Applied Wave Optics: Film Waveguides

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All sources and vector graphics are available on the GitHub repository
`https://github.com/reincas/AWO_course`.

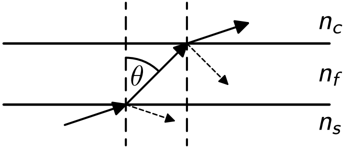
Film Waveguides

Waveguiding requires at least two interfaces. The film waveguide with two parallel interfaces consisting of three material layers **substrate**, **film** and **cladding** is the simplest kind of dielectric waveguide. Utilisation of total internal reflection requires the highest refractive index for the film layer. We denote the other layers according to

$$n_f > n_s \geq n_c$$

The range of propagation angles θ in the film splits into three cases. Small angles result in normal reflection at both interfaces. This is called **free wave**:

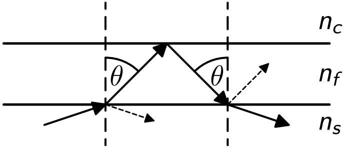
$$\frac{n_s}{n_f} \geq \frac{n_c}{n_f} > \sin\theta$$



Film Waveguides (cont.)

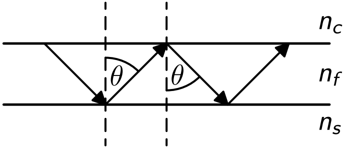
If substrate and cladding have different refractive indices, intermediate angles can result in total internal reflection at the cladding only. This is called **substrate wave**:

$$\frac{n_s}{n_f} > \sin\theta > \frac{n_c}{n_f}$$



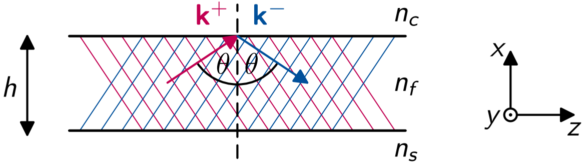
Waveguiding requires total internal reflection at both interfaces. The **film wave** is trapped between both interfaces. It can enter and leave the film only via its end faces:

$$\sin\theta > \frac{n_s}{n_f} \geq \frac{n_c}{n_f}$$



Reflected Wave

For each guided wave with wave vector $\mathbf{k}^{+} = k(\cos\theta, 0, \sin\theta)$ there is also its reflected counterpart with wave vector $\mathbf{k}^{-} = k(-\cos\theta, 0, \sin\theta)$:



With $\mathbf{r} = (x, y, z)$ the expressions for these two waves are

$$\mathbf{E}^{\pm}(t, \mathbf{r}) = \hat{\mathbf{E}}\, \mathrm{e}^{i(\omega t - n_f \mathbf{k}^{\pm}\mathbf{r})} = \hat{\mathbf{E}}\, \mathrm{e}^{i[\omega t - k n_f(\pm x\cos\theta + z\sin\theta)]}$$

Standing Wave

The superposition of both waves results in a sum wave **standing** in x-direction and **travelling** in z-direction along the waveguide:

$$\begin{aligned}
\mathbf{E}(t,\mathbf{r}) &= \mathbf{E}^{+}(t,\mathbf{r}) + \mathbf{E}^{-}(t,\mathbf{r}) \\
&= \hat{\mathbf{E}}\left[\mathrm{e}^{i[\omega t - kn_f(-x\cos\theta + z\sin\theta)]} + \mathrm{e}^{i[\omega t - kn_f(x\cos\theta + z\sin\theta)]}\right] \\
&= \hat{\mathbf{E}}\left(\mathrm{e}^{ikn_f x\cos\theta} + \mathrm{e}^{-ikn_f x\cos\theta}\right)\mathrm{e}^{i(\omega t - kn_f z\sin\theta)} \\
&= 2\,\hat{\mathbf{E}}\cos(kn_f x\cos\theta)\,\mathrm{e}^{i(\omega t - \beta z)}
\end{aligned}$$

with $\beta = kn_{\text{eff}}$ and the **effective refractive index** $n_{\text{eff}} = n_f \sin\theta$ of the propagation in z-direction. The value of this effective refractive index always lies between the film and the surrounding materials:

$$n_f \geq n_{\text{eff}} \geq n_s \geq n_c$$

Characteristic Equation

Note: The derivation requires **constructive interference** of both waves. The phase delay of a full round trip in x-direction must sum up to an integer multiple of 2π:

$$kn_f h \cos\theta - 2\Phi_c(\theta) + kn_f h \cos\theta - 2\Phi_s(\theta) = m\,2\pi \qquad \text{with } m = 0, 1, 2, \ldots$$

The propagation angle θ of a guided wave must satisfy the **characteristic equation**

$$kn_f h \cos\theta - \Phi_c(\theta) - \Phi_s(\theta) = m\pi$$

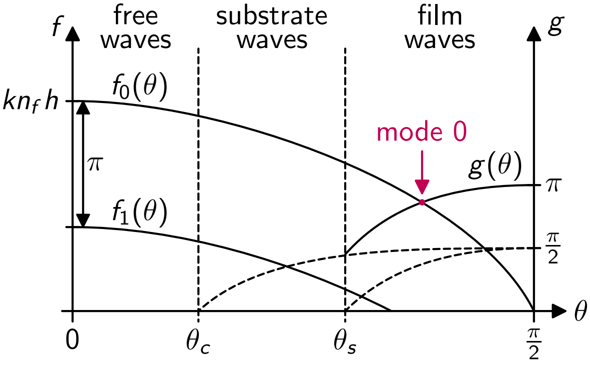
with the polarisation dependent phase shifts of total internal reflection

$$\text{TE: } \tan\Phi_i(\theta) = \sqrt{\frac{n_{\text{eff}}^2(\theta) - n_i^2}{n_f^2 - n_{\text{eff}}^2}} \qquad \text{TM: } \tan\Phi_i(\theta) = \frac{n_f^2}{n_i^2}\sqrt{\frac{n_{\text{eff}}^2(\theta) - n_i^2}{n_f^2 - n_{\text{eff}}^2}}$$

with $i \in [s, c]$. Such guided waves are called **transversal waveguide modes**.

Waveguide Modes

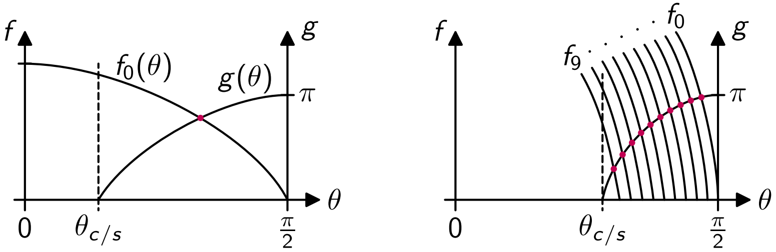
There exists no analytical solution of the characteristic equation. For a graphical solution we split it into two functions $f_m(\theta) = kn_f h\cos\theta - m\pi$ and $g(\theta) = \Phi_s(\theta) + \Phi_c(\theta)$. Guided modes thus correspond to intersections of both functions:



with the critical angles $\sin\theta_c = n_c/n_f$ and $\sin\theta_s = n_s/n_f$.

Special Cases

Symmetric waveguides with $n_c = n_s$ are always guiding at least one mode:



For thick films with $h \gg \lambda$ the function $f_m(\theta)$ becomes very steep, resulting in many intersections. Such waveguides are called **multimode waveguides**.

Universal Solution

The V-number as dimensionless **frequency parameter** combines the external input parameters of geometry(h), material (NA) and signal (λ) using the definition of the numerical aperture $NA = \sqrt{n_f^2 - n_s^2}$:

$$V = kh\sqrt{n_f^2 - n_s^2} = 2\pi \frac{h}{\lambda} NA \in [0, \infty]$$

The **asymmetry parameter** α is zero for symmetric waveguides with $n_c = n_s$. For TE modes it is given by:

$$\alpha_{TE} = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2} \in [0, \infty]$$

The **phase parameter** B_m describes a waveguide mode m based on the propagation angle θ_m as output parameter:

$$B_m = \frac{n_{\text{eff}}^2 - n_s^2}{n_f^2 - n_s^2} = \frac{n_f^2 \sin^2\theta_m - n_s^2}{n_f^2 - n_s^2} \in [0, 1]$$

Universal Solution (cont.)

With these definitions, the terms of the characteristic equation translate to:

$$k n_f h \cos\theta_m = V\sqrt{1-B_m} \qquad \tan\Phi_{s,TE} = \sqrt{\frac{B_m}{1-B_m}} \qquad \tan\Phi_{c,TE} = \sqrt{\frac{B_m+\alpha_{TE}}{1-B_m}}$$

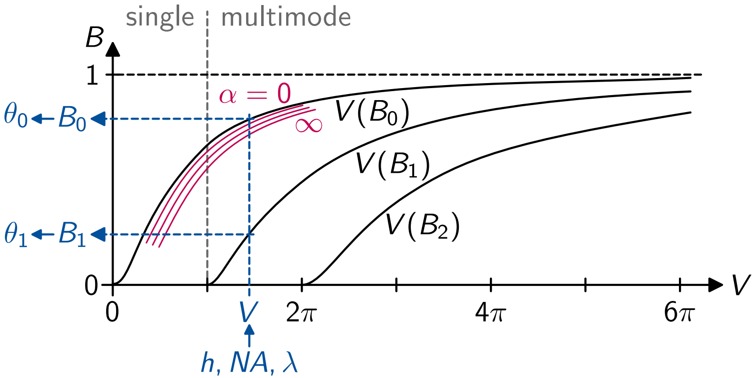
The universal characteristic equation is thus

$$V(B_m) = \frac{1}{\sqrt{1-B_m}}\left(m\pi + \arctan\sqrt{\frac{B_m}{1-B_m}} + \arctan\sqrt{\frac{B_m+\alpha_{TE}}{1-B_m}}\right)$$

There again exists no analytical solution, but V is a monotonous function of B_m and has certain properties, which make a numerical inversion simple and straight forward. This is used to determine the phase parameter for each mode.

Numerical Algorithm

We start with the calculation of the V-number based on the parameters of the waveguide. Its value determines the number of modes. For each mode m the phase parameter B_m between 0 and 1 is found using the bisection method on the monotonous function $V(B_m)$. Finally, the mode angle θ_m is directly calculated from B_m.



Number of Modes

For each mode m the curve starts at a certain point V_m, which determines the number of modes for a given V-number:

$$V_m = V|_{B_m=0} = m\pi + \arctan\sqrt{\alpha_{TE}}$$

Taking into account that each TE mode comes with a respective TM mode, this **number of modes** N is two times the first integer above V/π in case of a symmetric waveguide with $\alpha_{TE} = 0$:

$$N = 2\,\mathrm{ceil}\left(\frac{V}{\pi}\right) = 2\,\mathrm{ceil}\left(\frac{2h}{\lambda}\sqrt{n_f^2 - n_s^2}\right)$$

Cut-Off Wavelength

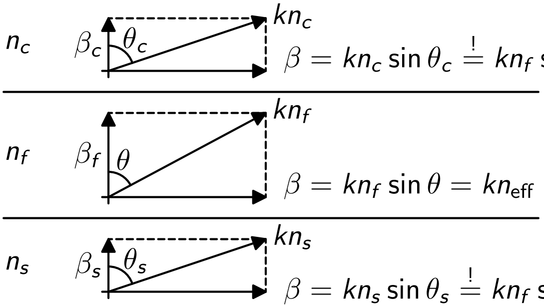
For a given waveguide and a given mode number m there exists a **cut-off wavelength** λ_m, which is the largest signal wavelength allowed for mode m to exist:

$$2\pi \frac{h}{\lambda_m} \sqrt{n_f^2 - n_s^2} = m\pi$$
$$\lambda_m = \frac{2h}{m} \sqrt{n_f^2 - n_s^2}$$

In particular, this means that a given waveguide is **single-mode** for signal wavelengths $\lambda > \lambda_1$ and the the same waveguide is **multimode** for $\lambda < \lambda_1$.

Wave Vectors

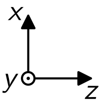
The component β of the wave vector in the direction of the waveguide is the same in all three layers, due to Snell's law:



$$\beta = kn_c \sin\theta_c \stackrel{!}{=} kn_f \sin\theta = kn_{\text{eff}}$$

$$\beta = kn_f \sin\theta = kn_{\text{eff}}$$

$$\beta = kn_s \sin\theta_s \stackrel{!}{=} kn_f \sin\theta = kn_{\text{eff}}$$



Wave Vectors (cont.)

The component β_i perpendicular to the interfaces with $i \in [s, f, c]$ can be calculated using Pythagoras' theorem:

$$\beta_i = \sqrt{n_i^2 k^2 - \beta^2} = k\sqrt{n_i^2 - n_f^2 \sin^2\theta}$$

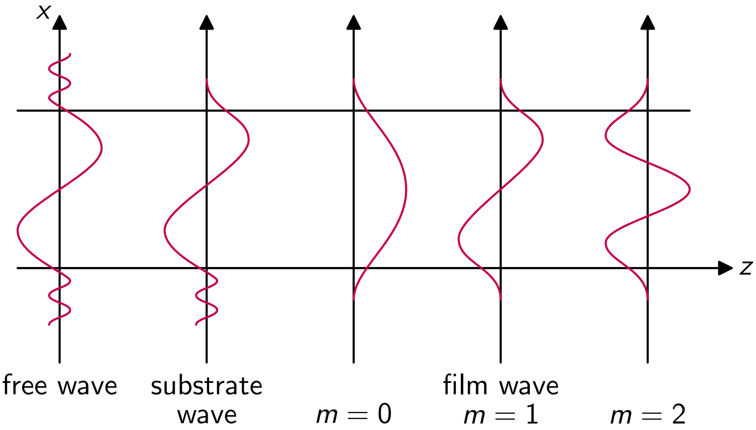
It is related to the **spatial period** Λ_i of the lateral mode shape by $\beta_i = 2\pi/\Lambda_i$. In case of total internal reflection, the square root is imaginary in the substrate and/or cladding:

$$\alpha_i = ik\sqrt{n_f^2 \sin^2\theta - n_i^2}$$

and the mode shape follows an exponential decay instead of a spatial oscillation. This tail of the mode protruding into the substrate and/or cladding is called **evanescent field**.

Mode Shapes

The lateral shape of the waveguide modes consists of piecewise defined sine and/or exponential functions. The shape as well as its first derivative is continuous at both interfaces. These conditions are fixing the lateral phase offsets in each layer.



Summary of Mode Calculations

- The **initial parameters** h, n_s, n_f, n_c, λ, and E_f are given by the waveguide design and the experimental conditions.
- Fresnel equations determine the amplitudes E_s and E_c.
- Solving the characteristic equation provides the number of modes N as well as their propagation angles θ_m.
- The propagation angles determine the lateral wave numbers β_i or exponential decay parameters α_i.
- Based on the previous results the continuity conditions at the interfaces deliver the lateral phase offset in each material layer.
- **Result:** Completely defined shape and propagation parameters for all guided and unguided modes.