Problem Set #1
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Exercise 1.3
$\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open}\}$ is not an algebra.
Proof:
Let $B \in \mathcal{G}_1$. Then B is open, and its complement B^c is closed. Therefore, $B^c \notin \mathcal{G}_1$, so $\mathcal{G}_1$ is not closed under complements and is not an algebra. $\square$

$\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$ is an algebra, but not a σ-algebra.
Proof:

1. $\emptyset \in \mathcal{G}_2$
2. Let $B \subset \mathcal{G}_2$. Then its complement B^c is also of the form $(a, b], (-\infty, b], \text{ and } (a, \infty)$. Therefore, $B^c \in \mathcal{G}_2$, so $\mathcal{G}_2$ is closed under complements.
3. Let $E_1, E_2, ..., E_n \in \mathcal{G}_2$. Then their finite union $\cup_{i=1}^n E_i \in \mathcal{G}_2$, so $\mathcal{G}_2$ is closed under finite unions.
4. Let $E_1, E_2, ... \in \mathcal{G}_2$. Then their countable union $\cup_{i=1}^\infty E_i \notin \mathcal{G}_2$, so $\mathcal{G}_2$ is not closed under countable unions.

   Therefore $\mathcal{G}_2$ is an algebra, but not a σ-algebra. $\square$

$\mathcal{G}_3 = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$ is a σ-algebra.
Proof:

1. $\emptyset \in \mathcal{G}_3$
2. Let $B \subset \mathcal{G}_3$. Then its complement B^c is also of the form $(a, b], (-\infty, b], \text{ and } (a, \infty)$. Therefore, $B^c \in \mathcal{G}_3$, so $\mathcal{G}_3$ is closed under complements.
3. Let $E_1, E_2, ..., E_n \in \mathcal{G}_3$. Then their finite union $\cup_{i=1}^n E_i \in \mathcal{G}_3$, so $\mathcal{G}_3$ is closed under finite unions.
4. Let $E_1, E_2, ... \in \mathcal{G}_3$. Then their countable union $\cup_{i=1}^\infty E_i \in \mathcal{G}_3$, so $\mathcal{G}_3$ is closed under countable unions.

   Therefore $\mathcal{G}_2$ is a σ-algebra. $\square$

Exercise 1.7
$\{\emptyset, X\}$ is the smallest σ-algebra.
Proof:
Let $\mathcal{A}$ be a σ-algebra. By definition, $\emptyset \in \mathcal{A}$. Then $\emptyset^c = X \in \mathcal{A}$. $\square$

$\mathcal{P}(X)$ is the largest σ-algebra.
Proof:

Suppose $\mathcal{P}(X)$ is the not largest σ-algebra. Then there exists a set $B \subset X$ such that $B \notin \mathcal{P}(X)$. This is a contradiction. Therefore $\mathcal{P}(X)$ is the largest σ-algebra. $\square$

Exercise 1.10
Let $\{\mathcal{S}_\alpha\}$ be a family of σ-algebras on X. Then $\cap_\alpha \mathcal{S}_\alpha$ is also a σ-algebra.
Proof:

1. $\emptyset \in \mathcal{S}_\alpha \forall \alpha \implies \emptyset \in \cap_\alpha \mathcal{S}_\alpha$ (contains $\emptyset$)
2. $S \in \cap_\alpha \mathcal{S}_\alpha \implies S \in \mathcal{S}_\alpha \forall \alpha \implies S^c \in \mathcal{S}_\alpha \forall \alpha \implies S^c \in \cap_\alpha \mathcal{S}_\alpha$ (closed under complements)
3. $S_1, S_2, ... \in \cap_\alpha \mathcal{S}_\alpha \implies S_1, S_2, ... \in \mathcal{S}_\alpha \forall \alpha \implies \cup_{i=1}^\infty S_i \in \mathcal{S}_\alpha \forall \mathcal{S}_\alpha \implies \cup_{i=1}^\infty S_i \in \cup_{i=1}^\infty \mathcal{S}_\alpha$ (closed under finite and countable unions) $\square$

Exercise 1.17
Let $(X, \mathcal{S}, \mu)$ be a measure space. Then μ is monotone and countably subadditive.
Proof:

1. Let $A, B \in \mathcal{S}$, and let $A \subset B$. Then $A \cup (B \cap A^c) = B$. These sets are disjoint, so $\mu(A) + \mu(B \cap A^c) = \mu(B) \implies \mu(A) \leq \mu(B)$.
2. Let $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$. Then $\cup_{i=1}^\infty A_i = A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap A_1^c \cap A_2^c) \cup \cdots$. Since these sets are disjoint, $\mu(\cup_{i=1}^\infty A_i) = \mu(A_1) + \mu(A_2 \cap A_1^c) + \mu(A_3 \cap A_1^c \cap A_2^c) + \cdots \leq \sum_{i=1}^\infty \mu(A_i)$ $\square$

Exercise 1.18
Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda : \mathcal{S} \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure $(X, \mathcal{S})$.
Proof:

1. $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$
2. Let $\{A_i\}_{i=1}^\infty \in \mathcal{S}$ s.t. $A_i \cap A_j = \emptyset, \forall i \neq j$.
$\lambda(\cup_{i=1}^\infty A_i) = \mu((\cup_{i=1}^\infty A_i) \cap B) = \mu((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots) = \mu(A_1 \cap B) + \mu(A_2 \cap B) + \cdots = \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i)$ $\square$

Exercise 1.20
Let μ be a measure on $(X, \mathcal{S})$. Then it is continuous from below in the sense that: $(A_1 \supset A_2 \supset A_2 \supset \cdots, A_i \in \mathcal{S}, \mu(A_1) < \infty) \implies (\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{i=1}^\infty A_i))$
Proof:
Let $B_n = A_n$. Note that $\cap_{i=1}^n A_i = B_n$.
$\mu(\cap_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^n A_i) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_n)$ $\square$

Exercise 2.10
The theorem states that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. The (*) in the theorem could be replaced by $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$, because we have that

$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ by the definition of the outer measure μ^*. $\square$

Exercise 2.14 Let $\mathcal{O}$ denote the collection of open sets of $\mathbb{R}$. Then $\sigma(\mathcal{O}) = \mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open sets of $\mathbb{R}$. $\sigma(\mathcal{A})$ is the σ-algebra generated by the family $\mathcal{A}$ that include $\mathcal{O}$. Therefore, $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A}) \subset \mathcal{M}$. $\square$

Exercise 3.1
Let $a \in \mathbb{R}$. Then $\{a\} \subset [a-\epsilon, a+\epsilon]$ $\forall \epsilon > 0$. Then $\bar{\mu}(a) \leq \bar{\mu}([a-\epsilon, a+\epsilon]) = 2\epsilon \implies \bar{\mu}(a) = 0$ $\forall a \in \mathbb{R}$. Let $A = \{a : a \in \mathbb{R}\} = \cup_{n=1}^{\infty}\{a_n\}$. Then $\bar{\mu}(A) = \bar{\mu}(\cup_{n=1}^{\infty}\{a_n\}) = \sum_{n=1}^{\infty} \bar{\mu}(a_n) = 0$. Therefore every countable subset of the real line has Lebesgue measure 0. $\square$

Exercise 3.4
Let $\{x \in X : f(x) < a\}$ be measurable in $\mathcal{M}$.
The set $\cap_{n=0}^{\infty}\{x \in X : f(x) < a + \frac{1}{n}\} = \{x \in X : f(x) \leq a\}$ is measurable since $\mathcal{M}$ is closed under countable intersection.
The sets $\{x \in X : f(x) < a\}^c = \{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) \leq a\}^c = \{x \in X : f(x) > a\}$ are also measurable since $\mathcal{M}$ closed under complements. $\square$

Exercise 3.7
The measurability of $f+g$, $f \cdot g$, and $|f|$ follow from the measurability of $F(f(x), g(x))$. The measurability of $\max(f,g)$ and $\min(f,g)$ follow from the fact that $\sup_{n\in\mathbb{N}} f_n(x)$ and $\inf_{n\in\mathbb{N}} f_n(x)$ are measurable. $\square$

Exercise 3.14
Let $\epsilon > 0$. Since f is bounded, $\exists M \in \mathbb{N}$ s.t. $f < M$. Then $\frac{1}{2^N} < \epsilon$ and for all $n \geq N, |f(x) - s_n(x) < \epsilon$ $\forall x$. Therefore the convergence in (1) is uniform. $\square$

Exercise 4.13
By Property 4.5, since $||f|| < M$ on $E \in \mathcal{M}$ and $\mu(E) < \infty$, we know that $0 \leq \int_E ||f|| d\mu < M\mu(E) < \infty$. Then $\int_E ||f||^+ d\mu$ and $\int_E ||f||^- d\mu$ are finite, so $||f||$ is absolutely integrable with respect to μ. $\square$

Exercise 4.14
Since $f \in \mathscr{L}^1(\mu, E)$, we know that both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite. Therefore, f must be finite almost everywhere on E. $\square$

Exercise 4.15
Since $f, g \in \mathscr{L}^1(\mu, E)$ and $f \leq g$, we have that $\int_E f^- d\mu \leq \int_E g^- d\mu$ and $\int_E f^+ d\mu \leq \int_E g^+ d\mu \implies \int_E f d\mu \leq \int_E g d\mu$. $\square$

Exercise 4.16
Since $f \in \mathscr{L}^1(\mu, E)$, we have that $\int_E f^- d\mu$ and $\int_E f^+ d\mu$ are finite. Since $A \subset E$, $E = A \cup (A^c \cap E) \implies \int_{A\cup(A^c\cap E)} f^- d\mu$ and $\int_{A\cup(A^c\cap E)} f^+ d\mu$ are finite $\implies$

$\int_A f^- d\mu + \int_{A^c \cap E}$ and $\int_A f^+ d\mu + \int_{A^c \cap E} f^+ d\mu$ Since $f \in \mathscr{L}^1(\mu, E)$, we have that $\int_E f^- d\mu$ and $\int_E f^+ d\mu$ are finite. Since $A \subset E$, $E = A \cup (A^c \cap E) \implies \int_{A \cup (A^c \cap E)} f^- d\mu$ and $\int_{A \cup (A^c \cap E)} f^+ d\mu$ are finite $\implies \int_A f^- d\mu + \int_{A^c \cap E} f^- d\mu$ and $\int_A f^+ d\mu + \int_{A^c \cap E} f^+ d\mu$ are finite. Therefore we know that $\int_A f^- d\mu$ and $\int_A f^+ d\mu$ are finite, and thus $f \in \mathscr{L}^1(\mu, A)$. $\square$

Exercise 4.21

Since $f \in \mathscr{L}^1$, we can define measures $\lambda_1(A) := \int_A f^+ d\mu$ and $\lambda_2(A) := \int_A f^- d\mu$ $\implies \lambda(A) = \lambda_1(A) - \lambda_2(A) = \int_A f d\mu$. Then since $\beta \subset A$, $A = B \cup (A - B) \implies \lambda_i(A) = \lambda_i(B) + \lambda_i(A - B)$. By hypothesis, $\lambda_i(A - B) = 0 \implies \int_A f d\mu = \lambda(A) = \lambda_1(A) - \lambda_2(A) = \lambda_1(B) - \lambda_2(B) = \lambda(B) = \int_B f d\mu$. $\square$