Problem Set #2
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Exercise 1
(i) *Proof:*

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2) \\
&= \frac{1}{4}(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle) \\
&= \frac{1}{4}(\langle \mathbf{x}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x}+\mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{x}-\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x}-\mathbf{y} \rangle) \\
&= \frac{1}{4}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle) \\
&= \frac{1}{4}(\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle) \\
&= \frac{1}{4}(2\langle \mathbf{x}, \mathbf{y} \rangle + 2\langle \mathbf{y}, \mathbf{x} \rangle) \\
&= \frac{1}{4}(4\langle \mathbf{x}, \mathbf{y} \rangle) \\
&= 4\langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

(ii) *Proof:*

$$\begin{aligned}
||\mathbf{x}||^2 + ||\mathbf{y}||^2 &= \frac{1}{2}(||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2) \\
&= \frac{1}{2}(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{x}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle) \\
&= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2
\end{aligned}$$

Exercise 2
Proof:

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{y}\rangle &= \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2 + i||\mathbf{x}-i\mathbf{y}||^2 - i||\mathbf{x}+i\mathbf{y}||^2) \\
&= \frac{1}{4}(||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2) + \frac{1}{4}(i||\mathbf{x}-i\mathbf{y}||^2 - i||\mathbf{x}+i\mathbf{y}||^2) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(i||\mathbf{x}-i\mathbf{y}||^2 - i||\mathbf{x}+i\mathbf{y}||^2) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(i\langle \mathbf{x}-i\mathbf{y}, \mathbf{x}-i\mathbf{y}\rangle - i\langle \mathbf{x}+i\mathbf{y}, \mathbf{x}+i\mathbf{y}\rangle) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(i\langle \mathbf{x}, \mathbf{x}-i\mathbf{y}\rangle + i^2\langle \mathbf{y}, \mathbf{x}-i\mathbf{y}\rangle - i\langle \mathbf{x}, \mathbf{x}+i\mathbf{y}\rangle + (-i)^2\langle \mathbf{y}, \mathbf{x}+i\mathbf{y}\rangle) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(i\langle \mathbf{x}, \mathbf{x}\rangle + i^2\langle \mathbf{x}, \mathbf{y}\rangle + i^2\langle \mathbf{y}, \mathbf{x}\rangle + i^3\langle \mathbf{y}, \mathbf{y}\rangle \\
&\qquad - i\langle \mathbf{x}, \mathbf{x}\rangle + (-i)^2\langle \mathbf{x}, \mathbf{y}\rangle + (-i)^2\langle \mathbf{y}, \mathbf{x}\rangle + (-i^3)\langle \mathbf{y}, \mathbf{y}\rangle) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(i^2\langle \mathbf{x}, \mathbf{y}\rangle + i^2\langle \mathbf{y}, \mathbf{x}\rangle + (-i)^2\langle \mathbf{x}, \mathbf{y}\rangle + (-i)^2\langle \mathbf{y}, \mathbf{x}\rangle) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}(-\langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle) \\
&= \langle \mathbf{x}, \mathbf{y}\rangle
\end{aligned}
$$

Exercise 3
(i) $\cos\ \theta = \frac{\langle x, x^5\rangle}{||x||\cdot||x^5||}$
$\langle x, x^5\rangle = \int_0^1 x \cdot x^5\ dx = \frac{1}{7}$
$\langle x, x\rangle = \int_0^1 x \cdot x\ dx = \frac{1}{3} \implies ||x|| = \frac{1}{\sqrt{3}}$
$\langle x^5, x^5\rangle = \int_0^1 x^5 \cdot x^5\ dx = \frac{1}{11} \implies ||x^5|| = \frac{1}{\sqrt{11}}$
$\implies \theta = \cos^{-1}(\frac{\sqrt{33}}{7})$

(ii) $\cos\ \theta = \frac{\langle x^2, x^4\rangle}{||x^2||\cdot||x^4||}$
$\langle x^2, x^4\rangle = \int_0^1 x^2 \cdot x^4\ dx = \frac{1}{7}$
$\langle x^2, x^2\rangle = \int_0^1 x^2 \cdot x^2\ dx = \frac{1}{5} \implies ||x^2|| = \frac{1}{\sqrt{5}}$
$\langle x^4, x^4\rangle = \int_0^1 x^4 \cdot x^4\ dx = \frac{1}{9} \implies ||x^4|| = \frac{1}{3}$
$\implies \theta = \cos^{-1}(\frac{3\sqrt{5}}{7})$

Exercise 8
(i) $\langle \cos t, \sin t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \sin t\ dt = 0$
$\langle \cos t, \cos 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \cos 2t\ dt = 0$
$\langle \cos t, \sin 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \sin 2t\ dt = 0$
$\langle \sin t, \cos 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin t \cos 2t\ dt = 0$
$\langle \sin t, \sin 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \sin 2t\ dt = 0$
$\langle \cos 2t, \sin 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos 2t \sin 2t\ dt = 0$

$\langle \cos t, \cos t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \cos t\ dt = 1$
$\langle \sin t, \sin t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin t \sin t\ dt = 1$
$\langle \cos 2t, \cos 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos 2t \cos 2t\ dt = 1$
$\langle \sin 2t, \sin 2t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin 2t \sin 2t\ dt = 1$

(ii) $||t|| = \langle t, t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \; dt = \frac{2\pi^2}{3}$

(iii)

$$\begin{aligned}
proj_X(\cos 3t) &= \langle \cos t, \cos 3t\rangle \cdot \cos t + \langle \sin t, \cos 3t\rangle \cdot \sin t \\
&\quad + \langle \cos 2t, \cos 3t\rangle \cdot \cos 2t + \langle \sin 2t, \cos 3t\rangle \cdot \sin 2t \\
&= \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \cos t \cos 3t \; dt\right) \cdot \cos t + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \sin t \cos 3t \; dt\right) \cdot \sin t \\
&\quad + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \cos 2t \cos 3t \; dt\right) \cdot \cos 2t + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \sin 2t \cos 3t \; dt\right) \cdot \sin 2t \\
&= 0
\end{aligned}$$

(iv)

$$\begin{aligned}
proj_X(t) &= \langle \cos t, t\rangle \cdot \cos t + \langle \sin t, t\rangle \cdot \sin t \\
&\quad + \langle \cos 2t, t\rangle \cdot \cos 2t + \langle \sin 2t, t\rangle \cdot \sin 2t \\
&= \left(\frac{1}{\pi}\int_{-\pi}^{\pi} (\cos t) \; t \; dt\right) \cdot \cos t + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} (\sin t) \; t \; dt\right) \cdot \sin t \\
&\quad + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} (\cos 2t) \; t \; dt\right) \cdot \cos 2t + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} (\sin 2t) \; t \; dt\right) \cdot \sin 2t \\
&= 2\sin t - \sin(2t)
\end{aligned}$$

Exercise 9

The rotation matrix R_θ is orthonormal since $R_\theta R_\theta^H = I$:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 10

(i) $Q \in M_n(\mathbb{F}^n)$ is an orthonormal matrix if and only if $Q^H Q = QQ^H = I$.

Proof:

Let Q be an orthonormal matrix. Then we have that $\langle \mathbf{x}, \mathbf{y}\rangle = \langle Q\mathbf{x}, Q\mathbf{y}\rangle = (Q\mathbf{x})^H Q\mathbf{y} = \mathbf{x}^H Q^H Q\mathbf{y}$. Since we know that $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^H\mathbf{y}$, it must be that $Q^H Q = I$. Also, $QQ^H = I$ since Q is invertible.

Now suppose that for a matrix Q, we have that $Q^H Q = QQ^H = I$. Consider the inner product $\langle Q\mathbf{x}, Q\mathbf{y}\rangle = (Q\mathbf{x})^H Q\mathbf{y} = \mathbf{x}^H Q^H Q\mathbf{y} = \mathbf{x}^H\mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle$. Therefore, Q must be an orthonormal matrix.

(ii) If $Q \in M_n(\mathbb{F})$ is an orthonormal matrix, then $||Q\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{F}^n$.

Proof:

$||Q\mathbf{x}|| = \sqrt{\langle Q\mathbf{x}, Q\mathbf{y}\rangle} = \sqrt{\langle \mathbf{x}, \mathbf{y}\rangle} = ||\mathbf{x}||$.

(iii) If $Q \in M_n(\mathbb{F}^n)$ is an orthonormal matrix, then so is Q^{-1}.

Proof:

Assume Q is an orthonormal matrix. Then we know that $Q^H Q = QQ^H = I$, so $Q^{-1} = Q^H$. To prove that Q^H is orthonormal, we can show that $(Q^H)^H Q^H = Q^H (Q^H)^H = I$.

(iv) The columns of an orthonormal matrix $Q \in M_n(\mathbb{F}^n)$ are orthonormal.
Proof:
Assume that Q is orthonormal. Consider standard basis vectors $e_i, e_j \in \mathbb{F}^n$. The ith column of Q is $Q\mathbf{e_i}$. Since Q is orthonormal, we have that $\langle Q\mathbf{e_i}, Q\mathbf{e_j}\rangle = \langle e_i, e_j\rangle = \delta_{ij}$. Therefore the columns of Q must be orthonormal.

(v) If $Q \in M_n(\mathbb{F}^n)$ is an orthonormal matrix, then $|det(Q)| = 1$. *Proof:*
Since Q is orthonormal, we know that $QQ^H = I$. Then $det(QQ^H) = det(I) = 1$. By the properties of determinants, we have that $det(QQ^H) = det(Q) \cdot det(Q^H) = 1$. Using the fact that Q and Q^H have the same determinants, we conclude that $|det(Q)| = 1$.
No. Consider the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(vi) If $Q_1, Q_2 \in M_n(\mathbb{F}^n)$ are orthonormal matrices, then the product $Q_1 Q_2$ is also an orthonormal matrix.
Proof:
$(Q_1 Q_2)^H Q_1 Q_2 = Q_2^H Q_1^H Q_1 Q_2 = I$
$Q_1 Q_2 (Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = I$.
Therefore $Q_1 Q_2$ is an orthonormal matrix.

Exercise 11
Let $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$ be a linearly dependent set. Assume that the vector $\mathbf{x}_k$ is linearly dependent on $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{k-1}\}$, i.e. $\mathbf{x}_k \in X = Span(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{k-1}\})$. When we apply the Gram-Schmidt orthonormalization process, we have that $\mathbf{p}_{k-1} = proj_X(\mathbf{x}_k) = \mathbf{x}_k \implies \mathbf{q}_k = \mathbf{0}$. If we remove all the zero vectors from the set, we have an orthonormal set of linearly dependent vectors.

Exercise 16
(i) Let $Q_1 R_1$ and $Q_2 R_2$ be distinct QR decompositions of a matrix A s.t. $Q_2 = Q_1 D$ and $R_2 = D^{-1} R_2$ where D is a diagonal matrix with all its diagonal entries ± 1. Then Q_2 is orthonormal since Q_1 is orthonormal. Also, R_2 is upper triangular since R_1 is upper triangular.

(ii) Assume that $A = Q_1 R_1 = Q_2 R_2$ s.t. Q_i is orthonormal and R_i is upper triangular with only positive diagonal entries. Thus R_i is invertible and $Q_i Q_i^H = I$. Then we have that $R_1 R_2^{-1} = Q_1^H Q_2$. Since the R_i are upper triangular with only positive entries on the diagonal, $R_1 R_2^{-1}$ must be upper triangular with only positive entries on the diagonal as well. Also, since the Q_i are orthonormal, $Q_1^H Q_2$ must be orthonormal. Then, $R_1 R_2^{-1} = Q_1^H Q_2 = I$, so $R_1 = R_2$ and $Q_1 = Q_2$.

Exercise 17
Let $A = \hat{Q}\hat{R}$.
$A^H A\mathbf{x} = A^H\mathbf{b}$
$\implies (\hat{Q}\hat{R})^H\hat{Q}\hat{R}\mathbf{x} = (\hat{Q}\hat{R})^H\mathbf{b}$
$\implies \hat{R}^H\hat{Q}^H\hat{Q}\hat{R}\mathbf{x} = \hat{R}^H\hat{Q}^H\mathbf{b}$
$\implies \hat{Q}\hat{R}\mathbf{x} = \mathbf{b}$
$\implies \hat{R}\mathbf{x} = \hat{Q}^H\mathbf{b}$

Exercise 23
By the triangle inequality,

$$||\mathbf{x}|| = ||(\mathbf{x}-\mathbf{y})+\mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}|| \Leftrightarrow ||\mathbf{x}|| - ||\mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}||.$$

Similarly and by scale preservation,

$$||\mathbf{y}|| = |-1|\cdot||\mathbf{y}|| = ||-\mathbf{y}|| = ||(\mathbf{x}-\mathbf{y})+\mathbf{x}|| \leq ||\mathbf{x}-\mathbf{y}||+||\mathbf{y}|| \Leftrightarrow ||\mathbf{y}||-||\mathbf{x}|| \leq ||\mathbf{x}-\mathbf{y}||$$
$$\implies |||\mathbf{x}|| - ||\mathbf{y}||| \leq ||\mathbf{x}-\mathbf{y}||$$

Exercise 24
(i) $||f||_{L^1} = \int_a^b |f(t)|dt$
1. Positivity: $|f(t)| \geq 0$ for all $x \implies \int_a^b |f(t)|dt \geq 0$.
Also, $\int_a^b |f(t)|dt = 0$ if and only if $|f(t)| = 0$.

2. Scale preservation: $||af||_{L^1} = \int_a^b |af(t)|dt = |a|\int_a^b |f(t)|dt = |a|||f||_{L^1}$

3. Triangle inequality: Consider $f, g \in C[a,b]$.

$$\begin{aligned} ||f+g||_{L^1} &= \int_a^b |f(t)+g(t)|dt \\ &\leq \int_a^b (|f(t)|+|g(t)|)dt \\ &= \int_a^b |f(t)|dt + \int_a^b |g(t)|dt \\ &= ||f||_{L^1} + ||g||_{L^1} \end{aligned}$$

(ii) $||f||_{L^2} = (\int_a^b |f(t)|^2 dt)^{\frac{1}{2}}$
1. Positivity: $|f(t)|^2 \geq 0$ for all $x \implies (\int_a^b |f(t)|^2 dt)^{\frac{1}{2}} \geq 0$.
Also, $(\int_a^b |f(t)|^2 dt)^{\frac{1}{2}} = 0$ if and only if $|f(t)| = 0$.

2. Scale preservation: $||af||_{L^2} = (\int_a^b |af(t)|^2 dt)^{\frac{1}{2}} = |a|(\int_a^b |f(t)|^2 dt)^{\frac{1}{2}} = |a|||f||_{L^2}$

3. Triangle inequality: Consider $f, g \in C[a, b]$.

$$\begin{aligned}
||f+g|| &= \Big(\int_a^b |f(t)+g(t)|^2 dt\Big)^{\frac{1}{2}} \\
&\leq \Big(\int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt\Big)^{\frac{1}{2}} \\
&\leq ||f||_{L^2} + ||g||_{L^2}
\end{aligned}$$

(iii) $||f||_{L^\infty} = \sup_{x\in[a,b]|f(x)|}$
1. Positivity: $|f(x)| \geq 0$ for all $x \implies \sup_{x\in[a,b]} |f(x)| \geq 0$.
Also, $\sup_{x\in[a,b]} |f(x)| = 0$ if and only if $|f(x)| = 0$.

2. Scale preservation: $||af||_{L^\infty} = \sup_{x\in[a,b]} |a||f(x)| = |a| \sup_{x\in[a,b]} |f(x)| = |a|||f||_{L^\infty}$

3. Triangle inequality: For any $x \in [a, b]$

$$\begin{aligned}
|f(x)+g(x)| &\leq |f(x)| + |g(x)| \\
&\leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| \\
&= ||f||_{L^\infty} + ||g||_{L^\infty} \\
&\implies \sup_{x\in[a,b]} |f(x)+g(x)| = ||f+g||_{L^\infty} \leq ||f||_{L^\infty} + ||g||_{L^\infty}
\end{aligned}$$

Exercise 26
Prove that topological equivalence is an equivalence relation. We check that it satisfies the three conditions:

1. Reflexivity: $||\cdot||_a \sim ||\cdot||_a$
We see that $m||\mathbf{x}||_a \leq ||\mathbf{x}||_a \leq M||\mathbf{x}||_a$ holds for a choice of constants $m = 1$ and $M = 2$.

2. Symmetry: $||\cdot||_a \sim ||\cdot||_b \implies ||\cdot||_b \sim ||\cdot||_a$
If $m||\mathbf{x}||_a \leq ||\mathbf{x}||_b \leq M||\mathbf{x}||_a$, then the relation holds for $\frac{1}{M}||\mathbf{x}||_b \leq ||\mathbf{x}||_a \leq \frac{1}{m}||\mathbf{x}||_b$.

3. Transitivity: $||\cdot||_a \sim ||\cdot||_b$ and $||\cdot||_b \sim ||\cdot||_c \implies ||\cdot||_a \sim ||\cdot||_c$
If $m_1||\mathbf{x}||_a \leq ||\mathbf{x}||_b \leq M_1||\mathbf{x}||_a$ and $m_2||\mathbf{x}||_b \leq ||\mathbf{x}||_c \leq M_2||\mathbf{x}||_b$, then the relation holds for $m_1m_2||\mathbf{x}||_a \leq ||\mathbf{x}||_c \leq M_1M_2||\mathbf{x}||_a$.

(i) Show that $||\mathbf{x}||_2 \leq ||\mathbf{x}||_1 \leq \sqrt{n}||\mathbf{x}||_2$.

$$||\mathbf{x}||_2 \leq ||\mathbf{x}||_1 \leq \sqrt{n}||\mathbf{x}||_2$$

$$\Big(\sum_{j=1}^n |x_j|^2\Big)^{\frac{1}{2}} \leq \sum_{j=1}^n |x_j| \leq \sqrt{n}\Big(\sum_{j=1}^n |x_j|^2\Big)^{\frac{1}{2}}$$

(ii) Show that $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2 \leq \sqrt{n}||\mathbf{x}||_\infty$.

Exercise 28

Exercise 29

Exercise 30

Exercise 37
We define $\mathcal{S} = 1, x, x^2$ to be the basis of the space V. Then we can evaluate L on the basis vectors: $L[1] = 0, L[x] = 1, L[x^2] = 2$. Every function $p \in V$ can be written as a linear combination of the basis vectors:

$$p = a_1 + a_2 x + a_3 x^2 \implies L[p] = a_1 L[1] + a_2 L[x] + a_3 L[x^2]$$

$$L[p] = \begin{bmatrix} L[1] & L[x] & L[x^2] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \langle (0, 1, 2), x \rangle$$

Exercise 38
We again define $\mathcal{S} = 1, x, x^2$ to be the basis of the space V, and we evaluate D on the basis vectors: $D[1](x) = 0, D[x](x) = 1, D[x^2](x) = 2x$. The matrix representation of D with respect to this basis is

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Using intergration by parts,

$$\langle q, D[p] \rangle = \int_{-\infty}^{\infty} q(x) p'(x) dx = -\int_{-\infty}^{\infty} q'(x) p(x) dx = -\langle D[q], [p] \rangle.$$

Therefore the matrix representation of the adjoint of D with respect to the basis is

$$D^* = -D = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 39

Exercise 40

Exercise 44

Exercise 45

Exercise 46

Exercise 47
Let $P = A(A^H A)^{-1} A^H$
(i) $P^2 = P$
Proof:

$$\begin{aligned} P^2 &= A(A^H A)^{-1} A^H A (A^H A)^{-1} A^H \\ &= A I_n (A^H A)^{-1} A^H \\ &= A(A^H A)^{-1} A^H \end{aligned}$$

(ii) $P^H = P$
Proof:

$$\begin{aligned} P^H &= (A(A^H A)^{-1} A^H)^H \\ &= A((A^H A)^{-1})^H A^H \\ &= A((A^H A)^H)^{-1} A^H \\ &= A(A^H A)^{-1} A^H \end{aligned}$$

(iii) $rank(P) = n$
Exercise 48

Exercise 50