Applied Wave Optics: Maxwell Equations

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Maxwell's equations

The most general differential form of Maxwell's equations is

$$\begin{aligned} \nabla \mathbf{D} &= \rho && \text{Gauss's law} \\ \nabla \mathbf{B} &= 0 && \text{Gauss's law for magnetism} \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t && \text{Faraday's law} \\ \nabla \times \mathbf{H} &= \mathbf{j} + \partial \mathbf{D}/\partial t && \text{Ampère-Maxwell law} \end{aligned}$$

with the electric field vector **E** $[\mathrm{V/m}]$ and its material dependent form, the displacement field **D** $[\mathrm{As/m^2}]$ as well as the magnetic field vector **H** $[\mathrm{A/m}]$ and its material dependent form, the magnetic induction **B** $[\mathrm{Vs/m^2}]$. The scalar electrical charge density is ρ $[\mathrm{C/m^3}]$ and the local flux of electrical charges is taken into account by the current density vector **j** $[\mathrm{A/m^2}]$.

Polarisation

The material dependent relationship between **D** and **E** is

$$\mathbf{D}(\mathbf{E}) = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E})$$

with the permittivity $\varepsilon_0 = 8.854 \cdot 10^{-12}\ \frac{\mathrm{As}}{\mathrm{Vm}}$ and the polarisation vector **P**. Its Taylor expansion gives

$$\mathbf{P}(\mathbf{E})|_{\mathbf{E}=0} = \mathbf{P}_0 + \varepsilon_0 \chi_e^{(1)} \mathbf{E} + \varepsilon_0 \chi_e^{(2)} \mathbf{E}^2 + \dots$$

where the first term is a **static** polarisation, characteristic for ferroelectric materials, the second term describes the **linear** behaviour and all following terms **non-linear** dependencies with the electric susceptibility tensors $\chi_\mathrm{e}^{(n)}$.

Magnetisation

The material dependent relationship between **H** and **B** is

$$\mathbf{B}(\mathbf{H}) = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}(\mathbf{H})$$

with the permeability $\mu_0 = 4\pi \cdot 10^{-7}\ \frac{\mathrm{Vs}}{\mathrm{Am}}$ and the magnetisation vector **M**. Its Taylor expansion gives

$$\mathbf{M}(\mathbf{H})|_{\mathbf{H}=0} = \mathbf{M}_0 + \chi_m^{(1)} \mathbf{H} + \chi_m^{(2)} \mathbf{H}^2 + \dots$$

where the first term is a **static** magnetisation, characteristic for ferromagnetic materials, the second term describes the **linear** behaviour and all following terms **non-linear** dependencies with the magnetic susceptibility tensors $\chi_m^{(n)}$.

Linear Case

For isotropic materials, all susceptibilities are scalars and the linear cases simplify to

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$$
$$\mathbf{B} = \mu\mu_0\mathbf{H}$$

with the two dimensionless scalar quantities called **relative permittivity** $\varepsilon = 1 + \chi_{\mathrm{e}}^{(1)}$ and **relative permeability** $\mu = 1 + \chi_m^{(1)}$.

Note

We always assume this case in the following.

Continuity Equation

Identity relation from vector algebra for an arbitrary field $\mathbf{A}$:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{A}) = 0$$

It must therefore also be true for the magnetic field and wie can utilize Maxwell's equations:

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{H}) &= \boldsymbol{\nabla}\left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\right) \\
&= \boldsymbol{\nabla}\mathbf{j} + \frac{\partial}{\partial t}\boldsymbol{\nabla}\mathbf{D} \\
&= \boldsymbol{\nabla}\mathbf{j} + \frac{\partial \varrho}{\partial t} \stackrel{!}{=} 0 \\
\boldsymbol{\nabla}\mathbf{j} &= -\frac{\partial \varrho}{\partial t}
\end{aligned}$$

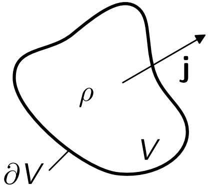
This is the continuity equation for electrical charges in its differential (local) form.

Continuity Equation (cont.)

The meaning of the continuity equation becomes more obvious, when it is integrated:

$$\int_{\partial V} \mathbf{j}\, d\mathbf{A} = -\frac{dQ}{dt}$$

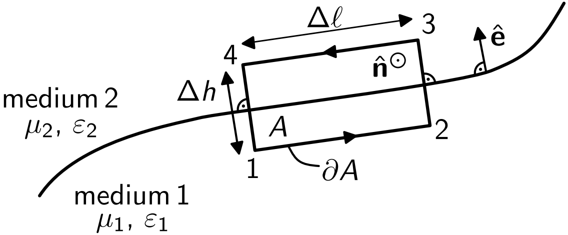
The left hand side is the total electrical current through the surface of a volume V and the right hand side the temporal variation of the total electrical charge $Q = \int \rho\, dV$ inside this volume.



Maxwell's equations guarantee that both quantities are identical.

Parallel Fields at Interfaces

Interface between two different materials:



Differentials for a line integration along the path ∂A:

$$
\begin{aligned}
1 \rightarrow 2: &\quad d\mathbf{r} = -\hat{\mathbf{n}} \times \hat{\mathbf{e}}\ dx \\
2 \rightarrow 3: &\quad d\mathbf{r} = \hat{\mathbf{e}}\ dx \\
3 \rightarrow 4: &\quad d\mathbf{r} = \hat{\mathbf{n}} \times \hat{\mathbf{e}}\ dx \\
4 \rightarrow 1: &\quad d\mathbf{r} = -\hat{\mathbf{e}}\ dx
\end{aligned}
$$

Parallel Fields at Interfaces (cont.)

Stokes theorem from vector analysis:

$$\int_A (\boldsymbol{\nabla} \times \mathbf{E}) d\mathbf{A} = \oint_{\partial A} \mathbf{E}\ d\mathbf{r}$$

We shrink the area $A \to 0$, and use the fact that for $\Delta\ell \to 0$ the line integrals $2 \to 3$ and $4 \to 1$ are identical, but with opposite sign:

$$\lim_{A\to 0} \int_A (\boldsymbol{\nabla} \times \mathbf{E}) d\mathbf{A} = \lim_{\Delta h, \Delta\ell \to 0} \left[-\int_1^2 (\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \mathbf{E}_1\ dx + \int_3^4 (\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \mathbf{E}_2\ dx \right]$$

Now we insert Faraday's law on the left and on the right the scalar triple products extract the field components parallel to the interface:

$$-i\omega \lim_{A\to 0} \int_A \mathbf{B}\ d\mathbf{A} = \lim_{\Delta h, \Delta\ell \to 0} \left[-\int_1^2 E_{1\|}\ dx + \int_3^4 E_{2\|}\ dx \right]$$

Parallel Fields at Interfaces (cont.)

The left side is obviously zero and on the right side we use the fact that for $\Delta\ell \to 0$ the fields can be treated as constant:

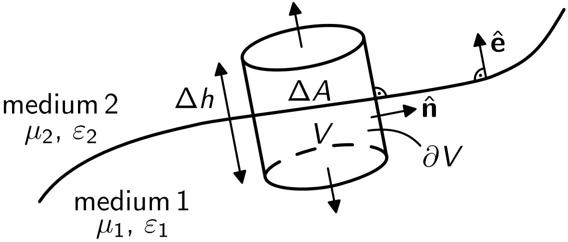
$$0 = \left(E_{1\|} - E_{2\|}\right) \Delta\ell$$

We get the same result when we carry out this calculation for the magnetic field and thus end with the following boundary conditions for field components parallel to the interface plane:

$$E_{1\|} = E_{2\|}$$
$$H_{1\|} = H_{2\|}$$

Normal Fields at Interfaces

Interface between two different materials:



Differentials for an integration over the surface ∂V:

$$
\begin{aligned}
\text{top base}: &\quad d\mathbf{A} = \hat{\mathbf{e}}\ dA \\
\text{lateral}: &\quad d\mathbf{A} = \hat{\mathbf{n}}\ dA \\
\text{bottom base}: &\quad d\mathbf{A} = -\hat{\mathbf{e}}\ dA
\end{aligned}
$$

Normal Fields at Interfaces (cont.)

Gauss's theorem from vector analysis:

$$\int\limits_V \nabla \mathbf{D} \; dV = \oint\limits_{\partial V} \mathbf{D} \; d\mathbf{A}$$

We shrink the volume $V \to 0$, and use the fact that in this case the lateral surface integrals in both materials vanish, because the field is constant:

$$\lim_{V \to 0} \int\limits_V \nabla \mathbf{D} \; dV = \lim_{\Delta A, \Delta h \to 0} \left[\int\limits_{\Delta A_1} \mathbf{D}_1 \hat{\mathbf{e}} \; dA - \int\limits_{\Delta A_2} \mathbf{D}_2 \hat{\mathbf{e}} \; dA \right]$$

Now we insert Gauss's law on the left and on the right the scalar products extract the component of the displacement field perpendicular to the interface:

$$\lim_{V \to 0} \int\limits_V \varrho \; dV = \lim_{\Delta A, \Delta h \to 0} \left[\int\limits_{\Delta A} D_{1\perp} dA - \int\limits_{\Delta A} D_{2\perp} dA \right]$$

Normal Fields at Interfaces (cont.)

The left side is obviously zero and on the right side we use the fact that for $\Delta A \to 0$ the fields can be treated as constant:

$$0 = (D_{1\perp} - D_{2\perp})\ \Delta A$$

We get the same result when we carry out this calculation for the magnetic induction and thus end with the following boundary conditions for field components perpendicular to the interface plane:

$$\begin{aligned} D_{1\perp} &= D_{2\perp} & \qquad \varepsilon_1 E_{1\perp} &= \varepsilon_2 E_{2\perp} \\ B_{1\perp} &= B_{2\perp} & \qquad \mu_1 H_{1\perp} &= \mu_2 H_{2\perp} \end{aligned}$$