frequencies; and the periodic solutions of these systems contain a large amount of third harmonic in addition to the fundamental. It is suggested that the nonlinearity introduces sufficient phase shift at both these frequencies to cause the system to oscillate in the absence of any periodic input.

Acknowledgment

The author would like to thank Prof. R. W. Brockett of The Massachusetts Institute of Technology, Cambridge, for contributing numerous ideas and suggestions to his research.

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An Application of Half-Cycle Posicast

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***Abstract*—The problem of compensating a fourth-order, lightly-damped linear feedback system by means of half-cycle Posicast [1] is examined and the results of analog computer simulations are shown. Transient and frequency responses are presented. Although half-cycle Posicast is ideally suited for driving a second-order system, the resulting behavior of the fourth-order system is seen to be quite good with half-cycle Posicast bringing about a fast step response with almost no ripple and eliminating completely any resonant peaking in the frequency response. The sensitivity of behavior to variations in system parameters is examined. Also the system is tested with a quasi-random input signal. In both cases, results are favorable.**

Introduction

The problem of compensating a feedback system which is very lightly damped has long confronted control engineers. Numerous schemes have been utilized with varying degrees of success. This paper investigates and applies one such scheme, half-cycle Posicast, which was introduced by O. J. M. Smith [1], [3]–[5]. This scheme has several advantages. It reduces overshoot and resonant peaking thus allowing higher forward gain to be used. This in turn reduces steady-state errors. All of these factors are of importance to the control engineer.

The purpose of this paper is to discuss half-cycle Posicast control, apply it to the design of a tracking radar antenna and so illustrate its advantages in a practical situation. Although there are many types of Posicast, only half-cycle Posicast will be considered here.

Theory of Posicast

Posicast is a linear controller used in series with the plant to which it is applied. Half-cycle Posicast operates upon step input signals in such a manner as to cause the output of the system to reach and maintain its desired value in one-half the natural period of the plant—thus the name half-cycle Posicast.

Manuscript received November 29, 1965; revised April 11, 1966. The research reported here was supported in part by the National Science Foundation.
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As a simple example, consider the suspended weight shown in Fig. 1. The objective is to move the weight from position 1 to position 2 without exciting oscillations. First the support of the weight is moved one-half the way from position 1′ to 2′. The weight swings from position 1 to 2, one-half a cycle, and stops before beginning its motion back in the direction of position 1. At the instant in which the weight is stopped, the support is quickly moved to position 2′, relaxing the system or removing all energy from it. The weight now having neither driving force nor momentum remains at position 2, the desired position.

The input signal here is a step from position 1 to position 2. Half-cycle Posicast operates upon this signal and gives the control sequence just described for the support position. By knowing the natural frequency of the system and the damping (here the damping is zero), a nonoscillatory response which reaches the desired value in one-half the natural period of the system is achieved. The system is initially allowed to see only enough of the input to cause its output to just reach the desired value. When the output reaches this value, then the input is adjusted such that the output maintains this desired value.

As another example of half-cycle Posicast, consider the unit step response of a lightly damped system shown in Fig. 2(a) where T_n is the damped natural period of this system. Half-cycle Posicast must act to subtract a portion of the input signal so that the first output peak instead of having the value $1+\delta$ will have the value 1. This means that half-cycle Posicast must allow only $1/(1+\delta)$ of the input signal to be seen by the system initially. Then at time $T_n/2$ seconds later, when the system output has reached the value 1 and stopped

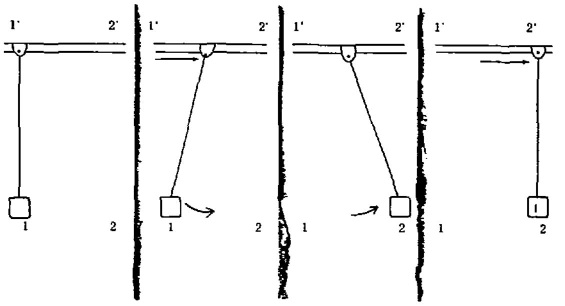


Fig. 1. Pictorial description of Posicast.

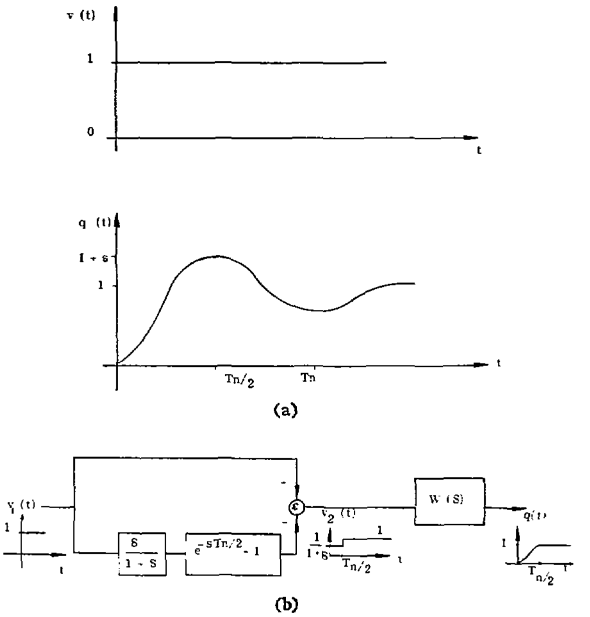


Fig. 2. (a) Step response of a typical lightly damped system. (b) A system utilizing half-cycle Posicast.

prior to swinging back toward a lesser value, half-cycle Posicast must act to bring the input seen by the system to 1 and thus reduce the actuating error to zero. A system utilizing half-cycle Posicast is shown in Fig. 2(b). If the system $W(s)$ is second order, the following equation holds

$$Kv_2(t) = J\ddot{q}(t) + B\dot{q}(t) + Kq(t). \tag{1}$$

Here v_1 is a step input of magnitude 1. Posicast causes v_2 to be a step input of magnitude $1/(1+\delta)$. At time $T_n/2$

$$\dot{q}(t) = 0 \tag{2}$$

and

$$q(t) = 1. \tag{3}$$

At this time half-cycle Posicast requires that

$$v_2(t) = 1 \tag{4}$$

which means

$$v_2(T_n/2) = q(T_n/2). \tag{5}$$

From (1), (2), and (5) we obtain

$$\ddot{q}(T_n/2) = 0. \tag{6}$$

Thus the system is at equilibrium and remains so provided $W(s)$ is stable.

Consider a system of higher order than 2, for example the fourth-order system

$$Av_2(t) = E\ddddot{q}(t) + D\dddot{q}(t) + C\ddot{q}(t) + B\dot{q}(t) + Aq(t). \tag{7}$$

If v_2 is a step input of magnitude $1/(1+\delta)$, then at time $T_n/2$ when

$$\dot{q}(t) = 0, \tag{8}$$

half-cycle Posicast requires that

$$v_3(t) = 1 \tag{9}$$

which means

$$v_2(T_n/2) = q(T_n/2). \tag{10}$$

Combining (7), (8), and (10) we obtain

$$0 = E\ddddot{q}(T_n/2) + D\dddot{q}(T_n/2) + C\ddot{q}(T_n/2) \tag{11}$$

and depending on the relative magnitudes of the coefficients in the differential equation, various degrees of oscillation will be exhibited in the system output. A more complicated form of Posicast would be required to eliminate these oscillations [6].

To see how half-cycle Posicast behaves when the input signal is other than a step function, one can examine this controller in the complex frequency domain. In the frequency domain, half-cycle Posicast can be shown to have an infinite number of zeros. Setting

$$\frac{1}{1+\delta} + \frac{\delta}{1+\delta} e^{-sT_n/2} = 0 \tag{12}$$

yields

$$\sigma = -\frac{2}{T_n} \ln \frac{1}{\delta} \tag{13}$$

and

$$\omega = \frac{2\Pi}{T_n}(1 + 2n) \tag{14}$$

where n is any integer.

The first pair of zeros cancels the pair of dominant poles in $W(s)$ and, if the system is second order, will eliminate any resonant peaking. The higher frequency pairs of zeros produce a pronounced ripple in the frequency response.

System Investigated

The plant to which half-cycle Posicast was applied is shown in Fig. 3(a). The load is a simplified tracking radar antenna. The spring-friction coupling accounts for resilient load members with damping between the drive and the load. An ideal torque source was assumed, and the entire system was assumed linear over the range of operations.

The equations of motion are as follows:

$$K_M I_A = T, \tag{15}$$

$$T = J_M\ddot{\theta}_M + B_1(\dot{\theta}_M - \dot{\theta}_L) + K_1(\theta_M - \theta_L), \tag{16}$$

and

$$J_L\ddot{\theta}_L = B_1(\dot{\theta}_M - \dot{\theta}_L) + K_1(\theta_M - \theta_L). \tag{17}$$

A preliminary task was to alter the system in such a manner that half-cycle Posicast could be used advantageously. One objective was to make the system as fast as possible (i.e., least possible rise time for a step input) even if it meant causing an oscillatory behavior (for hopefully the application of half-cycle Posicast would eliminate the oscillations). The system was speeded up considerably by including rate feedback from the torque source and the load, and then increasing the forward gain. Also, it was found that this configuration and combination of gains caused the natural period and overshoot to remain fairly insensitive to variations in B_1 and K_1. This was a necessity if the same half-cycle Posicast were to be effective for all values of B_1 and K_1. (Note from Fig. 4 the range over which B_1 and K_1 can vary.)

A block diagram of the system in a feedback loop is shown in Fig. 3(b). The new equations of motion are

$$K_0(\theta_2 - \theta_L) - K_2\dot{\theta}_M - K_3\ddot{\theta}_L = I_A, \tag{18}$$

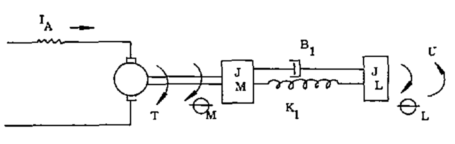
$$K_M I_A = T, \tag{19}$$

$$T = J_M\ddot{\theta}_M + B_1(\dot{\theta}_M - \dot{\theta}_L) + K_1(\theta_M - \theta_L), \tag{20}$$

and

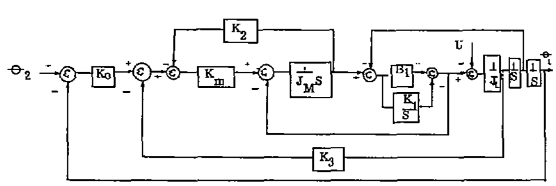
$$B_1(\dot{\theta}_M - \dot{\theta}_L) + K_1(\theta_M - \theta_L) = J_L\ddot{\theta}_L. \tag{21}$$

In relation to our earlier notation, θ_L corresponds to q and θ_2 corresponds to v_2.



Motor Inertia, $J_M = 1,6 \times 10^5$ ft. lb. sec^2/Rad.
Load Inertia. $J_L = .6 \times 10^5$ ft. lb. sec^2/Rad.
Friction Coefficient. $B_1 = 1.09 \times 10^5$ to 17.4×10^5 ft. lb. Rad/sec.
Spring Constant. $K_1 = 1.09 \times 10^8$ to 17.4×10^8 ft. lb./Rad.

(a)



$K_0 = 1$ Amp/Rad
$K_m = 10 \times 10^8$ ft. lb/Amp
$K_2 = 1 \times 10^2$ Amp Sec/Rad
$K_3 = 9.4 \times 10^4$ Amp sec^2/ Rad

(b)

Fig. 3. (a) Lightly damped system to which Posicast was applied. (b) System with feedback before Posicast was added.

Results

The system was simulated on an analog computer. Figure 4(a) shows the step response of the system before Posicast was added. The extreme values of K_1 and B_1 were used to demonstrate the insensitivity of natural period and overshoot to changes in these parameters. One sees from Fig. 4(a) that

$$T_n \simeq 0.205 \text{ second} \tag{22}$$

and

$$\delta \simeq 0.786. \tag{23}$$

These parameters define the half-cycle Posicast function.

The reason for the insensitivity of T_n and δ to variations in B_1 and K_1 can be understood if one examines the location of the poles of $W(s)$ as B_1 and K_1 are varied. These poles are:

for $B=1.09\times10^5$ and $K=1.09\times10^8$,

$$s_1 = -2.41 + j30.67,$$
$$s_2 = -2.41 + j30.67,$$
$$s_3 = -35.4 + j103.6,$$

and

$$s_4 = -35.4 - j103.6;$$

for $B=1.74\times10^6$ and $K=1.74\times10^9$,

$$s_1 = -4.22 + j29.15,$$
$$s_2 = -4.22 - j29.15,$$
$$s_3 = -32.2 + j437.6,$$

and

$$s_4 = -32.2 - j437.6;$$

for $B=1.09\times10^5$ and $K=1.74\times10^9$,

$$s_1 = -4.23 + j29.15,$$
$$s_2 = -4.23 - j29.15,$$
$$s_3 = -33.6 + j455.8,$$

and

$$s_4 = -33.6 - j455.8;$$

and for $B=1.74\times10^6$ and $K=1.09\times10^8$,

$$s_1 = -4.22 + j29.6,$$
$$s_2 = -4.22 - j29.6,$$
$$s_3 = -62.7,$$

and

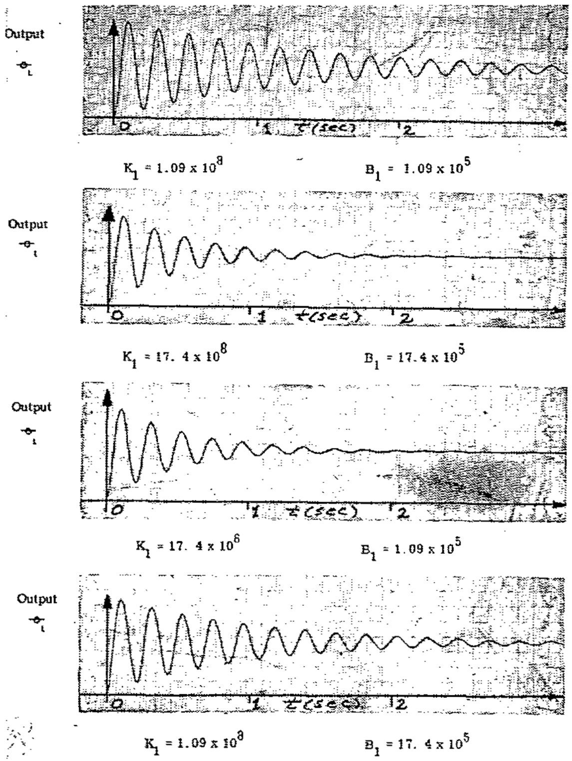
$$s_4 = -205.5.$$

In each case the first two poles predominate, making a second-order approximation quite accurate and indicating that half-cycle Posicast should work reasonably well. One also notes that the imaginary portions of the first two poles remain fairly constant for all four cases. This explains the constancy of the natural period in the four cases.

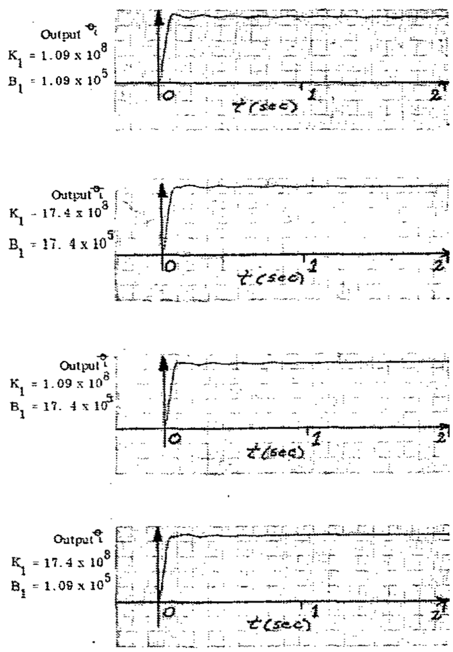
It can be shown that for a second-order linear system with poles at $-\sigma \pm j\omega$, the percentage overshoot in the step response is equal to $e^{-\sigma\pi/\omega}$. For the four cases studied here, a second-order approximation yields the following four values for $\sigma\Pi/\omega$: 0.247, 0.456, 0.458, and 0.241. One then obtains the following values for $e^{-\sigma\pi/\omega}$: 0.782, 0.634, 0.633, and 0.786, which are in fair agreement with the percentage overshoot as measured experimentally. The fact that there is not complete correspondence must be attributed to the influence of the third and fourth poles.

Figure 4(b) shows the step response of the system after half-cycle Posicast has been added. The time delay was achieved by means of a tape recorder writing and reading simultaneously with separate write and read heads. It is seen that the slight oscillations are on the order of 4 percent of the input signal. In Fig. 5 are shown frequency response characteristics. The notch at $\omega=\omega_n$ brought about by the first set of zeros of half-cycle Posicast has eliminated any resonant peaking.

The system was tested with a quasi-random signal. This signal was achieved by manually varying the amplitude and frequency of a triangle wave generator. The frequency was varied over twice the bandwidth of the system. The results of this test are shown in Fig. 6(a).



(a)



(b)

Fig. 4. (a) Step response of the system shown in Fig. 3(b). (b) Step response of the system shown in Fig. 3(b) with Posicast added.

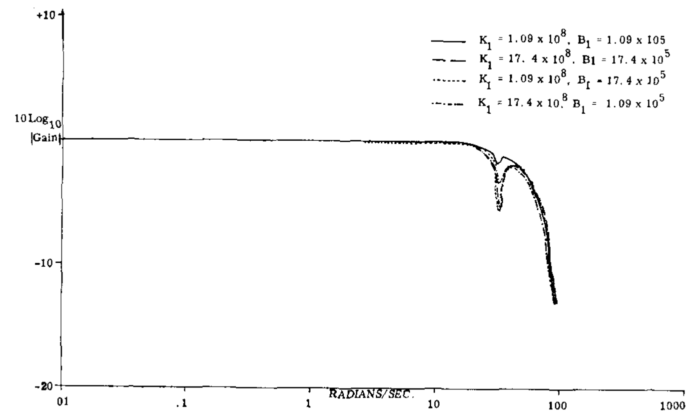
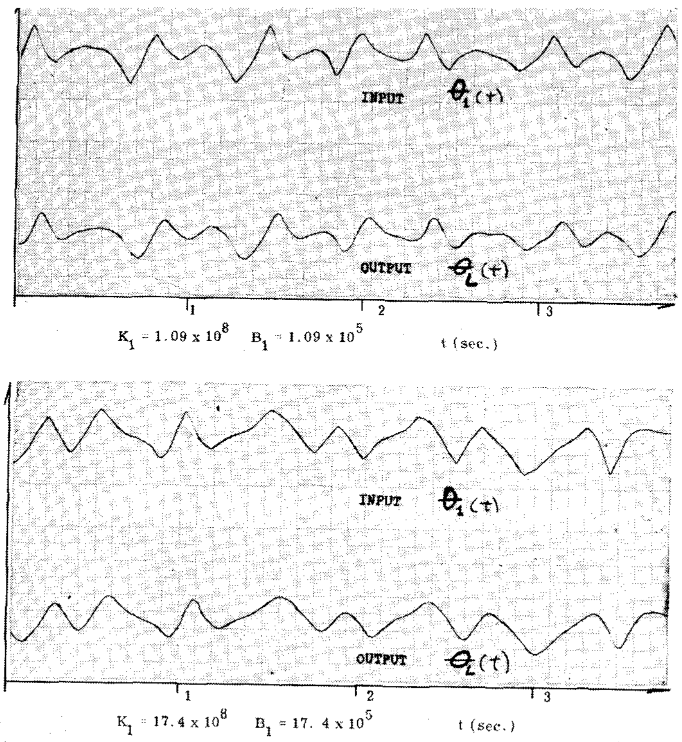
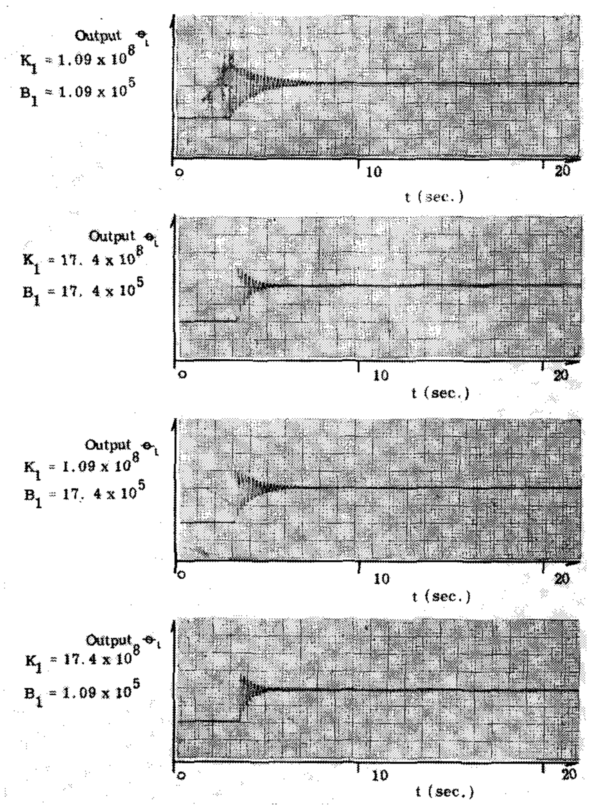


Fig. 5. Frequency response of the system shown in Fig. 3(b) with Posicast added.

Fig. 6. (a) Response of Posicast system to quasi-random input signal. (b) Step load disturbance response of Posicast system.



(a)



(b)

A final test was to input a stepwise load disturbance. The response is shown in Fig. 6(b). The oscillatory nature of the system is very evident here, for the load disturbance goes directly into the system and is not operated on by Posicast. One should keep in mind that Posicast does not increase the damping in a system, but rather eliminates oscillations by not permitting the natural frequencies of the system to be excited from the input. Posicast can also be utilized in such a manner as to eliminate oscillations caused by load disturbances. The reader is referred to [1, p. 341] for further discussion of this.

Conclusion

Half-cycle Posicast has been applied to a fourth-order linear system and this system tested through analog computer simulation. Half-cycle Posicast is ideally suited for second-order sytems; however, the results indicate that if a higher order system can be well approximated as a second-order system, then half-cycle Posicast can be used effectively here also. For the fourth-order system studied, half-cycle Posicast reduced overshoot and eliminated resonant peaking permitting higher forward gains to be used. Steady-state velocity errors and steady-state load disturbance errors were made small although the transient response to a load disturbance was not improved. Variations in system parameters over a wide range had no serious adverse effects. The response of the system to a quasi-random input signal was well behaved.

Acknowledgment

The author wishes to express his thanks to Prof. G. C. Newton, Jr., of the Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, for his constructive criticism.

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