18.S097 Problem Set #1

Due: Friday, January 8 at 11:59 PM EST

Feel free to work with other students, but make sure to submit your own solutions and list all collaborators. Note there may be multiple solutions for any given problem. Unless stated otherwise, you may cite any results or theorems from lecture – the notes can be found on the Canvas page. If you're confused about any of the expectations or notation, feel free to come to office hours or ask us by email!

1. Write the following using (i) set builder notation and (ii) the intersection, union, and/or complement of sets, by defining $A = \{\text{the set of all apples}\}$, $B = \{\text{the set of all blue things}\}$, $C = \{\text{the set of all colorful things}\}$, $G = \{\text{the set of all green things}\}$.
   (a) The set of all apples that are green
   (b) The set of all things that are an apple, not blue, and colorful
2. Write the following using predicate and/or quantifier logic, using the same definitions of A, B, C, G as in 1:
   (a) If something is an apple, it is not blue.
   (b) Anything that is blue is colorful.
   (c) If all apples are green, then no apples are blue.
3. Describe the following sets in English, using the same definitions of A, B, C, G as in 1:
   (a) $\{x \mid x \in C \cap (A - G) \cup B\}$
   (b) $\{x \mid x \in \mathbb{Z} \wedge x^2 + 5x + 4 > 0\}$
4. (More on truth tables) Truth tables are a useful tool in logic as well as the design of Boolean circuits in computer science. A truth table for a proposition has *input* columns for all possible combinations of truth values of all of the predicate variables that appear, as well as any intermediate truth values we need to evaluate the entire expression. Here's an example on organizing your truth table for $P \to \neg Q$:

P	Q	$\neg Q$	$P \to \neg Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

(a) (De Morgan's Laws) Use truth tables to show that (i) $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ and (ii) $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$. (Bonus: write down the equivalent expressions using set intersections, unions, and complements.)

(b) (Law of Contrapositives) Use truth tables to show that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$.

(c) Use truth tables to show that $(\neg P \wedge Q) \to (Q \vee R)$ is a *tautology*, that is, it is true regardless of the truth values of the predicates P, Q, R.

Remark: truth tables are especially important for simplifying the structure of a propositional statement. When we build computers, we often want to use as few logical steps as possible both to take up less space and to speed up computation. Here, we can replace four "steps" (not, and, or, implies, etc.) with a single true input!

The questions below ask for **fully rigorous proofs**, so the focus is on being precise with mathematical language and using axioms, definitions, and assumptions clearly. (So if you're not sure whether you can assume something, cite it!) Try to write out your reasoning as clearly as you can, and we'll do our best to give feedback in grading (on the mathematics, but also importantly your writing style and rigor).

5. Taking the notation from lecture 2, prove that $S(0) = 1$ is both the left and right multiplicative identity, i.e. $a \cdot 1 = 1 \cdot a = a$ for any $a \in \mathbb{N}$. Hint: to show that $S(0)$ is the left multiplicative identity, use induction.

6. Using induction, prove that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. (Make sure to include all of the different components of an inductive proof!)

7. The **factorial** of n, denoted $n!$, is the product $1 \cdot 2 \cdot \cdots \cdot n$. (For example, $3! = 3 \cdot 2 \cdot 1 = 6$.)

   (a) Prove that $n! > 2^n$ for all $n \geq 4$.

   (b) (Challenge problem) Prove that for any positive integer k, we have $n! > k^n$ for large enough n. In other words (this is a good concept to understand for future lectures!), show that $n! > k^n$ as long as $n \geq C_k$, for a constant C_k depending on k.

8. (Challenge problem) Prove the well-ordering principle (WOP) is equivalent to the axiom of induction. (Hint: to prove the bijection, you must prove that WOP with the other Peano axioms implies the axiom of induction, and that the Peano axioms imply WOP.)