same point, then the unit-speed reparametrizations of α and γ are the *same* parametrized curve. Essentially this is because if two particles move along the same path, starting at the same point, at the same time, with the same speed, then at each time both particles must be at the same place.

4 Curvature

Given a parametric curve, we can reparametrize it to be "unit speed" as long as it doesn't slow down to a stop anywhere. Let's assume we've done that, so we only deal with unit speed curves.

We saw that with unit speed curves, there is no "tangential acceleration". That is, the *speed* doesn't change, so there's no acceleration *in the direction of motion* (like you'd get if you stepped on the gas or hit the brakes). Acceleration is change in *velocity*, and the velocity may change without the speed changing, by changing *direction*. The changing direction of the velocity vector, when moving at constant speed, is entirely a result of the curving path the particle is taking, not the manner it's moving along that path. So it reflects the geometry of the image. It's due to the road, not the driver.

We therefore define the *curvature* of a unit-speed curve to be the *length of the acceleration vector.* If γ is a unit-speed curve, then the curvature of γ at time t is

$$\kappa(t) = ||\gamma''(t)||.$$

Let's look at the curvature of a circle. Circles are the natural standards for curvature, since they curve "the same way" at every point.

Take $\gamma(t) = (R\cos(t), R\sin(t))$ to parametrize the circle of radius R centered at the origin. What is the speed of γ ?

We have $\gamma'(t) = R(-\sin(t), \cos(t))$. Now that's a vector of length R. So we have $\ell(t) = Rt$ — we have identified the function ℓ as simply *multiplication* by R, so its inverse ϕ is *division* by R. So the unit-speed reparametrization is (as you might have expected) $\alpha(t) = \gamma(t/R) = R(\cos(t/R), \sin(t/R))$. You can check that the speed is now 1, so the curvature of α is

$$\kappa(t) = ||\alpha''(t))|| = ||R(-\cos(t/R)/R^2, -\sin(t/R)/R^2)|| = 1/R$$

This says that at any point on the circle, its curvature is the same, $1/R$. In particular, the larger the circle, the smaller the curvature.

Because of this connection, sometimes instead of curvature $\kappa(t)$, the reciprocal $1/\kappa(t)$ is quoted, and called the "radius of curvature" at $\alpha(t)$. This just compares a curve to a circle as standard for curvature. The radius of curvature of any curve α, $1/\kappa(t)$, gives the radius of the circle that "best fits" the curve at the point $\alpha(t)$. See exercise .

For a general parametric curve*, the curvature is defined to be the curvature of the unit-speed reparametrization. It is possible to work out formulas not directly involving the reparametrization, but we won't do that here.

[part 2 continues with integrals over curves]

*"General" is a poor choice of word, as we need the velocity vector to be differentiable here.