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Homework 7

Problem 7.1

$$\begin{cases} x' = 4x-3y \\ y' = 3x+4y \\ x(0)=1 \\ y(0)=9 \end{cases}$$

$x' = 4x-3y$

$3y = 4x - x'$

$y = \frac{4}{3}x - \frac{1}{3}x'$

$y' = \frac{4}{3}x' - \frac{1}{3}x''$

$\frac{4}{3}x' - \frac{1}{3}x'' = 3x+4y$

$\frac{4}{3}x' - \frac{1}{3}x'' = 3x+4\left(\frac{4}{3}x - \frac{1}{3}x'\right)$

$\frac{4}{3}x' - \frac{1}{3}x'' = 3x + \frac{16}{3}x - \frac{4}{3}x'$

$4x' - x'' = 9x + 16x - 4x'$

$-x'' + 8x' - 25x = 0$

$x'' - 8x' + 25x = 0$

$r^2 - 8r + 25 = 0$

$r_{1,2} = 4 \pm 3i$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{8 \pm \sqrt{64-100}}{2}$$

$$= \frac{8 \pm \sqrt{-36}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

$$= 4 \pm 3i$$

$x(t) = c_1 \exp((4+3i)x) + c_2 \exp((4-3i)x)$

$y = \frac{4}{3}x - \frac{1}{3}x'$

$= \frac{4}{3}\left(c_1 \exp((4+3i)x) + c_2 \exp((4-3i)x)\right) - \frac{1}{3}\left(c_1 \exp((4+3i)x) + c_2 \exp((4-3i)x)\right)'$

$= \frac{4}{3}\left(c_1 \exp((4+3i)x) + c_2 \exp((4-3i)x)\right) - \frac{1}{3}\left((4+3i)c_1 \exp((4+3i)x) + (4-3i)c_2 \exp((4-3i)x)\right)$

$= -i\, c_1 \exp((4+3i)x) + i\, c_2 \exp((4-3i)x)$

$$\begin{cases} x(t) = c_1 \exp((4+3i)x) + c_2 \exp((4-3i)x) \\ y(t) = -i\, c_1 \exp((4+3i)x) + i\, c_2 \exp((4-3i)x) \end{cases}$$

$$\begin{cases} x(0) = 1 = c_1 + c_2 \\ y(0) = 9 = -i c_1 + i c_2 \end{cases}$$

problem 7.2

$$\begin{cases} x' = x - 3y \\ y' = 3x + 7y \\ x(0) = 8 \\ y(0) = 4 \end{cases}$$

$x' = x - 3y$

$3y = x - x'$

$y = \frac{1}{3}x - \frac{1}{3}x'$

$y' = \frac{1}{3}x' - \frac{1}{3}x''$

$\frac{1}{3}x' - \frac{1}{3}x'' = 3x + 7y$

$\frac{1}{3}x' - \frac{1}{3}x'' = 3x + 7\left(\frac{1}{3}x - \frac{1}{3}x'\right)$

$\frac{1}{3}x' - \frac{1}{3}x'' = 3x + \frac{7}{3}x - \frac{7}{3}x'$

$x' - x'' = 9x + 7x - 7x'$

$-x'' + 8x' - 16x = 0$

$x'' - 8x' + 16x = 0$

$r^2 - 8r + 16 = 0$

$(r-4)(r-4) = 0$

$r = 4$ multiplicity 2

$x(t) = c_1 \exp(4x) + c_2 x \exp(4x)$

$y = \frac{1}{3}x - \frac{1}{3}x'$

$= \frac{1}{3}(c_1 \exp(4x) + c_2 x \exp(4x)) - \frac{1}{3}(c_1 \exp(4x) + c_2 x \exp(4x))'$

$= \frac{1}{3}(c_1 \exp(4x) + c_2 x \exp(4x)) - \frac{1}{3}(4c_1 \exp(4x) + c_2 \exp(4x) + 4x c_2 \exp(4x))$

$= -c_1 \exp(4x) - c_2 x \exp(4x) - \frac{1}{3} c_2 \exp(4x)$

$$\begin{cases} x(t) = c_1 \exp(4x) + c_2 x \exp(4x) \\ y(t) = -c_1 \exp(4x) - c_2 x \exp(4x) - \frac{1}{3} c_2 \exp(4x) \end{cases}$$

$$\begin{cases} x(0) = 8 = c_1 + c_2 \\ y(0) = 4 = -c_1 - c_2 \end{cases}$$

Problem 7.3

$$X' = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} X \qquad \text{same as} \qquad \begin{aligned} x' &= x - 2y \\ y' &= 3x + 4y \end{aligned}$$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -2 \\ 3 & 4-\lambda \end{pmatrix}$$

$$= (1-\lambda)(4-\lambda) - (-6)$$

$$= 4 - 5\lambda + \lambda^2 + 6$$

$$= \lambda^2 - 5\lambda + 10$$

$$= 0$$

$$\lambda_{1,2} = \frac{5 \pm 5i\sqrt{15}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 - 4(100)}}{2}$$

$$= \frac{5 \pm \sqrt{-375}}{2}$$

$$= \frac{5 \pm \sqrt{25 \cdot 15 \cdot -1}}{2}$$

$$= \frac{5 \pm 5i\sqrt{15}}{2}$$

$$\lambda_1 = \frac{5 + 5i\sqrt{15}}{2}$$

$$\begin{pmatrix} 1 - \left(\frac{5+5i\sqrt{15}}{2}\right) & -2 \\ 3 & 4 - \left(\frac{5+5i\sqrt{15}}{2}\right) \end{pmatrix} \begin{pmatrix} v_{1a} \\ v_{1b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(1 - \left(\frac{5+5i\sqrt{15}}{2}\right)\right) v_{1a} - 2 v_{1b} = 0$$

$$V_1 = \begin{pmatrix} 1 \\ (1 - (\frac{5+5i\sqrt{15}}{2}))/2 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 1 \\ (1 - (\frac{5+5i\sqrt{15}}{2}))/2 \end{pmatrix} \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)t\right)$$

$$\lambda_2 = \frac{5 - 5i\sqrt{15}}{2}$$

$$\begin{pmatrix} 1 - \left(\frac{5-5i\sqrt{15}}{2}\right) & -2 \\ 3 & 4 - \left(\frac{5-5i\sqrt{15}}{2}\right) \end{pmatrix} \begin{pmatrix} v_{2a} \\ v_{2b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(1 - \left(\frac{5-5i\sqrt{15}}{2}\right)\right) v_{2a} - 2 v_{2b} = 0$$

$$V_2 = \begin{pmatrix} 1 \\ (1 - (\frac{5-5i\sqrt{15}}{2}))/2 \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ (1 - (\frac{5-5i\sqrt{15}}{2}))/2 \end{pmatrix} \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)t\right)$$

$$X(t) = c_1 \begin{pmatrix} 1 \\ (1 - (\frac{5+5i\sqrt{15}}{2}))/2 \end{pmatrix} \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)t\right) + c_2 \begin{pmatrix} 1 \\ (1 - (\frac{5-5i\sqrt{15}}{2}))/2 \end{pmatrix} \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)t\right)$$

Another method:

$$x' = x - 2y$$
$$y' = 3x + 4y$$

$$x' = x - 2y$$
$$2y = x - x'$$
$$y = \frac{1}{2}x - \frac{1}{2}x'$$
$$y' = \frac{1}{2}x' - \frac{1}{2}x''$$

$$\frac{1}{2}x' - \frac{1}{2}x'' = 3x + 4y$$
$$\frac{1}{2}x' - \frac{1}{2}x'' = 3x + 4\left(\frac{1}{2}x - \frac{1}{2}x'\right)$$
$$\frac{1}{2}x' - \frac{1}{2}x'' = 3x + 2x - 2x'$$
$$x' - x'' = 6x + 4x - 4x'$$
$$-x'' + 5x' - 10x = 0$$
$$x'' - 5x' + 10x = 0$$
$$r^2 - 5r + 10 = 0$$
$$r_{1,2} = \frac{5 \pm 5i\sqrt{15}}{2}$$

$$x(t) = c_1 \exp\left(\left(\frac{5 + 5i\sqrt{15}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{5 - 5i\sqrt{15}}{2}\right)x\right)$$

$$y = \frac{1}{2}x - \frac{1}{2}x'$$
$$= \frac{1}{2}\left(c_1 \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)x\right)\right) - \frac{1}{2}\left(c_1 \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)x\right)\right)'$$
$$= \frac{1}{2}\left(c_1 \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)x\right)\right) - \frac{1}{2}\left(\left(\frac{5+5i\sqrt{15}}{2}\right)c_1 \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)x\right) + \left(\frac{5-5i\sqrt{15}}{2}\right)c_2 \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)x\right)\right)$$
$$= \left(\frac{1}{2} - \frac{1}{2}\left(\frac{5+5i\sqrt{15}}{2}\right)\right)c_1 \exp\left(\frac{5+5i\sqrt{15}}{2}x\right) + \left(\frac{1}{2} - \frac{1}{2}\left(\frac{5-5i\sqrt{15}}{2}\right)\right)c_2 \exp\left(\frac{5-5i\sqrt{15}}{2}x\right)$$

$$\begin{cases} x(t) = c_1 \exp\left(\left(\frac{5+5i\sqrt{15}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{5-5i\sqrt{15}}{2}\right)x\right) \\ y(t) = \left(\frac{1}{2} - \frac{1}{2}\left(\frac{5+5i\sqrt{15}}{2}\right)\right)c_1 \exp\left(\frac{5+5i\sqrt{15}}{2}x\right) + \left(\frac{1}{2} - \frac{1}{2}\left(\frac{5-5i\sqrt{15}}{2}\right)\right)c_2 \exp\left(\frac{5-5i\sqrt{15}}{2}x\right) \end{cases}$$