# DSGE Problem Set

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# DSGE

## Exercise 1

$$
\begin{aligned}
K_{t+1} &= Ae^{z_t} K_t^{\alpha} \\
K_{t+2} &= Ae^{z_{t+1}} K_{t+1}^{\alpha} \\
&= Ae^{z_{t+1}} (Ae^{z_t} K_t^{\alpha})^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{e^{z_t} K_t^{\alpha} - Ae^{z_t} K_t^{\alpha}} &= \beta E_t \Big\{ \frac{\alpha e^{z_{t+1}} (Ae^{z_t} K_t^{\alpha})^{\alpha-1}}{e^{z_{t+1}} (Ae^{z_t} K_t^{\alpha})^{\alpha} - Ae^{z_{t+1}} (Ae^{z_t} K_t^{\alpha})^{\alpha}} \Big\} \\
\frac{1}{e^{z_t} K_t^{\alpha} (1-A)} &= \beta E_t \Big\{ \frac{\alpha}{Ae^{z_t} K_t^{\alpha} (1-A)} \Big\} \\
A &= \alpha\beta
\end{aligned}
$$

## Exercise 2

$$
\begin{aligned}
& c_t = (1-\tau)[w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \\
& c_t^{-1} = \beta E_t \{ c_{t+1}^{-1} [(r_{t+1} - \delta)(1-\tau) + 1] \} \\
& \frac{a}{1-\ell_t} = c_t^{-1} w_t (1-\tau) \\
& r_t = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} \\
& w_t = (1-\alpha) e^{z_t} k_t^{\alpha} \ell_t^{-\alpha} \\
& \tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \\
& z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)
\end{aligned}
$$

## Exercise 3

$$
\begin{aligned}
& c_t = (1-\tau)[w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \\
& c_t^{-\gamma} = \beta E_t \{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1-\tau) + 1] \} \\
& \frac{a}{1-\ell_t} = c_t^{-\gamma} w_t (1-\tau) \\
& r_t = \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha} \\
& w_t = (1-\alpha) e^{z_t} k_t^{\alpha} \ell_t^{-\alpha} \\
& \tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \\
& z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)
\end{aligned}
$$

**Exercise 4**

$$
\begin{aligned}
&c_t = (1-\tau)[w_t\ell_t + (r_t-\delta)k_t] + k_t + T_t - k_{t+1}\\
&c_t^{-\gamma} = \beta E_t\{c_{t+1}^{-\gamma}[(r_{t+1}-\delta)(1-\tau)+1]\}\\
&a(1-\ell_t)^{-\xi} = c_t^{-\gamma} w_t(1-\tau)\\
&r_t = \frac{\alpha e_{z_t} k_t^{\eta-1}}{[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{1/\eta}}\\
&w_t = \frac{(1-\alpha)e_{z_t}\ell_t^{\eta-1}}{[\alpha k_t^\eta + (1-\alpha)\ell_t^\eta]^{1/\eta}}\\
&\tau[w_t\ell_t + (r_t-\delta)k_t] = T_t\\
&z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0,\sigma_z^2)
\end{aligned}
$$

**Exercise 5**
Characterizing equations:

$$
\begin{aligned}
&c_t = (1-\tau)[w_t + (r_t-\delta)k_t] + k_t + T_t - k_{t+1}\\
&c_t^{-\gamma} = \beta E_t\{c_{t+1}^{-\gamma}[(r_{t+1}-\delta)(1-\tau)+1]\}\\
&0 = c_t^{-\gamma} w_t(1-\tau)\\
&r_t = \alpha k_t^{\alpha-1}(e^{z_t})^{1-\alpha}\\
&w_t = (1-\alpha)k_t^{\alpha}(e^{z_t})^{1-\alpha}\\
&\tau[w_t + (r_t-\delta)k_t] = T_t\\
&z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0,\sigma_z^2)
\end{aligned}
$$

Steady state:

$$
\begin{aligned}
&\bar{c} = (1-\tau)[\bar{w} + (\bar{r}-\delta)\bar{k}] + \bar{T}\\
&1 = \beta E_t[(\bar{r}-\delta)(1-\tau)+1]\\
&0 = \bar{c}^{-\gamma}\bar{w}(1-\tau)\\
&\bar{r} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha}\\
&\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha}\\
&\tau[\bar{w} + (\bar{r}-\delta)\bar{k}] = \bar{T}
\end{aligned}
$$

Solve for the steady state value of $k$:

$$
\begin{aligned}
&1 = \beta[(\bar{r}-\delta)(1-\tau)+1]\\
&1 = \beta[(\alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha}-\delta)(1-\tau)+1]\\
&1 = \beta(\alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha}-\delta)(1-\tau)+\beta\\
&\frac{1-\beta}{\beta(1-\tau)} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha}-\delta\\
&\bar{k} = \left(\frac{1}{\alpha(e^{\bar{z}})^{1-\alpha}}\left[\frac{1-\beta}{\beta(1-\tau)}+\delta\right]\right)^{\frac{1}{\alpha-1}}
\end{aligned}
$$

**Exercise 6**
Characterizing equations:

$$
\begin{aligned}
&c_t = (1-\tau)[w_t\ell_t + (r_t-\delta)k_t] + k_t + T_t - k_{t+1}\\
&c_t^{-\gamma} = \beta E_t\{c_{t+1}^{-\gamma}[(r_{t+1}-\delta)(1-\tau)+1]\}\\
&a(1-\ell_t)^{-\xi} = c_t^{-\gamma} w_t(1-\tau)\\
&r_t = \alpha k_t^{\alpha-1}(\ell_t e^{z_t})^{1-\alpha}\\
&w_t = (1-\alpha)k_t^{\alpha}(e^{z_t})^{1-\alpha}\ell_t^{-\alpha}\\
&\tau[w_t\ell_t + (r_t-\delta)k_t] = T_t\\
&z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0,\sigma_z^2)
\end{aligned}
$$

Steady state:

$$
\begin{aligned}
&\bar{c} = (1-\tau)[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}] + \bar{T}\\
&1 = \beta E_t[(\bar{r}-\delta)(1-\tau)+1]\\
&a(1-\bar{\ell})^{-\xi} = \bar{c}^{-\gamma}\bar{w}(1-\tau)\\
&\bar{r} = \alpha\bar{k}^{\alpha-1}(\bar{\ell}e^{\bar{z}})^{1-\alpha}\\
&\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha}\bar{\ell}^{-\alpha}\\
&\tau[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}] = \bar{T}
\end{aligned}
$$

## Linearization
**Exercise 3**

$$
\begin{aligned}
&E_t\{FX_{t+1} + GX_t + HX_{t-1} + LZ_{t+1} + MZ_t\} = 0\\
&E_t\{F(PX_t + QZ_{t+1}) + G(PX_{t-1} + QZ_t) + HX_{t-1} + L(NZ_t + \epsilon_t) + MZ_t\} = 0\\
&E_t\{FPX_t + FQZ_{t+1} + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + L\epsilon_t + MZ_t\} = 0\\
&E_t\{FP(PX_{t-1} + QZ_t) + FQ(NZ_t + \epsilon_t) + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + L\epsilon_t + MZ_t\} = 0\\
&E_t\{FP^2X_{t-1} + FPQZ_t + FQNZ_t + FQ\epsilon_t + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + L\epsilon_t + MZ_t\} = 0\\
&FP^2X_{t-1} + FPQZ_t + FQNZ_t + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + MZ_t = 0\\
&(FP^2 + GP + H)X_{t-1} + (FPQ + FQN + FQ + LN + M)Z_t = 0\\
&[(FP+G)P + H]X_{t-1} + [(FQ+L)N + (FP+G)Q + M]Z_t = 0
\end{aligned}
$$

Sorry that I didn't finish this problem set. I was really sick this week, and I decided to prioritize health and sleep. Thank you for three wonderful lectures, and again, I apologize for my unfinished work. -Reiko