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Homework 3

Problem 3.1

$P' = -\alpha P(M-P) = \alpha P(P-M)$

$\frac{dP}{dt} = \alpha P(P-M)$

$\frac{dP}{P(P-M)} = \alpha dt$

$\int \frac{dP}{P(P-M)} = \int \alpha \, dt$

$\int_{P_0}^{P} \frac{dP}{P(P-M)} = \int_0^{T_P} \alpha \, dt$

$\rightarrow$ Partial fraction decomposition:

$\frac{1}{P(P-M)} = \frac{A}{P} + \frac{B}{P-M}$

$1 = A(P-M) + B(P)$

Let $P=0 \Rightarrow 1 = A(-M)$

$A = -\frac{1}{M}$

Let $P=M \Rightarrow 1 = BM$

$B = \frac{1}{M}$

$\frac{1}{P(P-M)} = \frac{-1/M}{P} + \frac{1/M}{P-M}$

$\int_{P_0}^{P} \left( \frac{-1/M}{P} + \frac{1/M}{P-M} \right) dP = \int_0^{T_P} \alpha \, dt$

$P_0 = 1000$
$M = 100$
$\alpha = 0.001$

$\int_{1000}^{P} \left( \frac{-1/100}{P} + \frac{1/100}{P-100} \right) dP = \int_0^{T_P} 0.001 \, dt$

$\frac{1}{100} \int_{1000}^{P} \left( -\frac{1}{P} + \frac{1}{P-100} \right) dP = 0.001 \int_0^{T_P} dt$

$\frac{1}{100} \int_{1000}^{P} \left( \frac{1}{P-100} - \frac{1}{P} \right) dP = 0.001 \int_0^{T_P} dt$

$\frac{1}{100} \left( \ln(P-100) - \ln(P) \Big|_{1000}^{P} \right. = 0.001 \left( t \Big|_0^{T_P} \right.$

$\frac{1}{100} \left( \ln\left( \frac{P-100}{P} \right) \Big|_{1000}^{P} \right. = 0.001 T_P$

$\ln\left( \frac{P-100}{P} \Big|_{1000}^{P} \right. = 0.1 T_P$

$\ln\left( \frac{P-100}{P} \right) - \ln\left( \frac{1000-100}{1000} \right) = 0.1 T_P$

$\ln\left( \frac{P-100}{P} \right) - \ln\left( \frac{900}{1000} \right) = 0.1 T_P$

$\ln\left( \frac{P-100}{P} \right) - \ln(0.9) = 0.1 T_P$

$P(t) \to \infty \quad \Rightarrow \quad \lim_{P \to \infty} \frac{P-100}{P} = 1$

$\ln(1) - \ln(0.9) = 0.1 T_P$

$T_P = \frac{\ln(1) - \ln(0.9)}{0.1}$

$T_P = \frac{0 + 0.1053605157}{0.1}$

$T_P = 1.053605157$

Problem 3.2

Let $T_1$ = temperature of water after $t=0 \Rightarrow t=19$

$$\frac{-T_1+100}{19-0} = 0.1\left(\frac{100+T_1}{2} - 29\right)$$

$$\frac{-T_1+100}{19} = 0.1\left(\frac{100+T_1}{2} - 29\right)$$

$$10\left(\frac{-T_1+100}{19}\right) = \frac{100+T_1}{2} - 29$$

$$20(-T_1+100) = 19(100+T_1) - 29(38)$$

$$-20T_1 + 2000 = 1900 + 19T_1 - 1102$$

$$-39T_1 + 2000 = 1900 - 1102$$

$$-39T_1 + 2000 = 798$$

$$-39T_1 = -1202$$

$$T_1 = 30.82051282 = 30.82°C$$

Let $T_m$ = temperature of mixing water at $t=19$

$T_1 = 30.82°C = 303.97\ K$

$T_2 = 0°C = 273.15\ K$

$m_1 = 3$ gallons

$m_2 = 6$ gallons

$$T_m = \frac{m_1T_1 + m_2T_2}{m_1+m_2} = \frac{3(303.97)+6(273.15)}{3+6} = 283.4233333 = 283.42\ K$$

$T_m = 283.42\ K = 10.27°C$

Let $T_f$ = temperature of water at $t=29$

$$\frac{T_f - 10.27}{29-19} = 0.1\left(29 - \frac{10.27+T_f}{2}\right)$$

$$\frac{T_f - 10.27}{10} = 0.1\left(29 - \frac{10.27+T_f}{2}\right)$$

$$T_f - 10.27 = 29 - \frac{10.27+T_f}{2}$$

$$2T_f - 20.54 = 58 - 10.27 - T_f$$

$$3T_f - 20.54 = 58 - 10.27$$

$$3T_f - 20.54 = 47.73$$

$$3T_f = 68.27$$

$$T_f = 22.75666667$$

$$T_f = 22.76°C$$

Problem 3.3

Phase1:

$$\begin{cases} m\frac{dv}{dy}v = mg - kv \\ v(y=0)=0 \\ v(y=h_1)=v_1 \end{cases}$$

$$\int_0^{v_0} m\frac{dv}{mg-kv}v = \int_0^{h_1} dy$$

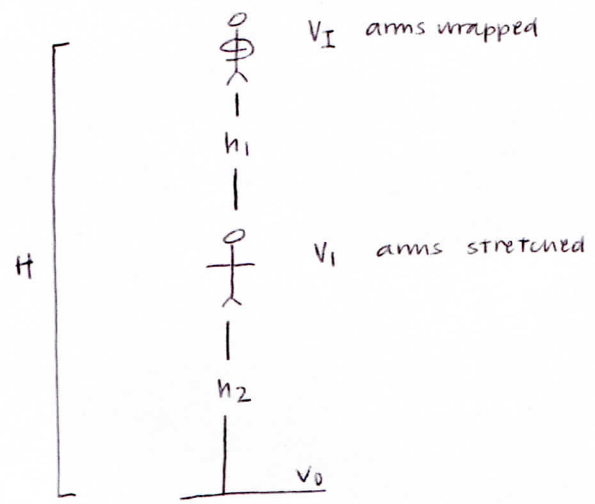
$$-\frac{m}{k}\int_0^{v_0}\left(1+\frac{mg}{k}\cdot\frac{1}{v-\frac{mg}{k}}\right)dv = \int_0^{h_1} dy$$

Let $v_T = \frac{mg}{k}$.

$$-\frac{v_T}{g}\int_0^{v_0}\left(1+\frac{v_T}{v-v_T}\right)dv = \int_0^{h_1} dy$$

$$v_1 + v_T\ln\left|\frac{v_1}{v_T}-1\right| = -\frac{gh_1}{v_T}$$

visualization:




Phase 2:

$$\begin{cases} m\frac{dv}{dy}v = mg - \beta kv \\ v(y=h_1)=v_1 \\ v(y=H)=v_0 \end{cases}$$

$$\int_{v_1}^{v_0} m\frac{dv}{mg-\beta kv}v = \int_{h_1}^{H} dy$$

$$(v_0-v_1)+\frac{v_T}{\beta}\ln\left|\frac{\beta v_0-v_T}{\beta v_1-v_T}\right| = -\frac{\beta g(H-h_1)}{v_T}$$

$$\begin{cases} v_1+v_T\ln\left|\frac{v_1}{v_T}-1\right| = -\frac{gh_1}{v_T} & \text{Phase 1} \\ (v_0-v_1)+\frac{v_T}{\beta}\ln\left|\frac{\beta v_0-v_T}{\beta v_1-v_T}\right| = -\frac{\beta g(H-h_1)}{v_T} & \text{Phase 2} \end{cases}$$

Total travel falling time:

Phase 1:

$$\begin{cases} m\frac{dv}{dt} = mg-kv \\ v(t=0)=0 \\ v(t=t_1)=v_1 \end{cases}$$

$$-\frac{v_T}{g}\int_0^{v_1}\frac{1}{v-v_T}dv = \int_0^{t_1} dt$$

$$t_1 = -\frac{v_T}{g}\ln\left|\frac{v_1}{v_T}-1\right|$$

Phase 2:

$$\begin{cases} m\frac{dv}{dt} = mg-\beta kv \\ v(t=0)=v_1 \\ v(t=t_2)=v_0 \end{cases}$$

$$\int_{v_1}^{v_0}\frac{m}{mg-\beta kv}dv = \int_0^{t_2} dt$$

$$t_2 = \frac{v_T}{\beta g}\ln\left|\frac{\beta v_1-v_T}{\beta v_0-v_T}\right|$$

Total falling time $= T = t_1+t_2 = -\frac{v_T}{g}\ln\left|\frac{v_1}{v_T}-1\right| + \frac{v_T}{\beta g}\ln\left|\frac{\beta v_1-v_T}{\beta v_0-v_T}\right|$

Problem 3.4

First block: resistance proportional to bullet speed, v

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

Let $x_M$ = farthest distance bullet can travel

$$\begin{cases} m\frac{dv}{dx}v = -k_1 v \\ v(x=0) = v_0 \\ v(x_M) = 0 \end{cases}$$

$$-\int_{v_0}^{0} \frac{m}{k_1}\,dv = \int_0^{x_M} dx$$

$$x_M = \frac{m}{k_1} v_0$$

Second block: resistance proportional to $v^{3/2}$

$$\begin{cases} m\frac{dv}{dx}v = -k_2 v^{3/2} \\ v(x=0) = v_0 \\ v(x_M) = 0 \end{cases}$$

$$-\int_{v_0}^{0} \frac{m}{k_2} v^{1-3/2}\,dv = \int_0^{x_M} dx$$

$$x_M = \frac{m}{k_2} 2\sqrt{v_0}$$

Third block: resistance proportional to $v^2$

$$\begin{cases} m\frac{dv}{dx}v = -k_3 v^2 \\ v(x=0) = v_0 \\ v(x_M) = 0 \end{cases}$$

$$-\int_{v_0}^{0} \frac{m}{k_3}\frac{dv}{v} = \int_0^{x_M} dx$$

$$x_M = -\frac{m}{k_3} \ln v \Big|_{v_0}^{0} \to \infty$$