# Applied Wave Optics: Waves at Interfaces

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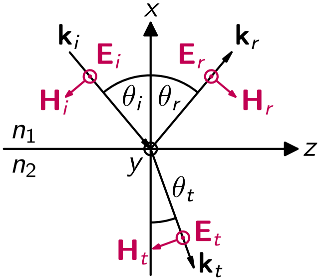
# TE Waves at Material Interfaces

An **s-polarized** wave is polarized perpendicular to the plane of incidence:




The incident wave $i$ is split into a reflected wave $r$ and a transmitted wave $t$ at the interface. With $k = 2\pi/\lambda$, the wave vectors are:

$$\mathbf{k}_i = n_1 k(-\cos\theta_i, 0, \sin\theta_i)$$
$$\mathbf{k}_r = n_1 k(\cos\theta_r, 0, \sin\theta_r)$$
$$\mathbf{k}_t = n_2 k(-\cos\theta_t, 0, \sin\theta_t)$$

# TE Waves at Material Interfaces (cont.)

When we skip the common phase $\mathrm{e}^{i\omega t}$ and the complex conjugate term, the electric field vectors are given by

$$\mathbf{E}_i = \hat{\mathbf{y}} E_i \ \mathrm{e}^{ikn_1 x \cos\theta_i - ikn_1 z \sin\theta_i}$$
$$\mathbf{E}_r = \hat{\mathbf{y}} E_r \ \mathrm{e}^{-ikn_1 x \cos\theta_r - ikn_1 z \sin\theta_r}$$
$$\mathbf{E}_t = \hat{\mathbf{y}} E_t \ \mathrm{e}^{ikn_2 x \cos\theta_t - ikn_2 z \sin\theta_t}$$

and the magnetic field vectors are

$$\mathbf{H}_i = -(\hat{\mathbf{x}} \sin\theta_i + \hat{\mathbf{z}} \cos\theta_i) \frac{n_1}{Z_0} E_i \ \mathrm{e}^{ikn_1 x \cos\theta_i - ikn_1 z \sin\theta_i}$$
$$\mathbf{H}_r = -(\hat{\mathbf{x}} \sin\theta_r - \hat{\mathbf{z}} \cos\theta_r) \frac{n_1}{Z_0} E_r \ \mathrm{e}^{-ikn_1 x \cos\theta_r - ikn_1 z \sin\theta_r}$$
$$\mathbf{H}_t = -(\hat{\mathbf{x}} \sin\theta_t + \hat{\mathbf{z}} \cos\theta_t) \frac{n_2}{Z_0} E_t \ \mathrm{e}^{ikn_2 x \cos\theta_t - ikn_2 z \sin\theta_t}$$

# TE Fresnel Equations

At the material interface $x = 0$ the boundary conditions $E_{1\|} = E_{2\|}$ and $H_{1\|} = H_{2\|}$ apply, resulting in

$$E_i \mathrm{e}^{-ikn_1 z \sin\theta_i} + E_r \mathrm{e}^{-ikn_1 z \sin\theta_r} = E_t \mathrm{e}^{-ikn_2 z \sin\theta_t}$$

$$n_1 \cos\theta_i E_i \mathrm{e}^{-ikn_1 z \sin\theta_i} - n_1 \cos\theta_r E_r \mathrm{e}^{-ikn_1 z \sin\theta_r} = n_2 \cos\theta_t E_t \mathrm{e}^{-ikn_2 z \sin\theta_t}$$

These equations are directly derived from Maxwell's equations and must therefore be valid everywhere for any incident field. This is is only possible if all **phases** are equal:

$$n_1 \sin\theta_i = n_1 \sin\theta_r = n_2 \sin\theta_t$$

resulting in

$$\theta_r = \theta_i \qquad \text{law of reflection}$$

$$n_1 \sin\theta_i = n_2 \sin\theta_t \qquad \text{Snell's law}$$

# TE Fresnel Equations (cont.)

For the **amplitudes** the boundary conditions result in

$$E_i + E_r = E_t$$
$$n_1 \cos\theta_i E_i - n_1 \cos\theta_i E_r = n_2 \cos\theta_t E_t$$

These equations are used to derive the **Fresnel equations** for TE waves:

$$r_E = \frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} = -\frac{H_r}{H_i} = -r_H$$
$$t_E = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{n_1 H_t}{n_2 H_i} = \frac{n_1}{n_2} t_H$$

where the impedance relations $E_r/H_r = -n_1/Z_0$ and $E_t/H_t = n_2/Z_0$ lead to the respective Fresnel equations for magnetic fields.

# Total Internal Reflection

We use Snell's law $n_1 \sin\theta_i = n_2 \sin\theta_t$ and the trigonometric identity relation $\sin^2\theta_t + \cos^2\theta_t = 1$ to express $\theta_t$ by the given $\theta_i$ in the Fresnel equations:

$$\cos\theta_t = \pm\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

For $n_2 < n_1$ there is a **critical angle** of incidence $\sin\theta_c = n_2/n_1$ separating the range of real and imaginary values of $\cos\theta_t$:

$$\cos\theta_t = \begin{cases} +\sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i} & \text{for } \theta_i < \theta_c \text{ (normal reflection)} \\ -i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1} & \text{for } \theta_i > \theta_c \text{ (total internal reflection)} \end{cases}$$

**Note:** The sign of the square root cannot be decided mathematically. Physical arguments based on the normal phase $\exp(ikn_2 x\cos\theta_t)$ are used instead.

# Total Internal Reflection (cont.)

In case of total internal reflection, numerator and denominator of $r_E$ are complex conjugates of each other resulting in a pure phase factor with $|r_E| = 1$:

$$r_E = \frac{E_r}{E_i} = \frac{n_1 \cos\theta_i + i n_2 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}}{n_1 \cos\theta_i - i n_2 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}}$$

$$= \frac{\left(n_1 \cos\theta_i + i n_2 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}\right)^2}{n_1^2 + n_2^2} \stackrel{!}{=} \mathrm{e}^{2i\Phi_{TE}}$$

Total internal reflection thus comes with a **phase shift** $2\Phi_{TE}$ for the reflected wave with

$$\tan\Phi_{TE} = \frac{\sqrt{n_1^2 \sin^2\theta_i - n_2^2}}{n_1 \cos\theta_i}$$

# Poynting Vector

Identity relation from vector analysis applied to electro-magnetic fields:

$$\nabla(\mathbf{E} \times \mathbf{H}) = \mathbf{H}(\nabla \times \mathbf{E}) - \mathbf{E}(\nabla \times \mathbf{H})$$

We define the **Poynting vector** $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ with the SI unit $\mathrm{W/m^2}$ of an energy flux or intensity and we insert Faradays's and Ampère's law:

$$\nabla\mathbf{S} = -\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} - \mathbf{jE} - \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}$$

Replacing **B** and **D** by **H** and **E** we get

$$\nabla\mathbf{S} = -\mathbf{jE} - \left(\mu\mu_0\mathbf{H}\frac{\partial \mathbf{H}}{\partial t} + \varepsilon\varepsilon_0\mathbf{E}\frac{\partial \mathbf{E}}{\partial t}\right)$$

and finally

$$\nabla\mathbf{S} = -\mathbf{jE} - \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\mu\mu_0\mathbf{H}^2 + \varepsilon\varepsilon_0\mathbf{E}^2\right)\right]$$

# Poynting Vector (cont.)

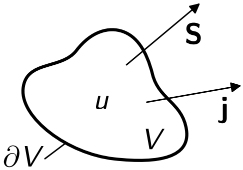
With the energy density

$$u = \frac{1}{2}\left(\mu\mu_0 \mathbf{H}^2 + \varepsilon\varepsilon_0 \mathbf{E}^2\right)$$

of the electro-magnetic field with the SI unit $\mathrm{J}/\mathrm{m}^3$ we get the fundamental equation

$$-\frac{\partial u}{\partial t} = \nabla \mathbf{S} + \mathbf{j}\mathbf{E}$$

It shows that two effects alter the energy density inside a given volume: Radiation $\nabla \mathbf{S}$ and charge transport by an electric current $\mathbf{jE}$ through the surface $\partial V$ of the volume:



# Radiation Intensity

Poynting vector of a plane wave:

$$\begin{aligned}\mathbf{S} = \mathbf{E} \times \mathbf{H} &= \frac{1}{2}\left(\hat{\mathbf{E}}\,\mathrm{e}^{i(\omega t - \mathbf{kr})} + \hat{\mathbf{E}}^{*}\mathrm{e}^{-i(\omega t - \mathbf{kr})}\right) \times \frac{1}{2}\left(\hat{\mathbf{H}}\,\mathrm{e}^{i(\omega t - \mathbf{kr})} + \hat{\mathbf{H}}^{*}\mathrm{e}^{-i(\omega t - \mathbf{kr})}\right) \\ &= \frac{1}{4}\left[(\mathbf{E} \times \mathbf{H})\,\mathrm{e}^{2i(\omega t - \mathbf{kr})} + (\mathbf{E}^{*} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^{*}) + (\mathbf{E}^{*} \times \mathbf{H}^{*})\,\mathrm{e}^{-2i(\omega t - \mathbf{kr})}\right]\end{aligned}$$

The radiation **intensity** is defined as absolute value of the temporal average of $\mathbf{S}$:

$$I = |\langle \mathbf{S} \rangle| = \frac{1}{4}|\mathbf{E}^{*} \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^{*}| = \frac{1}{2}|\Re(\mathbf{E}^{*} \times \mathbf{H})|$$

and is thus directly connected to the measured radiation power.

# Intensity at the Interface

Intensity of the incident, reflected, and transmitted waves normal to the interface:

$$I_i = \frac{1}{2}|\Re(\mathbf{E}^*_{i\|} \times \mathbf{H}_{i\|})| = \frac{1}{2}|\Re(E^*_i H_i \cos\theta_i)| = \frac{1}{2}|E_i|^2 \frac{n_1}{Z_0} \cos\theta_i$$
$$I_r = \frac{1}{2}|\Re(\mathbf{E}^*_{r\|} \times \mathbf{H}_{r\|})| = \frac{1}{2}|\Re(E^*_r H_r \cos\theta_r)| = \frac{1}{2}|E_r|^2 \frac{n_1}{Z_0} \cos\theta_i$$
$$I_t = \frac{1}{2}|\Re(\mathbf{E}^*_{t\|} \times \mathbf{H}_{t\|})| = \frac{1}{2}|\Re(E^*_t H_t \cos\theta_t)| = \frac{1}{2}|E_t|^2 \frac{n_2}{Z_0} |\Re(\cos\theta_t)|$$

For **total internal reflection** with $\theta_i > \theta_c$ we have an imaginary $\cos\theta_t$ and thus $I_t = 0$:

$$R = \frac{I_r}{I_i} = \frac{|E_r|^2}{|E_i|^2} = |r_E|^2 = \left|\frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}\right|^2 = 1$$
$$T = \frac{I_t}{I_i} = 0$$

# Intensity at the Interface (cont.)

For **normal reflection** with $\theta_i < \theta_c$ we obtain

$$R = \frac{I_r}{I_i} = \frac{|E_r|^2}{|E_i|^2} = \left| \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \right|^2$$

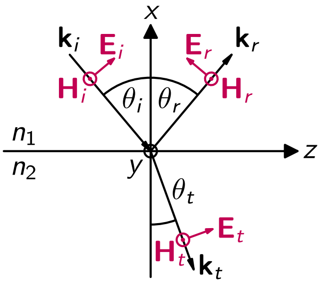
$$T = \frac{I_t}{I_i} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i} \frac{|E_t|^2}{|E_i|^2} = \frac{n_1 n_2 \cos\theta_i \cos\theta_t}{(n_1 \cos\theta_i + n_2 \cos\theta_t)^2}$$

with $R + T = 1$, and in case of normal incidence with $\theta_i = 0$ the reflection factor reduces to

$$R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

# TM Waves at Material Interfaces

An **p-polarized** wave is polarized parallel to the plane of incidence:




The incident wave $i$ is split into a reflected wave $r$ and a transmitted wave $t$ at the interface in exactly the same way as for the TE wave:

$$\mathbf{k}_i = n_1 k(-\cos\theta_i, 0, \sin\theta_i)$$
$$\mathbf{k}_r = n_1 k(\cos\theta_r, 0, \sin\theta_r)$$
$$\mathbf{k}_t = n_2 k(-\cos\theta_t, 0, \sin\theta_t)$$

# TM Waves at Material Interfaces (cont.)

When we skip the common phase $\mathrm{e}^{i\omega t}$ and the complex conjugate term, the electric field vectors are given by

$$\mathbf{E}_i = (\hat{\mathbf{x}} \sin\theta_i + \hat{\mathbf{z}} \cos\theta_i) E_i \; \mathrm{e}^{ikn_1 x \cos\theta_i - ikn_1 z \sin\theta_i}$$
$$\mathbf{E}_r = (\hat{\mathbf{x}} \sin\theta_r - \hat{\mathbf{z}} \cos\theta_r) E_r \; \mathrm{e}^{-ikn_1 x \cos\theta_r - ikn_1 z \sin\theta_r}$$
$$\mathbf{E}_t = (\hat{\mathbf{x}} \sin\theta_t + \hat{\mathbf{z}} \cos\theta_t) E_t \; \mathrm{e}^{ikn_2 x \cos\theta_t - ikn_2 z \sin\theta_t}$$

and the magnetic field vectors are

$$\mathbf{H}_i = \hat{\mathbf{y}} \frac{n_1}{Z_0} E_i \; \mathrm{e}^{ikn_1 x \cos\theta_i - ikn_1 z \sin\theta_i}$$
$$\mathbf{H}_r = \hat{\mathbf{y}} \frac{n_1}{Z_0} E_r \; \mathrm{e}^{-ikn_1 x \cos\theta_r - ikn_1 z \sin\theta_r}$$
$$\mathbf{H}_t = \hat{\mathbf{y}} \frac{n_2}{Z_0} E_t \; \mathrm{e}^{ikn_2 x \cos\theta_t - ikn_2 z \sin\theta_t}$$

# TM Fresnel Equations

With the boundary conditions $E_{1\|} = E_{2\|}$ and $H_{1\|} = H_{2\|}$ at $x = 0$, we get

$$\cos\theta_i E_i \mathrm{e}^{-ikn_1 z \sin\theta_i} - \cos\theta_r E_r \mathrm{e}^{-ikn_1 z \sin\theta_r} = \cos\theta_t E_t \mathrm{e}^{-ikn_2 z \sin\theta_t}$$
$$n_1 E_i \mathrm{e}^{-ikn_1 z \sin\theta_i} + n_1 E_r \mathrm{e}^{-ikn_1 z \sin\theta_r} = n_2 E_t \mathrm{e}^{-ikn_2 z \sin\theta_t}$$

The phases must be equal again, resulting in the law of reflection and Snell's law again.

$$\cos\theta_i E_i - \cos\theta_i E_r = \cos\theta_t E_t$$
$$n_1 E_i + n_1 E_r = n_2 E_t$$

These equations are used to derive the **Fresnel equations** for TM waves:

$$r_E = \frac{E_r}{E_i} = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = -\frac{H_r}{H_i} = -r_H$$
$$t_E = \frac{E_t}{E_i} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} = \frac{n_1 H_t}{n_2 H_i} = \frac{n_1}{n_2} t_H$$

# Total Internal Reflection

The critical angle for total internal reflection is the same for TM waves. Numerator and denominator of $r_E$ are again complex conjugates of each other:

$$r_E = \frac{E_r}{E_i} = \frac{n_2 \cos\theta_i + i n_1 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}}{n_2 \cos\theta_i - i n_1 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}}$$

$$= \frac{\left(n_2 \cos\theta_i + i n_1 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}\right)^2}{n_2^2 \cos^2\theta_i + \frac{n_1^2}{n_2^2}\left(n_1^2 \sin^2\theta_i - n_2^2\right)} \overset{!}{=} \mathrm{e}^{2i\Phi_{TM}}$$

The **phase shift** $2\Phi_{TM}$ for the reflected wave in case of total internal reflection thus is

$$\tan\Phi_{TM} = \frac{n_1^2}{n_2^2} \frac{\sqrt{n_1^2 \sin^2\theta_i - n_2^2}}{n_1 \cos\theta_i} = \frac{n_1^2}{n_2^2} \tan\Phi_{TE}$$

# Brewster Angle

Fresnel's equations for TM waves allow for a certain angle of incidence $\theta_i = \theta_B$ with vanishing reflection $r_E(\theta_B) = 0$:

$$n_2 \cos\theta_B = n_1 \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B}$$

$$\frac{n_2^2}{n_1^2}\cos^2\theta_B = 1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B$$

We use the trigonometric identity $\sin^2\theta_B + \cos^2\theta_B = 1$ and collect the $\sin^2\theta_B$ terms on the left side of the equation:

$$\left(\frac{n_1^2}{n_2^2} - \frac{n_2^2}{n_1^2}\right)\sin^2\theta_B = 1 - \frac{n_2^2}{n_1^2}$$

$$\frac{n_1^4 - n_2^4}{n_1^2 n_2^2}\sin^2\theta_B = \frac{n_1^2 - n_2^2}{n_1^2}$$

# Brewster Angle (cont.)

Using the identity $n_1^4 - n_2^4 = (n_1^2 + n_2^2)(n_1^2 - n_2^2)$ and expressing $\sin^2\theta_B$ by $\tan^2\theta_B$ results in

$$\sin^2\theta_B = \frac{n_2^2}{n_1^2 + n_2^2}$$

$$\frac{\tan^2\theta_B}{1 + \tan^2\theta_B} = \frac{n_2^2/n_1^2}{1 + n_2^2/n_1^2}$$

A comparison of both sides reveals the final expression for the **Brewster angle**:

$$\tan\theta_B = \frac{n_2}{n_1}$$

Note that a Brewster angle exists only for p-polarisation.

# Power Reflection Diagram

**Summary:** Power reflection factor and phase shift for the reflection of an electro-magnetic wave at a material with lower or higher refractive index.