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Homework 5

Problem 5.1

$y''' + y'' + y' + y = 1 + e^{x} + e^{-x} + e^{2x} + e^{-2x}$

$y''' + y'' + y' + y = 0$

$r^3 + r^2 + r + 1 = 0$

$(r^2+1)(r+1) = 0$

$r = \pm i \, ; \, r = -1$

$y_c = c_1 \exp(-x) + c_2 \exp(ix) + c_3 \exp(-ix)$

$y_p = A\exp(x) + Bx\exp(-x) + C\exp(2x) + D\exp(-2x) + E$

$y_p' = A\exp(x) + (-Bx\exp(-x) + B\exp(-x)) + 2C\exp(2x) - 2D\exp(-2x)$

$= A\exp(x) - Bx\exp(-x) + B\exp(-x) + 2C\exp(2x) - 2D\exp(-2x)$

$y_p'' = A\exp(x) - (-Bx\exp(-x) + B\exp(-x)) + (-B\exp(-x)) + 4C\exp(2x) + 4D\exp(-2x)$

$= A\exp(x) + Bx\exp(-x) - B\exp(-x) - B\exp(-x) + 4C\exp(2x) + 4D\exp(-2x)$

$= A\exp(x) + Bx\exp(-x) - B\exp(-x) - B\exp(-x) + 8C\exp(2x) - 8D\exp(-2x)$

$y_p''' = A\exp(x) + (-Bx\exp(-x) + B\exp(-x)) + B\exp(-x) + B\exp(-x) + 8C\exp(2x) - 8D\exp(2x)$

$= A\exp(x) - Bx\exp(-x) + 3B\exp(-x) + 8C\exp(2x) - 8D\exp(2x)$

$4A = 1$

$A = \frac{1}{4}$

$Bx - Bx + B + Bx - B - B - Bx + 3B = 1$

$2B = 1$

$B = \frac{1}{2}$

$C + 2C + 4C + 8C = 1$

$15C = 1$

$C = \frac{1}{15}$

$D - 2D + 4D - 8D = 1$

$-5D = 1$

$D = -\frac{1}{5}$

$E = 1$

$y_p = \frac{1}{4}\exp(x) + \frac{1}{2}x\exp(-x) + \frac{1}{15}\exp(2x) - \frac{1}{5}\exp(-2x) + 1$ PS

GS:

$y = c_1\exp(-x) + c_2\exp(ix) + c_3\exp(-ix) + \frac{1}{4}\exp(x) + \frac{1}{2}x\exp(-x) + \frac{1}{15}\exp(2x) - \frac{1}{5}\exp(-2x) + 1$

Problem 5.2

$$y^{(4)} - y^{(3)} - y^{(2)} - y^{(1)} - 2y = 1 + x + x^2 + x^3$$

$$y^{(4)} - y^{(3)} - y^{(2)} - y^{(1)} - 2y = 0$$

$$r^4 - r^3 - r^2 - r - 2 = 0$$

$$(r+1)(r-2)(r^2+1) = 0$$

$$r = -1, 2, \pm i$$

$$y_c = C_1 \exp(-x) + C_2 \exp(2x) + C_3 \exp(ix) + C_4 \exp(-ix)$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

$$y_p''' = 6A$$

$$y_p'''' = 0$$

$$\text{LHS} = y_p^{(4)} - y_p^{(3)} - y_p^{(2)} - y_p^{(1)} - 2y_p$$

$$= 0 - 6A - 6Ax - 2B - 3Ax^2 - 2Bx - C - 2Ax^3 - 2Bx^2 - 2Cx - 2D$$

$$\text{RHS} = x^3 + x^2 + x + 1$$

$$-2A = 1$$
$$A = -\tfrac{1}{2}$$

$$-3A - 2B = 1$$
$$-3(-\tfrac{1}{2}) - 2B = 1$$
$$\tfrac{3}{2} - 2B = 1$$
$$3 - 4B = 2$$
$$-4B = -1$$
$$B = \tfrac{1}{4}$$

$$-6A - 2B - 2C = 1$$
$$-6(-\tfrac{1}{2}) - 2(\tfrac{1}{4}) - 2C = 1$$
$$3 - \tfrac{1}{2} - 2C = 1$$
$$6 - 1 - 4C = 2$$
$$5 - 4C = 2$$
$$-4C = -3$$
$$C = \tfrac{3}{4}$$

$$-6A - 2B - C - 2D = 1$$
$$-6(-\tfrac{1}{2}) - 2(\tfrac{1}{4}) - \tfrac{3}{4} - 2D = 1$$
$$3 - \tfrac{1}{2} - \tfrac{3}{4} - 2D = 1$$
$$12 - 2 - 3 - 8D = 4$$
$$7 - 8D = 4$$
$$-8D = -3$$

$D = \frac{3}{8}$

$A = -\frac{1}{2}$

$B = \frac{1}{4}$

$C = \frac{3}{4}$

$D = \frac{3}{8}$

$y_p = -\frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{3}{4}x + \frac{3}{8}$ PS

GS:

$$y = C_1 \exp(-x) + C_2 \exp(2x) + C_3 \exp(ix) + C_4 \exp(-ix) - \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{3}{4}x + \frac{3}{8}$$

Problem 5.3

$x^3 y''' + 6x^2 y'' + 4xy' - 4y = 3 + 2x + x^2 \qquad \forall x > 0$

$x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0$

$y(x) = x^r$

$y'(x) = (x^r)' = r x^{r-1}$

$y''(x) = (x^r)'' = r(r-1)x^{r-2}$

$y'''(x) = (x^r)''' = r(r-1)(r-2)x^{r-3}$

$x^3 (x^r)''' + 6x^2 (x^r)'' + 4x(x^r)' - 4(x^r) = 0$

$x^3 r(r-1)(r-2)x^{r-3} + 6x^2 r(r-1)x^{r-2} + 4x r x^{r-1} - 4x^r = 0$

$r(r-1)(r-2)x^r + 6r(r-1)x^r + 4r x^r - 4x^r = 0$

$x^r(r^3 - 3r^2 + 2r) + x^r(6r^2 - 6r) + x^r 4r - x^r 4 = 0$

$x^r(r^3 - 3r^2 + 2r + 6r^2 - 6r + 4r - 4) = 0$

$x^r(r^3 + 3r^2 - 4) = 0$

$x^r(r-1)(r+2)(r+2) = 0$

$r = 1, -2$ (multiplicity 2)

$y = c_1 x + c_2 x^{-2} + c_3 \ln(x) x^{-2}$ ← homo part of the answer

(I got stuck on this problem.)

Problem 5.4

$x^2 y'' - x(x-1)y' + (x-1)y = 0 \qquad \forall x > 2 \text{ given } y_1(x) = x$

$y(x) = ux$
$y'(x) = u + u'x$
$y''(x) = u' + u' + u''x = 2u' + u''x$

$x^2 y'' - x(x-1)y' + (x-1)y = 0$

$y'' = \dfrac{x(x-1)y' - (x-1)y}{x^2}$

$2u' + u''x = \dfrac{x(x-1)(u+u'x) - (x-1)ux}{x^2}$

$2u' + u''x = \dfrac{(x^2-x)(u+u'x) - (x-1)ux}{x^2}$

$2u' + u''x = \dfrac{ux^2 + u'x^3 - ux - u'x^2 - ux^2 + ux}{x^2}$

$2u' + u''x = \dfrac{u'x^3 - u'x^2}{x^2}$

$2u' + u''x = u'x - u'$

$u''x = u'x - 3u'$

$u'' = \dfrac{u'x - 3u'}{x}$

Let $v = u'$
$v' = u''$

$v' = \dfrac{vx - 3v}{x}$

$v' = \dfrac{v(x-3)}{x}$

$\dfrac{v'}{v} = \dfrac{x-3}{x}$

$\int \dfrac{v'}{v} = \int \dfrac{x-3}{x}\,dx$

$\ln v = \int 1 - \frac{3}{x}\,dx$

$\ln v = x - 3\ln x + c$

$e^{\ln v} = e^{x - 3\ln x + c}$

$v = e^x e^{-3\ln x + c}$

$v = e^x e^{\ln x^{-3}} e^c$

$v = e^x x^{-3} c_1$

$v = \dfrac{c_1 e^x}{x^3}$

$u' = \dfrac{c_1 e^x}{x^3}$

$\int u' = \int \dfrac{c_1 e^x}{x^3}\,dx$

$u = c_1 \int \dfrac{e^x}{x^3}\,dx$

$$u = c_1\left(\frac{e^x}{-2x^2} - \int \frac{e^x}{-2x^2}\,dx\right) + c_2$$

$$\int uv' = uv - \int u'v$$

$u = e^x$ $\quad v = -\frac{1}{2x^2}$

$u' = e^x$ $\quad v' = \frac{1}{x^3}$

$$u = c_1\left(-\frac{e^x}{2x^2} + \frac{1}{2}\int \frac{e^x}{x^2}\,dx\right) + c_2$$

$u = e^x$ $\quad v = -\frac{1}{x}$

$u' = e^x$ $\quad v' = \frac{1}{x^2}$

$$u = c_1\left(-\frac{e^x}{2x^2} + \frac{1}{2}\left(\frac{e^x}{-x} - \int \frac{e^x}{-x}\right)\right) + c_2$$

$$u = c_1\left(-\frac{e^x}{2x^2} + \frac{1}{2}\left(-\frac{e^x}{x} + \int \frac{e^x}{x}\,dx\right)\right) + c_2$$

$$y(x) = \left(c_1\left(-\frac{e^x}{2x^2} + \frac{1}{2}\left(-\frac{e^x}{x} + \int \frac{e^x}{x}\,dx\right)\right) + c_2\right)x$$