CMSC 25025
Homework 1

Reid McIlroy-Young

April 6, 2017

1 Statistical refresher

1.1

$$\begin{aligned}
F(X) &= \int_{-\infty}^{X} f_x(t)dt \\
\Rightarrow f_x &= \frac{dF}{dX}
\end{aligned}$$

$$\begin{aligned}
X &= F^{-1}(U) \\
F(X) &= F(F^{-1}(U)) \\
F(X) &= U \\
\Rightarrow X &\sim F
\end{aligned}$$

1.2

$$X, Y \sim Uniform(0,1) \tag{1}$$
$$f_{X,Y}(x,y) = 1 \tag{2}$$

$$
\begin{aligned}
Z &= X - Y \\
F_Z(z) &= P(Z \leq z) \\
&= P(X \leq Y + z) \\
&= \int_{-\infty}^{\infty} \int_{y-z}^{\infty} f_{X,Y}(x,y) dxdy \\
&= \int_{0}^{1} \int_{y-z}^{1} 1 dxdy \\
&= \int_{0}^{1} [x]_{y-z}^{1} dy \\
&= \int_{0}^{1} 1 - y - z dy \\
&= 1 - z + 2zU(-z)
\end{aligned}
$$

$$
\begin{aligned}
Z &= min\{X, Y\} \\
F_Z(z) &= P(Z \leq z) \\
&= P(min\{X, Y\} \leq z) \\
&= 1 - P(min\{X, Y\} \geq z) \\
&= 1 - P(X \geq z)P(Y \geq z) \\
&= 1 - (1 - z)(1 - z) \\
&= 2z - z^2
\end{aligned}
$$

1.3

$$
\begin{aligned}
x &\sim N(0, 1) && (3) \\
Y &= e^X && (4) \\
u^{-1}(Y) &= log(Y) && (5) \\
\frac{du^-1}{dY} &= \frac{1}{Y} && (6) \\
\Rightarrow f_Y(y) &= \frac{1}{y\sqrt{2\pi}} e^{\frac{log^2(y)}{2}} && (7)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(Y) &= \int_{-\infty}^{\infty} y \frac{1}{y\sqrt{2\pi}} e^{\frac{log^2(y)}{2}} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{log^2(y)}{2}} dy \\
&= e^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 \frac{1}{y\sqrt{2\pi}} e^{\frac{log^2(y)}{2}} dy \\
&= \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}} e^{\frac{log^2(y)}{2}} dy \\
&= e^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= \mathbf{E}(Y^2) - [\mathbf{E}(Y)]^2 \\
&= e^2 - [e^{\frac{1}{2}}]^2 \\
&= e^2 - e
\end{aligned}
$$

1.4

$$
\begin{aligned}
Var(Y) &= \mathbf{E}(Y^2) - [\mathbf{E}(Y)]^2 \\
&= \mathbf{E}(\mathbf{E}[Y^2|X]) - [\mathbf{E}(\mathbf{E}[Y|X])]^2 \\
&= \mathbf{E}(Var(Y|X) + [\mathbf{E}(Y|X)]^2) - [\mathbf{E}(\mathbf{E}[Y|X])]^2 \\
&= \mathbf{E}(Var(Y|X)) + \mathbf{E}([\mathbf{E}(Y|X)]^2) - [\mathbf{E}(\mathbf{E}[Y|X])]^2 \\
&= \mathbf{E}(Var(Y|X)) + Var(\mathbf{E}(Y|X))
\end{aligned}
$$

2 SVD

3 Basic Regression

$$H = X(X^TX)^{-1}X^T \tag{8}$$

$$\hat{\beta} = (X^TX)^{-1}X^T\mathbf{y} \tag{9}$$

$$\tag{10}$$

3.1

$$\begin{aligned} H\mathbf{y} &= (X(X^TX)^{-1}X^T)\mathbf{y} \\ &= X((X^TX)^{-1}X^T\mathbf{y}) \\ &= X\hat{\beta} \end{aligned}$$

3.2

$$\begin{aligned} HX &= X(X^TX)^{-1}X^TX \\ &= XI \\ &= X \end{aligned}$$

3.3

$$\begin{aligned} H^T &= (X(X^TX)^{-1}X^T)^T \\ &= X^{T^T}(X^TX)^{-1^T}X^T \\ &= X(X^TX^{T^T})^{-1}X^T \\ &= X(X^TX)^{-1}X^T) \\ &= H \end{aligned}$$

3.4

$$\begin{aligned} H^2 &= (X(X^TX)^{-1}X^T)^2 \\ &= (X(X^TX)^{-1}X^T)(X(X^TX)^{-1}X^T) \\ &= X(X^TX)^{-1}IX^T \\ &= (X(X^TX)^{-1}X^T)^2 \\ &= H \end{aligned}$$

3.5

3.6

$$\begin{aligned} trace(H) &= trace(X(X^TX)^{-1}X^T) \\ &= trace((X^TX)^{-1}X^TX) \\ &= trace(I_d) \\ &= d = rank(X) \quad \because \text{H is a projection} \end{aligned}$$