Local Information Scrambling in Random Quantum Circuits

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Random quantum circuits are an effective model for the behavior of chaotic quantum systems, including the phenomenon of information scrambling. While it is known that local random circuits scramble information in a time linear in system size, for applications in condensed matter, it is often impractical to allow evolution time to scale with the size of a physical system. We define local forms of scrambling, in which information in the system appears scrambled when only a region of fixed size can be accessed, and characterize the conditions for which it is achieved by a shallow local random circuit. We find that up to the second moment, local scrambling of a product state input occurs in log depth—requiring circuit depth proportional to the logarithm of the size of the region—and provide intuition for the result. We also describe applications to classifying topological phases and characterizing the entanglement structure of quantum matter.

I. INTRODUCTION

Random quantum circuits are a recent topic of interest due to their applications as models for both defining theoretical concepts as well as physical phenomena. In particular, as random quantum circuits are instances sampled from a distribution over unitary operators, they provide a framework for describing the evolution of complex quantum systems and understanding quantum chaos [1, 2]. These methods have been applied to measuring the complexity of classical simulations of quantum circuits [3, 4], developing classical protocols to efficiently learn quantum states [5], understanding the behavior of condensed matter systems [1, 2, 6], and addressing fundamental questions such as in quantum statistical mechanics and black hole physics [7–12].

Previous work has shown that for a one-dimensional circuit architecture, scrambling of the entire system occurs in linear time, requiring circuit depth which scales proportionally with the size of the system [13, 14]. However, prior to the complete dissipation of information, the system is expected to exhibit *local scrambling*, so that information cannot be obtained from accessing a local subsystem. Although local scrambling is not as widely studied, it is relevant to applications in condensed matter physics when only part of the system can be accessed or when describing the behavior of large systems under a fixed timescale [1, 2]. Characterizing local forms of scrambling is therefore important in understanding thermalization and emergent randomness as well as describing quantum phases of matter [15, 16].

In this report we summarize background on random quantum circuits and averaging over the unitary group as well as related work before giving formal definitions for local scrambling of an input state based on the absorption of a local unitary. For the shallow circuit regime, we provide upper bounds which are explicitly independent of total system size. We show that up to the second moment, local scrambling of a product state by a local one-dimensional random circuit occurs in log time, requiring depth proportional to the logarithm of the size of the local region which is accessed. We also provide intuition for this result by considering its connection to the unitary design definition of scrambling. Finally, we describe opportunities for further work, including applications to classifying topological phases and classical shadow tomography schemes for reconstructing quantum states.

II. PRELIMINARIES

A. Unitary designs

We consider circuits on n qudits of local dimension q, for which the system is represented by a Hilbert space $\mathcal{H}$ of dimension $d = q^n$. Formally, the Haar measure is given as the unique left- and right-invariant measure

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over the unitary group. For any operator X acting on $\mathcal{H}^{\otimes k}$, the k-th moment over the Haar distribution can be defined via the twirling operators

$$T^{(k)}_{\text{Haar}}(X) = \int_{\text{Haar}} dU\, U^{\otimes k} X (U^\dagger)^{\otimes k}. \tag{1}$$

These can be computed by using the Schur-Weyl duality and the symmetries of the unitary group to express $T^{(k)}_{\text{Haar}}(X)$ as a linear combination of permutation operators S_k on the different copies of the Hilbert space, and solving the resulting linear system of equations. We focus on the second moment, which is a linear combination of identity ($\mathbb{1}$) and swap (S) operations on the two copies of the Hilbert space and is given by

$$T^{(2)}_{\text{Haar}}(X) = \int_{\text{Haar}} dU\, U^{\otimes 2} X (U^\dagger)^{\otimes 2} = \frac{1}{d^2-1}\left(\mathbb{1}\operatorname{Tr} X - \frac{1}{d}\mathbb{1}\operatorname{Tr} SX + S\operatorname{Tr} SX - \frac{1}{d} S \operatorname{Tr} X\right). \tag{2}$$

Due to the exponential scaling of the Hilbert space, it is often useful to consider ensembles of unitaries which capture some of the properties of the Haar distribution. An ensemble $\varepsilon = \{p_i, U_i\}$ is called a *unitary* k-*design* if it reproduces the first k moments of the Haar distribution, so that

$$T^{(k')}_{\varepsilon}(X) = \int_{\varepsilon} dU\, U^{\otimes k'} X (U^\dagger)^{\otimes k'} = T^{(k')}_{\text{Haar}}(X) \tag{3}$$

for all X acting on $\mathcal{H}^{\otimes k'}$ and $1 \leq k' \leq k$ [14]. In general it is difficult to find explicit constructions of k-designs for arbitrary k; however, constructions have been given for approximate designs in connection to stochastic Hamiltonians and random circuits [9]. In particular, we focus on the approximate k-design condition for scrambling, and adapt it to to describe a form of local scrambling.

B. Random quantum circuits

We restrict to a local one-dimensional architecture consisting of two-site gates on neighboring qudits, which are sampled uniformly over the Haar distribution. We are interested in studying the scaling relationship between the behavior of random circuits and their *depth*, the minimum number of non-overlapping layers in the circuit. We can therefore consider circuits with alternating layers of gates up to some depth L:

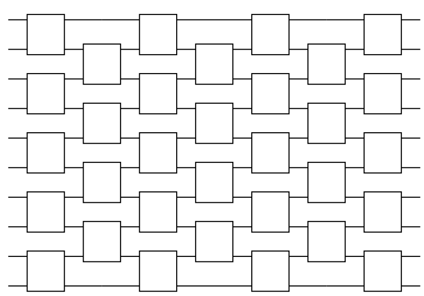


FIG. 1: Local one-dimensional random circuit architecture with $n = 10$, $L = 7$

It was previously shown that local random circuits acting on n qudits form approximate k-designs in $O(n \cdot \text{poly}(k))$ [14], where an ensemble $\varepsilon = \{p_i, U_i\}_i$ is an ϵ-approximate k-design if the diamond norm of the difference of the k-th moment is bounded by

$$\left\| T^{(k)}_{\varepsilon} - T^{(k)}_{\text{Haar}} \right\|_{\diamond} \leq \epsilon, \tag{4}$$

where the diamond norm of a channel is defined as

$$\|T\|_{\diamond} \equiv \sup_{d} \|T \otimes \mathbb{1}_d\|_{1\to 1}. \tag{5}$$

Here the superoperator norm $\|\cdot\|_{p\to p}$ is defined for $p \geq 1$ via a supremum over non-trivial operators

$$\|T\|_{p\to p} \equiv \sup_{\mathcal{O}\neq 0} \frac{\|T(\mathcal{O})\|_p}{\|\mathcal{O}\|_p} = \sup_{\mathcal{O}\neq 0} \frac{(\mathrm{Tr}\,|T(\mathcal{O})|^p)^{1/p}}{(\mathrm{Tr}\,|\mathcal{O}|^p)^{1/p}}. \tag{6}$$

This result on approximate designs can also be taken as a condition for scrambling in random circuits, since integrating over the Haar measure erases the effect of any unitary applied on the system. More concretely, for any $V \in U(d)$, we have that

$$\int_{\mathrm{Haar}} dU\,(VU)^{\otimes k} X (U^\dagger V^\dagger)^{\otimes k} = \int_{\mathrm{Haar}} dU\,U^{\otimes k} X (U^\dagger)^{\otimes k} = T^{(k)}_{\mathrm{Haar}}(X), \tag{7}$$

since $V^{\otimes k}$ is absorbed into the integral due to the left- and right-invariance of the Haar measure. Thus information contained in V is completely scrambled by a circuit which perfectly samples from the Haar distribution.

III. PREVIOUS WORK

A. Unitary designs from statistical mechanics

Hunter-Jones [14] showed the unitary design condition for scrambling is achieved in linear time by applying a mapping from integrals over local random circuits to an Ising-like model. While the diamond norm is difficult to work with directly, it can be related to another quantity known as the frame potential. For any ensemble of unitaries ε, the k-th frame potential is given by

$$\mathcal{F}^{(k)}_{\varepsilon} = \int_{\varepsilon} dU dV \left|\mathrm{Tr}\,U^\dagger V\right|^{2k}. \tag{8}$$

The frame potential for any ensemble is lower bounded by the Haar value, and furthermore the difference between the values bounds the diamond norm distance via

$$\left\|T^{(k)}_{\varepsilon} - T^{(k)}_{\mathrm{Haar}}\right\|_{\diamond} \leq d^{2k}(\mathcal{F}^{(k)}_{\varepsilon} - \mathcal{F}^{(k)}_{\mathrm{Haar}}). \tag{9}$$

By taking advantage of an exact mapping to a lattice model, the second frame potential for a local one-dimensional random circuit was shown to converge asymptotically to the Haar value, so that local random circuits achieve a 2-design in $O(n)$ depth; the result generalizes to higher k in the limit of infinite local dimension.

In this work, we are interested primarily in the action of a random circuit on an input state and in conditions for local scrambling which are asymptotically independent of system size. While the frame potential analysis is not directly useful for describing local scrambling due to the factor of $d^{2k} = q^{2kn}$, we can apply a similar lattice mapping technique in order to bound the second moment, corresponding to accessing two copies of the input state.

B. Anti-concentration in random circuits

Dalzell, Hunter-Jones, and Brandão [3] studied the related property of anti-concentration in various local random circuit architectures, where an ensemble of states is defined to be anti-concentrated if the expected collision probability is at most a constant factor times that for a uniform distribution. Here collision probability is defined as the probability that two identical copies of the circuit produce the same outcome when measured; for measurements with respect to the computational basis, it is given by

$$Z = q^n \mathbb{E}[p(|0\rangle)^2], \tag{10}$$

where $|0\rangle$ denotes the computational basis state in which all qudits are initialized to 0. The collision probability for the Haar-random distribution, which samples uniformly over all unitary transformations on the system, is given by

$$Z_H = \frac{2}{q^n + 1}. \tag{11}$$

Since the expected collision probability is a second moment quantity, satisfying the approximate 2-design property guarantees that the output of the circuit also satisfies the anti-concentration property; however, in general anti-concentration is weaker than the approximate 2-design condition. It was shown that for the one-dimensional ring as well as complete graph architectures, the output of a local random circuit reaches anti-concentration in log depth, faster than the circuit forms an approximate 2-design. In particular, for a one-dimensional random circuit with depth L, the collision probability is bounded by

$$Z_L = Z_H \left(1 + (e-1)n\left(\frac{2q}{q^2+1}\right)^{L-1}\right). \tag{12}$$

We provide intuition for the difference in these two measures of scrambling and discuss further connections as well as potential for future work related to state designs in Section VI.

IV. LOCAL SCRAMBLING

The observation that the k-th moment of an operator with respect to the Haar distribution must commute with and thereby absorb the effect of any unitary motivates the definition of local scrambling via absorption of a unitary acting on a local region. We expect that this holds for $|A| \ll L$, and in particular that the maximum size of the region absorbed scales asymptotically with L, independent of the size of the entire system.

More concretely, let μ_L denote the ensemble generated by a local random quantum circuit consisting of L layers, and let A be a local region. For a constant U_A, we are interested in the condition

$$\int d\mu_L U^{\otimes k} X (U^\dagger)^{\otimes k} \approx \int d\mu_L (U_A U)^{\otimes k} X (U^\dagger U_A^\dagger)^{\otimes k}. \tag{13}$$

Note that for the case $k = 1$, the twirling channel produces the completely mixed state, and thus information in the system is scrambled at constant depth $d = 1$.

We restrict to the simplest nontrivial case $k = 2$, for which $T^{(2)}_{\mu_L}$ consists of terms with either the identity or swap operation on each qudit, and consider the action of the random circuit on an input state ρ. We therefore consider the following definition for local scrambling, up to the second moment:

Definition 1 (Local scrambling) *Consider a random circuit of depth L on n qudits with input ρ and a geometrically connected subsystem A. Let μ_L denote the ensemble consisting of instances of the random circuit. Then, up to some $\epsilon > 0$, the region A of ρ is scrambled by the circuit if for any unitary U_A which acts trivially on the subsystem $\overline{A}$,*

$$\left\| \int d\mu_L U^{\otimes 2} \rho^{\otimes 2} (U^\dagger)^{\otimes 2} - \int d\mu_L (U_A U)^{\otimes 2} \rho^{\otimes 2} (U^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \leq \epsilon.$$

Recall that the trace norm or Schatten 1-norm of an operator A is defined via

$$\|A\|_1 := \operatorname{Tr}|A| = \operatorname{Tr}\sqrt{AA^\dagger}. \tag{14}$$

Our strategy is to apply the triangle inequality to the quantity of interest in Definition 1 and bound the coefficients in terms of L. While naively summing over the coefficients gives a factor of q^{2n} after taking the trace norm, fortunately, it is possible to reduce the problem to one which is asymptotically independent of n. From there, we apply the lattice mapping technique [14] in order to upper bound the coefficients of contributing terms.

A. Spin lattice model

From Eq. 2, integrating over a two-site unitary on qudits $j, j+1$ which are not entangled with the rest of the system yields

$$\frac{1}{q^4-1}\left(\mathbb{1}\,\mathrm{Tr}\,\rho_{j,j+1}^{\otimes 2} - \frac{1}{q^2}\mathbb{1}\,\mathrm{Tr}\,S\rho_{j,j+1}^{\otimes 2} + S\,\mathrm{Tr}\,S\rho_{j,j+1}^{\otimes 2} - \frac{1}{q^2}S\,\mathrm{Tr}\,\rho_{j,j+1}^{\otimes 2}\right)\otimes\rho_{\{i\}_1^n\setminus\{j,j+1\}}^{\otimes 2}, \tag{15}$$

where $\rho_{\{i\}_1^n\setminus\{j,j+1\}}$ is the reduced density matrix on the rest of the system. By linearity, the final "output state" can be represented as a sum of terms in which the action of each gate is either identity or swap, with some coefficient depending on $\mathrm{Tr}\,\rho_{j,j+1}^{\otimes 2}$ or $\mathrm{Tr}\,S\rho_{j,j+1}^{\otimes 2}$. Applying the statistical mechanics mapping technique in [14], we can represent these terms by replacing each of the gates with a "spin" which is assigned one of the possible permutation operators, resulting in an Ising-like model with domain walls separating regions of different spins. Note that these configurations do not correspond to physical instances of the circuit, and in general are not valid quantum states. See Figure 2 for an example of one of these configurations.

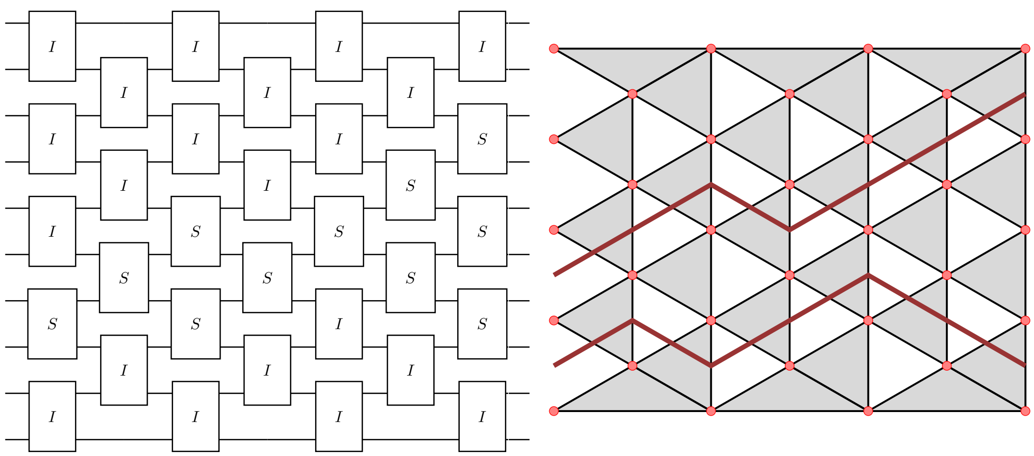


FIG. 2: Example of a domain wall configuration in the spin lattice model (right) corresponding to one of the terms in the output (left), where each gate is replaced by a vertex and domain walls have been drawn in between regions of identity and swap.

The weighting for each term can therefore be computed via a partition function on the lattice. Let Γ denote all possible configurations γ. Then the second moment is given by

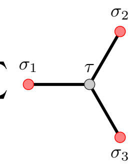
$$T_{\mu_L}^{(2)}(\rho^{\otimes 2}) = \int d\mu_L U^{\otimes 2}\rho^{\otimes 2}(U^\dagger)^{\otimes 2} = \sum_{\gamma\in\Gamma} W(\gamma)\gamma_L, \tag{16}$$

where γ_L denotes the configuration of the last layer. Each of the coefficients $W(\gamma)$ can be expressed in terms of the triangular plaquettes:

$$W(\gamma) = \prod_{\triangleleft\in\gamma} J_{\sigma_2\sigma_3}^{\sigma_1}, \tag{17}$$

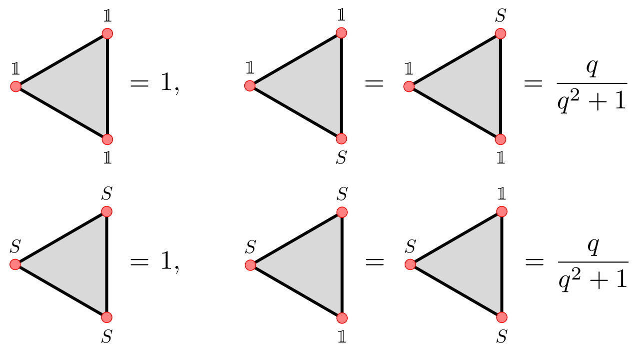
where for a plaquette with spins $\sigma_1, \sigma_2, \sigma_3$ on its three vertices, the $J_{\sigma_2\sigma_3}^{\sigma_1}$ are computed by summing over the $\mathrm{Tr}\, \tau_{j,j+1}\rho_{j,j+1}^{\otimes 2}$ coefficients, where τ represents the "output" of the gate associated with the σ_1 vertex:

$$J^{\sigma_1}_{\sigma_2\sigma_3} = \sum_\tau$$



$$= \sum_\tau Wg(\sigma_1^{-1}\tau, q^2) q^{\ell(\tau^{-1}\sigma_2)} q^{\ell(\tau^{-1}\sigma_3)}. \tag{18}$$

Here Wg denotes the Weingarten function for the unitary group, and $\ell(\sigma)$ denotes the number of closed cycles in a permutation σ. For $k = 2$, we have $\sigma_i, \tau \in \{\mathbb{1}, S\}$, so the coefficients for all non-zero plaquette configurations are given by



(19)

Since plaquettes with $\sigma_2 = \sigma_3$ and $\sigma_1 \neq \sigma_2$ result in a coefficient of 0, only lattice configurations which do not have pairs of annihilating domain walls contribute to the second moment. Furthermore, a domain wall contributes a factor of $\frac{q}{q^2+1}$ for each layer it propagates, so that configurations with less domain walls have larger weights.

B. Reduced trapezoidal circuit

In the shallow circuit regime, we can simplify the quantity of interest in Definition 1 by considering the "lightcone" of U_A, which consists of the trapezoidal region of size $|A| + 2L$. We will assume for the remainder of this section that $|A| + 2L \leq n$, so that the lightcone is a trapezoidal region. It is clear that the rest of the circuit commutes with U_A; to see this, consider the unitary $U_B = U^\dagger U_A U$ which is constructed by pushing U_A through the circuit. Since the architecture consists of alternating layers of two-site gates, for each layer U_A must be pushed through neighboring gates which act on two additional qudits, so that U_B acts nontrivially on at most $|A| + 2L$ qudits. Since the random circuit is a mixed channel, we can show that the original condition is upper bounded by its value for the trapezoidal circuit.

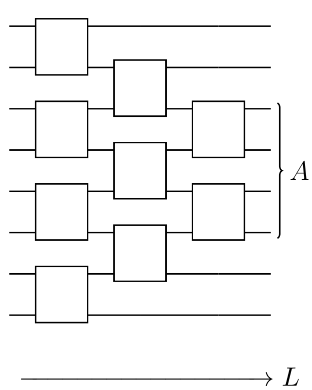


FIG. 3: Trapezoidal circuit consisting of gates in the "lightcone" of the region A

Theorem 1 (Trapezoidal circuit reduction) *Suppose that $n \geq |A| + 2L$, and let μ_* denote the ensemble of unitaries generated by the trapezoidal region of gates in the lightcone of A. Then for any U_A which acts trivially on $\overline{A}$, we have*

$$\begin{aligned}
&\left\| \int d\mu_L U^{\otimes 2} \rho^{\otimes 2} (U^\dagger)^{\otimes 2} - \int d\mu_L (U_A U)^{\otimes 2} \rho^{\otimes 2} (U^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
\leq &\left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_L (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 .
\end{aligned}$$

Proof. Let μ_{rest} denote the ensemble generated by the rest of the circuit which is not included in the trapezoidal region. Then for any $U \in \mu_L$, we can write $U = U_{\text{rest}} U_*$ for some $U_{\text{rest}} \in \mu_{\text{rest}}$ and $U_* \in \mu_*$. We can then rewrite the left hand side in terms of the two parts of the circuit:

$$\begin{aligned}
&\left\| \int d\mu_L U^{\otimes 2} \rho^{\otimes 2} (U^\dagger)^{\otimes 2} - \int d\mu_L (U_A U)^{\otimes 2} \rho^{\otimes 2} (U^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
= &\left\| \int d\mu_* d\mu_{\text{rest}} (U_{\text{rest}} U_*) \rho^{\otimes 2} (U_*^\dagger U_{\text{rest}}^\dagger)^{\otimes 2} - \int d\mu_* d\mu_{\text{rest}} (U_A U_{\text{rest}} U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_{\text{rest}}^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
= &\left\| \int d\mu_{\text{rest}} U_{\text{rest}}^{\otimes 2} \left[\int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right] (U_{\text{rest}}^\dagger)^{\otimes 2} \right\|_1 . \\
\leq &\int d\mu_{\text{rest}} \left\| U_{\text{rest}}^{\otimes 2} \left[\int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right] (U_{\text{rest}}^\dagger)^{\otimes 2} \right\|_1
\end{aligned}$$

Since the operator norm of a unitary U must satisfy $\|U\|_\infty = 1$, we can apply Hölder's inequality to obtain

$$\begin{aligned}
&\left\| \int d\mu_L U^{\otimes 2} \rho^{\otimes 2} (U^\dagger)^{\otimes 2} - \int d\mu_L (U_A U)^{\otimes 2} \rho^{\otimes 2} (U^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
\leq &\int d\mu_{\text{rest}} \left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
= &\left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 .
\end{aligned} \tag{20}$$

We have thus reduced the definition to a condition explicitly independent of the total system size, as desired.

C. Bounds for product state input

We now consider the simplest case, in which we are given two copies of a product state as input. We observe that U_A must commute with terms that correspond to configurations whose last layer contain all identity or all swap in the region A. Our strategy is therefore to apply the triangle inequality to the right hand side of Eqs. 16, 20 and bound the coefficients of the contributing configurations via a mapping between the set of configurations with a domain wall in A and the set of all configurations; we will then independently bound the sum of all configuration weights by adapting an anti-concentration result previously proved in [3].

We begin by establishing an upper bound for the quantity of interest in terms of the sum of all trapezoidal lattice configuration weights:

Theorem 2 (Correspondence counting bound) *Consider a trapezoidal circuit of depth L which forms the lightcone of a region A. Let Γ denote all trapezoidal lattice configurations, and $W(\gamma)$ denote the weight assigned to a given configuration γ. Then the right hand side of Eq. 20 can be bounded in terms of a sum over all configurations as follows:*

$$\left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \leq 2|A| q^{2|A|+4L} \left(\frac{2q}{q^2+1} \right)^{L-1} \sum_{\gamma \in \Gamma} W(\gamma).$$

Proof. Using the weight function, we can rewrite the quantity of interest as

$$\left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 = \left\| \sum_{\gamma \in \Gamma} W(\gamma) \left[\gamma - U_A^{\otimes 2} \gamma (U_A^\dagger)^{\otimes 2} \right] \right\|_1 .$$

Recall that for any U_A which acts trivially on $\overline{A}$ and $\gamma = \sigma^{\otimes |A|} \otimes \gamma_{\overline{A}}$, we have $U_A^{\otimes 2} \gamma (U_A^\dagger)^{\otimes 2} = \gamma$. Then letting Γ^* denote trapezoidal lattice configurations with a domain wall in A, we can simplify this as

$$\begin{aligned} &\left\| \sum_{\gamma \in \Gamma^*} W(\gamma) \left[\gamma - U_A^{\otimes 2} \gamma (U_A^\dagger)^{\otimes 2} \right] \right\|_1 \\ \leq &\sum_{\gamma \in \Gamma^*} W(\gamma) \left[\|\gamma\|_1 + \left\| U_A^{\otimes 2} \gamma (U_A^\dagger)^{\otimes 2} \right\|_1 \right] \\ \leq & \, 2 \sum_{\gamma \in \Gamma^*} W(\gamma) \|\gamma\|_1, \end{aligned}$$

where we have again applied Hölder's inequality on U_A. Then since the single-qudit $\mathbb{1}$ and S each have q^2 eigenvalues with magnitude 1, we have

$$\left\| \int d\mu_* U_*^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger)^{\otimes 2} - \int d\mu_* (U_A U_*)^{\otimes 2} \rho^{\otimes 2} (U_*^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \leq 2q^{2|A|+4L} \sum_{\gamma \in \Gamma^*} W(\gamma). \tag{21}$$

We now construct a mapping between $F : \Gamma^* \to \Gamma$ as follows: for any $\gamma_1 \in \Gamma^*$, we map it to the configuration $\gamma_0 \in \Gamma$ constructed by deleting the first domain wall that ends in A. Then consider the preimage $F^{-1} : \Gamma \to \Gamma^*$. Note that

$$\bigcup_{\gamma_0 \in \Gamma} F^{-1}(\gamma_0) = \Gamma^*, \tag{22}$$

so that all contributing configurations are counted by the union of preimages. We observe that for any $\gamma_0 \in \Gamma$, we can choose at most $|A|$ locations to add a domain wall, and at most two ways to propagate it backwards for each layer. Since domain walls are forbidden from annihilating in the direction of propagation, for any $\gamma_1 \in F^{-1}(\gamma_0)$, we must have

$$W(\gamma_1) = W(\gamma_0) \left(\frac{q}{q^2+1} \right)^{L-1} .$$

We can therefore bound the right hand side of Eq. 21 via

$$\begin{aligned} 2q^{2|A|+4L} \sum_{\gamma \in \Gamma^*} W(\gamma) &\leq 2q^{2|A|+4L} \sum_{\gamma_0 \in \Gamma} \sum_{\gamma \in F^{-1}(\gamma_0)} W(\gamma) \\ &\leq 2q^{2|A|+4L} \sum_{\gamma \in \Gamma} |A| \left(\frac{2q}{q^2+1} \right)^{L-1} W(\gamma). \end{aligned} \tag{23}$$

It is not immediately obvious that the right hand side of Eq. 23 should decrease in L; however, since the second moment must approach that of the Haar distribution, it can be shown by applying the results of [3] that these weights are in fact exponentially small in the size of the lightcone.

Lemma 1 *For a trapezoidal circuit with depth L and terminal region A which is initialized to a product state input, the sum of lattice configuration weights is bounded by a constant times $1/q^{|A|+2L}(q^{|A|+2L}+1)$,* *and satisfies*

$$\sum_{\gamma\in\Gamma} W(\gamma) \leq \frac{2z}{q^{|A|+2L}(q^{|A|+2L}+1)}\left(1+(e-1)(|A|+2L)\left(\frac{2q}{q^2+1}\right)^{L-1}\right)\exp\left[\frac{1}{1-(2q/(q^2+1))^2}\right],$$

where we have defined $z := 1+\left(\frac{q}{q^2+1}\right)^{L-1}$.

Proof. Consider the set Γ' which consists of all trapezoidal lattice configurations which contain only domain walls that exits through the side of the trapezoid, the set Γ_0 of all trapezoidal lattice configurations which do not contain a domain wall exiting through the side, as well as the set Γ_r which consists of all linear ring circuit configurations with dimension $|A|+2L$ and depth L. For any trapezoidal configuration $\gamma \in \Gamma$, we can decompose γ into unique $\gamma' \in \Gamma'$ and $\gamma_0 \in \Gamma_0$, with $W(\gamma) = W(\gamma')W(\gamma_0)$. Thus we have that

$$\sum_{\gamma\in\Gamma} W(\gamma) \leq \sum_{\gamma'\in\Gamma'} W(\gamma') \sum_{\gamma_0\in\Gamma_0} W(\gamma_0). \tag{24}$$

We first show that $\sum W(\gamma')$ is bounded even as L increases to infinity. Note that for each layer from $t=1$ to L, we may have domain walls exiting from neither, one, or both sides of the trapezoid. Thus the sum of weights from exiting domain walls is upper bounded by

$$\begin{aligned}\sum_{\gamma'\in\Gamma'} W(\gamma') &\leq \prod_{t=0}^{\infty}\left(1+\left(\frac{2q}{q^2+1}\right)^t\right)^2 \\ &\leq \prod_{t=0}^{\infty}\exp\left[\left(\frac{2q}{q^2+1}\right)^{2t}\right] \\ &= \exp\left[\sum_{t=0}^{\infty}\left(\frac{2q}{q^2+1}\right)^{2t}\right] \\ &= \exp\left[\frac{1}{1-(2q/(q^2+1))^2}\right]. \end{aligned}\tag{25}$$

We now consider the relationship between $\sum W(\gamma_0)$ and $\sum W(\gamma_r)$. Note that due to periodic boundary conditions, Γ_r contains only configurations with an even number of domain walls in each layer. Since domain wall parity is conserved, we can consider two separate cases; one in which $\gamma_0 \in \Gamma_0$ has an even number of domain walls, and one in which it has an odd number. We construct a mapping $F_r : \Gamma_0 \to \Gamma_r$ as follows: if $\gamma_0 \in \Gamma_0$ is even, it is mapped to the unique configuration $\gamma_r \in \Gamma_r$ with the rest of the circuit filled in such that no domain walls are added, and $W(\gamma_0) = W(\gamma_r)$. Otherwise, it is mapped to the unique configuration $\gamma_r \in \Gamma_r$ such that one domain wall is added along the upper side of the trapezoid. Thus each configuration $\gamma_r \in \Gamma_r$ is mapped to by at most one odd or one even configuration, and

$$\sum_{\gamma_0\in\Gamma_0} W(\gamma_0) \leq \left[1+\left(\frac{q}{q^2+1}\right)^{L-1}\right]\sum_{\gamma_r\in\Gamma_r} W(\gamma_r). \tag{26}$$

Finally, we can rewrite the norm of the output for the ring lattice and apply Eq. 12:

$$\begin{aligned}&\frac{1}{q^{|A|+2L}}\left\|\int d\mu_r U_r^{\otimes 2}(|0\rangle\langle 0|)^{\otimes 2}(U_r^\dagger)^{\otimes 2}\right\|_1 \\ &= \frac{1}{q^{|A|+2L}}\operatorname{Tr}\left|\int d\mu_r U_r^{\otimes 2}(|0\rangle\langle 0|)^{\otimes 2}(U_r^\dagger)^{\otimes 2}\right| \\ &= \frac{1}{q^{|A|+2L}}\sum_{i=0}^{q^{|A|+2L}-1}\int d\mu_r\, \langle i|^{\otimes 2}\, U_r^{\otimes 2}(|0\rangle\langle 0|)^{\otimes 2}(U_r^\dagger)^{\otimes 2}\,|i\rangle^{\otimes 2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q^{|A|+2L}} \sum_{i=0}^{q^{|A|+2L}-1} E_{\mu_r}[p(|i\rangle)^2] = E_{\mu_r}[p(|0\rangle)^2] \\
&= \frac{1}{q^{|A|+2L}} \sum_{i=0}^{q^{|A|+2L}-1} \langle i|^{\otimes 2} \left(\sum_{\gamma_r \in \Gamma_r} W(\gamma_r)\gamma_r \right) |i\rangle^{\otimes 2} = \sum_{\gamma_r \in \Gamma_r} W(\gamma_r) \\
&\leq \frac{2}{q^{|A|+2L}(q^{|A|+2L}+1)} \left(1 + (e-1)(|A|+2L) \left(\frac{2q}{q^2+1} \right)^{L-1} \right).
\end{aligned} \tag{27}
$$

Substituting into Eq. 24 yields the desired result.

We now apply the previous results to give an upper bound for local scrambling of a product state input by a random circuit.

Theorem 3 (Product state scrambling) *Consider a linearly connected system which is initialized to a product state. In the regime in which the circuit depth and size of the region are small compared to the size of the entire system, a connected subsystem A is scrambled by a local random circuit in $O(\log |A|)$ depth.*

Proof. Substituting Theorems 1-2 and Lemma 1 into Definition 1, we have

$$
\begin{aligned}
&\left\| \int d\mu_L U^{\otimes 2} (|0\rangle\langle 0|)^{\otimes 2} (U^\dagger)^{\otimes 2} - \int d\mu_L (U_A U)^{\otimes 2} (|0\rangle\langle 0|)^{\otimes 2} (U^\dagger U_A^\dagger)^{\otimes 2} \right\|_1 \\
&\leq \frac{4|A|szq^{2|A|+4L}}{q^{|A|+2L}(q^{|A|+2L}+1)} \left(\frac{2q}{q^2+1} \right)^{L-1} \left(1 + (e-1)(|A|+2L) \left(\frac{2q}{q^2+1} \right)^{L-1} \right) \\
&\leq 4|A|sz \left(\frac{2q}{q^2+1} \right)^{L-1} \left(1 + (e-1)(|A|+2L) \left(\frac{2q}{q^2+1} \right)^{L-1} \right),
\end{aligned} \tag{28}
$$

where for simplicity we have defined

$$
z := 1 + \left(\frac{q}{q^2+1} \right)^{L-1}, \quad s := \exp \left[\frac{1}{1 - (2q/(q^2+1))^2} \right]. \tag{29}
$$

Taking $q \geq 2$, for fixed $|A|$ and $L \gg 1/(\ln(q^2+1) - \ln 2q)$ the right hand side of Eq. 28 scales as

$$
O\left(|A| \left(\frac{2q}{q^2+1} \right)^{L-1} \right).
$$

Thus in the regime where the circuit is shallow relative to the size of the system, the depth required to locally scramble a product state input up to some $\epsilon > 0$ scales as $O(\log |A|/\epsilon)$.

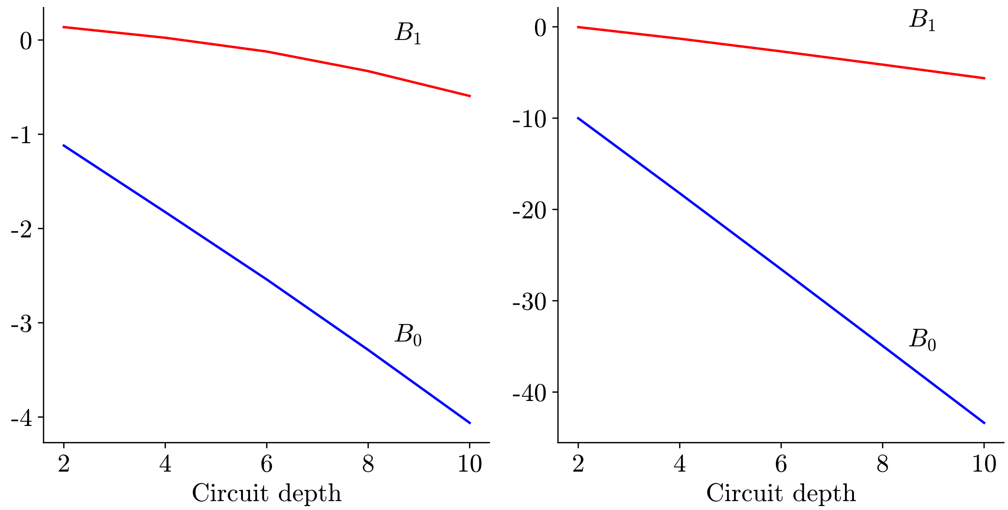
V. NUMERICAL RESULTS

While we have not provided an analytic lower bound for the local scrambling depth, we can infer the tightness of our bound and the true scaling of the quantity in Definition 1 from numerical simulations. In particular, we computed the exact quantities

$$
B_0 = \left\| \int d\mu_L U^{\otimes 2} (|0\rangle\langle 0|)^{\otimes 2} (U^\dagger)^{\otimes 2} \right\|_1, \quad B_1 = q^{2|A|+4L} \sum_{\gamma \in \Gamma^*} W(\gamma) \tag{30}
$$

for $q = 2, 4$ and $|A| = 4$, up to depth $L = 10$. Recall Γ^* is defined as the set of all configurations which have a domain wall in the region A and therefore contribute to the quantity in Definition 1, and W denotes the weights for a product state input. Note that here B_0 is computed for a ring architecture using $n = |A|+2(t-1)$ for the total size of the system at depth t; however the mixed channel analysis still applies.

We observe that B_0 appears to decrease exponentially in the depth of the circuit, as expected. On the other hand, B_1 appears to decay more slowly and does not approach B_0, suggesting that our analytic bound is fairly loose as it is an upper bound for B_1. However, we recover the asymptotically exponential decay, suggesting that the log depth scaling is correct. See Figure 4 for a log plot of both bounds with circuit depth.



(a) $q = 2, |A| = 4$

(b) $q = 4, |A| = 4$

FIG. 4: Comparison of the natural logarithm of numerically computed values for each of the bounds given in Eq. 30, with two different values of q

VI. DISCUSSION

We have shown that up to the second moment, a large system with no initial entanglement is locally scrambled by a random circuit in log depth, i.e. the depth required scales asymptotically with the logarithm of the size of the region of interest. Based on related work with the statistical mechanics mapping, we conjecture that this result generalizes to more general states, as well as higher moments. We further note that in the regime in which the circuit is not shallow relative to the size of the system, the analysis becomes similar to that for the anti-concentration result in [3].

We emphasize that this notion of state scrambling should be distinguished from operator scrambling, which is often defined using approximate designs and is expected to occur in linear depth in random circuits. One potential reason to expect the difference in scaling is the possibility of operators which entangle the two copies of the Hilbert space in the definition of a unitary design. We can consider the decomposition of any such operator into permutations on the copies of the Hilbert space, and unitary transformations on each of the copies. The same instance of a random circuit is applied to each copy, and thus cannot directly "undo" the entanglement between the different copies of the Hilbert space.

The formal definition of local scrambling presented in this paper can be applied to a variety of problems, offering potential directions for future work:

- State scrambling can be applied to study the properties of topological order. In particular, topological phases can be defined as the region of the Hilbert space which can be accessed by varying a Hamiltonian for a finite time, which is modeled by a shallow random circuit. It is known that two topological phases cannot be distinguished by any single measurement on a state [16]; more precisely, the expectation value of any operator on the two topological phases must be equal since states are scrambled in constant depth up to the first moment. Our results suggest that when more than one copy of the state is provided, the ability to distinguish two phases is limited by the localization of the measurement.

- Classical shadows are constructed using an ensemble of random unitary transformations, and can be used to efficiently reconstruct certain properties of a quantum state depending on the types of transformations performed [5]. In particular, the amount of long-range entanglement generated limits the measurement of long-range correlations in the system. Although strategies for the two limiting cases of measuring single-qudit and global properties are understood, recent work such as [15] has focused on interpolating between the two cases by applying finite-depth random circuits; importantly, the ensemble chosen to measure a local property only needs to locally scramble the information contained in the state. Although our results are not directly applicable to the rate of convergence of classical shadow tomography, which depends on a third-moment quantity, we conjecture that a similar circuit depth scaling may be achieved.

- Techniques such as the statistical mechanics mapping can be extended to studying scrambling in certain physical systems with conserved quantities; in particular, the convergence of different coefficients can be analyzed for any subgroup of unitary operators which is equipped with a Weingarten calculus. This strategy is complementary to hydrodynamics approaches such as [17].

Finally, although this work does not generalize directly to the diamond norm bound for channels, it may be possible to apply similar techniques to analyze a local form of scrambling on operators. In particular, it is expected that the "local" form of the approximate design property will be satisfied before the system is completely scrambled, requiring depth scaling linearly with the size of a local region.

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