Applied Wave Optics: Coherence

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Coherence

Origin: The Latin verb *cohaerere* = to stick together.

A wave field is called coherent, when it behaves identical at every place and time except of a certain phase shift. Coherence is the foundation of all **interference** phenomena.
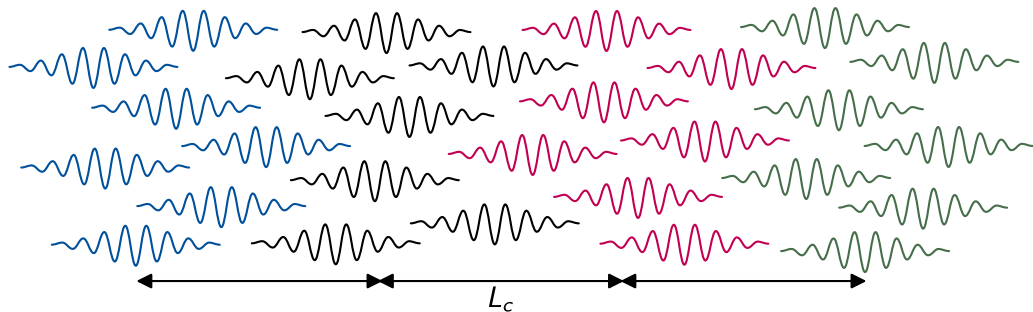
The photons in a laser beam can be described as a stream of wave packets. Each initial seed photon was spontaneously emitted with an arbitrary phase. The subsequent amplification by **stimulated emission** keeps this phase.

There are two kinds of coherence

- ▶ **temporal coherence** along the optical axis z
- ▶ **spatial coherence** perpendicular to the axis in x and y direction

Coherence (cont.)

The average distance between the coherent sets of photons originating from the same seed is called **coherence length** L_c:

Superposition

The frequency and wavelength of all photons of a laser beam is (almost) the same.
Superposition of two plane waves from two sets with phase difference $\Delta\varphi$:

$$E = \frac{1}{2}\left[\left(E_1 + E_2 e^{i\Delta\varphi}\right) e^{i(\omega t - \mathbf{kr})} + \left(E_1 + E_2 e^{-i\Delta\varphi}\right) e^{-i(\omega t - \mathbf{kr})}\right]$$

The phase difference results in an interference term in the equation for the total intensity

$$\begin{aligned}
I &= \frac{1}{2Z}|E|^2 \\
&= \frac{1}{2Z}(E_1^2 + E_2^2 + 2E_1 E_2 \cos\Delta\varphi) \\
&= I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\Delta\varphi
\end{aligned}$$

Coherence Factor

We introduce the coherence factor γ and the **contrast function** K:

$$K = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\, \gamma$$

With the total incoherent intensity $I_0 = I_1 + I_2$ we get the interference equation

$$I = I_0(1 + K \cos \Delta\varphi)$$

The coherence factor allows to describe the general case of **partial coherence**

- ▶ incoherent: $\gamma = 0$
- ▶ partially coherent: $0 < \gamma < 1$
- ▶ coherent: $\gamma = 1$

Interference Pattern

The general phase shift between two waves may result from either a temporal offset τ and/or a spatial offset $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$:

$$\Delta\varphi = \omega\tau - k\Delta\mathbf{r}$$

For two waves with **equal intensity** $I_1 = I_2 = I_0/2$ the contrast function reduces to $K = \gamma$. The cosine function in the interference equation $I = I_0(1 + \gamma\cos\Delta\varphi)$ leads to the characteristic oscillation of interference patterns with local minima and maxima:
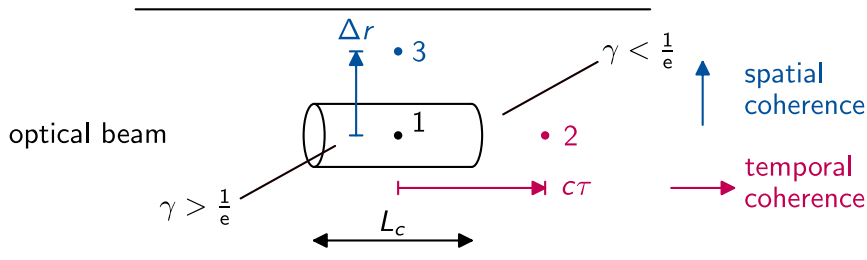
$$I_{min} = I_0(1 - \gamma)$$
$$I_{max} = I_0(1 + \gamma)$$

This allows to determine the local coherence factor from the measurement of the interference pattern $I(\tau, \Delta\mathbf{r})$:

$$\gamma = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

Coherence Volume

The self-superposition of an optical beam without temporal and spatial offset always results in a maximum of $\gamma = 1$ even for otherwise incoherent beams. Aside this point, the coherence factor decreases for increasing τ or Δr. The value $\gamma = 1/e$ defines a **coherence volume**. In axial direction its size is called **coherence length** L_c or **coherence time** $t_c = L_c/c$:

Correlation Function

The mathematical description of coherence uses the cross-correlation function:

$$\Gamma_{12}(\tau) = \langle E(\mathbf{r}_1, t)\, E^*(\mathbf{r}_2, t + \tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int\limits_{-T/2}^{+T/2} E(\mathbf{r}_1, t)\, E^*(\mathbf{r}_2, t + \tau)\, dt = 2Z\sqrt{I_1 I_2}\,\gamma(\tau)$$

The spatio-temporal contrast function is defined as normalised cross-correlation:

$$K_{12}(\tau) = \left| \frac{2\Gamma_{12}(\tau)}{\Gamma_{11}(0) + \Gamma_{22}(0)} \right| = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\,\gamma(\tau)$$

with the maxima of the auto-correlation functions

$$\Gamma_{11}(0) = 2Z\, I_1 \qquad \Gamma_{22}(0) = 2Z\, I_2$$

Correlation Function (cont.)

The case of pure temporal coherence is given for $\mathbf{r}_1 = \mathbf{r}_2$. The contrast function is a normalised auto-correlation function then:
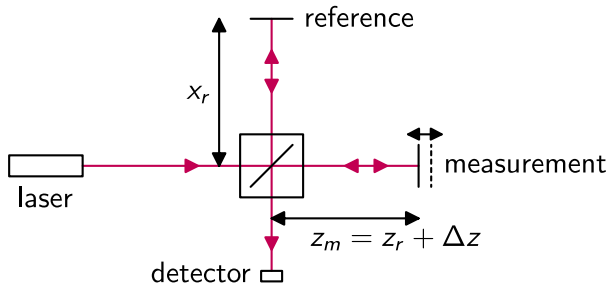
$$K(\tau) = \left| \frac{\Gamma(\tau)}{\Gamma(0)} \right|$$

Pure spatial coherence is given for $\tau = 0$. The contrast function is the temporal maximum of the normalised cross-correlation function in this case:
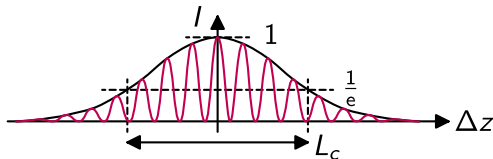
$$K_{12} = \left| \frac{2\Gamma_{12}}{\Gamma_{11} + \Gamma_{22}} \right|$$

Temporal Coherence

A **Michelson interferometer** can be used to measure the coherence length of a laser:

Interference pattern of the interferometer:

Wiener-Khinchin Theorem

The Wiener-Khinchin theorem states that the **power spectral density** of a statistical process is identical to the Fourier transform of its **autocorrelation function**:

$$S(\omega) = \int\limits_{-\infty}^{+\infty} \Gamma(\tau)\, e^{-i\omega t} d\tau$$

with the autocorrelation function

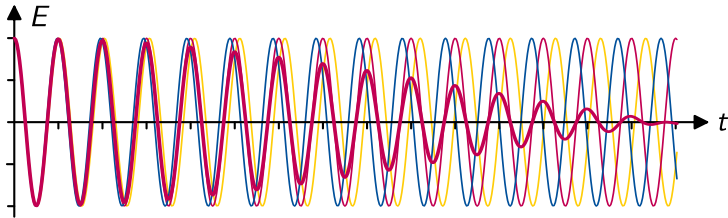$$\Gamma(\tau) = \langle E(t)E^*(t+\tau)\rangle$$

From this follows the relationship between the spectral linewidth $\Delta\lambda$ of an optical beam and its coherence length L_c. Due to $\lambda = c/\nu$ this leads also to a simple relationship between the spectral bandwidth $\Delta\nu$ of a signal and the coherence time t_c:
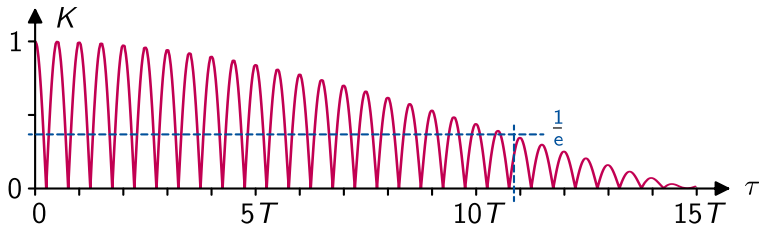
$$L_c = \frac{\lambda^2}{\Delta\lambda} \qquad\qquad t_c = \frac{1}{\Delta\nu}$$

Spectral Bandwidth

Superposition of waves with three different frequencies around a centre frequency ν_0 and with the spectral bandwidth $\Delta\nu = 0.09\,\nu_0$:

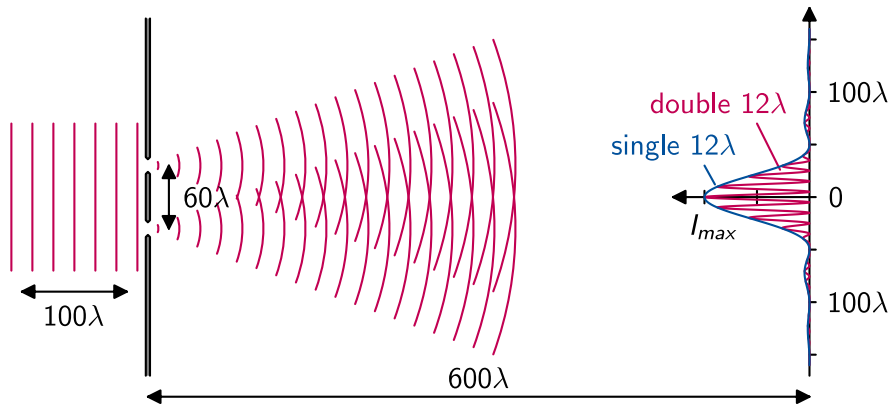The respective contrast function $K(\tau) = |\Gamma(\tau)/\Gamma(0)|$ is

A list of optical radiation sources with increasing coherence:

Source	λ	$\Delta\lambda$	$\Delta\nu$	L_c	t_c
white light	400-800 nm	400 nm	400 THz	800 nm	2.5 fs
red LED	640 nm	40 nm	30 THz	10 µm	30 fs
FP diode laser	1.3 µm	4 nm	710 GHz	420 µm	1.4 ps
sodium-vapor lamp	589 nm	600 pm	500 GHz	600 µm	2 ps
HeNe laser	633 nm	2 pm	1.5 GHz	200 mm	670 ps
DFB diode laser	1.3 µm	100 fm	20 MHz	15 m	50 ns
stabilised HeNe laser	633 nm	200 am	150 kHz	2 km	7 µs
DFB fibre laser	1.55 µm	24 am	3 kHz	100 km	330 µs

Spatial Coherence

Young's **double slit experiment** can be used to measure the spatial coherence of a laser:

The maximum intensity I_{max} decreases with increasing slit distance and reaches $1/e$ of its single slit value for the spatial coherence length.

van Cittert–Zernike Theorem

The van Cittert–Zernike theorem states that the **angular intensity distribution** of a source is identical to the spatial Fourier transform of its **cross-correlation function**:

$$I(k_x, k_y) = \int\limits_{-\infty}^{+\infty}\!\!\!\int \Gamma_{12} \, e^{i(k_x x + k_y y)} dx \, dy$$

with the cross-correlation function

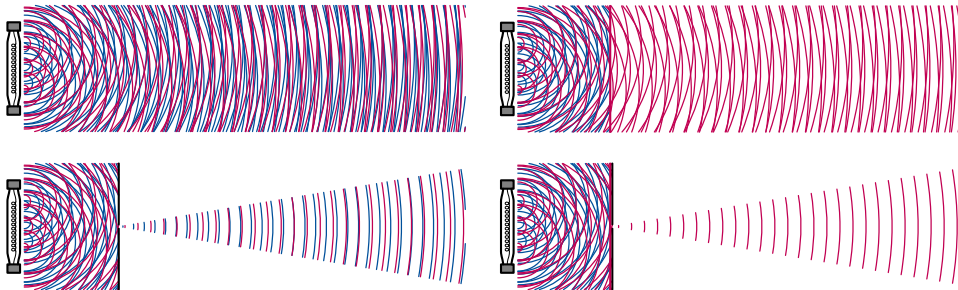$$\Gamma_{12} = \langle E(\mathbf{r}_1) \, E^*(\mathbf{r}_2) \rangle$$

and $x = x_2 - x_1$ and $y = y_2 - y_1$. From this follows a condition for spatial coherence of a light source in the far field:

$$2d \sin \alpha \ll \lambda$$

with the source aperture width d and half exit angle α.

Coherence Filters

Generating coherent light from an incoherent source (incandescent bulb):

- ▶ **Incoherence** (no filter): many wavelengths, many directions
- ▶ **Spatially coherence** (pinhole): many wavelengths, single direction
- ▶ **Temporal coherence** (spectral filter): single wavelength, many directions
- ▶ **Full coherence** (both filters): single wavelength, single direction