HW-04 Part TWO

Abraham J. Reines

October 14, 2023

1 4.3.39

1.1 Part a: Standard Factored Form for a^3

Given a number a in its standard factored form:

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$$

The standard factored form for a^3 can be obtained by raising each factor in a to the third power.

$$a^3 = (p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k})^3 = p_1^{3e_1} \cdot p_2^{3e_2} \cdot \ldots \cdot p_k^{3e_k}$$

1.2 Part b: Least Positive Integer k for a Perfect Cube

A number qualifies as a *perfect cube* if and only if it can be expressed in the form a^3, where $a \in \mathbb{Z}$. Put simply, a number is a perfect cube if its cube root belongs to the set of integers.

Let's scrutinize the expression $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$:

1. The prime factor 2 manifests with an exponent of 4. To conform this factor to our perfect cube condition, the exponent should be raised to the nearest multiple of 3 greater than 4, which is 6. Consequently, k must incorporate an additional $2^2 = 4$.

2. The prime factor 3 is raised to the exponent 5. Following the same logic, we identify the nearest multiple of 3 greater than 5 to be 6. This necessitates the inclusion of $3^1 = 3$ in k.

3. The prime factor 7 appears with an exponent of 1. To meet the criteria for a perfect cube, k must be supplemented with $7^2 = 49$.

4. The prime factor 11 is presented with an exponent of 2. The closest multiple of 3 greater than this exponent is 3, thus requiring an additional $11^1 = 11$ in k.

To compute k, we multiply these additional factors:

$$k = 2^2 \cdot 3^1 \cdot 7^2 \cdot 11^1 = 4 \times 3 \times 49 \times 11 = 6468$$

Therefore, multiplication of the original expression by 6468 yields a perfect cube. This elegant outcome emanates from the intrinsic properties of exponents in concert with the unique prime factorization of integers.

2 4.4.30

2.1 Part a: The Quotient-Remainder Theorem

For any given integer n and any positive integer d, there exist unique integers q and r such that:

$$n = dq + r \quad \text{and} \quad 0 \leq r < d$$

2.2 Special Case: $d = 4$

Given $d = 4$ and any integer n, the Quotient-Remainder Theorem allows us to express n in one of the following four forms:

$$n = 4q \quad \text{or} \quad n = 4q + 1 \quad \text{or} \quad n = 4q + 2 \quad \text{or} \quad n = 4q + 3$$

for some integer q.

2.3 Scenario 1: $n = 4q$

In this scenario, the product of two consecutive integers can be expressed as:

$$n(n+1) = (4q)(4q+1) = 16q^2 + 4q = 4k$$

where $k = 4q^2 + q$.

Given that q is an integer, it is evident that $k = 4q^2 + q$ will also be an integer.

2.4 Scenario 2: $n = 4q + 1$

In this case, the product of two consecutive integers can be written as:

$$n(n+1) = (4q+1)(4q+2) = 16q^2 + 12q + 2 = 4k + 2$$

where $k = 4q^2 + 3q$.

Again, given that q is an integer, $k = 4q^2 + 3q$ must also be an integer.

2.5 Scenario 3: $n = 4q + 2$

For this case, the product of two consecutive integers can be represented as:

$$n(n+1) = (4q+2)(4q+3) = 16q^2 + 20q + 6 = 4k + 2$$

where $k = 4q^2 + 5q + 1$.

Since k is a sum of integers, it is necessarily an integer.

2.6 Scenario 4: $n = 4q + 3$

In this case, the product of two consecutive integers can be written as:

$$n(n+1) = (4q+3)(4q+4) = 16q^2 + 28q + 12 = 4k$$

where $k = 4q^2 + 7q + 3$.

Since k is a sum of integers, it is necessarily an integer.

2.7 Conclusion

In summary, the product of any two consecutive integers will have one of the forms $4k$ or $4k + 2$, thereby demonstrating the power and utility of the Quotient-Remainder Theorem when $d = 4$.

3 Part b: Using Mod Notation

For any integer n and positive integer d, n can be expressed as:

$$n \equiv r \mod d$$

where $0 \leq r < d$.

3.1 Special Case for $d = 4$

When $d = 4$, any integer n can be represented as one of the following forms:

$$n \equiv 0 \mod 4 \quad \text{or} \quad n \equiv 1 \mod 4 \quad \text{or} \quad n \equiv 2 \mod 4 \quad \text{or} \quad n \equiv 3 \mod 4$$

3.2 Scenario 1: $n \equiv 0 \mod 4$

The product of two consecutive integers can be expressed as:

$$n(n+1) \equiv 0 \mod 4$$

3.3 Scenario 2: $n \equiv 1 \mod 4$

The product of two consecutive integers can be expressed as:

$$n(n+1) \equiv 2 \mod 4$$

3.4 Scenario 3: $n \equiv 2 \mod 4$

The product of two consecutive integers can be expressed as:

$$n(n+1) \equiv 2 \mod 4$$

3.5 Scenario 4: $n \equiv 3 \mod 4$

The product of two consecutive integers can be expressed as:

$$n(n+1) \equiv 0 \mod 4$$

3.6 Conclusion

In conclusion, the product of any two consecutive integers will either be congruent to 0 or 2 modulo 4, which aligns well with the Quotient-Remainder Theorem in a modular context.