

$$\text{Im}[\chi_{e,\omega}^{\text{abc}}] = \frac{\pi|e|^3}{2\hbar^2} \int \frac{d^3k}{8\pi^3} \sum_{vc} \sum_{q \neq (v,c)} \frac{1}{\omega_{cv}^\Sigma} \left[ \frac{\text{Im}[\mathcal{V}_{qc}^{\Sigma,a} \{r_{cv}^b r_{vq}^c\}]}{(2\omega_{cv}^\Sigma - \omega_{cq}^\Sigma)} - \frac{\text{Im}[\mathcal{V}_{vq}^{\Sigma,a} \{r_{qc}^c r_{cv}^b\}]}{(2\omega_{cv}^\Sigma - \omega_{qv}^\Sigma)} \right] \delta(\omega_{cv}^\Sigma - \omega)$$

$$\text{Im}[\chi_{e,2\omega}^{\text{abc}}] = -\frac{\pi|e|^3}{2\hbar^2} \int \frac{d^3k}{8\pi^3} \sum_{vc} \frac{4}{\omega_{cv}^\Sigma} \left[ \sum_{v' \neq v} \frac{\text{Im}[\mathcal{V}_{vc}^{\Sigma,a} \{r_{cv'}^b r_{v'v}^c\}]}{2\omega_{cv'}^\Sigma - \omega_{cv}^\Sigma} - \sum_{c' \neq c} \frac{\text{Im}[\mathcal{V}_{vc}^{\Sigma,a} \{r_{cc'}^c r_{c'v}^b\}]}{2\omega_{c'v}^\Sigma - \omega_{cv}^\Sigma} \right] \delta(\omega_{cv}^\Sigma - 2\omega)$$