

Applied Wave Optics: Film Waveguides

Reinhard Caspary

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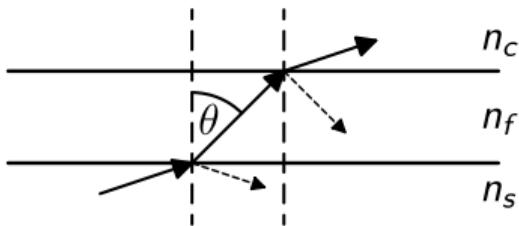
Film Waveguides

Waveguiding requires at least two interfaces. The film waveguide with two parallel interfaces consisting of three material layers **substrate**, **film** and **cladding** is the simplest kind of dielectric waveguide. Utilisation of total internal reflection requires the highest refractive index for the film layer. We denote the other layers according to

$$n_f > n_s \geq n_c$$

The range of propagation angles θ in the film splits into three cases. Small angles result in normal reflection at both interfaces. This is called **free wave**:

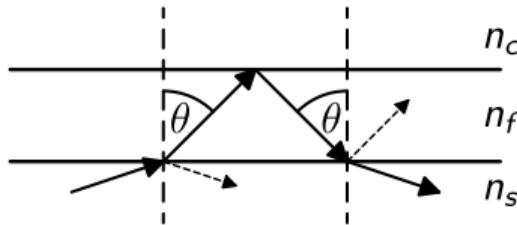
$$\frac{n_s}{n_f} \geq \frac{n_c}{n_f} > \sin \theta$$



Film Waveguides (cont.)

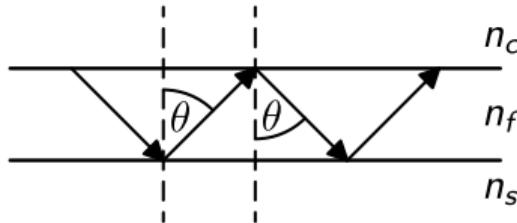
If substrate and cladding have different refractive indices, intermediate angles can result in total internal reflection at the cladding only. This is called **substrate wave**:

$$\frac{n_s}{n_f} > \sin \theta > \frac{n_c}{n_f}$$



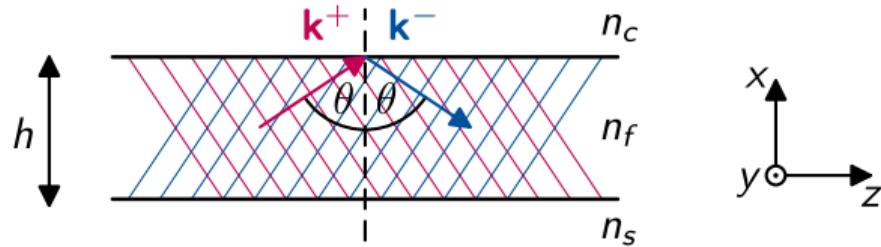
Waveguiding requires total internal reflection at both interfaces. The **film wave** is trapped between both interfaces. It can enter and leave the film only via its end faces:

$$\sin \theta > \frac{n_s}{n_f} \geq \frac{n_c}{n_f}$$



Reflected Wave

For each guided wave with wave vector $\mathbf{k}^+ = k(\cos \theta, 0, \sin \theta)$ there is also its reflected counterpart with wave vector $\mathbf{k}^- = k(-\cos \theta, 0, \sin \theta)$:



With $\mathbf{r} = (x, y, z)$ the expressions for these two waves are

$$\mathbf{E}^\pm(t, \mathbf{r}) = \hat{\mathbf{E}} e^{i(\omega t - n_f \mathbf{k}^\pm \cdot \mathbf{r})} = \hat{\mathbf{E}} e^{i[\omega t - kn_f (\pm x \cos \theta + z \sin \theta)]}$$

Standing Wave

The superposition of both waves results in a sum wave **standing** in x -direction and **travelling** in z -direction along the waveguide:

$$\begin{aligned}\mathbf{E}(t, \mathbf{r}) &= \mathbf{E}^+(t, \mathbf{r}) + \mathbf{E}^-(t, \mathbf{r}) \\ &= \hat{\mathbf{E}} \left[e^{i[\omega t - kn_f(-x \cos \theta + z \sin \theta)]} + e^{i[\omega t - kn_f(x \cos \theta + z \sin \theta)]} \right] \\ &= \hat{\mathbf{E}} \left(e^{ikn_f x \cos \theta} + e^{-ikn_f x \cos \theta} \right) e^{i(\omega t - kn_f z \sin \theta)} \\ &= 2 \hat{\mathbf{E}} \cos(kn_f x \cos \theta) e^{i(\omega t - \beta z)}\end{aligned}$$

with $\beta = kn_{\text{eff}}$ and the **effective refractive index** $n_{\text{eff}} = n_f \sin \theta$ of the propagation in z -direction. The value of this effective refractive index always lies between the film and the surrounding materials:

$$n_f \geq n_{\text{eff}} \geq n_s \geq n_c$$

Characteristic Equation

Note: The derivation requires **constructive interference** of both waves. The phase delay of a full round trip in x -direction must sum up to an integer multiple of 2π :

$$kn_f h \cos \theta - 2\Phi_c(\theta) + kn_f h \cos \theta - 2\Phi_s(\theta) = m 2\pi \quad \text{with } m = 0, 1, 2, \dots$$

The propagation angle θ of a guided wave must satisfy the **characteristic equation**

$$kn_f h \cos \theta - \Phi_c(\theta) - \Phi_s(\theta) = m\pi$$

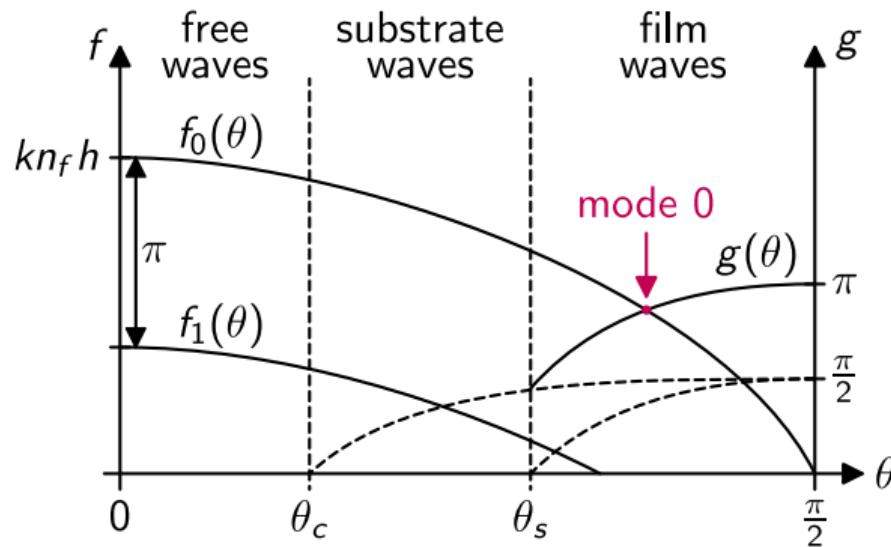
with the polarisation dependent phase shifts of total internal reflection

$$\text{TE: } \tan \Phi_i(\theta) = \sqrt{\frac{n_{\text{eff}}^2(\theta) - n_i^2}{n_f^2 - n_{\text{eff}}^2}} \quad \text{TM: } \tan \Phi_i(\theta) = \frac{n_f^2}{n_i^2} \sqrt{\frac{n_{\text{eff}}^2(\theta) - n_i^2}{n_f^2 - n_{\text{eff}}^2}}$$

with $i \in [s, c]$. Such guided waves are called **transversal waveguide modes**.

Waveguide Modes

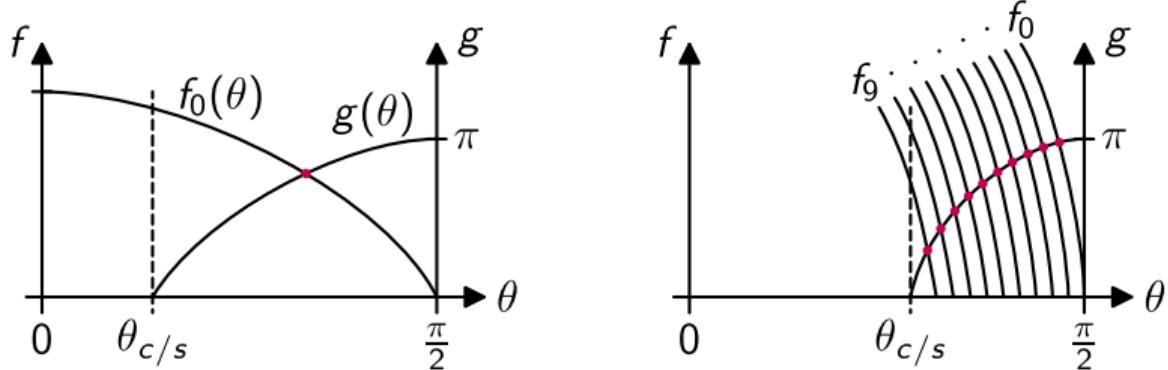
There exists no analytical solution of the characteristic equation. For a graphical solution we split it into two functions $f_m(\theta) = kn_f h \cos \theta - m\pi$ and $g(\theta) = \Phi_s(\theta) + \Phi_c(\theta)$. Guided modes thus correspond to intersections of both functions:



with the critical angles $\sin \theta_c = n_c/n_f$ and $\sin \theta_s = n_s/n_f$.

Special Cases

Symmetric waveguides with $n_c = n_s$ are always guiding at least one mode:



For thick films with $h \gg \lambda$ the function $f_m(\theta)$ becomes very steep, resulting in many intersections. Such waveguides are called **multimode waveguides**.

Universal Solution

The V -number as dimensionless **frequency parameter** combines the external input parameters of geometry(h), material (NA) and signal (λ) using the definition of the numerical aperture $NA = \sqrt{n_f^2 - n_s^2}$:

$$V = kh\sqrt{n_f^2 - n_s^2} = 2\pi \frac{h}{\lambda} NA \in [0, \infty]$$

The **asymmetry parameter** α is zero for symmetric waveguides with $n_c = n_s$. For TE modes it is given by:

$$\alpha_{TE} = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \in [0, \infty]$$

The **phase parameter** B_m describes a waveguide mode m based on the propagation angle θ_m as output parameter:

$$B_m = \frac{n_{\text{eff}}^2 - n_s^2}{n_f^2 - n_s^2} = \frac{n_f^2 \sin^2 \theta_m - n_s^2}{n_f^2 - n_s^2} \in [0, 1]$$

Universal Solution (cont.)

With these definitions, the terms of the characteristic equation translate to:

$$kn_f h \cos \theta_m = V \sqrt{1 - B_m} \quad \tan \Phi_{s,TE} = \sqrt{\frac{B_m}{1 - B_m}} \quad \tan \Phi_{c,TE} = \sqrt{\frac{B_m + \alpha_{TE}}{1 - B_m}}$$

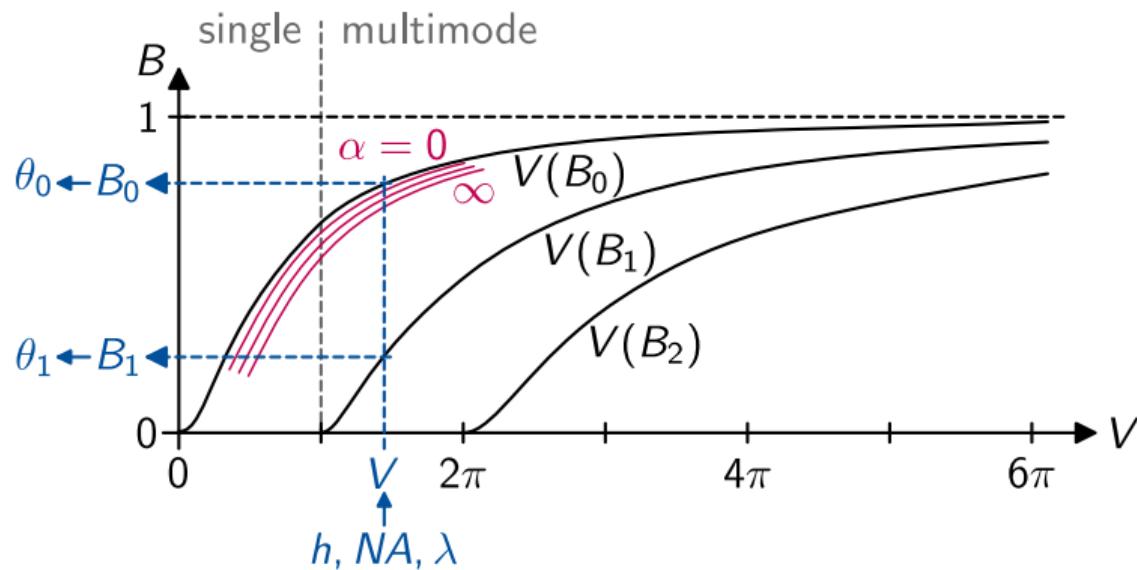
The universal characteristic equation is thus

$$V(B_m) = \frac{1}{\sqrt{1 - B_m}} \left(m\pi + \arctan \sqrt{\frac{B_m}{1 - B_m}} + \arctan \sqrt{\frac{B_m + \alpha_{TE}}{1 - B_m}} \right)$$

There again exists no analytical solution, but V is a monotonous function of B_m and has certain properties, which make a numerical inversion simple and straight forward. This is used to determine the phase parameter for each mode.

Numerical Algorithm

We start with the calculation of the V -number based on the parameters of the waveguide. Its value determines the number of modes. For each mode m the phase parameter B_m between 0 and 1 is found using the bisection method on the monotonous function $V(B_m)$. Finally, the mode angle θ_m is directly calculated from B_m .



Number of Modes

For each mode m the curve starts at a certain point V_m , which determines the number of modes for a given V -number:

$$V_m = V|_{B_m=0} = m\pi + \arctan \sqrt{\alpha_{TE}}$$

Taking into account that each TE mode comes with a respective TM mode, this **number of modes** N is two times the first integer above V/π in case of a symmetric waveguide with $\alpha_{TE} = 0$:

$$N = 2 \operatorname{ceil} \left(\frac{V}{\pi} \right) = 2 \operatorname{ceil} \left(\frac{2h}{\lambda} \sqrt{n_f^2 - n_s^2} \right)$$

Cut-Off Wavelength

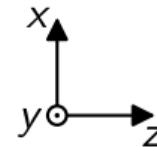
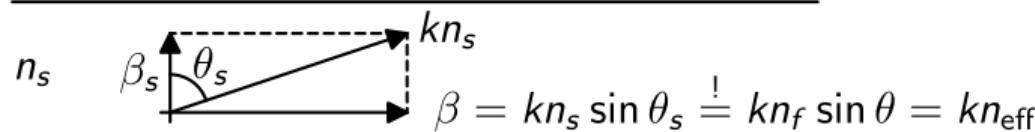
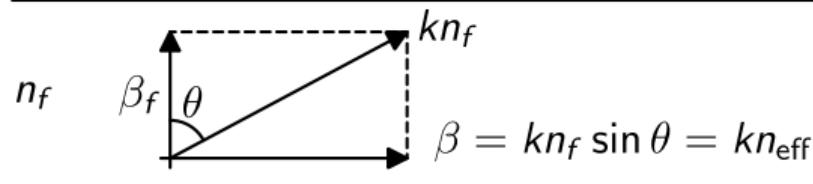
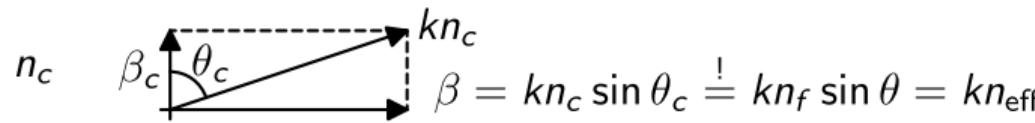
For a given waveguide and a given mode number m there exists a **cut-off wavelength** λ_m , which is the largest signal wavelength allowed for mode m to exist:

$$2\pi \frac{h}{\lambda_m} \sqrt{n_f^2 - n_s^2} = m\pi$$
$$\lambda_m = \frac{2h}{m} \sqrt{n_f^2 - n_s^2}$$

In particular, this means that a given waveguide is **single-mode** for signal wavelengths $\lambda > \lambda_1$ and the same waveguide is **multimode** for $\lambda < \lambda_1$.

Wave Vectors

The component β of the wave vector in the direction of the waveguide is the same in all three layers, due to Snell's law:



Wave Vectors (cont.)

The component β_i perpendicular to the interfaces with $i \in [s, f, c]$ can be calculated using Pythagoras' theorem:

$$\beta_i = \sqrt{n_i^2 k^2 - \beta^2} = k \sqrt{n_i^2 - n_f^2 \sin^2 \theta}$$

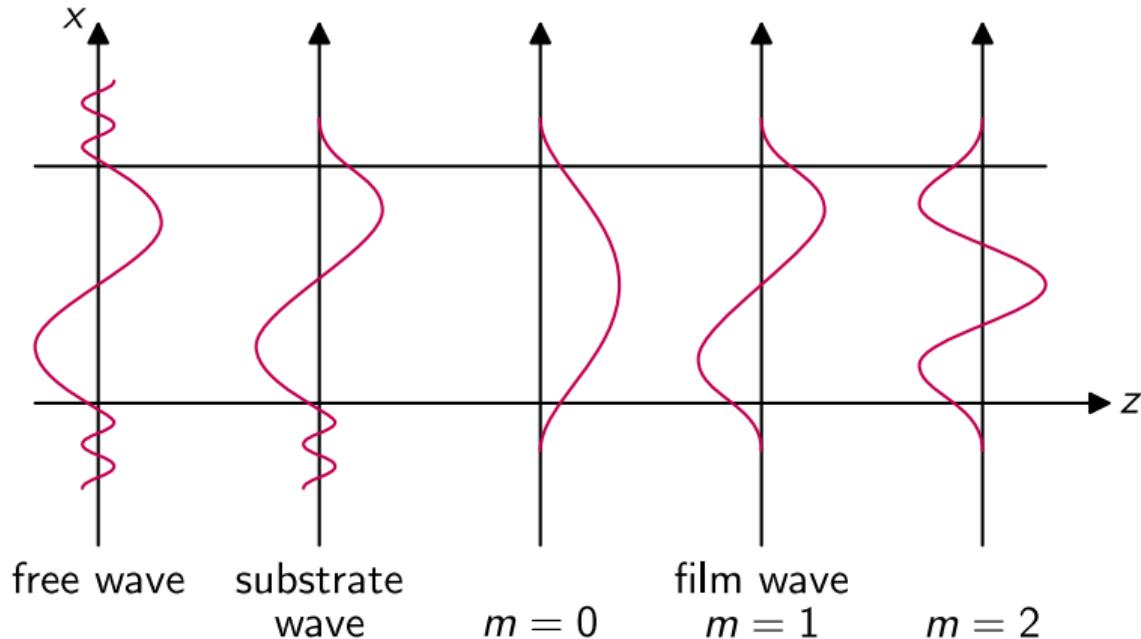
It is related to the **spatial period** Λ_i of the lateral mode shape by $\beta_i = 2\pi/\Lambda_i$. In case of total internal reflection, the square root is imaginary in the substrate and/or cladding:

$$\alpha_i = ik \sqrt{n_f^2 \sin^2 \theta - n_i^2}$$

and the mode shape follows an exponential decay instead of a spatial oscillation. This tail of the mode protruding into the substrate and/or cladding is called **evanescent field**.

Mode Shapes

The lateral shape of the waveguide modes consists of piecewise defined sine and/or exponential functions. The shape as well as its first derivative is continuous at both interfaces. These conditions are fixing the lateral phase offsets in each layer.



Summary of Mode Calculations

- ▶ The **initial parameters** h , n_s , n_f , n_c , λ , and E_f are given by the waveguide design and the experimental conditions.
- ▶ Fresnel equations determine the amplitudes E_s and E_c .
- ▶ Solving the characteristic equation provides the number of modes N as well as their propagation angles θ_m .
- ▶ The propagation angles determine the lateral wave numbers β_i or exponential decay parameters α_i .
- ▶ Based on the previous results the continuity conditions at the interfaces deliver the lateral phase offset in each material layer.
- ▶ **Result:** Completely defined shape and propagation parameters for all guided and unguided modes.