The Quantized Electromagnetic Field

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The quantization of the electromagnetic field has vast implications for interactions between matter and radiation, and is a necessary step in the development of modern quantum field theory. This article presents the derivation of the Hamiltonian of the field by applying the framework of canonical quantization. The properties of the solutions are discussed, as well as the resulting consequences. Spontaneous emission is introduced as an example of an effect which cannot be explained by interactions between the atom and the electromagnetic field in the classical theory of radiation.

I. INTRODUCTION

Although Planck first introduced the concept of quantized radiation to model blackbody radiation [1], the photon itself is often attributed to Einstein, who explained the photoelectric effect by assuming "heuristically" that light was composed of independent quanta with energy given by

$$E = h\nu \tag{1}$$

This example is often cited to illustrate the "wave-particle duality" in modern physics pedagogy. [2] However, the theory of quantum mechanics based on the Schrödinger equation does not impose such postulates on the electromagnetic field, and relies on classical field equations to describe the behavior of massive particles. [3] In fact, inelastic scattering phenomena such as the photoelectric effect can be accurately described within the framework of quantum mechanics without the need for a quantized field.

However, this model fails for spontaneous emission, in which a system initially in an excited state transitions to a state with lower energy, emitting a photon in the process. The inability of semiclassical quantum mechanics to predict this phenomenon is made clear by considering the Hamiltonian of a hydrogenic atom in the absence of external fields:

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{\hat{r}^2} + \hat{H}'$$

where H' represents terms corresponding to the fine and hyperfine structure. Since the eigenstates of the hydrogenic atom are orthogonal, the probability of transitions between different energy levels is apparently zero. Yet it was well-known that all excited atoms decay spontaneously to their ground states [4]; thus, the classical description of the electromagnetic field cannot be correct.

Although the rates of spontaneous and stimulated emission can be calculated using the concept of statistical equilibrium and were predicted earlier [5], the dynamics of the process could not be explained until Dirac's derivation of the quantization of the electromagnetic field in 1927. [6]

In this paper we will apply a procedure known as *second quantization* to the field, so called to distinguish from *first quantization*, in which the physical quantities position and momentum are promoted to quantum operators. We will then derive the Hamiltonian of the electromagnetic field in free space, as well as discuss the properties of the resulting solutions. Finally, we will revisit spontaneous emission and explore other consequences of a quantized field theory.

II. CANONICAL FORMALISM

We begin by reexamining the classical Hamiltonian formalism. Consider an arbitrary system whose state at any point in time is fully specified by the generalized coordinates $\mathbf{q}$ and its conjugate momentum $\mathbf{p}$, and whose energy is given by $H(\mathbf{q}, \mathbf{p}, t)$. The coordinates satisfy Hamilton's equations:

$$\frac{dq_i}{dt} = \{q_i, H\} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

where the Poisson bracket is defined by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Furthermore, it can be shown that these coordinates satisfy the relations

$$\{q_i, q_j\} = 0,$$
$$\{p_i, p_j\} = 0,$$
$$\{q_i, p_j\} = \delta_i j$$

Following the procedure of canonical quantization, we promote classical variables to operators by imposing the mapping

$$\{f, g\} \mapsto \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

giving us the familiar commutation relations and algebraic structure of quantum mechanics. [9]

III. SECOND QUANTIZATION

A. Expansion in free space

We are interested in a description of the electromagnetic field in free space. We begin with Maxwell's equations in a vacuum:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{2}$$

In order to simplify these equations, the fields are written in terms of the electromagnetic potentials:

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \end{aligned}$$

It is convenient to adopt the Coulomb gauge, specified by $\nabla \cdot \mathbf{A} = 0$. From (2), we have Φ is constant; we can therefore choose $\Phi = 0$. We are then left with

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{3}$$

The solutions to Maxwell's equations in a vacuum must be a linear combination of plane waves of the form $e^{i\mathbf{k}\cdot\mathbf{x}}$. We can impose boundary conditions by working inside a finite box of dimension L, giving us:

$$k_i = \frac{2\pi n_i}{L}$$

Since we have chosen $\mathbf{A}$ to be a divergence-free field, it can be written as a spatial Fourier expansion over normal modes in the transverse direction:

$$\mathbf{A} = \sum_\ell \boldsymbol{\varepsilon}_\ell \tilde{A}_\ell(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\boldsymbol{\varepsilon}_\ell(\mathbf{k})$ denotes the polarization. By convention, $\boldsymbol{\varepsilon}_\ell \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_x \pm \boldsymbol{\varepsilon}_y)$, so that the two $\boldsymbol{\varepsilon}_i$ are orthogonal to each other and to $\mathbf{k}$. Using Eqs. 2, 3 we obtain the expansion of the electric and magnetic fields as well:

$$\begin{aligned} \mathbf{E} &= \sum_\ell \boldsymbol{\varepsilon}_\ell \tilde{E}_\ell(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \sum_\ell -\boldsymbol{\varepsilon}_\ell \left(\frac{d}{dt}\tilde{A}_\ell(\mathbf{k}, t)\right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{B} &= \sum_\ell ik\boldsymbol{\varepsilon}'_\ell \tilde{A}_\ell(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$

where $\boldsymbol{\varepsilon}'_\ell = \frac{1}{k}(\mathbf{k} \times \boldsymbol{\varepsilon}_\ell)$. Finally, substituting the coefficients $\tilde{A}_{\mathbf{k}}$ back into Maxwell's equations yields

$$\begin{aligned} \frac{d}{dt}\tilde{A}_\ell &= -\tilde{E}_\ell \\ \frac{d}{dt}\tilde{E}_\ell &= \omega_\ell^2 \tilde{A}_\ell \end{aligned}$$

where $\omega_\ell = ck_\ell$.

B. Quantization of the electromagnetic field

We now define the quantity

$$a_\ell \equiv \omega_\ell \tilde{A}_\ell - i\tilde{E}_\ell$$

allowing us to define the conjugate variables

$$\begin{aligned} Q_\ell &\equiv \sqrt{\frac{1}{2\omega_\ell}}(a_\ell + a_\ell^*) \\ P_\ell &\equiv -i\sqrt{\frac{\omega_\ell}{2}}(a_\ell - a_\ell^*) \end{aligned}$$

Note that by construction we have $\{Q_\ell, Q_{\ell'}\} = \{P_\ell, P_{\ell'}\} = 0$, since the normal modes are mutually independent. We then promote Q_ℓ and P_ℓ to quantum operators by imposing the quantization condition

$$[\hat{Q}_\ell, \hat{P}_{\ell'}] = i\hbar\delta_{\ell,\ell'} \tag{4}$$

Furthermore, we will set $[\hat{a}_\ell, a_{\ell'}^\dagger] = \delta_{\ell,\ell'}$, which yields

$$\begin{aligned} \hat{a}_\ell &= \sqrt{\frac{\omega_\ell}{2\hbar}}\hat{Q} + i\sqrt{\frac{1}{2\hbar\omega_\ell}}\hat{P} \\ \hat{a}_\ell^\dagger &= \sqrt{\frac{\omega_\ell}{2\hbar}}\hat{Q} - i\sqrt{\frac{1}{2\hbar\omega_\ell}}\hat{P} \end{aligned} \tag{5}$$

These operators are analogous to the ladder operators of the harmonic oscillator.

C. Free space hamiltonian

We return briefly to the classical electromagnetic field:

$$\begin{aligned} \mathbf{A} &= \sum_\ell \frac{1}{\omega_\ell}\boldsymbol{\varepsilon}_\ell[a_\ell e^{i\mathbf{k}\cdot\mathbf{x}} + a_\ell^* e^{-i\mathbf{k}\cdot\mathbf{x}}] \\ \mathbf{E} &= \sum_\ell i\boldsymbol{\varepsilon}_\ell[a_\ell e^{i\mathbf{k}\cdot\mathbf{x}} - a_\ell^* e^{-i\mathbf{k}\cdot\mathbf{x}}] \\ \mathbf{B} &= \sum_\ell \frac{i}{c}\boldsymbol{\varepsilon}'_\ell[a_\ell(t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_\ell^* e^{-i\mathbf{k}\cdot\mathbf{x}}] \end{aligned} \tag{6}$$

These expansions can be substituted into the classical energy density:

$$\begin{aligned} H &= \frac{\epsilon_0}{2}\int_V d^3x \left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\ &= -\frac{\epsilon_0}{2}\sum_{\ell,\ell'}\int_V d^3x \Big[\boldsymbol{\varepsilon}_\ell \cdot \boldsymbol{\varepsilon}_{\ell'}(a_\ell e^{i\mathbf{k}\cdot\mathbf{x}} - a_\ell^* e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ &\qquad (a_{\ell'} e^{i\mathbf{k}'\cdot\mathbf{x}} - a_{\ell'}^* e^{-i\mathbf{k}'\cdot\mathbf{x}}) + \\ &\qquad \boldsymbol{\varepsilon}'_\ell \cdot \boldsymbol{\varepsilon}'_{\ell'}(a_\ell e^{i\mathbf{k}\cdot\mathbf{x}} + a_\ell^* e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ &\qquad (a_{\ell'} e^{i\mathbf{k}'\cdot\mathbf{x}} + a_{\ell'}^* e^{-i\mathbf{k}'\cdot\mathbf{x}})\Big] \\ &= \epsilon_0 V \sum_\ell (a_\ell a_\ell^* + a_\ell^* a_\ell) \end{aligned}$$

where we have used the fact that the modes are orthogonal unless $\mathbf{k}' = \mathbf{k}, \ell' = \pm\ell$, as well as the convention that $\boldsymbol{\varepsilon}'_\ell = -\boldsymbol{\varepsilon}'_{-\ell}$. [10] We now fix the mapping

$$a_\ell(\mathbf{k},t) \mapsto \sqrt{\frac{\hbar}{2\omega_\ell \epsilon_0 V}}\hat{a}_\ell$$

Using Eq. 4, we can then write the analogous quantum Hamiltonian in the form

$$\hat{H} = \sum_\ell \hbar\omega_\ell \left(\hat{a}_\ell^\dagger \hat{a}_\ell + \frac{1}{2}\right) \tag{7}$$

IV. SOLUTIONS OF THE FREE HAMILTONIAN

We have shown that the free space Hamiltonian of the electromagnetic field can be written as a sum of oscillators in independent modes. The eigenvectors can therefore be represented by a tensor product of the oscillator states $|n_\ell\rangle$ over all modes, taking the form $|n_1, n_2, n_3, ...\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes ....$ As with the harmonic oscillator, we can apply the operators $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$ to raise and lower the energy of the state.

The photon thus emerges as the smallest possible increment to the energy of the field. The aptly named creation and annihilation operators can be thought of as inserting or deleting a photon from the field, respectively; note that we thus recover the relationship between energy and frequency given by (1). Furthermore, we can easily verify that the expectation values of the fields are zero for the ground state, which corresponds to the electrodynamic vacuum.

We can consider photons to be particles insofar as they undergo collisions with other particles and have properties analogous to those of macroscopic objects. However, unlike particles in semiclassical quantum mechanics, photons are freely emitted or absorbed, and particle number is no longer conserved. As we have defined in this paper, the photon corresponds to an increment in the intensity of a plane wave solution to Maxwell's equations. In this sense, the photon cannot be localized, and does not have a well-defined position space representation, though we may identify its contribution to the field by the plane wave

$$\psi(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{3/2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

Notice that this definition is distinct from that used in experimental quantum optics, in which the photon is taken to be a particle with finite spread in position and momentum. In this context, the photon typically denotes a coherent state or other minimum-uncertainty superposition of a range of frequencies. This type of photon therefore has no definite wavelength or energy, but can be thought to travel along a path approximating a classical trajectory through space.

A. Photon momentum

To obtain an expression for the momentum of a photon, we substitute (6) into the classical momentum of electromagnetic radiation, which is proportional to the volume integral of the Poynting vector:

$$\begin{aligned}\mathbf{P}_r &= \epsilon_0 \int_V d^3x\, \mathbf{E} \times \mathbf{B} \\ &= \frac{\epsilon_0 V}{c}\sum_\ell \frac{\mathbf{k}}{k}(a_\ell a_\ell^* + a_\ell^* a_\ell + a_\ell a_{-\ell} + a_\ell^* a_{-\ell}^*)\end{aligned}$$

The last two terms cancel out when summing over $\pm\ell$, since $\mathbf{k}_{-\ell}$ carries the opposite sign. We therefore have

$$\mathbf{P}_r = \epsilon_0 V \sum_\ell \frac{\mathbf{k}}{\omega_\ell}(a_\ell a_\ell^* + a_\ell^* a_\ell)$$

which maps to the quantum momentum operator

$$\hat{\mathbf{P}}_r = \sum_\ell \hbar\mathbf{k}\hat{a}_\ell^\dagger \hat{a}_\ell \tag{8}$$

where we have again used the fact that the $\mathbf{k}_{\pm\ell}$ are of opposite sign to cancel the $1/2$. Experimental observations confirm that photon momentum is indeed conserved. [2]

B. Photon mass

Given our conception of the photon as a particle, it is reasonable to ask whether it has mass. However, it is straightforward to show that the rest mass of the particle is in fact 0.

The relativistic energy and momentum are given respectively by

$$\begin{aligned}E^2 &= \frac{m_0^2 c^4}{1 - v^2/c^2}, \\ p^2 &= \frac{m_0^2 v^2}{1 - v^2/c^2}\end{aligned}$$

where m_0 is the rest mass. We can solve for the mass, using the expression for the momentum from (8):

$$\begin{aligned}m_0^2 c^4 &= E^2 - p^2 c^2 \\ &= \hbar^2\omega_\ell^2 - c^2\hbar^2 k^2 \\ &= 0\end{aligned}$$

Alternatively, since the energy is proportional to the momentum, the mass of the photon is not well-defined according to classical mechanics. Note that the photon therefore cannot be described nonrelativistically, which matches our intuition.

C. Infinite self-energy

Upon closer inspection, it is apparent that the ground state energy diverges, since we must sum over all possible modes. This model thus carries the implication that the energy density at every point in space is infinite. We offer an interpretation as well as a couple of remarks on this result:

1. It is only possible to measure differences between energy levels of a system, rather than its "absolute" energy.

2. In practice, the ground state energy can be subtracted from calculations to produce a finite result. The insertion and deletion of a finite number of photons will always result in a finite increment in energy.

V. SPONTANEOUS EMISSION

We will now calculate the rate of spontaneous emission and show that it matches Einstein's result derived using the principle of statistical equilibrium. [5] We again consider an atom placed in a vacuum, so that the expected value of the electric and magnetic fields are zero. However, unlike in the semiclassical picture, we cannot take $\hat{\mathbf{A}}$ to be identically zero in the Coulomb gauge. The total energy of the system is given by

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\hat{\mathbf{v}}^2 + \frac{\epsilon_0}{2}\int_V d^3x\left(\hat{\mathbf{E}}^2 + c^2\hat{\mathbf{B}}^2\right)\\ &= \frac{1}{2m}(\hat{\mathbf{p}} - q\hat{\mathbf{A}})^2 + \frac{\epsilon_0}{2}\int_V d^3x\left(\hat{\mathbf{E}}^2 + c^2\hat{\mathbf{B}}^2\right)\end{aligned}$$

From (7), the contribution from the transverse components of the field is given by

$$\hat{H}_\perp = \sum_\ell \hbar\omega_\ell\left(\hat{a}_\ell^\dagger\hat{a}_\ell + \frac{1}{2}\right)$$

In the presence of free charges, we must also account contributions from the longitudinal components of the electromagnetic field:

$$\hat{H}_\parallel = \frac{\epsilon_0}{2}\int_V d^3x\left|\hat{\mathbf{E}}\right|_\parallel^2 = \hat{V}_e$$

We thus have

$$\begin{aligned}\hat{H} &= \frac{1}{2m}(\hat{\mathbf{p}} - q\hat{\mathbf{A}})^2 + \hat{V}_e + \sum_\ell \hbar\omega_\ell\left(\hat{a}_\ell^\dagger\hat{a}_\ell + \frac{1}{2}\right)\\ &= \frac{1}{2m}\hat{\mathbf{p}}^2 + \left[\frac{q^2}{2m}\hat{\mathbf{A}}^2 - \frac{q}{m}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}\right]\\ &\quad + \left[\hat{V}_e + \sum_\ell \hbar\omega_\ell\left(\hat{a}_\ell^\dagger\hat{a}_\ell + \frac{1}{2}\right)\right]\\ &= \hat{H}_a + \hat{H}_I + \hat{H}_r\end{aligned}$$

where $\hat{H}_I$ represents the interaction between the system and the electromagnetic field, while $\hat{H}_a$ and $\hat{H}_r$ act only on the electron and the field, respectively. We now expand each of the terms in $\hat{H}_I$:

$$\begin{aligned}\hat{H}_{I1} &\equiv -\frac{q}{m}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}\\ &= -\frac{q}{m}\sum_\ell\sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell V}}\hat{\mathbf{p}}\cdot\boldsymbol{\varepsilon}_\ell(\hat{a}_\ell + \hat{a}_\ell^\dagger),\\ \hat{H}_{I2} &\equiv \frac{q^2}{2m}\hat{\mathbf{A}}^2\\ &= \frac{q^2\hbar}{4m\epsilon_0 V}\sum_{\ell,\ell'}\frac{\boldsymbol{\varepsilon}_\ell\cdot\boldsymbol{\varepsilon}_{\ell'}}{\sqrt{\omega_\ell\omega_{\ell'}}}(\hat{a}_\ell\hat{a}_{\ell'} + \hat{a}_\ell\hat{a}_{\ell'}^\dagger + \hat{a}_\ell^\dagger\hat{a}_{\ell'} + \hat{a}_\ell^\dagger\hat{a}_{\ell'}^\dagger)\end{aligned}$$

Since $\hat{H}_{I1}$ is linear in $\hat{a}_\ell$, it must induce transitions between states of opposite parity, for which the number of photons differs by one; similarly, $\hat{H}_{I2}$ induces transitions of the same parity, in which the number of photons differs by two. The rate of spontaneous emission is therefore determined by the matrix elements of $\hat{H}_{I1}$. From Fermi's golden rule, we have

$$R_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f|\,\hat{H}_{I1}\,|i\rangle\right|^2$$

We will write the states in the form $|\psi; n_\ell\rangle = |\psi\rangle\otimes|n_\ell\rangle$, where $|\psi\rangle$ and $|n_\ell\rangle$ denote the electronic configuration of the atom and the number of photons in a specified mode, respectively. The relevant matrix elements are given by

$$\begin{aligned}&\langle\psi_f; n_\ell + 1|\,\hat{H}_{I1}\,|\psi_i; n_\ell\rangle\\ &= -\frac{q}{m}\sqrt{\frac{\hbar(n_\ell+1)}{2\epsilon_0\omega_\ell V}}\,\langle\psi_f|\,\hat{\mathbf{p}}\cdot\boldsymbol{\varepsilon}_\ell\,|\psi_i\rangle\end{aligned}$$

From the commutation relations, we obtain

$$\begin{aligned}[\hat{\mathbf{x}}, \hat{H}_a] &= \frac{i\hbar}{m}\hat{\mathbf{p}}\\ \langle\psi_f|\,\hat{\mathbf{p}}\,|\psi_i\rangle &= -i\omega_\ell m\,\langle\psi_f|\,\hat{\mathbf{x}}\,|\psi_i\rangle\end{aligned}$$

where we have used $E_f - E_i = \hbar\omega_\ell$, allowing us to write the matrix elements in the form

$$\langle f|\,\hat{H}_{I1}\,|i\rangle = iq\sqrt{\frac{\hbar\omega_\ell(n_\ell+1)}{2\epsilon_0 V}}\,\langle\psi_f|\,\hat{\mathbf{x}}\cdot\boldsymbol{\varepsilon}_\ell\,|\psi_i\rangle$$

The density of states in the mode is given by

$$\rho_\ell(\theta,\phi) = \frac{\omega_\ell^2 V}{8\pi^3\hbar c^3}$$

giving us the transition rate per solid angle

$$\begin{aligned}\frac{d\Gamma}{d\Omega} &= \frac{2\pi}{\hbar}\left|\langle f|\,\hat{H}_{I1}\,|i\rangle\right|^2\rho_\ell(\theta,\phi)\\ &= \frac{q^2}{2\pi}\frac{\omega_\ell^3}{\hbar c^3}(n_\ell+1)|\langle\psi_f|\,\hat{\mathbf{x}}\cdot\boldsymbol{\varepsilon}_\ell\,|\psi_i\rangle|^2\end{aligned}$$

Note that the transition rate explicitly depends on the number of photons available; atoms preferentially decay via more populated channels.

In order to compute the rate of spontaneous emission in a vacuum, we take $n_\ell = 0$. We now apply the dipole approximation and integrate over all angles to get

$$\begin{aligned} \frac{d\Gamma}{d\Omega} &= \frac{q^2}{2\pi}\frac{\omega_\ell^3}{\hbar c^3}|\langle\psi_f|\,\hat{\mathbf{x}}\,|\psi_i\rangle|^2\sin^2\theta \\ \Gamma &= \frac{4\alpha\omega_\ell^3|\langle f|\,\hat{\mathbf{x}}\,|i\rangle|^2}{3c^2} \end{aligned} \tag{9}$$

which agrees with Einstein's result. [11]

VI. DISCUSSION

The consequences of the quantized electromagnetic field are directly relevant to quantum optics, with modern applications in quantum information circuits. For example, the Jaynes-Cumming model of a two-level system in an optical cavity is central to experimental optics. One of the major predictions of the model is a ladder of quantized energy levels scaling with $\sqrt{n}$, a strictly quantum effect.

The non-classical nature of the field is also responsible for a variety of other behavior, famously including the Lamb shift and corrections to the magnetic dipole moment of the electron. In particular, the zero-point energy of the electromagnetic vacuum has consequences for physics at varying scales, from cosmology to nanoscale fabrication. For instance, it is now known that a net force can be observed between two objects even in the absence of a field, a phenomenon known as the Casimir effect. [7]

The quantization procedure presented in this paper can be generalized to other classical interactions. Although the electromagnetic field was the first to be treated this way, similar methods have been applied to quantum many-body systems, such as in condensed matter theory. These tools have thus enabled the study of a variety of quasiparticles which like the photon can be thought of as excitations in their respective fields.

Moreover, although we have presented a nonrelativistic framework for electrodynamics, the quantum formalism can be extended to produce the relativistic equations of motion. The accuracy of the resulting predictions has made quantum electrodynamics one of the great successes of modern physics; consequently, it has served as a model for subsequent field theories. The quantization of the electromagnetic field thus represents a decisive departure from the classical understanding of physics and the continuous nature of macroscopic phenomena.

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