HW-05 Part ONE

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Statement 5.1.39

We are given the summation

$$\sum_{m=1}^{n+1} m(m+1)$$

and tasked with rewriting this expression by separating off the final term. The focus is to employ the recursive definition of summation to accomplish this.

1 Recursive Definition of Summation

The recursive definition of summation can be mathematically expressed as:

1. $$\sum_{k=m}^{m} a_k = a_m,$$
2. $$\sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n$$

This holds for all integers $n > m$.

2 Separating the Final Term

Using the recursive definition of summation, we can rewrite the original summation as:

$$\sum_{m=1}^{n+1} m(m+1) = \left(\sum_{m=1}^{n} m(m+1)\right) + [(n+1)((n+1)+1)]$$

Upon simplification, this becomes:

$$\sum_{m=1}^{n+1} m(m+1) = \left(\sum_{m=1}^{n} m(m+1)\right) + [(n+1)(n+1+1)]$$

Further simplifying, we get:

$$\sum_{m=1}^{n+1} m(m+1) = \left(\sum_{m=1}^{n} m(m+1)\right) + [(n+1)(n+2)]$$

3 Conclusion

The original summation $\sum_{m=1}^{n+1} m(m+1)$ can be rewritten by separating off the final term $(n+1)(n+2)$, while the remaining part of the sum goes from $m = 1$ to n. This separated form adheres to the recursive definition of summation.

Problem Statement 5.2.16

Prove the following statement by mathematical induction:

For all integers $n \geq 2$,

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

4 Base Case: $n = 2$

First, let's verify the base case, where $n = 2$:

$$\text{Left-hand side} = (1 - \frac{1}{2^2}) = \frac{3}{4}$$

$$\text{Right-hand side} = \frac{2+1}{2 \times 2} = \frac{3}{4}$$

Both sides are equal, so the base case holds.

5 Inductive Step

Assume k is an integer ≥ 0 such that $P(k)$ is true for some arbitrary $n = k$:

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{k^2}) = \frac{k+1}{2k}$$

We need to show $P(k+1)$ is true for $n = k+1$:

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{(k+1)^2}) = \frac{(k+1)+1}{2(k+1)}$$

5.1 Proving the Inductive Step

Starting with the left-hand side of the expression for $n = k+1$:

$$\left((1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{k^2})\right)(1 - \frac{1}{(k+1)^2})$$

Substitute the inductive hypothesis $\frac{k+1}{2k}$:

$$= \frac{k+1}{2k} \times (1 - \frac{1}{(k+1)^2})$$

$$= \frac{k+1}{2k} \times \frac{(k+1)^2 - 1}{(k+1)^2}$$

$$= \frac{k+1}{2k} \times \frac{k^2 + 2k}{(k+1)^2}$$

$$= \frac{(k+1) \times (k^2+2k)}{2k \times (k+1)^2}$$

$$= \frac{(k+1) \times k \times (k+2)}{2k \times (k+1)^2}$$

$$= \frac{(k+2)}{2(k+1)}$$

This proves the left-hand side equals the right-hand side for $n = k+1$:

$$L.H.S = \frac{(k+1)+1}{2(k+1)} = \frac{k+2}{2(k+1)} = R.H.S$$

6 Conclusion

By mathematical induction, the statement

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

is true for all integers $n \geq 2$.