HW-05 Part TWO

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1 Preliminaries 5.6.12

Definition 1. *A sequence* $s_0, s_1, s_2, \ldots \{s_n\}$ *is defined as follows:*

$$s_n = \frac{(-1)^n}{n!}, \quad \textit{for all integers } n \geq 0 \qquad \textit{\textbf{...(1)}}$$

2 Main Result

Theorem 1. *The sequence* $\{s_n\}$ *satisfies the recurrence relation:*

$$s_k = \frac{-s_{k-1}}{k}, \quad \textit{for all integers } k \geq 1$$

3 Proof of Theorem

Let k be an arbitrary integer such that $k \geq 1$.

1. **Initialization**: We begin by substituting $n = k - 1$ into Equation (1) to acquire a base expression for s_{k-1}:

$$s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!} \qquad \textbf{...(2)}$$

2. **Recursion Step**: Next, we substitute $n = k$ into Equation (1) to acquire an expression for s_k:

$$s_k = \frac{(-1)^k}{k!}$$

3. **Manipulation**: Utilizing the property $n! = n \cdot (n-1)!$ and the laws of exponents $a^m \cdot a^n = a^{m+n}$, we rewrite s_k:

$$s_k = \frac{(-1)^k}{k \cdot (k-1)!} = \frac{-1}{k} \cdot \frac{(-1)^{k-1}}{(k-1)!}$$

4. **Substitution**: Using Equation (2), we rewrite s_k in terms of s_{k-1}:

$$s_k = \frac{-1}{k} \cdot s_{k-1}$$

Hence, $s_k = \frac{-s_{k-1}}{k}$, for all integers $k \geq 1$. This confirms the sequence $s_0, s_1, s_2, \dots \{s_n\}$ satisfies the stated recurrence relation for $k \geq 1$, thereby completing the proof.

4 Remarks

Remark 1. *This proof elucidates the mathematical structure underlying the sequence $s_0, s_1, s_2, \dots \{s_n\}$, providing insights into its behavior as n varies. Such analyses are crucial in the broader context of discrete mathematics and its applications.*

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Theorem 2. *For all integers $k \geq 1$, the following identity holds:*

$$F_{k+1}^2 - F_k^2 = F_{k-1} \cdot F_{k+2}$$

Proof. Consider the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$.

We will prove the given identity by expanding the left-hand side using the algebraic identity $x^2 - y^2 = (x-y)(x+y)$:

$$F_{k+1}^2 - F_k^2 = (F_{k+1} + F_k)(F_{k+1} - F_k) \tag{1}$$

From the recurrence relation of the Fibonacci sequence, we have:

$$F_{k+1} = F_k + F_{k-1} \tag{2}$$

and

$$F_{k+2} = F_{k+1} + F_k \tag{3}$$

Substituting the value from equation (2) into the expression $F_{k+1} - F_k$, we obtain:

$$F_{k+1} - F_k = F_k + F_{k-1} - F_k = F_{k-1} \tag{4}$$

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Equation (4) is derived from the fundamental recurrence relation which defines the Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}$$

This relation tells us each term in the sequence is the sum of the two preceding terms, starting with $F_0 = 0$ and $F_1 = 1$.

For equation (4), we focus on the terms F_{k+1} and F_k. The recurrence relation for F_{k+1} is:

$$F_{k+1} = F_k + F_{k-1}$$

Now, to express $F_{k+1} - F_k$, we simply subtract F_k from both sides of this equation:

$$\begin{aligned} F_{k+1} - F_k &= (F_k + F_{k-1}) - F_k \\ &= F_k - F_k + F_{k-1} \\ &= 0 + F_{k-1} \\ &= F_{k-1} \end{aligned}$$

The result of this manipulation is the difference between the consecutive Fibonacci numbers F_{k+1} and F_k is the prior Fibonacci number F_{k-1}. This is what equation (4) represents:

$$F_{k+1} - F_k = F_{k-1}$$

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Substituting the expressions from equations (3) and (4) into equation (1), we get:

$$(F_{k+1} + F_k)(F_{k+1} - F_k) = F_{k-1} \cdot F_{k+2}$$

This simplifies to the identity we set out to prove:

$$F_{k+1}^2 - F_k^2 = F_{k-1} \cdot F_{k+2}$$

Therefore, by direct application of the properties of the Fibonacci sequence and algebraic manipulation, the theorem is proven without the need for mathematical induction. $\square$