# Applied Wave Optics: Coherence

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All sources and vector graphics are available on the GitHub repository `https://github.com/reincas/AWO_course`.

# Coherence

**Origin:** The Latin verb *cohaerere* = to stick together.

A wave field is called coherent, when it behaves identical at every place and time except of a certain phase shift. Coherence is the foundation of all **interference** phenomena.

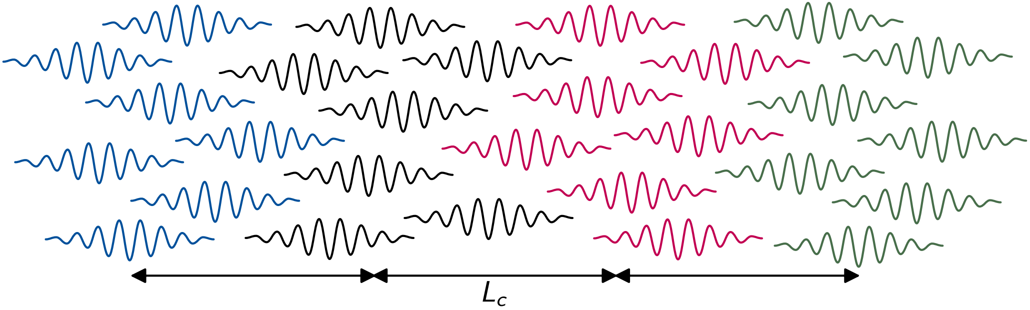
The photons in a laser beam can be described as a stream of wave packets. Each initial seed photon was spontaneously emitted with an arbitrary phase. The subsequent amplification by **stimulated emission** keeps this phase.

There are two kinds of coherence

- **temporal coherence** along the optical axis $z$
- **spatial coherence** perpendicular to the axis in $x$ and $y$ direction

# Coherence (cont.)

The average distance between the coherent sets of photons originating from the same seed is called **coherence length** $L_c$:



# Superposition

The frequency and wavelength of all photons of a laser beam is (almost) the same.
Superposition of two plane waves from two sets with phase difference $\Delta\varphi$:

$$E = \frac{1}{2}\left[\left(E_1 + E_2 \mathrm{e}^{i\Delta\varphi}\right)\mathrm{e}^{i(\omega t - \mathbf{kr})} + \left(E_1 + E_2 \mathrm{e}^{-i\Delta\varphi}\right)\mathrm{e}^{-i(\omega t - \mathbf{kr})}\right]$$

The phase difference results in an interference term in the equation for the total intensity

$$\begin{aligned} I &= \frac{1}{2Z}|E|^2 \\ &= \frac{1}{2Z}(E_1^2 + E_2^2 + 2E_1 E_2 \cos\Delta\varphi) \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\varphi \end{aligned}$$

# Coherence Factor

We introduce the coherence factor $\gamma$ and the **contrast function** $K$:

$$K = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \gamma$$

With the total incoherent intensity $I_0 = I_1 + I_2$ we get the interference equation

$$I = I_0(1 + K \cos \Delta\varphi)$$

The coherence factor allows to describe the general case of **partial coherence**

- incoherent: $\gamma = 0$
- partially coherent: $0 < \gamma < 1$
- coherent: $\gamma = 1$

# Interference Pattern

The general phase shift between two waves may result from either a temporal offset $\tau$ and/or a spatial offset $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$:

$$\Delta\varphi = \omega\tau - k\Delta\mathbf{r}$$

For two waves with **equal intensity** $I_1 = I_2 = I_0/2$ the contrast function reduces to $K = \gamma$. The cosine function in the interference equation $I = I_0(1 + \gamma \cos \Delta\varphi)$ leads to the characteristic oscillation of interference patterns with local minima and maxima:

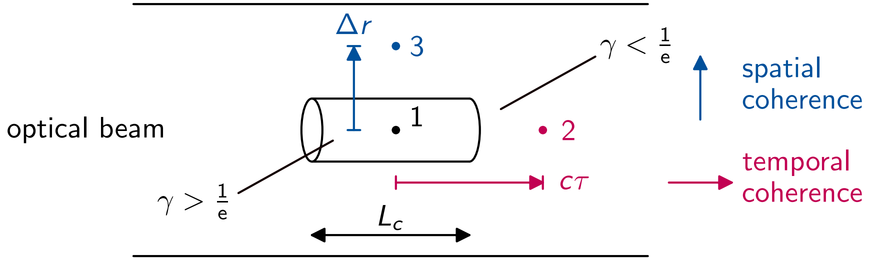
$$I_{min} = I_0(1 - \gamma)$$
$$I_{max} = I_0(1 + \gamma)$$

This allows to determine the local coherence factor from the measurement of the interference pattern $I(\tau, \Delta\mathbf{r})$:

$$\gamma = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

# Coherence Volume

The self-superposition of an optical beam without temporal and spatial offset always results in a maximum of $\gamma = 1$ even for otherwise incoherent beams. Aside this point, the coherence factor decreases for increasing $\tau$ or $\Delta r$. The value $\gamma = 1/\mathrm{e}$ defines a **coherence volume**. In axial direction its size is called **coherence length** $L_c$ or **coherence time** $t_c = L_c/c$:



# Correlation Function

The mathematical description of coherence uses the cross-correlation function:

$$\Gamma_{12}(\tau) = \langle E(\mathbf{r}_1, t)\, E^*(\mathbf{r}_2, t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} E(\mathbf{r}_1, t)\, E^*(\mathbf{r}_2, t+\tau) dt = 2Z\sqrt{I_1 I_2}\,\gamma(\tau)$$

The spatio-temporal contrast function is defined as normalised cross-correlation:

$$K_{12}(\tau) = \left| \frac{2\Gamma_{12}(\tau)}{\Gamma_{11}(0) + \Gamma_{22}(0)} \right| = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\,\gamma(\tau)$$

with the maxima of the auto-correlation functions

$$\Gamma_{11}(0) = 2Z\, I_1 \qquad\qquad \Gamma_{22}(0) = 2Z\, I_2$$

# Correlation Function (cont.)

The case of pure temporal coherence is given for $\mathbf{r}_1 = \mathbf{r}_2$. The contrast function is a normalised auto-correlation function then:

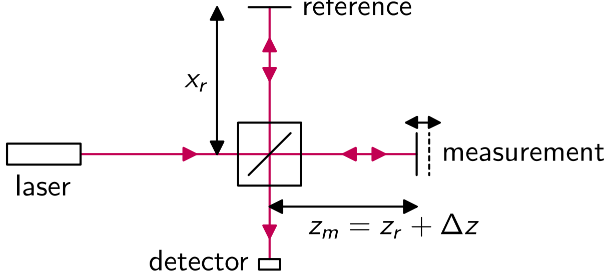
$$K(\tau) = \left| \frac{\Gamma(\tau)}{\Gamma(0)} \right|$$

Pure spatial coherence is given for $\tau = 0$. The contrast function is the temporal maximum of the normalised cross-correlation function in this case:

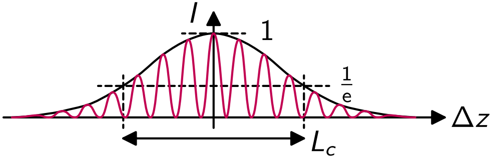
$$K_{12} = \left| \frac{2\Gamma_{12}}{\Gamma_{11} + \Gamma_{22}} \right|$$

# Temporal Coherence

A **Michelson interferometer** can be used to measure the coherence length of a laser:




Interference pattern of the interferometer:



# Wiener-Khinchin Theorem

The Wiener-Khinchin theorem states that the **power spectral density** of a statistical process is identical to the Fourier transform of its **autocorrelation function**:

$$S(\omega) = \int_{-\infty}^{+\infty} \Gamma(\tau)\, \mathrm{e}^{-i\omega t} d\tau$$

with the autocorrelation function

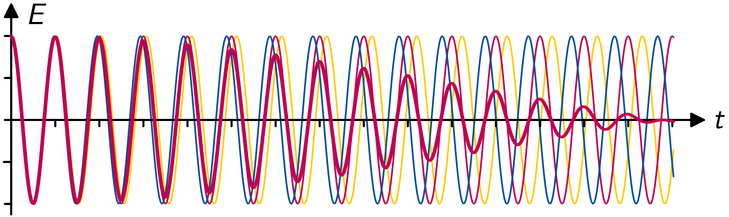
$$\Gamma(\tau) = \langle E(t) E^*(t+\tau) \rangle$$

From this follows the relationship between the spectral linewidth $\Delta\lambda$ of an optical beam and its coherence length $L_c$. Due to $\lambda = c/\nu$ this leads also to a simple relationship between the spectral bandwidth $\Delta\nu$ of a signal and the coherence time $t_c$:

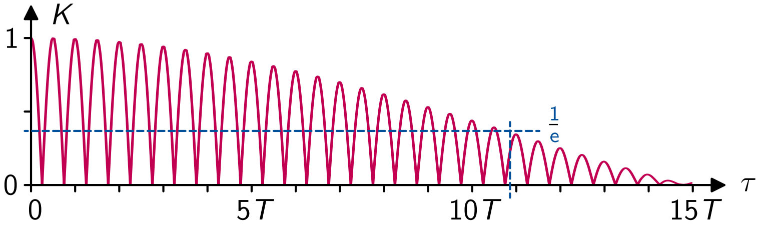
$$L_c = \frac{\lambda^2}{\Delta\lambda} \qquad\qquad t_c = \frac{1}{\Delta\nu}$$

# Spectral Bandwidth

Superposition of waves with three different frequencies around a centre frequency $\nu_0$ and with the spectral bandwidth $\Delta\nu = 0.09\,\nu_0$:




The respective contrast function $K(\tau) = |\Gamma(\tau)/\Gamma(0)|$ is



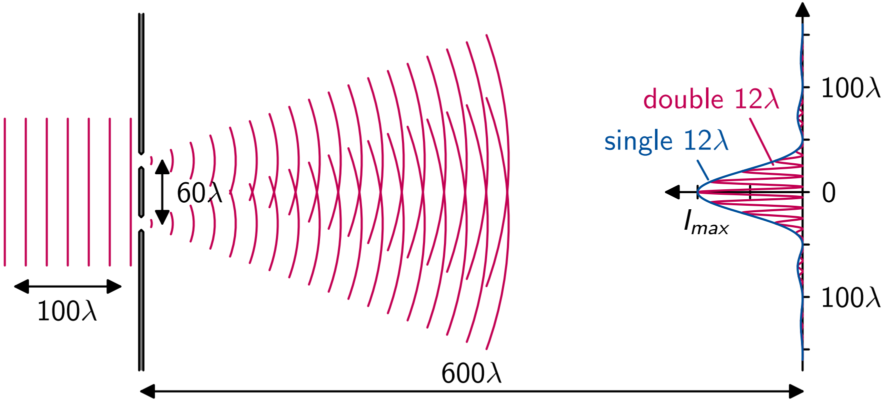
# Spectral Bandwidth (cont.)

A list of optical radiation sources with increasing coherence:

| Source | $\lambda$ | $\Delta\lambda$ | $\Delta\nu$ | $L_c$ | $t_c$ |
|---|---|---|---|---|---|
| white light | 400-800 nm | 400 nm | 400 THz | 800 nm | 2.5 fs |
| red LED | 640 nm | 40 nm | 30 THz | 10 µm | 30 fs |
| FP diode laser | 1.3 µm | 4 nm | 710 GHz | 420 µm | 1.4 ps |
| sodium-vapor lamp | 589 nm | 600 pm | 500 GHz | 600 µm | 2 ps |
| HeNe laser | 633 nm | 2 pm | 1.5 GHz | 200 mm | 670 ps |
| DFB diode laser | 1.3 µm | 100 fm | 20 MHz | 15 m | 50 ns |
| stabilised HeNe laser | 633 nm | 200 am | 150 kHz | 2 km | 7 µs |
| DFB fibre laser | 1.55 µm | 24 am | 3 kHz | 100 km | 330 µs |

# Spatial Coherence

Young's **double slit experiment** can be used to measure the spatial coherence of a laser:




The maximum intensity $I_{max}$ decreases with increasing slit distance and reaches $1/\mathrm{e}$ of its single slit value for the spatial coherence length.

# van Cittert–Zernike Theorem

The van Cittert–Zernike theorem states that the **angular intensity distribution** of a source is identical to the spatial Fourier transform of its **cross-correlation function**:

$$I(k_x, k_y) = \iint_{-\infty}^{+\infty} \Gamma_{12}\, \mathrm{e}^{i(k_x x + k_y y)} dx\, dy$$

with the cross-correlation function

$$\Gamma_{12} = \langle E(\mathbf{r}_1)\, E^*(\mathbf{r}_2) \rangle$$

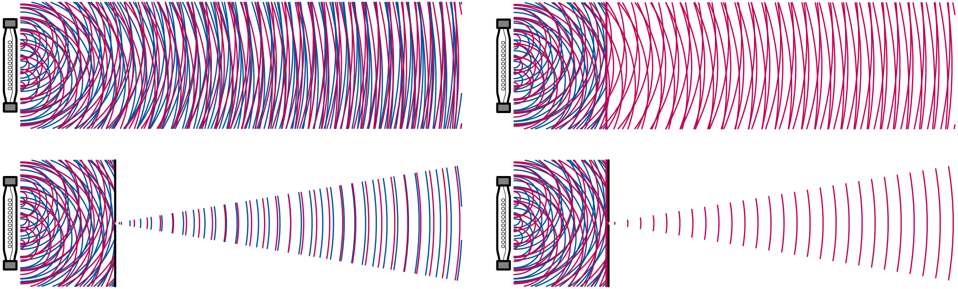
and $x = x_2 - x_1$ and $y = y_2 - y_1$. From this follows a condition for spatial coherence of a light source in the far field:

$$2d \sin \alpha \ll \lambda$$

with the source aperture width $d$ and half exit angle $\alpha$.

# Coherence Filters

Generating coherent light from an incoherent source (incandescent bulb):



- **Incoherence** (no filter): many wavelengths, many directions
- **Spatially coherence** (pinhole): many wavelengths, single direction
- **Temporal coherence** (spectral filter): single wavelength, many directions
- **Full coherence** (both filters): single wavelength, single direction