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Introduction to Fractals: Self-similar Recursive Patterns

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Assignment #3

Abstract

The term *fractal* was first coined by the father of fractals, Benoit Mandelbrot, in 1975. But before Mandelbrot introduced fractals, the studies of other early mathematicians, such as Karl Weierstrass and David Hilbert, have already contributed to the field of fractal geometry. Fractals are self-similar patterns consisting of infinitely, repeating geometric figures. Some famous examples include the Mandelbrot set, Koch snowflake, Sierpinski triangle, and Menger sponge. Fractals that are infinitely self-similar are defined as pure; however, fractals that are finitely self-similar and only have a few iterations are defined as limited. Limited fractals can be found in nature because the patterns in nature do not repeat indefinitely.

Introduction

The use of the term *fractal* was first introduced and created by Benoit Mandelbrot in 1975 (Turner, 1998). The term *fractal* originates from the Latin word *fractus*, which means to break (Latdict, n.d.). Before Mandelbrot began referring the self-similar patterns as fractals, other mathematicians already started studying fractals. In the late 19th century, the mathematics community believed that all continuous functions must be differentiable in at least one place (Turner, 1998). However, Karl Weierstrass created a non-differentiable continuous function. Since there was no use for continuous, but not differentiable functions at the time, Weierstrass's functions contributed to the study of fractals. David Hilbert also contributed to the early study of fractals with his space filling curves. The Hilbert space filling curve is a one-dimensional curve that is created by rotating and iterating a shape to fill a two-dimensional space (Turner, 1998). These continuous curves did not have continuous inverses. Figure 1 is a Hilbert space filling curve. The non-differentiable, but continuous functions, and space filling curves studied by Weierstrass and Hilbert are defined as fractals.

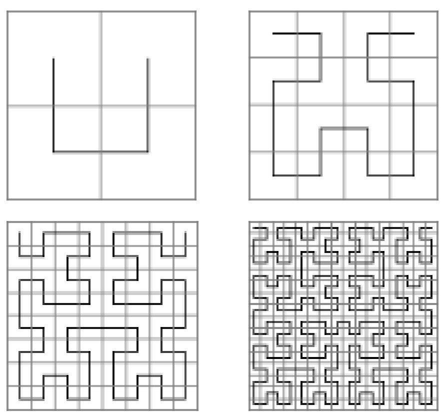

Figure 1. Example of Hilbert space filling curve.

Definition

Fractals are infinite, recursive patterns with self-similar properties. Fractals are self-similar structures because zooming in or out of a part of the fractal results in a smaller or bigger copy of the whole. Iterating equations creates the repeating patterns and self-similarity in fractals. Fractals can also be created using simple patterns.

Pure fractals have exact self-similar properties in every iteration (Haggit, 2011). In contrast, limited fractals are not exact in their self-similarity. Limited fractals do not iterate indefinitely; these fractals only display a few iterations.

Fractal Geometry

Fractal geometry defines and measures the complexity of a fractal by measuring the roughness of its surface (Haggit, 2011). Fractal dimensions are used to measure the complexity of fractals. Fractals have irrational numbers as a dimension (Mathigon, n.d.). For example, a line has a dimension of 1 and when scaled by a factor of 2, its length increases by a factor of 2. A square has a dimension of 2: length and width. When scaled by a factor of 2, its area increases by a factor of 4. A cube has a dimension of 3: length, width, and height. When scaled by a factor of 2, its volume increases by a factor of 8. When the Sierpinski Triangle is scaled by a factor of 2, its area increases by a factor of 3. The dimension of the Sierpinski Triangle can be calculated using logarithms and is about 1.585.

One method to determine a fractal's dimension is called the box-counting, or Minkoski-Bouligand Dimension method, as seen in (1) (Haggit, 2011). A fractal is placed on a piece of grid paper. The larger the fractal and more detailed the grid paper, the more accurate the calculation will be. However, compared to other methods, the Minkoski-Bouligand Dimension method is not always the most accurate.

$$\boldsymbol{D = \log(N)/\log(1/h)} \tag{1}$$

D is the fractal dimension. N is the number of grid boxes that contain some part of the fractal inside. h is the number of grid boxes the fractal spans on the grid paper.

Another method to determine the fractal dimension is called the Hausdorff Dimension method, as seen in (2).

$$\boldsymbol{D = \log(N)/\log(s)} \tag{2}$$

D is the fractal dimension. N is the number of parts a fractal produces from each segment. s is the size of each new part compared to the original segment.

Famous Examples

One of the famous fractals is the Mandelbrot set. The Mandelbrot set is named after Benoit Mandelbrot. The Mandelbrot set is created by the recursive equation in (3).

$$\boldsymbol{z_{n+1} = z_n^2 + C} \tag{3}$$

z_n is the current iteration. z_{n+1} is the next iteration. C is a complex number. The choice of C will determine the fractal (Francis, n.d.). The Mandelbrot set consists of all possible choice for C, so the iterations (z_n) will be less than or equal to 2 (Francis, n.d.). In the Mandelbrot set, z_n starts at zero.

The Mandelbrot set is drawn in the complex plane, so it may be difficult to draw by hand. Figure 2 shows the Mandelbrot set.

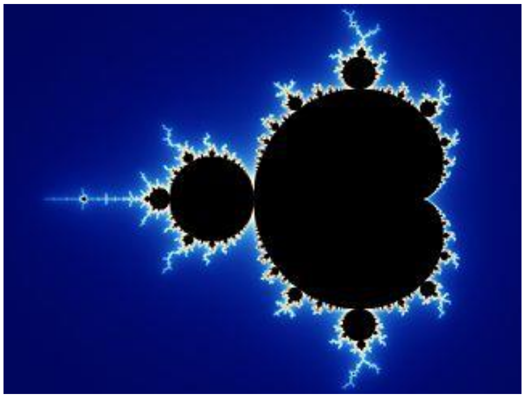

Figure 2. Mandelbrot set.

Another example is the Koch snowflake named after Helge von Koch. The Koch snowflake is created by starting with an equilateral triangle. For each iteration, turn the center of each edge from the previous iteration into a triangular bump. The first five iterations of the Koch snowflake are shown in Figure 3.

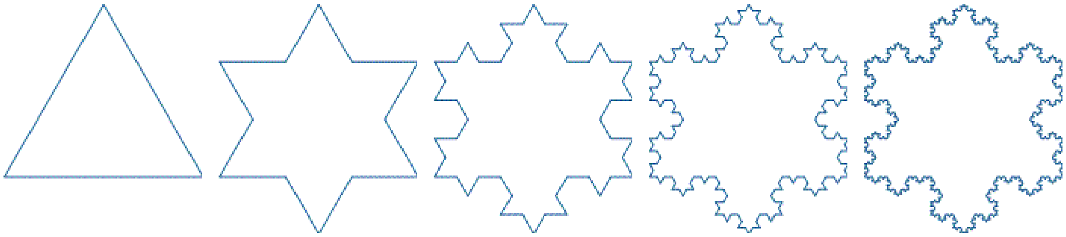

Figure 3. First few iterations of the Koch snowflake.

Another famous fractal example is the Sierpinski Triangle named after Waclaw Sierpinski. This fractal introduced by Sierpinski in 1915 (Parsons, n.d.). The Sierpinski Triangle is created by starting with an equilateral triangle. The recursive pattern is to place a smaller equilateral triangle inside each of the equilateral triangles from the previous iteration. Visually, the Sierpinski Triangle fractal is triangles within triangles. Figure 4 shows the first five iterations of the Sierpinski Triangle.

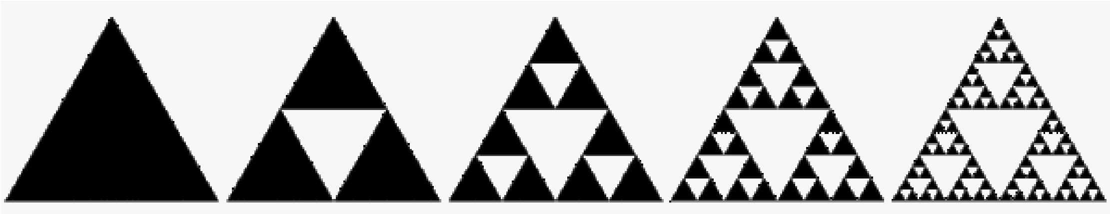

Figure 4. First few iterations of the Sierpinski Triangle.

Another example of famous fractals is the Menger Sponge. The Menger Sponge is a three-dimensional fractal named after Karl Menger. The Menger Sponge is similar to the Sierpinski because the recursive pattern is embedded inside the overall bigger shape. The Menger Sponge is created by starting with a cube. In each iteration, a smaller cube is placed inside each of the cubes from the previous iteration. Visually, there are cubes within cubes for the Menger Sponge fractal. The first four iterations of the Menger Sponge are shown in Figure 5.

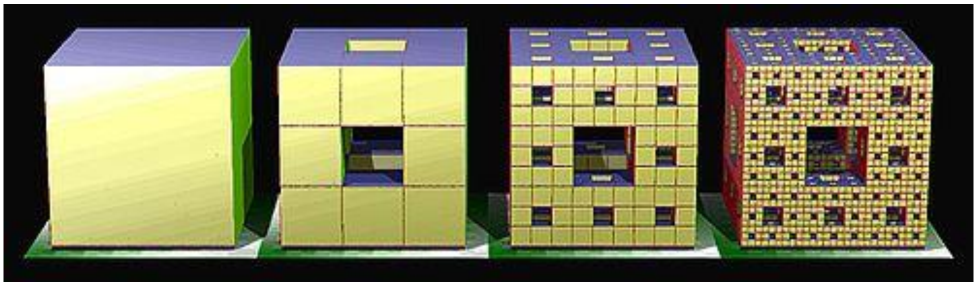

Figure 5. First few iterations of the Menger Sponge.

Applications

Lewis Fry Richardson was one of the first people to apply the study of fractals (Haggit, 2011). In the early 20th century, Richardson studied the length of coastlines and argued that the length depends on the measurement tool; a more detailed measurement tool measures irregularity better. Richardson concluded an indefinitely long coastlines contains a finite space (Haggit, 2011). Similarly, the Koch snowflake has an infinite perimeter while containing a finite area.

Another application of the study of fractals is in image compression (Shannon, 2012). The self-similar properties of fractals can be used to transform and scale images. Fractals are used for image compression because the quality of the image is the same, but the size of the image is smaller.

Real-World Connections

Pure fractals are difficult to find in nature because patterns in nature are not infinite and the iterating pattern would eventually stop. However, many things in nature are limited fractals and fractal nature. Some examples of limited fractals in nature are in trees and lightning bolts. Tree branches branch out into smaller branches and twigs. Leaf veins branch out into smaller veins. Lightning bolts are composed of smaller lightning bolts. Other examples include Romanesco broccoli, ferns, rivers, mountains, coastlines, clouds, and snowflakes (McNally, 2010).

Fractals are also found in popular culture. Fractals are mentioned in the song *Let It Go* in Disney's *Frozen*: "my soul is spiraling in frozen fractals all around." This lyric refers to snowflake fractals. The snowflake fractal pattern is formed when a water droplet freezes and crystallizes (Stiefel, 2017). Also, the *Triforce* in Nintendo's *The Legend of Zelda* series is a fractal. The Triforce is made up of four equilateral triangles to form a large equilateral triangle. The Triforce symbol in Figure 6 resembles the first iteration of the Sierpinski Triangle.

Figure 6. Triforce, from Nintendo's The Legend of Zelda series, resembles the Sierpinski Triangle.

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