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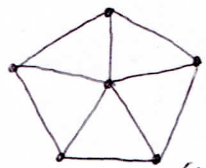
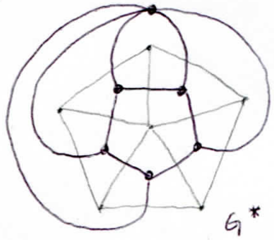
Chapter 4

22. Find the dual of this graph and verify Lemma.



G

Dual graph:



G^* (in pen)

verify Lemma.

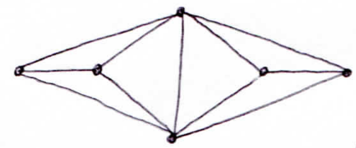
G:

$n = 6$
$m = 10$
$f = 6$

G^*:

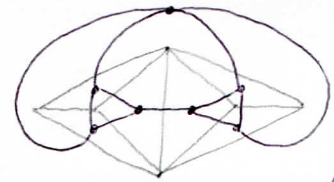
$n^* = 6 = f$
$m^* = 10 = m$
$f^* = 6 = n$

23. Find the dual graph and verify Lemma.



G

Dual graph:



G^* (in pen)

verify Lemma.

G:

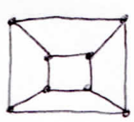
$n = 6$
$m = 11$
$f = 7$

G^*:

$n^* = 7 = f$
$m^* = 11 = m$
$f^* = 6 = n$

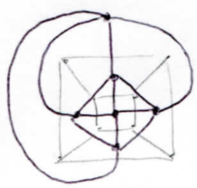
24. Show that the dual of the cube graph is the octahedron graph, and that the dual of the dodecahedron graph is the icosahedron graph.

Cube graph: Dual of cube graph = octahedron graph



G

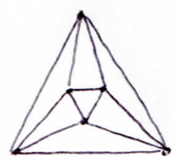
$n = 8$
$m = 12$
$f = 6$



G^*

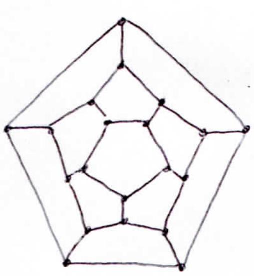
$n^* = 6 = f$
$m^* = 12 = m$
$f^* = 8 = n$

$\Rightarrow$



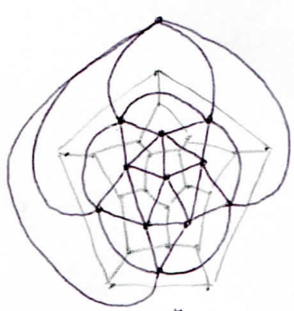
$n = 6$
$m = 12$
$f = 8$

Dodecahedron graph: Dual of dodecahedron graph = icosahedron graph



G

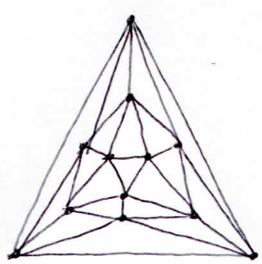
$n = 20$
$m = 30$
$f = 12$



G^*

$n^* = 12 = f$
$m^* = 30 = m$
$f^* = 20 = n$

$\Rightarrow$

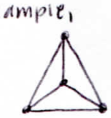


$n = 12$
$m = 30$
$f = 20$

25. Show that that the dual of a wheel is a wheel.

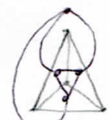
For a wheel, $n = f$, so $n^* = f^* = n = f$. $m^* = m$.

For example,



W_4

$n = f = 4$
$m = 6$



W_4^*

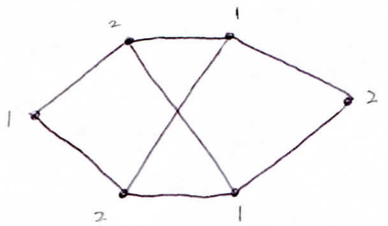
$n^* = f^* = 4$
$m = 6$

$\Rightarrow$

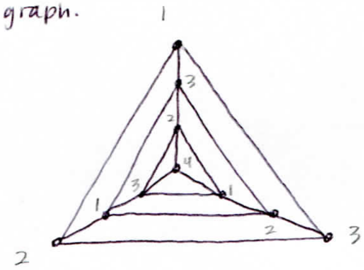
The dual of W_4 is W_4.

Chapter 5

1. Find the chromatic number of each graph.

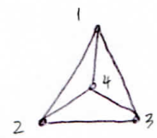


2-chromatic

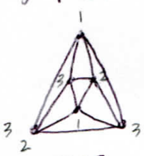


4-chromatic

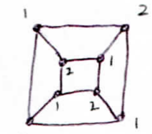
4. (i) What is the chromatic number of each of the Platonic graphs?



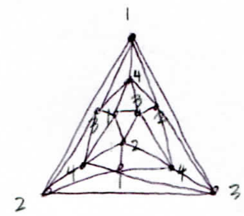
$\chi = 4$



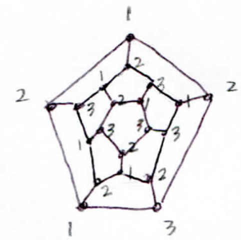
$\chi = 3$



$\chi = 2$



$\chi = 4$

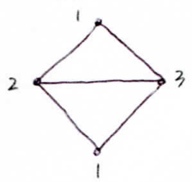


$\chi = 3$

4. (ii) What is the chromatic number of the complete tripartite graph $K_{r,s,t}$?

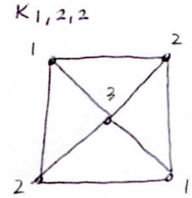
In a complete tripartite graph, there are no two vertices within the same set that are adjacent. Every vertex of each set is adjacent to every vertex of the other two sets. So, in a complete tripartite graph, there are 3 sets. Since there are a bunch a triangles, the chromatic number is 3.

ex. $K_{1,1,2}$



$\chi = 3$

$K_{1,2,2}$



$\chi = 3$

7. Let G be a simple graph with n vertices, which is regular of d. By considering the number of vertices that can be assigned the same colour, prove that $\chi(G) \geq n/(n-d)$.

$$\chi(G) \geq \frac{n}{n-d}$$

$$(n-d)\,\chi(G) \geq n$$

Let $(n-d) \geq$ max # of vertices of a common color

The # of vertices that can be assigned the same color is a vertex and its non-neighboring vertices.

d is the # of vertices a vertex is adjacent to.

The largest possible independent set : a vertex and its non-adjacent vertices, which is the upper bound of each color class

8. (i) Let G be a simple planar graph containing no triangles. Using Euler's formula, show that G contains a vertex of degree at most 3.

bdy ≥ 4 because there are no triangles.

$$4f \leq \sum \text{bdy} = 2m$$

$$4f \leq 2m$$

$$f \leq \tfrac{1}{2}m$$

$$f = m - n + 2$$

$$m - n + 2 \leq \tfrac{1}{2}m$$

$$\tfrac{1}{2}m - n + 2 \leq 0$$

$$m - 2n + 4 \leq 0$$

$$m \leq 2n - 4$$

Assume the conclusion ($\deg \leq 3$) is false. So, $\deg > 4$.

$$4n \leq \sum \deg = 2m$$

$$4n \leq 2m$$

$$4n \leq 2(2n-4)$$

$$4n \leq 4n - 4$$

is a contradiction. So, G contains a vertex of degree at most 3.

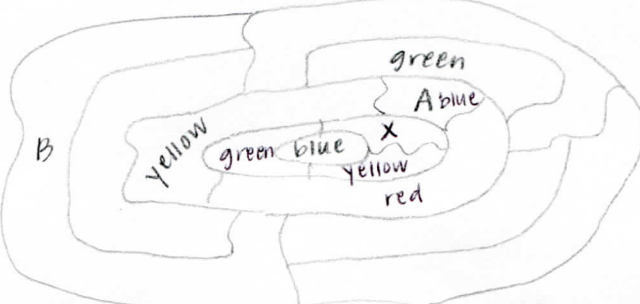
8. (ii) Use induction to deduce that G is 4-colorable.

The theorem is true for any G with $(n-1)$ vertices. Assume that it is true with graphs. If you take out a vertex of degree 3 from a graph that you know is 4-colorable, and now you add the vertex you took out back to the graph, then building on that assumption by adding on vertices with degree 3, G is 4-colorable.

19. Consider the map in which the countries are to be colored red, blue, green, and yellow.

(i) Show that country A must be red.

By contradiction:

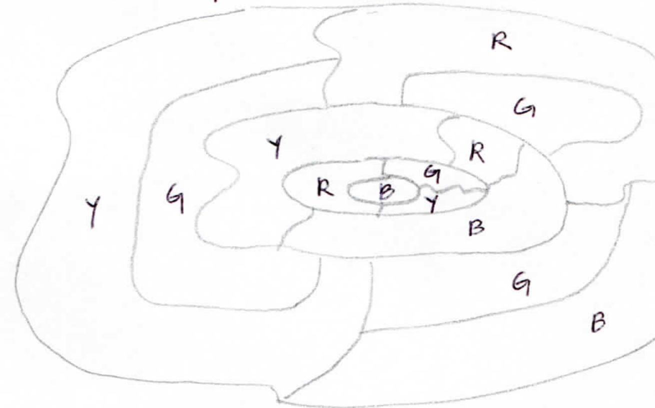


A must be red because by contradiction, if A is blue, then the "X" area below country A cannot be red, blue, green, or yellow.

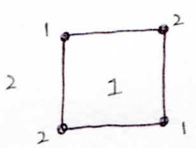
(ii) What color is country B?

yellow

Let R = red
B = blue
G = green
Y = yellow



21. Give an example of a plane graph that is 2-colorable (f) and 2-colorable (v).



Theorem states:
A map G is 2-colorable (f) if and only if G is an Eulerian graph (even degree).

This is an Eulerian graph.

A connected planar graph without loops is 2-colorable (v) iff it is bipartite.

This graph is bipartite.

Therefore, this square planar graph is 2-colorable (f) and 2-colorable (v).

22. The plane is divided into a finite number of regions by drawing infinite straight lines in an arbitrary manner. Show that these regions can be 2-colored.

The theorem states that G is 2-colorable iff every vertex has even degree.

For this problem, every line that intersects or goes through a vertex, each edge contributes 2 to its degree. Therefore, the vertices will have even degree. And therefore, it will be 2-colorable.

24. Let G be a simple plane graph with fewer than 12 faces, and suppose that each vertex of G has degree at least 3.

(i) Prove that G is 4-colorable (v).

If bdy = 3,

contract the face with bdy = 3 to a vertex.
now there is 1 less face and G can be 4-colored.
now expand the vertex back to a face.
all the other faces are colored. the boundaries between the face are still there, but now they have a boundary in common.
the new face is not colored.
every time you contract a face and color the graph, and then expand the vertex back to a face, you have 3 colors to avoid since the boundary is 3, which is okay because G is 4-colorable.

(ii) Dualize the result.

If G and its subgraphs have a vertex of deg ≤ 4, then the graph is 4-colorable.

Suppose G is a simple plane graph with n vertices and all simple plane graphs with (n-1) vertices are 4-colorable. G contains a vertex of degree of at most 4.

Suppose that deg(v) = 4. Four vertices are adjacent to v. If these vertices are mutually adjacent, the contract 2 edges. The resulting graph is 4-colorable. Put back the edges and color them with the same color as v. Color v with a color different from the (at most 3) colors assigned to the other vertices.