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Homework 6

Problem 6.1

$$\left(x\frac{d}{dx}-1\right)^{2020}, \quad y(x)=0$$

Let $x=\exp(t)$

$$\left(\exp(t)\frac{d}{\exp(t)\,dt}-1\right)^{2020} y=0$$

$$\left(\frac{d}{dt}-1\right)^{2020} y=0$$

$$(r-1)^{2020}=0$$

There are 2020 identical real roots.

$$y_c(t)=(c_1+c_2t+c_3t^2+\dots+c_{2020}t^{2019})\exp(t)$$
$$=\left(\sum_{k=1}^{2020} c_k t^{k-1}\right)\exp(t).$$

$$y_c(x)=\left(\sum_{k=1}^{2020} c_k (\ln x)^{k-1}\right)x$$

$$y_p=At^{2020}\exp(t)$$

$$LHS=\left(\frac{d}{dt}-1\right)^{2020}(At^{2020}\exp(t))$$
$$=\left(\frac{d}{dt}-1\right)^{2019}\left(\frac{d}{dt}-1\right)(At^{2020}\exp(t))$$
$$=\left(\frac{d}{dt}-1\right)^{2019}\left(\frac{d}{dt}(At^{2020}\exp(t))-At^{2020}\exp(t)\right)$$
$$=\left(\frac{d}{dt}-1\right)^{2019}(2020At^{2019}\exp(t))$$
$$=\left(\frac{d}{dt}-1\right)^{2018}\left(\frac{d}{dt}-1\right)(2020At^{2019}\exp(t))$$
$$=\left(\frac{d}{dt}-1\right)^{2018}\left(\frac{d}{dt}(2020At^{2019}\exp(t))-2020At^{2019}\exp(t)\right)$$
$$=\left(\frac{d}{dt}-1\right)^{2018}2020A(2019)t^{2018}\exp(t)$$
$$=\dots$$
$$=2020!\,A\exp(t)$$

$$\left(\frac{d}{dt}-1\right)^{2020}(At^{2020}\exp(t))=\exp(t)$$

$$A(2020!)\exp(t)=\exp(t)$$

$$\Rightarrow A=\frac{1}{!}$$

GS: $y(t)=\exp(t)(c_1+c_2t+c_3t^2+\dots+c_{2020}t^{2019})+\frac{1}{2020!}t^{2020}\exp(t)$

$y(x)=x(c_1+c_2\ln x+c_3\ln^2 x+\dots+c_{2020}(\ln x)^{2019})+\frac{1}{2020!}(\ln x)^{2020}x$

Problem 6.2

$$y''' + y'' + y' + y = 1 + \cos x + \sin x + e^{-x} + e^{x}$$

$y''' + y'' + y' + y = 0$

$r^3 + r^2 + r + 1 = 0$

$r^2(r+1) + (r+1) = 0$

$(r^2+1)(r+1) = 0$

$r = \pm i, -1$

$y_c = c_1 \exp(-x) + c_2 \exp(ix) + c_3 \exp(-ix)$

$y_p = A + x(B\cos x + C\sin x) + D\cos x + E\sin x + Fx\exp(-x) + G\exp(x)$

$y_p' = x(-B\sin x + C\cos x) + B\cos x + C\sin x - D\sin x + E\cos x - Fx\exp(-x) + F\exp(-x) + G\exp(x)$

$y_p'' = x(-B\cos x - C\sin x) - B\sin x + C\cos x - B\sin x + C\cos x - D\cos x - E\sin x + Fx\exp(-x) - F\exp(-x) - F\exp(-x) + G\exp(x)$

$y_p''' = x(B\sin x - C\cos x) - B\cos x - C\sin x - B\cos x - C\sin x - B\cos x - C\sin x + D\sin x - E\cos x - Fx\exp(-x) + F\exp(-x) + F\exp(-x)$
$F\exp(-x) + G\exp(x)$

$A + x(B\cos x + C\sin x) + D\cos x + E\sin x + Fx\exp(-x) + G\exp(x) + x(-B\sin x + C\cos x) + B\cos x + C\sin x - D\sin x + E\cos x - Fx\exp(-x) +$
$F\exp(-x) + G\exp(x) + x(-B\cos x - C\sin x) - B\sin x + C\cos x - B\sin x + C\cos x - D\cos x - E\sin x + Fx\exp(-x) - F\exp(-x) - F\exp(-x) +$
$G\exp(x) + x(B\sin x - C\cos x) - B\cos x - C\sin x - B\cos x - C\sin x - B\cos x - C\sin x + D\sin x - E\cos x - Fx\exp(-x) + F\exp(-x) +$
$F\exp(-x) + F\exp(-x) + G\exp(x) = 1 + \cos x + \sin x + e^{-x} + e^{x}$

$A = 1$

$D + B + E + C + C - D - B - B - B - E + Bx + Cx - Bx - Cx = 1$

$-2B + 2C = 1$

$E + C - D - B - B - E - C - C - C + D + Cx - Bx - Cx + Bx = 1$

$-2C - 2B = 1$

$-2B + 2C = 1$

$-2B - 2C = 1$

$-4B = 2$

$B = -\frac{1}{2}$

$C = 0$

$Fx - Fx + F + Fx - F - F - Fx + F + F + F = 1$

$2F = 1$

$F = \frac{1}{2}$

$4G = 1$

$G = \frac{1}{4}$

GS: $c_1\exp(-x) + c_2\exp(ix) + c_3\exp(-ix) + 1 + x\left(-\frac{1}{2}\cos x\right) + \frac{1}{2}x\exp(-x) + \frac{1}{4}\exp(x)$

Problem 6.3

$$x^2y'' + xy' + 19^2 y = \cos(19\ln x) + \sin(19\ln x)$$

$$y(x) = x^r$$
$$y'(x) = rx^{r-1}$$
$$y''(x) = r(r-1)x^{r-2}$$

$$x^2 r(r-1)x^{r-2} + x r x^{r-1} + 19^2 x^r = 0$$
$$x^r(r^2-r) + x^r r + x^r 19^2 = 0$$
$$x^r(r^2 - r + r + 19^2) = 0$$
$$x^r(r^2 + 19^2) = 0$$
$$r = \pm 19i$$

$$y_c = c_1 \exp(19ix) + c_2 \exp(-19ix)$$

$$y_p = x(A\cos(19\ln x) + B\sin(19\ln x))$$
$$y_p' = x\left(-A\sin(19\ln x)\left(\tfrac{19}{x}\right) + B\cos(19\ln x)\left(\tfrac{19}{x}\right)\right) + (A\cos(19\ln x) + B\sin(19\ln x))$$
$$= x\left(-\frac{19A\sin(19\ln x)}{x} + \frac{19B\cos(19\ln x)}{x}\right) + (A\cos(19\ln x) + B\sin(19\ln x))$$
$$y_p'' = x\left(\frac{x(-19A\cos(19\ln x)(\frac{19}{x})) + (19A\sin(19\ln x))}{x^2} + \frac{x(-19B\sin(19\ln x)(\frac{19}{x})) + (19B\cos(19\ln x))}{x^2}\right) +$$
$$\left(-\frac{19A\sin(19\ln x)}{x} + \frac{19B\cos(19\ln x)}{x}\right) + \left(-A\sin(19\ln x)\left(\tfrac{19}{x}\right)\right) + B\cos(19\ln x)\left(\tfrac{19}{x}\right)$$
$$= x\left(\frac{-19^2A\cos(19\ln x) + (19A\sin(19\ln x))}{x^2} + \frac{-19^2B\sin(19\ln x) + 19B\cos(19\ln x)}{x^2}\right) +$$
$$\left(\frac{-19A\sin(19\ln x)}{x} + \frac{19B\cos(19\ln x)}{x}\right) + \left(\frac{-19A\sin(19\ln x)}{x} + \frac{19B\cos(19\ln x)}{x}\right)$$
$$= \frac{-19^2A\cos(19\ln x) + 19A\sin(19\ln x) - 19^2B\sin(19\ln x) + 19B\cos(19\ln x) - 38A\sin(19\ln x) + 38B\cos(19\ln x)}{x}$$

$$= \frac{-19^2A\cos(19\ln x) - 19A\sin(19\ln x) - 19^2B\sin(19\ln x) + 57B\cos(19\ln x)}{x}$$

$$x^2y'' + xy' + 19^2y = x(-19^2A\cos(19\ln x) - 19A\sin(19\ln x) - 19^2B\sin(19\ln x) + 57B\cos(19\ln x)) + x(-19A\sin(19\ln x) + 19B\cos(19\ln x)) +$$
$$x(A\cos(19\ln x) + B\sin(19\ln x)) + 19^2x(A\cos(19\ln x) + B\sin(19\ln x))$$
$$= -19^2xA\cos(19\ln x) - 19xA\sin(19\ln x) - 19^2xB\sin(19\ln x) + 57xB\cos(19\ln x) - 19xA\sin(19\ln x) +$$
$$19xB\cos(19\ln x) + xA\cos(19\ln x) + xB\sin(19\ln x) + 19^2xA\cos(19\ln x) + 19^2xB\sin(19\ln x)$$
$$= -38xA\sin(19\ln x) + xB\sin(19\ln x) + 76xB\cos(19\ln x) + xA\cos(19\ln x)$$

$$-38A + B = 1$$
$$76B + A = 1$$
$$A = \frac{-25}{963}$$
$$B = \frac{39}{963}$$

GS: $y(x) = C_1\exp(19ix) + C_2\exp(-19ix) + x\left(-\frac{25}{963}\cos(19\ln x) + \frac{39}{963}\sin(19\ln x)\right)$

Problem 6.4

$y'' + y = \tan(19x)$

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$ $\quad y_1 = \cos(x) \quad y_2 = \sin(x)$

$y_c = c_1 \cos(x) + c_2 \sin(x)$

$y_1' = -\sin(x) \quad y_2' = \cos(x)$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f(x)$

$f(x) = \tan(19x)$

$$K(x,t) = \frac{y_2(x)\, y_1(t) - y_1(x)\, y_2(t)}{W[t]}$$

$$= \frac{\sin x \cos t - \cos x \sin t}{1}$$

$$= \sin(x-t)$$

$$y_p(x) = \int^x K(x,t)\, f(t)\, dt$$

$$= \int^x \sin(x-t) \tan(19t)\, dt$$

$$= -\cos x \ln|\sec x + \tan x|$$

GS: $c_1 \cos(x) + c_2 \sin(x) - \cos x \ln|\sec x + \tan x|$