HW-04 Part TWO

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4.5.21: Proposition

Let n be an odd integer. It follows $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$.

Proof

1. **Representation of Odd Integer n:**

   Given n is an odd integer, we can express it as $n = 2m + 1$ for some integer m.

2. **Evaluation of $\lceil \frac{n}{2} \rceil$:**

   Consider $\lceil \frac{n}{2} \rceil$.

$$\lceil \frac{2m+1}{2} \rceil = \lceil \frac{2m}{2} + \frac{1}{2} \rceil = \lceil m + \frac{1}{2} \rceil = m + 1$$

   Here, we utilize the definition and properties of the ceiling function.

3. **Expression for $n + 1$:**

   Since $n = 2m + 1$, we have:

$$n + 1 = 2m + 2 = 2(m + 1)$$

   This implies $m + 1 = \frac{n+1}{2}$.

4. **Final Conclusion:**

   Combining these results, we find $\lceil \frac{n}{2} \rceil = m + 1 = \frac{n+1}{2}$.

Thus, the proposition is proven.

4.6.22: Statement

For any real number r, if r^2 is irrational, then r is also irrational.

Proof by Contradiction

In the contradiction, there exists a real number r such that r^2 is irrational but r is rational. We assume our premise is true: n is true and m is false.

Let n and m be integers.

Using the definition of rational numbers, $r = \frac{n}{m}$ where $m \neq 0$.

Squaring both sides, we get:

$$(r)^2 = \left(\frac{n}{m}\right)^2 \quad \Rightarrow \quad r^2 = \frac{n^2}{m^2}$$

As products of integers, n^2 and m^2 are integers. Moreover, m^2 is non-zero.

Thus, by definition, $\frac{n^2}{m^2}$ is rational, implying r^2 is rational.

This contradicts the initial statement r^2 is irrational. Hence, we conclude that if r^2 is irrational, then r must also be irrational. Therefore, the statement is a contradiction.

The contradiction demonstrates our original statement is true. Thus, the proof is complete.

Proof by Contraposition

Let r be a real number such that r is not irrational, i.e., r is rational. Let n and m be integers. The original statement is of the form "If P then Q", where P is "r^2 is irrational" and Q is "r is irrational". The contrapositive of this statement is "If not Q then not P", which translates to "If r is not irrational (i.e., r is rational), then r^2 is not irrational (i.e., r^2 is rational)".

Using the definition of rational numbers, $r = \frac{n}{m}$ where $m \neq 0$. Squaring both sides, we get:

$$(r)^2 = \left(\frac{n}{m}\right)^2 \quad \Rightarrow \quad r^2 = \frac{n^2}{m^2}$$

Since n^2 and m^2 are both products of integers, they are integers. Moreover, m^2 is non-zero. Thus, by definition, $\frac{n^2}{m^2}$ is rational, implying r^2 is rational, not irrational, which proves the contrapositive.

Therefore, the original statement "For every real number r, if r^2 is irrational then r is irrational" must also be true. The proof by contraposition is thus complete.