1. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses, then it has dinner together with probability 0.2. What proportion of games results in a team dinner?

The process is in:
state 0 if the team wins a game
state 1 if the team loses a game

The transition probabilities P_{ij} $(i,j = 0,1)$:
P_{00} = probability that if the team wins a game, the team wins the next game
= 0.8
P_{01} = probability that if the team wins a game, the team loses the next game
= 1-0.8=0.2
P_{10} = probability that if the team loses a game, the wins the next game
= 0.3
P_{11} = probability that if the team loses a game, the loses the next game
= 1-0.3=0.7

This is a Markov chain with three states 0, 1 with transition probability matrix **P**:

$$\mathbf{P} = \begin{Vmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{Vmatrix} = \begin{Vmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{Vmatrix}$$

The long-run proportions π_i:

$\pi_0 = 0.8\pi_0 + 0.3\pi_1$

$\pi_1 = 0.2\pi_0 + 0.7\pi_1$

$\pi_0 + \pi_1 = 1$

π_0 = long-run proportion of the team winning a game
π_0 =0.6

π_1 = long-run proportion of the team losing a game
π_1 =0.4

P(if the team wins a game, then it has dinner together) =
= P(team dinner given that the team wins) =
= P(team dinner | team wins) =
= 0.7

P(if the team loses a game, then it has dinner together) =
= P(team dinner given that the team loses) =
= P(team dinner | team loses) =
= 0.2

Proportion of games resulting in a team dinner =
= (long-run proportion of the team winning a game) * [P(if the team wins a game, then it has dinner together)] + (long-run proportion of the team losing a game) * [P(if the team loses a game, then it has dinner together)]
$= \pi_0 * P(team\ dinner \mid team\ wins) + \pi_1 * P(team\ dinner \mid team\ loses)$
$= (0.6) * (0.7) + (0.4) * (0.2)$

$= 0.42 + 0.08$
$= 0.50$

A: The proportion of games that results in a team dinner is 0.50

2. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are all exponential with rate μ, what is the expected amount of time you will spend in the bank?

$Expected\ service\ time\ for\ customer\ being\ served = \frac{1}{\mu}$

$Expected\ service\ time\ for\ first\ customer\ on\ line = \frac{1}{\mu}$

$Expected\ service\ time\ for\ second\ customer\ on\ line = \frac{1}{\mu}$

$Expected\ service\ time\ for\ third\ customer\ on\ line = \frac{1}{\mu}$

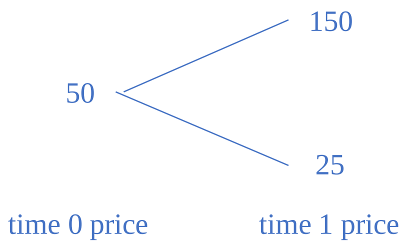
$Expected\ service\ time\ for\ fourth\ customer\ on\ line = \frac{1}{\mu}$

$Expected\ service\ time\ for\ fifth\ customer\ on\ line\ (me\ waiting\ at\ the\ end\ of\ the\ line) = \frac{1}{\mu}$

$$\begin{aligned}
&Total\ expected\ time\ I\ will\ spend\ in\ the\ bank \\
&\qquad = Expected\ service\ time\ for\ customer\ being\ served \\
&\qquad + Expected\ service\ time\ for\ first\ customer\ on\ line \\
&\qquad + Expected\ service\ time\ for\ second\ customer\ on\ line \\
&\qquad + Expected\ service\ time\ for\ third\ customer\ on\ line \\
&\qquad + Expected\ service\ time\ for\ fourth\ customer\ on\ line \\
&\qquad + Expected\ service\ time\ for\ fifth\ customer\ on\ line\ (me) \\
&= \frac{1}{\mu} + \frac{1}{\mu} + \frac{1}{\mu} + \frac{1}{\mu} + \frac{1}{\mu} + \frac{1}{\mu} \\
&= \frac{6}{\mu}
\end{aligned}$$

A: The expected amount of time I will spend in the bank is $\frac{6}{\mu}$

4. A stock is presently selling at a price of \$50 per share. After one time period, its selling price (in present value dollars) be either \$150 or \$25. An option to purchase y units of the stock at time 1 can be purchased at cost cy.
   a. What should c be in order for there to be no sure win?
   b. If c=4, explain how you could guarantee a sure win.
   c. If c=10, explain how you could guarantee a sure win.
   d. Use the arbitrage theorem to verify your answer to part (a).



Suppose at time 0, we buy x units of stock and buy y units of options.

$$value = \begin{cases} 150x + 25y & if\ price\ is\ \$150\ at\ time\ 1 \\ 25x & if\ price\ is\ \$25\ at\ time\ 1 \end{cases}$$

$Given\ x, select\ y\ such\ that\ 150x + 25y = 25x$
$y = -5x$

$By\ choosing\ y = -5x, the\ value\ of\ our\ position\ at\ time\ 1\ will\ always\ be\ given\ by\ the\ value = 25x.$
$value(at\ time\ 1) = 25x$

$original\ cost = 50x + cy$
$= 50x + c(-5x)$ $\qquad (y = -5x)$
$= 50x - 5cx$

$gain = value - original\ cost$
$= 25x - (50x - 5cx)$
$= 25x - 50x + 5cx$
$= -25x + 5cx$
$= x(-25 + 5c)$

$-25 + 5c = 0$
$5c = 25$
$c = 5$

<u>A:</u> In order for there to be no sure win, c should be 5

$c = 4$

$Since\ c \neq 5, we\ can\ select\ the\ sign\ of\ x\ such\ that\ gain = x(-25x + 5c) > 0$
$by\ letting\ x\ be\ negative\ when\ c < 5, and\ letting\ x\ be\ positive\ when\ c > 5.$

$In\ this\ case, c = 4 < 5, so\ we\ will\ let\ x\ be\ negative.$

$gain = x(-25x + 5c) > 0$
$x\big(-25 + 5(4)\big) > 0$
$x(-25 + 20) > 0$
$x(-5) > 0$
$-5x > 0$

Then by selling 1 unit of stock ($x = -1$) for $50, and buying 5 units of options ($y = -5x = -5(-1) = 5$) for 5*4=$20, we can guarantee a sure win.

$initial\ gain = 50x - 5cx$
$= 50 - 20$
$= 30$

$value(at\ time\ 1) = 25x$
$= 25(-1)$
$= -25$

$gain = initial\ gain + value(at\ time\ 1) =$
$= 30 - 25$
$= 5$

<u>A:</u> Selling 1 unit of stock and buying 5 units of options can guarantee a sure win/profit of $5

$c = 10$

$Since\ c \neq 5, we\ can\ select\ the\ sign\ of\ x\ such\ that\ gain = x(-25x + 5c) > 0$
$by\ letting\ x\ be\ negative\ when\ c < 5, and\ letting\ x\ be\ positive\ when\ c > 5.$

$In\ this\ case, c = 10 > 5, so\ we\ will\ let\ x\ be\ positive.$

$gain = x(-25x + 5c) > 0$
$x\big(-25 + 5(4)\big) > 0$
$x(-25 + 20) > 0$
$x(-5) > 0$
$-5x > 0$

Then by buying 1 unit of stock ($x = 1$) for $50, and selling 5 units of options ($y = -5x = -5(1) = -5$) for 5*10=$50, we can guarantee a sure win.

$initial\ cost = original\ cost$
$= 50 - 50$
$= 0$

$value(at\ time\ 1) = 25x$
$= 25(1)$
$= 25$

$gain = value(at\ time\ 1) - original\ cost$
$= 25 - 0$

$= 25$

A: Buying 1 unit of stock and selling 5 units of options can guarantee a sure win/profit of \$25

There are m=2 possible outcomes.

$Let\ X = (outcome\ at\ time\ 1)$

$$X = \begin{cases} 1 & if\ price\ is\ \$150\ at\ time\ 1 \\ 2 & if\ price\ is\ \$25\ at\ time\ 1 \end{cases}$$

There are n=2 wagers:
To buy (or sell) the stock at time 0 (i=1)
To buy (or sell) the option at time 0 (i=2)

By the arbitrage theorem, there is no arbitrage, provided there exists a vector $\vec{p} = (p_1, p_2) = (p_1, 1 - p_1)$ such that the expected return on each wager is zero. That is, $\vec{p}$ must satisfy:

$$E_{\vec{p}}(r_i(X)) = \sum_{j=1}^{2} p_j r_i(j)$$
$$= p_1 r_i(1) + p_2 r_i(2)$$
$$= p_1 r_i(1) + (1 - p_1) r_i(2)$$
$$= 0, \quad for\ all\ i = 1,2$$

return/gain on 1 unit of stock:

$$r_1(j) = \begin{cases} 150 - 50 & if\ j = 1 \\ 25 - 50 & if\ j = 2 \end{cases}$$

$$r_1(j) = \begin{cases} 100 & if\ j = 1 \\ -25 & if\ j = 2 \end{cases}$$

return/gain on 1 unit of option:

$$r_2(j) = \begin{cases} 25 - c & if\ j = 1 \\ -c & if\ j = 2 \end{cases}$$

If the expected return on one unit of stock is zero, then

$$0 = E_{\vec{p}}(r_1(X))$$
$$= \sum_{j=1}^{2} p_j r_1(j)$$
$$= p_1 r_1(1) + p_2 r_1(2)$$
$$= p_1 r_1(1) + (1 - p_1) r_1(2)$$
$$= p_1(100) + (1 - p_1)(-25)$$
$$p_1 = \frac{1}{5} = 0.2$$

Then,

$$p_2 = (1 - p_1)$$
$$= 1 - \frac{1}{5}$$

$= \frac{4}{5} = 0.8$

Then,

$\vec{p} = (p_1, p_2) = \left(\frac{1}{5}, \frac{4}{5}\right) = (0.2, 0.8)\ is\ a\ probability\ vector\ such\ that\ E_{\vec{p}}\big(r_1(X)\big) = 0$

Then to avoid arbitrage, it must also be true that for $p_1 = \frac{1}{5}$ and $p_2 = \frac{4}{5}$

$0 = E_{\vec{p}}\big(r_2(X)\big)$

$= \sum_{j=1}^{2} p_j r_2(j)$

$= p_1 r_2(1) + p_2 r_2(2)$

$= p_1(25 - c) + p_2(-c)$

$= \left(\frac{1}{5}\right)(25 - c) + \left(\frac{4}{5}\right)(-c)$

$= 5 - \frac{c}{5} - \frac{4c}{5}$

$= 5 - \frac{5c}{5}$

$= 5 - c$

$c = 5$ to avoid arbitrage ■