Q1 (a). $Y = F(X)$.

$F_Y(y) = P(Y \le y)$

$= P(F(X) \le y)$

$= P(X \le F^{-1}(y))$

$= F(F^{-1}(y))$

$= y.$

$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = 1.$

$\therefore$ Y follows a uniform distribution.

(a') known $X = F^{-1}(U)$

$\{F^{-1}(U) \le x\} = \{U \le F(x)\}$.

let $a = F(x)$. in $P(U \le a) = a$.

$P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$

$\therefore X \sim F$

(b). $Z = X - Y$

$P(X - Y < z)$

$= \int_0^1 \int_0^y f_{XY}(x,y)\, dx\, dy.$

$= \int_0^1 \int_{y-z}^{y+z} f_X(x) \cdot f_Y(y)\, dx\, dy.$

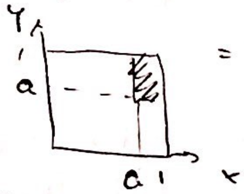
$= \int_0^1 \int_{y-z}^{y+z} f_X(y)\, dx\, dy$

$$\int_{y-z}^{y+z} f_X(x)\, dx = \begin{cases} y+z & 0 \le y < z \\ 1-2z & z \le y < 1-z \\ 1+z-y. & 1-z \le y < 1 \end{cases}$$

$$F_Z(z) = \begin{cases} \int_0^z y+z\, dy = \frac{3}{2}z^2 & 0 < y < z. \\ \int_z^{1-z} 1-2z\, dy = 4z^2-4z+1 & z < y < 1-z. \\ \int_{1-z}^1 1-z-y\, dy = \frac{3}{2}z^2 & 1-z < y < 1. \end{cases}$$

$$f_Z(z) = \begin{cases} 3z & 0 \le y < z. \\ 8z-4 & z < y < 1-z \\ 3z & 1-z \le y < 1. \end{cases}$$

(b') $Z = \min\{X, Y\}$

$P(Z \ge a) :$  $= (1-a)^2$

$\therefore F_Z'(a) = P(Z \le a)$

$= 1-(1-a)^2$

$\therefore f_Z(a) = F_Z'(a) = 2(1-a)$ for $a \in (0,1)$

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(c). $Y = e^X = g(x)$. $g'(x) = e^x$

$X = \ln Y$.

$X \sim N(0,1)$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \cdot \frac{1}{y}.$$

$$= \frac{1}{y\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}}$$

$$E(Y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(\ln y)^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2}} e^x dx$$

$$= e^{\frac{1}{2}}.$$

$$E(Y^2) = \frac{1}{\sqrt{2\pi}} \int_0^\infty y \cdot e^{-\frac{(\ln y)^2}{2}} dx$$

$$= e^2$$

$\therefore Var(Y) = E(Y^2) - (E(Y))^2 = e^2 - (e^{\frac{1}{2}})^2 = e^2 - e.$

(d). $Var(Y|X) = E([Y - E(Y|X)]^2|X)$

$= E(Y^2|X) - (E(Y|X))^2$

$\therefore E(Var(Y|X)) = E(E(Y^2|X) - (E(Y|X))^2)$

$= E(Y^2) - E([E(Y|X)]^2)$

$Var(E(Y|X)) = E([E(Y|X)]^2) - [E(E(Y|X))]^2$

$= E([E(Y|X)]^2) - [E(Y)]^2$

$\therefore E(Var(Y|X)) + Var(E(Y|X))$

$= E(Y^2) - [E(Y)]^2$

$= Var(Y)$

$\therefore Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$

Q2 (a) $\hat{\beta} = (X^TX)^{-1}X^Ty$.

$\therefore X\hat{\beta} = X \cdot \left((X^TX)^{-1}X^Ty \right) = \left(X(X^TX)^{-1}X^T \right) y$

$= Hy$.

(b) HX

$= (X(X^TX)^{-1}X^T)X$

$= X(X^TX)^{-1}X^TX$

$= X(X^TX)^{-1}(X^TX)$

$= XI$

$= X$

(c) $H^T = [X(X^TX)^{-1}X^T]^T$

$= X[(X^TX)^{-1}]^TX^T$

$\because (X^T)^{-1} = (X^{-1})^T$

$[(X^TX)^{-1}]^T = [(X^TX)^T]^{-1} = (X^TX)^{-1}$

$\therefore H^T = X(X^TX)^{-1}X^T$

$= H$.

(d) $H \cdot H = (X(X^TX)^{-1}X^T)(X(X^TX)^{-1}X^T)$

$= X(X^TX)^{-1}(X^TX)(X^TX)^{-1}X^T$

$= X(X^TX)^{-1}X^T$

$= H$

(e) $\text{rank}(H) = \text{rank}(X(X^TX)^{-1}X^T)$

$= \text{rank}((X^TX)^{-1})$

$= \text{rank}(X^TX)$

$= \text{rank}(X)$

$\therefore$ Colum space of H is ~~the~~ equal to X, L

$Col(H) = Col(X) = L$

$\therefore \hat{y} = Hy$ is the projection of y onto column space L.

(f) $\text{trace}(H) = \text{trace}(X(X^TX)^{-1}X^T)$

$= \text{trace}((X^TX)^{-1}X^T \cdot X)$

$= \text{trace}((X^TX)^{-1}(X^TX))$

$= \text{trace}(I_d)$

$= d$

~~= ra~~

$\text{rank}(X) = \text{rank}(X^TX) = d.$

$\therefore \text{trace}(H) = \text{rank}(X) = d.$

. 3.

(a). $X = U\Sigma V^T$ with eigenvalues $\sigma_1, \sigma_2 \dots \sigma_r$.

$X^TX = (U\Sigma V^T)^T(U\Sigma V^T)$

$= V\Sigma U^T \cdot U\Sigma V^T$

$= V\Sigma I \Sigma V^T$

$= V\Sigma^2 V^T$

$\because$ Let $\sigma_1', \sigma_2' \dots \sigma_r'$ denote eigenvalues of (X^TX)

$\therefore \sigma_i' = \sigma_i^2$

$\because$ X^TX and XX^T have same eigenvalues.

$\therefore$ Eigenvalues of XX^T are also $\sigma_1', \dots \sigma_r'$ where $\sigma_i' = \sigma_i^2$

(b). $X = \sum_i^n \sigma_i u_i v_i^T$

$\therefore Xv_i = \sum_i^n \sigma_i u_i v_i^T \cdot v_i = \sigma_i u_i$

$X^T u_i = (\sum \sigma_i u_i v_i^T)^T u_i$

$= \sum \sigma_i v_i (u_i^T u_i)$

$= \sigma v_i$

(c) $\|X\|_F = \|U\Sigma V\|_F$.

$\because$ U, V are orthogonal

$\therefore$ it satisfies univariate invariant.

$\therefore \|X\|_F = \|\Sigma\|_F = \sqrt{\sum \sigma_i^2}$

(d) $|\det(X)| = |\det(U\Sigma V)|$ orth

$= |\det(U)| \cdot |\det(\Sigma)| \cdot \det|(V)|$.

$= |\det(\Sigma)|$

$= \prod \sigma_i$

(e). $H = X(X^TX)^{-1}X^T$

$= (U\Sigma V^T)\left[(U\Sigma V^T)^T(U\Sigma V^T)\right]^{-1}(U\Sigma V^T)^T$

$= (U\Sigma V^T)\left(V\Sigma U^T U\Sigma V^T\right)^{-1}(U\Sigma V^T)^T$

$= (U\Sigma V^T)\,(V\Sigma^2 V^T)^{-1}\,(V\Sigma U^T)$

$= U\Sigma V^T \; V\Sigma^{-2} V^T V\Sigma U^T$

$= U\Sigma(V^TV)\;\Sigma^{-2}\;(V^TV)\Sigma U^T$

$= UU^T$

(f). $X_k = U\Sigma_k V^T$

$\hat{\beta} = (X_k^TX_k)^{-1}X_k^Ty$

$= \left((U\Sigma_k V^T)^T(U\Sigma_k V^T)\right)^{-1}(U\Sigma_k V^T)^Ty$

$= (V\Sigma_k^2 V)^{-1}(U\Sigma_k V^T)^T y$

$= (V\Sigma_k^2 V)^{-1}\cdot V\Sigma_k U^Ty$

$= V\Sigma_k^{-2} V^T V\Sigma_k U^T y$

$= V\Sigma_k^{-2}\Sigma_k U^Ty$