Math 172: Power series

June 2, 2016

Abstract

These notes rely heavily on the previous notes on series. Last edited: 3 June 2015

A series like

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots \tag{1}$$

where x is an "indeterminate" (that is, just a blank) is called a *power series*. This is similar to the concept of polynomial, except that the sum is allowed to go on forever. Another (very important) example is the series

$$\sum_{n=0}^{\infty} x^n \tag{2}$$

which is essentially the geometric series with "indeterminate ratio" x.

The actual definition of power series won't be a surprise:

Definition 1. A *power series* is an expression

$$\sum_{n=0}^{\infty} a_n x^n$$

where (a_n) is a sequence of real numbers and x is an indeterminate. The sequence a_n is called the *coefficient sequence*.*

Example 1. The series (1) has coefficient sequence $a_n = 1/n!$ while the series (2) has $a_n = 1$ for all n.

Notice that if we attempt to evaluate a power series by plugging a number in for x, we get an *ordinary series of numbers* — that is, *another expression*. We only get a *number* if the resulting series converges.

We will say that the set of real numbers we can substitute for x in a given power series and have a convergent series result is the *domain of convergence* of that power series.

*We can have power series in more than one indeterminate, but we don't consider them here.

Example 2. From what we know about geometric series, the domain of convergence of the power series (2) is the interval $(-1, 1)$.

Example 3. The domain of convergence of the power series (1) will turn out to be *all real numbers*: we will see this in the next section.

These examples give a faithful picture of the domain of convergence of a power series in general:

Theorem 1. *The domain of convergence of a power series is always an interval* $(-r, r)$ *(with either or both endpoints possibly included) or* $\{0\}$.

Remark 1. The entire set of real numbers is a possible domain of convergence, so we need to allow $r = \infty$. We interpret $r = 0$ to mean that the domain of convergence is simply $\{0\}$.

Definition 2. The number r in the theorem (or ∞) is called the *radius of convergence* of the power series.

Example 4. The power series (2) has radius of convergence 1 while the power series (1) has radius of convergence ∞.

The key fact behind Theorem 1 is:

Fact 1. *If* $\sum a_n x^n$ *converges for a real number* x_0 *then it converges* absolutely *for all real numbers* x *with* $|x| < |x_0|$.

[picture]

Proof. Since the series of real numbers $\sum a_n x_0^n$ converges, its terms must have limit 0 (see notes on series). In particular they must be bounded, so there is some M so that $|a_n x_0^n| \le M$ for all n. Now if x is a real number with $|x| < |x_0|$ then

$$|a_n x^n| = |a_n x_0^n| \cdot |\frac{x}{x_0}|^n \le M r^n$$

where $r = |x/x_0| = |x|/|x_0| < 1$. So $\sum |a_n x^n|$ converges by comparison to the geometric series $\sum M r^n = M \sum r^n$. This says $\sum a_n x^n$ converges *absolutely.* □

One aspect of the argument above is worth emphasizing:

Exercise 1. Let s_n be the sequence of partial sums of a series $\sum M a_n$ (where M is a real number) and let t_n be the sequence of partial sums of the series $\sum a_n$. Show that $s_n = M t_n$ for all n. Conclude that the limit of s_n is M times the limit of t_n. What does this translate to in terms of sums of series?

As a consequence of Fact 1, we see why the domain of convergence is an interval centered at 0.

Exercise 2. Can a power series $\sum a_n x^n$ converge for $x = -2$ but diverge for $x = 1$?

1 Some examples

Example 5. The series (1) converges for all real numbers, so its radius of convergence is ∞. This is because

Fact 2. *If x is a real number, then the sequence $x^n/n!$ has limit* 0.

This is often described by saying "factorials grow faster than exponentials" or "factorial growth is beyond geometric". (You may recall from school that a "geometric progression" is a sequence where you get from a term to the next by multiplying by a fixed factor, for example $2, 10, 50, 250, 1250, \ldots$ That is, the *ratio* of successive terms is constant. So the terms are $a_n = a_0 r^n$. This is the sense of "geometric" used here.)

Proof. The claim is that

$$\frac{x \cdot x \cdot x \cdots x}{1 \cdot 2 \cdot 3 \cdots n}$$

becomes smaller than any prespecified positive threshold. How large does n need to be to make this less than $1/1000$, say? We don't need the exact answer, remember, just an index beyond which it's true. Of course, that index will depend on x.

The factors x/k will be less than, say, $1/2$ when $k > 2x$. So from that point on, all factors are less than $1/2$. Multiplying by factors of $1/2$ repeatedly gives us something with limit 0. So let $N = \lceil 2x \rceil$ (the smallest integer larger than $2x$). Then, replacing the factors x/k which are less than $1/2$ by $1/2$ we have the inequality

$$\frac{x^n}{n!} < \frac{x^N}{N!} \left(\frac{1}{2}\right)^{n-N} = C \left(\frac{1}{2}\right)^n$$

for $n \geq N$. (Here $C = (2x)^N/N!$ does not depend on n). As the latter sequence has limit 0 as $n \to \infty$, so does the former. $\square$

Exercise 3. Show that

$$\frac{1000^n}{n!} = \frac{1000^N}{N!} \cdot \frac{1000^{n-N}}{(N+1)(N+2)\cdots n}$$

(for any N). Find N so that $1000/k < 1/2$ for $k > N$ and show that for this N:

$$\frac{1000^{n-N}}{(N+1)(N+2)\cdots n} < \left(\frac{1}{2}\right)^{n-N} \tag{3}$$

Now use this inequality to find an M so that all terms of the sequence

$$\frac{1000^n}{n!}$$

with $n \geq M$ are less than 0.00001. (The answers here are two actual integers. One of them is probably too big for a calculator to deal with, so don't frustrate yourself needlessly — use what you know about exponents. To get an idea for the size, 100! is almost 10^{158}. $1000^{100} = (10^3)^{100} = 10^{300}$ so $1000^{100}/100!$ is about 10^{142}, so not small yet*... You may find that the fact that $2^{10} > 10^3$ is useful in converting powers of 10 to powers of 2...)

Is there anything preventing you from doing the same thing if the 0.00001 was replaced by another positive number? Answer: no! That says that the limit of the sequence is 0 (make sure you understand why). What if the 1000 were replaced by another positive number, say Avogadro's number? Does this stop you? Answer: no again! That's what Fact 2 says.

Back to Example 1 OK, so we've seen that the *sequence of terms* $x^n/n!$ has limit 0; but why does this mean the *series*

$$\sum \frac{1}{n!}x^n$$

converges? Have we forgotten the lesson of the harmonic series? WHAT ABOUT THE HARMONIC SERIES??!!??!!

Ahah! We can use the same trick! To show that $\sum \frac{1}{n!}x^n$ converges for a certain x, we use the fact that $(2x)^n/n!$ has limit 0. After all, we can plug *any number at all* in for x and the limit is still 0. So we plug in $2x$. That says that $2^n x^n/n!$ has limit 0, so is bounded by some M, so $x^n/n! < M2^{-n}$. Therefore $\sum x^n/n!$ converges by comparison with the geometric series with ratio $1/2$. Hah!

Notice the sloppy inequalities. (By that I mean that there's a lot of number line between the left-hand side and the right-hand side of (3). Check for $n = 3000$ for example!) This shows that $x^n/n!$ goes to zero *really* quickly, so much so that sloppy inequalities still work; there's always sumpin' to spare!

Ummm... OK, having emphasized an important point, let me admit that there's a slicker way to do this. *Just use the ratio test.* The ratio of successive terms in (1) is

$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0 \quad \text{as } n \to \infty$$

So by the limit version of the ratio test, this converges for any x. Moreover, as convergent series have terms with limit 0, we get $x^n/n! \to 0$ for any x for free! Suh-wheet!

Example 6. $\sum nx^n$ converges for x with $|x| < 1$. It definitely *doesn't* converge for $x = 1$ so the radius of convergence is $r = 1$.

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*I'm going for the "understatement of the year" award.

Exercise 4. Prove this using the idea above. Let x have $|x| < 1$. Choose x' so that $|x| < |x'| < 1$. Then factor $|nx^n|$ as $n|x'|^n|x/x'|^n$. Then $n|x'|^n \to 0$ (use L'Hôpital's rule with $f(t) = t/e^{ct}$ and interpret the limit as $\lim_{t\to+\infty} f(t)$). So $n|x'|^n$ is bounded. Now what?)

Exercise 5. Use the ratio test to prove that the radius of convergence is 1 for $\sum nx^n$.

Example 7. $\sum \frac{1}{n}x^n$ has domain of convergence $[-1, 1)$. It converges for $|x| < 1$ by comparison with the geometric series with ratio $|x|$; it doesn't converge for $x = 1$ as that gives us the harmonic series; it converges for $x = -1$ as that gives us an alternating series which converges (absolute value of terms decreasing and having limit 0). It cannot converge for any value of x with $|x| > 1$ because if so it would have to converge at $x = 1$.

Example 8. The power series $\sum n^n x^n$ has radius of convergence 0. This is because for any given $x > 0$ there is an integer n_0 so that $n_0 x > 1$. Then for $n > n_0$ we have

$$n^n x^n = (nx)^n > (n_0 x)^n \to +\infty$$

.

Example 9. The power series $\sum 2^n x^n$ has radius of convergence $r = 1/2$. That's because it's a geometric series with ratio $2x$ and this ratio needs to be less than 1 in absolute value for convergence: $|2x| < 1$. So $2|x| < 1$ i.e. $|x| < 1/2$.

Example 10. The power series $\sum \sin(n)x^n$ has radius of convergence $r \geq 1$, because $|\sin(t)| \leq 1$. To show that it is exactly 1 it would be enough to show that there are infinitely many values of n for which $|\sin(n)| \geq \varepsilon > 0$ for some ε. That's because then we could compare to a series with terms εx^{n_k} which wouldn't have limit 0 for $x > 1$, so $\sum \sin(n)x^n$ can't converge absolutely for $x > 1$.

Exercise 6. Finish up the example of the radius of convergence of $\sum \sin(n)x^n$. If it's false that "there are infinitely many values of n for which $|\sin(n)| \geq \varepsilon$" what would this mean? If it's false for every $\varepsilon > 0$, what does that mean?

Exercise 7. The behavior of $\sin(n)$ seems harder to get a grasp on than it should be. When you think about it, it seems quite reasonable that any value between -1 and 1 should be approached arbitrarily closely by $\sin(n)$ for arbitrarily large n. This would mean you could find an increasing sequence n_k of natural numbers so that $\sin(n_k) \to t$ (where t is a given real number between -1 and 1). Can you do this?

Exercise 8. You can also show that $\sum \sin(n)x^n$ cannot have radius of convergence > 1 by noting that if it did, it would have to converge for $x = 1$ and this would imply $\sin(n) \to 0$. Showing that $\sin(n)$ doesn't have limit 0 would finish it, then.

It is enough to show that for any N there is an $n > N$ with $\sin(n) > 1/2$. Here is a simple argument: $\sin(\pi/2) = 1$ and by periodicity $\sin(2\pi k + \pi/2) = 1$. When x is within $\pi/4$ of $\frac{\pi}{2}$ we have $\sin(x) \geq \sin(\frac{\pi}{4}) > 0.7$. Now $\frac{\pi}{4} > \frac{1}{2}$ and any number at all is within $1/2$ of an integer, so choose k so that $2\pi k + \pi/2 > N$ and then choose n to be the integer closest to $2\pi k + \pi/2$.

2 Laurent series

A series like a power series but allowing negative powers of x as well is called a *Laurent series*. Much of what we do with power series works for Laurent series as well. We won't give them special attention, though. Just note that Laurent series can't be evaluated at $x = 0$ (so their domains of convergence are slightly more complicated). Laurent series are best studied with complex numbers (in a course on complex analysis).

Exercise 9. Show that the domain of convergence of the Laurent series $\sum_{n=-2}^{\infty} x^n$ is $(-1, 0) \cup (0, 1)$.

3 Series in powers of $x - a$

Everything we say about power series works for series of the form

$$\sum a_n (x - a)^n$$

where a is a real number (having nothing to do with the coefficient sequence a_n). These are sometimes called "power series centered at $x = a$". Their domains of convergence are symmetric intervals centered at a rather than 0. We will see these arise naturally when we describe approximation of functions by polynomials (Taylor's theorem).

Here is an illuminating exercise. Please make sure you are very comfortable with it.

Exercise 10. Write the following polynomials as series in $x - 1$. That is, find the coefficients a_n so the given polynomial is equal to $\sum a_n (x - 1)^n$.

Hint: you can use algebra (the method of equating coefficients, or if you want to get fancy, polynomial division) or calculus. Make sure you understand both.

1. x^2
2. $x^2 + 2x$
3. x^3

4. $x^3 - 3x^2 + 4x + 1$
5. $x^2 - 2x + 1$

Exercise 11. Using the "key fact" (1), formulate and prove a version of Theorem 1 for power series centered at a. (What happens if you use $x - a$ and $x_0 - a$ in place of x in (1)?)

It's worth noting that if $\sum a_n(x-2)^n$ converges in $(2-r, 2+r)$ to a function f then the function $x \mapsto f(x+2)$ is the sum of $\sum a_n x^n$ for x in $(-r, r)$. So little is lost in considering power series in x only.

4 Operations on power series

The familiar operations with polynomials work with power series as well.

Given two power series in the same indeterminate x, we can do basic arithmetic:

$$\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n \tag{4}$$

$$-\sum a_n x^n = \sum (-a_n) x^n \tag{5}$$

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{6}$$

Your reaction to this last formula may be "Whuh?". So let's see why it's the same thing you know and love, the bread and butter of algebra classes 'round the world:

$$(a_0 + a_1 x)(b_0 + b_1 x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_1 b_1 x^2$$

(Cover this paper up right now, write down a product of two degree 1 polynomials, and express the product as a polynomial. Then compare what you have with this. After uncovering the paper again, of course.)

Exercise 12. a) Write the following products of polynomials in x as polynomials in x. Be very systematic to avoid losing track of things. b) Verify that the result of substituting coefficient values in for the indeterminate coefficients in the fourth product is the same as the result you got for the first three products.

1. $(1 + 2x - 3x^2)(2 - x + x^2 - x^3)$
2. $(3 - 2x + x^2 - x^3)(2 + x + x^2 + x^3)$
3. $(1 + x + x^2 + x^3 + x^4)(1 - x)$
4. $(a + bx + cx^2 + dx^3 + ex^4)(p + qx + rx^2 + sx^3)$

Division Division works like this:

$$\frac{\sum a_n x^n}{\sum b_n x^n} = \sum c_n x^n \text{ where } c_n \text{ is given by the following system of equations} \tag{7}$$

$$\begin{aligned} c_0 b_0 &= a_0 & \dashrightarrow \quad & c_0 = a_0/b_0 \\ c_0 b_1 + c_1 b_0 &= a_1 & \dashrightarrow \quad & c_1 = (a_1 - c_0 b_1)/b_0 \\ c_0 b_2 + c_1 b_1 + c_2 b_0 &= a_2 & \dashrightarrow \quad & c_2 = (a_2 - c_0 b_2 - c_1 b_1)/b_0 \\ &\vdots \end{aligned} \tag{8}$$

so to divide, you solve a system of linear equations, triangular as you'd expect. (8).

Remark 2. The same is true for polynomials, but you'll notice in that case there are too many equations for the number of unknowns. That's where remainders come in. But here we have infinitely many unknowns, so there's not necessarily a problem. In fact, all we have to be able to do is divide by b_0; that is, as long as $b_0 \neq 0$ we can divide.

Since many of us like our polynomials with integer coefficients, it's worth asking:

Exercise 13. If power series $\sum a_n x^n$ and $\sum b_n x^n$ have integer coefficients and $b_0 \neq 0$, when does the quotient power series have integer coefficients? Answer: when b_0 divides all the coefficients a_n. In particular, when $b_0 = \pm 1$.

Exercise 14. Determine the coefficients of

$$\frac{\sum a_n x^n}{1 - x}$$

directly by solving the equations (8). As an operation on the coefficient sequence, how would you describe this?

So power series division is always possible if the denominator has nonzero constant term.

Formal derivative The *formal derivative* of a power series is "what you get when you pretend the sum rule works for infinite sums". To be more precise, there is no difference quotient or limit involved; we simply define an operator $\mathfrak{D}$ on power series in x by

$$\mathfrak{D}\left(\sum a_n x^n\right) = \sum n a_n x^{n-1} = \sum (n+1) a_{n+1} x^n \tag{9}$$

Remark 3. To answer a natural question: there really *is* a difference here. For example the power series $\sum n! x^n$ has radius of convergence 0, so *cannot be thought of as a function at all*, and therefore *it is not even possible to ask whether it has a derivative anywhere*. So the notion of "derivative" does not even remotely apply to this expression. The *formal* derivative, however, is *completely* oblivious to this. All it needs is a power series. Function? What's a function?

4.1 Convergence

The definitions above are purely *formal*, in that there is no claim or requirement that anything converge or that evaluation of both sides will give the same value. However, it's not hard to check, by looking at the sequences of partial sums, that if both series on the left in (4) converge for a particular x_0 then so does the one on the right, and to the same value. Similarly for (5).

The case of products is a little more involved, as the "collecting like terms" has rearranged the terms, so it's messy to relate the partial sums. But the notion of convergence introduced in the notes on series deals with this very smoothly. The situation is the following:

Claim 1. If $\sum u_n$ and $\sum v_m$ are *absolutely* convergent, and we define their product to be the sum of all terms $u_n v_m$ (picture a table extended out indefinitely), then that sum converges in the unordered sense.

Proof. This is not difficult, but a little involved. Here is the table referred to:

$$\begin{array}{cccccc}
 & v_0 & v_1 & v_2 & v_3 & v_4 \quad \cdots \\
u_0 & u_0v_0 & u_0v_1 & u_0v_2 & u_0v_3 & \\
u_1 & u_1v_0 & u_1v_1 & u_1v_2 & & \\
u_2 & u_2v_0 & u_2v_1 & & & \\
u_3 & u_3v_0 & & & & \\
u_4 & & & & & \\
\vdots & & & & &
\end{array}$$

To show that the sum of all these terms converges in the unordered sense, we have to show that given any $\varepsilon > 0$ we can find a finite set S of indices so that for any finite set $T \supset S$, we have

$$|\sum_{(n,m)\in T\setminus S} u_n v_m| < \varepsilon \tag{10}$$

We are given that $\sum u_n$ and $\sum v_n$ are absolutely convergent, and we need to use this. Given any $\eta > 0$ we get an N and an M so that the tails sum to less than η:

$$\begin{aligned}
\sum_{n>N} |u_n| &< \eta \\
\sum_{m>M} |v_m| &< \eta
\end{aligned}$$

So take $S = \{(n,m)\, n \le N,\ m \le M\}$. Then

$$\begin{aligned}
\Big|\sum_{(n,m)\in T\setminus S} u_n v_m\Big| &\le \sum_{(n,m)\in T, n>N, m>M} |u_n v_m| \\
&\le \sum_{n>N, m>M} |u_n||v_m| \\
&= \left(\sum_{n>N} |u_n|\right)\left(\sum_{m>M} |v_m|\right) \\
&< \eta^2
\end{aligned}$$

So we need only choose $\eta = \sqrt{\varepsilon}$, then the resulting N and M and S defined as above will give (10). □

The claim is the main ingredient in showing that whenever the power series on the left of (6) converge, then so does the one on the right, and their sums are equal.

Here's why. Recall that power series converge *absolutely* within their radius of convergence. So if x is within both radii of convergence, we can apply the claim above (with $u_n = a_n x^n$ and $v_m = b_m x^m$) to see that the sum of all $a_n b_m x^n x^m$ converges in the unordered sense. Thanks to this, we can group and order as we please, without affecting the sum. How do we please to group and order? We please to *collect like terms*, that is, terms with the same value of $n+m$. That gives the formula for c_n in (6). In the table it's simply the sum along upward-sloping diagonals.

Fact 3. *Suppose* $\sum a_n x^n$ *and* $\sum b_n x^n$ *have radius of convergence* r_a *and* r_b *respectively. Then the power-series product* $\sum c_n x^n$*, where the coefficient sequence* c_n *is the convolution*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \tag{11}$$

has radius of convergence $r \ge \min(r_a, r_b) = r_0$*. Moreover, if we define functions on* $(-r_0, r_0)$ *by the sums of the three power series, say* f, g*, and* h*, then* $f \cdot g = h$*.*

Remark 4. The idea is simply that $fg = h$ where it makes sense, namely where all three series converge, and that the radius of convergence of $\sum c_n x^n$ is at least the smaller of r_a and r_b.

Example 11. This is a counterexample. The formula we used for power series could be used for *any* series $\sum a_n$ and $\sum b_m$ to produce a series $\sum c_n$ (the series $\sum c_n$ is sometimes called the *Cauchy product* of $\sum a_n$ and $\sum b_n$). Does it always converge, and if so does it converge to the product of the sums $\sum a_n$ and $\sum b_m$?

Take $a_n = (-1)^{n+1}/\sqrt{n}$ and $b_m = (-1)^{m+1}/\sqrt{m}$. What is c_n (defined by (11))?

Exercise 15. Extra credit! Show that the sequence $|c_n|$ is larger than a sequence with limit 2, so the sum cannot converge.

Exercise 16. Extra credit! Show that if $a_n = b_n = (-1)^{n+1}/n$ and c_n is the Cauchy product as above, then $\sum c_n$ actually *does* converge, and the limit is $\log(2)^2$. You need to know that $\sum a_n = \log(2)$ — that's a consequence of Abel's theorem (below).

Remark 5. The radius of convergence of a product can be larger than the minimum of the radii of the factors. For example, $(1-x)(1+x+x^2+\cdots) = 1$ which has infinite radius of convergence, even though the geometric series has radius of convergence 1.

Quotient The quotient of two power series is a little mysterious in that we don't give a radius of convergence. The reason is that the radius of convergence depends on where the denominator is zero. The mystery is that even *complex* zeros of the denominator count.

Fact 4. *If $\sum a_n x^n$ has radius of convergence $r > 0$ and $\sum b_n x^n$ has $b_0 \neq 0$ and radius of convergence $r' > 0$ then the quotient $\sum c_n x^n$ has positive radius of convergence.*

For example, $1/(1+x^2)$ is a quotient of two (very simple!) power series with infinite radius of convergence, but has only radius of convergence 1 (you can check that it is $1 - x^2 + x^4 - x^6 + \cdots$). The mystery is resolved with complex numbers: the denominator is 0 at $x = i$ which is 1 unit from 0.

If x is within the interval of convergence, we know from the convergence of products that the sum of $\sum c_n x^n$ (call it h) times the sum of $\sum b_n x^n$ (call it g) is the sum of the power-series product, and by definition of the c_n (by the right-hand equations of (8)) and the power-series product, that product is $\sum a_n x^n$. Call this sum f. So we have $hg = f$ and so $f/g = h$ at points where g is not zero. In other words: the power-series quotient $\sum c_n x^n$ is convergent and converges to the quotient of functions f/g.

Formal derivative As for the convergence of the formal derivative:

Fact 5. *If $\sum a_n x^n$ has radius of convergence $r > 0$ then $\mathfrak{D}(\sum a_n x^n)$ has the same radius of convergence.*

Proof. The easiest way to see this is to use Fact 1. If $\sum a_n x^n$ converges for $x = x_0$ and x has $|x| < |x_0|$ then as in exercise (4) you can factor $n a_n x^{n-1} = \frac{1}{x} n a_n x_0^n (x/x_0)^n \leq M (x/x_0)^n$ where M is an upper bound for $n a_n x_0^n / x$ — see the exercise for why it has an upper bound) so the series converges (absolutely) by comparison with the geometric series with ratio $|x/x_0| < 1$. $\square$

We will see that the formal derivative of a power series converges to the actual derivative of the function defined by the sum of power series within its radius of convergence. So again, everything works about as well as it could.

Example 12. Another counterexample. Consider $\sum \frac{1}{n}\sin(nx)$. This series (a trigonometric series, not a power series) converges to a 2π-periodic function which is differentiable except at integer multiples of 2π (it's $f(x) = (\pi - x)/2$ on $(0, 2\pi)$). You might think its derivative should be the sum of the series

$$\sum D(\frac{1}{n}\sin(nx)) = \sum \cos(nx)$$

but in fact *this series doesn't converge to a function at all.* So the "sum rule" *doesn't necessarily work* for infinite sums. (However, this series can be interpreted as representing a "distribution", a very useful Fields-Medal winning idea of Laurent Schwartz).

5 Real analytic functions

A power series with radius of convergence $r > 0$ defines a function on its domain of convergence. What kind of function is it? Are any of our favorite functions sums of power series?

A function f on a domain D is called *real-analytic* near a point a in D if it is the sum of a convergent power series in $x - a$ on $(a - r, a + r)$ for some $r > 0$. So the functions we are asking about are real analytic near 0. It is difficult to give an accurate representation of this situation, but real-analytic functions are extremely special among functions (even among functions with nice graphs). We'll see some hints of this soon.

We will see that calculus works very well with power series: in many situations, just as for polynomials. The reason has to do with the nature of their convergence, which is hiding in the argument used to prove Fact 1. Let's repeat the key idea of that argument:

Idea if $\sum a_n x^n$ converges for x_0 and $|x| < |x_0|$ then we can write $a_n x^n = a_n x_0^n (x/x_0)^n$ and $a_n x_0^n$ is bounded (in fact, has limit 0) while $|x/x_0| < 1$.

The important fact hiding here is this. Suppose we take a closed interval $[-b, b]$ where $b = \frac{1}{2}|x_0|$ say and require x to be in this interval. Then the ratio $|x/x_0|$ above is less than $1/2$ and so we have that the tail $\sum_{n=N}^{\infty} a_n x^n$ is less than say $\varepsilon(\frac{1}{2})^N(2)$, *independently* of x as long as x is in $[-b, b]$. We could replace the $\frac{1}{2}$ by any number less than 1 and get a similar result.

Contrast this with the sequence of functions $f_n(x) = x^n$ on $[0, 1]$. This sequence converges to the function

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1 \end{cases}$$

because $r^n \to 0$ if $|r| < 1$. But if you ask how large n has to be in order that f_n is within $1/1000$ of f you can't give an answer: the closer x is to 1, the larger n has to be to have $x^n < 1/1000$. In fact, n needs to be at least $\log(1000)/\log(1/x)$ and the closer x gets to 1 the larger this gets, without bound.

The property of being able to choose a single n to get f_n within a prescribed distance from f is called *uniformity* of convergence.

One easy-to-use criterion for uniform convergence for series of functions is:

Criterion 1. *If each function f_n:$I \to \mathbf{R}$ is bounded on I : $|f_n(x)| \leq M_n$ for all $x \in I$, then if $\sum M_n$ converges, then $\sum f_n$ converges* uniformly *on I.*

This is called the "Weierstrass M-test". Weierstrass was one of the first to recognize the importance of uniform convergence (in the late 1800s).

The condition given defines what has been called *normal convergence*. Using normal convergence allows us to use our tests for convergence on ordinary series rather than going back to the definition of limit, so we will do so whenever possible.

Note. It's probably clear from the above, but *power series converge normally* in any interval $(-s, s)$ where $s < r$ =radius of convergence. That's what the key fact says. It says that the constants M_n can be taken as geometrically decreasing (with ratio s/r).

Let's start by noting that

Fact. *if $\sum f_n$ and $\sum g_n$ converge normally, then so does $\sum(f_n + g_n)$.*

Proof. $|f_n + g_n| \leq |f_n| + |g_n| \leq$the sum of the Weierstrass M_n bounds for $|f_n|$ and $|g_n|$, so we can take as Weierstrass bound the sum of the bounds. By the rules for limits of sequences, the partial sums of this series converge, so $\sum(f_n + g_n)$ converges normally. □

Theorem 2. *If $f_n : I \to \mathbf{R}$ are continuous functions and the series $\sum f_n$ converges normally, then the sum is continuous on I.*

Proof. Since this requires us to roll up our sleeves just a skosh, it's in the appendix. □

We'll need a little more to prove that power series work like polynomials:

Theorem 3. *If $f_n : I \to \mathbf{R}$ are differentiable functions on a bounded interval I, with continuous derivatives, and*

1. *for some $a \in I$ the series $\sum f_n(a)$ converges, and*

2. the series of derivatives $\sum f_n'$ *converges NORMALLY,*

then $\sum f_n$ *converges normally on* I *to a differentiable function* f*, and* $\sum f_n' = f'$*.*

What good is this? Since the formal derivative of a power series is another power series, with the same radius of convergence, we can see that this power series must have sum f'. In other words, we can "differentiate power series like polynomials", that is, we can "differentiate power series term-by-term". Of couse we can do what we want, but the implied claim here is that this gives us a power series whose sum is the actual derivative of the function.

Proof. We first need to show that $\sum f_n(x)$ converges for all x in I. Use the FTOC and properties of the integral to get (this is why we need f_n' continuous)

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \tag{12}$$

We have $\sum f_n(a)$ convergent, and as $\sum f_n'$ converges normally, there are numbers M_n so that $|f_n'(x)| \le M_n$ and $\sum M_n$ converges.

So using the triangle inequality for integrals*,

$$|\int_a^x f_n'| \le \int_a^x |f_n'(x)|dx \le \int_a^x M_n\,dx = M_n(x-a) \le M_n \cdot \text{(length of } I\text{)}$$

so $\sum \int_a^x f_n'$ is normally convergent on I. Hence $\sum f_n(x)$ is normally convergent.

Now rearranging the previous inequality and using (12) for the integral, we have

$$\left| \frac{f_n(x) - f_n(a)}{x-a} \right| \le M_n$$

for all $x \ne a$. Define

$$q_n(x) = \begin{cases} \frac{f_n(x)-f_n(a)}{x-a} & x \ne a \\ f_n'(a) & x = a \end{cases}$$

so q_n is continuous on I. By continuity, $|q_n(x)| \le M_n$. So $\sum q_n$ is a normally convergent series of continuous functions; the sum is therefore continuous. But what is the sum? When $x \ne a$ it is $\frac{f(x)-f(a)}{x-a}$ just by rules for limits. As the sum is continuous, it has a limit as $x \to a$ which means that f is differentiable at a. Moreover, that limit is then by definition $f'(a)$. *But it is also the sum of* $\sum q_n(a) = \sum f_n'(a)$. Therefore, $\sum f_n'(a) = f'(a)$. Here a could have been any point, so we conclude that $f' = \sum f_n'$. □

*For this to be literally correct, we need to interpret the integrals as unoriented. Otherwise we need more absolute value bars if $x < a$.

We can use this fact to get a lot of information about functions which were previously a bit mysterious.

5.1 Mystery power series #1 revealed!

For example, consider the power series

$$\sum \frac{1}{n!} x^n$$

again. We saw it had infinite radius of convergence, so its sum defines a function $f(x)$ for all real x. What is f'? Well, the formal derivative of the power series is

$$\sum n \cdot \frac{1}{n!} x^{n-1} = \sum \frac{1}{(n-1)!} x^{n-1}$$

This is the same series again (don't let indexing fool you — write out a few terms). The theorem says that this has sum $f'(x)$ for any real x, but on the other hand we know its sum is $f(x)$, so we have $f' = f$. By now you should know that

Fact. *Any function equal to its own derivative is a multiple of* e^x*!*

So what multiple of e^x is our function f? We only need to check the value $f(0)$, which is 1, so we have the amazing formula

$$e^x = \sum \frac{1}{n!} x^n$$

5.2 Mystery power series #2

How about the power series

$$\sum \frac{1}{n} x^n?$$

This has radius of convergence 1, so defines a function f on $(-1, 1)$. The formal derivative is

$$\sum n \cdot \frac{1}{n} x^{n-1} = \sum x^{n-1}$$

which starts at $n = 1$ so is simply the geometric series. We know this has sum $(1-x)^{-1}$ and according to the theorem, this is f'. So f must be an antiderivative of $(1-x)^{-1}$. So $f(x) = -\log(1-x) + c$ for some constant c. Evaluating at $x = 0$ gives $c = 0$ so we have the formula

$$\log(1-x) = -\sum \frac{1}{n} x^n = -x - x^2/2 - x^3/3 - \cdots$$

for x in $(-1, 1)$.

5.3 Newton's binomial series

Newton used an extension of the Binomial Theorem in his work. He noted that the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

still make sense when n isn't a positive integer if you use the rightmost version. (When $k = 0$ we interpret the numerator as having *no factors at all*, so the coefficient is 1.)

It turns out these coefficients arise in a power series for *any* power of $1 + x$:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \tag{13}$$

Note that when α is a positive integer, $\dbinom{\alpha}{k} = 0$ when $k > \alpha$ because there is a factor 0 in the numerator. So in that case we get the usual Binomial Theorem expressing $(1 + x)^n$ as a *polynomial* of degree n.

Also note that when $\alpha = -1$ we get the series expansion of $(1 + x)^{-1} = \frac{1}{1+x}$. We already know this — it's the geometric series again:

$$\sum (-x)^n = \sum (-1)^n x^n$$

If we take $\alpha = 1/2$ we get a series expansion we haven't seen yet:

$$\sqrt{1+x} = \sum \binom{\frac{1}{2}}{n} x^n$$

You will work out an expression for the coefficients $\binom{1/2}{n}$ below.

Proof of Newton's Binomial Theorem. We can prove (13) as follows. First determine the radius of convergence. It's 1. Second, call $f(x)$ the sum of the power series for $|x| < 1$. We need to show that $f(x) = (1 + x)^\alpha$. We'd like this to be expressed in terms that are easy to see with power series, such as a simple condition relating f' and f for example. By the generalized power rule, $(1+x)^\alpha$ has derivative $\alpha(1+x)^{\alpha-1}$ as long as $\alpha \neq 0$. So we will show that

$$(1+x)f'(x) = \alpha f(x) \tag{14}$$

and then use uniqueness for solutions of this differential equation.

The power series for f' is the formal derivative of (13), namely

$$f'(x) = \sum_{k=0}^{\infty} k\binom{\alpha}{k}x^{k-1} = \sum_{j=0}^{\infty}(j+1)\binom{\alpha}{j+1}x^j$$

(reindexing) and $(1+x)f'(x) = f'(x) + xf'(x)$ so the power series for this has coefficient of x^n equal

$$\begin{aligned}(n+1)\binom{\alpha}{n+1} + n\binom{\alpha}{n} &= (n+1)\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!} + n\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \\ &= \alpha\binom{\alpha-1}{n} + \alpha\binom{\alpha-1}{n-1} \\ &= \alpha\binom{\alpha}{n}\end{aligned}$$

the last step due to Pascal's Triangle (which still works with any real number α, exercise!). These numbers are the coefficients of x^n in the power series for $\alpha f(x)$. So $(1+x)f'(x) = \alpha f(x)$.

To conclude that $f(x) = (1+x)^\alpha$ we need a *uniqueness* result for the differential equation (14). All we know right now is that $(1+x)^\alpha$ solves this equation, as does f. If we know there is only one solution (subject to some condition both of our functions meet, perhaps) then we can conclude they're equal.

So suppose g is another solution of (14). Look at the quotient g/f defined near 0 ($f(0) = 1$ and f is continuous so $f(x) > 0$ on some interval around 0). Take the derivative to get

$$\frac{g'f - gf'}{f^2} = \frac{(1+x)g'f - (1+x)gf'}{(1+x)f^2} = \frac{\alpha gf - g\alpha f}{(1+x)f^2} = 0$$

so g/f is constant, that is, $g = cf$ for some constant c.

OK, since $f(0) = 1$ (from the definition of $\binom{\alpha}{0} = 1$) and $(1+0)^\alpha = 1$ we have that $f(x) = (1+x)^\alpha$. QED!

Exercise 17. Show that

$$\binom{\alpha}{k-1} + \binom{\alpha}{k} = \binom{\alpha+1}{k}$$

from the definition.

Exercise 18. Write out the first six terms of Newton's series for $\sqrt{1+x}$.

Exercise 19. Using the same approach as for e^x and the others, show that the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots \tag{15}$$

has sum equal to $\cos(x)$ and the power series

$$\sum \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1} = x - x^3/3! + x^5/5! - x^7/7! + \cdots \tag{16}$$

has sum equal to $\sin(x)$. You will want to use what you know about uniqueness from a previous assignment.

Exercise 20. Find a power series for $\arcsin(x)$. (Hint: would Newton's binomial series for $(1+u)^{-\frac{1}{2}}$ be of any help?)

5.4 Euler's Formula

Since power series only involve basic arithmetic (and limits) we can plug in anything for which addition and multiplication and limits make sense. In particular, we can plug in *complex numbers*. So we can make sense out of e^{ix} (where i is the "imaginary unit" with $i^2 = -1$) as

$$e^{ix} = \sum \frac{1}{n!}(ix)^n \tag{17}$$

Using the fact that the powers of i repeat, you can find the real part of the sum as the even index terms, and the imaginary part as the odd index terms. Do these look familiar? Yes, if you did the previous exercise: they're the series for $\cos(x)$ and $\sin(x)$! You can also check that the series (17) converges absolutely (in place of absolute value, use the modulus of a complex number). So if we *define* e^{ix} by (17) (it's not like there's another way we were thinking of defining a complex power, right?) we get the amazing formula:

$$e^{ix} = \cos(x) + i\sin(x)$$

which works for all real numbers. This is Euler's formula (one of many).

5.5 Uniqueness

If f is the sum of a power series, $f(x) = \sum a_n x^n$ with a positive radius of convergence r, then f is differentiable on $(-r, r)$ and $f'(x)$ is given by the sum of the formal derivative $\sum n a_n x^{n-1}$ (which we have shown also has radius of convergence r). As this is also a power series, we can keep going. Conclusion: f is *infinitely* differentiable, that is, we can differentiate as many times as we like.

This allows us to recover the coefficients a_n from the values of the derivatives of f at 0. The idea is that $f(0)$ is equal to the constant term of the power series. Applying to the various derivatives, $f'''(0)$ is the constant term of the power series for the third derivative. What is that? You can see it comes from the term a_3x^3 in the series for f. As you differentiate, you get what were powers as factors: $a_3x^3 \dashrightarrow 3a_3x^2 \dashrightarrow 2\cdot 3a_3x^1 \dashrightarrow 1\cdot 2\cdot 3a_3$. So $f'''(0) = 3!a_3$. Solving for the coefficients,

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= f'(0) \\ a_2 &= f''(0)/2 \\ a_3 &= f'''(0)/3! \\ &\vdots \end{aligned}$$

and in general $a_n = f^{(n)}(0)/n!$ where the notation $f^{(n)}$ means nth derivative of f.

This says there is *only one* power series (at 0) which can converge to f. This means that if two functions, f and g, are sums of power series (are real-analytic) and agree on a neighborhood of 0, *no matter how small*, then $f = g$ on the *entire domain of convergence.*

Even more is true: if f and g are real-analytic at each point in an interval I, and $f(x_n) = g(x_n)$ for all terms of a sequence x_n in I having a limit in I, then $f = g$ on all of I. For example, if there is a function f real-analytic on $\mathbf{R}$ and $f(1/n) = \sin(1/n)$ for all n then f must be the sine function. You can draw many graphs through the points $(n, \sin(1/n))$ but *none of the others represent real-analytic functions.*

This is a slight weakening of a property polynomials have: recall that if two polynomials of degree $< n$ agree on n points, then they're equal. This is because their difference is a polynomial of degree less than n having n roots, so can only be the zero polynomial. So if two polynomials of any degree agree on an infinite set of points, then they're equal.

This is another reason power series are thought of as "infinite polynomials".

6 Abel's theorem

If $\sum a_nx^n$ has finite radius of convergence $r > 0$ then $f(x) = \sum a_nx^n$ defines a function on the domain of convergence. This may include either or both endpoints of the interval. *But our results don't include those endpoints,* only points in $(-r, r)$ (look again at the key fact).

For example, we saw that $\log(1-x) = -\sum \frac{1}{n}x^n$ for x in $(-1, 1)$. If you put $x = -1$ in to the series, you get the convergent alternating series

$$\sum(-1)^{n+1}\frac{1}{n}$$

It's natural to ask: does this converge to $\log(1-(-1)) = \log(2)$? *Our results so far say nothing about this.*

Let $f(x) = -\sum \frac{1}{n}x^n$ for x in $[-1, 1)$. We are asking if the sum f is continuous (from the right) at -1. Since $\log(1-x)$ is continuous at $x = -1$, this would show $f(-1) = \lim_{x\to -1+} f(x) = \lim_{x\to -1}\log(1-x) = \log(2)$.

Abel proved that the answer is yes. Here is a simple way to state his theorem:

Theorem 4. *If $\sum a_n x^n$ converges at $x = 1$ then the function f defined on $(-1, 1)$ by the sum has limit $\sum a_n$ as $x \to 1^-$.*

This settles the example of $\log(2)$: substitute $-x$ for x in $-\sum \frac{1}{n}x^n$. We know the limit of $\log(1-(-x)) = \log(1+x)$ is $\log(2)$ as $x \to 1^-$ because log is continuous at 2. According to the theorem, it is $-\sum(-1)^n/n = \sum(-1)^{n+1}\frac{1}{n}$. So we have the following concrete computational formula for the otherwise enigmatic number $\log(2)$:

$$\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Remark. This is subtle. Note first that it doesn't hold as a matter of definition. For example, take the series (not a power series) $1 + \sum_{n=1}^{\infty} x^{n-1}(x-1)$ on $(-1, 1)$. Define $f(x)$ to be the sum of the series for $x \in (-1, 1)$. Check that $f(x) = 0$, so $\lim_{x\to 1^-} f(x) = 0$ but plugging $x = 1$ into the series gives 1, not 0.

Proof. One clever approach uses the fact that division by $1 - x$ gives a new power series whose coefficients are the partial sums of the sequence of coefficients (you proved this in exercise 14). We twist this a little. Letting $L = \sum a_n$ we get

$$\frac{L - \sum a_n x^n}{1-x} = \sum t_n x^n$$

where $t_n = \sum_{k=n+1}^{\infty} a_k$ is the tail sequence for the series $\sum a_n$. (Exercise!). This gives

$$L - \sum a_n x^n = (1-x)\sum t_n x^n \tag{18}$$

as power series, and by our results on power series arithmetic, this holds for any x in $(-1, 1)$. So if we could show that the sum $\sum t_n x^n$ is bounded to the left of $x = 1$ we'd be done: limit zero times bounded = limit zero!

However, $\sum t_n x^n$ need *not* be bounded on $(0,1)$! So this is a more sublime situation than just the "rules for limits". What we need to prove comes down to: $\square$

Fact. *If* (c_n) *is a sequence with limit* 0*, let* $g(x)$ *be the sum of* $\sum c_n x^n$ *for* $x \in (-1,1)$*. Then* $(1-x)g(x) \to 0$ *as* $x \to 1-$*.*

Proof. This requires a somewhat more detailed analysis than usual, so it's relegated to the appendix. $\square$

Let's see how to use this to prove Abel's theorem.

Proof. The idea is that since $\sum a_n$ converges, then its tail sequence $t_n \to 0$. Use the fact above, with $c_n = t_n$. This says that the right-hand side of (18) has limit 0 as $x \to 1-$. Therefore so does the left-hand side! $\square$

Exercise 21. Find a power series for the arctangent (inverse tangent) and use Abel's theorem to get a series expression for $\frac{\pi}{4}$.

Exercise 22. If $\sum a_n$ converges, let $L = \sum a_n$ and $s_n = \sum_{k=0}^{n} a_k$ and let t_n be the tail sequence, $t_n = \sum_{k=n+1}^{\infty} a_k$. Show that

$$\frac{L - \sum a_n x^n}{1-x} = \frac{L}{1-x} - \sum s_n x^n = \sum L x^n - \sum s_n x^n = \sum t_n x^n$$

(Use Exercise 14.)

Exercise 23. Let $c_n = 1/n$ and explain the following:

1. Why the power series $\sum c_n x^n$ has radius of convergence 1;
2. What familiar function the sum $g(x) = \sum c_n x^n$ is for x in $(-1,1)$;
3. Why $g(x)$ has limit ∞ as $x \to 1-$;
4. Why L'Hôpital's rule says $(1-x)g(x)$ has limit 0 as $x \to 1-$;

This exercise gives a good example of what Abel's theorem deals with. To see that it is a real example, we'd like to know what sequence (a_n) would have tail sequence $1/n$. That's easy: since $t_{n-1} - t_n = a_n$ we get $a_n = \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$. The next exercise uses this to illustrate Abel's theorem.

Exercise 24. Let the sequence (a_n) be defined by $a_n = \frac{1}{n(n-1)}$. What is the radius of convergence of the power series $\sum a_n x^n$? Let its sum be $f(x)$; identify f as an antiderivative of $\log(1-x)$ on $(-r,r)$. Evaluate the constant term to get a formula for f; then find $\lim_{x\to 1-} f(x)$. Show that this is equal to $\sum a_n$ without using Abel's theorem. (Hint: $\sum a_n$ can be expressed as a "telescoping" series.)

6.1 Summability

Abel's theorem has another interpretation. We've seen series which don't converge, but in some sense seem like there's a reasonable notion of "sum" anyway. For example, we might feel that the series $1-1+1-1+\cdots$ should have sum $1/2$ as the partial sums are $1, 0, 1, 0, 1, \ldots$ alternating between 1 and 0.

If we take a positive number $r<1$ and sum $\sum(-1)^n r^n = \sum(-r)^n$ we get $\frac{1}{1-(-r)} = \frac{1}{1+r}$. Now take the limit as $r \to 1^-$. That gives $\frac{1}{2}$ as a sort of sum of the series $\sum(-1)^n$. This way of attaching a sum to a divergent series is called "Abel's summability method".

Another approach is to take the sequence of partial sums and form a new sequence of cumulative averages:

$$s_1, (s_1+s_2)/2, (s_1+s_2+s_3)/3, \ldots$$

For our sequence, the terms go $n/(2n-1), n/(2n)$ so the limit is $1/2$ again. This is called "Cesaro's summability method".

A good property for a summability method is that if a series actually converges, then the summability method should give the same limit. This is called "regularity". Abel's theorem can then be seen as saying that Abel summability has this property.

Exercise 25. If the sequence a_n has limit L, show that the cumulative averages

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$$

also have limit L. Hint: why is it true that

$$A_n - L = \frac{1}{n}\sum_{k=1}^{n}(a_k - L)$$

and how can you use it? You probably have to split the sum into two as in Abel's Theorem. (Congratulations, you just proved regularity of Cesaro summability!)

7 Appendix

This is an important fact about uniform convergence. Here is the proof for normal convergence, which is a little less technical (and all we need). But it still requires some ε slinging, so it's here in the appendix.

Theorem. *If $f_n : I \to \mathbf{R}$ are continuous functions and $\sum f_n$ converges normally on I then the sum is continuous on I as well.*

Proof. Normal convergence means that there is a sequence of numbers M_n so that $|f_n(x)| \leq M_n$ for all x in I, and $\sum M_n$ converges. This means that $\sum f_n(x)$ converges *absolutely*, by comparison, for any x. Let $f(x)$ be the sum. To show that f is continuous at a point a in I we need to show that, if ε is any positive number, then we can find a $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. Write

$$f(x) - f(a) = \sum_{n=0}^{N} (f_n(x) - f_n(a)) + t_N(x)$$

where $t_N(x)$ is the tail of the $f_n(x) - f_n(a)$ series (which therefore depends on x). This series converges normally, as it's a sum of normally convergent series, so we can find N so that $|t_N(x)| < \varepsilon/2$ no matter what x is. As the partial sums are continuous functions, we can find $\delta > 0$ so that if $|x - a| < \delta$ then $\left|\sum_{n=0}^{N}(f_n(x) - f_n(a))\right| < \varepsilon/2$. Check that this δ does the trick! $\square$

The main fact from Abel's theorem uses a similar analysis.

Theorem 5. *Let (c_n) be a sequence with limit 0 and let $g(x)$ be the sum of the power series $\sum c_n x^n$ for $|x| < 1$. Then $(1 - x)g(x) \to 0$ as $x \to 1-$.*

Proof. Since $c_n \to 0$, given any positive ε, we can find an N so that $|c_n| < \varepsilon/2$ (or any positive number we like, but we like this one) for all $n \geq N$. Now break the sum above up at term N:

$$\sum c_n x^n = p_N(x) + \sum_{n=N+1}^{\infty} c_n x^n \tag{19}$$

Here $p_N(x) = \sum_{k=0}^{N} c_n x^n$ is simply a polynomial of degree N. The sum on the right can be bounded for $x < 1$:

$$\begin{aligned} \left|\sum_{n=N+1}^{\infty} c_n x^n\right| &\leq \sum_{n=N+1}^{\infty} |c_n| \cdot x^n \\ &\leq x^{N+1} \sum_{k=0}^{\infty} \frac{\varepsilon}{2} x^k \\ &= x^{N+1} \frac{\varepsilon}{2} \sum_{k=0}^{\infty} x^k \\ &= \frac{\varepsilon}{2} x^{N+1} \frac{1}{1-x} \end{aligned}$$

Now this is *not* obviously bounded near $x = 1$! But it's just good enough. Multiply both sides of (19) by $1-x$ and use this bound:

$$|(1-x)g(x)| \le (1-x)\,|p_N(x)| + x^{N+1}\frac{\varepsilon}{2}$$

for $x<1$.

To show that the limit as $x \to 1^-$ is 0 we need to show that for any $\varepsilon > 0$ we can find a $\delta > 0$ so that if $x \in (1-\delta, 1)$, then $|(1-x)g(x)| < \varepsilon$. From our inequality above, it is now enough to show that $(1-x)|p_N(x)| < \varepsilon/2$. The polynomial $p_N(x)$ is bounded on $[0,1]$, say by M_N (just by continuity). Then if we choose $\delta = \varepsilon/(2M_N)$ we are done:

$$|(1-x)g(x)| \le (1-x)M_N + x^{N+1}\varepsilon/2 \le \delta M_N + \varepsilon/2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since $x > 1-\delta$ means $1-x<\delta$.

The logical structure here is: given any $\varepsilon > 0$ we find N to break the series into two parts, one part simply polynomial, the other part arbitrarily small independently of x. Let M_N be an upper bound for the polynomial part on $[0,1]$. Determine δ by $\varepsilon/(2M_N)$. Then if $x \in (1-\delta, 1)$ we have $|(1-x)g(x)| < \varepsilon$.

That is exactly what it means that $(1-x)g(x) \to 0$ as $x \to 1^-$. □

Example 13. Here is an example to show how special power series are. Let functions $f_n : D \to \mathbf{R}$ be defined by

$$f_n(x) = \frac{x^{n-1}}{(1-x^n)(1-x^{n+1})}$$

where D is the set of real numbers other than ± 1.