On Measurement: The Relativity of Information Frames

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Abstract

1 Introduction

2 Motivation - The Relativity of Information Frames

Consider two physical observers, Alice and Bob, each equipped with a clock and a ruler. To infer a particle's momentum, they make two position measurements and record the elapsed time.

However,

- If they agree on the spatial separation, they must disagree on the elapsed time;
- If they agree on the elapsed time, the measured spatial separation must differ.

Their interactions with the world differ — and so does what each can resolve as an event.

What Alice calls "particle at position x at time t" is determined by her interaction channels and detection thresholds.

Thus there is no global, frame-independent σ-algebra of events. Every physical system carries its own information frame: a σ-algebra of distinguishable outcomes accessible through its interactions.

Einstein taught that coordinate descriptions are relative while causal order is invariant. We extend this principle.

Relativity of Information Frames (RIF)
Nature does not privilege one information frame over another. What is physical is what all information frames can agree upon.

Measurement is not the revelation of a pre-existing global state; it is the joint refinement (and, when necessary, coarse-graining) of information frames when systems interact. From this symmetry, quantum state update, pointer bases, and even causal geometry follow as consequences.

3 Background

3.1 Measure Theory

A full account of measure theory and probability theory is outside the scope of this paper; for a standard reference, see [1]. In this section we introduce only the notion of a probability space and the measure-theoretic concepts that will be explicitly used later. The exposition is intentionally brief and self-contained, though some familiarity with the subject is helpful.

Probability Spaces

In measure-theoretic probability, a probability space is given by a triple:

$$(\Omega, \mathcal{F}, \mu)$$

where each component encodes a distinct element of a probality model.

The Sample Space Ω

The sample space Ω is the set of all possible outcomes of an experiment. It's elements $\omega \in \Omega$ represent individual realizations or trials. In general, Ω is endowed with additional structure beyond serving as the underlying space from which outcomes are drawn.

Depending on the context, elements of Ω may correspond to coin toss outcomes, experiment runs, or realizations of an abstract physical system.

The σ-algebra $\mathcal{F}$

The σ-algebra $\mathcal{F}$ specifies which subsets of Ω are considered **events**. Events are thus sets of outcomes $\omega \in \Omega$ to which probabilities may be assigned.

Definition 3.1 (σ-algebra)**.** A σ-algebra $\mathcal{F}$ is a collection of subsets of Ω such that:

1. $,\Omega \in \mathcal{F}$
2. Closed under complemetns. $F \in \mathcal{F} \to F^c \in \mathcal{F}$
3. Closed under countable unions.

$$\{F_i\}_{i \in \mathbb{N}} \in \mathcal{F} \to \bigcup_{i \in \mathbb{N}} F_i \in \mathcal{F}$$

The σ-algebra encodes the events that are meaningfully distinguishable within the model and therefore forms the central structural component of a probability space.

The Probability Measure μ

A probability measure assigns probabilities to events in $\mathcal{F}$.

Definition 3.2 (Probability Measures)**.** A probability measure is a function $\mu : \mathcal{F} \to [0, 1]$ satisfying:

1. Total probabilities:
$$\mu(\emptyset) = 0 \quad \mu(\Omega) = 1$$

2. For any countable collection of pairwise disjoint sets $\{F_i\}_{i\in\mathbb{N}}$, with $F_i \cap F_j = \emptyset$ for all $i \neq j$.
$$\mu\left(\bigcup_{i\in\mathbb{N}} F_i\right) = \sum_{i\in\mathbb{N}} \mu(F_i)$$

In this work, probability measures will be intrepreted as states on a measurable space. Encoding how probabilities are distributed over the events in $\mathcal{F}$.

Measurable Functions

A **measurable function**, often refered to in probability theory as a **random variable**, is a function between measurable spaces
$$f : \Omega_1 \to \Omega_2$$
such that:
$$\forall A \in \mathcal{F}_2 \qquad f^{-1}(A) \in \mathcal{F}_1$$
That is, the preimage of every event in $\mathcal{F}_2$ is a event in $\mathcal{F}_1$.

Pushfoward Measure

Given a measurable function
$$f : \Omega_1 \to \Omega_2$$
And a probability measure μ in $(\Omega_1, \mathcal{F}_1)$, we can define a probability measure on $(\Omega_2, \mathcal{F}_2)$ by
$$(f_*\mu)(A) := \mu\big(f^{-1}(A)\big), \qquad A \in \mathcal{F}_2$$
The measure $f_*\mu$ is called the **pushfoward** of μ along f.

The pushfoward measure represents the probability distribution induced on Ω_2 by the map f when the underlying space is described by the measure μ.

Embeddings

We will be particulary interested in certain classes of measurable maps that preserve the structure of measurable spaces.

Definition 3.3 (Measurable Isomorphism)**.** A measurable function $T : \Omega_1 \to \Omega_2$ is called a **measurable isomorphism** if it is **bijective** and it's inverse $T^{-1} : \Omega_2 \to \Omega_1$ is also measurable. In this case, the measurable spaces are **isomorphic**, denoted:
$$(\Omega_1, \mathcal{F}_1) \cong \big(\Omega_2, \mathcal{F}_2\big)$$
Such maps preserve the full measurable structure.

Definition 3.4 (Measurable Space Automorphisms)**.** For a measurable space $(\Omega, \mathcal{F})$, the group of measurable automorphisms is

$$\mathrm{Aut}\,(\Omega, \mathcal{F}) := \left\{T : \Omega \to \Omega \;\mid\; T \text{ is bijective, } T \text{ and } T^{-1} \text{ are measurable.}\right\}$$

While measurable isomorphisms preserve the entire structure of a space, our work requires maps that allow a measurable space to be faithfully represented within a larger one.

Definition 3.5 (Measurable Embedding)**.** A **measurable embedding** is a injective measurable function

$$\iota : \Omega_1 \to \Omega_2$$

such that the inverse map

$$\iota^{-1} : \iota(\Omega_1) \to \Omega_1$$

is measurable, where $\iota(\Omega_1)$ is equipped with the sub-σ-algebra induced from $\mathcal{F}_2$.

In this case, ι identifies $(\Omega_1, \mathcal{F}_1)$ with a measurable subspace of $(\Omega_2, \mathcal{F}_2)$, preserving the σ-algebra of the source space.

Definition 3.6 (Embedding-induced σ-algebra)**.** Let $\iota : \Omega_1 \to \Omega_2$ be a measurable embedding. The *embedding-induced σ-algebra* of ι is the sub-σ-algebra on $\iota(\Omega_1) \subset \Omega_2$ induced from $\mathcal{F}_2$, defined by

$$\mathcal{F}^{\iota} \;:=\; \mathcal{F}_2 \!\restriction_{\iota(\Omega_1)} =\; \{\, A \cap \iota(\Omega_1) \mid A \in \mathcal{F}_2 \,\}.$$

The same sub-σ-algebra used in the definition above.

3.2 Contextuality

A central concept underlying the theory developed in this work is that of **contextuality**. The definitions presented here are adapted from the sheaf-theoretic formulation of contextuality introduced in [**?**], to a measure-theoretic framework.

Labels and Contexts

We begin by introducing the notion of a measurement label. Intuitively, a measurement label represents a physical distinction that can be probed in a system, that is, a question that may be asked of the system. Measurement labels encode the primitive degrees of freedom of the model.

Measurement Labels

Definition 3.7 (Measurement labels)**.** A **measurement label** is an abstract symbol m associated with an measurable space $(\Omega_m, \mathcal{F}_m)$, representing the possible outcomes of measuring m. We write

$$m \longmapsto (\Omega_m, \mathcal{F}_m)$$

The collection of all measurement labels cosidered by a model considers is denoted $\mathcal{M}$.

Global Space

Definition 3.8 (Global measurable space)**.** The **global space**, representing the joint space of all measurement labels, is the product measurable space

$$(\Omega_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}}) := \left(\prod_{m \in \mathcal{M}} \Omega_m \, , \bigotimes_{m \in \mathcal{M}} \mathcal{F}_m \right)$$

Here $\bigotimes_{m \in \mathcal{M}} \mathcal{F}_m$ denotes the product σ-algebra generated by cylinder sets. For two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ the product σ-algebra is given by

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma\left(\{F_1 \times F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}\right)$$

We emphasize that no probability measure is specified on the global space at this stage. Our interest here lies in the measurable structure itself, independently of any particular choice of global state.

Contexts

We now introduce the notion of a context. A context represents a collection of measurement labels that are jointly accessible to an observer, or equivalently, a set of degrees of freedom that can be meaningfully considered together. Intuitively, a context specifies the questions an observer may simultaneously ask of the system.

Definition 3.9 (Context)**.** A **context** is a finite subset $C \subseteq \mathcal{M}$ of measurement labels. To each context we associate a measurable space

$$(\Omega_C, \mathcal{F}_C) := \left(\prod_{m \in C} \Omega_m \, , \bigotimes_{m \in C} \mathcal{F}_m \right)$$

Canonical Context Projections

Within a context, we define canonical projection maps onto the outcome spaces of individual measurament labels.

Definition 3.10 (Canonical context projections)**.** For a context C and a label $m \in C$, the canonical projection is the measurable function

$$\pi_{C \to \{m\}} : \Omega_C \to \Omega_m$$

defined by

$$\pi_{C \to \{m\}}(\omega) = \omega_m \qquad \forall \omega \in \Omega_C$$

More generally, for any subcontext $D \subseteq C$, we define the projection

$$\pi_{C \to D} : \Omega_C \to \Omega_D$$

by restriction to the coordinates indexed by D.

Intuitively, the projection $\pi_{C \to \{m\}}$ the **perspective** C has on the measurament label m. The inverse images of projections define canonical measurable subsets of a context space.

Definition 3.11 (Cylinder Sets)**.** Let C be a context and $m \in C$. For any $A \in \mathcal{F}_m$, the corresponding **cylinder set** in Ω_C is defined by

$$\pi^{-1}_{C \to \{m\}}(A) := \{\omega \in \Omega_C \ \mid \ \omega_m \in A\}$$

Since π is a measurable function, all cylinder sets belong to $\mathcal{F}_C$.

Contexts do not introduce independent information: all events in a context arise as pullbacks of events associated with its measurement labels via the canonical projections. In this sense, a context contains no information that does not originate from its constituent labels.

Empirical Model

We now introduce the first concept that explicitly involves probability measures: the notion of an **empirical model**.

Intuitively, an empirical model represents a particular realization of the system as accessed through different contexts. Equivalently, it may be viewed as a family of probability distributions describing the observable statistics associated with each context, subject to consistency on overlaps.

Definition 3.12 (Empirical model)**.** An **empirical model** is a family

$$\{\mu_C\}_{C \in \mathcal{C}}$$

of probability measures, where for each context C,

$$\mu_C \text{ is a probability measure on } (\Omega_C, \mathcal{F}_C)$$

These measures are required to satisfy the following *compatibility condition*: For all $C, C' \in \mathcal{C}$ and all subcontexts $D \subseteq C \cap C'$,

$$(\pi_{C \to D})_* \, \mu_C \ = \ (\pi_{C' \to D})_* \, \mu_{C'}$$

This condition expresses the requirement that the probability distributions assigned to different contexts agree on their common measurement labels, and hence represent consistent marginals of a single underlying empirical situation.

Contextuality

Empirical models allow us to define a central notion driving the framework developed in this work, that of **contextuality**. Informally, contextuality captures the failure of different observational perspectives to arise as consistent restrictions of a single global description.

Definition 3.13 (Contextuality)**.** Let $\{\mu_C\}_{C \in \mathcal{C}}$ be an empirical model. The empirical model is said to be **contextual** if there exists no probability measure μ on the global measurable space $(\Omega_\mathcal{M}, \mathcal{F}_\mathcal{M})$ such that

$$(\pi_{\mathcal{M} \to C})_* \, \mu \ = \ \mu_C \qquad \forall C \subset \mathcal{C}$$

If such a probability measure exists, the empirical model is called **non-contextual**.

Intuitively, non-contextuality means that the probabilistic data obtained from all contexts can be understood as arising from a single joint probability distribution on the global space, with each context revealing only a partial perspective of that global state.

Contextuality, by contrast, indicates that no such global probability measure exists: although each context admits a well-defined probabilistic description, these descriptions cannot be combined into a single coherent global model.

The existence of contextual empirical models is not merely a formal possibility. It is well established that there are experimental scenarios for which no non-contextual global description exists; a detailed analysis can be found in [**?**]. The notion introduced here corresponds to probabilistic contextuality in the sense of [**?**], expressed in measure-theoretic language.

Standing Assumptions

All measurable spaces considered in this work are assumed to be standard Borel spaces. Probability measures are taken to be Borel probability measures satisfying the usual regularity conditions required for the constructions used below.

4 Structural Contextuality

To get a better understanding on RIF we need to look at other formulations of contextuality. We want to understand contextuality without particular probability distributions.

For this, we adapt the sheaf like condition, than just the failure of probabilities to have the correct marginals, this in particular matches the **strong contextuality** defined in [**?**]. First we define a **Information Frame**.

Definition 4.1 (Information Frame)**.** A information frame is a probability space over a context $C \subseteq \mathcal{M}$ with a particular event algebra $\mathcal{F} \subseteq \bigotimes_{m \in C} \mathcal{F}_m$ corresponding to the set of distinctions that frame can see about the labels m.

$$\mathscr{I}_{C,\mathcal{F}} := \left(\prod_{m \in C} \Omega_m, \mathcal{F} \right)$$

A information frame could be seen as a perspective on the world. A view of what it can in principle see about a system. A probability measure there represents a particular state of the world. For notation convinience we will use $\mathscr{I}_1 = \mathscr{I}_{C1,\mathcal{F}_1}$

In general, Information Frames are defined on the support of some probability measure, meaning we get rid of events that a context does not see as possible.

An important piece we have in such spaces are shared events. We will often refer to shared events E as follows:

Definition 4.2 (Shared Event)**.** We call an event E a shared event of contexts C and C' when $C \cap C' = L \neq \emptyset$, the shared labels, and E is a event only on those labels.

Then we can define, using the **Cylinder embeddings**:

$$E_1 = \pi^{-1}_{C \to L}(E) \quad E_2 = \pi^{-1}_{C' \to L}(E)$$

We will often write $E \in \mathcal{F}_C \cap F_{C'}$, and even treat E as an event in those algebras. But it always corresponds to the events of the **Cylinder embedding**.

We can now do a definition of contextuality based only on the structure of the σ-algebras.

Definition 4.3 (Structural Contextuality)**.** Given a family of contexts $\mathcal{C}$ of $\mathcal{M}$ and corresponding information frames $\mathscr{I}_{C,\mathcal{F}_C}$ and a global space $(\Omega, \mathcal{F})$.

The family is said to be **structuraly non-contextual** if there is a family of corresponding **embeddings** and some sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ such that:

$$\iota_{C \in \mathcal{C}} : \mathscr{I}_{C,\mathcal{F}_C} \to (\Omega, \mathcal{G})$$

Such that for every pair C, C' and every event $E \in \mathcal{F}_C \cap \mathcal{F}_{C'}$, that is events on shared labels, we have:

$$\iota_C(E) = \iota_{C'}(E)$$

Formally, for $L = C \cap C'$ that requires:

$$\iota_C(\pi^{-1}_{C \to L}(E)) = \iota_{C'}(\pi^{-1}_{C \to L}(E))$$

If no such embedding exists then the family is called **structuraly contextual**.

The existence of **strong contextuality** as seem in [**?**] guarantees structural contextuality also exists, in particular the results in [**?**] show that, if we try to keep the full σ-algebra event structure, structural contextuality will eventually happen. This shows contextuality is essentially inevitable in sufficiently complex logics.

In general, in the product space, this can be seen as the failure of cylinder sets to hold all the complexity of all frames event structure. We can think of this in the simple case of two frames. Consider:

$$\iota_1(x) = (x, f(x)) \quad \iota_2(y) = (g(y), y)$$

Here we simply represent the embedding as the identity on the labels it controls, that forces part of the embedding structure, the other part, corresponding to labels the original space does not see, is free to be any measurable function.

The issue is when the events have non-trivial intersections. It is not always possible to match the functions and identity correctly that is precisely **structural contextuality**.

5 The Relativity Of Information Frames

5.1 Interaction

Now we work on making precise the meaning of the **Relativity of Information Frames (RIF)**. To do so, the central objects of study will be **Information Frames** as in defintion 4 .1. Information frames are the only ontic objects in this theory, everything works on their interactions.

We now turn our attetion to making the definition of interaction precise. The first step is to define the **Joint Frame**. A information frame where the interaction takes place. Before this we will note that when we write:

$$\iota(\mathcal{F}_C) := \mathcal{G}$$

That is, there is some sub-σ-algebra $\mathcal{G}$ on the codomain where that embedding exits. We also note that there can be many such embeddings. These differences are not of particular interest to us, they are related by the automorphism symmetry defined in Definition 3.7, so we can treat them as equivalent.

Definition 5.1 (Joint Frame)**.** Given a family of information frames $\{\mathscr{I}_i\}_{i\in I}$ we can construct their joint frame as:

$$\mathcal{J}_I := (\Omega_{\mathcal{J}_I},\, \mathcal{F}_{\mathcal{J}_I}) := \left(\prod_{m \in \{\bigcup_{i\in I} C_i \subseteq \mathcal{M}\}} \Omega_m,\ \bigotimes_{i\in I} \iota_i\left(\mathcal{F}_i\right) \right)$$

The space that hold all distinctions of all information frames.

While such a frame is always definable, contextuality tells us it is not always possible to assign it a probability measure that is consistent with the frames that generated it.

Structural contextuality tells us when it is impossible to make a non-contextual joint frame. Furthermore, indentification of events becomes difficult, there can be different events with the same "shadow". In particular, shared events are events we want to identify as the same, but in contextual cases in the joint space, we can't.

One important thing to note however, is that once we generate such a joint space, the embeddings are fixed, we are not allowed to change embeddings midway. While embeddings can be different, the only the part that preserves the σ-algebra structure of it's frame is of relevance to us, so all embeddings can be treated as the same.

5.2 The Structure of the contexts

To make our realitivity principle precise we need a way to compare the structure of contexts in the joint frame.The problem is once we fix a joint frame $\mathcal{J}_I$ through some family of embeddings $\iota_{i\in I}$ we still can't properly compare the contexts as they do not live in the global space.

A way to do this is to consider the following map

Definition 5.2 (Local Projections)**.** Given a family of embeddings ι_i that generated a particular frame $\mathcal{J}_I$. The local projections e_i are

$$e_i := e_{C_i} := \left(\iota_i \circ \pi_{\mathcal{J}_I \to C_i}\right) : \Omega_{\mathcal{J}_I} \to \Omega_{\mathcal{J}_I}$$

These e_i represent the full event structure of the frame $\mathscr{I}_i$ in the joint frame. They can also be seen as **idempotent coarse-graining maps**, looking from the perspective of a given context at an event.

5.3 The Symmetry Of Information

Privilege

We are finally ready to introduce the concept of **The Relativity of Information Frames**. We begin with the definition of **Privilege** in a **Joint Frame**.

Definition 5.3 (Privilege)**.** We say a joint frame $\mathcal{J}_I$ privileges $\mathscr{I}_1$ over $\mathscr{I}_2$ for some event $E \in \mathcal{F}_{\mathcal{J}_I}$ if, for its local projections e_1, e_2 we have:

$$e_2(E) \neq e_2(e_1(E))$$

Basically we say that the joint frame privileges $\mathscr{I}_1$ over $\mathscr{I}_2$ if the local projection of $\mathscr{I}_1$ changes what $\mathscr{I}_2$ sees. That is, coarse-graining through $\mathscr{I}_1$ perspectives alters $\mathscr{I}_2$ perspective.

In particular, its worth noting that the local projections do not commute when privilege happens:

$$e_1 \circ e_2(E) \neq e_2 \circ e_1(E)$$

We also note that, if $\mathscr{I}_1$ is privileged over $\mathscr{I}_2$ on event E, it is still possible for $\mathscr{I}_2$ to be privileged over $\mathscr{I}_1$ on E. This is a bidirectional definition despite the name somewhat implying a ordering structure.

The Relativity of Information Frames

We can now state the relativity of information frames precisely.

Axiom 1 (The Relativity Of Information Frames). *After interaction, a event is physically admissible if and only if it does not privilege one information frame over another. Equivalently, in the physically admissible joint frame, no information frame is privileged over another for any event in its σ-algebra.*

More explicitly for any pair of frames i, j of the joint frame and any shared event E we have:

$$e_i(E) = e_i(e_j(E)) \quad \text{and} \quad e_j(E) = e_j(e_i(E))$$

That is, the set of physical events is:

$$\mathcal{F}^{\text{phys}} := \{E \in \mathcal{F}_{\mathcal{J}_I} \mid e_i \circ e_j(E) = e_i(E) \quad \forall \mathscr{I}_i, \mathscr{I}_j\}$$

Proposition 5.4. *The sets in $\mathcal{F}^{phys}$ form a σ-algebra.*

Proof. If $E \in \mathcal{F}^{\text{phys}}$ then, for every i, j:

$$e_i(e_j(E^c)) = e_i(e_j(E)^c) = e_i(e_j(E))^c = e_i(E)^c = e_i(E^c)$$

So $E^c \in \mathcal{F}^{\text{phys}}$.

And let E_n be a countable collection of RIF-valid events, that is:

$$e_i(e_j(E_n)) = e_i(E_n) \quad \forall n, i, j$$

Since:

$$e_i\left(\bigcup_n E_n\right) = \bigcup_n e_i(E_n)$$

Given any $i, j \in I$ we have:

$$e_i\left(e_j\left(\bigcup_n E_n\right)\right) = e_i\left(\bigcup_n e_j(E_n)\right) = \bigcup_n e_i \circ e_j(E_n) = \bigcup_n e_i(E_n) = e_i\left(\bigcup_n E_n\right)$$

□

To match quantum mechanics we will call this σ-algebra the **pointer algebra**.

Definition 5.5 (Pointer Algebra)**.** The algebra of physically admissible events is called the pointer algebra.

$$\mathcal{F}_{\text{ptr}} = \mathcal{F}^{\text{phys}}$$

5.4 Relation to Contextuality

Now we prove a few important properties of the pointer algebra. We already saw how closely related to contextuality the pointer algebra is, that is RIF fails exactly when contextuality is present.

First we show that, when a joint frame with the pointer algebra is defined, we have a particular type of non-contextuality. It is important to note here, that our structural contextuality definition no longer applies. This is because after throwing away events we no longer have embeddings, the maps there might be many to one. So for this we equip the information frames with **empirical probability families**, and show that we can construct a well behaved measure on the **joint pointer frame**.

Lemma 5.6 (The Joint Pointer Frame is non-contextual)**.** *Let $\{\mu_i\}_{i\in I}$ be a empirical family on the contexts of the information frames $\{\mathscr{I}_i\}_{i\in I}$, then there exists a probability measure μ in the **Pointer Joint Frame** such that:*

$$\mu(E) = \mu_i(\pi_{\mathcal{J}_I \to \mathcal{C}_i}(E))$$

For every event E and every choice of i.

Proof. For some $i \in I$ and for each $E \in \mathcal{F}_{\text{ptr}}$, define:

$$\mu(E) := \mu_i(\pi_i(E))$$

Let $E \in \mathcal{F}_{\text{ptr}}$ and let C_j be some other context. If $C_j \cap C_i = \emptyset$, no consistency condition is required. If $C_j \cap C_i \neq \emptyset$ RIF implies that either both are trivial or they have the exact same projection onto $C_i \cap C_j$. In the later case the consistency conditions imply:

$$\mu(E) = \mu_i(\pi_i(E)) = \mu_j(\pi_j(E))$$

So μ is a consistent probability measure on the joint pointer frame. □

We now see that, for contextual families, RIF is not satisfied.

Proposition 5.7 (Contextuality violates RIF)**.** *If a family of contexts given by I is **structurally contextual**. Then the naive joint frame $\mathcal{J}_I$ violates RIF for some event.*

Proof. For simplicity consider only two frames that disagree. We know there is a shared event E for which

$$\iota_1(\pi_1(E)) \neq \iota_2(\pi_2(E))$$

Now let $A = \iota_1(\pi_1(E))$ and $E = \iota_2(\pi_2(E))$. This makes it clear that:

$$\pi_2(E) \neq \pi_2(A)$$

Since otherwise, because the embeddings are injective we would have $A = E$. This already implies:

$$e_2 \circ e_1(E) \neq e_2(E)$$

So RIF is violated for the event E. □

Now we show that, using a larger σ-algebra then RIF, allows us to construct a contextual empirical family.

Lemma 5.8 (Any larger σ-algebra is contextual)**.** *Suppose $\mathcal{G}$ is a σ-algebra on a joint frame such that $\mathcal{F}_{ptr} \subsetneq \mathcal{G}$. Then there exists a empirical family $\{\mu_i\}_{i\in I}$ for which no consistent probability measure μ exists on the joint frame with $\mathcal{G}$.*

Proof. □

Both together allow us to conclude

Theorem 5.9 (The Pointer Algebra is the maximally informative non-contextual σ-algebra)**.** *The algebra $\mathcal{F}_{ptr}$ is maximally informative. That is, any other event from the interaction frame, allows contextuality.*

Proof. Immediate from both lemmas. □

6 Quantum Mechanics

Here we endevor to trace the parallels with standard quantum mechanics. Where each ingredient fits in the picture of information frames we have built and how are they related.

6.1 Observables, Measurements and Operators

Measurement Device

We begin by defining the measurement devices. In this theory a measurement device is given by a pure information frame $\mathscr{I}_{\text{mes}}$. That is a measurement device is represented by $(\Omega_{\text{mes}}, \mathcal{F}_{\text{mes}})$, it can be seen as what the measurement device sees about the world, the events it is capable of recognizing.

Observables

An observable, for a particular measurement device i, is a random variable $O_i : \Omega_i \to \mathcal{O}$. Where $\mathcal{O}$ is the outcome space, it can be $\mathbb{R}$ or other such spaces.

Each observable through the original embeddings, induces a global random variable on the joint context, which we will denote with $(\Omega, \mathcal{F})$. To define the global random variables we need the partial inverse:

$$\pi_i : \iota_i(\Omega_i) \to \Omega_i$$

Then the representative of the observable in the joint frame is:

$$\widetilde{O}_i := O_i \circ \pi_i : \Omega \to \mathcal{O}$$

The σ-algebra generated by $\widetilde{O}_i$ represents the set of events distinguishable by the observable O_i. We say observables are **compatible** if their algebras remain jointly Boolean.

Observable Operators

To see how incompatible observables do not commute, we can look at a strategy similar to what we did we the full algebras, it also the representation that matches operators more directly. Start with a event E in the global space, for observables of different contexts O_i and O_j we can define:

$$e_i^{O_i} := \iota_i(O_i^{-1}(O_i(\pi_i(E)))) \quad \text{and} \quad e_j^{O_j} := \iota_j(O_j^{-1}(O_j(\pi_j(E))))$$

If they are incompatible, we will have:

$$e_i^{O_i} \circ e_j^{O_j} \neq e_j^{O_j} \circ e_i^{O_i}$$

Similarly to what we saw for local projections.

Unitary Evolution

Basicaly it is automorphism selection.

General Operators

Other operators, those not associated with observables such as stochastic maps, or Hilbert-space operators without direct observational intrepetation appear only as a transformation built from the basic local projections of Definition 5.2 and their compositions and mixtures.

In general, in the Hilbert space representation, the contextual transformations are represented by completely positive maps associated with the corresponding measurement.

Measurement

A measurement of an observable O_i corresponds to evaluating its lifted form $\widetilde{O}_i : \Omega \to \mathcal{O}$. However, the collapse theorem restricts the physically admissible events to those lying in the pointer σ-algebra $\mathcal{F}_{\text{ptr}}$.

Consequently, only those values of $\widetilde{O}_i$ whose events are compatible with $\mathcal{F}_{\text{ptr}}$ can occur as outcomes. In this sense, measurement reduces to evaluating the observable on the pointer algebra. The observable possible values are exactly those that survive the collpase into $\mathcal{F}_{\text{ptr}}$.

Once a definite outcome happens after sampling. The frames carry the information of that outcome. Effectively, the algebras are conditioned on the outcome A. That is

$$\mathcal{F} \mid A := \{E \cap A : E \in \mathcal{F}\}$$

That is, the perspectives of the frames are filtered to the outcome produced.

6.2 Born Rule Martingale

In this section we explore how the Born rule appears in this framework. We do not claim, with the current tools, to recover the quadratic nature of the born rule or its usual form. That is left to a reconstruction of the Hilbert space.

Setup

First we consider a particular context's frame $\mathscr{I}_i$. We define a probability measure representing that frame μ_0. Th

In the joint frame we can look at the pushfoward measure given by the original embedding.

$$\mu := \mu_0 \pi_i = \mu_0(\pi_i(E)) \quad E \in \mathcal{F}_{\mathcal{J}}$$

We note that this probability measure does not, nescessarly, represent all contexts of the joint frame. In fact, in contextual cases, that is not possible. This is simply the probability of the context i transported to the global frame.

In fact, this pushfoward probability is the probability distribution given by the quantum trace rule for the observable of that context. To see this, simply consider the observable model we had before and note:

$$\mu(\widetilde{O}_i^{-1}(B)) = \mu_0(O_i^{-1}(B))$$

In the Hilbert space, this looks like:

$$tr(\rho E_B) = \rho(E_b)$$

The Collapse Filtration

The collpase theorem, in particular for each partial $\mathcal{F}_{\text{ptr}} \subsetneq \mathcal{G} \subseteq \mathcal{F}_{\mathcal{J}}$, is a sub-$\sigma$ of $\mathcal{F}_{\mathcal{J}}$.

So we can define the reverse filtration converging to the pointer basis.

$$\mathcal{F}_{\mathcal{J}} =: F_0 \supseteq F_1 \supseteq F_2 \supseteq ...\mathcal{F}_{\text{ptr}}$$

This filtration can be seen as the joint space being coarse grained to the pointer basis.

The Collapse Martingale

We can now use μ as a probability measure to define, for any event $A \in \mathcal{F}_{\mathcal{J}}$ we define:

$$M_n := \mathbb{E}_\mu \left[\mathbf{1}_A \mid \mathcal{F}_n\right]$$

As seen [1] we know this is a Martingale on the reverse filtration, and by Doob convergence theorem we have:

$$M_n \to \mathbb{E}_\mu \left[\mathbf{1}_A \mid F_{\text{ptr}}\right] \quad \text{(a.s.)}$$

Which means that the probability of event A in the pointer filtration is exactly $\mu(A)$.

6.3 Wigner's Friend Consistency

We will now turn our attention to looking at how Wigner's Friend paradox looks like in this framework.

Setup

For the Wigner friend scenario we acutually have two interactions. First the interaction of the friend, which we will map to the joint frame $\mathcal{J}_{\text{friend}}$. Which represents the interaction between the system and the friend.

Once that interaction goes through the system has collapsed to the pointer frame $\mathcal{J}^{\text{ptr}}_{\text{friend}}$. Then Wigner comes and interacts with that joint frame, forming a new frame $\mathcal{J}_{\text{Wigner}}$, which again must collapse to $\mathcal{J}^{\text{ptr}}_{\text{Wigner}}$.

We use the fact that, these algebras can all be seen as filtrations of each other to show that, Wigner cannot assign inconsistant probabilities to the events it can observe.

The Sequence of Filtrations

When the friend interacts with the system, he generates the joint frame $\mathcal{J}^{\text{ptr}}_{\text{friend}}$. When Wigner interacts with that frame, a new joint frame must be built. That joint frame starts by lifting the algebras through the embeddings ι_{friend} and ι_{Wigner}.

The nature of embeddings mean we can consider this all a sequence of filtrations on the same space. Namely:

$$\mathcal{F} \supseteq \mathcal{F}^{\text{ptr}}_{\text{friend}} \supseteq F^{\text{ptr}}_{\text{Wigner}}$$

With these filtrations in place, we can look at the same style of probability assignments we had in the born rule.

6.3.1 The Probabilities

Now let $A \in \mathcal{F}^{\text{ptr}}_{\text{Wigner}}$ be a event that exists in the final pointe algebra. That is, a event that all systems involved can talk about.

We can determine the friend's probaility for event A as we did for the born rule martingale:

$$M_{\text{friend}} := \mathbb{E}_\mu \left[\mathbf{1}_A \mid \mathcal{F}^{\text{ptr}}_{\text{friend}}\right]$$

Now for Wigner, we have the same:

$$M_{\text{Wigner}} := \mathbb{E}_\mu \left[\mathbf{1}_A \mid \mathcal{F}^{\text{ptr}}_{\text{Wigner}}\right]$$

But since we have

$$F^{\text{ptr}}_{\text{Wigner}} \subseteq \mathcal{F}^{\text{ptr}}_{\text{friend}} \subseteq \mathcal{F}$$

The tower law for martingales gives:

$$\mathbb{E}_\mu \left[M_{\text{friend}} \mid \mathcal{F}^{\text{ptr}}_{\text{Wigner}} \right] = \mathbb{E}_\mu \left[\mathbb{E}_\mu \left[\mathbf{1}_A \mid \mathcal{F}^{\text{ptr}}_{\text{friend}} \right] \mid \mathcal{F}^{\text{ptr}}_{\text{Wigner}} \right] = \mathbb{E}_\mu \left[\mathbf{1}_A \mid \mathcal{F}^{\text{ptr}}_{\text{Wigner}} \right]$$

Thus, when the friend's description is updated to the Wigner's pointer frame is exactly Wigner's own description.

6.4 The Hilbert Space Realization

We will not attempt a full Hilbert space reconstruction in this paper. But we note the relations to previous work that fits this picture and where some structure might help narrow some things.

The Orthomodular Lattice

We have Boolean σ-algebras corresponding to each context $\mathcal{F}_i$. We have their embeddings ι_i into the joint frame.

We can then define on the joint frame:

$$\mathcal{L} := \text{ the closure of } \bigcup_i \iota_i(\mathcal{F}_i)$$

Events here have:

- events complement $E \to E^c$,
- events meet $E \wedge F = E \cap F$,
- events join $E \vee F = \mathrm{cl}_{\mathcal{L}}(E \cup F)$,

Proposition 6.1 ($\mathcal{L}$ is a orthomodular lattice)**.** *The event structure $\mathcal{L}$, generated by all frame embeddings in the joint frame is an orthomodular lattice. Each frame embeds as a maximal Boolean subalgebra of $\mathcal{L}$. Contextuality is exactly the failure of distributivity in $\mathcal{L}$ and what makes it orthomodular.*

Proof. The relation to contextuality allows us to use the standard theorems such as [**?**] to see how this algebra represents a orthomodular lattice. □

The Complex Field

While we don't attempt to prove the uniqueness of the choice of the complex field in this work we note that, accumulation of amplitudes and its removal through the RIF principle leaves space to narrow down amplitudes that fit $\mathbb{C}$ over something like $\mathbb{H}$.

The Hilbert Space

Under the usual regularity assumptions on $\mathcal{L}$, the standard representation theorems of quantum logic apply (e.g [**?**], [**?**], [**?**]). Therefore there exists a Hilbert space $\mathcal{H}$ and a lattice isomorphism.

$$\mathcal{L} \cong \mathsf{Proj}(\mathcal{H})$$

The Pointer Algebra

Each frame corresponds to a maximal commuting family of projections in $\mathcal{H}$. The pointer algebra $\mathcal{F}_{\text{ptr}}$ becomes the lattice of projections associated with a single classical basis (a maximal distributive subalgebra).

Thus the pointer corresponds to the diagonal projections in a distinguished basis, the pointer basis.

States in the Hilbert Space

Since $\mathcal{L}$ carries empirical probability measures representing the pushfoward from the frames preparations as we done previously, Gleanson's theorem [**?**] implies that each measure corresponds uniquely to a density operator ρ on $\mathcal{H}$.

The born rule then, as we have seen appears as the familiar trace rule:

$$tr(\rho E_B) = \rho(E_b)$$

7 Dynamics

In this version of the framework, dynamics is not introduced as a primitive law like a Hamiltonian flow or differential equation. Instead, dynamics arises from the structure of interactions between information frames.

Each interaction between contexts induces:

- A creation of a joint frame,
- followed by collapse to its pointer algebra,
- and a pushfoward updated of the preparation measure.

Thus, from a fixed frame point of view, the time evolution of it's description of the world is simply a sequence of coarse-grained σ-algebras obtained from sucessive interations, effectively a filtration:

$$\mathcal{F}_{\text{frame}} =: \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots \supseteq \mathcal{F}_n \supseteq \ldots$$

This point of view makes several dynamical properties appear naturally:

1. Arrow of Time

   Because each contextual interaction corresponds to additional coarse-graining, the evolution of σ-algebras is monotone (information losing). This monotonicity defines a natural direction of time.

2. Locality Graph

   Interactions occur only between specific frames. The pattern of which frame couples to which determines a graph structure, which plays the role of space locality.

3. Maximum Speed of Influence

In contextual scenarios, incompatibility forces coarse-graining. The maximal number of interaction steps before contextuality appears bounds how "fast" influence can propagate along the locality graph.

4. No Signaling

   Because interactions only merge σ-algebras along edges of the locality graph, and collapse happens locally on the joint frame, no frame can influence another without a mediated interaction.

None of these dynamical features require any additional axioms. They follow from Axiom 1 and the definitions of how interactions are built.

7.1 Arrow of Time

In this framework, time is not an external parameter. Instead we use the ordering of interactions between information frames to induce a canonical direction: each interaction generates a joint frame where the initial frame algebra is embedded, collapse forces a coarse graining of its algebra, this monotone loss of distinguishability defines a natural arrow of time for that frame.

Interaction-induced evolution of σ-algebras

To begin, we first fix a information frame $\mathscr{I}_0$. It then proceeds to interact with several frames $\mathscr{I}_{j_1}, \mathscr{I}_{j_2}$ and so on.

At each step, a joint frame is first created to host the interaction, Axiom 1 and the collapse theorem then converge into a physically realizable joint frame with a sigma algebra

$$\mathcal{F}^{\mathrm{ptr}}_{\mathcal{J}_n}$$

That is, the pointer frame on the n-th joint frame. Since this is always constructed by embedding the original σ-algebra $\mathcal{F}_0$, we can track its evolution along the interactions.

We can look at the events of $\mathcal{F}_0$ that survive in the pointer frame $\mathcal{F}^{\mathrm{ptr}}_{\mathcal{J}_1}$ as the σ-algebra of $\mathscr{I}_0$ at step 1. That is:

$$\mathcal{F}_1 := \mathcal{F}^{\mathrm{ptr}}_{\mathcal{J}_1} \cap \iota_1(\mathcal{F}_0)$$

This gives a natural sequence:

$$\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_1 \supseteq ...$$

Each step removes events that are incompatible with the new join. The future σ-algebras representations of $\mathcal{F}_0$ are strictly coarser algebras. There is no way physical way in the framework to restore the lost distinctions.

7.2 Locality Graph

In this theory, **locality** is not a geometric primitive yet. It is the combinatorial structure that records which frames have interacted and must therefore share a joint pointer algebra.

The Locality Graph

Let $\{\mathscr{I}_i\}_{i\in I}$ be a family of relevant information frames. We define the graph:

$$G = (V, E)$$

$$(i, j) \in E \quad \iff \quad \text{frames } \mathscr{I}_i \text{ and } \mathscr{I}_j \text{ have interacted.}$$

And at every point, an edge represents there is a joint frame where the collapse has taken place between $(i, j) \in E$.

Locality as constraint on the pointer basis

If frames i and j have interacted, then their local projections e_i and e_j both act on the same joint frame. Namely the pointer basis there must satisfy:

$$\mathcal{F}_{\text{ptr}} \subseteq \text{Fix}(e_i) \cap \text{Fix}(e_j)$$

And if they have not interacted their projetions do not constrain the same σ-algebra, which means:

- Frames in the same subgraph in G influence one another through constraints in the same pointer algebra
- Frames in different subgraphs remain completely independent. No constraints propagate between them.

This reproduces the operational notion of locality, influence propagates only along paths in the interaction graph.

Dynamics respects the locality graph

When a new interaction occurs between frames i and j we:

1. add an edge (i, j) to G;
2. build the joint frame of all contexts in the connected component containing i and j
3. collapse using all local projections in that component.

Therefore every subgraph of G behaves as local regions, every interaction affects only the σ-algebra generated by its own components.

Locality therefore appears as:

- graph locality where edges are interactions
- probabilistic locality conditional expectations factor across components of the sub graph

Intrepretation

This provides a structural notion of spacetime locality:

- Information frames as ”positions”.

- Interactions as "lightlike" adjacency.
- Paths in G are the only channels through which constraints can propagate.

There is no geometry at this stage, geometry would be additional structure placed on top of the locality graph.

We emphasize that "locality" and "time" in this theory are entirely structural notions. They arise from the patterns of interactions between frames, not from any pre-assumed geometric spacetime. Any system that can be expressed through interacting information frames possesses a well-defined locality graph and a induced temporal order, even when no traditional spacetime background is specified.

7.3 Maximum Speed

In this framework, influence can propagate only through interactions between frames, and such interactions are encoded by edges in the locality graph G.

However contextuality imposes an additional constraint: after a sufficiently long chain of interactions on mixed degrees of freedom, contextual incompatibility nescessarely appears. Forcing a coarse-graining of the joint σ-algebra. This defines a **time step**, the length of the chain in that time step is always bounded by contextuality.

Contextuality forces coarse-graining

A fundamental structural fact of contextuality ([**?**]) is that for a sufficiently large family of information frames on shared degrees of freedom, their combined σ-algebra on the joint frame cannot remain non-contextual.

While this is not a formal statement, as we could stay replaying interactions on non-contextual frames, in realistic scenarios contextuality will show up. At such step, the pointer algebra will nescessarly coarse grain for some frame. That frame then sees the length of the chain over the time step as the first interaction step.

Over several interactions, such chains which are always finite, have varying length. But any such interaction will have a finite upper bound.

Maximum speed for frame i

Once we fix a frame i we see its interaction path in G as:

$$i = i_0 \to i_1 \to \ldots \to i_n = j$$

In the step i_n is when the first coarse graining happens. We can then define:

$$v_i := \sup n : \mathcal{F}_n = \mathcal{F}_0$$

That is, the largest number of sequential interactions along which the σ-algebra of i remains untouched.

Intrepetation

This pehnomenon should not be interpreted as the emergence of a specific relativistic sped such as a "speed of light". It is a structural consequence of contextuality: in any sufficiently rich system of interacting information frames, repeated interactions inevitably produce contextual incompatibilities, which force coarse-graining and thereby prevent influence from propagating indefinitely without degradation.

The resulting bound is a maximum interaction speed inherent to the information structure itself. Although different contexts may have different local bounds, a common finite upper bound always exists for any fixed system.

7.4 No Signaling

No signaling is an immediate structural consequence of the locality graph and Axiom 1. Because interactions are encoded as edges in the graph, and Axiom 1 applies only on the joint frames of the interacting frames, no frame can influence another without a path of interaction connecting them.

Local Independence

Let i and j represent two frames. If they have never interacted, they lie on different subgraphs of G. Then no local projection e_i of i acts on any algebra observed by j.

Thus any mutual time step between then leaves $\mathcal{F}_j$ untouched, that is:

$$\mathcal{F}_j^{n+1} = \mathcal{F}_j^n$$

When that time step operates on i, quite simply that is, a frame cannot alter the algebra of a frame unless they interact.

Causal Separation

If i and j lie in different subgraphs, let e_k bet the local projection of a new interaction on the subgraph of i. Since there is no path from k to j the projection e_k has no overlap with the algebras j can see. Therefore, e_k has no effect on $\mathcal{F}_j$.

Therefore

- Effects of interactions propagate only along paths in the locality graph.
- Only frames lying in the same subgraph can influence each other.

This is a purely structural notion of causal separation.

Intrepretation

No signaling in this framework is almost tautological, information cannot be transmitted between frames that have not interacted.

8 Intrepetation and Comparisons

Here we explore the theory in the context of the broader literature on measurement and study some of its features, including the implications it has on ontology. We begin with a informal example of the framework applied to the spin of a electron. To illustrate how this framework maps to standard Quantum Mechanics.

8.1 Spin Example

Consdier a concreate quantum scenario, that of a single spin-$\frac{1}{2}$ system.

We examine two incompatible measurements:

- $\mathscr{I}_z$: spin along the z-axis.
- $\mathscr{I}_x$: spin along the x-axis.

Each frame is represented by a two-outcome algebra:

$$\mathcal{F}_z = \{\emptyset, \{\uparrow_z\}, \{\downarrow_z\}, \Omega_z\} \quad \mathcal{F}_x = \{\emptyset, \{\uparrow_x\}, \{\downarrow_x\}, \Omega_x\}$$

These are Boolean σ-algebras describen the events accessible to measurements in their respective basis.

We begin with a free particle $\mathscr{I}_{\text{part}}$. This particle makes no distinctions about its own internal spin. So for this experiment, its algebra is considered trivial.

The particle then interacts first with $\mathscr{I}_x$. This generates a joint frame where the interaction can take place with algebra exactly:

$$\mathcal{F}_1^{\text{ptr}} = \mathcal{F}_x = \{\emptyset, \{\uparrow_x\}, \{\downarrow_x\}, \Omega_x\}$$

The observable is sampled there and obtains event $\uparrow_x$. Then the structure of the joint frame is conditioned on this event. That is, the actual state is:

$$\mathcal{F}_1 \mid \{\uparrow_x\} := \{E \cap \{\uparrow_x\} : E \in \mathcal{F}_1\}$$

Then, if we were to measure again on the same basis, this encodes running the same observable on the joint space. The only possibility in the restricted algebra is $\uparrow_x$.

Intuitively, the frame saw $\uparrow$ so the events, even in the measurement apparatus frame are conditioned on $\{\uparrow_x\}$. Now a new apparatus $\mathscr{I}_z$ comes and interacts with the system.

That leads to a new joint frame being built, and a new pointer basis, but here the observables are contextual so the pointer basis is given by:

$$\mathcal{F}_2^{\text{ptr}} = \{E \mid e_z \circ e_x(E) = e_z(E)\}$$

But this pointer algebra is still conditioned on $\{\uparrow_x\}$. So effectively the state is:

$$\mathcal{F}_2^{\text{ptr}} \mid \{\uparrow_x\}$$

The Hilbert space representation of the spin-x on the basis of z show that there are events that surive the z measurement in the pointer basis. Furthermore the fact that the algebra is conditioned on $\{\uparrow_x\}$ means we get:

$$\begin{array}{lll} \uparrow_z & \text{with probability} & \frac{1}{2} \\ \downarrow_z & \text{with probability} & \frac{1}{2} \end{array}$$

Whatever outcome we get here, will again condition $\mathcal{F}_2^{\text{ptr}}$. If we measure again on z we will only be able to get definite outcomes.

But if we were to measure on x on the joint frame conditioned on the event $\{\uparrow_x\}$ there would be two possible outcomes, with probability defined by the z basis.

8.2 The Ontology of Information Frames

8.3 Comparisons - Collapse Models

8.4 Comparisons - Intrepetations

9 Discussion

9.1 Geometry and Dynamics

By taking the Markov view in section 5.5 we can explore this framework from the point of view of the fisher metric. When contextuality happens, the ricci curvature of the fisher metric blows up. That should trigger a back action on the tensor to adjust for the pointer basis.

Known derivations of the schrodiger equation using the quantum fisher metric can be explained and made canonical with this framework.

Furthermore, one can define a action principle of interactions. Using information geometry, we believe that can give a new way to model physical systems.

9.2 The Complex Field

9.3 Reconstruction Programs

9.4 Spacetime

10 Conclusion

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