Problem 4.1

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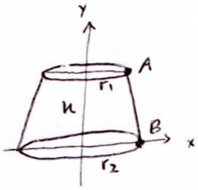
Homework 4

cross-sectional area:

$A(y) = \pi r^2$
$= \pi\left(\frac{r_1+r_2}{h}y + r_2\right)^2$



Torricelli's draining:

$$\begin{cases} \frac{A(y)\,dy}{dt} = -k\sqrt{y} \\ y(t=0) = y_0 \end{cases}$$

$$\frac{A(y)\,dy}{dt} = -k\sqrt{y}$$

$$\frac{A(y)\,dy}{\sqrt{y}} = -k\,dt$$

$$\int_h^0 \frac{A(y)\,dy}{\sqrt{y}} = \int_0^{T_1} -k\,dt$$

$$\int_h^0 \frac{\pi\left(\frac{r_1+r_2}{h}y+r_2\right)^2}{\sqrt{y}}\,dy = \int_0^{T_1} -k\,dt$$

$$\int_h^0 \frac{\pi\left(\left(\frac{r_1+r_2}{h}y\right)^2 + r_2^2 + 2r_2\left(\frac{r_1+r_2}{h}\right)y\right)}{\sqrt{y}}\,dy = \int_0^{T_1} -k\,dt$$

$$\int_h^0 \frac{\pi y^2\left(\frac{r_1+r_2}{h}\right)^2 + \pi r_2^2 + 2\pi r_2\left(\frac{r_1+r_2}{h}\right)y}{\sqrt{y}}\,dy = \int_0^{T_1} -k\,dt$$

$$\int_h^0 \pi y^{3/2}\left(\frac{r_1+r_2}{h}\right)^2 + \pi r_2^2\frac{1}{\sqrt{y}} + 2\pi r_2\sqrt{y}\left(\frac{r_1+r_2}{h}\right)dy = \int_0^{T_1} -k\,dt$$

$$\left[\pi\left(\frac{r_1+r_2}{h}\right)^2\frac{2}{5}y^{5/2} + \pi r_2^2(2\sqrt{y}) + 2\pi r_2\left(\frac{r_1+r_2}{h}\right)\frac{2}{3}y^{3/2}\right]_h^0 = -kt\Big|_0^{T_1}$$

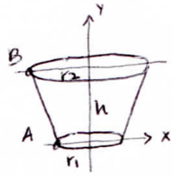
$$0 - \left(\pi\left(\frac{r_1+r_2}{h}\right)^2\frac{2}{5}h^{5/2} + \pi r_2^2\,2\sqrt{h} + 2\pi r_2\left(\frac{r_1+r_2}{h}\right)\frac{2}{3}h^{3/2}\right) = -T_1 k$$

$$-\pi\left(\frac{r_1+r_2}{h}\right)^2\frac{2}{5}h^{5/2} - \pi r_2^2\,2\sqrt{h} - 2\pi r_2\left(\frac{r_1+r_2}{h}\right)\frac{2}{3}h^{3/2} = -T_1 k$$

$$T_1 = \frac{\frac{2}{5}\pi\left(\frac{r_1+r_2}{h}\right)^2 h^{5/2} + 2\pi r_2^2\sqrt{h} + \frac{4}{3}\pi r_2 h^{3/2}\left(\frac{r_1+r_2}{h}\right)}{k}$$

cross-sectional area:

$A(y) = \pi r^2$
$= \pi\left(\left(\frac{r_2-r_1}{h}\right)y + r_1\right)^2$



Torricelli's draining:

$$\begin{cases} \frac{A(y)\,dy}{dt} = -k\sqrt{y} \\ y(t=0) = y_0 \end{cases}$$

$$\int_h^0 \frac{\pi\left(\frac{r_2-r_1}{h}y + r_1\right)^2}{\sqrt{y}}\,dy = \int_0^{T_2} -k\,dt$$

$$\int_h^0 \frac{\pi\left(\left(\frac{r_2-r_1}{h}y\right)^2 + r_1^2 + 2r_1 y\left(\frac{r_2-r_1}{h}\right)\right)}{\sqrt{y}}\,dy = \int_0^{T_2} -k\,dt$$

$$\int_h^0 \frac{\pi y^2 \left(\frac{r_2-r_1}{h}\right)^2 + \pi r_1^2 + 2\pi r_1 y \left(\frac{r_2-r_1}{h}\right)}{\sqrt{y}}\, dy = \int_0^{T_2} -k\, dt$$

$$\int_h^0 \pi y^{3/2} \left(\frac{r_2-r_1}{h}\right)^2 + \pi r_1^2 \frac{1}{\sqrt{y}} + 2\pi r_1 \sqrt{y} \left(\frac{r_2-r_1}{h}\right) dy = \int_0^{T_2} -k\, dt$$

$$\left[\pi \left(\frac{r_2-r_1}{h}\right)^2 \frac{2}{5} y^{5/2} + \pi r_1^2 2\sqrt{y} + 2\pi r_1 \left(\frac{r_2-r_1}{h}\right) \frac{2}{3} y^{3/2} \right]_h^0 = -kt \Big|_0^{T_2}$$

$$0 - \left(\pi \left(\frac{r_2-r_1}{h}\right)^2 \frac{2}{5} h^{5/2} + \pi r_1^2 2\sqrt{h} + 2\pi r_1 \left(\frac{r_2-r_1}{h}\right) \frac{2}{3} h^{3/2} \right) = -T_2 k$$

$$-\pi \left(\frac{r_2-r_1}{h}\right)^2 \frac{2}{5} h^{5/2} - \pi r_1^2 2\sqrt{h} - 2\pi r_1 \left(\frac{r_2-r_1}{h}\right) \frac{2}{3} h^{3/2} = -T_2 k$$

$$T_2 = \frac{\frac{2}{5}\pi \left(\frac{r_2-r_1}{h}\right)^2 h^{5/2} + 2\pi r_1^2 \sqrt{h} + \frac{4}{3}\pi r_1 h^{3/2} \left(\frac{r_2-r_1}{h}\right)}{k}$$

If $r_1 = 0$:

$$T_1 = \frac{\frac{2}{5}\pi \left(\frac{r_1+r_2}{h}\right)^2 h^{5/2} + 2\pi r_2^2 \sqrt{h} + \frac{4}{3}\pi r_2 h^{3/2} \left(\frac{r_1+r_2}{h}\right)}{k}$$

$$= \frac{\frac{2}{5}\pi \left(\frac{r_2}{h}\right)^2 h^{5/2} + 2\pi r_2^2 \sqrt{h} + \frac{4}{3}\pi r_2 h^{3/2} \left(\frac{r_2}{h}\right)}{k}$$

$$= \frac{\frac{2}{5}\pi r_2^2 \sqrt{h} + 2\pi r_2^2 \sqrt{h} + \frac{4}{3}\pi r_2^2 \sqrt{h}}{k}$$

$$= \frac{\frac{56}{15}\pi r_2^2 \sqrt{h}}{k}$$

$$T_2 = \frac{\frac{2}{5}\pi \left(\frac{r_2-r_1}{h}\right)^2 h^{5/2} + 2\pi r_1^2 \sqrt{h} + \frac{4}{3}\pi r_1 h^{3/2} \left(\frac{r_2-r_1}{h}\right)}{k}$$

$$= \frac{\frac{2}{5}\pi \left(\frac{r_2}{h}\right)^2 h^{5/2}}{k}$$

$$= \frac{\frac{2}{5}\pi r_2^2 \sqrt{h}}{k}$$

Problem 4.2

$y'' - y' - 2y = 0$

TS: $y = \exp(rx)$
$y' = r\exp(rx)$
$y'' = r^2 \exp(rx)$

$r^2\exp(rx) - r\exp(rx) - 2\exp(rx) = 0$
$\exp(rx)\,(r^2 - r - 2) = 0$
$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r_1 = 2$
$r_2 = -1$

GS: $y = C_1\exp(2x) + C_2\exp(-x)$

Check: $y = C_1\exp(2x) + C_2\exp(-x)$
$y' = 2C_1\exp(2x) - C_2\exp(-x)$
$y'' = 4C_1\exp(2x) + C_2\exp(-x)$

$$y'' - y' - 2y = 4C_1\exp(2x) + C_2\exp(-x) - (2C_1\exp(2x) - C_2\exp(-x)) - 2(C_1\exp(2x) + C_2\exp(-x))$$
$$= 4C_1\exp(2x) + C_2\exp(-x) - 2C_1\exp(2x) + C_2\exp(-x) - 2C_1\exp(2x) - 2C_2\exp(-x)$$
$$= 0$$

GS: $y(x) = C_1\exp(2x) + C_2\exp(-x)$

Problem 4.3

$y'' + 2y' + 2y = 0$

TS: $y = \exp(rx)$

$y' = r\exp(rx)$

$y'' = r^2 \exp(rx)$

$r^2 \exp(rx) + 2r\exp(rx) + 2\exp(rx) = 0$

$\exp(rx)(r^2 + 2r + 2) = 0$

$r^2 + 2r + 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(2)}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$r_{1,2} = -1 \pm i$

$y_{1,2} = \exp((-1 \pm i)x)$

GS: $y = C_1 \exp((-1+i)x) + C_2 \exp((-1-i)x)$

$y = \exp(-x)(C_1 \exp(ix) + C_2 \exp(-ix))$

$\exp(\pm ix) = \cos x \pm i \sin x$

$y = \exp(-x)(C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x))$

$= \exp(-x)((C_1 + C_2)\cos x + (iC_1 - iC_2)\sin x)$

Let $c_1 = C_1 + C_2$

$c_2 = iC_1 - iC_2$

GS: $y(x) = \exp(-x)(c_1 \cos x + c_2 \sin x)$

Problem 4.4

$y'' + 2y' + y = 0$

TS: $y = \exp(rx)$

$y' = r\exp(rx)$

$y'' = r^2 \exp(rx)$

$r^2 \exp(rx) + 2r\exp(rx) + \exp(rx) = 0$

$\exp(rx)\,(r^2 + 2r + 1) = 0$

$r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$

$r_{1,2} = -1$

$y_1 = \exp(-x)$

$y_2 = x\exp(-x)$

Check:

$y_1 = \exp(-x)$

$y_1' = -\exp(-x)$

$y_1'' = \exp(-x)$

$y'' + 2y' + y = \exp(-x) + 2(-\exp(-x)) + \exp(-x)$

$= \exp(-x) - 2\exp(-x) + \exp(-x)$

$= 0$

$y_2 = x\exp(-x)$

$y_2' = -x\exp(-x)$

$y_2'' = x\exp(-x)$

$y'' + 2y' + y = x\exp(-x) + 2(-x\exp(-x)) + x\exp(-x)$

$= x\exp(-x) - 2x\exp(-x) + x\exp(-x)$

$= 0$

GS: $y(x) = C_1 \exp(-x) + C_2\, x\exp(-x)$

Problem 4.5

$y''' + y'' + y' + y = 0$

TS: $y = \exp(rx)$

$y' = r\exp(rx)$

$y'' = r^2 \exp(rx)$

$y''' = r^3 \exp(rx)$

$r^3 \exp(rx) + r^2 \exp(rx) + r\exp(rx) + \exp(rx) = 0$

$\exp(rx)(r^3 + r^2 + r + 1) = 0$

$r^3 + r^2 + r + 1 = 0$

$r^2(r+1) + r + 1 = 0$

$(r^2+1)(r+1) = 0$

$r = \pm i \quad r = -1$

$r_{1,2,3} = +i, -i, -1$

$y_1 = \exp(ix)$

$y_2 = \exp(-ix)$

$y_3 = \exp(-x)$

GS: $y(x) = C_1 \exp(ix) + C_2 \exp(-ix) + C_3 \exp(-x)$

$\exp(\pm ix) = \cos x \pm i\sin x$

$C_1 \exp(ix) = C_1(\cos x + i\sin x)$

$C_2 \exp(-ix) = C_2(\cos x - i\sin x)$

$y = C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x) + C_3\exp(-x)$

$= (C_1 + C_2)\cos x + (iC_1 - iC_2)\sin x + C_3\exp(-x)$

let $c_1 = C_1 + C_2$

$c_2 = iC_1 - iC_2$

GS: $y(x) = c_1 \cos x + c_2 \sin x + C_3 \exp(-x)$