Chapter 4

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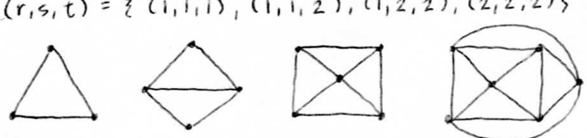
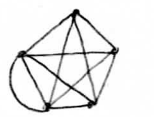
5. (ii) For which values of r, s, t is the complete tripartite graph $K_{r,s,t}$ planar?

We do not want a graph that is a subgraph of $K_{3,3}$, so we cannot have two 3's as the values. None of the values can be greater than 3. If one of the values is 3, then the other two values must be ones or twos. We can have three 2's as the values.

$(r,s,t) = \{(1,1,1), (1,1,2), (1,2,2), (2,2,2)\}$

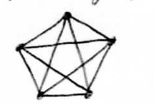


8. Give an example of:
(i) a non-planar graph that is not homeomorphic to K_5 or $K_{3,3}$.



This graph is not homeomorphic to K_5 or $K_{3,3}$ because it cannot be obtained by adding or deleting vertices of degree 2.

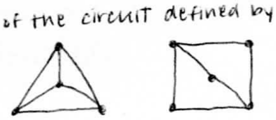
(ii) a non-planar graph that is not contractible to K_5 or $K_{3,3}$.



This graph is not contractible to K_5 or $K_{3,3}$.

Why does the existence of these graphs not contradict the theorems?
Because these graphs have a subgraph that is homeomorphic or contractible to K_5 or $K_{3,3}$

10. (i) Show that K_4 and $K_{2,3}$ are not outerplanar.
K_4 has circuits of length 3, so one vertex is left out of the circuit defined by the unbounded region.
$K_{2,3}$ has circuits of length 4, so one vertex is left out of the circuit defined by the unbounded region.



(ii) Show if G is an outerplanar graph, then G has no subgraph homeomorphic to K_4 or $K_{2,3}$.
By contradiction, if we replace a path in a subgraph that is not outerplanar with an edge, then the new graph is not outerplanar. K_4 and $K_{2,3}$ are not outerplanar. If a subgraph is homeomorphic to K_4 or $K_{2,3}$ then the subgraph is also not outerplanar.

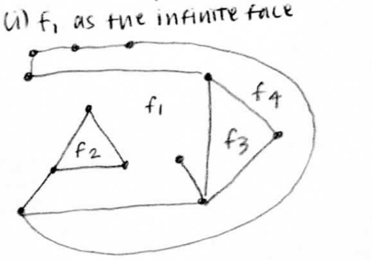
13. Verify Euler's formula for:
(i) the wheel W_8
$m = 14$
$n = 8$
$f = 8$ (7 bounded + 1 unbounded)
$n - m + f \stackrel{?}{=} 2$
$8 - 14 + 8 \stackrel{?}{=} 2$
$-6 + 8 \stackrel{?}{=} 2$
$2 = 2$ ✓ verified.

(ii) the graph of the octahedron
$m = 12$
$n = 6$
$f = 8$ (7 bounded + 1 unbounded)
$n - m + f \stackrel{?}{=} 2$
$6 - 12 + 8 \stackrel{?}{=} 2$
$-6 + 8 \stackrel{?}{=} 2$
$2 = 2$ ✓ verified.

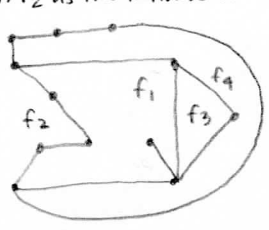
(iii) graph of Fig.
$m = 15$
$n = 9$
$f = 8$ (7 bounded + 1 unbounded)
$n - m + f \stackrel{?}{=} 2$
$9 - 15 + 8 \stackrel{?}{=} 2$
$-6 + 8 \stackrel{?}{=} 2$
$2 = 2$ ✓ verified.

(iv) $K_{2,7}$
$m = 14$
$n = 9$
$f = 7$
$n - m + f \stackrel{?}{=} 2$
$9 - 14 + 7 \stackrel{?}{=} 2$
$-5 + 7 \stackrel{?}{=} 2$
$2 = 2$ ✓ verified

14. Redraw the graph with:
(i) f_1 as the infinite face



(ii) f_2 as the infinite face



15. (i) $5f \le \sum \deg = 2m$
$5f \le 2m$
$f \le 2/5\, m$
$m - n + 2 \le 2/5\, m$
$3/5\, m - n + 2 \le 0$
$3m - 5n + 10 \le 0$
$m \le \frac{5n - 10}{3}$

(ii) $rf \le 2m$
$f \le \frac{2m}{r}$
$m - n + 2 \le \frac{2m}{r}$
$m \le \frac{2m}{r} + n - 2$
$m \le \frac{2m}{r} + \frac{nr}{r} - \frac{2r}{r}$
$m \le \frac{2m + nr - 2r}{r}$

Petersen graph: $15 \le \frac{5(10) - 10}{3}$ is not true, so Petersen graph is nonplanar.

16. (i) Show that G must have at least 12 pentagonal faces.

$n - m + f = 2$ $\quad n = m - f + 2$
$f = m - n + 2$
$\sum \deg = 2m$
$\sum bdy = 2m$
boundary is either 5 or 6
all deg ≥ 3
$3n \le \sum bdy = 2m$
$3n \le 2m$
$n \le 2/3\, m$
$m - f + 2 \le 2/3\, m$
$1/3\, m - f + 2 \le 0$
$m - 3f + 6 \le 0$
$m \le 3f - 6$

$m \le 3f - 6$
$\sum bdy = 2m = 2(3f - 6)$
$\sum bdy = 6f - 12$
Let f_5, f_6 = # of pentagon and hexagon faces
$f_5 + f_6 = f$
$5f_5 + 6f_6 \le \sum bdy = 6f - 12$
$5f_5 + 6f_6 \le 6f - 12$
$5f_5 + 6f_6 \le 6(f_5 + f_6) - 12$
$5f_5 + 6f_6 \le 6f_5 + 6f_6 - 12$
$f_5 \ge 12$

16. (ii) $3n = 2m$
$n = 2/3\, m$
$m - f + 2 = 2/3\, m$
$1/3\, m - f + 2 = 0$
$m - 3f + 6 = 0$
$m = 3f - 6$

$\sum bdy = 2m = 2(3f - 6)$
$\sum bdy = 6f - 12$
Let f_5, f_6 = # of pentagon and hexagon faces
$f_5 + f_6 = f$
$5f_5 + 6f_6 = 6f - 12$
$5f_5 + 6f_6 = 6(f_5 + f_6) - 12$
$5f_5 + 6f_6 = 6f_5 + 6f_6 - 12$
$f_5 = 12$

17. (i) Prove that G has a face bounded by at most 4 edges.

$bdy \geq 3$

$deg \geq 3$

$3n \leq \sum deg = 2m$

$3n \leq 2m$

$n \leq 2/3 m$

$m = f + 2 \leq 2/3 m$

$m \leq 3f - 6$

contradiction: assume $bdy \geq 5$

$5f \leq \sum bdy = 2m \leq 6f - 12$

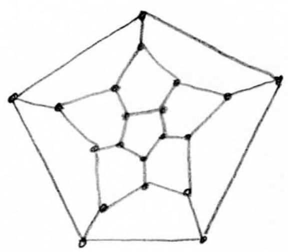
$5f \leq 6f - 12$

$0 \leq f - 12$

$f \geq 12$

contradiction because the problem said that $f < 12$

(ii) Give an example to show that the result in (i) is false if G has 12 faces.



The platonic solid, dodecahedron has 12 faces. The boundary is 5 edges for a dodecahedron.

18. (i) Let c_i = # of i-gons

$3c_3 + 4c_4 + 5c_5 + \ldots = \sum bdy = 2m$

$3c_3 + 4c_4 + 5c_5 + \ldots = 2(3f - 6)$

$3c_3 + 4c_4 + 5c_5 + \ldots = 6f - 12$

$c_3 + c_4 + c_5 + \ldots = f$

$3c_3 + 4c_4 + 5c_5 + \ldots = 6(c_3 + c_4 + c_5 + \ldots) - 12$

$3c_3 + 4c_4 + 5c_5 + \ldots = 6c_3 + 6c_4 + 6c_5 + \ldots - 12$

$3c_3 + 2c_4 + c_5 + \ldots = 12$

(ii) we can see in the formula

$3c_3 + 2c_4 + c_5 - c_7 - 2c_8 - 3c_9 - \ldots = 12$

that the positive coefficients stop at 5.

19. (i) Prove that G and G^c cannot both be planar.

$m \leq 3n - 6$

Let G have exactly 11 vertices

$n = 11$

$m \leq 3n - 6$

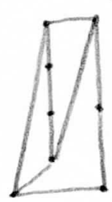
$m \leq 3(11) - 6$

$m \leq 33 - 6$

$m \leq 27$

There are 55 edges in K_{11}. But you cannot split 55 edges into 2 groups in which $m \leq 27$ is satisfied.

19. (ii) Find a graph with 8 vertices G and G^c are both planar.



G

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24. (a) $d_1 v = \sum deg = 2e = \sum bdy = d_2 r$

$e = \frac{d_1}{2} v$

$r = \frac{d_1}{d_2} v$

(b) $r = e - v + 2$

$\frac{d_1}{d_2} v = \frac{d_1}{2} v - v + 2$

$2d_1 v = d_1 d_2 v + 2($

$2d_1 v = d_1 d_2 v + 4d_2$

$2d_1 v + 2d_2 v - d_1 d_2 v = 4d_2$

$v(2d_1 + 2d_2 - d_1 d_2) = 4d_2$

(c) $v(2d_1 + 2d_2 - d_1 d_2) = 4d_2$

v is positive

$4d_2$ is positive

so $(2d_1 + 2d_2 - d_1 d_2)$ must be > 0

$d_1 d_2 - 2d_1 - 2d_2 < 0$

$d_1 d_2 - 2d_1 - 2d_2 + 4 < 4$

$(d_1 - 2)(d_2 - 2) < 4$

(d)

d_1	d_2
3	3
3	4
3	5
4	3
5	3