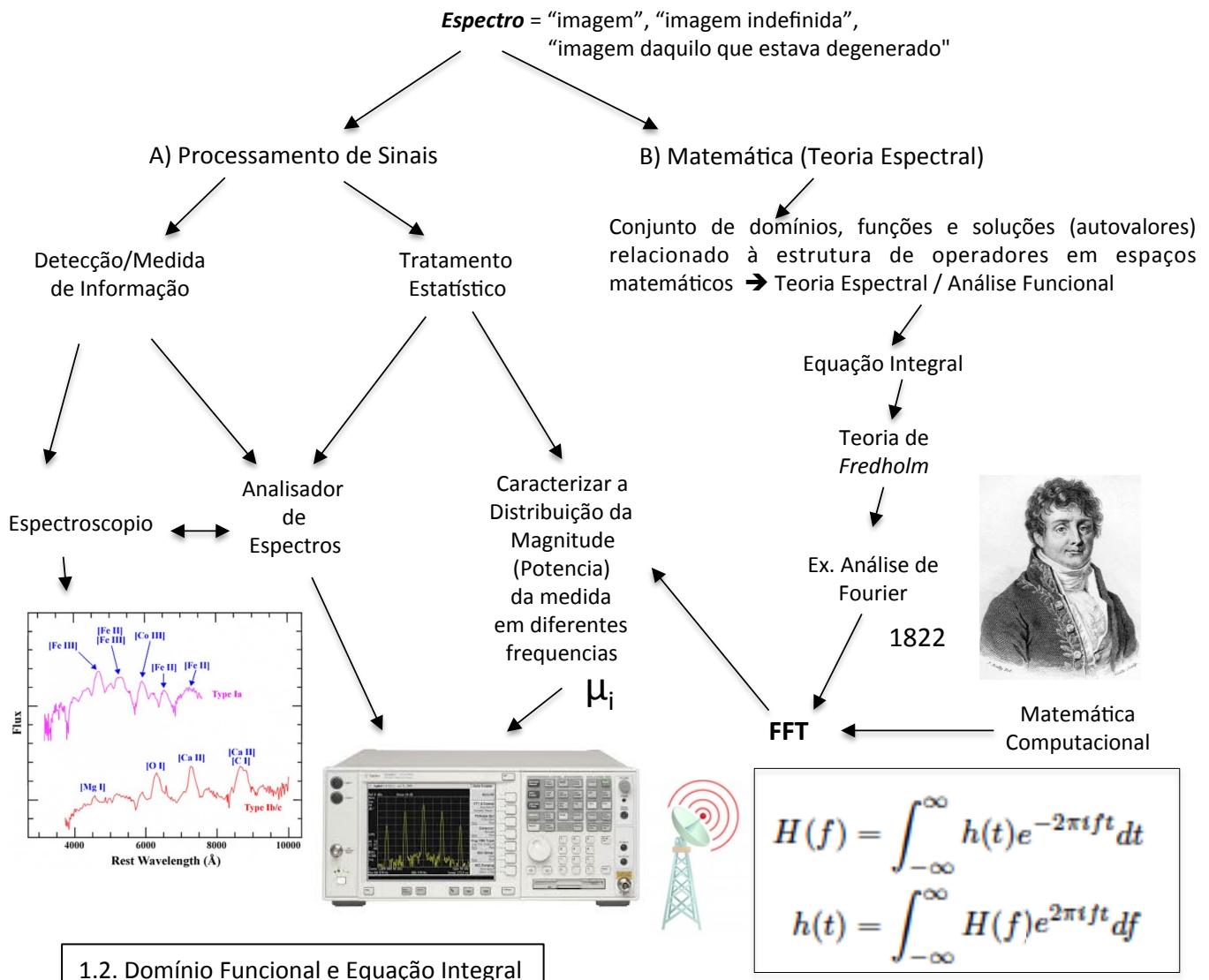


# CAP 239: A ALGEBRA DA ANÁLISE ESPECTRAL



Espaço de Hilbert (1895)  
 $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3$



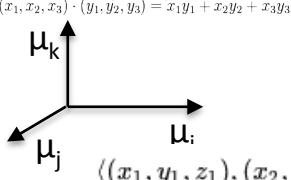
Equações Integrais de I. Fredholm (1903)



Operadores Integrais de V. Volterra (1906)



Transformadas Integrais de T. Lalescu (1908)



$$H_{\mu j} [h(\mu_i)] = h_{\mu i} + \lambda \int_a^{\mu_i} K(\mu_i, \mu_j) h(\mu_i) d\mu_i$$

$$\|\mu\| = \sqrt{\langle \mu, \mu \rangle}$$

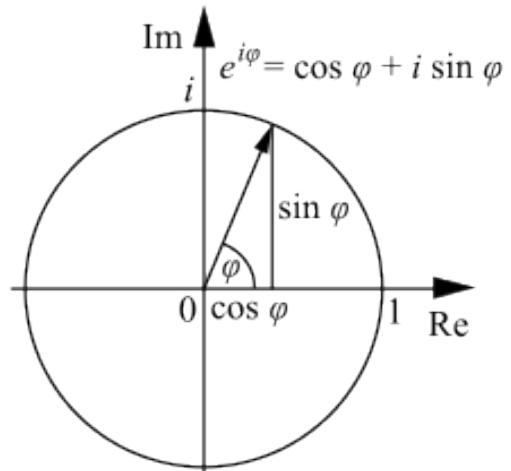
Transformada	Símbolo	Núcleo da transformada	$t_1$	$t_2$
Transformada de Fourier	$\mathcal{F}$	$\frac{e^{iut}}{\sqrt{2\pi}}$	$-\infty$	$\infty$
Transformada de Mellin	$\mathcal{M}$	$t^{u-1}$	0	$\infty$
Transformada de Laplace	$\mathcal{L}$	$e^{-ut}$	0	$\infty$
Transformada de Hankel	$\mathcal{K}$	$t J_\nu(ut)$	0	$\infty$
Transformada de Abel	$\mathcal{A}$	$\frac{t}{\sqrt{t^2 - u^2}}$	$u$	$\infty$
Transformada de Hilbert	$\mathcal{H}$	$\frac{1}{\pi} \frac{1}{u - t}$	$-\infty$	$\infty$
Transformada de cosseno	$\mathcal{C}$	$2\cos(ut)$	0	$\infty$
Transformada de wavelet	$\mathcal{W}$	$\left[ \frac{1}{\sqrt{s}} w\left(\frac{t-u}{s}\right) \right]^*$	$-\infty$	$\infty$

## Núcleo Complexo

$$\mathcal{F}(\delta(t-a)) = \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} dt = e^{-i\omega a}$$

Visite o site para descrição fasorial da FFT:  
<http://toxicdump.org/stuff/FourierToy.swf> e  
 aproxime uma onda quadrada de um  
 triângulo alterando a fase e a velocidade.

Copie os padrões inicial e o final e anote  
 as amplitudes dos fasores correspondentes.  
 Crie experimentos com os padrões de  
 pulso e randômico.



A Série de Fourier pode ser escrita nas suas formas *Harmonica* e *Complexa*, respectivamente, como:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos(w_n t - \theta_n)$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i w_n t}$$

## TRANFORMADAS DE FOURIER E ESPECTROS DE POTÊNCIA

Nos referimos à TF no plural pois trata-se de duas representações diferentes para uma mesma estrutura funcional. Além disso, trata-se de uma estrutura que pode ser apresentada na sua forma contínua e também na sua forma discreta (Transformadas Discretas de Fourier). O tratamento contínuo nos leva à um importante conjunto de propriedades e teoremas os quais formam a base do Espectro de Potências de um sinal  $A(\mu)$  ou  $H(t)$  que pode ser tratado através dessas transformadas.

Estrutura no Domínio Contínuo:

$$h(t) \longleftrightarrow H(f) \text{ ou } H(w)$$

Estrutura no Domínio Discreto:

$$H(n) \longleftrightarrow h(k)$$

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt \\ h(t) &= \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df \end{aligned}$$

rad/seg	ciclos/seg
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$\omega \equiv 2\pi f$

$H(\omega) \equiv [H(f)]_{f=\omega/2\pi}$

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \\ h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega \end{aligned}$$

$$\begin{aligned} h_k &= \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n / N} \\ H_n &\equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n / N} \end{aligned}$$

## PROPRIEDADES E TEOREMAS IMPORTANTES:

(1) TF é, algebricamente, uma Operação Linear:  $\mathcal{F}\{f + g\} = \mathcal{F}\{f\} + \mathcal{F}\{g\}$  e  $\mathcal{F}\{cf\} = c \mathcal{F}\{f\}$

(2) Simetrias (auxiliam na eficiência computacional  $\rightarrow$  FFT):

If...	then...
$h(t)$ is real	$H(-f) = [H(f)]^*$
$h(t)$ is imaginary	$H(-f) = -[H(f)]^*$
$h(t)$ is even	$H(-f) = H(f)$ [i.e., $H(f)$ is even]
$h(t)$ is odd	$H(-f) = -H(f)$ [i.e., $H(f)$ is odd]
$h(t)$ is real and even	$H(f)$ is real and even
$h(t)$ is real and odd	$H(f)$ is imaginary and odd
$h(t)$ is imaginary and even	$H(f)$ is imaginary and even
$h(t)$ is imaginary and odd	$H(f)$ is real and odd

(3) Pares de Transformação:  $h(t) \iff H(f)$

$$\begin{aligned} h(at) &\iff \frac{1}{|a|}H\left(\frac{f}{a}\right) && \text{"time scaling"} \\ \frac{1}{|b|}h\left(\frac{t}{b}\right) &\iff H(bf) && \text{"frequency scaling"} \\ h(t - t_0) &\iff H(f) e^{2\pi i f t_0} && \text{"time shifting"} \\ h(t) e^{-2\pi i f_0 t} &\iff H(f - f_0) && \text{"frequency shifting"} \end{aligned}$$

(4) **Teorema da Convolução:**

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau)h(t - \tau) d\tau$$

A TF da convolução é simplesmente igual ao produto das transformadas:

$$g * h \iff G(f)H(f)$$

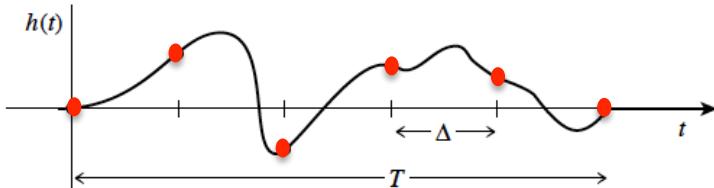
(5) **Teorema da Correlação:**

$$\text{Corr}(g, h) \equiv \int_{-\infty}^{\infty} g(\tau + t)h(\tau) d\tau$$

$$\text{Corr}(g, h) \iff G(f)H^*(f)$$

$$H(-f) = H^*(f)$$

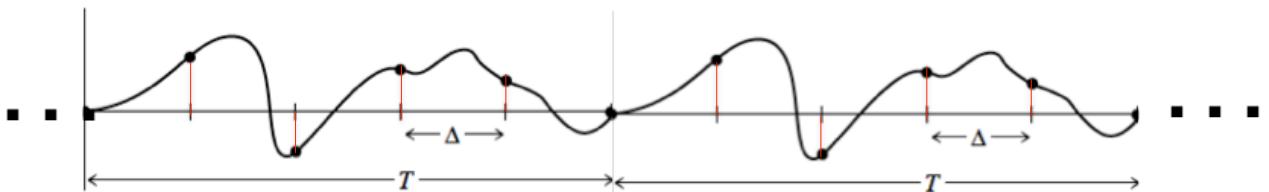
## TRANSFORMADAS RÁPIDAS DE FOURIER (FAST FOURIER TRANSFORMS-FFT)



$N$  amostras (“samples”)  $\rightarrow h(n)$

$h(0), h(1), h(2), \dots, h(5)$

$h(0), h(1), h(2), \dots, h(n), \dots, h(N-1)$



$$h_k \equiv h(t_k), \quad t_k \equiv k\Delta, \quad k = 0, 1, 2, \dots, N - 1$$

*Delta de Dirac*

$n \rightarrow k$

$$\mathcal{F}(\delta(t-a)) = \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} dt = e^{-i\omega a}$$

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N}$$

$$W \equiv e^{2\pi i/N}$$

$$H_n = \sum_{k=0}^{N-1} W^{nk} h_k$$

$W$  = matriz com  $n \times k$  elementos  
|  $w_{n,k}$  é uma constante gerada a partir  
do produto de  $n \times k$ , isto é,  $O(N^2)$ !

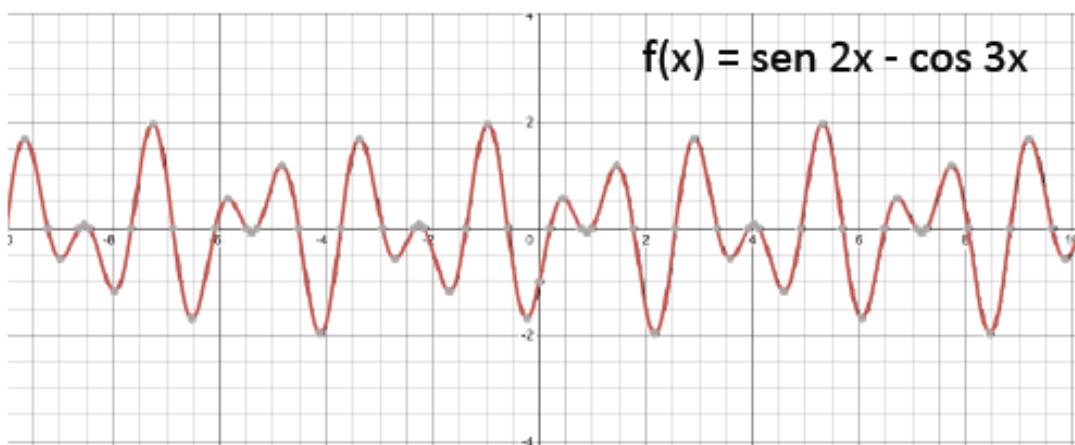
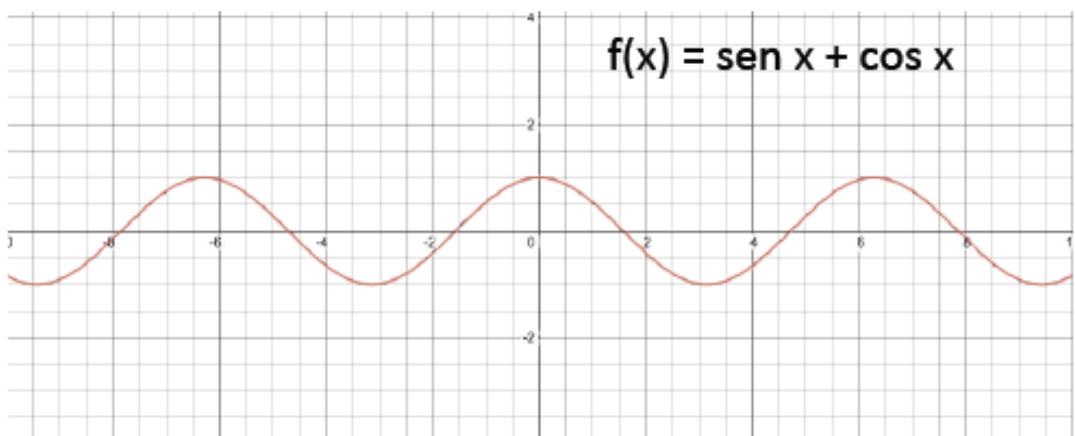
Como reduzir para  $O(N \log_2 N)$  operações?

De Gauss (1805) a Danielson & Lanczos (1942) e, então  
Cooley & Tukey (1960)

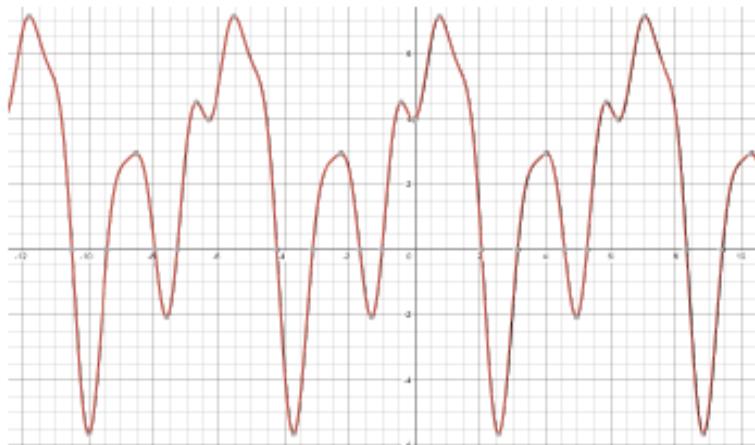
Metodologia via propriedade de paridade:

Algoritmo para reordenação de vetores com base na reversão de bits!!

$$O(N^2) \xrightarrow{H_{n,k}} O(N \log_2 N) \quad \text{FFT!!}$$



$$f(x) = 2 + \sin x + 3\sin 2x - 2\sin 3x + \cos x + 2\cos x - \cos 5x$$



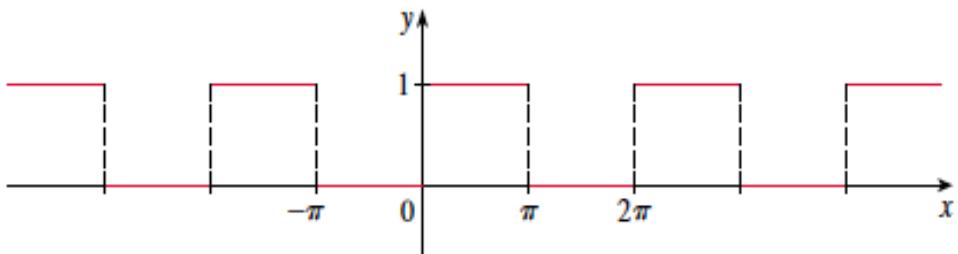
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

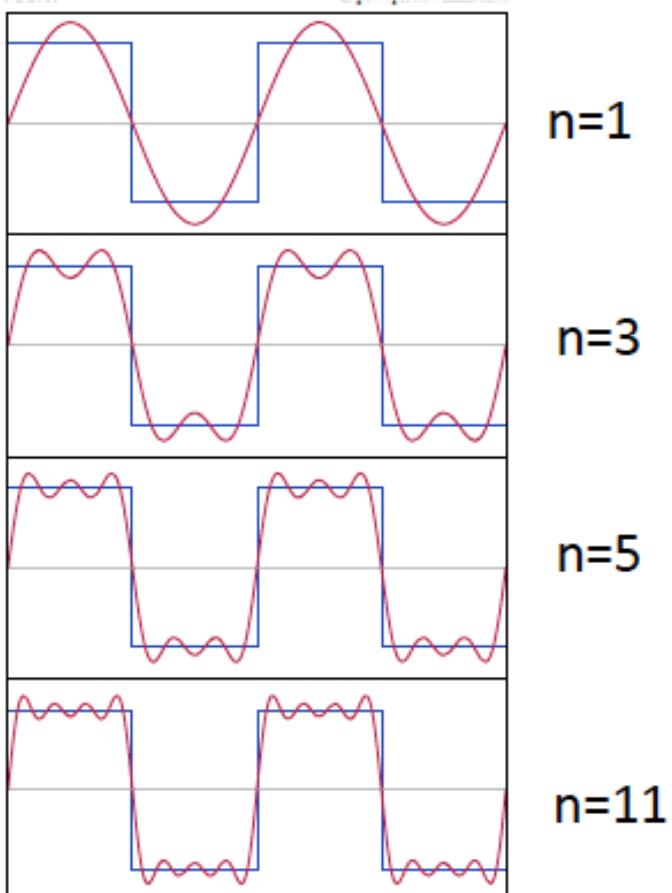
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

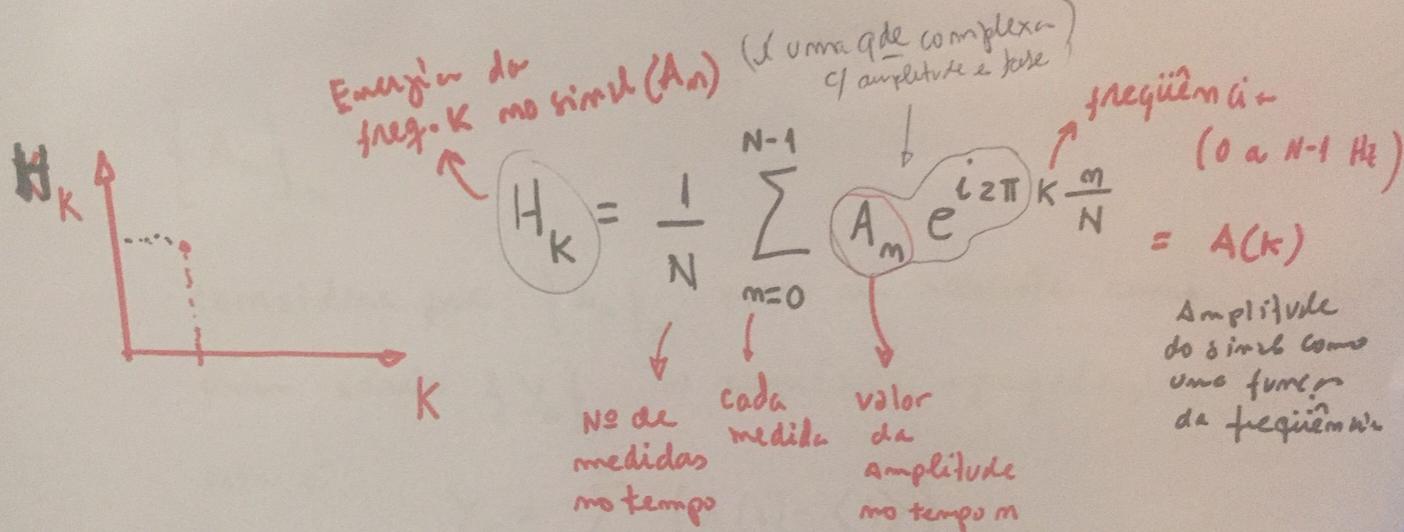
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x)$$

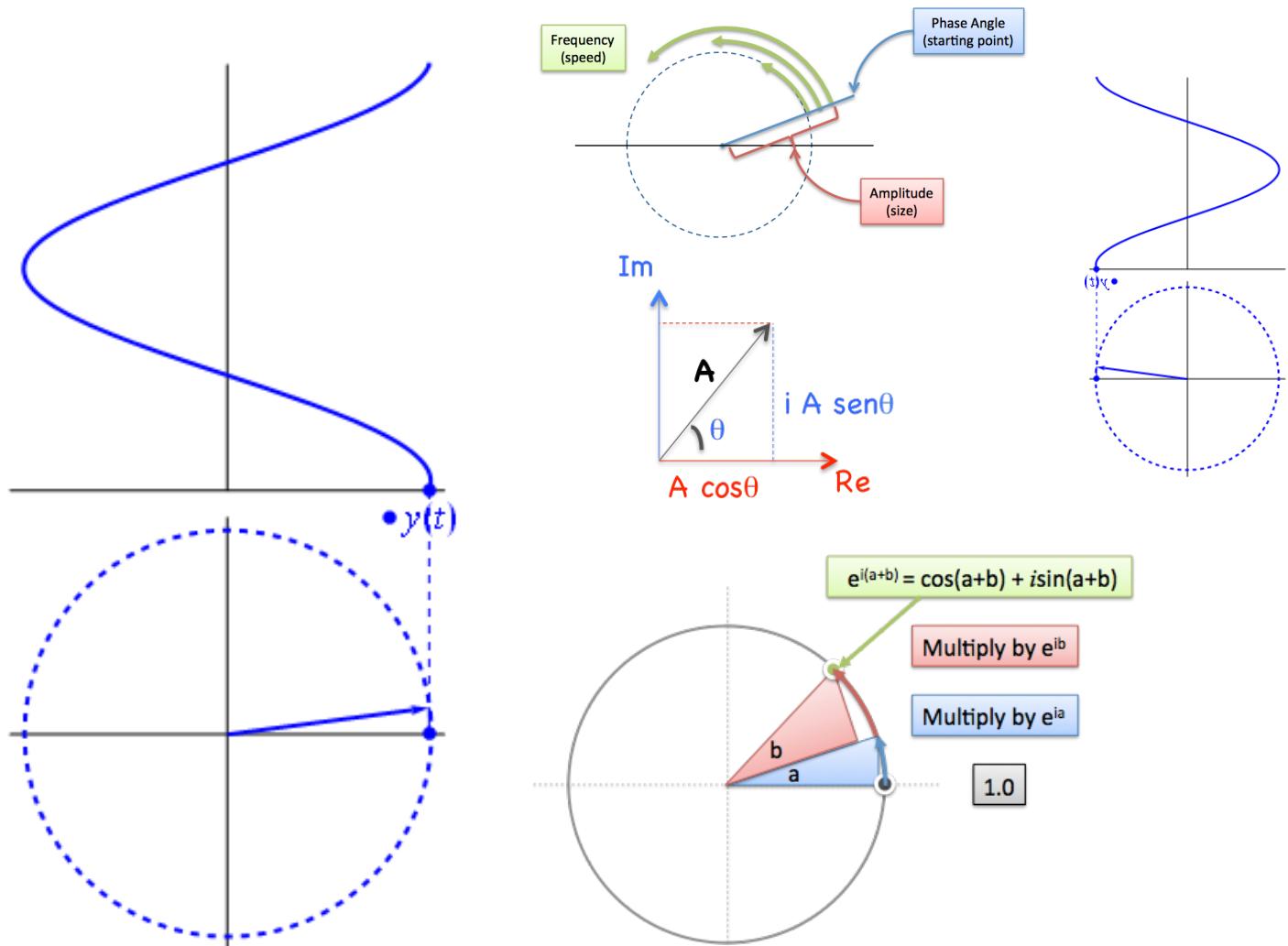


$$S_n(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \cdots + \frac{2}{n\pi} \sin nx$$





Para obter a "ENERGIA" p/ uma dada freqüência  $K$ ,  
gire o sinal no plano complexo na freqüência  $K$   
e tire a média  $\frac{1}{N} \sum_{m=0}^{N-1} \square \%$  sobre todos os pontos.



## ↓ TEOREMA DE WIEINER-KINCHIN

$$\frac{1}{N} \sum_{m=0}^{N-1} \langle A(m) \cdot A(m-\tau) \rangle \cdot e^{i 2\pi k \frac{m}{N}} = \langle |A(k)|^2 \rangle$$

↑ Lembrando que  $A(w) = \int A(t) e^{-iwt} dt$   $w = 2\pi k$

$$A^*(w) = \int A(t) e^{iwt} dt$$

↑ NORMALIZAÇÃO:  $\delta(w) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-iwt} dt = 1 \Rightarrow \delta(t) = \int_{-\infty}^{\infty} e^{iwt} dt$

$$\therefore \int_{-\infty}^{\infty} A(t) \delta(t - t_m) dt = A(t_m)$$

## ↓ TEOREMA DE PARSEVAL

$$\int_{-\infty}^{\infty} |A(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(w)|^2 dw \quad \hookrightarrow \int A(t) e^{-iwt} dt$$

Se  $A: \mathbb{R} \rightarrow \mathbb{R}$   
 $\exists \epsilon \mid \delta \in (0, 1]$

$$\epsilon \left\langle |A(t)|^2 \right\rangle = |A(t) - A(t-\delta)| \leq \epsilon |t - \tau|^\gamma \rightarrow \text{exp. de Holder/Hurst}$$

$\Rightarrow C(L) \sim L^{-\gamma}$

S/ MÉDIA !!

## DFA: Detrended Fluctuation Analysis

$$\{A_i\}_N$$

considere que  $\{A_i\}_N$  possa ser escrita como uma série  $\{Y_k\}_N$  de sequências apagadas, isto é:

$$Y_k = \sum_{i=1}^k (A(i) - \langle A \rangle)$$

Ex.  $\{A_i\}_5 = \{1, 2, 3, 2, 1\} \Rightarrow \langle A \rangle = \frac{9}{5} = 1.8$

$$\begin{aligned} \therefore Y_1 &= A_1 - 1.8 = -0.8 \quad \Rightarrow Y_2 = \sum_{i=1}^2 (A_i - \langle A \rangle) = -0.8 + (A_2 - \langle A \rangle) = \\ &\vdots \\ Y_5 &= Y_4 + (A_5 - \langle A \rangle) = 0 \quad = -0.8 + (2 - 1.8) \\ &= -0.6 \end{aligned}$$

$$\therefore \{Y_k\}_N = \{-0.8, -0.6, 0.6, 0.8, 0\}$$

ajustar polinômios  
de ordem  $q$   
em cada janela

$$F_q(m) = \sqrt{\frac{1}{N} \sum_{i=1}^N (A(i) - \tilde{Y}(i))^2}$$

$\tilde{Y}(i)$   
 $q = \text{ordem do polinômio}$

ajustar via  
janelas de tamanho  $m$   
e grande N medidas.

$$F_1(2) =$$

$$F_1(3) =$$

$$F_1(4) =$$

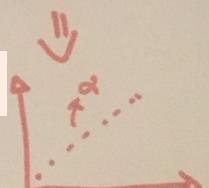
$$F_1(5) = \sqrt{\frac{0.2^2 + 1.4^2 + 2.4^2 + 1.2^2 + 1^2}{5}} = 1.43$$

$$\log F(n) \times \log n$$

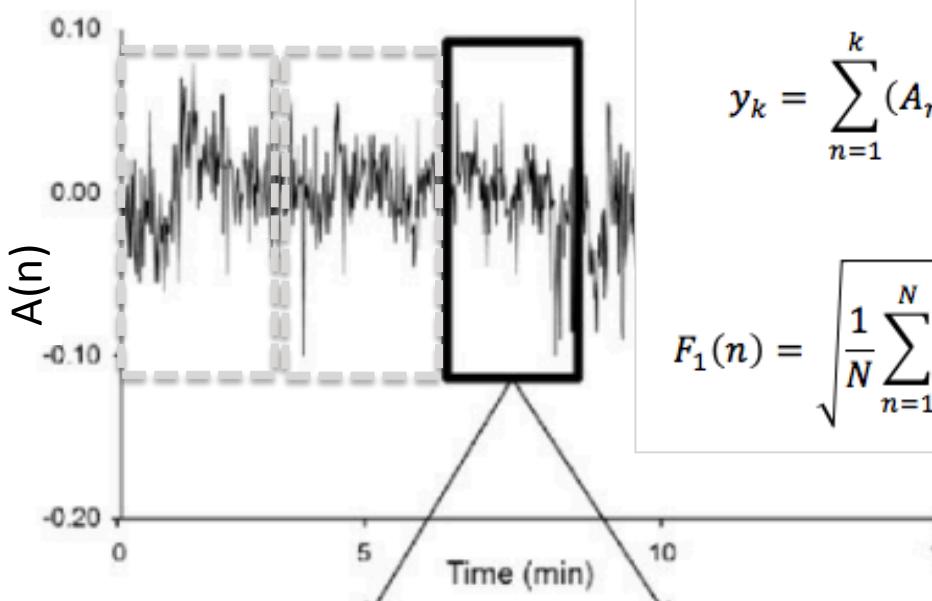
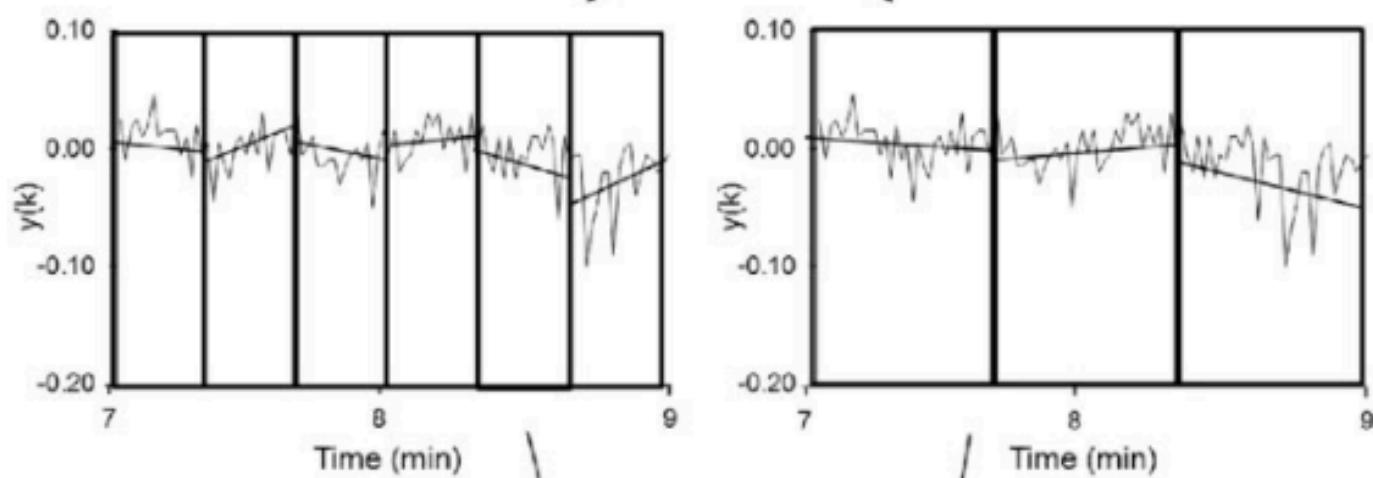
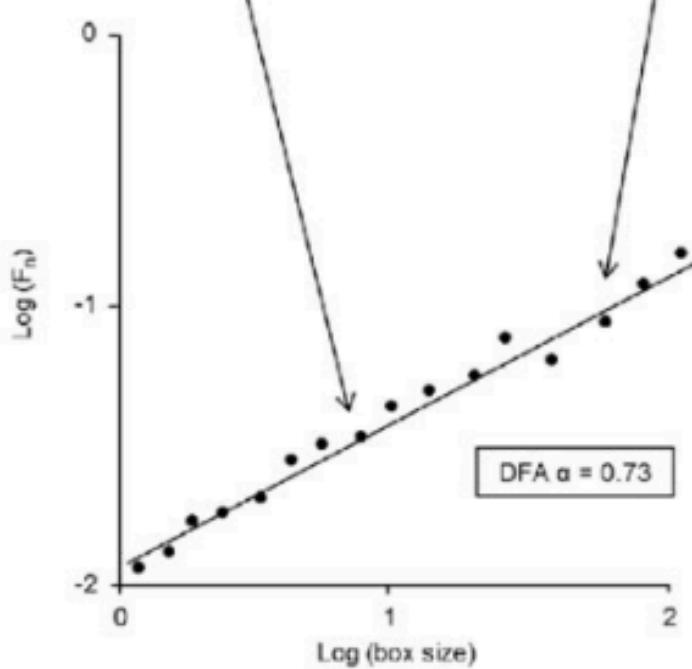
$$\text{como } C(L) \sim e^{-\beta L^\alpha} \sim L^\beta \Rightarrow \beta = 2 - 2\alpha$$

c/  $\boxed{\beta = 1 - \gamma}$

$$\begin{aligned} \beta &= 2\alpha - 1 \\ \gamma &= 1 - \beta \end{aligned}$$

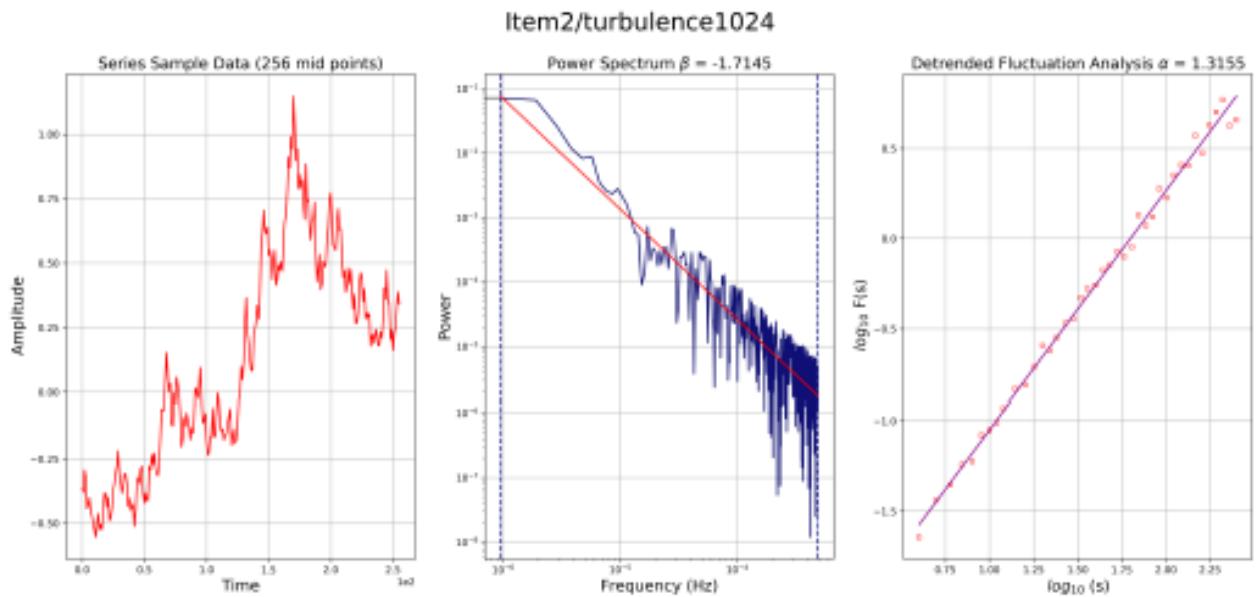


$$\log n$$

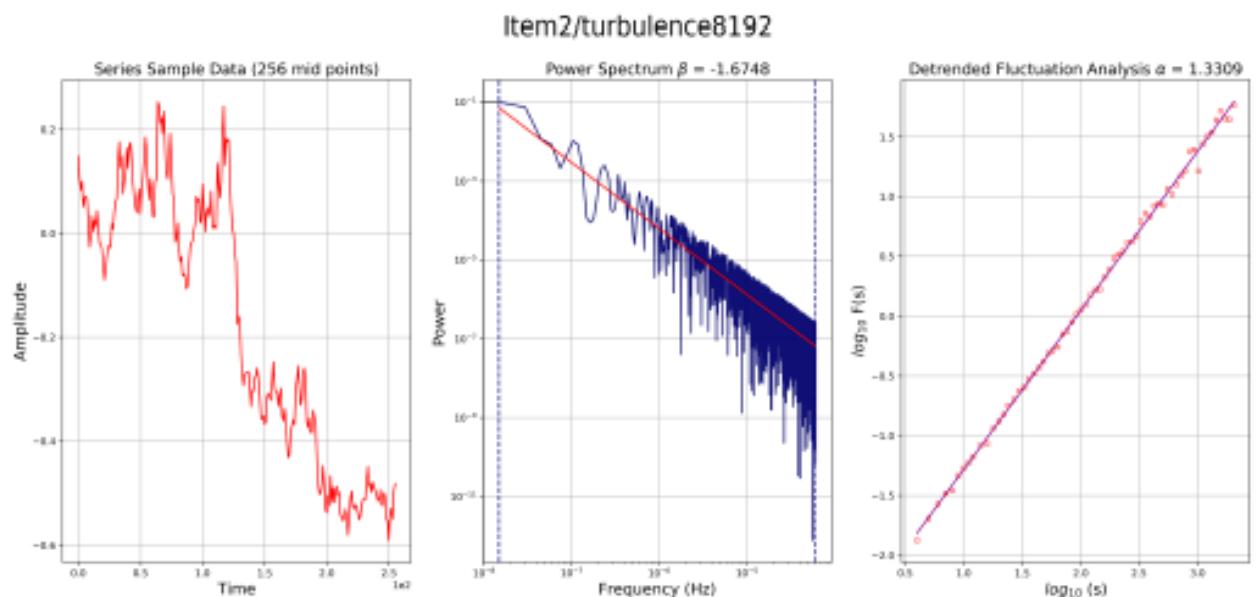
**A****B****C**

- Memória: dependencia imposta pela auto-correlacao  $\rightarrow$  Markoviano x Não-markoviano
- Natureza do modelo de distribuição (ruído): gaussiano x não-gaussiano
- Classe de Universalidade do Processo Subjacente: PSD ( $K$ )  $\approx K^{-\beta}$

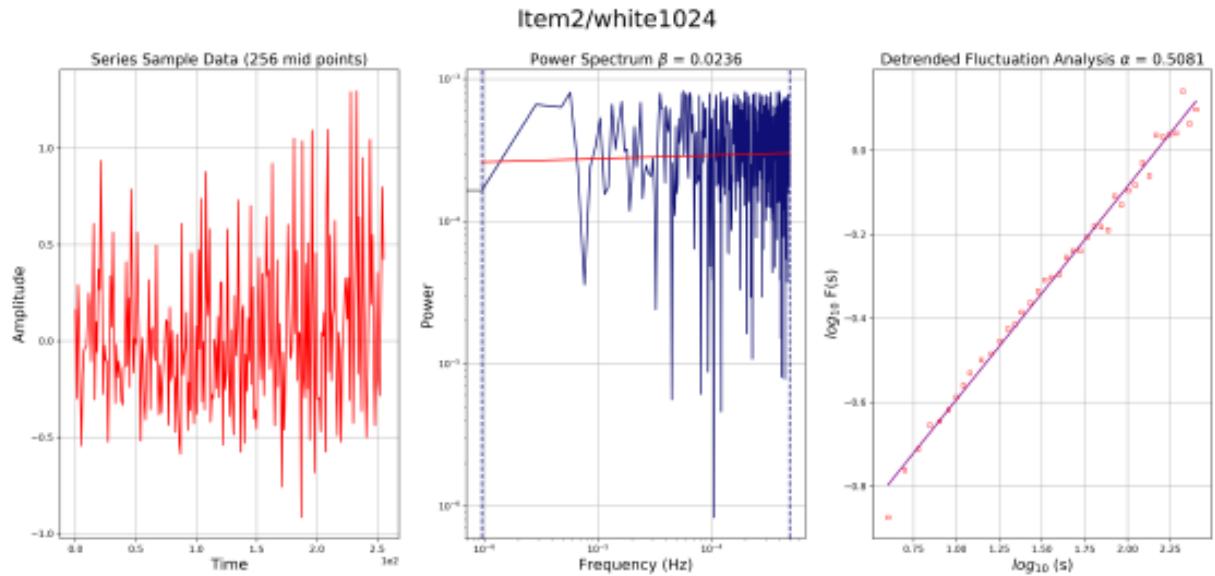
## Turbulência com 1024 pontos



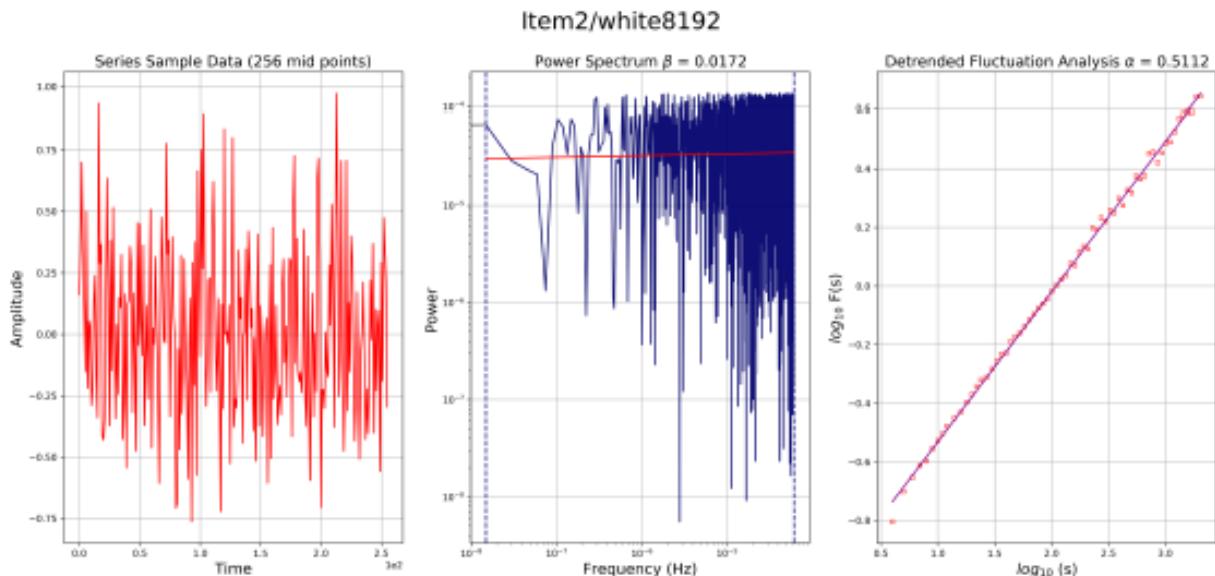
## Turbulência com 8192 pontos



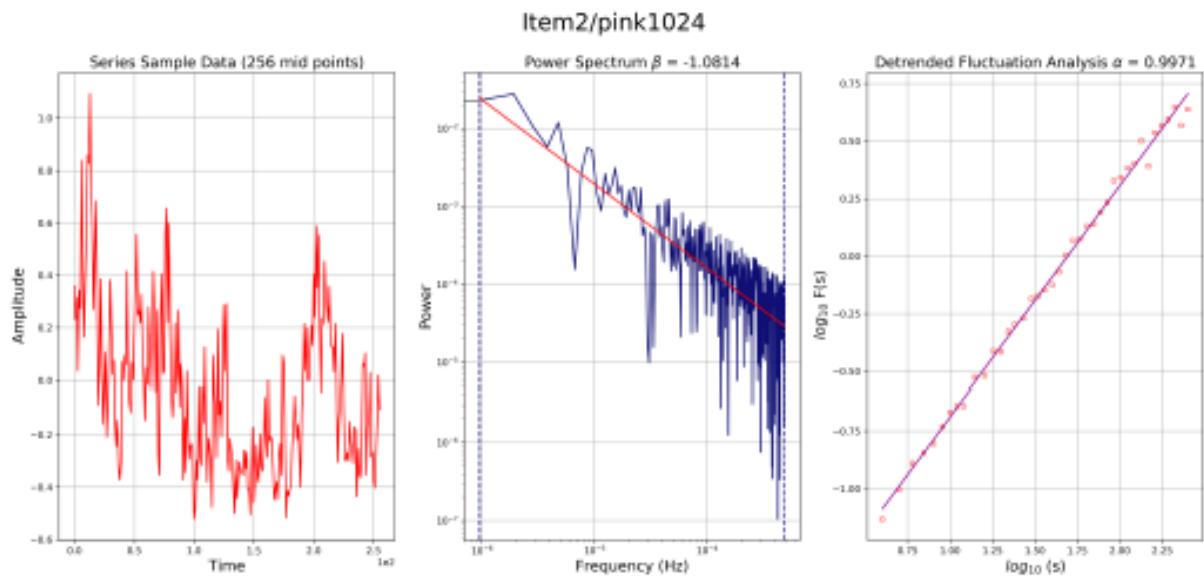
## Ruído Branco com 1024 pontos



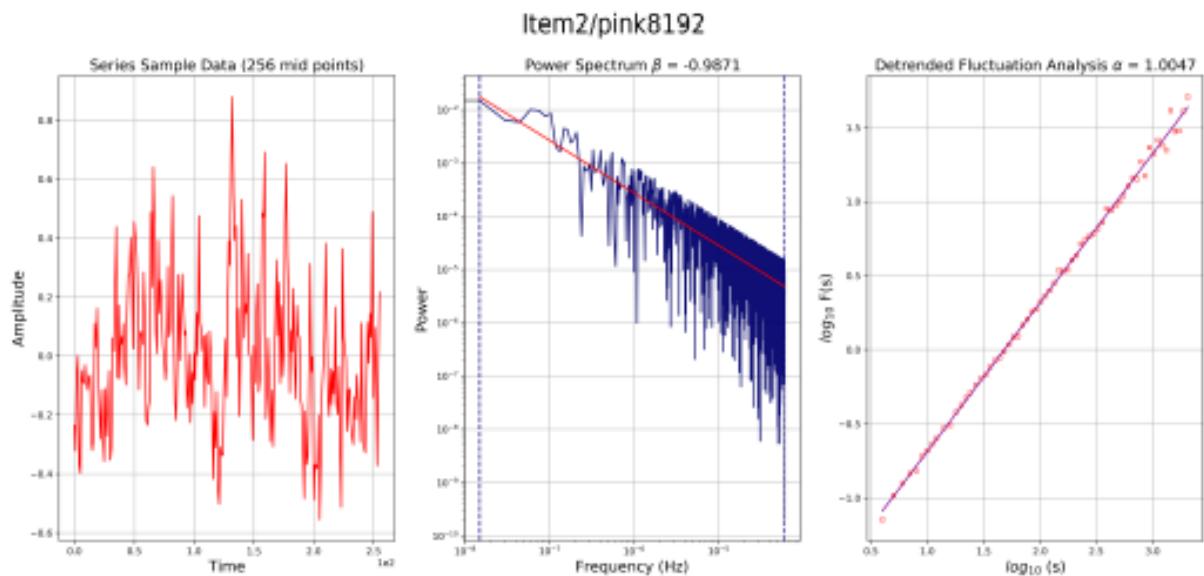
## Ruído Branco com 8192 pontos



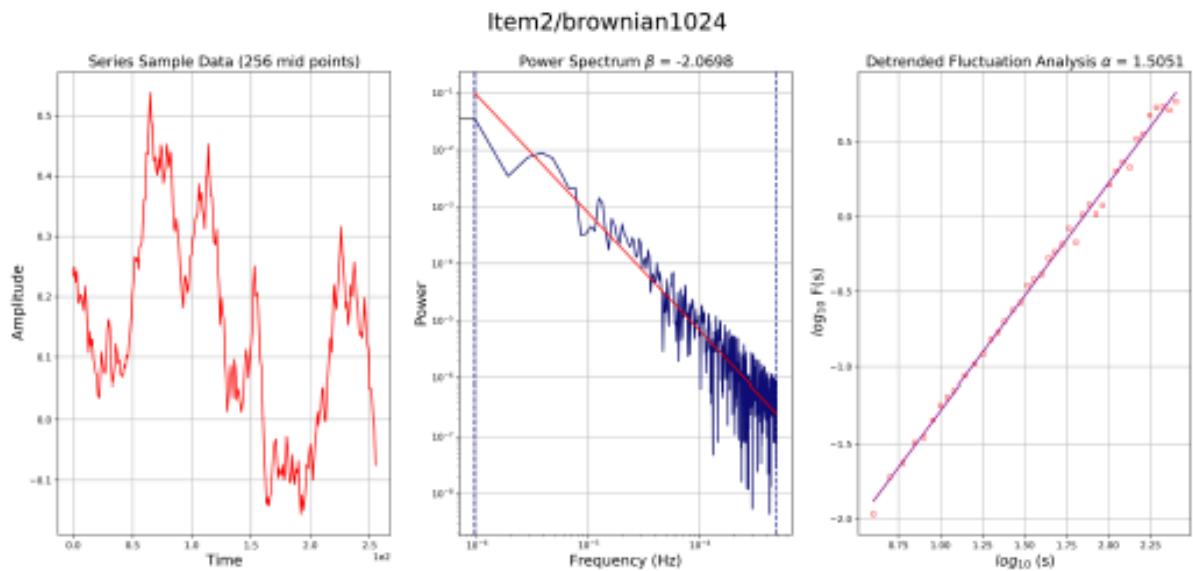
## Ruído Rosa com 1024 pontos



## Ruído Rosa com 8192 pontos



## Movimento Browniano com 1024 pontos



## Movimento Browniano com 8192 pontos

