Problem Set #5
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Exercise 8.1-8.2
See jupyter notebook

Exercise 8.3

$$\begin{aligned}
\text{maximize} \quad & 4b + 3j \\
\text{subject to} \quad & 15b + 10j \leq 1800 \\
& 2b + 2j \leq 300 \\
& j \leq 200 \\
& b, j \geq 0
\end{aligned}$$

Exercise 8.4

$$\begin{aligned}
\text{maximize} \quad & 2x_{AB} + 5x_{AD} + 5x_{BC} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 2x_{CF} + 4x_{DE} + 3x_{EF} \\
\text{subject to} \quad & x_{AB} + x_{AD} = 10 \\
& x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\
& x_{CF} - x_{BC} = -2 \\
& x_{DE} - x_{AD} - x_{BD} = -3 \\
& x_{EF} - x_{BE} - x_{DE} = 4 \\
& - x_{BF} - x_{CF} - x_{EF} = -10 \\
& 0 \leq x_{AB}, x_{AD}, x_{BC}, x_{BD}, x_{BE}, x_{BF}, x_{CF}, x_{DE}, x_{EF} \leq 6
\end{aligned}$$

Exercise 8.5
(i)

$$\begin{aligned}
\text{maximize} \quad & 3x_1 + x_2 \\
\text{subject to} \quad & x_1 + 3x_2 + w_1 = 15 \\
& 2x_1 + 3x_2 + w_2 = 18 \\
& x_1 - x_2 + w_3 = 4 \\
& x_1, x_2, w_1, w_2, w_3 \geq 0
\end{aligned}$$

$$
\begin{array}{rclrcrc}
\zeta & = & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2 \\
\hline\hline
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3 \\
\hline\hline
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3 \\
\end{array}
$$

Optimizer: $(6, 2)$
Optimum value: 20

(ii)

$$
\begin{aligned}
\text{maximize} \quad & 4x + 6y \\
\text{subject to} \quad & -x + 3x_2 + w_1 = 11 \\
& x + y + w_2 = 27 \\
& 2x + 5y + w_3 = 90 \\
& x, y, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{rclcccccc}
\zeta & = & & & 4x & + & 6y \\
\hline
w_1 & = & 11 & + & x & - & y \\
w_2 & = & 27 & - & x & - & y \\
w_3 & = & 90 & - & 2x & - & 5y \\
\hline
\zeta & = & 66 & + & 10x & - & 6w_1 \\
y & = & 11 & + & x & - & w_1 \\
w_2 & = & 16 & - & 2x & + & w_1 \\
w_3 & = & 35 & - & 7x & + & 5w_1 \\
\hline
\zeta & = & 116 & + & \frac{8}{7}w_1 & - & \frac{10}{7}w_3 \\
y & = & 16 & - & \frac{2}{7}w_1 & - & \frac{1}{7}w_3 \\
w_2 & = & 6 & - & \frac{3}{7}w_1 & + & \frac{2}{7}w_3 \\
x & = & 5 & + & \frac{5}{7}w_1 & - & \frac{1}{7}w_3 \\
\hline
\zeta & = & 132 & - & \frac{8}{3}w_2 & - & \frac{2}{7}w_3 \\
y & = & 12 & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \\
w_1 & = & 14 & - & \frac{7}{3}w_2 & + & \frac{2}{3}w_3 \\
x & = & 15 & - & \frac{5}{3}w_2 & + & \frac{1}{3}w_3 \\
\end{array}
$$

Optimizer: $(15, 12)$
Optimum value: 132

Exercise 8.6

$$
\begin{aligned}
\text{maximize} \quad & 4b + 3j \\
\text{subject to} \quad & 15b + 10j + w_1 = 1800 \\
& 2b + 2j + w_2 = 300 \\
& j + w_3 = 200 \\
& b, j, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{rclcrcr}
\zeta & = & & & 4b & + & 3j \\
\hline
w_1 & = & 1800 & - & 15b & - & 10j \\
w_2 & = & 300 & - & 2b & - & 2j \\
w_3 & = & 200 & - & j & & \\
\end{array}
$$

$$
\begin{array}{rclcrcr}
\zeta & = & 450 & + & b & - & \frac{3}{2}w_2 \\
\hline
w_1 & = & 300 & - & 5b & + & 5w_2 \\
j & = & 150 & - & b & - & \frac{1}{2}w_2 \\
w_3 & = & 50 & + & b & + & \frac{1}{2}w_2 \\
\end{array}
$$

$$
\begin{array}{rclcrcr}
\zeta & = & 510 & - & \frac{1}{5}w_1 & - & \frac{1}{2}w_2 \\
\hline
b & = & 60 & - & \frac{1}{5}w_1 & + & w_2 \\
j & = & 90 & + & \frac{1}{5}w_1 & - & \frac{3}{2}w_2 \\
w_3 & = & 110 & - & \frac{1}{5}w_1 & + & \frac{3}{2}w_2 \\
\end{array}
$$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls
Maximal profit: \$510

Exercise 8.7
(i)

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & -4x_1 - 2x_2 + x_3 = -8 \\
& -2x_1 + 3x_2 + x_4 = 6 \\
& x_1 + x_5 = 3 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Auxiliary problem:

$$
\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{subject to} \quad & -4x_1 - 2x_2 + x_3 - x_0 = -8 \\
& -2x_1 + 3x_2 + x_4 - x_0 = 6 \\
& x_1 + x_5 - x_0 = 3 \\
& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
x_5 & = & 3 & - & x_1 & & & + & x_0 \\
\hline
\zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & x_3 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & x_3 \\
x_4 & = & 14 & - & 2x_1 & - & 5x_2 & + & x_3 \\
x_5 & = & 11 & - & 5x_1 & - & 2x_2 & + & x_3 \\
\hline
\zeta & = & & & & & & - & x_0 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & - & \frac{1}{4}x_0 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & + & \frac{1}{2}x_0 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & + & \frac{5}{4}x_0 \\
\hline
\zeta & = & 2 & + & \frac{3}{2}x_2 & + & \frac{1}{4}x_3 & & \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & & \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & & \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & & \\
\hline
\zeta & = & 3 & + & 2x_2 & - & x_5 & & \\
\hline
x_1 & = & 3 & & & - & x_5 & & \\
x_4 & = & 12 & - & 3x_2 & - & 2x_5 & & \\
x_3 & = & 4 & + & 2x_2 & - & 4x_5 & & \\
\hline
\zeta & = & 11 & - & \frac{2}{3}x_4 & - & \frac{7}{3}x_5 & & \\
\hline
x_1 & = & 3 & & & - & x_5 & & \\
x_2 & = & 4 & - & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 & & \\
x_3 & = & 4 & - & \frac{2}{3}2x_4 & - & \frac{16}{3}x_5 & & \\
\end{array}
$$

Optimal point: $(3, 4)$
Optimal value: 11

(ii)

$$
\begin{aligned}
\text{maximize} \quad & 5x_1 + 2x_2 \\
\text{subject to} \quad & 5x_1 + 3x_2 + x_3 = 15 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& 4x_1 - 3x_2 + x_5 = -12 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Auxiliary problem:

$$\text{maximize} \quad -x_0$$
$$\text{subject to} \quad 5x_1 + 3x_2 + x_3 - x_0 = 15$$
$$3x_1 + 5x_2 + x_4 - x_0 = 15$$
$$4x_1 - 3x_2 + x_5 - x_0 = -12$$
$$x_0, x_1, x_2, x_3, x_4, x_5 \geq 0$$

ζ	$=$							$-$	x_0
x_3	$=$	15	$-$	$5x_1$	$-$	$3x_2$	$+$		x_0
x_4	$=$	15	$-$	$3x_1$	$-$	$5x_2$	$+$		x_0
x_5	$=$	-12	$-$	$4x_1$	$+$	$3x_2$	$+$		x_0
ζ	$=$	-12	$-$	$4x_1$	$+$	$3x_2$	$-$		x_5
x_3	$=$	27	$-$	x_1	$-$	$6x_2$	$+$		x_5
x_4	$=$	27	$+$	x_1	$-$	$8x_2$	$+$		x_5
x_0	$=$	12	$+$	$4x_1$	$-$	$3x_2$	$+$		x_5
ζ	$=$	$-\frac{15}{8}$	$-$	$\frac{29}{8}x_1$	$-$	$\frac{3}{8}x_4$	$-$		$\frac{5}{8}x_5$
x_3	$=$	$\frac{27}{4}$	$-$	$\frac{7}{4}x_1$	$+$	$\frac{3}{4}x_4$	$+$		$\frac{1}{4}x_5$
x_2	$=$	$\frac{27}{8}$	$+$	$\frac{1}{8}x_1$	$-$	$\frac{1}{8}x_4$	$+$		$\frac{1}{8}x_5$
x_0	$=$	$\frac{15}{8}$	$+$	$\frac{29}{8}x_1$	$+$	$\frac{3}{8}x_4$	$+$		$\frac{5}{8}x_5$

The original problem has no feasible solutions.

(iii)

$$\text{maximize} \quad -3x_1 + x_2$$
$$\text{subject to} \quad x_2 + x_3 = 4$$
$$-2x_1 + 3x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

ζ	$=$		$-$	$3x_1$	$+$	x_2
x_3	$=$	4			$-$	x_2
x_4	$=$	6	$+$	$2x_1$	$-$	$3x_2$
ζ	$=$	2	$-$	$\frac{7}{3}x_1$	$-$	$\frac{1}{3}x_4$
x_3	$=$	2	$-$	$\frac{2}{3}x_1$	$+$	$\frac{1}{3}x_4$
x_2	$=$	2	$+$	$\frac{2}{3}x_1$	$-$	$\frac{1}{3}x_4$

Optimal point: $(0, 2)$
Optimal value: 2

Exercise 8.8

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded, but where the objective function still has a unique feasible maximizer.

$$\text{maximize} \quad -x - y - z$$
$$\text{subject to} \quad x, y, z \geq 0$$

Maximizer: $(0, 0, 0)$

Exercise 8.9

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded and where the objective function has no maximizer.

$$\text{maximize} \quad x + y + z$$
$$\text{subject to} \quad x, y, z \geq 0$$

Exercise 8.10

Give an example of a three-dimensional linear problem where the feasible region is empty.

$$\text{maximize} \quad x + y + z$$
$$\text{subject to} \quad x + y + z \leq -1$$
$$x, y, z \geq 0$$

Exercise 8.11

Give an example of a three-dimensional linear problem where the feasible region is nonempty, closed, and unbounded, but **0** is not feasible.

$$\text{maximize} \quad x + y + z$$
$$\text{subject to} \quad x + y + z \geq 1$$
$$\text{subject to} \quad x + y + z \leq 3$$
$$x, y, z \geq 0$$

Auxiliary problem:

$$\text{maximize} \quad -w$$
$$\text{subject to} \quad -x - y - z - w \leq -1$$
$$\text{subject to} \quad x + y + z - w \leq 3$$
$$x, y, z, w \geq 0$$

Exercise 8.12

$$\text{maximize} \quad 10x_1 - 57x_2 - 9x_3 - 24x_4$$
$$\text{subject to} \quad 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0$$
$$0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0$$
$$x_1 + x_7 = 0$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

ζ	$=$			$10x_1$	$-$	$57x_2$	$-$	$9x_3$	$-$	$24x_4$
x_5	$=$		$-$	$0.5x_1$	$+$	$1.5x_2$	$+$	$0.5x_3$	$-$	x_4
x_6	$=$		$-$	$0.5x_1$	$+$	$5.5x_2$	$+$	$2.5x_3$	$-$	$9x_4$
x_7	$=$	1	$-$	x_1						
ζ	$=$		$-$	$27x_2$	$+$	x_3	$-$	$44x_4$	$-$	$20x_5$
x_1	$=$			$3x_2$	$+$	x_3	$-$	$2x_4$	$-$	$2x_5$
x_6	$=$			$4x_2$	$+$	$2x_3$	$-$	$8x_4$	$+$	x_5
x_7	$=$	1	$-$	$3x_2$	$-$	x_3	$+$	$2x_4$	$+$	$2x_5$
ζ	$=$	1	$-$	$30x_2$	$-$	$42x_4$	$-$	$18x_5$	$-$	x_7
x_1	$=$	1							$-$	x_7
x_6	$=$	2	$-$	$2x_2$	$-$	$4x_4$	$+$	$5x_5$	$-$	$2x_7$
x_3	$=$	1	$-$	$3x_2$	$+$	$2x_4$	$+$	$2x_5$	$-$	x_7

Optimal point: $(1, 0, 1, 0)$ Optimum value: 1

Exercise 8.15
If $\mathbf{x} \in \mathbb{R}^n$ is feasible for the primal and $\mathbf{y} \in \mathbb{R}^m$ is feasible for the dual, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof:
Let $\mathbf{x} \in \mathbb{R}^n$ be feasible for the primal and $\mathbf{y} \in \mathbb{R}^m$ be feasible for the dual. Then we know that $A\mathbf{x} \leq \mathbf{b}$ and $A^T\mathbf{y} \leq \mathbf{c}$, so

$$A\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x}^T A^T \leq \mathbf{b}^T$$
$$\mathbf{x}^T A^T \mathbf{y} \leq \mathbf{b}^T\mathbf{y}$$
$$\mathbf{x}^T \mathbf{c} \leq \mathbf{b}^T\mathbf{y}$$
$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

Exercise 8.17
For a linear optimization problem in standard form, the dual of the dual optimization problem is again the primal problem.

Proof:
Consider the primal problem:

$$\text{maximize} \ \ \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \ A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \succeq \mathbf{0}$$

with the dual problem:

$$\text{minimize} \ \ \mathbf{b}^T\mathbf{y}$$
$$\text{subject to} \ A^T\mathbf{y} \succeq \mathbf{c}$$
$$\mathbf{y} \succeq \mathbf{0}$$

The dual in standard form becomes:

$$\text{maximize} \ \ (-\mathbf{b}^T)\mathbf{y}$$
$$\text{subject to} \ (-A)^T\mathbf{y} \preceq -\mathbf{c}$$
$$\mathbf{y} \succeq \mathbf{0}$$

and the dual of the dual is:

$$\text{minimize} \ \ (-\mathbf{c}^T)\mathbf{x}$$
$$\text{subject to} \ ((-A)^T)^T\mathbf{x} \succeq -\mathbf{b}$$
$$\mathbf{x} \succeq \mathbf{0}$$

In standard form, this is:

$$\text{maximize} \ \ \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \ A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \succeq \mathbf{0}$$

which is the original primal problem.

Exercise 8.18
Primal:

$$\text{maximize} \ \ x_1 + x_2$$
$$\text{subject to} \ \ 2x_1 + x_2 + w_1 = 3$$
$$x_1 + 3x_2 + w_2 = 5$$
$$2x_1 + 3x_2 + w_3 = 4$$
$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

ζ	$=$		x_1	$+$	x_2
w_1	$=$	3	$- \ 2x_1$	$-$	x_2
x_2	$=$	5	$- \ x_1$	$-$	$3x_2$
x_5	$=$	4	$- \ 2x_1$	$-$	$3x_2$
ζ	$=$	$\frac{3}{2}$	$+ \ \frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
x_1	$=$	$\frac{3}{2}$	$- \ \frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
w_2	$=$	$\frac{7}{2}$	$- \ \frac{5}{2}x_2$	$+$	$\frac{1}{2}w_1$
w_3	$=$	1	$- \ 2x_2$	$+$	w_1
ζ	$=$	$\frac{7}{4}$	$- \ \frac{1}{4}w_1$	$-$	$\frac{1}{4}w_1$
x_1	$=$	$\frac{5}{4}$	$- \ \frac{3}{4}w_1$	$+$	$\frac{1}{4}w_3$
w_2	$=$	$\frac{9}{4}$	$- \ \frac{3}{4}w_1$	$+$	$\frac{5}{4}w_3$
x_2	$=$	$\frac{1}{2}$	$+ \ \frac{1}{2}w_1$	$-$	$\frac{1}{2}w_3$

Optimal point: $\left(\frac{5}{4}, \frac{1}{2}\right)$

Optimum value: $\frac{7}{4}$

Dual:

$$
\begin{aligned}
\text{minimize}\quad & 3y_1 + 5y_2 + 4y_3 \\
\text{subject to}\quad & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Dual in standard form:

$$
\begin{aligned}
\text{maximize}\quad & -3y_1 - 5y_2 - 4y_3 \\
\text{subject to}\quad & -2y_1 - y_2 - 2y_3 + v_1 - v_0 = -1 \\
& -y_1 - 3y_2 - 3y_3 + v_2 - v_0 = -1 \\
& y_1, y_2, y_3, v_1, v_2 \geq 0
\end{aligned}
$$

ζ	$=$								$-$	v_0
v_1	$=$	-1	$+$	$2y_1$	$+$	y_2	$+$	$2y_3$	$+$	v_0
v_2	$=$	-1	$+$	y_1	$+$	$3y_2$	$+$	$3y_3$	$+$	v_0
ζ	$=$	-1	$+$	$2y_1$	$+$	y_2	$+$	$2y_3$	$-$	v_1
v_0	$=$	1	$-$	$2y_1$	$-$	y_2	$-$	$2y_3$	$+$	v_1
v_2	$=$		$-$	y_1	$+$	$2y_2$	$+$	y_3	$+$	v_1
ζ	$=$								$-$	v_0
y_2	$=$	1	$-$	$2y_1$	$-$	$2y_3$	$+$	v_1	$-$	v_0
v_2	$=$	2	$-$	$5y_1$	$-$	$3y_3$	$+$	$3v_1$	$-$	$2v_0$
ζ	$=$	-2	$+$	y_1	$-$	$3y_2$	$-$	$2v_1$		
y_3	$=$	$\frac{1}{2}$	$-$	y_1	$-$	$\frac{1}{2}y_2$	$+$	$\frac{1}{2}v_1$		
v_2	$=$	$\frac{1}{2}$	$-$	$2y_1$	$+$	$\frac{3}{2}y_2$	$+$	$\frac{3}{2}v_1$		
ζ	$=$	$-\frac{7}{4}$	$-$	$\frac{3}{2}y_2$	$-$	$\frac{5}{4}v_1$	$-$	$\frac{1}{2}v_2$		
y_3	$=$	$\frac{1}{4}$	$-$	$2\frac{3}{2}y_2$	$-$	$\frac{1}{4}v_1$	$+$	$\frac{1}{2}v_2$		
y_1	$=$	$\frac{1}{4}$	$+$	$\frac{3}{2}y_2$	$+$	$\frac{3}{4}v_1$	$-$	$\frac{1}{2}v_2$		

Optimal point: $\left(\frac{1}{4}, 0, \frac{1}{4}\right)$

Optimum value: $\frac{7}{4}$