Assignment4: Image Compression

Linear Algebra

Due: December 23, 2024

Introduction

One of the image compression method is using SVD (Singular Value Decomposition). It treats an image as an $m \times n$ matrix A, and performs SVD on A. Assume $m > n$,

$$A = U\Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T,$$

where $\sigma_i > \sigma_{i+1}$ for $i = 1, 2, \ldots n-1$. The compression process only keeps the k largest singular values and their corresponding singular vectors

$$\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^T = \sum_{i=1}^{k} u_i \sigma_i v_i^T.$$

Figure 1 shows an example. The original picture is of size 267x189, as shown in 1.(a). After the compression process performs SVD, the new image only uses 60 singular values and singular vectors to reconstruct the image. The compressed image is shown in 1.(b). The compression ratio is

$$\rho = \frac{267 \times 50 + 50 + 50 \times 189}{267 \times 189} = 0.45,$$

because we only need to stored the trunked $\tilde{U}$, $\tilde{\Sigma}$, and $\tilde{V}$. The pixel-wise difference between the original image and the compressed image is given in 1.(c). If we compute the average relative difference, which is

$$\delta = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|A_{i,j} - \tilde{A}_{i,j}|}{A_{i,j} + \epsilon}, \epsilon = 10^{-8}$$

we get 0.041, which is about 4%. The result shows that we can store the image with some loss using only half of the storage. Figure 1.(d) plots the singular values of A, the matrix of the original picture.

Assignments

1. (40%) Find five different styled pictures whose sizes are between 256x256 to 1024x1024, and perform SVD compression on them.
2. (30%) For each picture, discuss the relation of compression ratio ρ and the average relative difference δ. For example, what δ would be if we want $\rho < 1$ or $\rho < 0.5$, or what ρ is if we want to keep $\delta < 5\%$.
3. (30%)
   (a) What kind of pictures can be compressed easily by SVD (smaller relative difference)? And why?
   (b) which means they have low ρ and δ? Try to find its relation with the distribution of singular values of the original pictures.
   (c) Which part of the pictures have the largest difference of the compressed images? And why?

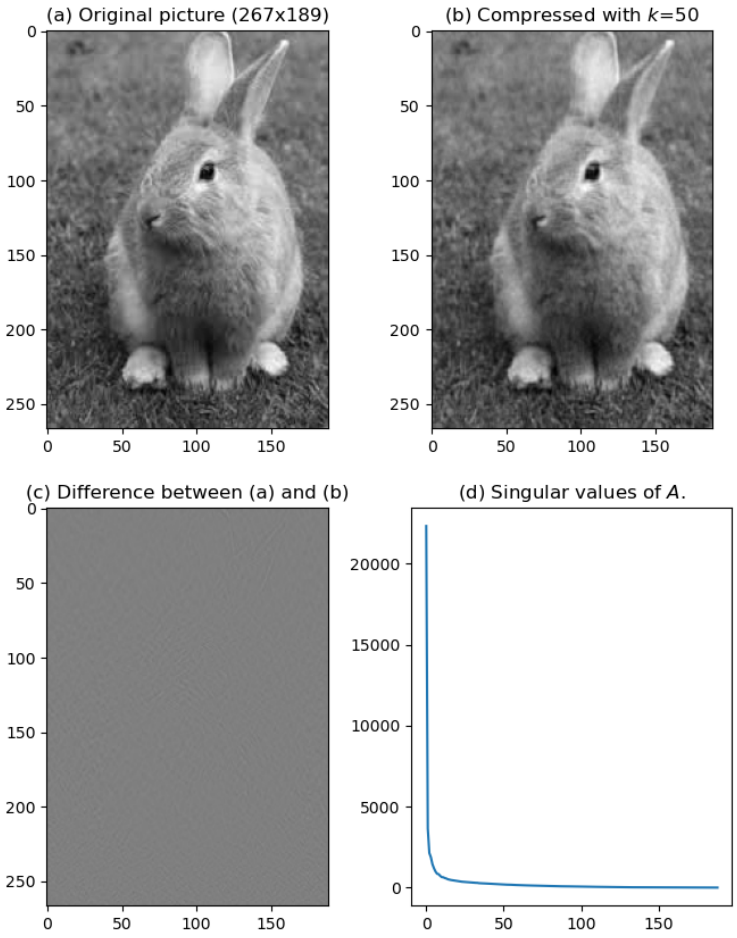

Figure 1: Bunny

Submission

Write a report in PDF file that includes your pictures, compressed results, singular values of the pictures, pixel-wise differences, and the discussions of question 2 and 3.

You need to submit:

1. HW4_studentID.ipynb
2. HW4_studentID.pdf