Hartshorne Solutions

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Contents

Chapter 0

Introduction

Solutions are intended to only rely on previous material introduced in Hartshorne as well as some basic abstract algebra and category theory, though we may occasionally break this restriction (especially when citing commutative algebra results). In some cases we refer the reader to other texts or resources, principally the Stacks Project. We will also occasionally remark about when certain assumptions that Hartshorne makes may be discarded - Hartshorne is a big fan of Noetherian assumptions which aren't always necessary. These are not too offensive, but sometimes removing them gets a little hairy.

Next, a warning: if you are attempting to learn algebraic geometry on your own, please do not rely too much on this document! The struggle to understand a problem is key to the process of learning. Do not deprive yourself of that struggle by referring to this document too early in your frustrations with a particular problem. Instead, try smaller cases, toy examples, simplifications of the problem, and the like. Once you feel like you've exhausted yourself, wait a bit longer and then read the solution until you encounter one (1) new idea. See how far you can get using this one new idea, and then repeat. The goal is to learn, not just to know.

This work owes a lot to the old solutions guide that Joe Cutrone and Nick Marshburn had floating around the internet at one point.

Our numbering conventions are the same as Hartshorne's: chapters are given in Roman numerals, appendices using letters, and sections and exercises in Arabic numerals. If you want to search for problem 3 in section 2 of chapter I, you can search for 'I.2.3' inside this document.

0.1 Issues with exercises

Here we catalog exercises which have errors or missing/unstated assumptions, rely on significant extra material not found in the text, or are just very hard.

0.1.1 Chapter I

- Exercise I.4.9 needs a stronger version of the theorem of the primitive element which is not mentioned in the text.

- Exercise I.5.6 has some issues with the field characteristic.
- Exercise I.7.7 requires a decent amount of intersection theory to solve completely, especially in the characteristic p case.

0.1.2 Chapter II

- Exercise II.3.11 should really be after the section on quasi-coherent sheaves, but it can be done at the level of technology available in this section.
- Exercise II.4.5(c) is quite difficult. EGA has a proof using Chow's lemma (exercise II.4.10) for a different approach.
- Exercise II.4.10, Chow's lemma, is quite difficult.
- Exercise II.5.12 has issues with the definition of an immersion, which can be fixed by adding a (locally) noetherian assumption.
- Exercise II.7.7(b) is missing a characteristic not two assumption.
- Exercise II.8.7 is made much easier by knowing that if Y is noetherian and Y_{red} is affine, then Y is also affine. This is exercise III.3.1 - the statement can be proven without the material of cohomology by directly constructing a gluing, but the machinery of cohomology makes that process much, much easier.

0.1.3 Chapter III

- Exercise III.2.6 uses 'finitely generated' in at best a misleading fashion, but this can be dealt with.
- Exercise III.7.4(a) is incompatible with Hartshorne's previous definition of the dualizing sheaf and trace map - the trace map is only defined up to unique isomorphism, so any scaling will work just as well, and not all of them give the desired result for this problem. This ruins parts (b) and (d) of this exercise too. My recommendation: skip this problem.
- Exercise III.9.4, openness of the flat locus, is a bit ridiculous to assign as an exercise. It is long, difficult, and relies on lots of commutative algebra that Hartshorne has not featured much in the text. Even if you do decide to make your own attempt, this result is important enough that you should read a complete proof in a textbook afterwards.
- Exercise III.9.5 is tough: the goal is to prove that one can commute the operations of taking fibers and cones, but this requires some real work to prove each time.
- Exercise III.9.10(c) is difficult and I could not solve it without invoking results from section III.11.

- Exercise III.10.7 has an incorrect classification of the curves in the linear system $\mathfrak{d}$: there are three types, not two.
- Exercise III.10.8 is missing a characteristic not two assumption.
- Exercise III.11.4 is very long as stated, but could be considerably shortened if one assumed that the fibers were geometrically connected instead of just connected.
- Exercise III.12.3 is incorrect as stated: both h^i in the problem statement are actually 0.

0.1.4 Chapter IV

- Exercise IV.1.8(c) requires some outside material about excellent rings to commute the operations of normalization and completion.
- Exercise IV.2.3(b) is somewhat difficult. It is hard to get your hands on an equation for the dual curve, so other methods are needed.
- Exercise IV.3.3 would be slightly easier if one could cite exercise II.8.4(e) to say that $\omega_X \cong \mathcal{O}_X(\sum d_i - n - 1)$ for a complete intersection $X = \bigcap_{i=1}^{n-1}$ in $\mathbb{P}^n_k$, but that problem contains an assumption that all the intermediate intersections are nonsingular. Instead, we use some facts about Koszul resolutions to reach the same conclusion.
- Exercise IV.4.6(b) is relatively challenging.
- Exercise IV.4.8 needs the results of exercise IV.4.15, which while independent, does come later in the text.
- Exercise IV.4.19 is quite computationally intensive.
- Exercise IV.5.5(b) is incorrect - one must allow cusps as well.
- Exercise IV.5.7(b), finding the automorphisms of the Klein quartic, is quite difficult.
- Exercise IV.6.7 requires a different definition of multisecant than has been used in the text so far.
- Exercise IV.6.8 wants you to mimic the proof of proposition IV.6.1, except the proof in the book is rather rough. We end up using a fair amount of outside material to patch this up.

0.1.5 Appendix C

- Exercise C.5.5 requires a bit of background in symmetric polynomials and algebraic combinatorics which you might not have. It also contains a computation where the details are a bit miserable to write down and we've omitted for now.

Chapter I

Varieties

We work over an algebraically closed field k except where noted. Please note that we follow all of Hartshorne's definitions and conventions here (except we write V instead of Z for talking about the zeros of an ideal and other alterations where noted) - some of these definitions and conventions will change as we move later in the book, especially things relating to the definition of a 'variety'. I make special warning to the reader of this strategy because the transition between the viewpoint in chapter I and chapter II can present some real challenges.

I.1 Affine Varieties

Exercise I.1.1.

a. Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial in one variable over k.

b. Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

c. (*) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Solution.

a. The morphism of rings $k[x, y] \to k[x]$ given by sending $x \mapsto x$ and $y \mapsto x^2$ has kernel exactly $(y - x^2)$. Thus $k[x, y]/(y - x^2) \cong k[x]$.

b. The morphism of rings $k[x, y] \to k[x, x^{-1}]$ given by sending $x \mapsto x$ and $y \mapsto x^{-1}$ has kernel exactly $(xy - 1)$. Thus $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$. This is not isomorphic to any polynomial ring over k because it has invertible elements not in the base field (x and x^{-1}, for example).

c. We prove some auxiliary results before beginning. First, if k is algebraically closed, every homogeneous polynomial in two variables x, y splits completely in to a product of homogeneous linear factors. To see this, dehomogenize our polynomial with respect to y, and now we have a polynomial in one variable over an algebraically closed field which must split completely. Rehomogenizing, we see the result. Second, an endomorphism of $k[x, y]$ given by $x \mapsto ax + by, y \mapsto cx + dy$ is an automorphism if and only if $ad - bc \neq 0$. This is because such an endomorphism is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and composition is given by multiplication of matrices. As any translation $x \mapsto x - e, y \mapsto y - f$ is an obvious automorphism, we see that the collection of affine maps $x \mapsto ax + by + e, y \mapsto cx + dy + f$ are automorphisms of $k[x, y]$ iff $ad - bc \neq 0$.

   To start the main portion of the proof, we can write f in terms of homogeneous components and rewrite the quadratic homogeneous term as a product of linear terms by our preliminary result. That is, $f = L_1L_2 + L_3 + c$ for L_i linear homogeneous polynomials and $c \in k$. Now we split in to cases based on whether L_1 and L_2 are linearly independent or not.

   If L_1 and L_2 are linearly independent, then the map $x \mapsto L_1$ and $y \mapsto L_2$ is an automorphism of $k[x, y]$. So we may assume our polynomial is, up to automorphism of $k[x, y]$, of the form $xy + (ax + by) + c$. By applications of the automorphisms $x \mapsto x - d$ and $y \mapsto y - e$ for $d, e \in k$ we may get rid of the $(ax + by)$ term. Now we're left with $xy + c$. By the assumption that our polynomial was irreducible, $c \neq 0$. So we may apply the automorphism $x \mapsto cx$ and end up with $cxy + c = c(xy + 1)$. Thus $k[x, y]/(f) \cong k[x, x^{-1}]$ when the homogeneous degree two part of f factors as a product of linearly independent forms.

In the case when L_1 and L_2 aren't linearly independent, then L_1 and L_3 must be linearly independent - otherwise, f is $p(L_1)$ where p is a quadratic in a single variable. As we are over an algebraically closed field, every positive single-variable polynomial is reducible. Thus we can write $f = aL_1^2 + L_3 + c$ for L_1, L_3 linearly independent forms. Since L_1 and L_3 are linearly independent, the map $x \mapsto \sqrt{a}L_1, y \mapsto L_3 + c$ is an automorphism of $k[x, y]$. So up to automorphisms our polynomial is $y - x^2$. Thus $k[x, y]/(f) \cong k[x]$ when the homogeneous degree two part of f does not factor as a product of linearly independent forms.

As a bonus, this proof is valid in every characteristic!

Exercise I.1.2. *The Twisted Cubic Curve.* Let $Y \subset \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We see that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = y^3$.

Solution. Let's make a list: we need to show that Y is closed, irreducible, compute $I(Y)$, and compute $A(Y)$. In order to show that Y is irreducible, it is enough to show that $I(Y)$ is prime, or that $A(Y)$ is a domain. In order to show that Y is closed, it suffices to show $Y = V(I(Y))$, since $V(I(S))$ is the closure of S for any set S.

We start by computing $I(Y)$. I claim that $y - x^2$ and $z - x^3$ are a generating set for $I(Y)$. Clearly $(y - x^2, z - x^3) \subset I(Y)$. Suppose $f \in I(Y)$ is a polynomial which vanishes on all points of the form (t, t^2, t^3). If $f = \sum \lambda_i x^{a_i} y^{b_i} z^{c_i}$ is in $I(Y)$, then $\sum \lambda_i x^{a_i+2b_i+3c_i}$ must also be in $I(Y)$ because their difference is in $(y - x^2, z - x^3)$. On the other hand, if $\sum \lambda_i x^{a_i+2b_i+3c_i}$ is nonzero, then it can only vanish at finitely many values of x, and therefore cannot vanish on all of Y, since Y contains infinitely many points with different x-coordinates (as every algebraically closed field is infinite). So for $f \in I(Y)$, we obtain $f \in (y - x^2, z - x^3)$, which shows that $I(Y) = (y - x^2, z - x^3)$. Thus $A(Y) \cong k[x, y, z]/I(Y) \cong k[x]$, which is a 1-dimensional domain, so we've finished except for showing that Y is closed.

Clearly $Y \subset V(I(Y))$, so it remains to check that every point satisfying $y - x^2$ and $z - x^3$ is in Y. Let (a, b, c) be such a point. Then if any of a, b, or c is zero, they all must be, and $(0, 0, 0)$ is in the image of our map. On the other hand, if each of a, b, c is nonzero, then we see that $b = a^2$ and $c = a^3$ so (a, b, c) is the image of a, and $V(I(Y)) \subset Y$. So Y is closed and we're done.

Exercise I.1.3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Solution. There are two ways to view this: we can either look for a primary decomposition of $(x^2 - yz, xz - x)$, or we can be a little more hands-on.

To be hands-on, we will think about what we can do to force both of our equations to be satisfied. The second equation factors as $x(z - 1) = 0$, so either $x = 0$ or $z = 1$. In the first case, our first equation becomes $yz = 0$, so we get two irreducible components: (x, y) and (x, z). In the second case, our first equation becomes $x^2 - y$ and we get one irreducible component: $(z - 1, x^2 - y)$.

The primary decomposition side is (unsurprisingly) similar. From the factorization $xz - x = x(z - 1)$, we see that $(x^2 - yz, xz - x) = (x^2 - yz, x) \cap (x^2 - yz, z - 1)$. The first ideal may be rewritten as (yz, x), which is $(x, y) \cap (x, z)$. The second can be rewritten as $(x^2 - y, z - 1)$. So $(x^2 - yz, xz - x) = (x, y) \cap (x, z) \cap (x^2 - y, z - 1)$ is a primary decomposition.

In total, we have two lines given by (x, y) and (x, z), and one parabola given by $(x^2 - y, z - 1)$.

Exercise I.1.4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topology on the two copies of $\mathbb{A}^2$.

Solution. The closed subsets of $\mathbb{A}^1$ are exactly the finite unions of points, the empty set, and the whole space. This means that nontrivial closed subsets of $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology are finite unions of points, horizontal lines, and vertical lines. In particular, the diagonal $\{(x, x) \mid x \in \mathbb{A}^1\}$ is not closed in the product topology. On the other hand, the diagonal is $V(x - y) \subset \mathbb{A}^2$.

Exercise I.1.5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution. The condition that B have no (nonzero) nilpotents is better known as being reduced.

$\Rightarrow$: Given an algebraic set $X \subset \mathbb{A}^n$, we take its ideal $I(X) \subset k[x_1, \cdots, x_n]$ and form the quotient ring $A(X) = k[x_1, \cdots, x_n]/I(X)$. As $I(X)$ is radical, the quotient is reduced and we are done.

$\Leftarrow$: Given a finitely-generated k-algebra B, there is a surjection $k[x_1, \cdots, x_n] \to B$, so we may write $B \cong k[x_1, \cdots, x_n]/I$ for some ideal I. As B is reduced, I is radical, so I is the ideal of $V(I)$, and B is the coordinate algebra of $V(I)$.

Exercise I.1.6. Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in the induced topology, then the closure $\overline{Y}$ is also irreducible.

Solution. Let X be irreducible and $Y \subset X$ a nonempty open. We may write $X = \overline{Y} \cup Y^c$, which is a union of closed subsets. By the assumption that X is irreducible, we must have that one of these sets is actually X. As Y is nonempty, $Y^c \neq X$, so $\overline{Y} = X$ and thus Y is dense.

Suppose $Y = A \cup B$ with $A, B \subset Y$ closed in the induced topology. By the definition of the induced topology, there are closed subsets $A', B' \subset X$ so that $A = A' \cap Y$ and $B = B' \cap Y$. So $A' \cup B'$ is a closed subset of X which contains Y. As $\overline{Y} = X$ by density and X is irreducible, this means either $A' = X$ or $B' = X$, which implies $A = Y$ or $B = Y$. Thus Y is irreducible.

Exercise I.1.7.

a. Show that the following conditions are equivalent for a topological space X: (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subset has a maximal element.

b. A noetherian toplogical space is *quasi-compact*, i.e., every open cover has a finite subcover.

c. Any subset of a noetherian topological space is noetherian in its induced topology.

d. A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Solution.

a. (i) $\Leftrightarrow$ (iii) is straightforwards: a descending chain of closed subsets is equivalent to an ascending chain of open sets by taking complements. One stabilizes if and only if the other does.

   One may also see that the statements of (ii) and (iv) are exactly dual in this fashion as well. So all that remains is to prove the equivalence of either (i) and (ii) or (iii) and (iv).

   (i) $\Rightarrow$ (ii): Suppose we have a nonempty family of closed subsets. Pick X_1 a member of this family. If X_1 is minimal, we are done. Else, there is some $X_2 \subsetneq X_1$ which is closed. We may repeat this process until we either find a minimal element, or we produce an infinite descending chain of proper inclusions of closed subsets. Since such a chain cannot exist by noetherianity, it must be the case that at some finite step X_n we actually found a minimal element.

   (ii) $\Rightarrow$ (i): Take a descending chain of closed subsets. By the conditions of (ii), this has a minimal element, and thus the chain must stabilize somewhere.

b. Suppose we have an open cover $X = \bigcup_{\alpha \in A} X_\alpha$. Pick an open subset X_1. If $X_1 \neq X$, then we can pick an open subset X_2 so that X_1 is properly contained within $X_1 \cup X_2$. Repeat: if this procedure stops at some point, then we've found a finite subcover. If this does not stop, then we have produced an infinite ascending chain of open subsets, which is equivalent to X non-noetherian by (a). So the procedure must stop and we must be able to find a finite subcover.

c. This is a straightforwards application of the induced topology. Let Y be a subset of X equipped with the induced topology and suppose X is noetherian. Pick a chain of closed subsets $V_1 \supset V_2 \supset V_3 \supset \cdots$ inside Y. Then $\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \cdots$ is a chain of closed subsets in X, and $\overline{V_i} \cap Y = V_i$. As the chain in X must stabilize, the chain in Y must stabilize, so Y is noetherian by (a).

d. Suppose first that X is an irreducible noetherian Hausdorff space. If X has two distinct points x, y, we can find disjoint open neighborhoods U_x, U_y inside X. But then $X = U_x^c \cup U_y^c$ is a decomposition of the irreducible X in to two proper nonempty closed subsets, which is impossible by the definition of an irreducible space. So X is a single point.

   Now I claim any noetherian topological space has finitely many irreducible components (an irreducible component is a maximal element of the collection of irreducible closed subsets). The proof is straightforwards: if X_1 is an irreducible component, then X_1^c is an open subset, and if $X = \bigcup X_i$ is a decomposition in to irreducible components, then $X = \bigcup X_i^c$ is an open cover of X. By (b), this has a finite refinement, which is exactly equivalent to having finitely many irreducible components.

   The combination of these two results show that if a topological space is noetherian Hausdorff, it must be finite discrete.

Exercise I.1.8. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume $Y \not\subset H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

Solution. Assume $Y \cap H \neq \emptyset$ in order that we might have something to prove - this implies that f is not a unit in $A(Y)$. The condition that $Y \not\subset H$ is equivalent to $f \notin I(Y)$, which means that $\overline{f}$, the image of f in $A(Y)$, is not a zero divisor as $I(Y)$ is radical. So all minimal primes over $(\overline{f}) \subset A(Y)$ are of height one by Krull's Hauptidealsatz (Theorem 1.11A). But these minimal primes exactly correspond to the irreducible components of $Y \cap H$, and by Theorem 1.8A, we see that every such irreducible component is of dimension $r-1$.

Exercise I.1.9. Let $\mathfrak{a} \subset A = k[x_1, \cdots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n-r$.

Solution. This may be solved by induction and applying exercise 1.8. The base case is $r=0$, whence $\mathfrak{a}=0$ and $V(\mathfrak{a}) = \mathbb{A}^n$, which is irreducible of dimension $n-0=n$.

For the inductive step, assume we have proven the result of r and we want to prove it for $r+1$. Write $\mathfrak{a} = (f_1, \cdots, f_r, f_{r+1}) = (f_1, \cdots, f_r) + (f_{r+1})$. Then $V(\mathfrak{a}) = V(f_1, \cdots, f_r) \cap V(f_{r+1})$, we can decompose $V(f_1, \cdots, f_r)$ as a union of irreducible components X_i of dimension $\geq n-r$. Now either $X_i \subset V(f_{r+1})$, in which case $X_i \cap V(f_{r+1}) = X_i$, so $\dim X_i \geq n-r \geq n-r-1$, or $X_i \not\subset V(f_{r+1})$. In the latter case, we may apply exercise 1.8 to see the intersection of each such irreducible component with $V(f_{r+1})$ is either empty or of dimension at least $n-r-1$, so the result is proven.

Exercise I.1.10.

a. If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

b. If X is a topological space which is covered by a family of open subsets $\{U_i\}_{i \in I}$, then $\dim X = \sup \dim U_i$.

c. Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

d. If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

e. Given an example of a noetherian topological space of infinite dimension.

Solution.

a. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ be a maximal chain of proper inclusions of irreducible closed subsets inside Y. By exercise 1.6, $\overline{Y_i}$ is irreducible inside X. Further, if $Y_i \subset Y_{i+1}$ is a proper inclusion, then $\overline{Y_i} \subset \overline{Y_{i+1}}$ is also a proper inclusion: $Y_i = \overline{Y_i} \cap Y$, so $\overline{Y_i} \setminus Y_i \subset \overline{Y} \setminus Y$, and thus if we have a point $y \in Y_{i+1} \setminus Y_i$, then y is also in $\overline{Y_{i+1}} \setminus \overline{Y_i}$. Thus $\overline{Y_0} \subset \overline{Y_1} \subset \cdots \subset \overline{Y_n}$ is a chain of proper inclusions of irreducible closed subsets inside X. By the definition of dimension as the supremum of length of such chains, we are done.

b. By (a), we have that $\dim X \geq \dim U_i$ for all i, so $\dim X \geq \sup \dim U_i$.

   We will finish by showing that there is an i so that $\dim U_i \geq \dim X$. Let $X_0 \subset X_1 \subset \cdots$ be chain of proper inclusions of irreducible closed subsets of X of maximal length. Now pick a U_i containing X_0, which we can do from the fact that the U_i form a cover. I claim that $(U_i \cap X_0) \subset (U_i \cap X_1) \subset \cdots$ is a chain of proper inclusions of irreducible closed subsets of U_i. The sets $U_i \cap X_j$ are closed in U_i by the definition of the induced topology, and they are irreducible by exercise 1.6 applied to $U_i \cap X_j \subset X_j$. It remains to see that these inclusions are proper.

   Suppose $U_i \cap X_j = U_i \cap X_{j+1}$. We note that $X_{j+1} \setminus X_j$ is a nonempty open subset of X_{j+1}, as is $U_i \cap X_{j+1}$. Recall that a dense set must meet every nonempty open: this implies that $U_i \cap X_{j+1}$ intersects $X_{j+1} \setminus X_j$, which means there's a point in $U_i \cap X_{j+1}$ not in $U_i \cap X_j$. So we see the inclusion is proper, thus $\dim U_i \geq \dim X$, and we're done.

c. Let $X = \{0, 1\}$ with the open sets $\tau = \{\emptyset, \{1\}, X\}$. X is irreducible and one-dimensional: $\{0\} \subset X$ is a chain of irreducible closed subsets of length one, and there are no other irreducible closed subsets (remember, irreducible means non-empty by definition). On the other hand, $\{1\}$ is open and dense (by exercise 1.6). But it's a singleton, which means it has dimension zero. (This is sometimes called the Sierpinski space, but it's better known to algebraic geometers as the spectrum of a discrete valuation ring. You'll see an explanation of the spectrum construction once you get to chapter 2, and it's a useful example to know because you can get very hands-on with it.)

d. Pick a chain of proper inclusions of irreducible closed subsets of Y of maximal length n: $Y_0 \subset Y_1 \subset \cdots \subset Y_n$. If $Y \neq X$, then we can extend this chain to $Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset X$ which shows that $\dim Y < \dim X$, contradicting our assumptions. (We used here that a closed subset of a closed subset is again a closed subset to conclude that Y_i are closed inside X and the induced topology plus exercise 1.6 to guarantee that Y_i are also irreducible as subset of X.)

e. Let X be $\mathbb{Z}_{\geq 0}$ where the nontrivial open sets are of the form $U_n = \{x \in \mathbb{N} \mid x > n\}$ over all choices of $n \in \mathbb{Z}_{\geq 0}$. Then every nontrival closed subset is of the form $\{0, 1, \cdots, n\}$, which is irreducible. So we have arbitrarily long chains of proper inclusions of irreducible closed subsets, and thus $\dim X = \infty$.

   On the other hand, any family of closed subsets has a minimal element: the set with the fewest elements. By 1.7(a), X is noetherian.

Exercise I.1.11. Let $Y \subset \mathbb{A}^3$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection* - cf. (Ex 2.17).

Solution. Here's what's on the menu: we show $I(Y)$ is prime, we show $I(Y)$ has height 2, we compute $I(Y)$, and we show that $I(Y)$ cannot be generated by 2 elements. We abbreviate $I(Y)$ as I during this solution for ease of typing.

To compute I, define a homomorphism $\varphi : k[x,y,z] \to k[t^3,t^4,t^5]$ given by $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. Then I is exactly the kernel of φ: it consists of functions which vanish when we plug in $x = t^3$, $y = t^4$, and $z = t^5$ for all values of t. So we have an injective map $k[x,y,z]/I \to k[t^3,t^4,t^5]$ by the first isomorphism theorem, and this map is clearly surjective, so it's an isomorphism. As $k[x,y,z]/I$ is a domain, I is prime. As $k[t^3,t^4,t^5]$ is one-dimensional, I is of height 2 by Theorem 1.8A.

Now we need to compute I. It is clear that $(xz - y^2, x^3 - yz, x^2y - z^2) \subset I$, and I claim this is actually an equality. In order to show this, pick $f \in I$, and we endeavor to show that $f \in (xz - y^2, x^3 - yz, x^2y - z^2)$. Our strategy will be similar to exercise 1.2: if we want to show $f \in (xz - y^2, x^3 - yz, x^2y - z^2)$, it suffices to show $f - p \in (xz - y^2, x^3 - yz, x^2y - z^2)$ for some $p \in (xz - y^2, x^3 - yz, x^2y - z^2)$. By making convenient choices of p, we may assume that f has some nice properties.

First, up to multiples of $x^2y - z^2$, we may assume f can be written as a linear combination of monomials $x^ay^bz^c$ with $c = 0$ or $c = 1$. Next, up to multiples of $xz - y^2$ and $yz - x^3$, we may assume that f can be written as λz plus a linear combination of monomials of the form x^ay^b. As $x^4 - y^3 = y(xz - y^2) + x(x^3 - yz)$, we can assume our monomials of the form x^ay^b have $b < 3$ at the cost of increasing the a. To recap, we've shown it's enough to analyze the case where $f = \lambda_1 z + \lambda_2 y p(x) + \lambda_3 y^2 q(x)$.

Now we compute image of this f under the homomorphism $\varphi : k[x,y,z] \to k[t^3,t^4,t^5]$ we used to define I two different ways. First, as $f \in I$, it has image 0. On the other hand, $\varphi(\lambda_1 z) = \lambda_1 t^5$, $\varphi(\lambda_2 y p(x)) = \lambda_2 t^4 p(t^3)$, and $\varphi(\lambda_3 y^2 q(x)) = \lambda_3 t^8 q(t^3)$. We note that $\varphi(\lambda_2 y p(x))$ is a linear combination of monomials of the form t^{3a+1} with $a > 0$ and $\varphi(\lambda_3 y^2 q(x))$ is a linear combination of monomials of the form t^{3b+2} with $b > 1$, and the collection of all such monomials along with t^5 are linearly independent. So we must have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and we get $I = (xz - y^2, x^3 - yz, x^2y - z^2)$.

In order to show that I cannot be generated by two elements, we define a nonstandard grading on $k[x,y,z]$. Let $\deg x = 3$, $\deg y = 4$, and $\deg z = 5$. Then I is a homogeneous ideal under this grading because it's generated by homogeneous elements, and the same conclusion applies to (x,y,z).

Now suppose we had two generators for our ideal: this would mean we have a surjection $k[x,y,z]^2 \to I \to 0$. Tensoring over $k[x,y,z]$ with $k[x,y,z]/(x,y,z)$, we would have that $k^2 \to I/(x,y,z)I$ is again a surjection of k-vector spaces, as tensor products are right-exact. This would imply that $I/(x,y,z)I$ is of dimension at most two as a k-vector space. Next, note that $(x,y,z)I$ is a graded submodule of the graded module I: this means the quotient $I/(x,y,z)I$ is also graded, and as it is of dimension two as a k-vector space, it can have at most two nonzero graded pieces. Since graded maps of graded modules can be analyzed on each graded piece, we see that the quotient map $I \to I/(x,y,z)I$ must send all but possibly two graded pieces of I to zero - in particular, there can be at most two integers d so that $((x,y,z)I)_d = 0$ but $I_d \neq 0$.

Let us look at the terms of minimal degree in $I(Y)$: we have $\deg(xz - y^2) = 8$, $\deg(x^3 - yz) = 9$, and $\deg(x^2y - z^2) = 10$, and there are no nonzero elements $f \in I$ with $\deg(f) < 8$ by our computation of the generating set earlier. So we have three graded degrees where $I_d \neq 0$ but $((x,y,z)I)_d = 0$, contradicting our work from the previous paragraph. Thus I cannot be generated by two elements.

Exercise I.1.12. Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $V(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible (cf. 1.4.2).

Solution. Let $f = (x^2 - 1)^2 + y^2$. This factors as $(x^2 - 1 + iy)(x^2 - 1 - iy)$ over $\mathbb{C}[x, y]$, and each of those polynomials are irreducible because they're of degree 1 in y. As neither of those polynomials are in $\mathbb{R}[x, y]$, we see that f is irreducible (we use the fact that polynomial rings over fields are UFDs here).

On the other hand, the zero set of f is $(\pm 1, 0)$, which is two distinct points and not irreducible.

(The essential failure here is that every naive affine variety over the reals is a hypersurface: if we have a collection of polynomials f_i, then they're all zero iff $\sum f_i^2$ is. Once you adopt a less-naive viewpoint such as the schematic viewpoint, this failure disappears.)

I.2 Projective Varieties

We make use of the affine cone construction from exercise 2.10 when solving exercises 2.2 and 2.4. You will be relieved to know that we do not use either 2.2 or 2.4 when solving 2.10, so there is no circular reasoning present.

Exercise I.2.1. Prove the "'homogeneous Nullstellensatz,"' which says if $\mathfrak{a} \subset S$ is a homogeneous ideal, and if $f \in s$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in V(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n+1)$ space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

Solution. The hint makes this problem trivial. $[a_0 : \cdots : a_n] \in \mathbb{P}^n$ is a point in $V_{\mathbb{P}^n}(\mathfrak{a})$ iff $(a_0, \cdots, a_n)$ is a point in $V_{\mathbb{A}^{n+1}}(\mathfrak{a})$, and f vanishes at this point as an element of the homogeneous coordinate ring iff it vanishses as an element of the usual coordinate ring. As $\mathfrak{a}$ and f are both homogeneous, they vanish at the origin, and we have satisfied the hypotheses for the usual Nullstellensatz.

Exercise I.2.2. For a homogeneous ideal $\mathfrak{a} \subset S = k[x_0, \cdots, x_n]$, show that the following conditions are equivalent:

(i). $V(\mathfrak{a}) = \emptyset$;

(ii). $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii). $\mathfrak{a} \supset S_d$ for some $d > 0$.

Solution. (i) $\Leftrightarrow$ (ii): Look at $V_{\mathbb{A}^{n+1}}(\mathfrak{a})$, the affine cone on $V(\mathfrak{a})$. $V(\mathfrak{a})$ is empty iff $V_{\mathbb{A}^{n+1}}(\mathfrak{a})$ is either empty or only the origin. By the usual Nullstellensatz, $V_{\mathbb{A}^{n+1}}(\mathfrak{a}) = \emptyset$ or $V_{\mathbb{A}^{n+1}}(\mathfrak{a}) = \{(0, \cdots, 0)\}$ iff $\sqrt{\mathfrak{a}}$ is either S or $(x_0, \cdots, x_n) = S_+$.

(ii) $\Rightarrow$ (iii): The stated conditions imply that $S_+ \subset \sqrt{\mathfrak{a}}$. Thus for each x_i there is an d_i so that $x_i^{d_i} \in \mathfrak{a}$ by the definition of the radical of an ideal. As there are finitely many x_i, set $d = \max_i(d_i)$. Now ever monomial of degree $(n+1)(d-1)+1$ is divisible by a monomial of the form x_i^d by the pigeonhole principle. Thus $S_{(n+1)(d-1)+1} \subset \sqrt{\mathfrak{a}}$.

(iii) $\Rightarrow$ (ii): If $S_d \subset \mathfrak{a}$, then $x_i^d \in \mathfrak{a}$ for all i, and so $x_i \in \mathfrak{a}$. So $S_+ \subset \sqrt{\mathfrak{a}}$, which implies that $\sqrt{\mathfrak{a}}$ is either S or S_+.

Exercise I.2.3.

a. If $T_1 \subset T_2$ are subsets of S^h, then $V(T_1) \supset V(T_2)$.

b. If $Y_1 \subset Y_2$ are subset of $\mathbb{P}^n$, then $I(Y_1) \supset I(Y_2)$.

c. For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d. If $\mathfrak{a} \subset S$ is a homogeneous ideal with $V(\mathfrak{a}) \neq \emptyset$, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

e. For any subset $Y \subset \mathbb{P}^n$, $V(I(Y)) = \overline{Y}$.

Solution.

a. Obvious.

b. Obvious.

c. Obvious.

d. Obvious.

e. Clearly $\overline{Y} \subset V(I(Y))$. On the other hand, if there is a point $P \in V(I(Y)) \setminus \overline{Y}$, then $I(\overline{Y}) \supsetneq I(\overline{Y} \cup \{P\})$. So there's a homogeneous polynomial vanishing on $\overline{Y}$ but not on P. But such a polynomial must also vanish on Y and is thus in $I(Y)$. So $P \notin V(I(Y))$, and therefore $V(I(Y)) \subset \overline{Y}$ and we're done.

Exercise I.2.4.

a. There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto V(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

b. An algebraic set $Y \subset \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c. Show that $\mathbb{P}^n$ itself is irreducible.

Solution.

a. This is exercise 2.3 above.

b. Look at the affine cone. The result follows from Corollary 1.4.

c. The ideal of $\mathbb{P}^n$ is (0), so the result follows by (b).

Exercise I.2.5.

a. $\mathbb{P}^n$ is a noetherian topological space.

b. Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution.

a. Let $X_0 \supset X_1 \supset \cdots$ be a descending chain of closed subset of $\mathbb{P}^n$. By exercise 2.3 above, this corresponds to an ascending chain of ideals $I(X_0) \subset I(X_1) \subset \cdots$ in the ring S. As S is noetherian, this chain of ideals must stabilize, and therefore the chain of closed subspaces must stabilize as well. So $\mathbb{P}^n$ is a noetherian topological space. (Alternatively, one may show that a space with a finite open cover by noetherian topological spaces is noetherian, and then $\mathbb{P}^n$ is noetherian because the standard affine opens are noetherian.)

b. See proposition 1.5.

Exercise I.2.6. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \to \mathbb{A}^n$ be he homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex. 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

Solution. Let us follow the hint. By exercise I.10(b), there is some i so that $\dim Y_i = \dim Y$, and for convenience we assume that $i = 0$. Any element of $S(Y)_{(x_0)}$ can be written as $\frac{F}{x_0^d} = f(1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$ where $F \in S$ is homogeneous of degree d. Such an element is exactly $\alpha(F) \in A(Y_i)$ where α is the map defined in proposition 2.2 and $\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}$ are the coordinates on $\mathbb{A}^n$. On the other hand, for any polynomial $f \in A(Y_i)$, we can homogenize it and then divide by an appropriate power of x_0 to get a polynomial $\frac{\beta(f)}{x_0^d} \in S(Y)$. It is easy to see that these are mutually inverse maps which define an isomorphism between $A(Y_i)$ and $S_{(x_0)}$.

By the mentioned results from section 1, this shows that $S(Y)_{(x_0)}$ has dimension $\dim Y_0 = \dim Y$. Now as x_0 is transcendental over $S(Y)_{(x_0)}$ by degree reasons and $S(Y)_{(x_0)}[x_0, x_0^{-1}] = S_{x_0}$, we see that by taking fraction fields we have that $\operatorname{trdeg} \operatorname{Frac}(S_{x_0}) = \operatorname{trdeg} \operatorname{Frac}(S_{(x_0)})$, and the latter quantity is exactly $\dim Y_0 = \dim Y$ by the isomorphism between $S_{(x_0)}$ and $A(Y_i)$. So we are done.

Exercise I.2.7.

a. $\dim \mathbb{P}^n = n$.

b. If $Y \subset \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$. [*Hint*: Use (Ex. 2.6) to reduce to (1.10).]

Solution.

a. Apply exercise 2.6 above with $S(\mathbb{P}^n) = k[x_0, \cdots, x_n]$ a ring of dimension $n + 1$.

b. Apply the hint. Intersecting Y with the standard open affines U_i, we get a collection of quasi-affine varieties $U_i \cap Y$, at least one of which is of dimension $\dim Y$ by exercise 1.10. By proposition 1.10 each $U_i \cap Y$ has the same dimension as $\overline{U_i \cap Y} \subset U_i$. But the $\overline{U_i \cap Y}$ form an open cover of $\overline{Y} \subset \mathbb{P}^n$, and by another application of exercise 1.10 we see the result.

Exercise I.2.8. A projective variety $Y \subset \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbb{P}^n$.

Solution. $S(Y)$ is a quotient of $S(\mathbb{P}^n)$ by a homogeneous prime ideal $\mathfrak{p}$. Since $\dim S(Y) = \dim Y + 1$ and $\dim S(\mathbb{P}^n) = n + 1$ by exercise 2.6, we have that $\mathfrak{p}$ is of height $\operatorname{codim} Y$ by theorem 1.8A. So $\mathfrak{p}$ is of height one iff $\dim Y = n - 1$, and by applying the proof of proposition 1.13 we see that $\mathfrak{p}$ is generated by a single element iff $\mathfrak{p}$ is of height one.

Exercise I.2.9. *Projective Closure of an Affine Variety.* If $Y \subset \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subset \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the *projective closure* of Y.

a. Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

b. Let $Y \subset \mathbb{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subset \mathbb{P}^3$ is called the *twisted cubic curve* in $\mathbb{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$ and use this example to show that if $f_1, \cdots, f_r$ generate $I(Y)$, then $\beta(f_1), \cdots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

Solution.

a. Assume Y is nonempty. Then, observe that $I(\overline{Y})$ can be generated by polynomials $F_1, \cdots, F_r$, none of which are divisible by x_0 because $\overline{Y}$ is not contained in $V(x_0)$. Next, note that $\alpha(F)$ vanishes on Y for any polynomial $F \in I(\overline{Y})$. As by definition of α and β, we get that $\beta(\alpha(F)) = F$ for $F \in S$ not divisible by x_0, this means that the F_r are $I(\overline{Y})$, and so they're β of something in $A(Y)$ by the previous sentence. So we've shown that $I(\overline{Y})$ is generated by $\beta(I(Y))$.

b. From the work done in exercise 1.2, we have that $I(Y) = (x^2 - y, x^3 - z)$. The embedding of the affine twisted cubic gives us the points $[1 : t : t^2 : t^3]$ inside $\mathbb{P}^3$ where $t \in k$. To find the projective closure, write $t = \frac{u}{v}$ and clear denominators: this gives us that the projective twisted cubic is the collection of points $[u^3 : u^2v : uv^2 : v^3]$ for $[u : v] \in \mathbb{P}^1$.

   Let's show first that $\beta(x^2 - y) = x_1^2 - x_2x_0$ and $\beta(x^3 - z) = x_1^3 - x_3x_0^2$ aren't enough to generate the ideal. The point $[0 : 0 : 1 : 1]$ satisfies both of these equations but isn't of the form $[u^3 : u^2v : uv^2 : v^3]$, so we see that $I(\overline{Y})$ must contain another generator.

   Now I claim that $I(\overline{Y}) = (x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2)$. Take a homogeneous polynomial $f \in I(\overline{Y})$ and write it as a sum of monomials $\sum \lambda_i x_0^{a_i} x_1^{b_i} x_2^{c_i} x_3^{d_i}$. Then $f \in (x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2)$ iff

$$f' = \sum \lambda_i x_0^{a_i+\min(b_i,c_i)} x_1^{b_i-\min(b_i,c_i)} x_2^{c_i-\min(b_i,c_i)} x_3^{d_i+\min(b_i,c_i)}$$

   is, because their difference is a multiple of $x_0x_3 - x_1x_2$. So now it is enough to consider f of the form

$$\sum \lambda_i x_0^{a_i} x_3^{d_i} + \sum \lambda_j x_0^{a_j} x_1^{b_j} x_3^{d_j} + \sum \lambda_k x_0^{a_k} x_2^{c_k} x_3^{d_k}$$

   and we can apply the same trick again with subtracting off multiples of $x_1x_3 - x_2^2$, $x_0x_2 - x_1^2$ and $x_0x_3 - x_1x_2$ so that all of the b_j and c_k are assumed to be zero or one. (We may have to do so repeatedly: consider x_1^4. Applying our trick using $x_0x_2 - x_1^2$ lets us consider instead $x_0x_1^2x_2$, and then we reduce to $x_0^2x_1x_3$ after an application of our trick using $x_0x_3 - x_1x_2$. We leave full formalization to the reader: the key idea is that every time you apply these rewriting rules the exponents on x_1 and x_2 drop.)

   Now we collect terms of a polynomial $\sum \lambda_i x_0^{a_i} x_1^{b_i} x_2^{c_i} x_3^{d_i}$ where $b_i + c_i \leq 1$ in a funny way: we can write $f' = P(x_0, x_3) + Q(x_0, x_3)x_1 + R(x_0, x_3)x_2$. Plugging in $x_0 = u^3$, $x_1 = u^2v$,

$x_2 = uv^2$, and $x_3 = v^3$, we see that $P(u^3, v^3)$ consists of terms where the exponent on u is a multiple of 3, $Q(u^3, v^3)u^2v$ consists of terms where the exponent on u is 1 modulo 3, and $R(u^3, v^3)uv^2$ consists of terms where the exponent on u is 2 modulo 3. So if $f \in I(\overline{Y})$, then this sum should be zero, which can only happen if $P = Q = R = 0$. Thus our claim about the generating set is proven.

Exercise I.2.10. *The Cone Over a Projective Variety.* Let $Y \subset \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} \setminus \{0, \cdots, 0\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \cdots, a_n)$ to the point with homogeneous coordinates $[a_0 : \cdots : a_n]$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \cdots, 0)\}.$$

a. Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \cdots, x_n]$.

b. $C(Y)$ is irreducible if and only if Y is.

c. $\dim C(Y) = \dim Y + 1$.

Solution.

a. As every element of $I(Y)$ is a sum of homogeneous elements, every element of $I(Y)$ vanishes at the origin in $\mathbb{A}^{n+1}$, so we have that $I(Y) \subset I(C(Y))$. To show the reverse inclusion, any element of $I(C(Y))$ vanishing at a point p must vanish at λp for all $\lambda \in k$, so it is a sum of homogeneous elements. But then each of these homogeneous elements define an element of $I(Y)$, so $I(C(Y)) \subset I(Y)$ and we have the desired equality.

b. Y is irreducible iff $I(Y)$ is prime, and $C(Y)$ is irreducible iff $I(C(Y))$ is prime. But $I(C(Y))$ is just $I(Y)$ by (a), so one is prime iff the other is.

c. We may assume that Y is irreducible by considering an irreducible component of maximal dimension. Then by exercise 2.6, $S(Y) = k[x_0, \cdots, x_n]/I(Y)$ is of dimension $\dim Y + 1$. On the other hand, $C(Y)$ is irreducible by (b) so $A(C(Y)) = k[x_0, \cdots, x_n]/I(C(Y))$ is of dimension $\dim C(Y)$. But as $A(C(Y)) = k[x_0, \cdots, x_n]/I(Y)$ by (a), so $A(C(Y))$ is of dimension $\dim Y + 1$, and thus we get the requested equality.

Exercise I.2.11. *Linear Varieties in $\mathbb{P}^n$.* A hypersurface defined by a linear polynomial is called a *hyperplane*.

a. Show that the following two conditions are equivalent for a variety Y in $\mathbb{P}^n$:

   (i) $I(Y)$ can be generated by linear polynomials.

   (ii) Y can be written as an intersection of hyperplanes.

   In this case we say that Y is a *linear variety* in $\mathbb{P}^n$.

b. If Y is a linear variety of dimension r in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

c. Let Y, Z be linear varieties in $\mathbb{P}^n$ with $\dim Y = r$, $\dim Z = s$. If $r+s-n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r+s-n$. (Think of $\mathbb{A}^{n+1}$ as a vector space over k, and work with it's subspaces.)

Solution.

a. The equivalence follows from the facts that $V(I_1 + \cdots + I_s) = V(I_1) \cap \cdots \cap V(I_s)$, $I(Y_1 \cap \cdots \cap Y_r) = \sqrt{I(Y_1) + \cdots + I(Y_r)}$, and that any ideal of a polynomial ring generated by linear terms is radical. (For the first two facts, these are a straightforwards exercise in the yoga of I and V: think about the inclusion-reversing correspondence and how that plays out on both sides. For the final fact, apply a linear change of coordinates so that the linear polynomials are given by the standard basis monomials, and then it's clear. Details left to the reader.)

b. Pick a generating set of linear polynomials $h_1, \cdots, h_s$ for $I(Y)$. Up to throwing away linearly dependent members, we may assume this set is linearly independent. This is equivalent to the existence of an invertible linear transformation taking h_i to x_i, so we may assume $I(Y)$ is generated by $(x_1, \cdots, x_s)$. But the variety determined by this ideal is obviously of dimension $n-s$, and $n-s=r$ iff $n-r=s$, so $I(Y)$ is generated by $n-r$ linearly independent polynomials.

c. Consider the affine cone. Then $C(Y)$ is a linear subspace of $\mathbb{A}^{n+1}$ of dimension $r+1$ and $C(Z)$ is a linear subspace of $\mathbb{A}^{n+1}$ of dimension $s+1$. By linear algebra, we see that $C(Y) \cap C(Z)$ has dimension at least $\min(0, (r+1)+(s+1)-(n+1)) = \min(0, r+s-n+1)$. If $r+s-n \geq 0$, then $\min(0, r+s-n+1) \geq 1$, so $C(Y) \cap C(Z) = C(Y \cap Z)$ is of dimension at least one, and thus contains a line through the origin, so $Y \cap Z$ contains a point and isn't empty. Furthermore, $C(Y \cap Z) = C(Y) \cap C(Z)$ is a linear space of dimension at least $r+s-n+1$, so it's ideal is generated by linear polynomials, and at most $2n-r-s$ of them. But by our work with the affine cone in 2.10, this means that $I(Y \cap Z)$ is generated by at most $2n-r-s$ linear polynomials and thus must be of dimension at least $r+s-n$.

Exercise I.2.12. *The d-Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \cdots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \cdots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ by sending the point $P = [a_0 : \cdots : a_n]$ to the point $\rho_d(P) = [M_0(a) : \cdots : M_N(a)]$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n=1$, $d=2$, then $N=2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

a. Let $\theta : k[y_0, \cdots, y_N] \to k[x_0, \cdots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $V(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

b. Show that the image of ρ_d is exactly $V(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

c. Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ on to the projective variety $V(\mathfrak{a})$.

d. Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

Solution.

a. Each y_i is sent to a term of the same degree, so this map is a map of graded rings, and thus the kernel is homogeneous. Since $k[y_0, \cdots, y_N]/\ker\theta$ is a subring of $k[x_0, \cdots, x_n]$, a domain, $\ker\theta$ is prime, and we're done.

   For a more explicit description, one may see that the ideal is generated by the polynomials of degree two $y_iy_j - y_ky_l$ where $M_iM_j = M_kM_l$. This is called a determinental ideal, and there's a lot of literature about these - take a gander!

b. Clearly we have that the d-uple embedding is contained inside $V(\mathfrak{a})$. It remains to show that any point in $V(\mathfrak{a})$ can be written as a point in the image of the d-uple embedding. Pick such a point p. It must have at least one nonzero coordinate, say corresponding to the degree-d monomial $\prod x_i^{a_i}$. Then as $(\prod x_i^{a_i})^d = \prod (x_i^d)^{a_i}$, we see that at least one monomial of the form x_i^d is nonzero. Now $[\frac{x_i^{d-1}x_0}{x_i^d} : \cdots : 1 : \cdots : \frac{x_i^{d-1}x_n}{x_i^d}]$ (with the 1 in the i^{th} spot) gives the point in $\mathbb{P}^n$ which has image p under the d-uple embedding, and we have shown the desired equality.

c. We've already verified that $V(\mathfrak{a})$ is a projective variety and that it's equal to the image of the d-uple embedding in parts (a) and (b). As ρ_d is a morphism, it's continuous, so it remains to show that it's injective. This is easy: suppose we have two points p and q which map to the same point under the d-uple embedding. By the assumption they map to the same point, we can find some coordinate (x_0, without loss of generality) which is nonvanishing for both. Then by the work in (b), we have uniquely determined the preimage, so $p = q$, the map is injective, and we're done.

d. This is immediate from the description as $[u^3 : u^2v : uv^2 : v^3]$ found in that problem.

Exercise I.2.13. Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the *Veronese surface*. If $Z \subset Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there existst a hypersurface $Y \subset \mathbb{P}^5$ such that $V \cap Y = Z$.

Solution. If $\{x_0, x_1, x_2\}$ are coordinates on $\mathbb{P}^2$, Y is given by $[x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$. For a homogeneous polynomial $f(x_0, x_1, x_2)$, f^2 is homogeneous of even degree and can be written as $g(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$ for some polynomial g. So $Z = V(f)$ is the intersection of $V(g)$ with Y. (Note that this is only an intersection as sets - once you get to schemes you'll be aghast at what's going on in this problem.)

Exercise I.2.14. *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \cdots, a_r) \times (b_0, \cdots, b_s)$ to $(\cdots, a_i b_j, \cdots)$ in lexicographic order, where $N = rs+r+s$. Note that ψ is well-defined and injective. It is called the *Segre embedding.* Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \cdots, r, j = 0, \cdots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \cdots, x_r, y_0, \cdots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{Im}\psi = V(\mathfrak{a})$.]

Solution. Follow the hint. We get an injective map $k[z_{ij}]/\mathfrak{a} \to k[x_0, \cdots, x_r, y_0, \cdots, y_s]$ where the target is a domain, so $\mathfrak{a}$ is a prime ideal. Clearly any point in the image of the embedding is in $V(\mathfrak{a})$, so it remains to see that every point in $V(\mathfrak{a})$ is in the image.

We use the same technique as with the d-uple embedding in problem 2.12. Given a point in $V(\mathfrak{a})$, there must be some coordinate which is nonzero, say the $x_i y_j$ coordinate. Then $x_i \neq 0$ and $y_j \neq 0$, so we can recover $\frac{x_k}{x_i}$ as $\frac{x_k y_j}{x_i y_j}$ and $\frac{y_l}{y_j}$ as $\frac{x_i y_l}{x_i y_j}$, and it is straightforwards to show that our point is the image of the point we construct in this fashion.

Exercise I.2.15. *The Quadratic Surface in* $\mathbb{P}^3$. Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

a. Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

b. Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$ each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, then $M_t \cap M_u = \emptyset$, and for all t, u, $L_t \cap M_u =$ one point.

c. Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

Solution.

a. From exercise 2.14, we see that the ideal cutting out the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is the kernel of the homomorphism $k[z_{00}, z_{01}, z_{10}, z_{11}] \to k[x_0, x_1, y_0, y_1]$ by $z_{ij} \mapsto x_i y_j$. This kernel is exactly $(z_{00}z_{11} - z_{01}z_{10})$, which is the same as $(xy - zw)$ after an appropriate choice of coordinates. (To see the statement about the kernel, pick a homogeneous element f, write it as a collection of monomials, and look at it's image under the homomorphism. Up to subtracting off multiples of $z_{00}z_{11} - z_{01}z_{10}$, we can take the monomials making up f to be of the form $z_{00}^a z_{11}^b z_{10}^c$ and $z_{00}^p z_{11}^q z_{01}^r$ with $a + b + c = p + q + r$ but $(a, b, c) \neq (p, q, r)$. But such monomials have image $x_0^a x_1^{b+c} y_0^{a+c} y_1^b$ and $x_0^{p+q} x_1^q y_0^p y_1^{q+r}$, and by looking at degrees we see that these are linearly independent unless $a = p + q$, $a + c = p$, $b + c = q$, and $b = q + r$, which imply that $c = q = 0$, $b = r = 0$, and thus $a = p$, contradicting our hypothesis.)

b. Clearly for any $t = [u, v] \in \mathbb{P}^1$ we have that $xu - zv$ and $yu - wv$ are lines in Q. The required intersection properties are clear.

c. The curve $x = y$ suffices. It is proper, not a finite union of points, and not of the form from (b), so it cannot be closed in the product of the Zariski topology (closed sets in the product topology are finite unions of products of closed sets in the two factors, and such products in this case are points, lines from (b), and the whole space).

Exercise I.2.16.

a. The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadratic surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

b. Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Solution.

a. If $w = 0$, then our equations are $x^2 = 0$ and $xy = 0$, which gives us the line $V(x, w)$. If $w \neq 1$, then dehomogenizing we have equations $X^2 - Y = 0$ and $XY - Z = 0$, which gives us the twisted cubic by $X = t$, $Y = t^2$, and $Z = t^3$.

b. It is clear that $y = 0$ implies that $x = 0$, so the intersection is the single point $[0 : 0 : 1]$ which has ideal (x, y). On the other hand, $I(C) + I(L) = (x^2 - yz, y)$, and $k[x, y, z]/(x^2 - yz, y) \cong k[x, z]/(x^2)$, which isn't a domain, so $(x^2 - yz - y)$ isn't a prime ideal, while (x, y) is. Thus the two ideals are not equal. (The appropriate fix here is either that the scheme-theoretic intersection is a double point, which we'll see in chapter 2, or to apply a radical to the sum as in exercise 2.11 above.)

Exercise I.2.17. *Complete Intersections.* A variety Y of dimension r in $\mathbb{P}^n$ is a (*strict*) *complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

a. Let Y be a variety in $\mathbb{P}^n$, let $Y = V(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geq n - q$.

b. Show that a strict complete intersection is a set-theoretic complete intersection.

c. (*) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

d. (**) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic complete intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.

Solution.

a. Consider $C(Y) \subset \mathbb{A}^{n+1}$. It is irreducible of dimension $\dim Y + 1$ with ideal $I(Y)$ by exercise 2.10. By exercise 1.9, $C(Y)$ has dimension at least $n+1-q$, so $\dim Y + 1 \geq n+1-q$, or $\dim Y \geq n-q$.

b. Take the hypersurfaces defined by the minimal generating set of the ideal.

c. First, we recall the definition of a homogeneous ideal: an ideal $I \subset R$ with $R = \bigoplus_{d\in D} R_d$ is homogeneous iff $I = \bigoplus_{d\in D} I \cap R_d$. This means that if f is in I, then for any d we have that $f^{(d)}$, the homogeneous degree-d part of f, is in I as well. Now let I be the ideal of the twisted cubic in projective space. By our computation of the ideal of the twisted cubic in exercise 2.9, we know $I = (x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2)$. So I contains no linear polynomials, and thus the minimal d so that any element in I has a nonzero homogeneous part of degree d is 2.

   Now suppose that there were two generators f, g for I. Then there are polynomials p_1, p_2, p_3, q_1, q_2, q_3 in $k[x_0, x_1, x_2, x_3]$ so that

$$p_1 f + q_1 g = x_0x_3 - x_1x_2,$$

$$p_2 f + q_2 g = x_0x_2 - x_1^2,$$

   and

$$p_3 f + q_3 g = x_1x_3 - x_2^2.$$

   As f and g are in I, we know that $f^{(1)} = g^{(1)} = 0$ by the sentence at the end of the previous paragraph. So $f = f^{(2)} + f'$ and $g = g^{(2)} + g'$. Write $p_i = p_i^{(0)} + p_i'$ and $q_i = q_i^{(0)} + q_i'$ as well. Now we can write the left hand side of the prominent equalities above as $p_i^{(0)} f^{(2)} + q_i^{(0)} g^{(2)}$ as $p_i' f^{(2)} + p_i' f' + p_i^{(0)} f'$ must vanish by degree considerations (similarly for g, q_i).

   As the degree-zero elements of $k[x_0, x_1, x_2, x_3]$ are exactly the constants k and the three generators $x_0x_3 - x_1x_2$, $x_0x_2 - x_1^2$, and $x_1x_3 - x_2^2$ are linearly independent, this means that the k-span of $f^{(2)}$ and $g^{(2)}$ is a three-dimensional vector space. This is impossible, so I must be generated by at least three elements.

   (You may wonder why I was so careful here! One would like to say something like 'if I is generated by two polynomials, then it's generated by two homogeneous polynomials' and then use the linear independence argument. This statement is not true in full generality, though: here is an example I first learned about from Eric Wofsey at Math Stack Exchange question number 3578075.

   Let R be the graded ring where each graded piece R_n is $\mathbb{Z}/n\mathbb{Z}$ with addition defined componentwise and all products of homogeneous elements of positive degree are zero. Now consider a collection of pairwise coprime integers $n_1, \cdots, n_k$, and the element x which is 1 in degrees n_i and 0 elsewhere. Then x generates a homogeneous ideal, since each homogeneous part of x

can be written as mx for some m which is congruent to 1 mod n_i and zero mod n_j for $j \neq i$. But (x) must be generated by at least k homogeneous elements.

So the quoted, hoped-for claim is at least close enough to false that we should be somewhat careful.)

To see that the twisted cubic is a set-theoretic complete intersection, consider $V(x_0x_2 - x_1^2)$ and $V(x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2))$. Let $J = (x_0x_2 - x_1^2, x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2))$. As the intersection of these two hypersurfaces is $V(J)$, showing that these two hypersurfaces intersect in the twisted cubic is equivalent to showing that $\sqrt{J} = I$. Clearly $\sqrt{J} \subset \sqrt{I} = I$ since the two generators vanish on the twisted cubic. On the other hand,

$$(x_1x_3 - x_2^2)^2 = -x_3^2(x_0x_2 - x_1^2) - x_2(x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2))$$

and

$$(x_0x_3 - x_1x_2)^2 = -x_0(x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2)) - x_2^2(x_0x_2 - x_1^2)$$

show that $I \subset \sqrt{J}$.

d. This is still not known to be completely solved as of the preparation of this document. Good luck!

I.3 Morphisms

This section sucks to come back to (from the perspective of someone who usually uses and likes the scheme-theoretic viewpoint) because schemes make lots of these arguments much easier. If you're reading this for the first time, perhaps this will make you look forwards to chapter II, I hope?

Exercise I.3.1.

a. Show that any conic in $\mathbb{A}^2$ is isomorphic either to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$ (cf. Ex. 1.1).

b. Show that $\mathbb{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by (Ex. 6.7) below.)

c. Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

d. We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

e. If an affine variety is isomorphic to a projective variety, then it consists of only one point.

Solution.

a. This is exactly exercise 1.1c.

b. First, note that the proper closed subsets of $\mathbb{A}^1$ are exactly the finite collections of points: $A(\mathbb{A}^1) = k[x]$, which is a PID, so a nontrivial ideal is (f) for some polynomial f. As k is algebraically closed, f splits as a product of elements of the form $(x-a)$. So this shows the claim.

   Thus the proper open subsets of $\mathbb{A}^1$ are exactly the complements of finite collections of points. Say our proper open subset $X \subset \mathbb{A}^1$ is the complement of $\{a_1, \cdots, a_n\}$. Then we define a morphism $X \to V(x(y-a_1)\cdots(y-a_n)-1) \subset \mathbb{A}^2$ by $x \mapsto (x, \frac{1}{(x-a_1)\cdots(x-a_n)})$. This is an isomorphism with inverse $(x,y) \mapsto x$, so we have that $A(X) \cong k[x,y]/(x(y-a_1)\cdots(y-a_n))$.

   If $X \cong \mathbb{A}^1$, then their coordinate algebras would be isomorphic. But $k[x]$ has no non-units outside of k, and $A(X)$ does: x is invertible and outside k. So composing the isomorphisms we get that the image of $x \in A(X)$ under the map $A(X) \to k[x] \to A(X)$ is in k, a contradiction, and the claim is proven.

c. By the same logic as 1.1.1c (the affine coordinate changes we use there extend to projective coordinate changes because they're linear maps), our conic is isomorphic to the one cut out by $xy - z^2$ up to a linear change of variables. Now let $C = V(xy - z^2) \subset \mathbb{P}^2$. Consider $f : \mathbb{P}^1 \to C$ by $[u : v] \mapsto [u^2 : v^2 : uv]$, and $g : C \to \mathbb{P}^1$ by $[r : s : t] \mapsto [1 : t/r]$ when $r \neq 0$ and $[t/s : 1]$ when $s \neq 0$ (note these two conditions agree: when $r, s \neq 0$ then $\frac{t}{r} = \frac{s}{t}$ rearranges to $rs - t^2 = 0$, the defining equation of the variety). By the definition of a morphism of quasi-projective varieties, these are morphisms, and they are mutually inverse to each other. Thus they are isomorphisms, and the claim is proven.

d. I claim that any two codimension-one irreducible sets in $\mathbb{P}^2$ must intersect, but that not every pair of codimension-one irreducible sets in $\mathbb{A}^2$ intersect. If this is true, then as irreducibility and dimension are homeomorphism invariants, we see that $\mathbb{A}^2$ is not homeomorphic to $\mathbb{P}^2$.

   One portion of the claim is easy: consider $V(x)$ and $V(x+1)$ in $\mathbb{A}^2$. These clearly do not intersect. For the other direction, let $X, Y \subset \mathbb{P}^2$ be two varieties of dimension one (= irreducible closed sets of dimension one). By exercise I.2.8, these are both the zero sets of single polynomials, say f_X and f_Y. By exercise I.2.10, their cones $C(X)$ and $C(Y)$ in $\mathbb{A}^3$ are dimension 2, cut out by the ideals (f_X) and (f_Y), and intersect at the origin. Now (f_X, f_Y) is an ideal defining the intersection of these cones, so by exercise I.1.9 every irreducible component of the cone is at least one-dimensional, and thus $C(X) \cap C(Y) = C(X \cap Y)$ contains a line through the origin. So $X \cap Y$ is nonempty, and we are done.

e. Let S be our variety which is simultaneously affine and projective. By Theorem I.3.4, the global functions on S are exactly k. Now let X be the affine variety consisting of a point, which has $A(X) = \mathcal{O}(X) = k$. By proposition I.3.5, the maps from $X \to S$ are in bijection with k-algebra homomorphisms $k \to k$. But the only k-algebra homomorphism $k \to k$ is the identity, so there is a unique map $\{pt\} \to S$, which means that S consists of one point.

Exercise I.3.2. A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

a. For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

b. For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

Solution.

a. A two-sided inverse map of sets is given by $(x, y) \mapsto \frac{y}{x}$ when $(x, y) \neq (0, 0)$ and $(0, 0) \mapsto 0$, so this takes care of bijectivity. As φ is a morphism, it is continuous, and we see that it's also open (the complement of a finite set of points gets mapped to the complement of a finite set of points), so it is a homeomorphism as bijective, continuous, and either closed or open exactly characterize homeomorphisms.

   On the other hand, it is not an isomorphism: this would imply that $k[t] \cong k[x, y]/(y^2 - x^3)$. But the first is a unique factorization domain, and the second is not: x and y are both irreducibles, but $y^2 = x^3$ demonstrates two distinct factorizations of one element.

b. Over a field of characteristic p, we have $x^p - y^p = (x - y)^p$, which immediately shows injectivity. On the other hand, k is algebraically closed, so $x^p - c$ for fixed c must always have a solution, so the Frobenius is surjective. As it is a morphism, it is continuous, and it's closed as well: any proper closed subset is a finite set of points, and the finite sets of points are exactly mapped to finite sets of points as the map is a bijection.

To show that this is not an isomorphism, we note that the induced map on coordinate algebras is $k[x] \to k[x]$ by $x \mapsto x^p$, which is not surjective and thus cannot be an isomorphism.

Exercise I.3.3.

a. Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.

b. Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism and the induced map φ^* on local rings is an isomorphism, for all $P \in X$.

c. Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

Solution.

a. This is a straightforwards definition-chase. Any regular function on a neighborhood U of $\varphi(P)$ pulls back to a regular function on $\varphi^{-1}(U)$ by definition of a morphism, and the equivalence relation is preserved as well.

b. Clearly an isomorphism of varieties must be an isomorphism of the underlying topological spaces, also known as a homeomorphism. The inverse morphism induces a map the other way on the local rings which is a two-sided inverse to the map of local rings induced by φ, so we get an isomorphism on each local ring.

   On the other hand, if our morphism isn't an isomorphism, then either it's not a homeomorphism, or it is, and the topological inverse map φ^{-1} for the homeomorphism isn't a morphism of varieties. In the latter case, this means that there is a regular function f on an open set $U \subset X$ so that $f \circ \varphi^{-1} : \varphi(U) \to k$ isn't regular. But this means that the equivalence class of f in the local ring at any point in U isn't hit by the induced morphism of local rings, and thus the morphism of local rings cannot be an isomorphism. We are done.

c. Suppose there is a $P \in X$ so that φ_P^* is not injective, that is, it takes a function f which does not vanish on some nonempty open subset U to the zero function. Then $\varphi(Y) \cap U$ must be empty, which contradicts the assumption that $\varphi(Y)$ is dense: any dense subset must meet every nonempty open subset.

Exercise I.3.4. Show that the d-uple embedding of $\mathbb{P}^n$ (Ex. 2.12) is an isomorphism onto it's image.

Solution. The set-theoretic inverse from our method from exercise I.2.12 part (b) which showed that the image is exactly $V(\mathfrak{a})$ turns out to actually be a morphism, which shows the required isomorphism. It is easy to see that this map is locally around any point given by regular functions, and that any set-theoretic map of quasi-projective varieties which is locally given by regular functions must satisfy the definition of a morphism on page 14-15 because the composition of rational functions is again a rational function.

Exercise I.3.5. By abuse of language, we will say that a variety 'is affine' if it is isomorphic to an affine variety. If $H \subset \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n \setminus H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ and use the fact that $\mathbb{P}^N$ minus a hyperplane is affine.]

Solution. Follow the hint. Embed $\mathbb{P}^n \to \mathbb{P}^N$ by the d-uple embedding where d is the degree of the hypersurface H. Then $H \subset \mathbb{P}^n$ is the intersection of a hyperplane $H' \subset \mathbb{P}^N$ with $\mathbb{P}^n$. By proposition 3.3, the complement of H' is affine, so $\mathbb{P}^n \setminus H = \rho_d(\mathbb{P}^n) \cap (\mathbb{P}^N \setminus H')$ is an affine variety, being a closed irreducible subset of an affine variety.

Exercise I.3.6. There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x,y]$ and use (3.5). See (III, Ex 4.3) for another proof.]

Solution. First, a general fact: if X is an affine variety and f is an element of $A(X)$, then $X \setminus V(f)$ is again affine and has coordinate algebra $A(X)_f$, the localization at f. To see this, embed X as $V(I) \subset \mathbb{A}^n$ and then we can write $X \setminus V(f)$ as $V(I, fx_{n+1}-1) \subset \mathbb{A}^{n+1}$ where the maps $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, f)$ and $(x_1, \cdots, x_n, x_{n+1}) \mapsto (x_1, \cdots, x_n)$ define isomorphisms between the two varieties. It is straightforward to see that the latter variety is affine with coordinate ring $A(X)_f$. This is a very useful fact and will be used a lot without much reference from now on.

Suppose f was a global function on $X = \mathbb{A}^2 \setminus \{(0,0)\}$. Then the restriction of f to $D(f) = X \setminus V(x)$ is a regular function (since regularity is a local criteria) and thus we can write it as $\frac{g}{x^m}$ for some polynomial $g \in k[x,y]$ not divisible by x. Similarly, the restriction of f to $D(y) = X \setminus V(y)$ is regular and may be written $\frac{h}{y^n}$ for $h \in k[x,y]$ not divisible by x. On $D(xy)$, we have that the restrictions must coincide, so we have $y^n g = x^m h$. As $k[x,y]$ is a UFD and y does not divide the RHS, we have $n = 0$. Similarly, $m = 0$ too. So $f = g = h$ and therefore $\mathcal{O}_X = k[x,y]$, so if it were affine then we must have $X \cong \mathbb{A}^2$ by Corollary 3.7. But this cannot be: $V(x,y) \subset X$ and $V(1)$ are both the empty set, contradicting corollary I.1.4.

Exercise I.3.7.

a. Show any two curves in $\mathbb{P}^2$ have nonempty intersection.

b. More generally, show that if $Y \subset \mathbb{P}^n$ is a projective variety of dimension ≥ 1, and if H is a hypersurface, then $Y \cap H \neq \emptyset$. [*Hint*: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

Solution.

a. By the work in exercise I.3.1(d), we have that every two codimension-one irreducible algebraic sets in $\mathbb{P}^2$ intersect.

b. The same logic as exercise I.3.1(d) works again: move to the affine cones in $\mathbb{A}^{n+1}$ and now we are intersecting two varieties of dimensions ≥ 2 and n which meet at the origin. Their intersection will have dimension at least one and thus must contain a line through the origin, which translates in to their intersection in projective space being nonempty.

Exercise I.3.8. Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n \setminus (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbb{P}^n$.)

Solution. Rename $x_i = x$ and $x_j = y$. Notice $\mathbb{P}^n \setminus (H_x \cap H_y) = \mathbb{A}^n_x \cup \mathbb{A}^n_y$, where regular functions on the first are of the form $\frac{g}{x^m}$ for $\deg g = m$ and g not divisible by x and regular functions on the second are of the form $\frac{h}{y^n}$ for $\deg h = n$ and h not divisible by y. On the overlap we have $\frac{g}{x^m} = \frac{h}{y^n}$, so by cross-multiplying we see that $m = n = 0$ so g, h are just constants. (This is the same as 3.6, essentially, and it works for the same reasons: Hartog's lemma.)

Exercise I.3.9. The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbb{P}^1$ and let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$ (Ex 3.4). But show that $S(X) \ncong S(Y)$.

Solution. The homogeneous coordinate ring of the first is $k[x_0, x_1]$, a UFD. The homogeneous coordinate ring of the second is $k[x_0, x_1, x_2]/(x_0x_2 - x_1^2)$, which is not a UFD. One may also check the dimension of the first graded piece: 2 versus 3. (See the material on the d-uple embedding for a reminder about why the coordinate ring is what it is.)

Exercise I.3.10. *Subvarieties.* A subset of a topological space is *locally closed* if it is an open subset of its closure, or equivalently, if it is the intersection of an open subset with a closed subset.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the *induced structure* on Y, and we call Y a *subvariety* of X.

Now let $\varphi : X \to Y$ be a morphism, let $X' \subset X$ and $Y' \subset Y$ be irreducible locally closed subsets such that $\varphi(X') \subset Y'$. Show that $\varphi|_{X'} : X' \to Y'$ is a morphism.

Solution. This follows directly from the definitions - it should be an observation located in the preceeding section, not an exercise.

Exercise I.3.11. Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

Solution. First, I claim that for any open neighborhood U of a point P in an irreducible topological space X, there is a bijection between irreducible closed subsets of X passing through P and irreducible closed subsets of U passing through P. The bijection is given by intersecting with U and taking closures inside X. These operations clearly preserve irreducibility (see exercise I.1.6). For any $Y \subset X$ closed and irreducible containing P, we have that $Y \cap U$ is a dense open subset of Y, so $\overline{Y \cap U} = Y$. On the other hand, for any $V \subset U$ closed and irreducible, then $\overline{V} \cap U = V$ by definition of the subspace topology. So it suffices to show this claim in a neighborhood of P.

Next, I claim that any point in a quasi-projective variety has an affine open neighborhood. Embed $X \subset \mathbb{P}^n$ as a locally closed subset. If X is projective, then take some standard affine open U_0, and the intersection $X \cap U_0$ is affine. If X is merely quasi-projective, write $X = V(I) \setminus V(J)$ for two homogeneous ideals $I, J \subset k[x_0, \cdots, x_n]$. Now I claim that there is a homogeneous generator

$f \in J$ which does not vanish on P: if all such generators vanished on P, then $P \in V(J)$ which is obviously wrong. So $Y := V(I) \cap V(f)^c$ is an affine open neighborhood of P, being a closed subset of the affine variety $\mathbb{P}^n \setminus V(f)$ where we use exercise I.3.5 for affineness.

Finally, I claim that $\mathcal{O}_P$ is the same ring whether we consider $P \in X$ or $P \in Y$: this is more or less obvious from the definition of $\mathcal{O}_P$ via equivalence relations. (In fact, the calculation of $\mathcal{O}_P$ will be the same no matter what open neighborhood of P we start out with.)

Now everything is straightforwards: the local ring $\mathcal{O}_P$ is the same as the localization of $A(Y)$ at the maximal ideal picking out P by Theorem I.3.2(c), and the prime ideals of a localization at a maximal ideal are exactly the prime ideals contained inside that maximal ideal. Applying the bijection between irreducible closed subsets and prime ideals, we see the result on Y, which by the first paragraph is equivalent to the result on the whole of X.

Exercise I.3.12. If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use (3.2c).]

Solution. We use the observation that $\mathcal{O}_P$ may be calculated from any open neighborhood of P: this is clear from the definition by equivalence classes of functions agreeing where defined. By the work in exercise I.3.11 above, any point in a quasi-projective variety has an affine open neighborhood, which by Theorem I.3.2(c) is of dimension equal to $\dim \mathcal{O}_P$. Thus we get an open cover of our variety by sets of dimension $\dim \mathcal{O}_P$, which means that our variety had dimension $\dim \mathcal{O}_P$ by exercise I.1.10.

Exercise I.3.13. *The Local Ring of a Subvariety.* Let $Y \subset X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f \rangle$ where $U \subset X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U. We say $\langle U, f \rangle$ is equivalent to $\langle V, g \rangle$ if $f = g$ on $U \cap V$. Show $\mathcal{O}_{Y,X}$ is a local ring with residue field $K(Y)$ and dimension $\dim X = \dim Y$. It is the *local ring* of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

Solution. We may immediately make two reductions: first, we may assume X is affine by taking an affine open neighborhood in X of some point in Y (see exercise I.3.11 for a reminder about why we can do this). Next, we may assume Y is closed in X: the equivalence classes of functions on Y and functions on $\overline{Y}$ are the same via taking intersections with open subsets of Y. So we are in the situation where X is affine and $Y \subset X$ is closed irreducible, say equal to $V(I)$ for some prime ideal I.

Now I claim that $\mathcal{O}_{Y,X} = A(X)_I$. For any element $\frac{f}{s}$ of $A(X)_I$, we note that s cannot vanish on Y, so $\frac{f}{s}$ defines a regular function on an open set which has nonempty intersection with Y. Conversely, any function defined on an open subset of X meeting Y can be represented locally as $\frac{f}{s}$ for s not in I, and thus every function has a representative in $A(X)_I$.

By the description of $\mathcal{O}_{Y,X}$ as $A(X)_I$ it is immediate that it is a local ring and by a standard argument from commutative algebra we have $A(X)_I/I_I \cong \operatorname{Frac} A(X)/I$. Dimension is clear as well: the prime ideals of $A(X)_I$ are exactly the prime ideals of $A(X)$ contained in I, and by extending a maximal chain of prime ideals we see the result on relative dimension.

(I have no idea why Hartshorne feels the need to point out that there are local rings with non-algebraically closed residue fields - for instance, $\mathbb{Z}_{(p)}$ is an example already and it's much easier to construct.)

Exercise I.3.14. *Projection from a Point.* Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$. Define a mapping $\varphi : \mathbb{P}^{n+1} \setminus \{P\} \to \mathbb{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.

a. Show that φ is a morphism.

b. Let $Y \subset \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbb{P}^1$ (Ex. 2.12). If t, u are homogeneous coordinates on $\mathbb{P}^1$, we say that Y is the curve given *parametrically* by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$ and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

Solution.

a. Up to a linear change of coordinates, we may assume $\mathbb{P}^n$ is $V(x_{n+1})$ and $P = [0 : \cdots : 0 : 1]$. Now I claim that for a point $Q = [a_0 : \cdots : a_{n+1}] \neq P$, the projection is just $[a_0 : \cdots : a_n]$. We can see this by writing the line between P and Q as $[ta_0 : \cdots : ta_{n+1} + s]$, which intersects $V(x_{n+1})$ when $s = -ta_{n+1}$. This makes it clear that any regular function pulls back to a regular function, so by the definition of a morphism we are finished.

b. By (a), we want to look at the parameterized curve given by $[t^3 : t^2u : u^3]$. This is a 1-dimensional closed subset of $\mathbb{P}^2$, so it's given by one equation. By inspection, we see that it's $y^3 - x^2z$, which we recognize as the cuspidal cubic in the affine patch $z = 1$. (If you're interested more in projections, there's a lot of interesting material about them! One key term is elimination theory, which is how you remove variables from systems of equations. See MSE question 3509044 for some more material.)

Exercise I.3.15. *Products of Affine Varieties.* Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties.

a. Show that $X \times Y \subset \mathbb{A}^{n+m}$ with it's induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subset $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subset Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).

b. Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

c. Show that $X \times Y$ is a product in the category of varieties, i.e., show that (i) the projetions $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dotted, "f"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \end{tikzcd}$$

d. Show that $\dim X \times Y = \dim X + \dim Y$.

Solution.

a. Follow the hint. Since $\{x\} \times Y \cong Y$ for any $x \in X$, we see that $Z_i \cap (\{x\} \times Y)$ are closed subsets of an irreducible variety, so one of them must be the whole of $\{x\} \times Y$. Thus every point in X is either in X_1 or X_2, and so $X = X_1 \cup X_2$.

Now I claim that $X_1 = \bigcap_{y \in Y} Z_1 \cap (X \times \{y\})$, which will show that it is closed as it is the intersection of the closed subsets $Z_1 \cap (X \times \{y\})$ (we use the identification $X \times \{y\} \cong X$ in here without writing it down). Indeed, if $x \in X$ is in X_1, that means that $(x, y) \in Z_1$ for all y, so $X_1 \subset \bigcap_{y \in Y} Z_1 \cap (X \times \{y\})$. Conversely, if $x \notin X_1$, then there is some $y \in Y$ so that $(x, y) \notin Z_1$, so $X_1 \supset \bigcap_{y \in Y} Z_1 \cap (X \times \{y\})$. Thus X_1 and X_2 are closed.

As $X = X_1 \cup X_2$ is a decomoposition of an irreducible space as a union of two closed subsets, we see that at least one of X_1 or X_2 is equal to X, and we are done. (Note that we never used anything about the projection maps being closed or anything like that - this is important to avoid, because that's false! The example $V(xy - 1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ is instructive in that it is a closed but not open subset whose projection is open but not closed.)

b. First, note that given a regular function f on X and a regular function g on Y, the function fg which assigns to each point $(x, y) \in X \times Y$ the value $f(x)g(y)$ is regular. So we can define a map $A(X) \times A(Y) \to A(X \times Y)$ by $(f, g) \mapsto fg$. As this is bilinear over k, it descends to a map from the tensor product $A(X) \otimes_k A(Y) \to A(X \times Y)$ by $\sum f_i \otimes g_i \mapsto \sum f_i g_i$. As the restriction of the coordinate functions on $\mathbb{A}^{m+n}$ to $X \times Y$ are in the image (they're the image of $x_i \otimes 1$ and $1 \otimes y_j$), the map is surjective.

So all that's left to do to show this is an isomorphism is to show injectivity. Suppose $\sum f_i \otimes g_i$ is sent to zero: we will show it's already zero in $A(X) \otimes_k A(Y)$. By successively rewriting any g_j is in the linear span of the other g_i, either there is a single unique g_i and it's zero and we're done or we may assume that all the g_i are linearly independent in $A(Y)$. Now for any fixed x_0, we have that $\sum f_i(x_0) g_i = 0$, which is a linear dependence relation on the g_i. By the assumption that they're linearly independent, we must have $f_i(x_0) = 0$ for all x_0 and all f_i. But the only such element of $A(X)$ is the zero element, so every f_i is zero, and $\sum f_i \otimes g_i$ is zero as required and our map is injective.

c. Denote $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the projection morphisms, as well as $f_X : Z \to X$ and $f_Y : Z \to Y$ the given morphisms. Now I claim that $z \mapsto (f_X(z), f_Y(z))$ is the unique morphism $Z \to X \times Y$. Uniqueness is clear, so all that's left to check is the fact this is a morphism. This is handled by the fact that f_X and f_Y are individually morphisms and the work we carried out in (b): any regular function on $X \times Y$ can be written locally as a quotient of polynomials with nonvanishing denominator, which when pulled back along our unique map obviously gets taken to a quotient of polynomials with nonvanishing denominator.

d. Let us recall some concepts about field extensions from algebra. Let $F \subset K$ be a field extension and let $F \subset L, M$ be subextensions. We say that L/F and M/F are algebraically disjoint if for any collection of elements $S \subset L$ and $T \subset M$ which are algebraically independent over F, we have $S \cap T = \emptyset$ and $S \cup T$ is again algebraically independent over F when considered as elements of K.

   Now I claim that if S and T are transcendence bases of algebraically disjoint subextensions $F \subset L$ and $F \subset M$ of a common extension $F \subset K$, then $S \cup T$ is a transcendence basis for the compositum LM viewed as an extension of F. Clearly by definition $S \cup T$ is algebraically independent, so it suffices to prove that LM is algebraic over $F(S \cup T)$. But this is straightforwards: if L is generated by the algebraic elements $\{\alpha_i\}_{i \in I}$ over $F(S)$ and M is generated by the algebraic elements $\{\beta_j\}_{j \in J}$ over $F(T)$, then LM is generated over $F(S \cup T)$ by the union $\{\alpha_i\}_{i \in I} \cup \{\beta_j\}_{j \in J}$, and all of these elements remain algebraic. So our claim is proven.

   To apply this to the situation at hand, I claim that $k(X)$ and $k(Y)$ are algebraically disjoint subextensions of $k \subset k(X \times Y)$ so that $k(X)k(Y) = k(X \times Y)$. Algebraic independence is straightforwards to prove: as $k(X) \cap k(Y) = k$, we need only show that the union of any two collections of algebraically independent elements remain algebraically independent. Suppose $S \subset k(X)$ and $T \subset k(Y)$ are both algebraically independent over k, and that we have a nontrivial algebraic relation among the memebers of $S \cup T$. Then this implies that after clearing denominators, we have a nontrivial polynomial in $A(X \times Y)$ which vanishes on $X \times Y$, which is an impossibility. So $k(X)$ and $k(Y)$ are algebraically independent as subfields of $k(X \times Y)$.

   To see that their compositum is all of $k(X \times Y)$, we use part (b). $A(X) \otimes_k A(Y) = A(X \times Y)$, so $k(X)k(Y)$ contains all of $A(X \times Y)$, and $k(X \times Y) = \operatorname{Frac} A(X \times Y)$, so the claim is proven. By the result on the transcendence degree of the compositum, we have $\operatorname{trdeg} k(X \times Y) = \operatorname{trdeg} k(X) + \operatorname{trdeg} k(Y)$, which by theorem I.1.8A translates to $\dim X \times Y = \dim X + \dim Y$.

Exercise I.3.16. *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of a projective variety. Now for any two quasi-projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, consider $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$.

a. Show that $X \times Y$ is a quasi-projective variety.

b. If X, Y are both projective, show that $X \times Y$ is projective.

c. (*) Show that $X \times Y$ is a product in the category of varieties.

Solution.

a. Consider the projections $\pi_{\mathbb{P}^n} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\pi_{\mathbb{P}^m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$. Clearly we have $\pi_{\mathbb{P}^n}^{-1}(X) = X \times \mathbb{P}^m$ and $\pi_{\mathbb{P}^m}^{-1}(Y) = \mathbb{P}^n \times Y$, and the intersection of these quasi-projective varieties is $X \times Y$, so $X \times Y$ is quasi-projective because intersections of quasi-projective varieties are quasi-projective.

b. Same as (a), except replacing quasi-projective with projective.

c. This is the same as exercise I.3.15(c) with a few minor modifications. Details left to the reader.

Exercise I.3.17. *Normal Varieties.* A variety Y is *normal* at a point $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is normal if it is normal at every point.

a. Show that every conic in $\mathbb{P}^2$ is normal.

b. Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations Q_1: $xy = zw$, Q_2: $xy = z^2$ are normal (cf (II. Ex. 6.4) for the latter.)

c. Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ is not normal.

d. If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.

e. Let Y be an affine variety. Show that there is a normal affine variety $\widetilde{Y}$ and a morphism $\pi : \widetilde{Y} \to Y$ with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a *dominant* morphism (i.e. $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to \widetilde{Y}$ so that $\varphi = \pi \circ \theta$. $\widetilde{Y}$ is called the normalization of Y. You will need (3.9A) above.

Solution. A few reminders before we begin: local rings don't depend on the open neighborhood used to calculate them. Any UFD is normal (this is the rational root theorem). As localizations of UFDs are UFDs (providing you don't invert 0) and any polynomial ring over a field is a UFD, we have that any localization of a polynomial ring over a field is normal.

a. By exercise I.3.1(c), every conic is isomorphic to $\mathbb{P}^1$, so every local ring of a conic is isomorphic to a local ring of $\mathbb{P}^1$. But all local rings of $\mathbb{P}^1$ are isomorphic to $k[x]_{(x)}$, which is normal by our reminders above.

b. Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations Q_1: $xy = zw$, Q_2: $xy = z^2$ are normal (cf (II. Ex. 6.4) for the latter.)

   The first is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by exercise I.2.15(a), so any point is located in an affine coordinate patch of the form $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and thus has local ring isomorphic to a localization of a polynomial ring over a field (specifically, it's isomorphic to $k[x, y]_{(x,y)}$), so it's normal by the reminders above.

   For the second, we can look at the affine patches. On the affine patch where $x \neq 0$, our surface is $V(y = z^2) \subset \mathbb{A}^3$, which has coordinate algebra $k[y, w]$ and thus any local ring in this patch is normal (the same logic holds for the affine patch with $y \neq 0$). If $z \neq 0$, then $x, y \neq 0$ too, so any point in the patch $z \neq 0$ is in one of the above patches, and all that's left is to check the local ring at $[0 : 0 : 0 : 1]$.

   We work in the patch given by $w \neq 0$. Here our variety is $V(xy - z^2) \subset \mathbb{A}^3$ and we're interested in the local ring at the origin, $(k[x, y, z]/(xy - z^2))_{(x,y,z)}$. We'll show that $k[x, y, z]/(xy - z^2)$ is integrally closed, which will imply the result for the local ring based on the reminders. We see that $K = \operatorname{Frac}(k[x,y,z]/(xy-z^2))$ is a degree-two extension of $k(x,y)$, so we can write any element of K as $u+vz$ for $u,v \in k(x,y)$. Now $u+vz$ satisfies $P(T) = T^2 - 2uT - (v^2xy - u^2)$, and I claim that if $u+vz$ is integral over $k[x,y,z]/(xy-z^2)$ then the coefficients of this polynomial are in $k[x,y]$.

If $v = 0$, the minimal polynomial for $u \in k(x,y)$ is $T - u$, which must divide the polynomial which witnesses u as integral over $k[x,y,z]/(xy-z^2)$. We may assume this polynomial is actually in $k[x,y]$ by multiplying it by its conjugate under the action $z \mapsto -z$, so u is a root of a monic polynomial with coefficients in $k[x,y]$ and is thus integral over $k[x,y]$ and is in $k[x,y]$. If $v \neq 0$, then $P(T)$ is the minimal polynomial of $u+vz$ as an element of $K/k(x,y)$, so it must divide the polynomial which shows that $u+vz$ is integral over $k[x,y,z]/(xy-z^2)$. By the same argument with the conjugate as in the $v=0$ case, this means that the roots of $P(T)$ are integral over $k[x,y]$, so the coefficients of this polynomial must also be integral over $k[x,y]$ by Vieta's formulas. But the elements of $k(x,y)$ integral over $k[x,y]$ are exactly the elements of $k[x,y]$ because $k[x,y]$ is integrally closed in its field of fractions, and the claim is proven. (This generalizes to the statement that if $R \subset F$ is the inclusion of an integrally closed domain in to its field of fractions and $F \subset E$ is a finite field extension, then any element in E which is integral over R has minimal polynomial in $R[T]$, which always feels like a bit of magic every time I use it even though it's not a particularly difficult proof.)

Now if $\operatorname{char} k \neq 2$, we are quickly done: $2u \in k[x,y]$ implies $u \in k[x,y]$, and thus v^2xy and therefore v must also be in $k[x,y]$, so $k[x,y,z]/(xy-z^2)$ is integrally closed. (This generalizes to $k[x_i,z]/(z^2-f)$ being normal for k not of characteristic 2 and $f \in k[x_i]$ squarefree.)

If the characteristic of k is 2, things are a little more involved. We may start by assuming $v \neq 0$ in our element $u+vz \in K$ integral over $k[x,y]$. Our goal is to show that if $v^2xy + u^2 = r \in k[x,y]$, then $u,v \in k[x,y]$. Write $u = f/g$ and $v = p/q$ in lowest terms, and after expanding the equality, we get

$$g^2p^2xy = q^2(rg^2 - f^2).$$

Now I claim $g = q$ up to a multiplicative constant. Letting α be an irreducible factor of g appearing with exponent d in the factorization of g, α^{2d} must divide q^2 because g and $rg^2 - f^2$ are relatively prime. Therefore α^d must divide q. Conversely, if β is an irreducible factor of q appearing with exponent e in the factorization of q, β^{2e} must divide g^2xy because p and q are relatively prime. Since x and y are coprime irreducibles, β can only divide at most one of them to order at most one, so β^{2d-1} must divide g^2, and therefore β^d must divide g. Since $k[x,y]$ is a domain, our equation reduces to

$$cp^2xy = rg^2 - f^2$$

for $c \in k^\times$.

Now we need to do a little manipulation with r. Let $r = r_{00} + r_{10} + r_{01} + r_{11}$ be the decomposition in to polynomials where all entries in r_{ij} have all exponents of x equal to i

mod 2 and exponents of y equal to j mod 2. Since the LHS of our equation consists only of terms of type 11, and f^2 as well as g^2 consists only of terms of type 00, we see that $r_{01} = r_{10} = 0$, $r_{11}g^2 = p^2xy$, and $r_{00}g^2 = f^2$. By coprimality of f and g, we have either $g = 1$ or $r_{00} = 0$. By coprimailty of g and p, we have either $g = 1$ or $r_{11} = 0$ (here's where we use that xy is squarefree - any factor of g divides the RHS to order 2, and thus divide p^2 to order at least 1, which means that it must divide p, contradicting coprimality). So either $g = 1$ or $r = 0$, and in both cases we get $u, v \in k[x, y]$.

c. The coordinate algebra is isomorphic to $k[t^2, t^3]$ which is not integrally closed in $k(t)$, the field of fractions: this is because the polynomial $p(\lambda) = \lambda^2 - t^2$ is satisfied by $\frac{t^3}{t^2}$ which is not in $k[t^2, t^3]$.

d. First we show that a ring R being integrally closed implies any (nonzero) localization is integrally closed. Suppose $g \in R$ is integral over $S^{-1}R$, satisfying $P = x^d + \sum_{i<d} c_i x^i$ where $c_i \in S^{-1}R$. Pick some s so that $sc_i \in R$ for all i. Then sg satisfies $x^d + \sum_{i<d} s^i c_i x^i$, so it is integral over R and thus in R, so $g \in S^{-1}R$.

   Conversely, if every localization of a domain at a maximal ideal is integrally closed, then our ring is integrally closed. Recall from algebra that $R = \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$, where the intersection is taken in the field of fractions. So any element in $\operatorname{Frac} R$ integral over R is also integral over each $R_{\mathfrak{m}}$, thus in $R_{\mathfrak{m}}$ by integral closedness, and then by the intersection identity must be in R.

e. The idea is to work on the coordinate algebra side. First, a dominant morphism of affine varieties $Z \to Y$ induces an injective map on coordinate rings: if the image of Z is dense, then every function vanishing on the image of Z vanishes everywhere since the vanishing set of a function is closed. So since the only function in a domain which vanishes everywhere is the zero function, we see that $\ker(A(Y) \to A(Z)) = 0$. (The grown-up version of this statement is that $\operatorname{Spec} B \to \operatorname{Spec} A$ dominant is equivalent to the kernel of the map $A \to B$ being contained in the nilradical of A - see EGA I 1.2.7 and remember to forgive yourself for being in the land where we we're using the definition of an affine variety we currently have.)

   Thus our morphism $f^* : A(Y) \to A(Z)$ is injective and induces a map on fraction fields $f^* : K(Y) \to K(Z)$ which sends $A(Y)$ to $A(Z)$. By assumption, $A(Z)$ is integrally closed in $K(Z)$, so it must contain the image of the integral closure of $A(Y) \subset K(Y)$: any element of $K(Y)$ satisfying a monic polynomial with coefficients in $A(Y)$ gets taken to an element of $K(Z)$ satisfying a monic polynomial with coefficients in $f^*(A(Y)) \subset A(Z)$, and $A(Z)$ is integrally closed in $K(Z)$ by assumption. So our map $A(Y) \to A(Z)$ factors uniquely through $A(Y)^\nu$, the normalization of $A(Y)$.

   All that remains to do is to show that $A(Y)^\nu$ is the coordinate algebra of an affine variety: if this were true, by corollary I.3.8 we would get $\widetilde{Y}$ and the unique factorization of $Z \to Y$ as $Z \to \widetilde{Y} \to Y$ by the equivalence of categories. This claim may be proven by applying theorem I.3.9A, with $A = A(Y)$ and $K = L = K(Y)$ to see that $A(Y)^\nu$ is a finitely generated k-algebra, and as it's a subring of a domain, a domain. So by choosing generators, we have that $A(Y)^\nu = k[t_1, \cdots, t_n]/I$ where I is prime, and thus $A(Y)$ is the coordinate algebra of some affine variety over k.

Exercise I.3.18. *Projectively Normal Varieties.* A projective variety $Y \subset \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

a. If Y is projectively normal, then Y is normal.

b. There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic in $\mathbb{P}^3$ given parametrically by $(x,y,z,w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.

c. Show that the twisted quartic curve Y is isomorphic to $\mathbb{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

Solution.

a. By theorem I.3.4(b), the local rings of Y are localizations of $S(Y)$. If $S(Y)$ is normal, then any nonzero localization is too by the work we did in exercise I.3.17(d) and we're done.

b. Let's show that Y is not projectively normal first. The homogeneous coordinate ring is $k[t^4, t^3u, tu^3, u^4]$, and $x^4 - t^4u^4$ is satisfied by $t^2u^2 = \frac{t^6u^2}{t^4}$ which is in the fraction field but not the ring itself.

   On the other hand, if Y is isomorphic to $\mathbb{P}^1$, then it is normal since isomorphisms preserve local rings by exercise I.3.3(a) and the local rings of $\mathbb{P}^1$ are exactly $k[x]_{(x)}$, which are UFDs and thus normal (see exercise I.3.17 for a reminder). So we'll be done once we've finished part (c).

c. Define a map $\mathbb{P}^1 \to Y$ by $[t : u] \mapsto [t^4 : t^3u : tu^3 : u^4]$ and a map $Y \to \mathbb{P}^1$ by $[x : y : z : w] \mapsto [1 : y/x]$ if $x \neq 0$ and $[x : y : z : w] \mapsto [z/w : 1]$ if $w \neq 0$. These are clearly mutually inverse morphisms, so we have our desired isomorphism. The homogeneous coordinate ring of $\mathbb{P}^1 \subset \mathbb{P}^1$ is just $k[x,y]$ which is obviously normal as it is a UFD.

Exercise I.3.19. *Automorphisms of* $\mathbb{A}^n$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \cdots, f_n$ of n variables $x_1, \cdots, x_n$. Let $J = \det(\frac{\partial f_i}{\partial x_j})_{ij}$ be the *Jacobian* polynomial of φ.

a. If φ is an isomorphism (in which case we call φ an *automorphism* of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

b. The converse of (a) is an unsolved problem, even for $n = 2$.

Solution. If φ is an isomorphism with inverse ψ, then we have that $J_\varphi \cdot J_\psi = J_{\varphi\circ\psi}$ by the chain rule, and the RHS is 1. As the only invertible elements of $k[x_1, \cdots, x_n]$ are the units in k, we have the result.

Part (b) is still very open. I would recommend considering whether you want to bash your head against the same wall as everyone else has been for more than 50 years before spending too much time thinking about the problem.

Exercise I.3.20. Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Left f be a regular function on $Y \setminus P$.

a. Show that f extends to a regular function on Y.

b. Show this would be false for $\dim Y = 1$.

Solution.

a. It suffices to consider the affine case where our variety has coordinate algebra $A(Y)$ which is a domain and a finitely generated k-algebra. f defines an element in $K(Y)$, and we want to show that it is in $\mathcal{O}_P \subset K(Y)$. By the assumption that f is regular away from P and P is of codimension at least two, we see that our function f defines an element in the local ring of any subvariety of codimension one passing through P, or rather, for any prime ideal $\mathfrak{p}$ of height one contained in the ideal $\mathfrak{m}$ cutting out P, we have that $f \in A(Y)_{\mathfrak{p}}$.

   Now recall a result from algebra: for a noetherian normal domain R with fraction field K, we have $R = \bigcap R_{\mathfrak{q}}$ inside K, where $\mathfrak{q}$ runs over the ideals of height one in R. In our situation, take $R = \mathcal{O}_P$ and the $\mathfrak{q}$ runs over all primes $\mathfrak{p}$ of height one containing $\mathfrak{m}$, and then the equality gives us that $f \in \mathcal{O}_P$.

   (For a reference to this claim about noetherian normal domains, see Matsumura's Commutative Ring Theory, theorem 11.5.)

b. Let $Y = \mathbb{A}^1$ and $f = \frac{1}{x}$. Then f cannot be in the local ring at the origin: the local ring at the origin is the valuation ring inside $k(x)$ defined by the valuation with respect to x being non-negative, but the valuation of f with respect to x is -1.

Exercise I.3.21. *Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \to Y$.

a. The *additive group* $\mathbb{G}_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $\mu(a,b) = a + b$. Show it is a group variety.

b. The *multiplicative group* $\mathbb{G}_m$ is given by the variety $\mathbb{A}^1 \setminus \{0\}$ and the morphism $\mu(a,b) = ab$. Show it is a group variety.

c. If G is a group variety, and X is any variety, show that the set $\operatorname{Hom}(X, G)$ has a natural group structure.

d. For any variety X, show that $\operatorname{Hom}(X, \mathbb{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

e. For any variety X, show that $\operatorname{Hom}(X, \mathbb{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

Solution.

a. This is just the underlying additive group of k, and the inversion map is $x \mapsto -x$ which is clearly a morphism.

b. This is just the underlying multiplicative group of k^* and the inversion map is $x \mapsto \frac{1}{x}$ which is clearly a morphism.

c. Given two maps $f, g : X \to G$ we can define a new map h by $h(x) = f(x) \cdot g(x)$ where the product on the RHS is given by the group operation. The inversion morphism is given by post-composing with the inversion map on G, and since the composition of morphisms is again a morphism, we see that this gives the hom set the structure of a group.

d. By proposition I.3.5, this hom set is the same as $\mathrm{Hom}(k[t], \mathcal{O}(X))$ in the category of k-algebras. Such maps are uniquely determined by where t goes, and t can go anywhere, so the isomorphism is exactly given by $f \mapsto f(t)$.

e. By the same reasons as the previous part, this hom set is the same as $\mathrm{Hom}(k[t, t^{-1}], \mathcal{O}(X))$ in the category of k-algebras. The maps are still uniquely determined by where t goes, but this time t must be mapped to an invertible element. So the isomorphism between the hom set and $\mathcal{O}(X)^*$ is just given by $f \mapsto f(t)$ again.

I.4 Rational Maps

Regular maps come with a lot of structure and a lot of requirements, and it's occasionally useful to be able to relax a little bit and think about maps which are defined most places instead of necessarily all over. This is exactly what a rational map is!

Exercise I.4.1. If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a rational function on X, there is a largest open subset U of X on which f is represented by a regular function. We say that f *is defined* at the points of U.

Solution. This is obvious: define a function to be $f(x)$ if $x \in U$ and $g(x)$ if $x \in V$. Then regularity is clear by the fact that the function is regular on U and regular on V.

Exercise I.4.2. Same problem for rational maps. If φ is a rational map of X to Y, show that there is a largest open subset on which φ is represented by a morphism. We sat the rational map *is defined* at the points of that open set.

Solution. Again, obvious: glue as in exercise I.4.1 where the relevant open sets are where the various equivalence classes are defined.

Exercise I.4.3.

a. Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.

b. Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$, and let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

Solution.

a. This is defined on the standard affine open $D(x_0) = \mathbb{A}^2$ which has coordinates $\frac{x_1}{x_0}$ and $\frac{x_2}{x_0}$, and the regular function is exactly $\frac{x_1}{x_0}$.

b. This gives us the rational map $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$, so it's the projection, and it's defined everywhere except $[0 : 0 : 1]$.

Exercise I.4.4. A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).

a. Any conic in $\mathbb{P}^2$ is a rational curve.

b. The cuspidal cubic $y^2 = x^3$ is a rational curve.

c. Let Y be the nodal cubic curve $y^2z = x^2(x + z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0, 0, 1)$ to the line $z = 0$ (E. 3.14) induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

Solution.

a. By exercise I.3.1(b), any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, which implies birational to $\mathbb{P}^1$.

b. We computed the fraction field of this curve as $k(t)$ in exercise I.3.17(c), which shows that it is rational.

c. Let's work inside the affine given by $z \neq 0$. Geometrically, what's going on here is that the projection morphism takes a point on our curve which isn't the origin to the slope of the line passing through the origin and that point. Conversely, the unique intersection of a line through the origin (which isn't the y-axis) with the curve gives the inverse rational map. (Try it out yourself: grab your favorite graphing calculator on the internet and graph the two equations $y^2 = x^3 + x^2$ and $y = tx$ as t changes.)

   Now let us write this out in a little more detail so we can see everything here is actually a morphism. Away from the origin, $(x, y) \mapsto \frac{y}{x}$ defines a morphism of our curve to $\mathbb{A}^1$. Conversely, the morphism $t \mapsto (t^2 - 1, t^3 - t)$ gives a morphism from $\mathbb{A}^1$ to our curve. We see that these maps compose to the identity where both defined: $\frac{t^3-t}{t^2-1} = t$ when $t \neq \pm 1$, and $(\frac{y^2}{x^2} - 1, \frac{y^3}{x^3} - \frac{y}{x}) = (\frac{x^3+x^2}{x^2} - 1, \frac{y^3-yx^2}{x^3}) = (x, y)$ when $x \neq 0$ and the point (x, y) is on our curve (we use $y^2 = x^3 + x^2$ and $x^3 = y^2 - x^2$ as tricks here, just in case you didn't see that one go by). So we have a birational equivalence.

Exercise I.4.5. Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$ (cf. Ex. 2.15).

Solution. The quadric surface is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, which has $U_0 \times U_0 \cong \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$ as an open subset, just as $\mathbb{P}^2$ has $\mathbb{A}^2$ as an open subset. So they're clearly birational. On the other hand, there are two lines in Q which do not meet, and every two curves in $\mathbb{P}^2$ meet. So $Q \not\cong \mathbb{P}^2$.

Exercise I.4.6. *Plane Cremona Transformations.* A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona Transformations.* We give an example, called a *quadratic transformation.* It is the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$ where no two of a_0, a_1, a_2 are 0.

a. Show that φ is birational, and is its own inverse.

b. Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

c. Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms. See also (V, 4.2.3).

Solution.

a. φ^2 is the map $[a_0 : a_1 : a_2] \mapsto [a_0^2a_1a_2 : a_0a_1^2a_2 : a_0^2a_1a_2^2]$ which is equal to $[a_0 : a_1 : a_2]$ whenever $a_0a_1a_2 \neq 0$.

b. The open set given by the nonvanishing of $a_0a_1a_2$ from (a) works as both U and V.

c. φ and φ^{-1} are both defined everywhere we have two nonzero coordinates, so the complement of $\{[0:0:1],[0:1:0],[1:0:0]\}$.

Exercise I.4.7. Let X and Y be two varieties. Suppose there are point $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subset X$ and $Q \in V \subset Y$ and an isomorphism of U to V which sends P to Q.

Solution. An isomorphism of $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ induces an isomorphism of their fraction fields $K(X)$ and $K(Y)$. So X and Y are birational via the map on $K(X)$ and $K(Y)$ induced by this morphism of fraction fields, and this morphism and its inverse are defined at P and Q essentially by definition. We can then find open neighborhoods U, V as requested.

Exercise I.4.8.

a. Show that any variety of positive dimension over k has the same cardinality as k. [*Hints*: do $\mathbb{A}^n$ and $\mathbb{P}^n$ first. Then for any X, use induction on the dimension n. Use (4.9) to make X birational to a hypersurface $H \subset \mathbb{P}^{n+1}$. Use (Ex. 3.7) to show that the projection of H to $\mathbb{P}^n$ from a point not on H is finite-to-one and surjective.]

b. Deduce that any two *curves* over k are homeomorphic (cf Ex. 3.7).

Solution.

a. $\mathbb{A}^n = k^n$ as sets, and $|k^n|$ is just $|k|$ because k is infinite. $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$, so inductively we see that $|\mathbb{P}^n| = |k|$ as well.

   I claim that any nonempty open subset U of a positive dimensional hypersurface $H \subset \mathbb{P}^{n+1}$ is also of cardinality $|k|$. Clearly it's a subset of $\mathbb{P}^{n+1}$ which is of cardinality $|k|$ by the above and so $|U| \leq |k|$. To show $|k| \leq |U|$, we start by showing that the projection of any subvariety Z of dimension d is again of dimension at most d. The image of the projection of Z is irreducible and dense in its closure Z' (which is also irreducible), so we get a dominant map of varieties $Z \to Z'$ and thus an extension of function fields $k(Z') \subset k(Z)$, which implies $\dim Z \geq \dim Z'$. Now projecting U to $\mathbb{P}^n$ from $P \notin H$, we see that this must be surjective on the complement of some hypersurface containing the projection of $U^c \subset H$, since U^c projects to a proper subvariety for dimension reasons and we know the projection $H \to \mathbb{P}^n$ is surjective. So it's enough to prove the claim for the complement of a hypersurface $V(f) \subset \mathbb{P}^n$ where $f \neq 0$.

   To prove this, consider the intersection of $V(f)$ with all the lines $\mathbb{P}^1 \subset \mathbb{P}^n$. These intersections are closed subsets of $\mathbb{P}^1$, and it cannot be the case that they are all $\mathbb{P}^1$: if they were, then $V(f) = \mathbb{P}^n$ which implies $f = 0$, in contradiction to our assumption that $f \neq 0$. So the complement of $V(f)$ contains an open subset of $\mathbb{P}^1$. As the open subsets of $\mathbb{P}^1$ are the complements of finite sets, this means that any open subset of $\mathbb{P}^1$ also has cardinality $|k|$. So $|k| \leq |U|$ and we've shown $|U| = |k|$.

   Now by (4.9), any variety $|X|$ is birational to a hypersurface, which implies that it contains an open subset isomorphic to an open subset of a hypersurface of the same dimension. By the above work, this means that $|k| \leq |X|$, but on the other hand, X embeds in $\mathbb{P}^q$ for some q, so $|X| \leq |\mathbb{P}^q| = |k|$ and $|X| = |k|$.

b. Curves are topological spaces of cardinality $|k|$ with the cofinite topology. So any bijection gives a homeomorphism, and since they all have the same cardinality, we're done. (Once we hit chapter II and schemes, this is true no matter the base field.)

Exercise I.4.9. Let X be a projective variety of dimension r in $\mathbb{P}^n$ with $n \geq r+2$. Show that for a suitable choice of $P \notin X$ and a linear $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, the projection from P to $\mathbb{P}^{n-1}$ (Ex. 3.14) induces a *birational* morphism of X onto its image $X' \subset \mathbb{P}^{n-1}$. You will need to use (4.6A), (4.7A), and (4.8A). This shows in particular that the birational map of (4.9) can be obtained by a finite number of such projections.

Solution. Before we begin, we'll upgrade our statement of the theorem of the primitive element. If all you're familiar with is the usual existence statement, you might worry that primitive elements in finite separable field extensions E/F are rare: in fact, for F infinite, 'most' elements $\alpha \in E$ are primitive. This will make our lives much easier later on in this proof.

Theorem (Improved Theorem of the Primitive Element)**.** *Let E/F be a finite separable extension, generated by $\alpha_1, \cdots, \alpha_n$. Then the collection of primitive elements for E is the complement of a finite collection of proper F-subspaces of E (viewed as an F-vector space). Further, for any infinite subset $S \subset E$, we may find a primitive element of the form $\sum s_i\alpha_i$ for $s_i \in S$.*

Proof. We begin assuming the results of theorem I.4.6A. In particular, let $E = F(\alpha)$. Now I claim that there are finitely many subextensions between E and F. To prove this, consider the minimal polynomial $p(x)$ of α over F. Then after adjoining all the roots of $p(x)$ to F, we get a Galois extension $K \supset F$ which is of finite degree, and since subfields are in correspondence with subgroups of the Galois group which is finite of order $\deg K/F$, there are finitely many intermediate subfields between F and K, and thus there are finitely many intermediate subfields between E and F.

Now the claim is clear: if $F(\alpha) \neq E$, then α lies in a subextension, and so any α not in that finite collection of proper F-subspaces of E must be a primitive element.

To show the second claim, suppose we have a finite separable extension $E \subset E(\alpha, \beta)$, and let $c \in E$. If $E(\alpha + c\beta) \subsetneq E(\alpha, \beta)$, then there is a nontrivial E-linear embedding $\sigma : E(\alpha, \beta) \to \overline{E}$ which restricts to the identity on $E(\alpha + c\beta)$ but does not fix $E(\alpha, \beta)$. In particular, $\sigma(\alpha) + c\sigma(\beta) = \alpha + c\beta$, but $\sigma(b) \neq \beta$, so $c = \frac{\sigma(\alpha)-\alpha}{\beta - \sigma(\beta)}$, which can only take on finitely many values. Further, if $S \subset E$ is an infinite set, this implies we can find a primitive elements with coefficients in S.

■

Now let us return to the problem. Taking T_i as coordinates on $\mathbb{P}^n$, up to a permutation of coordinates we may assume that $X \cap D(T_0) \neq \emptyset$. This implies that $k(X)$ is generated by the images of $t_i = \frac{T_i}{T_0}$ under $k[D(T_0)] \to k[X \cap D(T_0)] \to k(X)$, so by theorem I.4.8A, the extension $k \subset k(X)$ is separably generated. By an application of theorem I.4.7A, $t_1, \cdots, t_n$ must contain a separating transcendence base, and up to a permutation of coordinates we may assume that this is exactly $t_1, \cdots, t_r$ so that $k(t_1, \cdots, t_r) \subset k(X)$ is a finite separable extension. As $k \subset k(t_1, \cdots, t_r)$ is an infinite subset, we may apply our upgraded theorem of the primitive element to the extension $k(t_1, \cdots, t_r) \subset k(X)$ to find a primitive element $\alpha = \sum_{r+1}^n a_i t_i$ with $a_i \in k$ so that

$k(X) = k(t_1, \cdots, t_r, \alpha)$. Up to a linear automorphism of $\mathbb{P}^n$ fixing all the coordinates T_0 through T_r, we can assume that $\alpha = t_{r+1}$. Now I claim that we can find a $P \in V(T_0, T_{r+1}) \cap D(T_{r+2})$ which isn't in X, and the projection π from P to $V(T_{r+2})$ induces an isomorphism of function fields $k(\pi(X)) \to k(X)$. The reason we can find such a $P \in V(T_0, T_{r+1})$ is the combination of our assumptions that $X \cap D(T_0) \neq \emptyset$ and $\dim X < n-1$: the first assumption means that $\dim X \cap V(T_0) < n-2$, so $X \cap V(T_0, T_{r+1})$ is a proper closed subset of $V(T_0, T_{r+1})$ and thus for $P \in V(T_0, T_{r+1})$ the conditions $P \notin X$ and $P \in D(T_{r+2})$ are both satisfied on dense open sets.

To verify that $k(\pi(X)) \to k(X)$ is an isomorphism, we start by computing the image of a point under the projection. Given a point $[x_0 : \cdots : x_n] \in \mathbb{P}^n$, its projection on to $V(T_{r+2})$ from $P = [0 : p_1 : \cdots : p_r : 0 : 1 : p_{r+3} : \cdots : p_n]$ is given by

$$[x_0 : x_1 - p_1 x_{r+2} : \cdots : x_r - p_r x_{r+2} : x_{r+1} : 0 : x_{r+3} - p_{r+3} x_{r+2} : \cdots : x_n - p_n x_{r+2}].$$

This means that for any i the pullback of the function $t_i \in k[\pi(X)]$ is given by $t_i - p_i t_{r+2}$ (in particular, t_{r+1} pulls back to t_{r+1}), so the image of the map on function fields contains $k(t_1 - p_1 t_{r+2}, t_2 - p_2 t_{r+2}, \cdots, t_r - p_r t_{r+2}, t_{r+1})$. As $k(X) = k(t_1, \cdots, t_r)(t_{r+1})$ by assumption, this means that $p_i t_{r+2}$ can be written as a polynomial in t_{r+1} for any i, and therefore $k(t_1 - p_1 t_{r+2}, t_2 - p_2 t_{r+2}, \cdots, t_r - p_r t_{r+2}, t_{r+1}) = k(t_1, \cdots, t_r)(t_{r+1})$, which is exactly $k(X)$. This means that $\pi(X)$ is birationally equivalent to X by corollary I.4.5.

Exercise I.4.10. Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point, and that $\widetilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \widetilde{Y} \to Y$ is bijective and bicontinuous, but it is not an isomorphism.

Solution. Let $[t : u]$ be coordinates on $\mathbb{P}^1$. The blowup of $\mathbb{A}^2$ at the origin is given by the vanishing locus of $xu = yt$ inside $\mathbb{A}^2 \times \mathbb{P}^1$. So the total transform (the preimage of the cuspidal cubic) is the variety determined by $xu = yt$ and $x^3 = y^2$ in $\mathbb{A}^2 \times \mathbb{P}^1$. Now we look on the affine open subsets $D(t)$ and $D(u)$.

When $t = 1$, we are looking at $V(xu = y, x^3 = y^2) \subset \mathbb{A}^3$ with coordinates x, y, u, which has two irreducible components: $V(x, y)$ and $V(x - u^2)$ (we get these via substituting $y = xu$ in to the second equation which gives $x^2(x - u^2) = 0$). The first is the exceptional divisor, and the second is the strict transform. When $u = 1$, we are looking at $V(x = yt, x^3 = y^2) \subset \mathbb{A}^3$ with coordinates x, y, t, which has two irreducible components: $V(x, y)$ and $V(t^3 y - 1)$ (we get these via substituting $x = yt$ in to the second equation which gives $y^2(t^3 y - 1) = 0$). The first is the exceptional divisor, and the second is the strict transform.

We observe that there is exactly one point of intersection: $(x, y, t, u) = (0, 0, 1, 0)$. To see that $\widetilde{Y} \cong \mathbb{A}^1$, observe that all the points on the strict transform lie in the first patch with $t = 1$: the strict transform in the second patch is $V(t^3 y - 1)$ which can't have any points with $t = 0$. So $V(xu - y, x - u^2) \subset \mathbb{A}^3$ is the strict transform, which is our good old friend the affine twisted cubic given by $u = a$, $x = a^2$, $y = a^3$, which is isomorphic to $\mathbb{A}^1$.

(In case you forgot, the map $\varphi : \widetilde{Y} \to Y$ is the same map as in exercise I.3.2(a) - the reason it is not an isomorphism is that one curve is singular and one isn't, which we'll see more about in the next section.)

I.5 Nonsingular Varieties

Hartshorne says 'blowings-up' instead of 'blowups' - the former is very rare these days, while the latter is more common (even though you might not want to say it at the airport). Hartshorne is also sometimes less than careful with characteristic assumptions in this section - there are going to be times where the exercises need adjustment to avoid certain issues which occur if the characteristic of the field we work over is 'bad' in some sense. We'll make note of this every time it comes up.

Exercise I.5.1. Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume $\operatorname{char} k \neq 2$). Which is which among the graphs below?

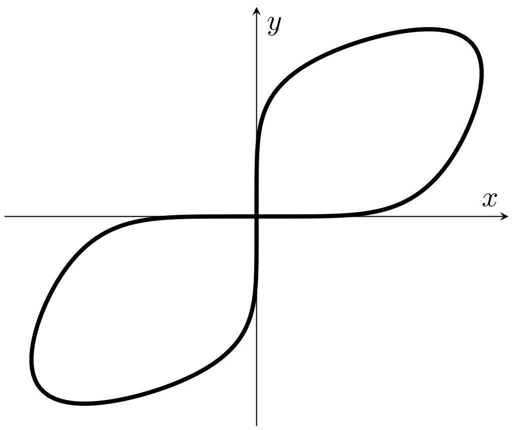
a. $x^2 = x^4 + y^4$

b. $xy = x^6 + y^6$

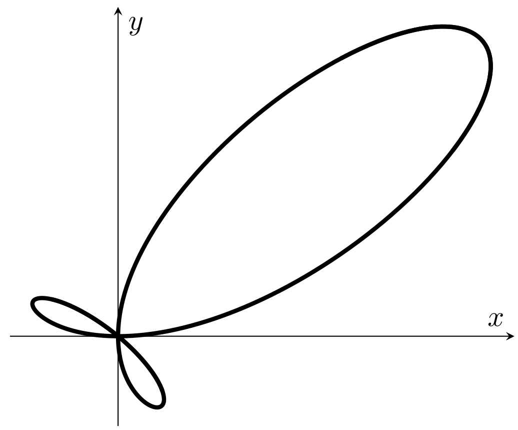
c. $x^3 = y^2 + x^4 + y^4$

d. $x^2y + xy^2 = x^4 + y^4$

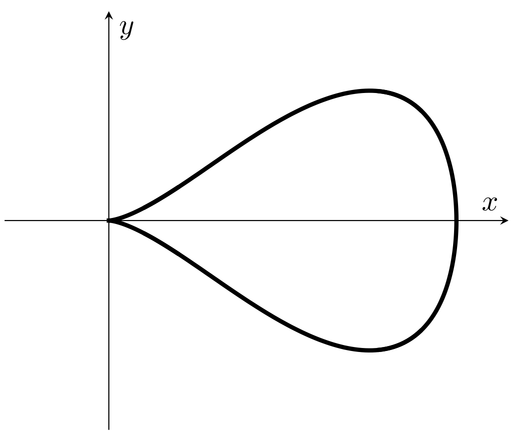
Singularities of plane curves.



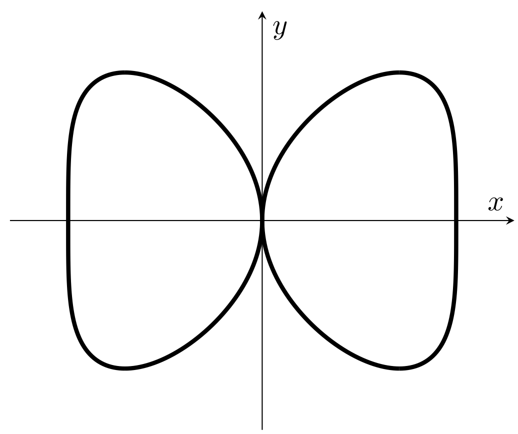
Node



Triple Point



Cusp



Tacnode

Solution.

a. This is the only curve invariant under $x \mapsto -x$ and $y \mapsto -y$, so it is the tacnode.

b. Near the origin when x and y are small, this equation looks like $xy = 0$, which means that it should have two distinct branches at the origin, so it is the node.

c. There are two ways to see this: either x can't be negative, and the only graph with $x \geq 0$ is the cusp, or the terms of smallest degree are $x^3 = y^2$, which is the usual cusp.

d. Near the origin when x and y are small, this equation looks like $xy(x + y) = 0$, which means that it should have three distinct branches at the origin, so it is the triple point.

Exercise I.5.2. Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume $\operatorname{char} k \neq 2$). Which is which in figure 5?

a. $xy^2 = z^2$

b. $x^2 + y^2 = z^2$

c. $xy + x^3 + y^3 = 0$

3D figures not present because they're hard. Look in the book.

Solution.

a. By process of elimination, one can see that this is the pinch point. (This figure is known as the Whitney Umbrella, and it's actually very interesting! I would recommend looking up some more about it - one example is that the real analytic space cut out by this equation has a non-coherent structure sheaf, which is a striking example!)

b. Obviously a cone, so this is the conical double point.

c. This has translational symmetry in the z-direction, so it's the double line.

Exercise I.5.3. *Multiplicities.* Let $Y \subset \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.

a. Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a nonsingular point of Y.

b. Find the multiplicity of each of the singular points in (Ex. 5.1) above.

Solution.

a. Multiplicity one is equivalent to a nonvanishing linear term of f, which is equivalent to the Jacobian matrix having a nonzero element, which is equivalent to it having corank 1, or our curve being smooth.

b. Every curve has multiplicity two besides the triple point which has multiplicity three.

Exercise I.5.4. *Intersection Multiplicity.* If $Y, Z \subset \mathbb{A}^2$ are two distinct curves, given by equations $f = 0$, $g = 0$, and if $P \in Y \cap Z$, we define the *intersection multiplicty* $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$.

a. Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z)_P \geq \mu_P(Y) \cdot \mu_P(Z)$.

b. If $P \in Y$, show that for almost all lines L through P (i.e., all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.

c. If Y is a curve of degree d in $\mathbb{P}^2$, and if L is a line in $\mathbb{P}^2$, $L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all points $P \in L \cap Y$, where $(Y \cdot Z)_P$ is defined using a suitable affine cover of $\mathbb{P}^2$.

Solution.

a. First, we may observe that $Y \cap Z$ is a finte set because it is a closed proper subset of Y, a curve. Without loss of generality, assume $P = (0,0)$. Since $Y \cap Z$ is a finite set, we can find an affine open neighborhood $U = D(h)$ of P inside $\mathbb{A}^2$ which contains no other points of intersection. By the nullstellensatz, we have $\sqrt{(f,g)} = \mathfrak{m}_P$ inside the ring $k[x,y]_h$ and thus inside the local ring at the origin.

Now to argue that this implies $\dim_k \mathcal{O}_P/(f,g) < \infty$. First, I claim that any element in the quotient can be written as a polynomial in x and y: we can write any element in $\mathcal{O}_P$ as $\frac{p}{1-q}$ for $p, q \in \mathfrak{m}$. Then as $\sqrt{(f,g)} = \mathfrak{m}$, we have that for any element in $\mathfrak{m}$, some power of it is in (f,g). So $(1-q)^{-1}$ exists and can be defined as $\sum_{i=1} q^i$ where the sum is taken over all i so that $q^i \neq 0$. By using this to rationalize the denominator, we can represent any element of $\mathcal{O}_P/(f,g)$ as a polynomial. On the other hand, there exist m,n positive integers so that $x^m, y^n \in (f,g)$ by the fact that $\sqrt{(f,g)} = \mathfrak{m}$. So the vector space of polynomials in $\mathcal{O}_P/(f,g)$ is at most mn-dimensional, and we've shown $\dim \mathcal{O}_P/(f,g) < \infty$.

(Alternatively, we could have used the fact that (f,g) is of height two and thus the quotient $\mathcal{O}_P/(f,g)$ is a zero-dimensional noetherian ring and therefore artinian, which implies it's finite-dimensional as a k-vector space. Depends on how much commutative algebra you know.)

Now to show $(Y \cdot Z)_P \geq \mu_P(Y) \cdot \mu_P(Z)$. By our work above, every element of $\mathcal{O}_P/(f,g)$ can be written as $p(x,y)$ for some polynomial p, which means that $\varphi : k[x,y] \to \mathcal{O}_P/(f,g)$ is surjective. Let $k[x,y]_{\leq d}$ be the vector space of polynomials of degree at most d. We'll look at how $\dim \varphi(k[x,y]_{<d})$ changes as d does. First, we see that φ is injective on $k[x,y]_{<\mu_P(Y)}$, so $\varphi(k[x,y]_{<\mu_P(Y)})$ is of dimension $\frac{\mu_P(Y)(\mu_P(Y)-1)}{2}$. When $\mu_P(Y) \leq d < \mu_P(Z)$, moving from $\varphi(k[x,y]_{<d-1})$ to $\varphi(k[x,y]_{<d})$, we get $d+1-B_d$ dimensions of new stuff, where B_d is the dimension of the space of polynomials of degree d which are the top homogeneous component of some multiple of f. The biggest we can ever make B_d is $d - \mu_P(Y)$, which happens in the case that f has some multiple in $\mathcal{O}_P$ which can be written as a homogeneous polynomial of degree $\mu_P(Y)$, so $\dim \varphi(k[x,y]_{<\mu_P(Z)}) - \dim \varphi(k[x,y]_{<\mu_P(Y)}) \geq (\mu_P(Z) - \mu_P(Y) + 1)\mu_P(Y)$. We can use the same logic when $\mu_P(Z) \leq d$ to see that the minimal dimension of the stuff added when moving from $\varphi(k[x,y]_{<d-1})$ to $\varphi(k[x,y]_{<d})$ is $d+1-(d-\mu_P(Y))-(d-\mu_P(Z)) = \mu_P(Y) + \mu_P(Z) - d + 1$. Summing, we find the minimum dimension of $\varphi(k[x,y]_{<\mu_P(Y)+\mu_P(Z)})$ is

$$1 + 2 + \cdots + m + m + \cdots + m + m + m - 1 + \cdots + 1 =$$
$$= \frac{\mu_P(Y)(\mu_P(Y)-1)}{2} + (\mu_P(Z) - \mu_P(Y) + 1)\mu_P(Y) + \frac{\mu_P(Y)(\mu_P(Y)-1)}{2} =$$
$$= \mu_P(Y)(\mu_P(Y) - 1 + \mu_P(Z) - \mu_P(Y) + 1) = \mu_P(Y) \cdot \mu_P(Z).$$

b. Again assume wihtout loss of generality that $P = (0,0)$. Suppose that $Y = V(f)$ for f irreducible with $f = f_r + f_{r+1} + \cdots$ and let our line through the origin be given as $V(L_1)$ for some linear form L_1. Pick another linear form L_2 so that L_1, L_2 generate $\mathfrak{m} \subset \mathcal{O}_P$. If

L_1 doesn't divide f_r, then we can write $f_r = L_2^r(1+p)$ for $p \in \mathfrak{m}$ inside $\mathcal{O}_P/(L_1)$, so that $\mathcal{O}_P/(f, L_1) \cong \mathcal{O}_P/(L_2^r(1+p), L_1)$. But $1+p$ is invertible by part (a), so $\mathcal{O}_P/(L_2^r(1+p), L_1) \cong \mathcal{O}_P/(L_2^r, L_1)$, and the multiplicity of Y at P is equal to the intersection multiplicity of Y and L_1 at P, assuming L_1 does not divide f_r.

It remains to show that almost all choices of L_1 do not divide f_r. But this is clear: f_r factors as a finite number of linear forms since it's a homogeneous polynomial in two variables over an algebraically closed field (refer to exercise I.1.1(c) for a proof of this statement), so this finite collection is all you have to avoid.

c. If you know Bezout, this is obvious, but we'll play along like we don't for the sake of the exercise. Up to a projective change of coordinates, we may assume that L is the line $y = 0$ and $Y = V(f)$ for some irreducible polynomial of degree d. Write $f = y\phi(x, y, z) + r(x, z)$ where p, r are homogeneous of degree $d-1, d$ respectively. By the assumption that Y is irreducible and distinct from L, we have that r is nonzero. Since it is a homogeneous polynomial in two variables over an algebraically closed field, it splits in to homogeneous linear factors: say $r = \prod(a_i x - b_i z)^{c_i}$.

Now consider the local ring at $P_i = [b_i : 0 : a_i]$. When $a_i \neq 0$ we have

$$\mathcal{O}_{P_i}/(y, f) \cong k[X, Y]_{(X - \frac{b_i}{a_i}, Y)}/(Y, Y\phi(X, Y, 1) + r(X, 1)) \cong$$

$$\cong k[X, Y]_{(X - \frac{b_i}{a_i}, Y)}/(Y, r(X, 1)) \cong k[X, Y]_{(X - \frac{b_i}{a_i}, Y)}/(Y, (X - \frac{b_i}{a_i})^{c_i})$$

which obviously has dimension c_i. When $a_i = 0$, we instead have

$$\mathcal{O}_{P_i}/(y, f) \cong k[Y, Z]_{(Z, Y)}/(Y, Z^{c_i})$$

which again has dimension c_i. So the sum of the intersection multiplicities is the sum of the c_i, which is equal to d.

Exercise I.5.5. For every degree $d > 0$, and every $p = 0$ or a prime number, give the equation of a nonsingular curve of degree d in $\mathbb{P}^2$ over a field k of characteristic p.

Solution. If p does not divide d, then $x^d + y^d + z^d$ works: in the standard affine patch where $z = 1$, we get an equation $x^d + y^d + 1 = 0$ which has partial derivatives dx^{d-1} and dy^{d-1}, which are both zero iff $x = y = 0$, and the point $(0, 0)$ is not on the curve. So our curve is nonsingular in the patch $z = 1$ and we may then conclude our entire curve is nonsingular via symmetry.

If p does divide d, then $xy^{d-1} + yz^{d-1} + zx^{d-1}$ works: in the standard affine patch where $z = 1$, we get an equation $xy^{d-1} + x^{d-1} + y$, which has partial derivatives $y^{d-1} - x^{d-2}$ and $1 - xy^{d-2}$. If $d = 2$, then $x = y = 1$ is the only way to make these partials simultaneously vanish, and $(1, 1)$ is not on our curve. So we may now assume $d \neq 2$, and note that $x, y \neq 0$ if both partials are to vanish.

Setting the second partial to zero, multiplying by y, and substituting the first partial we get $y = x^{d-1}$. Substituting this into the first partial derivative and setting to zero, we get that

$x^{(d-1)^2} = x^{d-2}$, which implies $x^{(d-1)^2-(d-2)} = 1$. On the other hand, substituting $y = x^{d-1}$ in to our original equation gives

$$x^{1+(d-1)^2} + 2x^{d-1} = 0$$

which factors as

$$x^{d-1}(x^{(d-1)^2-(d-2)} + 2) = 0.$$

As $x \neq 0$, we now have $x^{(d-1)^2-(d-2)} = 1$ and $x^{(d-1)^2-(d-2)} + 2 = 0$, an impossibility. So our curve is nonsingular in the patch $z = 1$ and we're again done by symmetry.

Exercise I.5.6. *Blowing Up Curve Singularities.*

a. Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\widetilde{Y}$ obtained by Y at $O = (0,0)$ is nonsingular (cf (4.9.1) and (Ex. 4.10)).

b. We define a *node* (also called an *ordinary double point*) to be a double point (i.e. a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\widetilde{Y}$. We say that 'blowing up P resolves the singularity at P'.

c. Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \widetilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

d. Let Y be the plane curve $y^3 = x^5$, which has a 'higher order cusp' at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Note: We will see later (V, 3.8) that any singular point of a plane curve can be resolved by a finite sequence of successive blowings-up.

Solution.

a. We start with the node, which has equation $xy = x^6+y^6$. Our plan is to show that the original curve Y is nonsingular everywhere except for the origin, which implies that $\widetilde{Y}$ is nonsingular everywhere outside it's intersection with the exceptional divisor (because the blowup is an isomorphism away from the exceptional divisor), and then handle the case of the points in the exceptional divisor.

   Taking partial derivatives of $x^6 + y^6 - xy = 0$ and setting them equal to zero, we see that any singular point must satisfy $y = 6x^5$ and $x = 6y^5$. If the characteristic of our field is either 2 or 3, then this immediately says that $x = y = 0$ is the only singular point. Now we assume that $\operatorname{char} k \neq 2, 3$. Substituting $y = 6x^5$ once in our original equation, we get $x^6 + y^6 - 6x^6 = 0$, or $y^6 = 5x^6$. By symmetry, we also have $x^6 = 5y^6$, so we get $x^6 = 25x^6$, or $24x^6 = 0$. But $24 \neq 0$ by the assumption that $\operatorname{char} k \neq 2, 3$, so $x = y = 0$ is the only singular point.

   Now for the blowup. The blowup of the plane at the origin can be covered by two open affines with coordinate algebras $k[x, y, t]/(y = tx) \cong k[x, t]$ and $k[x, y, u]/(x = uy) \cong k[y, u]$.

The equation of the total transform in each is given by $tx^2 = x^6 + t^6x^6$ and $uy^2 = y^6 + u^6y^6$ respectively, which implies the equation of the strict transform in each patch is $(1+t^6)x^4 - t$ and $(1+u^6)y^4 - u$, respectively. The partial derivative with respect to t is $6t^5x^4 - 1$, which is -1 on the exceptional divisor where $x = 0$, so by symmetry we get that any point on $\widetilde{Y}$ in the exceptional divisor is nonsingular.

The cusp with equation $x^3 = y^2 + x^4 + y^4$ has a slight issue: there are some characteristics where this claim is not true for this equation because there are singular points on the curve away from the origin which stay singular after blowing up. It will still be true that the points in $\widetilde{Y}$ lying over O are nonsingular, but the first part of our argument above fails in some specific cases which we'll note as we go through.

To find the singular points of the cusp, take partial derivatives and set them equal to zero, obtaining the equations $4x^3 - 3x^2 = x^2(4x-3) = 0$ and $2y + 4y^3 = 2y(1+2y^2) = 0$. Clearly $x = y = 0$ is a solution for any characteristic. In the case that $\operatorname{char} k = 2$, these reduce to $x^2 = 0$ and 0, which gives a singular point away from O at $(0,1)$. Now assume $\operatorname{char} k \neq 2$: the only non-$O$ candidates for possible singular points are $x = \frac{3}{4}$ and $y = \pm\sqrt{-\frac{1}{2}}$. Plugging these in, we get that such a point satisfies our equation iff $91 = 0$ in k, which is to say $\operatorname{char} k = 7$ or $\operatorname{char} k = 13$. So $\operatorname{char} k \neq 7, 13$ is equivalent to the only singular point being the origin.

To see that the points in $\widetilde{Y}$ lying over O are nonsingular, cover the blowup of the plane at O with two affine charts as in the first example. Then the strict transform has equation $(1+t^4)x^2 - x + t^2$ and $(1+u^4)y^2 - u^3y + 1$ in each of these patches. The partial derivative of the first equation with respect to x is nonvanishing when $x = 0$ for any t, so we only need to check the Jacobian of the second equation for $u = 0$. The partial derivative of the second equation with respect to y is $2y(1+u^4) - u^3$, which when evaluated at $u = 0$ is zero iff $y = 0$. But $y = u = 0$ is not on the curve, so all the points of the strict transform on the exceptional divisor are nonsingular.

b. Up to a projective change of coordinates, we may assume that neither of the tangent directions of our curve is a coordinate axis. Write $f(x,y) = f_2 + f_3 + \cdots$ and consider the standard two charts covering the blowup of the plane with coordinates x, y: $k[x,y,t]/(y = tx) \cong k[x,t]$ and $k[x,y,u]/(x = yu) \cong k[y,u]$. In the first chart, the strict transform is cut out by $f(x,tx)/x^2 = p(t) + xq(x,t)$, and $\frac{d}{dt}$ of this is $p_t(t) + xq_t(x,t)$. By the condition that the tangent directions aren't the axes, we see that up to scaling, $p(t) = (t-a)(t-b)$ with $a \neq b$ and $a, b \neq 0$. Now I claim the partial derivative $p_t(t) + xq_t(x,t)$ does not vanish on any point (x_0, t_0) in the intersection of the strict transform and the exceptional divisor: being on the exceptional divisor means $x_0 = 0$ and being on the strict transform means $p(t_0) + x_0q(x_0,t_0) = 0$, which means that $p(t_0) = 0$. On the other hand, $p_t(t) = 2t - a - b$ which does not vanish at $t = a$ or b by the assumption $a \neq b$, so there are two distinct nonsingular points in this patch. Using the other coordinate patch sends a, b to $\frac{1}{a}, \frac{1}{b}$, and we may verify that these are the only two points in $\widetilde{Y}$ mapping to O.

(For a geometric perspective, note that the distinct tangent directions get sent to distinct points in the exceptional divisor: so this matches what we expect.)

c. This isn't true in characteristic two, so we'll sidestep that issue for now by assuming $\operatorname{char} k \neq 2$ and write a little bit about that case at the end.

   We use the same patches we've been using. The strict transform is cut out in $\mathbb{A}^2$ with coordinates x, t by the equation $(1+t^4)x^2 - 1$ in the patch $u = 1$, and this has no intersection with the exceptional divisor cut out by $x = 0$. So we may focus our attention on the patch $t = 1$. Here, the strict transform is cut out in $\mathbb{A}^2$ with coordinates y, u by the equation $u^4y^2 + y^2 - u^2$, and degree-two portion factors as $(y-u)(y+u)$ which is indeed two distinct tangent directions as requested.

   In the case when $\operatorname{char} k = 2$, the equation $x^4 + y^4 - x^2$ can be written as $(x^2 + y^2 - x)$ which is the equation of a smooth curve (the partial derivative with respect to x is nonvanishing everywhere). Strange things can happen in characteristic p when p divides all the exponents, and you may be interested to go learn about this on your own independent of Hartshorne.

d. Hartshorne doesn't define a triple point, but if we interpret this to be a point of multiplicity 3, the conclusion is clear.

   On to the blowup: in the standard charts we've been using throughout this question, our strict transform is cut out by $u^5y^2 - 1$ and $x^2 - t^2$. We see that there are no points in the intersection of the strict transform and the exceptional divisor with $u = 0$, so we may focus our attention on the $u = 1$ patch. Here the equation of our strict transform is $x^2 - t^2$ which is an ordinary double point (assuming $\operatorname{char} k \neq 2$) and thus may be resolved by one further blowing up as in (b).

   (For the $\operatorname{char} k = 2$ case, the same comments from the tacnode case apply: stuff gets weird in this setting when we talk about algebraic sets cut out by polynomials where p divides all the exponents.)

Exercise I.5.7. Let $Y \subset \mathbb{P}^2$ be a nonsingular plane curve of degree > 1, defined by the equation $f(x, y, z) = 0$. Let $X \subset \mathbb{A}^3$ be the affine variety defined by f (this is the cone over Y; see (Ex. 2.10)). Let P be the point $(0, 0, 0)$ which is the *vertex* of the cone. Let $\varphi : \widetilde{X} \to X$ be the blowing-up of X at P.

a. Show that X has just one singular point, namely P.

b. Show that $\widetilde{X}$ is nonsingular (cover it with open affines).

c. Show that $\varphi^{-1}(P)$ is isomorphic to Y.

Solution.

a. Suppose we consider a point $p_0 = [a : b : c]$ with $c \neq 0$ on Y. This lies in the standard affine open with $c \neq 0$, and Y is nonsingular at p_0 iff $\frac{\partial f(x,y,1)}{\partial x}$ or $\frac{\partial f(x,y,1)}{\partial y}$ is nonvanishing. But then one of $\frac{\partial f(x,y,z)}{\partial x}$ or $\frac{\partial f(x,y,z)}{\partial y}$ is nonvanishing for any point of the form $(\lambda a, \lambda b, \lambda c) \in \mathbb{A}^3$ for $\lambda \in k^\times$ because the partial derivatives are homogeneous.

   On the other hand, all the partial derivatives of f are homogeneous of degree at least one, so they all vanish at P. Thus P is the only singular point.

b. The blowup of $\mathbb{A}^3$ at the origin is the subset of $\mathbb{A}^3 \times \mathbb{P}^2$ cut out by $xu = yt$, $yv = zu$, and $xv = zt$ where x, y, z are coordinates on $\mathbb{A}^3$ and t, u, v are homogeneous coordinates on $\mathbb{P}^2$. We can cover this by open affines corresponding to $t \neq 1$, $u \neq 1$, $v \neq 1$. Let us investigate the first such patch which has coordinate algebra $k[x, y, z, t, u, v]/(y = xu, z = xv) \cong k[x, u, v]$. The equation of $\widetilde{X}$ in any one of these open subsets is just $f(1, u, v)$, which is the same as the equation cutting out Y in the affine patch $x \neq 0$ inside $\mathbb{P}^2$, so $\widetilde{X}$ is smooth on this patch by the same logic as (a). By symmetry, this shows $\widetilde{X}$ is smooth.

c. This is clear from the observation that the equation of $\widetilde{X}$ in the affine patches is given by $f(1, u, v)$, $f(t, 1, v)$, and $f(t, u, 1)$: intersecting with the exceptional divisor we get a variety inside $\mathbb{P}^2$ which is cut out on each standard affine open by those equations, which are just the same as those cutting out Y.

Exercise I.5.8. Let $Y \subset \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \cdots, f_t \in S = k[x_0, \cdots, x_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogeneous coordinates $P = (a_0, \cdots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $|\frac{\partial f_i}{\partial x_j}(a_0, \cdots, a_n)|$ is $n - r$. [*Hint*: (a) Show that this rank is independent of the homogeneous coordinates chosen for P; (b) pass to an open affine $U_i \subset \mathbb{P}^n$ containing P and use the affine Jacobian matrix; (c) you will need Euler's lemma, which says that if f is a homogeneous polynomial of degree d, then $\sum x_i \frac{\partial f}{\partial x_i} = d \cdot f$.]

Solution. Assume $P \in U_0$, the standard affine open given by the nonvanishing of x_0. Further assume we've picked representatives for coordinates so that the x_0-coordinate of P is 1. Then for $i \neq 0$, we have $\frac{\partial f}{\partial x_i}(x_0, \cdots, x_n) = \frac{\partial f}{\partial x_i}(1, x_1, \cdots, x_n)$ which is the same as partial derivative of the dehomogenization of f with respect to x_0. So the affine Jacobian matrix of the variety cut out by the dehomogenizations of the f_j is the $n \times t$ submatrix of the projective Jacobian matrix of the variety cut out by the f_j obtained by deleting the column where we take the derivative with respect to x_0. But this column is in the span of the others by Euler's lemma, and thus deleting it does not change the rank.

Exercise I.5.9. Let $f \in k[x, y, z]$ be a homogenous polynomial, let $Y = Z(f) \subset \mathbb{P}^2$ be the algebraic set defined by f, and suppose that for every $P \in Y$, at least one of $\frac{\partial f}{\partial x}(P)$, $\frac{\partial f}{\partial y}(P)$, $\frac{\partial f}{\partial z}(P)$ is nonzero. Show that f is irreducible (and hence that Y is a nonsingular variety). [*Hint*: Use (Ex. 3.7).]

Solution. If f factored as gh for nonconstant g, h then $V(g)$ and $V(h)$ would intersect at some point P by exercise I.3.7 (as suggested by the hint). But at such a point, $\frac{\partial f}{\partial x_i} = g(P)\frac{\partial h}{\partial x_i}(P) + h(P)\frac{\partial g}{\partial x_i}(P) = 0$, a contradiction. So f does not factor, and it is irreducible.

Exercise I.5.10. For a point P on a variety X, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

a. For any point $P \in X$, $\dim T_P(X) \geq \dim X$, with equality if and only if P is nonsingular.

b. For any morphism $\varphi : X \to Y$, there is a natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

c. If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

Solution.

a. This is exactly proposition I.5.2A.

b. We get an induced local map of local rings $\mathcal{O}_{\varphi(P)} \to \mathcal{O}_P$ by pullback, which means we get an induced map $\mathfrak{m}_{\varphi(P)} \to \mathfrak{m}_P$ and then an induced map $\mathfrak{m}_{\varphi(P)}/\mathfrak{m}^2_{\varphi(P)} \to \mathfrak{m}_P/\mathfrak{m}^2_P$, which after dualizing gives the desired map.

c. This map is given affine locally by $k[t] \to k[x,y]/(x-y^2)$ with $t \mapsto x$. Using the obvious isomorphism $k[x,y]/(x-y^2) \cong k[y]$, we see the map turns in to $t \mapsto y^2$. On local rings, we get $k[t]_{(t)} \to k[y]_{(y)}$ by $t \mapsto y^2$, which induces the zero map on $(t)/(t^2) \to (y)/(y^2)$, and dualizing this again gives the zero map.

Exercise I.5.11. *The Elliptic Quartic Curve in* $\mathbb{P}^3$. Let Y be the algebraic set in $\mathbb{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y \setminus P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called the *elliptic quartic curve* in $\mathbb{P}^3$. Since it is defined by two equations it is another example of a complete intersections (Ex. 2.17).

Solution. In order to find a polynomial vanishing on the image of this projection, we take the resultant of our two polynomials $x^2 - xz - yw$ and $yz - xw - zw$ defining $Y \subset \mathbb{P}^3$ with respect to w: this gives us a polynomial which doesn't involve w and vanishes on the image, so this polynomial must also vanish on the projection. In this case, these polynomials are both degree one in w, which makes it easy: we take the determinant of the Sylvester matrix

$$\begin{pmatrix} -y & -x-z \\ x^2 - xz & yz \end{pmatrix}$$

to get $-y^2z + x^3 - xz^2$, which is exactly the equation defining the plane curve C we're supposed to be projecting to.

On the other hand, in C we have that $\frac{x^2-xz}{y} = \frac{yz}{x+z}$ because when we cross-multiply we get the equation of C. So we can define a morphism $C \setminus [1:0:-1] \to Y$ by $[x:y:z] \mapsto [x:y:z:\frac{x^2-xz}{y}] = [x:y:z:\frac{yz}{x+z}]$ and we see that this is an inverse to the projection where both are defined. Thus $C \setminus [1:0:-1] \cong Y \setminus P$.

Since $C \setminus [1:0:-1]$ is irreducible, $Y \setminus P$ is irreducible, which implies that $Y = \overline{Y \setminus P}$ is irreducible. We may verify smoothness of Y by checking that P is smooth and that all the points in $C \setminus [1:0:-1]$ are smooth. The first check is straightforwards: the (projective) Jacobian of Y is

$$\begin{pmatrix} 2x - z & -w & -x & -y \\ -w & z & y - w & -x - z \end{pmatrix}$$

which when evaluated at $P = [0:0:0:1]$ has a nonzero multiple of the identity matrix as it's first 2×2 minor. The second check is also not so hard: the (projective) Jacobian of C is

$$\begin{pmatrix} -3x^2 + z^2 & 2yz & y^2 + 2xz \end{pmatrix}.$$

In characteristic 2, the only point where all three equations simultaneously vanish is $[1:0:-1]$ (the Jacobian has entries $x^2 + z^2 = (x+z)^2$ and y^2), so $C \setminus [1:0:-1]$ is smooth. In other characteristics, either y or z must be zero. If $z = 0$, then $y = 0$ by the third entry. If $y = 0$, then $xz = 0$ from the third entry - if we take $z = 0$, then $x = 0$ from the equation of our curve. On the other hand, if we take $x = 0$, then $z = 0$ too by the first partial. Thus $C \setminus [1:0:-1]$ is smooth and we're done.

Exercise I.5.12. *Quadric Hypersurfaces.* Assume $\operatorname{char} k \neq 2$, and let f be a homogeneous polynomial of degree 2 in $x_0, \cdots, x_n$.

a. Show after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \cdots + x_r^2$ for some $0 \leq r \leq n$.

b. Show that f is irreducible if and only if $r \geq 2$.

c. Assume $r \geq 2$, and let Q be the quadratic hypersurface in $\mathbb{P}^n$ defined by f. Show that the singular locus $Z = \operatorname{Sing} Q$ of Q is a *linear* variety (Ex. 2.11) of dimension $n - r - 1$. In particular, Q is nonsingular if and only if $r = n$.

d. In case $r < n$, show that Q is a cone with axis Z over a nonsingular quadric hypersurface $Q' \subset \mathbb{P}^r$. (This notion of cone generalizes the one defined in (Ex.2.10). If Y is a closed subset of $\mathbb{P}^r$, and if Z is a linear subspace of dimension $n - r - 1$ in $\mathbb{P}^n$, we embed $\mathbb{P}^r$ in $\mathbb{P}^n$ so that $\mathbb{P}^r \cap Z = \emptyset$, and define the *cone over Y with axis Z* to be the union of all lines joining a point of Y to a point of Z.)

Solution.

a. We can define a symmetric matrix M associated to f by taking M_{ii} to be the coefficient of x_i^2 in f and taking M_{ij} to be half the coefficient of $x_i x_j$ in f. This makes M the matrix of a quadratic form Q associated to f, and the statement that there is a linear change of variables taking f to the desired form is exactly the statment that we can find a *partially orthogonal* basis for this quadratic form (to be precise, a basis $\{v_i\}_{i \in I}$ so that $Q(v_i, v_j) = 0$ if $i \neq j$ and $Q(v_i, v_i) = 0$ or 1). (Partially orthogonal is not necessarily standard nomenclature, but it's what I call it here.)

   In the case that $Q = 0$, any basis will do. If $Q \neq 0$, then pick a vector v_1 so that $Q(v_1, v_1) = c_1 \neq 0$. Now we replace v_1 by $\frac{1}{\sqrt{c_1}} v_1$ so that $Q(v_1, v_1) = 0$, which we can do because k is algebraically closed and thus $\sqrt{c_1}$ exists. Now suppose we've found a basis $v_1, \cdots, v_n$ for a subspace $V \subset E$ of dimension n which is partially orthogonal. Now pick any vector v_{n+1} in $E \setminus V$: by subtracting $Q(v_{n+1}, v_i) \cdot v_i$ from v_{n+1}, we may ensure that $Q(v_{n+1}, v_i) = 0$ for

$i < n+1$. We note that this preserves the property that $v_{n+1} \notin V$: we're only subtracting elements of V. Finally, if $Q(v_{n+1}, v_{n+1}) \neq 0$, we may scale it so that $Q(v_{n+1}, v_{n+1}) = 1$ by the same method from earlier in the paragraph. Since we started with a finite dimensional vector space, this process eventually finishes and the claim is proven.

b. Clearly $r = 0$ and $r = 1$ are reducible, so it suffices to show that $r \geq 2$ implies f irreducible. Now we want to say that f irreducible in $k[x_0, \cdots, x_r]$ is equivalent to f irreducible in $k[x_0, \cdots, x_n]$. Clearly if f factors in the former, it must factor in the latter. On the other hand, if $f = gh$ factors in the latter, g and h don't have any terms involving x_j for $r < j \leq n$ with nonzero coefficients: if they both did, then the coefficient of x_j^2 in f would be nonzero since we're working in a domain. So only one can, and then the coefficient of $x_i x_j$ in f is nonzero, a contradiction.

   Thus it suffices to check irreducibility in $k[x_0, \cdots, x_r]$. But f defines a smooth hypersurface in $\mathbb{P}^r$ (one can see this immediately from the Jacobian). So by the same logic as exercise I.5.9, it must be irreducible.

c. The Jacobian is
$$\begin{pmatrix} 2x_0 & \cdots & 2x_r & 0 & \cdots & 0 \end{pmatrix}$$
   which clearly defines a linear variety of the requested dimension.

d. First, defining the terminology for the first time in a parenthetical after the question is bad practice - don't do this!

   As for the actual question, this is obvious: the variety cut out by f and $x_{r+1} = \cdots = x_n = 0$ is smooth in $\mathbb{P}^r$ and disjoint from the linear variety given by $x_0 = \cdots = x_r = 0$. Any point on a line between a point in the former and the latter is in $V(f)$, and we're done.

Exercise I.5.13. It is a fact that any regular local ring is an integrally closed domain (Matsumurua [2, Th. 36, p. 121]). Thus we see from (5.3) that any variety has a nonempty open subset of normal points (Ex. 3.17). In this exercise, show directly (without using (5.3)) that the set of nonnormal points of a variety is a proper closed subsets (you will need the finiteness of integral closure: see (3.9A)).

Solution. Without loss of generality, assume our variety X is affine with coordinate algebra A and function field K, and let $A \subset \overline{A} \subset K$ be the integral closure. By theorem I.3.9A, $\overline{A}$ is a finitely-generated k-algebra, and since it's a subring of a field, it is a domain. So $\overline{A}$ is the coordinate algebra of a variety Y, and the ring map $A \to \overline{A}$ defines a birational morphism $Y \to X$ of algebraic varieties over k, and since $\overline{A}$ is normal, every local ring of Y is normal. Thus on a nonempty open subset, we have all the local rings of X are normal, and so our variety X is normal at at least one point.

Now pick generators $f_1, \cdots, f_n$ for B as an A-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal corresponding to a point $P \in X$ and $\mathfrak{n} \subset \overline{A}$ a maximal ideal corresponding to a point $Q \in X$ with $Q \mapsto P$. This gives us a series of inclusions of rings
$$\mathcal{O}_{X,P} = A_\mathfrak{m} \subset \overline{A}_\mathfrak{m} \subset \overline{A}_\mathfrak{n} = \mathcal{O}_{Y,Q}.$$

Since integral closure commutes with localization (see the first paragraph of exercise II.3.8), if $A_\mathfrak{m}$ is integrally closed, then $\overline{A}_\mathfrak{m} = \overline{A_\mathfrak{m}}$ is already integrally closed and $\mathfrak{m} = \mathfrak{n}$, so all of the inclusions are actually equalities. If the first inclusion is strict, then $A_\mathfrak{m}$ is not integrally closed and P is a non-normal point, so P is a normal point exactly when $A_\mathfrak{m} = \overline{A}_\mathfrak{m}$. Since $\overline{A} = A[f_1, \cdots, f_n]$, this inclusion is an equality iff all the f_i are in $\mathcal{O}_{X,P}$.

Now our goal is to show that the set of $P \in X$ so that any of the f_i are not in $\mathcal{O}_{X,P}$ is closed. Since there are finitely many f_i, it suffices to show that for a rational function $f_i \in K$ the set of $P \in X$ so that $f_i \notin \mathcal{O}_{X,P}$ is closed. Let $I(f_i)$ be the ideal of A given by $\{a \in A \mid af_i \in A(Y)\}$: this is known as the ideal of denominators of f_i. I claim that the vanishing locus of $I(f_i)$ is exactly the set of points where $f_i \notin \mathcal{O}_{Y,y}$.

To show this, suppose $\mathfrak{m} \subset A$ is a maximal ideal corresponding to $P \in X$. Then the statement $f_i \in \mathcal{O}_{X,P} = A_\mathfrak{m}$ is equivalent to $f_i = \frac{b}{a}$ for some $a \in A \setminus \mathfrak{m}$, so $I(f_i) \subset \mathfrak{m}$ iff $f_i \notin \mathcal{O}_{X,P}$. This finishes the problem, since it tells us that the closed subset $V(I(f_i))$ is exactly the locus of points $P \in X$ where $f_i \notin \mathcal{O}_{X,P}$.

Exercise I.5.14. *Analytically Isomorphic Singularities.*

a. If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).

b. Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \cdots \in k[[x, y]]$, and if the leading form of f_r factors as a $f_r = g_s h_t$, where g_s, h_t are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series

$$g = g_s + g_{s+1} + \cdots$$

$$h = h_t + h_{t+1} + \cdots$$

in $k[[x, y]]$ such that $f = gh$.

c. Let Y be defined by the equation $f(x, y) = 0$ in $\mathbb{A}^2$, and let $P = (0, 0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r+$ higher terms. We say that P is an *ordinary r-fold* point if f_r is a product of r *distinct* linear factors. Show that any two ordinary double points are analytically isomorphic. Ditto for ordinary triple points. But show that there is a one-parameter family of mutually nonisomorphic ordinary 4-fold points.

d. (*) Assume char $k \neq 2$. Show that any double point of a plane curve is analytically isomorphic to the singularity at $(0, 0)$ of the curve $y^2 = x^r$, for a uniquely determined $r \geq 2$. If $r = 2$, it is a node (Ex. 5.6). If $r = 3$, we call it a *cusp*; if $r = 4$ a *tacnode*. See (V, 3.9.5) for further discussion.

Solution.

a. We offer an alternate characterization of the multiplicity of a plane curve Y at a point $P=(a,b)$: the dimension of $k[x-a,y-b]/(x-a,y-b)^n$ is $\frac{n(n+1)}{2}$, and for $n<\mu_P(Y)$ we have that $k[x-a,y-b]/(x-a,y-b)^n=(k[x-a,y-b]/(f))/(x-a,y-b)^n$ since $f\in(x-a,y-b)^n$. On the other hand, for $n=\mu_P(Y)$ we get that $\dim(k[x-a,y-b]/(f))/(x-a,y-b)^n=\frac{n(n+1)}{2}-1$, so we can define the multiplicity at P to be the smallest power n of the maximal ideal so that $\dim A(Y)/\mathfrak{m}_p^n\neq\frac{n(n+1)}{2}$.

If R is a ring with ideal I, then $R^\wedge/I^n\cong R/I^n$. In particular, we have that

$$(k[[x,y]]/(f))/(x,y)^n\cong(k[[x,y]]/(g))/(x,y)^n$$

where f and g are the polynomials cutting out Y and Z respectively. So the above characterization of multiplicity in terms of dimensions of quotients applies here and we're done.

b. First, observe that

$$gh=g_sh_t+(g_sh_{t+1}+g_{s+1}h_t)+\cdots+(\sum_{i=0}^n g_{s+i}h_{t+n-i})+\cdots$$

so being able to write $f=gh$ is equivalent to simultaneously solving the system of equations $f_{r+n}=\sum_{i=0}^n g_{s+i}h_{t+n-i}$ for all $n\geq 0$. We proceed inductively: clearly the assumption that $f_r=g_sh_t$ means we have a solution for $n=0$. Now assuming we've found a solution for all n up to some fixed n_0, this means we need to solve $f_{r+n}-\sum_{i=1}^{n_0} g_{s+i}h_{t+n-i}=g_sh_{t+n_0+1}+g_{s+n_0+1}h_t$ for g_{s+n_0+1} and h_{t+n_0+1}, and the left hand side of this equation is completely determined by our previous choices.

Let P_d denote the vector space of homogeneous polynomials in two variables of degree d. To show that we can always solve this equation for g_{s+n_0+1} and h_{t+n_0+1}, we'll show that the map $P_{t+n}\times P_{s+n}\to P_{s+t+n}$ given by $(a,b)\mapsto ag_s+bh_t$ is surjective assuming g_s and h_t are relatively prime.

I claim it is enough to prove surjectivity for $n=0$. Any standard basis monomial M in P_{s+t+n} can be written as x^iy^j times some standard basis monomial M' in P_{s+t}. If we can find p,q so that $pg_s+qh_t=M'$, then $(x^iy^jp)g_s+(x^iy^jq)h_t=M$, and this shows that every standard basis monomial in P_{s+t+n} is in the image of our map, so it is surjective.

To prove surjectivity when $n=0$, consider the matrix of our map in the standard monomial basis. This is exactly the Sylvester matrix associated to the homogeneous resultant of g_s and h_t. But the homogeneous resultant of g_s and h_t is nonzero iff they are coprime, so we are done.

c. We start with a lemma about automorphisms of $k[[x,y]]$.

Lemma. *Let $\varphi:k[[x,y]]\to k[[x,y]]$ be the map given by sending a power series*

$$f(x,y)\mapsto f(\sum_{i,j>0}a_{ij}x^iy^j,\sum_{k,l>0}b_{kl}x^ky^l).$$

Then φ is an automorphism iff $a_{10}b_{01}-a_{01}b_{10}\neq 0$, and all k-linear automorphisms are of this form.

Proof. We construct an inverse by induction. Let ψ be the map given by

$$f(x,y) \mapsto f(\sum_{p,q>0} c_{pq}x^p y^q, \sum_{r,s>0} d_{rs}x^r y^s).$$

For the base case, we find after some brief calculations that the condition on the coefficients of the degree-one portion of $\phi \circ \psi(x)$ and $\phi \circ \psi(y)$ being equal to x and y respectively is the same as the matrix equation

$$\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which we can solve for the c_{pq} and d_{rs} iff the first matrix has nonzero determinant, which is exactly the condition $a_{10}b_{01} - a_{01}b_{10} \neq 0$.

For the inductive step, assume we've found the coefficients c_{pq} and d_{rs} for $p+q, r+s < n$ and we now want to find the coefficients in the case that $p+q = r+s = n$. Supposing $p+q = n$, we find that the coefficient of $x^p y^q$ in $\phi \circ \psi(x)$ is $a_{10}c_{pq} + a_{01}d_{pq}$ plus a δ, sum of products of as and $c_{p'q'}$ and $d_{p'q'}$ for $p' + q' < n$, and the coefficient of $x^p y^q$ in $\phi \circ \psi(y)$ is $b_{10}c_{pq} + b_{01}d_{pq}$ plus η, a sum of bs and $c_{p'q'}$ and $d_{p'q'}$ for $p' + q' < n$. So we obtain a linear system

$$\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \begin{pmatrix} c_{pq} \\ d_{pq} \end{pmatrix} = \begin{pmatrix} -\delta \\ -\eta \end{pmatrix}$$

and the condition about the nonvanishing of the determinant implies that we may always solve this system for c_{pq} and d_{pq}.

So we've shown that $\phi \circ \psi = id$. We see that they must compose in the other order to the identity by constructing a compositional inverse χ for ψ satisfying $\psi \circ \chi = id$ and considering $\chi = \phi \circ \psi \circ \chi = \phi$.

To see that every automorphism is of this form, note that an automorphism must map $(x,y) \to (x,y)$, so it is continuous in the $\mathfrak{m}$-adic topology and is determined by what it does to $k[x,y] \subset k[[x,y]]$ since the former is dense in the $\mathfrak{m}$-adic topology. This means that our automorphism is determined by it's action on x and y, which must be of the form above.

This lemma generalizes: any endomorphism of a power series ring over a field is an automorphism iff the linear terms are linearly independent.

■

Now suppose P is an ordinary double point of f. Then $f_2 = L_1L_2$ factors as a product of independent linear terms, so $f = (L_1 + \cdots)(L_2 + \cdots)$ by (b), and the automorphism $x \mapsto L_1, y \mapsto L_2$ from the lemma gives that $k[[x,y]]/(f) \cong k[[x,y]]/(xy)$. So every ordinary double point is analytically isomorphic to the one given by xy.

For ordinary triple points, things are slightly more involved. Supposing $f_3 = L_1L_2L_3$ for mutually independent linear terms, we get that $f = (L_1 + \cdots)(L_2 + \cdots)(L_3 + \cdots)$ factors by (b), and the automorphism given by $x \mapsto L_1 + \cdots, y \mapsto L_2 + \cdots$ gives that $k[[x,y]]/(f) \cong k[[x,y]]/(xy(ax+by)+p)$ where p is a power series with no terms of order less than four and $a, b \neq 0$.

Up to the automorphism sending $x \mapsto bx$ and $y \mapsto ay$, we may assume that $a = b = 1$, and now our goal is to eliminate p. Finding an automorphism of $k[[x,y]]$ which does this is equivalent to solving a collection of linear systems in the same fashion as in (b): the degree-n portion after substituting in $x \mapsto x + \sum_{r>1} p_r(x,y)$ and $y \mapsto y + \sum_{r>1} q_r(x,y)$ where p_r, q_r are homogeneous of degree r can be written as a linear combination of products of p_i and q_j for $i, j < n$ plus $p_n(y^2+2xy) + q_n(x^2+2xy)$. The map from $P_n \times P_n \to P_{n+2}$ which gives the contribution of the terms p_n, q_n to the degree-$n+2$ homogeneous part of our power series after substitution is $(p_n, q_n) \mapsto p_n(y^2+2xy) + q_n(x^2+2xy)$ which by the same argument as in part (b) can be seen to be surjective because $y^2+2xy = y(y+2x)$ and $x^2+2xy = x(x+2y)$ are coprime. So we can always solve for p_n, q_n to eliminate the higher-order terms, and any ordinary triple point is analytically isomorphic to the one given by $xy(x+y)$.

For ordinary quadruple points, the same logic from part (b) gives us that $k[[x,y]]/(f) \cong k[[x,y]]/(xy(ax+by)(cx+dy)+p)$ where p is a power series with no terms of order less than five, and $ax+by$ and $cx+dy$ determine lines in the plane which are distinct from each other and the axes. By scaling x and multiplying the whole equation by a constant, we can get to $k[[x,y]]/(xy(x+y)(x+ty)+p)$ for some $t \in k$ not equal to 0 or 1. The same strategy for removing p from the triple point case will again apply here: this time, the map $P_n \times P_n \to P_{n+3}$ explaining the contribution of the terms p_n, q_n to the degree-$n+3$ portion of our power series after substituting is given by $(p_n, q_n) \mapsto (3x^2y + (2t+2)xy^2 + ty^3)p_n + (x^3 + (2t+2)x^2y + 3txy^2)q_n$. It is quick to check that these polynomials are coprime and thus every ordinary quadruple point is isomorphic to one given by $xy(x+y)(x+ty)$ for some $t \in k$, $t \neq 0, 1$.

To see that there is a one-parameter family of non-isomorphic configurations, look at the action of $PGL(2)$ on the space of four distinct points inside $\mathbb{P}^1$, aka lines in the plane. Since the parameter space is of dimension four and the group acting on it is three-dimensional, there must be at least a one-dimensional family of non-isomorphic configurations. We've shown that there is at most a one-dimensional family of such configurations by putting them in to the form $xy(x+y)(x+ty)$ above, so we have the result.

d. We'll need the Weierstrass preparation theorem for this problem. Here's one version, from Bourbaki's commutative algebra:

Theorem (Weierstrass Preparation Theorem - Bourbaki's *Commutative Algebra*, VII.3.8 prop 6). *Let A be a complete local ring. If $f = \sum_{n=0}^{\infty} a_n t^n \in A[[t]]$ so that not all a_i are in the maximal ideal of A, there is a unique unit $u \in A[[t]]$ and a unique polynomial F of the form*

$$F = t^s + b_{s-1}t^{s-1} + \cdots + b_0$$

with $b_i \in \mathfrak{m}$ so that $f = uF$.

Our goal is to show that if $f \in k[x,y]$ is an irreducible polynomial with $\mu_{(0,0)}(f) = 2$, then $k[[x,y]]/(f) \cong k[[x,y]]/(y^2 - x^r)$ for some $r \geq 2$. Write $f = f_2 + \cdots$ where f_2 is homogeneous of degree two.

If f_2 factors as $L_1 L_2$ for L_1, L_2 independent linear forms, then by a change of variables we may assume $f_2 = xy$. By part (b), this means $f = (x+p)(y+q)$ for power series p, q with all terms of degree two or more. But $x \mapsto x - p$ and $y \mapsto y - q$ define an automorphism of $k[[x,y]]$ which sends f to xy, so we have $k[[x,y]]/(f) \cong k[[x,y]]/(xy)$, and by a further change of variables $x \mapsto (y - x)$ and $y \mapsto (x + y)$ we see that $k[[x,y]]/(f) \cong k[[x,y]]/(y^2 - x^2)$.

If f_2 does not factor as a product of independent linear forms, then change variables so that $f_2 = y^2$. Now apply Weierstrass preparation to f with $A = k[[x]]$ and $t = y$. Thus $f = u(y^2 + y \cdot v(x) + w(x))$ for a unit u and power series $v(x), w(x) \in xk[[x]]$, and after making the substitution $y \mapsto y - \frac{1}{2}v(x)$, we may assume that $v(x) = 0$ (this is where we use the assumption that $\operatorname{char} k \neq 2$). So $k[[x,y]]/(f) \cong k[[x,y]]/(y^2 - w'(x))$, where $w'(x) = x^n(c + xW(x))$ for $c \in k^*$ and $W(x) \in k[[x]]$. Scaling x by $c^{-1/n}$, we have that $k[[x,y]]/(f) \cong k[[x,y]]/(y^2 - x^n(1 + xW(x)))$.

Now I claim that we may construct the power series $\sqrt{1+x}$ in $k[[x]]$ when k is a field of characteristic not two. The power series of $\sqrt{1+x}$ from calculus is $\sum_{n=0} \binom{1/2}{n} x^n$, which makes sense whenever $\operatorname{char} k \neq 2$. Therefore we may plug in $xW(x)$ and define an automorphism of $k[[x,y]]$ by $x \mapsto x$ and $y \mapsto y\sqrt{1 + xW(x)}$: this gives that $k[[x,y]]/(f) \cong k[[x,y]]/((1 + xW(x))(y^2 - x^n)) \cong k[[x,y]]/(y^2 - x^n)$. As $w'(x)$ was uniquely determined by the Weierstrass Preparation theorem and the transformation $y \mapsto y - \frac{1}{2}v(x)$, and all further transformations preserved the order of w', we see that n is uniquely determined.

Exercise I.5.15. *Families of Plane Curves.* A homogeneous polynomial f of degree d in three variables x, y, z, has $\binom{d+2}{2}$ coefficients. Let these coefficients represent a point in $\mathbb{P}^N$, where $N = \binom{d+2}{2} - 1 = \frac{1}{2}d(d+3)$.

a. Show that this gives a correspondence between points of $\mathbb{P}^N$ and algebraic sets in $\mathbb{P}^2$ which can be defined by an equation of degree d. The correspondence is $1-1$ except in some cases where f has a multiple factor.

b. Show under this correspondence that the (irreducible) nonsingular curves of degree d correspond $1-1$ to the points of a nonempty Zariski-open subset of $\mathbb{P}^N$. [*Hints*: (1) use elimination theory (5.7A) applied to the homogeneous polynomials $\frac{\partial f}{x_0}, \cdots, \frac{\partial f}{x_N}$; (2) use the previous (Ex. 5.5, 5.8, 5.9) above.]

Solution.

a. A point $[a_0 : \cdots : a_N]$ in $\mathbb{P}^N$ corresponds to the polynomial $f = \sum a_i M_i$ up to a nonzero scaling, where M_i is the i^{th} monomial of degree d in the variables x, y, z. Such polynomials correspond to algebraic sets definable as the vanishing locus of a polynomial of degree d by taking $V(f)$.

b. If f is reducible or $V(f)$ is irreducible but singular, there is a point where $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ by exercises I.5.8 and I.5.9. These conditions produce a list of polynomial equations with integer coefficients which are satisfied by the points of $\mathbb{P}^N$ exactly when these conditions are met by theorem I.5.7A, so the set of such points is closed in $\mathbb{P}^N$. By exercise I.5.5, it is not the whole of $\mathbb{P}^N$, so it must be of codimension at least one, and thus it's complement is open dense and we are done.

I.6 Nonsingular Curves

This entire section exists in a funny in-between land: the concept of an abstract nonsingular curve is not quite like the varieties we've been talking about before, and it's not quite like the schemes we'll see later. This technology only really works in the case of curves - if one is interested in birational classification of varieties of higher dimensions, the key phrase to go look up is the minimal model program. This is a rich and deep area!

An annoying error in Hartshorne (perhaps only a typo) is that he writes $PGL(1)$ for the automorphism group of $\mathbb{P}^1$ instead of $PGL(2)$. 2 is correct while 1 is incorrect.

Exercise I.6.1. Recall that a curve is *rational* if it is birationally equivalent to $\mathbb{P}^1$ (Ex. 4.4). Let Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$.

a. Show that Y is isomorphic to an open subset of $\mathbb{A}^1$.

b. Show that Y is affine.

c. Show that $A(Y)$ is a unique factorization domain.

Solution.

a. By corollary I.6.10, this is isomorphic to an open subset of a nonsingular projective curve, and since any proper open subset of $\mathbb{P}^1$ is contained in a copy of $\mathbb{A}^1$, the result follows.

b. Any open subset of $\mathbb{A}^1$ is affine. (This has been covered before, for instance exercise I.3.1.)

c. Any localization of a UFD is again a UFD, and the nonempty open subsets of $\mathbb{A}^1$ have coordinate algebras which are localizations of $k[t]$ at some element.

Exercise I.6.2. *An Elliptic Curve.* Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$, and assume that the characteristic of the base field k is $\neq 2$. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure transcendental extension of k.

a. Show that Y is nonsingular, and deduce that $A = A(Y) \cong k[x, y]/(y^2 - x^3 + x)$ is an integrally closed domain.

b. Let $k[x]$ be the subring of $K = K(Y)$ generated by the image of x in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

c. Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) = a \cdot \sigma(a)$. Show that $N(a) \in k[x]$, $N(1) = 1$, and $N(ab) = N(a) \cdot N(b)$ for any $a, b \in A$.

d. Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Show that A is *not* a unique factorization domain.

e. Prove that Y is not a rational curve (Ex. 6.1). See (II, 8.20.3) and (III, Ex. 5.3) for other proofs of this important result.

Solution.

a. The Jacobian is $\begin{pmatrix} 3x^2 - 1 & 2y \end{pmatrix}$. Since $2 \neq 0$, $y = 0$ for the second entry to be zero. But the points where $y = 0$ on the curve are $(-1, 0), (0, 0), (1, 0)$, none of which make the first entry zero, so our curve is smooth. Since smooth implies normal, we are done.

b. Since k is algebraically closed, x is transcendental over k, and $k[x]$ is a polynomial ring. y is integral over $k[x]$, since it satisfies $y^2 = x^3 - x$. So A is integral over $k[x]$ and integrally closed, so it is the integral closure of $k[x]$.

c. The automorphism $y \mapsto -y$ of $k[x, y]$ preserves the equation $y^2 = x^3 - x$, so it descends to an automorphism of A. Write an element of A as $yf(x) + g(x)$. Then $N(yf(x) + g(x)) = (yf(x) + g(x))(-yf(x) + g(x)) = -(yf(x))^2 + g(x)^2 = -(x^3 - x)f(x)^2 + g(x)^2$. $N(1) = 1$ is clear, and $N(ab) = ab\sigma(ab) = ab\sigma(a)\sigma(b) = a\sigma(a)b\sigma(b) = N(a)N(b)$ is too.

d. If a is a unit with inverse b, then $N(a)N(b) = N(ab) = N(1) = 1$ so $N(a)$ and $N(b)$ must be units too. On the other hand, any nonconstant element of A has norm which isn't a constant (look at the term of top degree). So the units of A are precisely the elements of $k^\times$.

   We note that no element of A has norm a polynomial of degree one. If $x = ab$, then $N(x) = N(a)N(b)$, but from the preceding observation, we must have that one of $N(a)$ or $N(b)$ is a unit, so x is irreducible. The same argument works for y and implies that if $y = ax$, then $N(y) = N(a)N(x)$ and so there is no such element a.

   To see that A is not a UFD, note that x divides $y^2 = x^3 - x$, which implies x divides y, in contradiction to the previous paragraph.

e. If Y were rational, then by exercise I.6.1(c) $A(Y)$ would be a UFD, but it is not.

Exercise I.6.3. Show by example that the result of (6.8) is false if either (a) $\dim X \geq 2$, or (b) Y is not projective.

Solution. In the first case, let $X = \mathbb{A}^2$ and let P be the origin. Define a map $X \setminus \{P\} \to \mathbb{P}^1$ by sending (a_1, a_2) to $[a_1 : a_2]$. By an application of proposition I.6.8 to the axes in X, we see that the origin must simultaneously be sent to $[1 : 0]$ and $[0 : 1]$, which is nonsense.

In the second case, let $X = \mathbb{A}^1$ and let P be the origin. Define a map $X \setminus \{P\} \to \mathbb{A}^1$ by $x \mapsto 1/x$. By an application of proposition I.6.8 to the composite morphism given by embedding the target $\mathbb{A}^1$ in $\mathbb{P}^1$, we see that this morphism extends by sending P to the point in $\mathbb{P}^1$ not in the embedded copy of $\mathbb{A}^1$ and we're done.

Exercise I.6.4. Let Y be a nonsingular projective curve. Show that every nonconstant rational function f on Y defines a surjective morphism $\varphi : Y \to \mathbb{P}^1$, and that for every $P \in \mathbb{P}^1$, $\varphi^{-1}(P)$ is a finite set of points.

Solution. Consider the map of function fields $k(t) \to k(Y)$ by $t \mapsto f$. This is a k-homomorphism between two function fields of dimension one, so by corollary I.6.12 it gives us a dominant (in particular, nonconstant) morphism $\varphi : Y \to \mathbb{P}^1$ which is given by $y \mapsto [1 : f(y)]$ where $f(y) \neq \infty$. If $\varphi(Y)$ misses a point, then by taking the copy of $\mathbb{A}^1$ which is the complement of that point, we can find a nonconstant regular function on $\mathbb{A}^1$ which pulls back to a nonconstant regular function on Y. This is impossible by theorem I.3.4, so $\varphi(Y) = \mathbb{P}^1$ and φ is surjective.

Note that since any point in $\mathbb{P}^1$ is closed, $\varphi^{-1}(P)$ is a closed set since morphisms are continuous. On the other hand, the fiber over any point must be a proper subset of Y, since φ is nonconstant. Since all proper closed subsets of a curve are finite (exercise I.4.8, for instance), we are done.

Exercise I.6.5. Let X be a nonsingular projective curve. Suppose that X is a (locally closed) subvariety of a variety Y (Ex. 3.10). Show that X is in fact a closed subset of Y. See (II, Ex. 4.4) for generalization.

Solution. First, we prove that any map out of a projective variety is closed.

To do this, we need to show that $\mathbb{P}^n \times Y \to Y$ is a closed map for any variety Y. By covering Y with affines, we may reduce to the case that Y is affine. Now suppose $Z \subset \mathbb{P}^n \times Y$ is closed: this means that it's cut out by a list of equations $f_1, \cdots, f_r$ which are homogeneous polynomials in the homogeneous coordinates on $\mathbb{P}^n$ with coefficients taken from $A(Y)$. By theorem I.5.7A, we may find polynomials $g_1, \cdots, g_s$ in the coefficients of these f_i which vanish iff all the f_i simultaneously vanish. But this exactly means that the image of $Z \subset \mathbb{P}^n \times Y$ under the projection $\mathbb{P}^n \times Y \to Y$ is closed, or that the projection $\mathbb{P}^n \times Y \to Y$ is closed.

Now consider a morphism $f : Z \to Y$ where Z is projective and $i : Z \to \mathbb{P}^n$ is the inclusion of Z in to some projective space (guaranteed by the definition of projective). We can factor this as $Z \to \mathbb{P}^n \times Y \to Y$ where the first map is $z \mapsto (i(z), f(z))$. To show the image of this map is closed, cover $\mathbb{P}^n \times Y$ by affine open subsets of the form $U_i \times Y$ for U_i the standard affine opens. Then on each of these opens, the set of points $(i(z), f(z))$ is closed because it's cut out by the equations for $Z \cap U_i$ and then also the equalities $y_i = f_i(z)$ where y_i are the coordinate functions on Y and f_i are the components of the map f. So the image of Z in $\mathbb{P}^n \times Y$ is closed and therefore the image of Z in Y is closed since the projection $\mathbb{P}^n \times Y \to Y$ is closed.

To apply this to our problem, write $i : X \to Y$ for the inclusion. Then by the above, $i(X)$ is closed in Y.

Exercise I.6.6. *Automorphisms of* $\mathbb{P}^1$. Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto (ax + b)/(cx + d)$ for $a, b, c, d \in k$, $ad - bc \neq 0$.

a. Show that a fractional linear transformation induces an *automorphism* of $\mathbb{P}^1$ (i.e., an isomorphism of $\mathbb{P}^1$ with itself). We denote the group of all these fractional linear transformations by $PGL(1)$.

b. Let $\operatorname{Aut} \mathbb{P}^1$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\operatorname{Aut} \mathbb{P}^1 \cong \operatorname{Aut} k(x)$, the group of k-automorphisms of the field $k(x)$.

c. Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $PGL(1) \to \operatorname{Aut} \mathbb{P}^1$ is an isomorphism.

Note: We will see later (II, 7.1.1) that similar result holds for $\mathbb{P}^n$: every automorphism is given by a linear transformation of the homogeneous coordinates.

Solution.

a. Clearly such a map extends to a morphism from $\mathbb{P}^1$ to itself by writing the map as $[x : y] \mapsto [ax+by : cx+dy]$. We exhibit a two-sided inverse: I claim the linear fractional transformation given by $x \mapsto \frac{dx-b}{-cx+a}$ is an inverse. To see this, compute the composition:

$$\frac{a(\frac{dx-b}{-cx+a})+b}{c(\frac{dx-b}{-cx+a})+d} = \frac{a(dx-b)+b(-cx+a)}{c(dx-b)+d(-cx+a)} = \frac{(ad-bc)x}{ad-bc}$$

which is clearly just $x \mapsto x$ when $ad - bc \neq 0$.

b. By the equivalence of categories between nonsingular curves and function fields of transcendence degree one, we immediately see the result: any isomorphism of $\mathbb{P}^1$ with itself induces an isomorphism of $K(\mathbb{P}^1) \cong k(t)$ with itself.

c. This is covered in many undergraduate abstract algebra books (for instance, Dummit and Foote exercise 13.2.18).

   First, any k-linear automorphism of $k(x)$ is determined by where it sends x: it must send x to some rational function $t = \frac{P(x)}{Q(x)}$ for polynomials P, Q relatively prime. To determine whether such a map is an automorphism, it suffices to determine the degree of the extension $k(t) \subset k(x)$.

   The polynomial $P(X) - tQ(X)$ is irreducible considered as an element of $k(X)[t]$ by degree reasons and the fact that P, Q are relatively prime. So by Gauss's lemma it is irreducible in $k[X][t] \cong k[t][X]$, and by another application of Gauss's lemma, it is irreducible in $k(t)[X]$. As this polynomial is satisfied by $X = x$, we see that it is the minimal polynomial of x over $k(t)$. By irreducibility, this is the minimal polynomial of t, and it has degree $\max(\deg P, \deg Q)$ in x. So $[k(x) : k(t)] = \max(\deg P, \deg Q)$, and the only way for $k(x)$ to equal $k(t)$ (and therefore for $x \mapsto t$ to be an automorphism) is for $\deg P = \deg Q = 1$. This is exactly the case of a fractional linear transformation, and we are finished.

Exercise I.6.7. Let $P_1, \cdots, P_r, Q_1, \cdots, Q_s$ be distinct points of $\mathbb{A}^1$. If $\mathbb{A}^1 \setminus \{P_1, \cdots, P_r\}$ is isomorphic to $\mathbb{A}^1 \setminus \{Q_1, \cdots, Q_s\}$, show that $r = s$. Is the converse true? Cf. (Ex. 3.1).

Solution. An isomorphism of these two varieties induces an isomorphism of their function fields and thus an isomorphism of the $\mathbb{P}^1$s they each embed in to. Such an isomorphism must send the P_i isomorphically on to the Q_j, so $r = s$.

The converse is false. By exercise I.5.14(c), the action of $PGL(2)$ on collections of four points in $\mathbb{P}^1$ is not transitive. As removing four points from $\mathbb{P}^1$ is the same as removing three points from $\mathbb{A}^1$, we see that there are choices of P_i and Q_j for $r = s = 3$ which are nonisomorphic.

I.7 Intersections in Projective Space

One of the key motivations for projective space is how intersections of subvarieties behave, and it's a treat to start getting acquainted with this structure here.

Exercise I.7.1.

a. Find the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ (Ex. 2.12). [Answer: d^n]

b. Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ (Ex.2.14). [Answer: $\binom{r+s}{r}$]

Solution.

a. By exercise I.2.12, the coordinate algebra of the d-uple embedding is isomorphic to the subalgebra of $k[x_0, \cdots, x_n]$ generated by the degree-d polynomials. The degree md portion of this subalgebra has dimension $\binom{md+n}{n}$ from our calculation of the dimensions of the graded pieces of $k[x_0, \cdots, x_n]$. The function which sends $m \mapsto \binom{md+n}{n}$ is given by

$$m \mapsto \frac{(md+n)(md+n-1)\cdots(md+1)}{n!}$$

and expanding, we see that the leading coefficient of m is $\frac{d^n}{n!}$ which gives the result.

b. By exercise I.2.14, the coordinate algebra of the Segre embedding is isomorphic to the subalgebra of $k[x_0, \cdots, x_r, y_0, \cdots, y_s]$ generated by the quadratic monomials of the form $x_i y_j$. The degree $2d$ portion of this subalgebra has dimension $\binom{d+r}{r}\binom{d+s}{s}$, since every monomial is a product of a degree d monomial in the x_i and the y_j, so the Hilbert function is $d \mapsto \binom{d+r}{r}\binom{d+s}{s}$, which can be re-expressed as

$$d \mapsto \frac{(d+r)\cdots(d+1)(d+s)\cdots(d+1)}{r!s!}$$

which is of degree $r+s$ in d. $(r+s)!$ times the the leading coefficient of this is exactly $\frac{(r+s)!}{r!s!} = \binom{r+s}{r}$ and we're done.

Exercise I.7.2. Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the arithmetic genus of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

a. Show that $p_a(\mathbb{P}^n) = 0$.

b. If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

c. More generally, if H is a hypersurface of degree d in $\mathbb{P}^n$, then $p_a(H) = \binom{d-1}{n}$.

d. If Y is a complete intersection (Ex. 2.17) of surfaces of degree a, b in $\mathbb{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

e. Let $Y^r \subset \mathbb{P}^n$, $Z^s \subset \mathbb{P}^m$ be projective varieties, and embed $Y \times Z \subset \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the Segre embedding. Show that

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

Solution. We make use of binomial coefficients with not-necessarily-positive entries here, and we should explain what that means: when we ask for $\binom{r}{s}$ with r negative, we extend the usual definition by declaring this to be equal to $\frac{r(r-1)(r-2)\cdots(r-s+1)}{s!}$.

a. The Hilbert polynomial of $\mathbb{P}^n$ is $d \mapsto \binom{n+d}{n}$, and as $\binom{n}{n} = 1$, the result is obvious.

b. By the computation in the proof of proposition I.7.6(d), we see that the Hilbert polynomial of a plane curve of degree d is $z \mapsto \binom{z+2}{2} - \binom{z-d+2}{2}$. Substituting $z = 0$, we get $1 - \frac{(-d+2)(-d+1)}{2}$, which when plugged in to the formula for the arithmetic genus gives $(-1)(1 - \frac{(-d+2)(-d+1)}{2} - 1) = \frac{(d-1)(d-2)}{2}$.

c. Same computation as (b).

d. We use the Koszul resolution for the coordinate algebra of Y. Let $S = k[x_0, \cdots, x_n]$ and $S(Y)$ be the coordinate algebra of Y, and suppose $I(Y) = (f, g)$ with $\deg f = a$ and $\deg g = b$. Then we have a resolution of $S(Y)$ by shifted copies of S as follows:

$$0 \to S(-a-b) \to S(-a) \oplus S(-b) \to S \to S(Y) \to 0$$

where the first morphism sends $x \mapsto (gx, -fx)$, the second morphism sends $(x, y) \mapsto fx + gy$, and the third morphism is the standard surjection. Hilbert polynomials add over exact sequences, so the Hilbert polynomial of $S(Y)$ is $d \mapsto \binom{d+3}{3} - \binom{d-a+3}{3} - \binom{d-b+3}{3} + \binom{d-a-b+3}{3}$. Plugging in $d = 0$ we get $\frac{ab}{2}(-a - b + 4)$, which gives the desired arithmetic genus.

e. If we knew that $H_{Y\times Z}(d) = H_Y(d)H_Z(d)$, then we would be done: writing $H_X = 1 + (-1)^{\dim X} p_a(X)$, plugging in, and solving for $p_a(Y \times Z)$, we get the desired result. To see that this equality of Hilbert polynomials is indeed true, think about the Segre embedding: it sets things up so that the degree d functions on the image of the embedding are exactly the products of degree d functions on each of the factors.

   To formalize this argument about the Hilbert polynomials a little bit, consider $k[y_0, \cdots, y_n] \otimes_k k[z_0, \cdots, z_m] \to S(Y) \otimes_k S(Z)$, the tensor product of the surjections $k[y_0, \cdots, y_n] \to S(Y)$ and $k[z_0, \cdots, z_m] \to S(Z)$. By the definition of the graded tensor product, this exhibits $S(Y) \otimes_k S(Z)$ as the sum of $S(Y)_i \otimes_k S(Z)_j$. The morphism $k[x_0, \cdots, x_N] \to k[y_0, \cdots, y_n, z_0, \cdots, z_m]$ which corresponds to the Segre embedding picks out the terms which are of equal y and z degree in this product, and combining this with the fact that $k[y_0, \cdots, y_n, z_0, \cdots, z_m] \cong k[y_0, \cdots, y_n] \otimes_k k[z_0, \cdots, z_m]$ and the above surjection gives that the coordinate algebra of $Y \times Z$ viewed as a subset of $\mathbb{P}^N$ under the Segre embedding is exactly $\bigoplus S(Y)_i \otimes S(Z)_i$, which is exactly what we wanted.

Exercise I.7.3. *The Dual Curve.* Let $Y \subset \mathbb{P}^2$ be a curve. We regard the set of lines in $\mathbb{P}^2$ as another projective space, $(\mathbb{P}^2)^*$, by taking (a_0, a_1, a_2) as homogeneous coordinates of the line $L : a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1. This is the *tangent line* to Y at P. Show that the mapping $P \mapsto T_P(Y)$ defines a *morphism* of $\operatorname{Reg} Y$ (the set of nonsingular points of Y) into $(\mathbb{P}^2)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subset (\mathbb{P}^2)^*$ of Y.

Solution. Up to a linear change of coordinates on $\mathbb{P}^2$, we may assume $P = [0 : 0 : 1]$. The assumption that P is smooth is equivalent to P being of multiplicity one on Y by exercise I.5.3, or that the polynomial f cutting out Y on the affine open patch given by $z \neq 0$ can be written as $f = f_1 + \cdots$ where f_1 is a nonzero linear form in x and y. By exercise I.5.4(b), we may conclude that every line through P which is not $V(f_1)$ has intersection multiplicity 1 with Y at P, and so $V(f_1)$ is the unique line through P meeting Y with multiplicity greater than one.

To see that the mapping $P \mapsto T_P(Y)$ is a morphism, we describe it using polynomials. Suppose now that f is a homogeneous generator of $I(Y)$, and define a map $\operatorname{Reg} Y \to (\mathbb{P}^2)^*$ by sending $P \mapsto [\frac{\partial f}{\partial x}(P) : \frac{\partial f}{\partial y}(P) : \frac{\partial f}{\partial z}(P)]$. Since f is a polynomial, all of its partials are polynomials, and since the derivative is linear and any two choices of f are related by multiplication by a nonzero constant, this gives us a well-defined rational map $\mathbb{P}^2 \to (\mathbb{P}^2)^*$ when the three polynomials are not all zero. By the projective Jacobian criteria (exercise I.5.8), this is exactly $\operatorname{Reg} Y$.

Exercise I.7.4. Given a curve of degree d in $\mathbb{P}^2$, show that there is a nonempty open subset U of $(\mathbb{P}^2)^*$ in its Zariski topology such that for each $L \in U$, L meets Y in exactly d points. [*Hint*: Show that the set of lines in $(\mathbb{P}^2)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a proper closed subset.] This result shows that we could have defined the defgree of Y to be the number d such that almost all lines in $\mathbb{P}^2$ meet Y in d points, where 'almost all' referes to a nonempty open set of the set of lines, when this set is identified with the dual projective space $(\mathbb{P}^2)^*$.

Solution. Follow the hint. The condition of a line L passing through a point P is a nontrivial linear condition on the coefficients of L, so the collection of lines passing through any point is some copy of $\mathbb{P}^1$ in the dual projective space $(\mathbb{P}^2)^*$. Since any curve has only finitely many singular points (it's smooth on a dense open subset, which has finite complement), this shows that the collection of lines passing through the singular points is a proper closed subvariety of $(\mathbb{P}^2)^*$. To see that the tangent lines are contained in a proper closed subset, note that the tangent lines are exactly the dual curve of Y from the previous exercise. Since Y is irreducible, it's image is irreducible, and from the fact that it surjects on to its image we can find an inclusion of function fields $K(\operatorname{Im}(Y)) \hookrightarrow K(Y)$ which shows that $\dim \operatorname{Im}(Y) \leq \dim Y = 1$, so $\operatorname{Im}(Y)$ is also contained in a proper closed subset. Taking the union of these proper closed subsets, we're done.

Exercise I.7.5.

a. Show that an irreducible curve Y of degree $d > 1$ in $\mathbb{P}^2$ cannot have a point of multiplicity $\geq d$ (Ex. 5.3).

b. If Y is an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$, then Y is a rational curve (Ex. 6.1).

Solution.

a. Let P be a point of multiplicity $\geq d$ and let Q be any other point (this exists because Y is one-dimensional). Then the line through PQ intersects Y with multiplicity at least $d+1$, contradicting Bezout (corollary I.7.8).

b. Let P be our point of multiplicity $d-1$. Then the map from the $\mathbb{P}^1$ of lines through P to Y given by sending a line to the unique other intersection of our line with Y is a birational morphism from $\mathbb{P}^1$ to Y with inverse given by sending a point $Q \neq P$ to the line $\overline{PQ}$.

   To check that these really are morphisms and they really are birational inverses of each other, we work in the affine case where $P = (0,0)$ and $f = f_{d-1} + f_d$ for f_i homogeneous of degree i. Writing $y = tx$, f can be written as $x^d p(x,t)$ where p is degree-one in x. So we can write x (and therefore y) as a polynomial terms of t, so we have a map $\mathbb{A}^1 \to Y$. On the other hand, $t = \frac{y}{x}$ is a rational map from Y to $\mathbb{A}^1$, and it is easy to see that this defines a rational inverse where $x \neq 0$.

Exercise I.7.6. *Linear Varieties.* Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety (Ex. 2.11). [*Hint*: First, use (7.7) and treat the case $\dim Y = 1$. Then do the general case by cutting with a hyperplane and using induction.]

Solution. We prove an alternate characterization of a linear variety first: a variety Y is linear iff it contains the line between any two distinct points. This equivalence is straightforwards: Hartshorne's definition of a linear variety is one who's homogeneous ideal is generated by linear polynomials. But a linear polynomial vanishing at P and Q must vanish at all points of the form $c_1P + c_2Q$, so the forward direction is proven. On the other hand, any polynomial vanishing at P, Q but not at some point of the form $c_1P + c_2Q$ can't be linear. So in the case that $I(Y)$ vanishes at P, Q but not at some $c_1P + c_2Q$, then for any choice of generators of $I(Y)$, there must be at least one which does not vanish at $c_1P + c_2Q$, and thus there is no generating set of $I(Y)$ which consists of homogeneous linear polynomials and Y is not a linear variety.

With this in mind, suppose Y is one-dimensional and pick two points $P, Q \in Y$. Let H be a hyperplane through P and Q. By theorem I.7.7, Y must be contained in H: if not, then 1, the product of the degrees of Y and H, must be equal to some number greater than 1, since P and Q are irreducible components of $Y \cap H$. So Y is contained in every hyperplane passing through P and Q, and thus it is contained in the intersection of all of these hyperplanes. But this intersection is the line through P and Q, and Y is a closed one-dimensional subset of this line. As the only closed one-dimensional subset of $\mathbb{P}^1$ is $\mathbb{P}^1$, $Y = \overline{PQ}$ and thus it must be a linear variety.

For the induction step, suppose we know that any algebraic set of dimension $< n$ and degree 1 is linear. Let Y be a variety of dimension $n > 1$ and degree 1. Pick two points P, Q and select a hyperplane H through P, Q which does not contain Y (we can do this because the intersection of all hyperplanes through P, Q is just $\overline{PQ}$, and Y is not one-dimensional). Then $H \cap Y$ is of dimension $n-1$, and by theorem I.7.7, it is irreducibleand of degree one (note that since $i(Y, H; Z_j)$ and $\deg Z_j$ are both positive while $(\deg Y)(\deg H) = 1$, we must have that there is only one Z_j and the degree of it must be 1). So $H \cap Y$ is linear by our induction assumption and contains the line through P, Q. By varying P, Q we see that Y contains all lines between all points $P, Q \in Y$ and by our alternate characterization of linear varieties, it must be linear.

Exercise I.7.7. Let Y be a variety of dimension r and degree $d > 1$ in $\mathbb{P}^n$. Let $P \in Y$ be a nonsingular point. Define X to be the closure of the union of all lines PQ, where $Q \in Y$, $Q \neq P$.

a. Show that X is a variety of dimension $r + 1$.

b. Show that $\deg X < d$. [*Hint*: Use induction on $\dim Y$.]

Solution.

a. Consider the map $Y \times \mathbb{P}^1 \to \mathbb{P}^n$ given by sending $(Q, [t : u]) \to tP + uQ$. This surjects on to the collection of points R which are on a line of the form $\overline{PQ}$ for $Q \neq P$, of which X is the closure. As $Y \times \mathbb{P}^1$ is irreducible, it's image is irreducible, and the closure of it's image is irreducible, so X is irreducible and this map from $Y \times \mathbb{P}^1$ is dominant. Thus we have an inclusion of function fields $k(X) \hookrightarrow k(Y \times \mathbb{P}^1)$, so $\dim X \leq \dim Y + 1$. On the other hand, $Y \subset X$, so $\dim X \geq \dim Y$. All we need to do to show that $\dim X = \dim Y + 1$ is to find a point in X not in Y. (Many claimed solutions to this problem say something like 'we can construct a rational inverse' to the map defined above and skip over the existence of such a point - this is problematic, because without this point, one can't actually define such an inverse!)

   We will prove the claim by induction on $\dim Y$. If $\dim Y = 1$, then I claim that for any $Q \in Y$ distinct from P, the intersection $\overline{PQ} \cap Y$ is a proper subset of $\overline{PQ}$. If it is not, then $\overline{PQ} \subset Y$, and by irreducibility, $Y = \overline{PQ}$, which is a linear variety. But by exercise I.7.6, this implies Y has degree one, which is not the case. So $\overline{PQ}$ contains a point not in Y, and thus X contains a point not in Y.

   For the rest of the argument, we will show that if $\dim Y > 1$, then we can find a hyperplane H through P so that $Y \cap H$ is smooth at P. Then either $Y \cap H$ is irreducible, in which case we will show that $i(Y, H; Y \cap H) = 1$ and so we have that $Y \cap H$ is a variety of dimension $r - 1$, degree d, and $P \in Y \cap H$ is a smooth point, or $Y \cap H$ is reducible, in which case we can find a point in $X \setminus Y$. The combination of these two statements will prove our claim by induction, and the central difficulty will be in showing $i(Y, H; Y \cap H) = 1$.

   If $\dim Y > 1$, we first note that $r < n$: otherwise $Y = \mathbb{P}^n$, which is linear, contradicting the fact that $\deg Y > 1$. Now, up to a change of coordinates on $\mathbb{P}^n$ we may assume that $P = [1 : 0 : \cdots : 0]$ and work in the affine open U_0. Then $Y \cap U_0$ is cut out by an ideal $(g_1, \cdots, g_m) \subset k[x_1, \cdots, x_n]$, and the Jacobian criterion says that P is smooth if the Jacobian matrix at P, $\frac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_n)}(P)$, has rank r. As $r < n$, we can find a vector $(a_1, \cdots, a_n) \in k^n$ so that after appending this to our matrix, we get a matrix of rank $r + 1$. Letting $f = \sum a_i x_i$, $\overline{f} = \sum a_i X_i$, and $H = V(\overline{f})$ where the x_i are coordinates on U_0 and X_i are homogeneous coordinates on $\mathbb{P}^n$, I claim that $H \cap Y$ is smooth at P.

As we can check smoothness in local coordinates, it suffices to verify that $(k[Y\cap U_0]/\sqrt{(f)})_{\mathfrak{m}_P}$ is a regular local ring. Since this is just the quotient of $(k[Y\cap U_0]/(f))_{\mathfrak{m}_P}$ by the nilradical, it suffices to show that $(k[Y\cap U_0]/(f))_{\mathfrak{m}_P}$ is already a regular local ring: this will show that the nilradical is zero because regular local rings are domains, and so $(f)=\sqrt{(f)}$ as ideals of $k[Y\cap U_0]_{\mathfrak{m}_P}$. But we've already done this, more or less by construction of f: the Jacobian matrix of $(g_1,\cdots,g_m,f)$ has rank $n-r+1$ by construction, so by theorem I.5.1, the local ring $(k[x_1,\cdots,x_n]/(g_1,\cdots,g_m,f))_{\mathfrak{m}_P}$ is a regular local ring, hence a domain, and thus $(f)_{\mathfrak{m}_P}$ is radical and so this is the local ring of $Y\cap H\cap U_0=V(f)\subset Y\cap U_0$ at the point P. So $k[V(f)\subset Y\cap U_0]_{\mathfrak{m}_p}$ is a regular local ring of dimension $r-1$, and thus $V(f)\subset Y\cap U_0$ is smooth at P, so $Y\cap V(\overline{f})$ is smooth at P. In particular, there is only one irreducible component of $Y\cap H$ through P by exercise I.3.11, the fact that regular local rings are domains, and the correspondence between irreducible components and minimal primes.

Now I claim that this computation implies that $i(Y,H;Z)=1$ where Z is the unique irreducible component of $Y\cap H$ through P. To do this, we show that we can compute intersection multiplicity from the data of an affine open via a lemma.

Lemma. *Suppose S is a noetherian graded ring and M a graded S-module, $\mathfrak{p}$ a minimal homogeneous prime of M, and $f\in S_1$ a homogeneous element not in $\mathfrak{p}$. Then the length of $M_{(\mathfrak{p})}$ over $S_{(\mathfrak{p})}$ is the same as the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$.*

Proof. First we show that the graded localization functor $M\mapsto M_{(f)}$ is exact. Given an exact sequence of graded S-modules $0\to M'\to M\to M''\to 0$, we have that localization at a homogeneous element f gives an exact sequence of graded modules $0\to M'_f\to M_f\to M''_f\to 0$ because localization is exact and commutes with direct sums: the d^{th} graded piece of M_f is exactly the elements $\frac{m}{f^n}$ where $\deg m=d+n$. As our maps are maps of graded modules and therefore preserve degree, we can restrict to the elements of any fixed degree to get an exact sequence $0\to(M'_f)_d\to(M_f)_d\to(M''_f)_d\to 0$, so we have the claim.

Next, I claim that if $\mathfrak{p}\subset S$ is a graded prime ideal and f is a homogeneous element not in $\mathfrak{p}$, then $M_{(\mathfrak{p})}=(M_{(f)})_{\mathfrak{p}'}$ where $\mathfrak{p}'$ is $(\mathfrak{p}_{(f)})_0$. This is not hard to see: given an element $\frac{x}{g}\in M_{(\mathfrak{p})}$, we can write it as $(xg^{\deg(f)-1}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)})$ because $\deg(x)=\deg(g)$ and $g^{\deg(f)}/f^{\deg(g)}$ isn't in $\mathfrak{p}'$. Conversely, if we have an element $(x/f^n)/(g/f^m)\in(M_{(f)})_{\mathfrak{p}'}$, we can write it as $(xf^m)/(gf^n)$ and the composition both ways is the identity.

Finally, we see that because the length of $M_{\mathfrak{p}}$ is the number of times that $S/\mathfrak{p}$ appears in the filtration promised by proposition I.7.4 and the functor $M\mapsto M_{(\mathfrak{p})}$ sends $S/\mathfrak{p}$ to $(S/\mathfrak{p})_{(\mathfrak{p})}\cong S_{(\mathfrak{p})}/\mathfrak{p}_{(\mathfrak{p})}$. So there is a filtration of $M_{(\mathfrak{p})}$ by $S_{(\mathfrak{p})}$-modules so that the number of times $S_{(\mathfrak{p})}/\mathfrak{p}_{(\mathfrak{p})}$ shows up is the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$. On the other hand, by the usual Jordan-Holder theorem that proposition I.7.4 is the analogue for, the length of $M_{(\mathfrak{p})}$ over $S_{(\mathfrak{p})}$ is exactly this number as well.

■

The upshot of this is that $i(Y,H;Z)$ is equal to the length of $(k[Y\cap U_0]/(f))_{I_{Z\cap U_0}}$ over

$k[Y \cap U_0]_{I_{Z\cap U_0}}$. But since $I_{Z\cap U_0} \subset \mathfrak{m}_P$ and localization is transitive, we have that

$$(k[Y \cap U_0]/(f))_{I_{Z\cap U_0}} \cong ((k[Y \cap U_0]/(f))_{\mathfrak{m}_P})_{I_{Z\cap U_0}} \cong (k[Z \cap U_0]_{\mathfrak{m}_P})_{I_{Z\cap U_0}} \cong k[Z \cap U_0]_{I_{Z\cap U_0}},$$

which clearly has length one over itself, and thus we've proven that $i(Y, H; Z) = 1$. By an application of theorem I.7.7, this shows that if $Y \cap H$ is irreducible, then $Y \cap H$ is a variety of dimension $r-1$ and degree d with a smooth point P, fulfilling our promise from earlier.

It remains to check that if $Y \cap H$ is reducible then we can find a point in X not in Y. Let Z_1 be the unique irreducible component of $Y \cap H$ containing P and let Z_2 be an irreducible component of $Y \cap H$ not containing P: as $Z_1 \cap Z_2$ is a proper closed subset of Z_1 and Z_2, we can pick a point $Q \in Z_2 \setminus Z_1$. Now, $Y \cap \overline{PQ} = (Y \cap H) \cap \overline{PQ}$, and we can write $\overline{PQ} \cap Y$ as a union of the closed sets $\overline{PQ} \cap Z_i$. If $Y \cap \overline{PQ} = \overline{PQ}$, then it must be the case that one of these closed sets $\overline{PQ} \cap Z_i$ is all of $\overline{PQ}$ by irreducibility. But this is not the case: it can't be Z_j for $j > 1$, as no Z_j with $j > 2$ contains P, and Z_1 doesn't contain Q by our choice of Q. So $Y \cap \overline{PQ}$ is a reducible subset of $\overline{PQ}$, which means it's a proper subset of $\overline{PQ}$ and we're finished.

b. We will need to talk about X constructed on Y and P and X constructed on $Y \cap H$ and P in the solution, so we need a better notation - we'll denote the X as constructed in the problem statement to be $J(Y, P)$ so that we can make the dependence on the set Y more obvious. (This stands for the *join variety* of X with P: this can be defined in general for any two projective varieties $V, W \subset \mathbb{P}^n$ as the closure of the collection of points on lines of the form $\overline{PQ}$ where $P \in V$, $Q \in W$, and $P \neq Q$.) We generalize the problem by removing the irreducibility assumption and prove the following: if $Y \subset \mathbb{P}^n$ is a closed algebraic set of degree at least 2 and $P \in Y$ is a smooth point, then $\deg J(Y, P) < \deg Y$. We note that we may always assume that Y is pure-dimensional, as any irreducible component not of top dimension does not contribute to the degree; additionally, we may assume that Y is of codimension 2, else $J(Y, P) = \mathbb{P}^n$ which is of degree 1 and the conclusion is trivially satisfied.

If you wanted to follow the hint, we would prove this by induction on $\dim Y$. The base case is clear: if Y is a dimension-zero algebraic set of degree d, it's exactly d points, and $J(Y, P)$ is a collection of at most $d-1$ lines. For the inductive step, we'll cut $J(Y, P)$ with a well-chosen hyperplane H. Assume we can find a hyperplane H through P which:

- does not contain the tangent space to Y at P,
- does not contain any irreducible component of $J(Y, P)$ or Y,
- has intersection multiplicity 1 with $J(Y, P)$ along every irreducible component of $H \cap J(Y, P)$ and the same for Y.

Then by the fact that $\deg H = 1$ and theorem I.7.7, we have that

$$\deg J(Y, P) = (\deg H)(\deg J(Y, P)) = \sum_Z i(J(Y, P), H; Z) \cdot \deg Z,$$

as Z runs over the irreducible components of $J(Y,P)\cap H$. By our assumptions on H, all of these intersection multiplicities are one. Thus

$$\deg J(Y,P) = \sum_Z \deg Z = \deg J(Y,P)\cap H.$$

Now I claim that $J(Y,P)\cap H = J(Y\cap H, P)$. This is straightforwards: both are (the closure of) the collection of points on a line through $(Y\setminus P)\cap H$ and P. Our first assumption on H implies that P is a smooth point of $Y\cap H$ by the logic of part (a). By our second assumption, all the irreducible components of $Y\cap H$ are of dimension $\dim Y - 1$. Thus $Y\cap H$ satisfies the inductive hypothesis, so we have that

$$\deg J(Y,P)\cap H = \deg J(Y\cap H, P) < \deg Y\cap H.$$

By theorem I.7.7, we note that this is at most $\deg Y$, and combining this with our previous observation that $\deg J(Y,P) = \deg J(Y,P)\cap H$, we see that we have $\deg J(Y,P) < \deg Y$.

It remains to prove that we can always find a hyperplane H. The main tool for this sort of thing would be Bertini's theorem, presented as theorem II.8.18 in this text - it's not completely out of the question for you to come up with this yourself, but the stronger version of being able to pick a hyperplane through P and still have all these conditions satisfied is a little trickier, there are added difficulties in positive characteristic, and works in this vein (especially those treating the extra difficulties we identified) were still being published for years after Hartshorne's text: for instance, the paper *Strong Bertini Theorems* by Diaz and Harbater explaining these sorts of things was published in Transactions of the AMS in 1991, a full 14 years after this text!

Alternatively, if one is willing to engage in some nontrivial intersection theory, there is a perfectly fine characteristic-independent proof: given two irreducible projective varieties $X, Y\subset \mathbb{P}^n$ over an arbitrary field k, let $\pi : \widetilde{J}\to J(X,Y)$ be the (rational) projection from the abstract join of X and Y in $\mathbb{P}^{2n+1}$ to $J(X,Y)\subset\mathbb{P}^n$. Then

$$\deg X \deg Y = \deg V + \deg\pi \deg J(X,Y)$$

where V is the Vogel cycle, a special cycle supported on $X\cap Y$ (for details, see *Join Varieties and Intersection Theory*, H. Flenner, in *Recent Progress in Intersection Theory*, ed. Ellingsrud, Fulton, Vistoli.). In our case, $Y = V = P$ and thus $\deg X = 1 + \deg\pi\deg J(X,P)$ which shows that $\deg J(X,P) < \deg X$.

Exercise I.7.8. Let $Y^r\subset\mathbb{P}^n$ be a variety of degree 2. Show that Y is contained in a linear subspace of dimension $r+1$ in $\mathbb{P}^n$. Thus Y is isomorphic to a quadric hypersuraces in $\mathbb{P}^{r+1}$ (Ex. 5.12).

Solution. Find a nonsingular point $P\in Y$ by the material developed in I.5. By an application of exercise I.7.7(b) with this P and Y, we have that the X constructed there is of degree 1 and dimension $r+1$, so by exercise I.7.6, X is the required linear subspace of dimension $r+1$.

I.8 What is Algebraic Geometry?

There are no problems here! Time to write something insightful about what Algebraic Geometry is.

Chapter II

Schemes

Schemes form a much nicer technical framework for doing algebraic geometry in than the varieties of chapter I. They're a lot more powerful, and many proofs from chapter I which seem difficult or unwieldy actually end up being much nicer once we have access to the scheme-theoretic language. The material in chapter I (algebraic varieties over fields) is still useful to know as we move in to chapter II primarily as a source of examples to practice one's technique on.

It may be interesting to compare Hartshorne's roadmap for treating various aspects of scheme theory with other texts written in the years since this was published. Notably, Ravi Vakil's *Foundations of Algebraic Geometry* gets good reviews, and the reference works of EGA and the Stacks Project are also valuable.

II.1 Sheaves

This section introduces a bunch of important background knowledge about sheaves we'll need for technical portions of arguments later. It would almost be tempting to put this in its own prerequisite course as an instructor, but just like commutative algebra, there's a lot of obvious and close ties to algebraic geometry which would be useful in teaching both (compare homological algebra and your first algebraic topology course in which you see homology/cohomology for the first time).

As for the exercises, many will seem painfully obvious or second-nature in hindsight, but you need to know them and prove them once in your life and then you can just cite them forever.

Exercise II.1.1. Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.

Solution. This immediately follows from the definition of sheafification (proposition-definition II.1.2) as $\mathcal{A}$ satisfies the universal property of the sheafification of the constant presheaf: the value of this sheaf consists of locally constant maps $U \to A$ where A is equipped with the discrete topology.

Exercise II.1.2.

a. For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.

b. Show that φ is injective (respectively, surjective) if and only if the induced map on stalks φ_P is injective (respectively, surjective) for all P.

c. Show that a sequence $\cdots \mathcal{F}^{i-1} \stackrel{\varphi^{i-1}}{\to} \mathcal{F}^i \stackrel{\varphi^i}{\to} \mathcal{F}^{i+1} \to \cdots$ is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Solution.

a. Consider the relevant commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow{\varphi_P} & \mathcal{G}_P \end{array}$$

Let $s \in \mathcal{F}(U)$ be an element of the kernel. By commutativity, s_P must also be sent to zero, so $(\ker \varphi)(U) \subset \ker(\varphi_P)$ and this is compatible with restriction, so we get $(\ker \varphi)_P \subset \ker(\varphi_P)$.

To show the opposite inclusion, pick an element $\overline{s} \in \ker(\varphi_P)$. Let s be the preimage of $\overline{s}$ in $\mathcal{F}(U)$, so that $\varphi(U)(s) = t \in \mathcal{G}(U)$ is zero in the stalk $\mathcal{G}_P$ (we may have to shrink U in order to find such an s, but it is always possible by the definition of the stalk as a limit). By the definition of the stalk, we can find some open subset $V \subset U$ so that $t|_V = 0$, and so by replacing U with V in our diagram above, we have that $s|_V \in \ker \varphi(V)$, which shows that $\ker(\varphi_P) \subset (\ker \varphi)_P$.

For images, the same ideas apply: let $t \in \mathcal{G}(U)$ be an element in the image. Then t_P is in the image as well, so $\operatorname{im} \varphi(U) \subset \operatorname{im}(\varphi_P)$, and via considering restriction maps, we can see that $(\operatorname{im} \varphi)_P \subset \operatorname{im}(\varphi_P)$.

For the reverse inclusion, pick an element $\bar{t} \in \operatorname{im}(\varphi_P)$ with $\bar{s} \in \mathcal{F}_P$ mapping to it. Then on some open neighborhood U of P, we can find an $s \in \mathcal{F}(U)$ which maps to $\bar{s}$, and then $\varphi(U)(s)$ restricts to $\bar{t}$ by commutativity of the diagram, showing that $\operatorname{im}(\varphi_P) \subset (\operatorname{im} \varphi)_P$.

b. Consider the zero map $0 : 0 \to \ker \varphi$. By proposition II.1.1, this is an isomorphism iff $(\ker \varphi)_P = 0$ for all P. But $\ker \varphi = 0$ if and only if φ is injective by definition, so we are done.

   For surjectivity, consider the inclusion morphism $i : \operatorname{im} \varphi \to \mathcal{G}$. By proposition II.1.1, this is an isomorphism iff i_P is an isomorphism for all P. But this is an isomorphism iff φ is surjective by definition, so we are done.

c. By definition, such a sequence is exact iff $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Again by proposition II.1.1 and part (a), this statement is equivalent to $\ker(\varphi^i_P) = \operatorname{im}(\varphi^{i-1}_P)$, which is just the statement for exactness at all stalks.

Exercise II.1.3.

a. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}_{i \in I}$ of U, and there are elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i.

b. Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, and an open set U such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Solution.

a. We show that the given condition is equivalent to being surjective on stalks, which implies surjectivity by exercise II.1.2(b). In one direction, if the given condition holds, then for any $P \in X$ and any element $s_P \in \mathcal{G}_P$, we can find a U so that there is a lift of s_P to $\mathcal{G}(U)$, apply the condition, and find some $t \in \mathcal{F}(U')$ with $U' \subset U$ and $\varphi(U')(t) = s|_{U'}$, which implies that $\varphi_P(t_P) = s_P$, proving surjectivity on stalks. For the other direction, run the argument backwards: given $s_P \in \mathcal{G}_P$ which is the image of some $t_P \in \mathcal{F}_P$, then we can find a neighborhood U of P so that s_P, t_P lift to sections on U, and then we have that after potentially shrinking U, we get that $\varphi(U)(t) = s$. Applying this argument at every point, we get the required covering.

b. Let $\mathcal{F}$ be the sheaf of holomorphic functions on $\mathbb{C}$, and let $\mathcal{F}^*$ be the sheaf of nowhere-zero holomorphic functions, and define $\varphi : \mathcal{F} \to \mathcal{F}^*$ by $f \mapsto e^f$. This is surjective because we can always define a logarithm sufficiently locally, but it is not surjective on $\mathbb{C}^\times$ since z does not have a globally-defined logarithm.

A scheme-ier example you will appreciate in a couple sections is the following. Let $X = \operatorname{Spec}\mathbb{Z}$, let $i : \operatorname{Spec}\mathbb{Z}/(p) \to \operatorname{Spec}\mathbb{Z}$ be the inclusion of the closed point corresponding to the ideal (p) for a prime p, and let $\mathcal{F} = i_*\mathbb{Z}_{(p)}$, the skyscraper sheaf with value $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at the prime ideal (p)). Then $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}} \to \mathcal{F}$ is surjective, because both sheaves have stalk $\mathbb{Z}_{(p)}$ at (p) with the map on stalks being the identity map, and the latter sheaf has the zero stalk everywhere else and any map to the zero object is surjective. On the other hand, this map is not surjective on any open subset of the form $D(q)$ for $q \neq p$: $\mathbb{Z}_q$ is finitely generated as a $\mathbb{Z}$-algebra and $\mathbb{Z}_{(p)}$ isn't, for the same reasons $\mathbb{Q}$ isn't a finitely-generated $\mathbb{Z}$-algebra.

Exercise II.1.4.

a. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of associated sheaves is injective.

b. Use part (a) to show that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\operatorname{im}\varphi$ can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.

Solution.

a. By construction, the stalks of $\mathcal{F}$ and $\mathcal{F}^+$ are the same, and the assumption that $\varphi(U)$ is injective for all U implies the map on stalks is injective at every point. But by exercise II.1.2(b), this implies $\mathcal{F}^+ \to \mathcal{G}^+$ is injective.

b. We have an injective morphism $\varphi(\mathcal{F}) \to \mathcal{G}$ where $\varphi(\mathcal{F})$ is the image presheaf. Then by (a) and the fact that $\mathcal{G}$ is already a sheaf, we get an injective map $\varphi(\mathcal{F})^+ \to \mathcal{G}$.

Exercise II.1.5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Solution. This is immediate from proposition II.1.1 and exercise II.1.2(b).

Exercise II.1.6.

a. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

b. Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$, and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Solution.

a. This follows immediately from the relevant statement for stalks (which is an exercise you should complete in a basic abstract algebra course) and exercise II.1.2(c).

b. By definition of exactness, $\operatorname{im}(\mathcal{F}' \to \mathcal{F}) = \ker(\mathcal{F} \to \mathcal{F}'')$, so by exercise II.1.4(b), we have that $\operatorname{im}(\mathcal{F}' \to \mathcal{F})$ is isomorphic to a subsheaf of $\mathcal{F}$. On the other hand, by exercise II.1.7(a), $\operatorname{im}(\mathcal{F}' \to \mathcal{F}) \cong \mathcal{F}'/\ker(0 \to \mathcal{F}') \cong \mathcal{F}'$ and thus we have that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$.

   For the statement about $\mathcal{F}''$, again apply exercise II.1.7(a) and the definition of a surjective morphism of sheaves as one with the image equal to the target.

Exercise II.1.7. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

a. Show that $\operatorname{im}\varphi \cong \mathcal{F}/\ker\varphi$.

b. Show that $\operatorname{coker}\varphi \cong \mathcal{G}/\operatorname{im}\varphi$.

Solution.

a. We have a morphism $\mathcal{F}/\ker\varphi \to \operatorname{im}\varphi$ given by the sheafification of the obvious morphism on presheaves, so we can check whether it is an isomorphism by looking on stalks by proposition II.1.1. Since stalks commute with images and kernels by exercise II.1.2(a), we may apply the first isomorphism theorem to see that the stalks of both sides are isomorphic.

b. The same proof as part (a) works verbatim.

Exercise II.1.8. For any open subset $U \subset X$, show that the functor $\Gamma(U, -)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is an exact sequence of groups. The functor $\Gamma(U, -)$ need not be exact; see (Ex. 1.21) below.

Solution. First we check exactness at the the first spot, i.e. that $\mathcal{F}'(U) \to \mathcal{F}(U)$ is an injection. If $s \in \mathcal{F}'(U)$ is an element of the kernel, then it's image in $\mathcal{F}'_P$ for any $P \in U$ is zero, since the map $\mathcal{F}'_P \to \mathcal{F}_P$ is injective by exercise II.1.2(b). But this means it must actually be the zero element, so $\mathcal{F}'(U) \to \mathcal{F}(U)$ is injective.

Next, we need to check that $\ker(\mathcal{F}(U) \to \mathcal{F}''(U)) = \operatorname{im}(\mathcal{F}'(U) \to \mathcal{F}(U))$. On the one hand, the composite map $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is zero, which implies that the induced map $\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is zero (unwind the definition), so we have that $\ker \supset \operatorname{im}$. To show the reverse inclusion, pick $s \in \ker(\mathcal{F}(U) \to \mathcal{F}''(U))$. Then at a point $P \in U$, we have that there exists a $t_P \in \mathcal{F}'_P$ so that t_P maps to s_P by exactness on stalks, and that this t_P is unique since $\mathcal{F}' \to \mathcal{F}$ is injective. Then about each point $P \in U$ we can find an open neighborhood U_P with a lift $\widetilde{t_P}$ of t_P so that the image of $\widetilde{t_P}$ in $\mathcal{F}(U_P)$ is $s|_{U_P}$. By uniqueness, these glue in to a section of the image sheaf which has image s, and we are done.

For a more high-minded proof once we know a little more machinery, $\Gamma(-, U)$ is Hom from the constant sheaf $\mathbb{Z}$ over U, and then this follows from abstract nonsense about abelian categories.

Exercise II.1.9. *Direct Sum.* Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

Solution. We need to verify conditions (3) and (4) of the definition of a sheaf on page 61: condition (3) states that if a section is zero on a cover, it is the zero section, and condition (4) states that we can glue compatible sections from elements of a cover in to a section over the whole open set.

We begin with (3). Suppose U is an open set and $\{V_i\}$ an open covering of U with $(s,t) \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ an element so that $(s,t)|_{V_i} = 0$ for all V_i. This implies that $s|_{V_i} = 0$ and $t|_{V_i} = 0$ for all V_i, and by an application of condition (3) to the sheaves $\mathcal{F}$ and $\mathcal{G}$, we see that $s = 0$ and $t = 0$, so $(s,t) = 0$.

For (4), the same idea works. Suppose U is an open set and $\{V_i\}$ an open covering of U with $(s_i,t_i) \in \mathcal{F}(V_i) \oplus \mathcal{G}(v_i)$ a collection of elements so that $(s_i,t_i)|_{V_i \cap V_j} = (s_j,t_j)|_{V_i \cap V_j}$ for all i,j. Then by an application of condition (4), we have that the s_i glue to some s, the t_i glue to some t, and so the (s_i,t_i) glue to (s,t).

Showing that this satisfies the universal property of direct sums and products is similarly straightforward. The key fact is that a presheaf morphism of two presheaves which happen to be sheaves is a sheaf morphism (see the definition of a morphism of presheaves and sheaves on page 62). If we have a sheaf $\mathcal{H}$ with maps $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$, then there is a unique morphism of presheaves $\mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$ which makes the direct sum diagram commute, given by taking the map guaranteed by the direct sum of abelian groups on sections over open sets. But since $\mathcal{H}$ and $\mathcal{F} \oplus \mathcal{G}$ are sheaves, this is a morphism of sheaves. The proof for direct product is the same but with some arrows reversed and the fact that finite direct sums and direct products agree for abelian groups.

Exercise II.1.10. *Direct Limit.* Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathcal{F}_i\}$, denoted $\varinjlim \mathcal{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathcal{G}$, and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathcal{F}_i \to \mathcal{G}$ such that for each i, the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathcal{G}$.

Solution. Given a sheaf $\mathcal{G}$ and a collection of compatible morphisms $\mathcal{F}_i \to \mathcal{G}$, we get a unique morphism from the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ to $\mathcal{G}$ satisfying the relevant conditions by applying the universal property of the direct limit to the map on sections over any open set U. Then by proposition-definition II.1.2, we have a unique map from the sheafification of this presheaf to $\mathcal{G}$ which must also satisfy the relevant conditions. As we defined the direct limit sheaf to be the sheafification of this presheaf, we are finished.

Exercise II.1.11. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i)$.

Solution. We need to verify the standard two conditions, locality and gluing. The main idea here is that the necessary equality of elements happens at some finite stage inside our direct limit, and by noetherianity, there are only finitely many situations in which to verify this equality of elements, so we can take least upper bound of all these situations and obtain equality in the directed system of sheaves at that point.

We'll change the notation a little bit here to refer to the directed system by greek letters and the elements of the cover by roman letters. Let $\mathcal{P}$ denote the presheaf $U \mapsto \varinjlim \mathcal{F}_\alpha(U)$, and let $\varphi_\alpha : \mathcal{F}_\alpha \to \mathcal{P}$ denote the morphism from $\mathcal{F}_\alpha$ to the direct limit. Now suppose $U \subset X$ is an open subset with $\{U_i\}$ an open cover. By the fact that X is noetherian, we may assume that this cover is finite (ref exercise I.1.7).

To check locality, assume we have two sections $s, t \in \mathcal{P}(U)$ with $s|_{U_i} = t|_{U_i}$ for all i. By construction of the direct limit for abelian groups, we may find elements $x \in \mathcal{F}_\alpha(U)$ and $y \in \mathcal{F}_\beta(U)$ so that $\varphi_\alpha(U)(x) = s$ and $\varphi_\alpha(U)(y) = t$. Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}_\alpha(U) & \xrightarrow{\varphi_\alpha(U)} & \mathcal{P}(U) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle res} \\ \mathcal{F}_\alpha(U_i) & \xrightarrow{\varphi_\alpha(U_i)} & \mathcal{P}(U_i) \end{array}$$

The fact that $s|_{U_i} = t|_{U_i}$ implies that $\varphi_\alpha(x|_{U_i}) = \varphi_\beta(y|_{U_i})$ in the limit, so there exists some $\gamma_i \geq \alpha, \beta$ so that in $\mathcal{F}_{\gamma_i}(U_i)$, we have $x|_{U_i} = y|_{U_i}$ (where we've suppressed the maps inside the direct system to make the notation slightly less awful).

Now we can take γ to be the maximum of γ_i across the finite collection of i, and we get that in $\mathcal{F}_\gamma(U)$ we have that $x = y$ by locality, so we have that $s = \varphi_\gamma(x) = \varphi_\gamma(y) = t$ and our claim is proven.

Gluing uses the same ideas. Let $s_i \in \mathcal{P}(U_i)$ be a collection of sections so that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all pairs i, j. As before, pick elements $x_i \in \mathcal{F}_{\alpha_i}(U_i)$ so that $\varphi_{\alpha_i}(x_i) = s_i$. The assertion that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ implies that there is some $\gamma_{ij} \geq \alpha_i, \alpha_j$ so that in $\mathcal{F}_{\gamma_{ij}}(U_i \cap U_j)$ we have $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$.

Now we can take γ to be the maximum of γ_{ij} across the finite set of all pairs i, j, and we get that the (images of the) x_i form a compatible family of sections of $\mathcal{F}_\gamma$, so they glue. On the other hand, they have image s_i in $\mathcal{P}$, so they must glue there as well, and our claim is proven.

(Abstract-nonsense remark: there should also be a solution to this following from the fact that colimits commute with limits when the indexing sets are nice - in our case, the limit is sheafification and the colimit is the direct limit. I leave this to the reader for now and present a more hands-on approach in the hope of giving the reader a better grasp of what a direct limit 'really is'.)

Exercise II.1.12. *Inverse Limit.* Let $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}$, and is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Solution. We make the same notational change for our solution as in the previous exercise: the inverse system will be labeled by greek letters and the elements of our cover by roman letters. Let $\mathcal{P}$ denote the presheaf $U \mapsto \varprojlim \mathcal{F}_\alpha(U)$, and let $\varphi_\alpha : \mathcal{P} \to \mathcal{F}_\alpha$ denote the map from the inverse limit to each element of the inverse system. Clearly by the universal property of inverse limits of abelian groups applied to each open $U \subset X$, we see that $\mathcal{P}$ satisfies the universal property of inverse limits as a presheaf, which means its sheafification will satisfy the universal property of inverse limits as a sheaf, per proposition-definition II.1.2.

To show that $\mathcal{P}$ is a sheaf, we check the locality and gluing conditions. Let U be an open set and $\{U_i\}$ an open cover, and suppose $s, t \in \mathcal{P}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i. In particular, $\varphi_\alpha(U_i)(s|_{U_i}) = \varphi_\alpha(U_i)(t|_{U_i})$ for all α and all i, so for any α we have that $\varphi_\alpha(s) = \varphi_\alpha(t)$ by the fact that the $\mathcal{F}_\alpha$ are sheaves. By the definition of the inverse limit, this implies that $s = t$ and we are done.

The same strategy works for gluing: if we have an open set U with open cover $\{U_i\}$ and $s_i \in \mathcal{P}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all pairs i, j, then $\varphi_\alpha(U_i)(s_i)|_{U_i \cap U_j} = \varphi_\alpha(U_j)(s_j)|_{U_i \cap U_j}$ for all α, and by the fact that the $\mathcal{F}_\alpha$ are sheaves, then the $\varphi_\alpha(U_i)(s_i)$ glue to a section s_α on U for every α. But then $\{s_\alpha\}$ defines an element of the inverse limit $\mathcal{P}(U_i)$ which restricts to s_i on each U_i by construction, and we've glued the s_i.

(Abstract-nonsense remark: there should also be a solution to this following from the fact that limits commute with limits, since both the inverse limit and the sheafification process are limits. I leave this to the reader for now and present a more hands-on approach in the hope of giving the reader a better grasp of what a limit 'really is'.)

Exercise II.1.13. *Espace Etale of a Presheaf.* (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathcal{F}$ on X, we define a topological space $\text{Spé}(\mathcal{F})$, called the *espace etale* of $\mathcal{F}$, as follows. As a set, $\text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P$. We define a projection map $\pi : \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_P$ to P. For each open set $U \subset X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\overline{s} : U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \overline{s} = id_u$, in other worse, it is a 'section' of π over U. We now make $\text{Spé}(\mathcal{F})$ in to a topological space by giving it the strongest topology such that all the maps $\overline{s} : U \to \text{Spé}(\mathcal{F})$ for all U, and for all $s \in \mathcal{F}(U)$ are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subset X$, $\mathcal{F}^+(U)$ is the set of continuous sections of $\text{Spé}(\mathcal{F})$ over U. In particular, the original presheaf $\mathcal{F}$ was a sheaf if and only if for each U, $\mathcal{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U.

Solution. We need to do two things here: we need to show that any $s \in \mathcal{F}^+(U)$ is gives a continuous section $U \to \text{Spé}(\mathcal{F})$, and to show that any continuous section $s : U \to \text{Spé}(\mathcal{F})$ gives rise to a section of $\mathcal{F}^+(U)$.

Sections of $\mathcal{F}^+(U)$ come from sections of $\mathcal{F}(U)$ and gluing sections $s_i \in \mathcal{F}(U_i)$ agreeing on overlaps (i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$) where $\{U_i\}$ forms an open cover of U. The first type of sections of $\mathcal{F}(U)$ clearly give continuous sections $U \to \text{Spé}(\mathcal{F})$ by definition, and the second type also give us continuous sections: we can define a section $\overline{s} : U \to \text{Spé}(\mathcal{F})$ by $\overline{s}(P) = \overline{s_i}(P)$ whenever $P \in U_i$, which can immediately be verified to be well-defined, continuous, and sections of π by the definition of $\text{Spé}(\mathcal{F})$. So sections of $\mathcal{F}^+(U)$ give continuous maps $U \to \text{Spé}(\mathcal{F})$ which are sections of $\pi : \text{Spé}(\mathcal{F}) \to X$ over U.

Before proceeding, we need to show that for a section $s \in \mathcal{F}(U)$, the set $\overline{s}(U)$ is open in $\text{Spé}(\mathcal{F})$. Supposing τ is our topology on $\text{Spé}(\mathcal{F})$ so that any map of the form $\overline{t} : V \to \text{Spé}(\mathcal{F})$ is continuous for any $t \in \mathcal{F}(V)$, I claim that the topology given by adjoining $\overline{s}(U)$ to τ also has this property, which will show that $\overline{s}(U)$ must already be in τ since τ is defined to be the strongest such topology. To prove this claim, suppose that $s \in \mathcal{F}(U)$ is a fixed section and $t \in \mathcal{F}(V)$ is arbitrary. If $\overline{s}(P) = \overline{t}(P)$ for some P, then we have that $s_P = t_P$, and so there's a neighborhood W of P on which $s|_W = t|_W$. Taking the union of all of these neighborhoods, we see that the locus where $s_P = t_P$ is open, and thus $\overline{t}^{-1}(\overline{s}(U))$ is open for any $t \in \mathcal{F}(V)$ and the claim is proven: $\overline{s}(U)$ must be open for any $s \in \mathcal{F}(U)$.

Now suppose $\overline{s} : U \to \text{Spé}(\mathcal{F})$ is a continuous section. Then at a point P, $\overline{s}(P)$ gives an element $s_P \in \mathcal{F}_P$, which yields a section $t \in \mathcal{F}(V)$ for $V \subset U$ an open neighborhood of P. Since $\overline{t}(V)$ is open, $\overline{s}^{-1}(t(V))$ gives an open subset of U which is a neighborhood of P where $\overline{s} = \overline{t}$. This means that locally, any continuous section $U \to \text{Spé}(\mathcal{F})$ is representable by a section (t, V) of $\mathcal{F}$, and any two such representatives are equal on their common intersection - so they patch together to a unique section of $\mathcal{F}^+(U)$ by the sheaf axioms. This means that every continuous section of $U \to \text{Spé}(\mathcal{F})$ determines a unique section of $\mathcal{F}^+$, and by our earlier work, this gives a bijection between sections of $\mathcal{F}^+(U)$ and sections $U \to \text{Spé}(\mathcal{F})$.

Exercise II.1.14. *Support.* Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. The *support* of s, denoted $\operatorname{Supp} s$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. Show that $\operatorname{Supp} s$ is a closed subset of U. We define the *support* of a $\mathcal{F}$, $\operatorname{Supp} \mathcal{F}$ to be $\{P \in X \mid \mathcal{F}_P \neq 0\}$. It need not be a closed subset.

Solution. The complement of $\operatorname{Supp} s$ is the set $\{P \in U \mid s_P = 0\}$. What this says is that for every P in this set, there is an open neighborhood U of P such that $s|_U = 0$. As the union of open sets is open, we see that $\{P \in U \mid s_P = 0\}$ is open and thus the support of s is closed. To see that the support of a sheaf is not necessarily closed, refer to II.1.19b.

Exercise II.1.15. *Sheaf $\mathcal{H}om$.* Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subset X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathcal{F}$ in to $\mathcal{G}$, 'sheaf hom' for short, and is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Solution. Checking that $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|U)$ is an abelian group is not difficult: given two maps of sheaves $f, g : \mathcal{F}|_U \to \mathcal{G}|_U$ we can define $f + g$ to be the map of sheaves sending $s \in \mathcal{F}|_U(V)$ to $f(s) + g(s) \in \mathcal{G}|_U(V)$. Checking this is a map of sheaves is trivial, since the restriction maps are homomorphisms of abelian groups.

Let $\mathcal{P}$ be the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. To show this is a sheaf, we need to verify locality and gluing. To that end, suppose U is an open set and $\{U_i\}$ is an open cover of U.

Suppose $s, t\mathcal{P}(U)$ are two maps so that $s|_{U_i} = t|_{U_i}$ for all i, and pick an arbitrary element $e \in \mathcal{F}(U)$. Then $s|_{U_i}(e|_{U_i}) = t|_{U_i}(e|_{U_i})$ for all i, so $s(e) = t(e)$ by the fact that $\mathcal{G}$ is a sheaf. But e was arbitrary, so we have that $s = t$ and $\mathcal{P}$ satisfies locality.

Now suppose $s_i \in \mathcal{P}(U_i)$ are a family of maps with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and $e_i \in U_i$ any similarly compatible collection of sections gluing to an element $e \in \mathcal{F}(U)$. Then $s_i(e_i)|_{U_i \cap U_j} = s_j(e_j)|_{U_i \cap U_j}$, and so $s_i(e_i)$ glue to an element f of $\mathcal{G}(U)$. Defining $s(e) = f$, we see that this gives a section $s \in \mathcal{P}(U)$ which restricts to $s_i \in \mathcal{P}(U_i)$, and thus $\mathcal{P}$ satisfies gluing. So $\mathcal{P}$ is a sheaf.

Exercise II.1.16. *Flasque Sheaves.* A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion $V \subset U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

a. Show that a constant sheaf on an irreducible topological space is flasque. See (1, §1) for irreducible topological spaces.

b. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ of abelian groups is also exact.

c. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

d. If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, the $f_*\mathcal{F}$ is a flasque sheaf on Y.

e. Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the sheaf of *discontinuous sections* of $\mathcal{F}$ as follows. For each open set $U \subset X$, $\mathcal{G}(U)$ is the set of maps $s : U \to \bigcup_{P\in U} \mathcal{F}_P$ such that for each $P \in U$, $s(P) \in \mathcal{F}_P$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Solution.

a. If X is irreducible, then any two open sets meet, so any open set is connected. Thus the sections of a constant sheaf with value A on any open are just A (from the description of a constant sheaf as locally constant functions, see example II.1.0.3) and the restriction maps are the identity and thus surjective.

b. By exercise II.1.8 we know that $\Gamma(U, -)$ is left-exact, so we just need to prove surjectivity at $\mathcal{F}''(U)$. Denote the map $\mathcal{F} \to \mathcal{F}''$ by φ, let $s'' \in \mathcal{F}''(U)$, and consider the set of all pairs (V, s) where $s \in \mathcal{F}(V)$ is such that $\varphi(V)(s) = s''|_V$. This set is nonempty: since φ is surjective, it's surjective on stalks, so for any $x \in U$ we can find $s_x \in \mathcal{F}_x$ which maps to s''_x under φ_x. This implies that s_x has some representative s on a neighborhood U' of x which maps to $s''|_{U'}$. The set of (V, s) is ordered by the relation that $(V_1, s_1) \geq (V_2, s_2)$ if $V_1 \supset V_2$ and $s_1|_{V_2} = s_2$. Every chain $\{(V_i, s_i)\}_{i\in I}$ in this set has an upper bound: we can take as our open set $V = \bigcup_{i\in I} V_i$, and define s to be the section which is the gluing of all the s_i, since $\mathcal{F}$ is a sheaf and the s_i agree on overlaps. By Zorn's lemma, this set has maximal elements.

   Pick a maximal element (V_0, s_0). If $V_0 \neq U$, we can find a nonempty open subset $V_1 \subset U$ so that $V_1 \not\subset V_0$ and a section $s_1 \in \mathcal{F}(V_1)$ so that $\varphi(V_1)(s_1) = s$. On $V_1 \cap V_0$, we have that $s_0 - s_1 = d \in \mathcal{F}'(V_1 \cap V_0)$. Pick $\hat{d} \in \mathcal{F}'(V_1)$ which restricts to d, and then $s_1 + \hat{d}$ agrees with s_0 on $V_0 \cap V_1$. Therefore by gluing we get a section on $V_1 \cup V_0 \neq V_0$ which maps to s, contradicting maximality of (V_0, s_0). So $V_0 = U$, and we see that the map $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective.

c. By part (b), we have that the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact for any U when $\mathcal{F}'$ is flasque. Consider the following diagram of exact sequences, where the vertical maps are the restriction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) & \longrightarrow & 0 \end{array}$$

By surjectivity, we can lift any element of $\mathcal{F}''(V)$ to an element of $\mathcal{F}(V)$ and then an element of $\mathcal{F}(U)$. By commutativity of the right-hand square, this means that the image of this element along the maps $\mathcal{F}(U) \to \mathcal{F}''(U) \to \mathcal{F}''(V)$ is the same as the image along the maps $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}''(V)$, so $\mathcal{F}''(U) \to \mathcal{F}''(V)$ is also surjective and thus $\mathcal{F}''$ is flasque.

d. The restriction map $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ is the same as the restriction map $\mathcal{F}(f^{-1}(U)) \to \mathcal{F}(f^{-1}(V))$ by definition of the direct image. Since the latter is surjective, the former is as well and $f_*\mathcal{F}$ is flasque.

e. This sheaf of discontinuous sections just gives $\mathcal{G}(U) = \prod_{P \in U} \mathcal{F}_P$, from which the statements are obvious: the surjective map $\mathcal{G}(U) \to \mathcal{G}(V)$ is given by projection onto $\prod_{P \in V} \mathcal{F}_P$, and the injective morphism $\mathcal{F} \to \mathcal{G}$ is given by sending $s \in \mathcal{F}(U)$ to $\prod_{P \in U} s_P$. Injectivity follows from the fact that if two sections are equal in a stalk, they are equal in a neighborhood of that stalk, and then by locality we have that they must be equal as sections.

Exercise II.1.17. *Skyscrape Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$, and 0 elsewhere, where $\overline{\{P\}}$ denotes the closure of the set consisting of the point P. Hence the name 'skyscraper sheaf'. Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{P\}}$, and $i : \overline{\{P\}} \to X$ is the inclusion.

Solution. If Q has an open neighborhood U not containing P, then the stalk of $i_P(A)$ at Q is zero, as the open neighborhoods of Q not containing P form a cofinal subset of the set of all open neighborhoods of Q (intersect with U), and the value of $i_P(A)$ on any such neighborhood is zero. On the other hand, if $Q \in \overline{\{P\}}$, then there are no neighborhoods of Q which do not contain P, so for all U containing Q, we have $i_P(A)(U) = A$ with restriction maps the identity, so the stalk of $i_P(A)$ at Q is A.

To see that the skyscraper sheaf can be described as the direct image, let $U \subset X$ be an open subset. Then $i_*(A)(U) = A$ if U meets $\overline{\{P\}}$, which is equivalent to $P \in U$ (if $P \notin U$, then U^c is a closed set containing P, thus containing $\overline{\{P\}}$ and so $U \cap \overline{\{P\}} = \emptyset$). So the morphism $i_*(A) \to i_P(A)$ given by the identity map on sections over any open is an isomorphism of sheaves as it induces isomorphisms on stalks.

Exercise II.1.18. *Adjoint Property of f^{-1}.* Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

Solution. This question is just a big definition-chase, but it can be pretty long and confusing. One big idea we'll use here to make this more understandable is for a continuous map $f : X \to Y$ we'll consider the presheaf inverse image functor, f°, which for a (pre)sheaf $\mathcal{A}$ on Y gives a presheaf $f^\circ\mathcal{A}$ on X defined by $(f^\circ\mathcal{A})(U) = \lim_{V\supset f(U)} \mathcal{A}(V)$. In terms of the sheaf inverse image functor f^{-1}, we have that $f^{-1}(-) = (f^\circ(-))^+$. We will write $\mathrm{Hom}_{Sh(X)}$ for morphisms of sheaves on X, and $\mathrm{Hom}_{PrSh(X)}$ for morphisms of presheaves on X. (The reason why this idea is useful is that sections of the inverse image presheaf are easier to deal with than the sections of the inverse image sheaf - no sheafification is necessary. Combining this with the fact that a map from a presheaf to a sheaf factors through the sheafification by proposition-definition II.1.2, this lets us work with the inverse image presheaf and then make conclusions about the inverse image sheaf.)

We start by computing $(f^\circ f_*\mathcal{F})(U)$. By definition, this is

$$\lim_{V\supset f(U)} (f_*\mathcal{F})(V) = \lim_{V\supset f(U)} \mathcal{F}(f^{-1}(V)).$$

Since $V \supset f(U)$ implies that $f^{-1}(V) \supset f^{-1}(f(U)) \supset U$ and $f^{-1}(V)$ is open, we have a restriction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ for any $V \supset f(U)$, and these commute with the maps defining the limit because those maps are also restriction maps. Thus we get a map $\lim_{V\supset f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ for any open U which commutes with restriction from U to U', and thus we have a morphism of presheaves $\varepsilon : f^\circ f_*\mathcal{F} \to \mathcal{F}$. Further, this morphism is natural in $\mathcal{F}$ because it's defined in terms of restriction morphisms. By proposition-definition II.1.2, our morphism $\varepsilon : f^\circ f_*\mathcal{F} \to \mathcal{F}$ factors naturally as $f^\circ f_*\mathcal{F} \to f^{-1} f_*\mathcal{F} \to \mathcal{F}$, which gives a natural map of sheaves $\varepsilon^+ : f^{-1} f_*\mathcal{F} \to \mathcal{F}$.

Now we compute $(f_* f^\circ\mathcal{G})(U)$. By definition, this is

$$(f^\circ\mathcal{G})(f^{-1}(U)) = \lim_{V\supset f(f^{-1}(U))} \mathcal{G}(V).$$

Since $f(f^{-1}(U)) = U$, we have that U is a member of the directed set of opens V so that $V \supset f(f^{-1}(U))$ (in fact, it's the final element). So we can define a natural morphism of presheaves $\eta : \mathcal{G} \to f_* f^\circ\mathcal{G}$ by sending a section $s \in \mathcal{G}(U)$ to the section s in $\mathcal{G}(U)$ inside the limit defining $(f_* f^\circ\mathcal{G})(U)$. Composing this with $f_*\theta : f_* f^\circ\mathcal{G} \to f_* f^{-1}\mathcal{G}$ where $\theta : f^\circ\mathcal{G} \to f^{-1}\mathcal{G}$ is the sheafification morphism (see proposition-definition II.1.2) we have that the composite $\mathcal{G} \to f_* f^\circ\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is the natural map of sheaves $\eta^+ : \mathcal{G} \to f_* f^{-1}\mathcal{G}$ that we want.

Given our natural maps $\varepsilon^+ : f^{-1} f_*\mathcal{F} \to \mathcal{F}$ and $\eta^+ : \mathcal{G} \to f_* f^{-1}\mathcal{G}$, we can define maps

$$\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \quad \text{by} \quad (f^{-1}\mathcal{G} \xrightarrow{\varphi} \mathcal{F}) \mapsto (\mathcal{G} \xrightarrow{\eta^+} f_* f^{-1}\mathcal{G} \xrightarrow{f_*\varphi} f_*\mathcal{F})$$

and

$$\mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \quad \text{by} \quad (\mathcal{G} \xrightarrow{\psi} f_*\mathcal{F}) \mapsto (f^{-1}\mathcal{G} \xrightarrow{f^{-1}\psi} f^{-1} f_*\mathcal{F} \xrightarrow{\varepsilon^+} \mathcal{F})$$

and we wish to show that these are mutually inverse. To do this, we'll factor our maps to make the computations easier. Start with $\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$: we write it as

$$\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{PrSh(X)}(f^\circ\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$$

where the first arrow is is precomposition with the sheafification map $\theta : f^\circ\mathcal{G} \to f^{-1}\mathcal{G}$, and the second arrow sends a map $\varphi : f^\circ\mathcal{G} \to \mathcal{F}$ to the composite $\mathcal{G} \stackrel{\eta^+}{\to} f_*f^\circ\mathcal{G} \stackrel{f^\circ\varphi}{\to} f_*\mathcal{F}$. Continuing with $\mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F})$, we write it as

$$\mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_{PrSh(X)}(f^\circ\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F})$$

where the first arrow sends a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$ to the composition $f^\circ\mathcal{G} \stackrel{f^\circ\varphi}{\to} f^\circ f_*\mathcal{F} \stackrel{\varepsilon}{\to} \mathcal{F}$ and the second sends a map $f^\circ\mathcal{G} \to \mathcal{F}$ to the induced map from the sheafification, $f^{-1}\mathcal{G} \to \mathcal{F}$. It is straightforwards to see that both of these factorizations are equal to our original maps between hom-sets of sheaves.

Now, in order to show that we have the requested bijection on hom-sets of sheaves, I claim it is enough to show that the maps above between $\mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$ and $\mathrm{Hom}_{PrSh(X)}(f^\circ\mathcal{G}, \mathcal{F})$ are mutually inverse. This follows from the fact that the maps between $\mathrm{Hom}_{PrSh(X)}(f^\circ\mathcal{G}, \mathcal{F})$ and $\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ are mutually inverse by proposition-definition II.1.2. Now that we have made this reduction, let us be explicit - here are the maps we want to show are equal:

$$(f^\circ\mathcal{G} \stackrel{\varphi}{\to} \mathcal{F}) \stackrel{?}{=} (f^\circ\mathcal{G} \stackrel{f^\circ\eta}{\to} f^\circ f_*f^\circ\mathcal{G} \stackrel{f^\circ f_*\varphi}{\to} f^\circ f_*\mathcal{F} \stackrel{\varepsilon}{\to} \mathcal{F})$$

and

$$(\mathcal{G} \stackrel{\psi}{\to} f_*\mathcal{F}) \stackrel{?}{=} (\mathcal{G} \stackrel{\eta}{\to} f_*f^\circ\mathcal{G} \stackrel{f_*f^\circ\psi}{\to} f_*f^\circ f_*\mathcal{F} \stackrel{f_*\varepsilon}{\to} f_*\mathcal{F}).$$

In order to unravel these definitions, we'll start by writing down what f° and f_* do to a morphism $\alpha : \mathcal{A} \to \mathcal{B}$ of (pre)sheaves. When considering an element s of the sections of $(f^\circ\mathcal{A})(U)$, we will sometimes pick a representative as (s_1, V_1) where $s_1 \in \mathcal{A}(V_1)$, and we may rest assured that this choice won't matter because of the definition of the direct limit involved in defining $f^\circ\mathcal{A}$ as being given by restriction maps. If the morphism $\alpha : \mathcal{A} \to \mathcal{B}$ is given by $\alpha(U) : \mathcal{A}(U) \to \mathcal{B}(U)$ for an open set U, then the morphism $f^\circ\alpha : f^\circ\mathcal{A} \to f^\circ\mathcal{B}$ is given on an open set U as $(f^\circ\alpha)(U) : \lim_{V\supset f(U)} \mathcal{A}(V) \to \lim_{V\supset f(U)} \mathcal{B}(V)$ by sending a representative $(s_1, V_1) \mapsto (\alpha(V_1)(s_1), V_1)$. On the other hand, if we apply f_* to our morphism $\alpha : \mathcal{A} \to \mathcal{B}$, then the morphism $(f_*\alpha)(U) : \mathcal{A}(f^{-1}(U)) \to \mathcal{B}(f^{-1}(U))$ is given by $\alpha(f^{-1}(U))$.

Now let's analyze $f^\circ\mathcal{G} \stackrel{f^\circ\eta}{\to} f^\circ f_*f^\circ\mathcal{G} \stackrel{f^\circ f_*\varphi}{\to} f^\circ f_*\mathcal{F} \stackrel{\varepsilon}{\to} \mathcal{F}$. The map $\eta : \mathcal{G} \to f_*f^\circ\mathcal{G}$ is given on sections over U as $s \in \mathcal{G}(U) \mapsto (s, U) \in \lim_{V\supset f(f^{-1}(U))} \mathcal{G}(V)$, so the map $f^\circ\eta : f^\circ\mathcal{G} \to f^\circ f_*f^\circ\mathcal{G}$ is given on sections over U as

$$\lim_{V\supset f(U)} \mathcal{G}(V) \to \lim_{V\supset f(U)} \left(\lim_{W\supset f(f^{-1}(U))} \mathcal{G}(W) \right)$$

$$(s, V_1) \mapsto ((s, V_1), V_1).$$

The map $f^\circ f_*\varphi : f^\circ f_*f^\circ\mathcal{G} \to f^\circ f_*\mathcal{F}$ is given on sections over U as

$$\lim_{V\supset f(U)} \left(\lim_{W\supset f(f^{-1}(U))} \mathcal{G}(W) \right) \to \lim_{V\supset f(U)} \mathcal{F}(f^{-1}(V))$$

$$((s, W_1), V_1) \mapsto ((\varphi(f^{-1}(V_1))((s, W_1))), f^{-1}(V_1)).$$

The map $f^\circ f_*\mathcal{F} \to \mathcal{F}$ is given on sections over U as

$$\lim_{V\supset f(U)} \mathcal{F}(f^{-1}(U)) \to \mathcal{F}(U)$$

$$(s, f^{-1}(V_1)) \mapsto res_{f^{-1}(V_1),U}(s).$$

Putting all of these maps together, we get the following map on sections over U:

$$(s, V_1) \mapsto ((s, V_1), V_1) \mapsto (\varphi(f^{-1}(V_1))((s, V_1)), f^{-1}(V_1)) \mapsto res_{f^{-1}(V_1),U}\varphi(f^{-1}(V_1))((s, V_1)),$$

and $\varphi(f^{-1}(V_1))((s, V_1)) = \varphi(U)((s, V_1))$ by the definition of a morphism of presheaves. So $f^\circ\mathcal{G} \xrightarrow{f^\circ\eta} f^\circ f_* f^\circ\mathcal{G} \xrightarrow{f^\circ f_*\varphi} f^\circ f_*\mathcal{F} \xrightarrow{\varepsilon} \mathcal{F}$ is the same on sections over U as φ, and thus they are equal as maps of presheaves.

To analyze $\mathcal{G} \xrightarrow{\eta} f_*f^\circ\mathcal{G} \xrightarrow{f_*f^\circ\psi} f_*f^\circ f_*\mathcal{F} \xrightarrow{f_*\varepsilon} f_*\mathcal{F}$, we'll do the same thing. The map $\eta : \mathcal{G} \to f_*f^\circ$ is given on sections over U as

$$\mathcal{G}(U) \to \lim_{V\supset f(f^{-1}(U))} \mathcal{G}(V)$$

$$s \mapsto (s, U).$$

The map $f_*f^\circ\psi : f_*f^\circ\mathcal{G} \to f_*f^\circ f_*\mathcal{F}$ is given on sections over U as

$$\lim_{V\supset f(f^{-1}(U))} \mathcal{G}(V) \to \lim_{V\supset f(f^{-1}(U))} \mathcal{F}(f^{-1}(V))$$

$$(s, V_1) \mapsto (\psi(V_1)(s), V_1).$$

The map $f_*\varepsilon : f_*f^\circ f_*\mathcal{F} \to f_*\mathcal{F}$ is given on sections over U as

$$\lim_{V\supset f(f^{-1}(U))} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(U))$$

$$(s, V_1) \mapsto res_{f^{-1}(V_1),f^{-1}(U)}s.$$

Putting all of these maps together, we get the following map on sections over U:

$$s \mapsto (s, U) \mapsto (\psi(U)(s), U) \mapsto res_{f^{-1}(U),f^{-1}(U)}\psi(U)(s),$$

which is just $s \mapsto \psi(U)(s)$ as $res_{f^{-1}(U),f^{-1}(U)}$ is the identity map by the definition of a presheaf. So $\mathcal{G} \xrightarrow{\psi} f_*\mathcal{F}$ and $\mathcal{G} \xrightarrow{\eta} f_*f^\circ\mathcal{G} \xrightarrow{f_*f^\circ\psi} f_*f^\circ f_*\mathcal{F} \xrightarrow{f_*\varepsilon} f_*\mathcal{F}$ are equal on sections over arbitrary open sets, so they are equal as maps of presheaves, and we are finished.

Exercise II.1.19. *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X \setminus Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

a. Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say 'consider $\mathcal{F}$ as a sheaf on X,' when we mean 'consider $i_*\mathcal{F}$.'

b. Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(U)$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subset U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!\mathcal{F})_P$ is equal to $\mathcal{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_!\mathcal{F}$ the sheaf obtained by *extending $\mathcal{F}$ by zero* outside U.

c. Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

Solution. First we restate the gluing and locality conditions in an easier-to-use form. For a sheaf of abelian groups $\mathcal{F}$ on a space X, an open subset U with open cover $\{U_i\}$, we can state the sheaf condition as the requirement that

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, where the first map is restriction and the second map is the difference of the restrictions $s_i \mapsto s_i|_{U_i \cap U_j}$ and $s_i \mapsto s_i|_{U_j \cap U_i}$. (Equivalently, we can think of $\mathcal{F}(U)$ being the equalizer of the diagram $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$ where the two arrows are the two ways to restrict. This is useful if you're using a more category-theoretic perspective.)

a. In order to calculate the stalk at a point, we can first restrict to a nice open neighborhood of the point and calculate from there. For instance, if $P \notin Z$, then Z^c is an open subset of X where for all open subsets $A \subset Z^c$, $(i_*\mathcal{F})(A) = \mathcal{F}(i^{-1}(A)) = \mathcal{F}(\emptyset) = 0$, and thus $(i_*\mathcal{F})_P = 0$ for $P \notin Z$.

   The observation that $(i_*\mathcal{F})(A) = 0$ if $A \cap Z = \emptyset$ implies that if $A \supset B$ are two open subsets of X so that $A \cap Z = B \cap Z$, then $(i_*\mathcal{F})(A) = (i_*\mathcal{F})(B)$: write $A = B \cup (A \cap Z^c)$, and consider the exact sequence from above. Then $0 \to (i_*\mathcal{F})(A) \to (i_*\mathcal{F})(B) \to (i_*\mathcal{F})(B \cap Z^c) = 0$ is exact, or $(i_*\mathcal{F})(A) \cong (i_*\mathcal{F})(B)$. This means that if A, A' are two open subsets of X so that $A \cap Z = A' \cap Z$, then $(i_*\mathcal{F})(A) = (i_*\mathcal{F})(A \cap A') = (i_*\mathcal{F})(A')$, so only $A \cap Z$ matters for computing sections of $i_*\mathcal{F}$.

   This plus the fact that Z has the induced topology means that the diagram computing $(i_*\mathcal{F})_P = \lim_{p \in A}(i_*\mathcal{F})(A)$ is the same as the diagram computing $\mathcal{F}_P = \lim_{p \in B} \mathcal{F}(B)$, up to adding/collapsing terms with the same value and the same maps in and out of them. Thus $(i_*\mathcal{F})_P = \mathcal{F}_P$.

b. The computation of stalks is clear from the procedure of restricting to an appropriate open subset as in the previous part of the question. Uniqueness is clear from looking on stalks: any other sheaf with this property admits a map from $j_!\mathcal{F}$ which is an isomorphism on stalks and thus an isomorphism.

c. The sequence on stalks is either $0 \to 0 \to \mathcal{F}_P \to \mathcal{F}_P \to 0$ or $0 \to \mathcal{F}_P \to \mathcal{F}_P \to 0 \to 0$ depending on whether $P \in U$ or $P \in Z$, respectively, so exactness is clear.

Exercise II.1.20. *Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X,\mathcal{F})$ to be the subgroup of $\Gamma(X,\mathcal{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.

a. Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathscr{H}^0_Z(\mathcal{F})$.

b. Let $U = X \setminus Z$, and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X
$$0 \to \mathscr{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$
Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

Solution.

a. This will follow immediately from the fact that $\mathcal{F}$ is a sheaf. Suppose U is an open subset and $\{U_i\}$ an open covering. Let $s \in \Gamma_{Z\cap U}(U, \mathcal{F}|_U)$ be a section such that $s|_{U_i} = 0$ in $\Gamma_{Z\cap U_i}(U_i, \mathcal{F}|_{U_i})$. Then s as a section of $\mathcal{F}|_U$ restricts to zero on all the U_i, so since $\mathcal{F}|_U$ is a sheaf of abelian groups, this means that $s = 0$ as a section of $\mathcal{F}|_U(U)$ and thus $s = 0$ in $\Gamma_{Z\cap U}(U, \mathcal{F}|_U)$.

   If, on the other hand, we have $s_i \in \Gamma_{Z\cap U_i}(U_i, \mathcal{F}|_{U_i})$ so that $s_i|_{U_i\cap U_j} = s_j|_{U_i \cap U_j}$, then viewing s_i and s_j as sections of $\mathcal{F}|_U$ over U_i and U_j, we see that we can glue them as sections of $\mathcal{F}|_U$ and this gluing preserves the fact that their support is in Z, so we can glue them as sections of the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$. Thus our presheaf is a sheaf.

b. $\mathscr{H}^0_Z(\mathcal{F}) \to \mathcal{F}$ is injective on sections over any open $V \subset X$ because $\Gamma_{Z\cap V}(V, \mathcal{F}|_V) \subset \Gamma(V, \mathcal{F}|_V)$, and therefore we have that our map is injective on stalks and an injective map of sheaves. This proves exactness at $\mathscr{H}^0_Z(\mathcal{F})$. To prove exactness at $\mathcal{F}$, note that the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is restriction to U, and so the kernel of this map is exactly the sections supported on $U^c = Z$.

   If $\mathcal{F}$ is flasque, then $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective on sections, which implies surjectivity as a map of sheaves, and we're done.

Exercise II.1.21. *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. 1. Let $\mathcal{O}_X$ be the sheaf of regular functions on X (1.0.1).

a. Let Y be a closed subset of X. For each open set $U \subset X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

b. If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*\mathcal{O}_Y$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

c. Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathcal{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

d. Again let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$ where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

e. Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the 'first Cousin problem' in several complex variables. See Gunning and Rossi [1, p. 248].)

Solution.

a. This is the kernel of the morphism of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_Y$ where $i : Y \to X$ is the inclusion. Thus it is a sheaf, as kernels of sheaves are sheaves. (Alternatively, you can verify this directly.)

b. By (a) and the first isomorphism theorem we have that $\mathcal{O}_X/\mathcal{I}_Y \to i_*\mathcal{O}_Y$ is injective, so all we need to do is to show surjectivity. Hartshorne defines a regular function on a quasi-projective variety to be locally representable as a quotient of homogeneous polynomials in $k[x_0, \cdots, x_n]$, which immediately implies surjectivity.

c. First, we note that for any projective variety Z, we have that $\Gamma(Z, \mathcal{O}_Z) = \mathcal{O}(Z)$, where the right-hand side is as defined in Chapter I, section 3. By theorem I.3.4(a), $\mathcal{O}(Z) = k$ for any projective variety, so we have that $\Gamma(X, \mathcal{O}_X) = k$, $\Gamma(X, i_*\mathcal{O}_P) = \Gamma(P, \mathcal{O}_P) = k$ and similarly for Q. Thus the final map in our exact sequence is $k \to k^2$ by $1 \mapsto (1, 1)$, which is not surjective.

d. For any open subset U, we have an obvious injection $\mathcal{O}_X(U) \to \mathcal{K}(U) = K(X)$ given by sending a regular function $f \in \mathcal{O}_X(U)$ to the class $\langle U, f \rangle \in K(X)$ (see the definition of $K(X)$ on page 16 in I.3 if you're rusty on this), and this commutes with restriction maps by the definition of the function field. Thus we have a morphism of sheaves, and further, this map is injective because it's injective on every open.

To form the desired exact sequence, note that we have a map $\mathcal{K} \to \prod_{P \in X} i_P(I_P)$ given by sending an element of the function field to its value in I_P for each P. To show that this actually lands in the direct sum, represent an element $h \in \mathcal{K}$ as a ratio of two homogeneous polynomials $h = f(x_0, x_1)/g(x_0, x_1)$ with $g \neq 0$. Whenever $g(P) \neq 0$, we see that $h \in \mathcal{O}_{X,P}$, so h can only define a nonzero element of $i_P(I_P)$ if $g(P) = 0$. But g vanishes at finitely many points of $\mathbb{P}^1$, so h defines a nonzero element in finitely many $i_P(I_P)$ and thus our map lands in the direct sum as requested. Now, since $(\sum_{P\in X} i_P(I_P))_Q = i_Q(I_Q)$ because stalks commute with direct sums (colimits commute with colimits), we can check on stalks and see that

$$0 \to \mathcal{O}_{X,Q} \to \mathcal{K}_Q \to \left(\prod_{P\in X} i_P(I_P)\right)_Q = i_Q(I_Q) \to 0$$

is exact for any $Q \in \mathbb{P}^1$ by definition of I_Q.

(Caution: it's important to check that this lands in the direct sum before taking stalks, because while the stalk of a direct sum is the direct sum of the stalks as colimits commute with colimits, the stalk of a direct product may not be the direct product of the stalks since colimits do not commute with arbitrary limits. For instance, if we consider $\mathbb{R}$ with the standard topology, let $\mathcal{F}$ be the sheaf of continuous functions to $\mathbb{R}$, and consider $f \in \prod_{n=1}^{\infty} \mathcal{F}$ to have n^{th} component a bump function centered at $\frac{1}{n}$ vanishing outside of $(\frac{1}{n} - 2^{-n}, \frac{1}{n} + 2^{-n})$, then the stalk at 0 of each component in f is zero, but $f \neq 0$ in the stalk at 0 since there is no neighborhood of 0 where $f = 0$.)

e. Since the sections functor is left-exact, it remains to check that $\mathcal{G}(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O})$ is surjective. It suffices to show that for any point P and any element $f \in i_P(I_P)$, we can find a $f' \in K(X)$ which has image f in the map $K(X) \to i_P(I_P)$ and 0 in $i_Q(I_Q)$ when $Q \neq P$, as the collection of such elements as P varies form a generating set for the global sections $\Gamma(X, \mathcal{K}/\mathcal{O}) \cong \Gamma(X, \sum_{P\in X} i_P(I_P)) \cong \sum_{P\in X} I_P$.

   All we need to do to show this is to describe elements of $I_P = K(X)/\mathcal{O}_P$. Starting with an element $f = g/h \in K(X)$ with g, h homogeneous of equal degrees and coprime, we can apply a linear automorphism of $\mathbb{P}^1$ so that $P = 0$ and neither g nor h vanish at ∞. This implies we can dehomogenize and write $g = \prod(x - a_i)^{c_i}$ and $h = \prod(x - b_i)^{d_i}$ up to a constant factor. Now by partial fraction decomposition (aka Bezout's Identity), we can write $\frac{g}{h} = \sum \frac{p_i}{(x-b_i)^{d_i}}$, and since $\frac{p_i}{(x-b_i)^{d_i}} \in \mathcal{O}_P$ if $b_i \neq 0$, we have that $f = \frac{p}{x^d}$ in $\mathcal{K}/\mathcal{O}_P$. Thus $\frac{p}{x^d} \in K(X)$ defines an element which agrees with $f \in I_P$ and maps to zero in every other I_Q, and we're done.

Exercise II.1.22. *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = id$, and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphism $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by *glueing* the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.

Solution. Let us start by defining the sections of $\mathcal{F}$ over an arbitrary open set $W \subset X$. Define

$\mathcal{F}(W)$ to be the equalizer of the diagram

$$\prod_i \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod_{i \neq j} \mathcal{F}_i(W \cap U_i \cap U_j),$$

which is equivalent to

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

(See the solution to II.1.19 for discussion about equalizers and sections of sheaves.) We define the restriction mappings for an inclusion $W' \subset W$ by sending $(s_i)_{i \in I} \mapsto (res_{W,W'} s_i)_{i \in I}$, and it is immediate to see that this data defines a sheaf (the sheaf condition for $\mathcal{F}$ is implied by the sheaf condition for each $\mathcal{F}_i$).

To prove that $\mathcal{F}|_{U_i}$ maps isomorphically on toe $\mathcal{F}_i$, let $W \subset U_i$. In this case, the condition defining $\mathcal{F}(W)$ implies that a section $s \in \mathcal{F}(W)$ can be represented by (s_i) on $W \cap U_i$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. So there is a natural bijection between $\mathcal{F}(W)$ and $\mathcal{F}_i(W)$ which gives the required isomorphism of sheaves. Uniqueness follows easily from the universal property of equalizers.

II.2 Schemes

Here we start collecting some basic, fundamental knowledge of how schemes work.

Exercise II.2.1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subset X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

Solution. By proposition II.2.3, we have a morphism of ringed spaces $\operatorname{Spec} A_f \to \operatorname{Spec} A$ induced by the map $A \to A_f$ given by localization. As prime ideals of A_f are exactly the prime ideals of A not containing f, we see that this morphism is a bijection on to $D(f) \subset \operatorname{Spec} A$, and the image of any open set $D(f') \subset \operatorname{Spec} A_f$ is exactly $D(ff') \subset D(f) \subset \operatorname{Spec} A$, so this map is a homeomorphism. By the construction of part (b) of proposition II.2.3, we see that the induced maps on stalks are the identity, which gives us that the structure sheaves of $\operatorname{Spec} A_f$ and $D(f)$ are isomorphic, and thus $D(f) \cong \operatorname{Spec} A_f$ as locally ringed spaces.

Exercise II.2.2. Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subset X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

Solution. Clearly $(U, \mathcal{O}_X|_U)$ is a locally ringed space, so it suffices to prove that every point of U has an affine open neighborhood. Let $x \in U \subset X$ be a point, and let $\operatorname{Spec} A$ be an open affine neighborhood of $x \in X$. Then $U \cap \operatorname{Spec} A$ is an open neighborhood of x in the affine scheme $\operatorname{Spec} A$, which means that it's the complement of some $V(I)$ for an ideal $I \subset A$. Pick some $f \in I$ which does not vanish at x, which we can do because $x \notin V(I)$. Then $D(f)$ is an open neighborhood of x in $\operatorname{Spec} A$, and by exercise II.2.1 above, we have that $D(f) \cong \operatorname{Spec} A_f$ is an open affine neighborhood of x in U.

Exercise II.2.3. *Reduced Schemes.* A scheme $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

a. Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

b. Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{red}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$, where for any ring A, we deonte by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme. We call it the *reduced scheme* associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism on the underlying topological spaces.

c. Let $f : X \to Y$ be a morphism of schemes, and assume X is reduced. Show that there is a unique morphism $g : X \to Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \to Y$.

Solution.

a. If X is reduced, then for any $s \in \mathcal{O}_{X,P}$ which satisfies $s^n = 0$, we can lift s to $s_U \in \mathcal{O}_X(U)$ where U is an open containing P. But $\mathcal{O}_X(U)$ is reduced, so $s_U = 0$ and $s = 0$. Conversely, if all the rings $\mathcal{O}_{X,P}$ are reduced, then the image of any nilpotent section $s \in \mathcal{O}_X(U)$ in $\mathcal{O}_{X,P}$ is zero for $P \in U$. But this means that $s = 0$ in a neighborhood of P, so $s = 0$ by the sheaf condition.

b. $(X, (\mathcal{O}_X)_{red})$ is clearly a ringed space. Further, it is a locally ringed space: since sheafification preserves stalks, we have that the stalks of $(\mathcal{O}_X)_{red}$ are quotients of the stalks of $\mathcal{O}_X$, and thus they are local rings. To prove that $(X, (\mathcal{O}_X)_{red})$ is a scheme, it suffices to prove that $(\operatorname{Spec} A, (\mathcal{O}_{\operatorname{Spec} A})_{red})$ is an affine scheme. I claim that it is exactly $\operatorname{Spec} A_{red}$.

   By the correspondence theorem, we have a bijection between the prime ideals of A and A_{red}, as any nilpotent element is in every prime ideal. This gives us a bijection between the underlying topological spaces of $\operatorname{Spec} A$ and $\operatorname{Spec} A_{red}$. By the characterization of open sets as unions of $D(f)$ for some $f \in A$, we see that this bijection is in fact a homeomorphism: a prime ideal $\mathfrak{p} \subset A$ is in $D(f)$ if and only if $\mathfrak{p}_{red} \subset A_{red}$ is in $D(f_{red})$ (as $f \notin \mathfrak{p}$ is equivalent to $f + n \notin \mathfrak{p}$ for any nilpotent n), so our bijection interchanges $D(f)$ and $D(f_{red})$.

   It remains to show that the sheaves $(\mathcal{O}_{\operatorname{Spec} A})_{red}$ and $\mathcal{O}_{\operatorname{Spec} A_{red}}$ are isomorphic. To do this, consider the morphism $\operatorname{Spec} A_{red} \to \operatorname{Spec} A$ induced by the map of rings $A \to A_{red}$. We'll show that this factors as $\operatorname{Spec} A_{red} \to (\operatorname{Spec} A)_{red} \to \operatorname{Spec} A$, and that the first map is an isomorphism. The bijection constructed above along with the definition of $(\operatorname{Spec} A)_{red}$ shows that the map on underlying topological spaces is a homeomorphism, so all that remains is to show that the induced map $(\mathcal{O}_{\operatorname{Spec} A})_{red} \to \mathcal{O}_{\operatorname{Spec} A_{red}}$ is an isomorphism of sheaves.

   To do this, we look on stalks. We can calculate these stalks as the limit over the system of affine opens $D(f) = D(f_{red})$ containing a point $\mathfrak{p}$, since they form a basis for the topology on the (homeomorphic) underlying topological spaces of $\operatorname{Spec} A$ $\operatorname{Spec} A_{red}$. To calculate the stalk of $(\mathcal{O}_{\operatorname{Spec} A})_{red}$ at a point $\mathfrak{p}$, we calculate the stalk of the presheaf $U \mapsto (\mathcal{O}_{\operatorname{Spec} A})_{red}$ at $\mathfrak{p}$, and this gives us $\lim_{D(f) \ni \mathfrak{p}} (A_f)_{red}$. As $(A_f)_{red} \cong (A_{red})_{f_{red}}$ because localization is exact, the induced map on stalks by $(\mathcal{O}_{\operatorname{Spec} A})_{red} \to \mathcal{O}_{\operatorname{Spec} A_{red}}$ given by $\lim_{D(f) \ni \mathfrak{p}} (A_{\mathfrak{p}})_{red} \to \lim_{D(f_{red}) \ni \mathfrak{p}_{red}} (A_{red})_{f_{red}}$ is an isomorphism, and we're done.

c. Since Y_{red} has the same underlying topological space as Y, all that we need to do is to analyze the map on sheaves. If we have a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$, then this factors uniquely through the presheaf $U \mapsto (\mathcal{O}_Y(U))_{red}$, as any morphism $\mathcal{O}_Y(U) \to (f_*\mathcal{O}_X)(U)$ factors uniquely through $(\mathcal{O}_Y(U))_{red}$ because X is reduced and this factorization is compatible with restriction maps. As $(\mathcal{O}_Y)^\circ_{red} \to f_*\mathcal{O}_X$ is a morphism from a presheaf to a sheaf, it factors uniquely through the sheafification, and we get a unique morphism $\mathcal{O}_{Y_{red}} \to f_*\mathcal{O}_X$, which gives us the required unique morphism $X \to Y_{red}$ so that $X \to Y$ factors as $X \to Y_{red} \to Y$.

   (Conceptually, this is just a 'globalization' of the fact that ring maps $R \to S$ with S reduced factor uniquely through R_{red} - a proof more along these lines which I think is conceptually clearer will be available once we have the tools of quasi-coherent sheaves, introduced in section 5 of this chapter.)

Exercise II.2.4. Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf (I, 3.5) for an analogous statement about varieties).

Solution. We produce a two-sided inverse for α. Given a map of rings $f : A \to \Gamma(X, \mathcal{O}_X)$, define a map $\varphi : X \to \operatorname{Spec} A$ by sending a point $x \in X$ to the prime ideal $\mathfrak{p}$ which is the inverse image of $\mathfrak{m}_x$ under the composition $A \to \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$. To see that this is continuous, note that $D(g) \subset \operatorname{Spec} A$ has preimage $D(f(g)) \subset X$, which is open. To construct the map of sheaves $\mathcal{O}_{\operatorname{Spec} A} \to \varphi_*\mathcal{O}_X$, we first show that it is enough to construct this map on opens of the form $D(g) \subset \operatorname{Spec} A$ as these form a basis for the topology on $\operatorname{Spec} A$.

Lemma. *Let $B = \{B_i\}$ be a basis for the topology on a space X. Then to specify a sheaf or a map of sheaves, it suffices to do so on each B_i.*

Proof. Recall that the sections of a sheaf $\mathcal{F}$ over an open subset $U \subset X$ are given as the equalizer of the diagram

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

for an open cover $\{U_i\}$ of U. For an arbitrary open $U \subset X$, we may take $\{U_i\}$ to be a collection of open sets of the basis B (by the definition of a basis). As every $U_i \cap U_j$ can be covered by a collection U_{ijk} of opens belonging to our basis, we get injective maps $\mathcal{F}(U_i \cap U_j) \to \prod_k \mathcal{F}(U_{ijk})$ for each i, j, and thus $\mathcal{F}(U)$ can also be computed as the equalizer of

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i \neq j} \prod_k \mathcal{F}(U_{ijk}),$$

which shows that the sections of $\mathcal{F}$ on U are completely determined by its sections over the elements of the basis. Any morphism $\mathcal{F} \to \mathcal{G}$ defined on the basis $\{U_i\}$ induces corresponding maps $\mathcal{F}(U) \to \mathcal{G}(U)$ by the universal property of equalizers, and such morphisms can easily be checked to be compatible with restriction.

■

In order to define the map $\mathcal{O}_{\operatorname{Spec} A} \to \varphi_*\mathcal{O}_X$, we first note that $f(g)$ is invertible on $D(f(g)) = \varphi^{-1}(D(g)) \subset X$. This means that the morphism $A \xrightarrow{f} \mathcal{O}_X(X) \to \mathcal{O}_X(D(f(g)))$ factors through A_g by the universal property of localization, so we have a map $A_g \to \mathcal{O}_X(D(f(g)))$, and this is the map of sheaves we'll use. This map is clearly compatible with restrictions, and thus via the preceeding lemma, defines a map of sheaves $\mathcal{O}_{\operatorname{Spec} A} \to \varphi_*\mathcal{O}_X$ which gives the map $f : A \to \Gamma(\mathcal{O}_X, X)$ on global sections. Finally, by noticing that the map $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \to \mathcal{O}_{X,x}$ is local by construction we have shown that this is indeed a morphism of locally ringed spaces and thus a morphism of schemes. So we've shown that given a morphism of rings, we can produce a morphism of schemes which upon taking global sections gives us back our morphism of rings.

To complete the proof, we need to show the composition the other way is the identity: if we start with a morphism of schemes $X \to \operatorname{Spec} A$, obtain a map of rings $A \to \mathcal{O}_X(X)$, and then construct a morphism of schemes $X \to \operatorname{Spec} A$, we get back what we started with. For any $x \in X$, we have the following commutative diagram of rings induced by the morphism $f : X \to \operatorname{Spec} A$:

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_{f(x)} & \longrightarrow & \mathcal{O}_{X,x} \end{array}$$

and it is clear that the procedure for recovering $X \to \operatorname{Spec} A$ from $A \to \mathcal{O}_X(X)$ is exactly inverse to the creation of this diagram, which shows that we get the same map of underlying topological spaces back. By looking at the stalks, we also see that we get the same morphism of sheaves back, as if two morphisms of sheaves on a topological space are different, they must induce different maps on some stalk. Thus the composition the other way is the identity and we're finished.

(One may note that this fact is true even if X is just a locally ringed space and not a scheme - in fact, the original attribution of this result is to Tate, and may be found in EGAII Err_1, Proposition 1.8.1. A version of this proof is reproduced at Stacks 01I1, as well.)

Exercise II.2.5. Describe $\operatorname{Spec} \mathbb{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to $\operatorname{Spec} \mathbb{Z}$.

Solution. The prime ideals of $\mathbb{Z}$ are (0) and (p) for p a prime. (0) is open, while (p) is closed, and so $\operatorname{Spec} \mathbb{Z}$ is a one-dimensional topological space and every nonzero prime ideal is maximal - in this sense, it looks kind of like a curve over a field. The closed sets are $V(n)$ for integers n, and these consist of all the primes p dividing n. Any open subset can be given as $D(n)$ for an integer n, and the value of the structure sheaf on this is exactly $\mathbb{Z}[\frac{1}{n}]$.

By exercise II.2.4, we have that morphism $X \to \operatorname{Spec} \mathbb{Z}$ are in bijection with ring homomorphisms $\mathbb{Z} \to \Gamma(\mathcal{O}_X, X)$. But as $\mathbb{Z}$ is an initial object in the category of rings, there is only one such morphism $\mathbb{Z} \to \Gamma(\mathcal{O}_X, X)$ and so we get a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$.

Exercise II.2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Solution. The zero ring has no proper ideals, so it's spectrum is empty. The unique map $\operatorname{Spec} 0 \to X$ is the trivial map on the underlying topological space and the zero map on sheaves. (One should note that for sheaf $\mathcal{F}$ taking values in a category $\mathcal{C}$ with a final object, the value of $\mathcal{F}$ on the empty set is the final object - in our case, the final object in commutative rings is the zero ring.)

Exercise II.2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Solution. $\operatorname{Spec} K$ is a one-point topological space with structure sheaf taking the value K on the only nonempty set. Given a morphism $\operatorname{Spec} K \to X$, we get the data of a point $x \in X$ via the image of $(0) \in \operatorname{Spec} K$, and a local map of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{K,(0)}$. But $\mathcal{O}_{K,(0)} = K$, and the condition that this is a local map means $\mathfrak{m}_x$ must be sent to 0, so our map factors through $k(x) \to K$ and as all nonzero morphisms of fields are injective, we have an injection $k(x) \hookrightarrow K$. Thus given a map $\operatorname{Spec} K \to X$ we have a point $x \in X$ and an inclusion $k(x) \hookrightarrow K$.

Conversely, given a point x and an inclusion $k(x) \hookrightarrow K$, we define a morphism $f : \operatorname{Spec} K \to X$ as follows. Send $(0) \in \operatorname{Spec} K$ to $x \in X$, which is clearly continuous. Given an open subset $U \subset X$, if U does not contain x, then $(f_*\mathcal{O}_{\operatorname{Spec} K})(U) = 0$ since $f^{-1}(U) = \emptyset$, so we may define $\mathcal{O}_X(U) \to (f_*\mathcal{O}_{\operatorname{Spec} K})(U)$ as the zero map. On the other hand, if U contains x, we can define our map $\mathcal{O}_X(U) \to (f_*\mathcal{O}_{\operatorname{Spec} K})(U)$ by noting that $(f_*\mathcal{O}_{\operatorname{Spec} K})(U) = K$, and thus we can use the composite map

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \hookrightarrow K$$

where we use the natural map from an open to a stalk and the inclusion $k(x) \hookrightarrow K$. This is clearly compatible with restrictions by the definition of $\mathcal{O}_{X,x}$, so we've defined a morphism of schemes $f : \operatorname{Spec} K \to X$.

Exercise II.2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

Solution. One should point out that the definition of rational point here ought to be clarified a little bit: the text $k(x) = k$ is supposed to mean that $k(x) \cong k$ as k-algebras where the k-algebra structure on $k(x)$ is induced from the fact that X is a k-scheme, that is, we have a morphism $X \to \operatorname{Spec} k$. Sometimes this language can be a sticking point for people on their first time through, and this isn't helped by the fact that the resolution of the confusion is a fact that 'everybody knows' but sometimes isn't written down well. This clarification is my attempt to combat the issue.

Back to the problem. We note that $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is a one-point topological space (the only prime ideal is (ε)), so a k-morphism $\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$ is given by a choice of a point $x \in X$ and a local morphism of k-algebras $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$. As the morphism $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$ is local, we see that it gives a morphism $\mathcal{O}_{X,x}/\mathfrak{m}_x \to k[\varepsilon]/\varepsilon = k$, or an inclusion $k(x) \hookrightarrow k$. By the assumption that X is a scheme over k, $k(x)$ is an extension of k, and so by dimension reasons we see that $k(x) = k$ and thus x is a rational point. To obtain an element of T_x, note that by the fact our morphism $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$ is local, we have an induced k-linear map $\mathfrak{m}_x \to (\varepsilon)$, and thus an induced k-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to (\varepsilon)/(\varepsilon)^2 = k\langle\varepsilon\rangle$ as $\mathfrak{m}_x^2$ must land in $(\varepsilon)^2 = 0$. This gives us a k-linear map $(k\langle\varepsilon\rangle)^* \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = T_x$, and we take the image of the map sending $c\varepsilon \mapsto c$ as our element of T_x.

Conversely, given a k-rational point $x \in X$ and an element $t \in T_x$, we define a morphism on underlying topological spaces $f : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$ by sending $(\varepsilon) \mapsto x$. Just as in II.2.7, we define the map $\mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2}$ as the zero map on every open $U \subset X$ not containing x. If $U \subset X$ contains x, then we define a map

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2 = \mathcal{O}_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2}(f^{-1}(U))$$

by the canonical map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$, and the map $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$ is given by sending $1 \mapsto 1$, and for an element $m \in \mathfrak{m}_x$, $m \mapsto t(\overline{m})$, where $\overline{m}$ denotes the class of m in $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Exercise II.2.9. If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \overline{\{\zeta\}}$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Solution. First we prove the statement for X affine. I claim that if $X = \operatorname{Spec} A$ is affine and $S \subset X$ is an irreducible closed subset, then its underlying topological space may be given as $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. We start by noticing that via lemma II.2.1, if some set can be expressed as V of two different ideals I, J then $\sqrt{I} = \sqrt{J}$. In particular, if I, J are radical, then $I = J$. Writing $S = V(I)$ and $S = V(A) \cup V(B)$ for I, A, B radical and applying the condition that if $S = V(A) \cup V(B)$, we must have either $S = V(A)$ or $S = V(B)$, we see that this is equivalent to $I = \sqrt{AB} = A \cap B$ implies $I = A$ or $I = B$, which is exactly the characterization that I is prime (see Atiyah-MacDonald 1.11.ii if you need a refresher). This implies that $S \subset \operatorname{Spec} A$ an irreducible closed is given as $V(\mathfrak{p}) = \overline{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$, and if $\mathfrak{q}$ is any other radical ideal so that $V(\mathfrak{q}) = S$, we have $\mathfrak{p} = \mathfrak{q}$ by lemma II.2.1. So $\mathfrak{p}$ is the unique generic point of S.

Now suppose that X is arbitrary. Cover X by open affines $\{U_i\}_{i \in I}$. Suppose $i, j \in I$ so that $U_i \cap S$ and $U_j \cap S$ are both nonempty. I claim that the generic point of $U_i \cap S$ also lies in $U_j \cap S$: if not, then $S \cap U_i \cap U_j^c$ is a proper closed subset of $S \cap U_i$ which contains the generic point of $U_i \cap S$, which implies that the closure of the generic point of $U_i \cap S$ is not $U_i \cap S$, a contradiction. Thus any generic point of S lies in every open affine subset (in fact, every open subset), and by uniqueness of generic points for irreducible closed subsets of affine schemes, we have the result.

Exercise II.2.10. Describe $\operatorname{Spec} \mathbb{R}[x]$. How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?

Solution. As $\mathbb{R}[x]$ is a PID and the prime elements in $\mathbb{R}[x]$ are polynomials of the form $x - a$ for $a \in \mathbb{R}$ and $(x - z)(x - \overline{z})$ for $z \in \mathbb{C}$, the prime ideals of $\mathbb{R}[x]$ are exactly (0), $(x - a)$ for $a \in \mathbb{R}$ and $((x - z)(x - \overline{z}))$ for $z \in \mathbb{C}$. All of these are maximal ideals except (0), so they define closed points. The closed points of $\operatorname{Spec} \mathbb{R}[x]$ are in bijection with the orbits of $\mathbb{C}$ under the Galois action by complex conjugation - in our case, this looks like folding $\mathbb{C}$ over the real axis. (This perspective of viewing the closed points of $\mathbb{A}^n_k$ as the closed points of $\mathbb{A}^n_{\overline{k}}$ modulo the Galois action generalizes, and it's also fairly informative in my opinion.)

Exercise II.2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Solution. As $k[x]$ is a PID for any field k, the points are (0), which is open, and (f) for f an irreducible polynomial in $k[x]$, and these points are closed. The residue field at (0) is $k(x)$, and the residue field at (f) is $k[x]/(f) \cong \mathbb{F}_q$ for $q = p^{\deg f}$. In order to count the number of points with residue field $\mathbb{F}_{p^n}$, we'll recall some basic Galois theory. All extensions of finite fields are of the form $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ for q a power of a prime. Every such extension is Galois with Galois group $\mathbb{Z}/n\mathbb{Z}$, and the Galois group is generated by the Frobenius $x \mapsto x^q$. Any irreducible polynomial in $\mathbb{F}_q[x]$ of degree n splits completely over $\mathbb{F}_{q^n}$ and not over any smaller field, and the Galois group of $\mathbb{F}_{q^n}/\mathbb{F}_q$

is transitive on these roots. We therefore have a bijection between Galois orbits of size n and prime ideals (f) for $f \in k[x]$ irreducible of degree n, which are in bijection with closed points of residue field $\mathbb{F}_{p^n}$. We further note that any element of $\mathbb{F}_{p^n}$ which isn't in any subextension gives a Galois orbit of size n under the action of the Galois group.

Now all we have to do is count: p^n, the number of elements in $\mathbb{F}_{p^n}$, can be written as the sum over all subextensions $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$ of the number of elements which lie in $\mathbb{F}_{p^d}$ and not any smaller subextension. By Mobius inversion, this gives us that the number of elements in $\mathbb{F}_{p^n}$ not in any smaller subextension is

$$\sum_{d|n} \mu(d) p^{n/d},$$

and after dividing this by n for the action of the Galois group, we see that the number of points with residue field $\mathbb{F}_{p^n}$ is exactly

$$\frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}.$$

Exercise II.2.12. *Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subset X_j$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi : U_{ij} \to U_{ji}$ such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j$ on U_{ij}. We say that X is obtained by *glueing* the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod X_i$.

Solution. Define the topological space X to be the disjoint union of the X_i modulo the relation that $x_i \in X_i \sim x_j \in X_j$ if they're both the image of the same point $u \in U_{ij} \cong U_{ji}$ under the maps $U_{ij} \to X_i$, $U_{ij} \to X_j$. This is an equivalence relation (note the use of the cocycle condition for transitivity), and the open subsets of X are exactly the unions of open subsets of the X_i - in particular, the image of each X_i is open in X, and the obvious map $X_i \to X$ is a homeomorphism on to its image.

This implies that $\{X_i\}$ form an open cover of X, and for each i, we may consider the structure sheaf $\mathcal{O}_{X_i}$ as a sheaf on X_i, and we note that the compatibility conditions are exactly what we need to apply exercise II.1.22 on gluing sheaves. Let $\mathcal{O}_X$ denote the sheaf we get on X from applying II.1.22 to the sheaves $\mathcal{O}_{X_i}$ on each X_i.

As any point in each X_i has an open neighborhood contained entirely inside X_i isomorphic as a locally ringed space to some affine scheme, we can take the same neighborhood to show that $(X, \mathcal{O}_X)$ is also a scheme (here we use the facts that $X_i \to X$ is a homeomorphism on to its image and $\mathcal{O}_X|_{X_i} \cong \mathcal{O}_{X_i}$).

Exercise II.2.13. A topological space is *quasi-compact* if every open cover has a finite subcover.

a. Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.

b. If X is an affine scheme show that $\mathrm{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is *quasi-compact* if $\mathrm{sp}(X)$ is.

c. If A is a noetherian ring, show that $\mathrm{sp}(\operatorname{Spec} A)$ is a noetherian topological space.

d. Give an example to show that $\mathrm{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

Solution.

a. The results of exercise I.1.7(c) and I.1.7(b) imply that any open subset of a noetherian topological space (equipped with the induced topology) is quasi-compact. To go the other direction, suppose we have an ascending chain of open subsets $U_1 \subset U_2 \subset \cdots$ of X. Then $\bigcup U_i$ is open and covered by the U_i, and by the assumption that all opens are quasi-compact, we may find a finite subcover. But this means that $\bigcup U_i = U_n$ for some n, and thus our chain stablizies and X is noetherian.

b. We can refine any open cover of $\operatorname{Spec} R$ to a cover by principal affine open sets $D(f_\alpha)$ as α ranges over some index set A. The statement that $\bigcup_{\alpha\in A} D(f_\alpha) = \operatorname{Spec} R$ implies that $\bigcap_{\alpha\in A} V(f_\alpha) = \emptyset$, or that $\sum_{\alpha\in A}(f_\alpha) = (1)$. Since we define the sum of ideals by taking finite R-linear sums of the generators, this implies that there's a finite subset $A' \subset A$ so that $\sum_{\alpha\in A'} r_\alpha f_\alpha = 1$. Thus $\bigcap_{\alpha\in A'} V(f_\alpha) = \emptyset$, or $\bigcup_{\alpha\in A'} D(f_\alpha) = \operatorname{Spec} R$, and we've found a finite subcover of our original open cover.

   To show that affine schemes are not in general noetherian, consider $\operatorname{Spec} k[x_1, x_2, \cdots]$, the spectrum of the polynomial ring in infinitely many variables. Then the sequence of closed subsets $V(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, \chi_3) \supset \cdots$ is an infinite descending chain of closed subsets which does not stablize. Thus $\operatorname{Spec} k[x_1, x_2, \cdots]$ is not noetherian.

c. If $V(I_1) \supset V(I_2) \supset \cdots$ is a descending chain of closed subsets, then $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals. As the chain of ideals stablizies, the chain of closed subsets must stabilize, so $\mathrm{sp}(\operatorname{Spec} R)$ is noetherian.

d. Consider $R = k[x_1, x_2, \cdots]/(x_1, x_2, \cdots)^2$. This has a unique prime ideal $(x_1, x_2, \cdots)$: it's the nilradical of R, which is contained in every prime ideal, but $A/(x_1, x_2, \cdots) \cong k$, so by the correspondence theorem there are no prime ideals properly containing $(x_1, x_2, \cdots)$. Thus $\operatorname{Spec} R$ is a single point, so $\operatorname{Spec} R$ is a noetherian topological space. On the other hand, R is not Noetherian since the ideal $(x_1, x_2, \cdots)$ is not finitely generated.

Exercise II.2.14.

a. Let S be a graded ring. Show that $\operatorname{Proj} S = \emptyset$ if and only if every element of S_+ is nilpotent.

b. Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that φ determines a natural morphism $U \to \operatorname{Proj} S$.

c. The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

d. Let V be a projective variety with homogeneous coordinate ring S (I, §2). Show that $t(V) \cong \operatorname{Proj} S$.

Solution.

a. If every element of S_+ is nilpotent, then every prime ideal of S contains S_+, so $\operatorname{Proj} S$ is empty by definition. Conversely, if $\operatorname{Proj} S$ is empty, then for every homogeneous f of positive degree we have $\operatorname{Spec} S_{(f)}$ is empty, so $S_{(f)}$ must be the zero ring. This implies that f is nilpotent for every f homogeneous of positive degree, and as any sum of nilpotent elements is nilpotent, we see that every element of S_+ is nilpotent.

b. Pick a generating set $\{s_i\}$ for S_+. Then saying that $\mathfrak{p} \not\supset \varphi(S_+)$ is the same as saying that there's some s_i so that $\varphi(s_i) \notin \mathfrak{p}$, which is the condition that $\mathfrak{p}$ is in the union of the open sets $D(\varphi(s_i)) \subset \operatorname{Proj} T$. As unions of open sets are open, we see that U is open.

   To check that we have a morphism $U \to \operatorname{Proj} S$, we define it affine-locally and show that it patches together. For any s_i, we have a map of rings $S_{(s_i)} \to T_{(\varphi(s_i))}$ which induces a map of affine schemes $\operatorname{Spec} T_{(\varphi(s_i))} \to \operatorname{Spec} S_{(s_i)}$. We see that on overlaps we have maps $\operatorname{Spec} T_{(\varphi(s_i s_j))} \to \operatorname{Spec} S_{(s_i s_j)}$, and these are all compatible as maps of underlying topological spaces by the fact that localization is transitive. To see the necessary condition on sheaves, we can use the same arguments involving equalizers from exercise II.1.22 about gluing sheaves to produce a morphism of sheaves $\mathcal{O}_{\operatorname{Proj} S} \to f_* \mathcal{O}_U$, and thus we have a morphism of schemes $U \to \operatorname{Proj} S$.

c. If a homogeneous ideal $\mathfrak{p} \subset T$ doesn't contain T_+, then there's some nonzero homogeneous element t of degree $d \geq d_0$ which fails to be contained in $\mathfrak{p}$. But because φ is an isomorphism in degrees $\geq d_0$, this means there exists a unique nonzero $s \in S_d$ so that $\varphi(s) = t$, and so $\mathfrak{p}$ doesn't contain $\varphi(S_+)$. So $U = \operatorname{Proj} T$.

   To show that this is an isomorphism, we check affine-locally on open sets of the form $D(s)$ for s homogeneous of positive degree. First, we note that $D(s) = D(s^n)$ for any n, as $S_{(s)} = S_{(s^n)}$ because $\frac{g}{s^a} = \frac{s^b g}{s^{a+b}}$, and by picking appropriate b, we can make the denominator a power of n. Next, we note that by picking n large enough so that $n \deg s \geq d_0$, we see that $S_{(s^n)} \cong T_{(\varphi(s^n))}$ because elements of $S_{(s^n)}$ are fractions $\frac{g}{s^{nd}}$ where $g \in S_{nd \deg s}$, elements of $T_{(\varphi(s^n))}$ are fractions $\frac{h}{\varphi(s)^{nd}}$ where $h \in T_{nd \deg s}$, and $S_{nd \deg s} \cong T_{nd \deg s}$ by assumption. So $\operatorname{Spec} S_{(s)} \cong \operatorname{Spec} T_{(\varphi(s))}$, and this map is affine-locally an isomorphism. Thus it's an isomorphism.

d. By the proof of proposition II.2.6, we have that for an affine variety V with coordinate algebra $k[V]$, we have that $t(v) \cong \operatorname{Spec} k[V]$. Picking an embedding $V \hookrightarrow \mathbb{P}^n$, we see that this gives that V is covered by the $n+1$ affine open varieties $U_i = D(x_i)$ with coordinate algebras $S_{(x_i)}$, and the pairwise intersections of these are the affine varieties with coordinate algebras given by $S_{(x_i x_j)}$. Applying $t(-)$ to all of this, we get the gluing data of $\operatorname{Proj} S$ and so by exercise II.2.12, we have that $t(V) \cong \operatorname{Proj} S$.

Exercise II.2.15.

a. Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.

b. If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

c. Now show that if V, W are any two varieties over k, then the natural map

$$\operatorname{Hom}_{\mathfrak{Var}}(V, W) \to \operatorname{Hom}_{\mathfrak{Sch}/k}(t(V), t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

Solution. This is another point where the language 'residue field is k' could trip you up if you interpret it incorrectly. See exercise II.2.8 for the relevant discussion.

a. We start by noting that the closed points of an affine scheme are exactly the maximal ideals. A point P in $\operatorname{Spec} A$ is closed iff any point $Q \in \overline{\{P\}}$ is actually equal to P. By lemma 2.1, this says that a point P is closed iff for every prime ideal Q contained in P, we have $Q = P$. This is exactly the definition of a maximal ideal.

   A set $S \subset X$ is closed iff $S \cap U_i$ is closed for all elements of an open cover $\{U_i\}$. Pick an affine open cover $\{\operatorname{Spec} A_i\}$ of X. Then for any $\operatorname{Spec} A_i$ containing P, we have the natural morphism $A_i \to k(P) = k$ is surjective (it's a k-morphism with target k), so P is given by a maximal ideal in A_i. Thus P is closed in every $\operatorname{Spec} A_i$ it's in, and so P having residue field k implies P is closed.

   Conversely, if $P \in X$ is closed, then there's an affine open neighborhood $\operatorname{Spec} A \subset X$ which contains P and P is closed in $\operatorname{Spec} A$, so it suffices to consider the case X affine. In particular, we note the coordinate algebra of X is a finitely-generated k-algebra. As any closed point of an affine scheme is given by a maximal ideal by our first paragraph in this answer, we see that $k(P)$ is a finitely-generated extension of k. By Zariski's lemma, this is a finite (thus algebraic) extension of fields, and as k is algebraically closed, we see that $k(P) = k$.

b. Consider the composite map $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P} \to k(P) = k$. As the first map is local, we have that this map descends to the quotient $\mathcal{O}_{Y,f(P)}/\mathfrak{m}_{f(P)} = k(f(P))$ and so we have a map of k-algebras $k(f(P)) \to k(P) = k$. As both are fields, this must be injective, but on the other hand $k(f(P))$ is an extension of k. By dimension reasons, this must be an isomorphism, so $k(f(P)) = k$ as well.

c. Suppose φ and ψ are two maps of varieties (in the sense of chapter I) $V \to W$. Then these induce maps $t(\varphi)$ and $t(\psi)$, and these maps $t(\varphi)$ and $t(\psi)$ recover the map on closed points from $V \subset t(V)$ to $W \subset t(W)$ by construction of t. If $\varphi \neq \psi$, then there's some point $P \in V$ so that $\varphi(P) \neq \psi(P)$, and thus $t(\varphi)(P) \neq t(\psi)(P)$, so $t(\varphi) \neq t(\psi)$ and the map is injective.

   To show surjectivity, suppose we have a map $\varphi : t(V) \to t(W)$. By parts (a) and (b), we have that the image of any closed point in $t(V)$ is a closed point in $t(W)$, so we can define a continuous map of topological spaces $V \to W$ by restricting our map φ to the sets of closed points - this is continuous because both spaces are endowed with their subspace topologies. To finish, we need to show that this is a morphism of varieties by showing that our map pulls back regular functions to regular functions. Suppose that f is a regular function on W defined at some point $w \in W$. We may suppose that f is defined on an affine open subvariety $Y \subset W$ containing w, and let $X \subset V$ be any affine open subvariety lying inside the preimage of Y under $\varphi|_V$. Then the pullback of $f \in \mathcal{O}_W(Y) = \mathcal{O}_{t(W)}(t(Y))$ to X is given by the image of f in $\mathcal{O}_{t(V)}(t(X)) = \mathcal{O}_V(X)$ by the map of sheaves induced by $t(V) \to t(W)$ followed by restriction, and this is a regular function. By covering all of $(\varphi|_V)^{-1}(W)$ with open affine subvarieties and the domain of definition of f by open affine subvarieties per proposition I.4.3, we have the result in general.

Exercise II.2.16. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

a. If $U = \operatorname{Spec} B$ is an open *affine* subscheme of X, and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

b. Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [*Hint*: Use an open affine cover of X.]

c. Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if $\operatorname{sp}(X)$ is noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

d. With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Solution.

a. If f_x isn't contained in $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$, then as $f_x = (f|_U)_x$ for any open $U \subset X$ containing x, we have that $U \cap X_f = U_{f|_U}$ for any open U. Now assume $U = \operatorname{Spec} B$ is affine. Then for any point $\mathfrak{p} \in \operatorname{Spec} B$, we have that $f \in \mathfrak{p}$ is equivalent to $f \in \mathfrak{p}_\mathfrak{p}$, which is the maximal ideal of the local ring $B_\mathfrak{p} = \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}$, so $(\operatorname{Spec} B)_f = \operatorname{Spec} B \setminus V(f) = D(f)$, and thus X_f is open. (Also, note that $(\operatorname{Spec} B)_f = \operatorname{Spec}(B_f)$ - this will be useful later!)

b. Pick a finite open affine cover of X by $\operatorname{Spec} A_i$. By the proof of (a), we see that if a restricts to zero in X_f, it restricts to zero in each $(\operatorname{Spec} A_i)_f = \operatorname{Spec}(A_i)_f$. But by the definition of localization, this implies that there's an n_i so that $f^{n_i}a = 0$ in A_i. As we have a finite open cover, we can take $n = \max_i n_i$, and then $f^n a = 0$ in every A_i.

c. Letting $U_i = \operatorname{Spec} A_i$, we have that b can be written as $\frac{b_i}{f^{n_i}}$ on each $(U_i)_f = U_i \cap X_f$ for $b_i \in A_i$ Let $n = \max_i n_i$. Then multiplying by $\frac{f^{n-n_i}}{f^{n-n_i}}$ we can assume that $n_i = n$. Now consider $b_i - b_j$ on $U_i \cap U_j$: we have that $(b_i - b_j)|_{(U_i \cap U_j)_f} = 0$, so by part (b) there's an m_{ij} so that $f^{m_{ij}}(b_i - b_j) = 0$. Taking $m = \max m_i j$, we get that the sections $f^m b_i$ all agree on overlaps, so they glue to a global section $c \in \mathcal{O}_X(X)$. But $c - f^{n+m}b$ restricts to $f^m b_i - f^m b_j = 0$ on $U_i \cap X_f$, so $c = f^{n+m}b$ on X_f and we're done.

d. Consider the obvious morphism $A_f \to \mathcal{O}_{X_f}(X_f)$. If $\frac{a}{f^n}$ is in the kernel, then $a|_{X_f} = 0$ and by (b) we have that $f^m a = 0$ as global sections for some m, so $\frac{a}{f^n} = 0$ and $A_f \to \mathcal{O}_{X_f}(X_f)$ is injective. To show surjectivity, suppose we have a section $b \in \mathcal{O}_{X_f}(X_f)$ - by (c) there's an m so that $f^m b$ is the restriction of $c \in \mathcal{O}_X(X)$, and thus $\frac{c}{f^m}$ has image b.

The condition that the intersection of two quasi-compact opens is quasi-compact is also implied by the condition that X is *quasi-separated*, which is a useful thing to know later in life. Indeed, the statement that $\Gamma(X_f, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_f$ for X quasi-compact and quasi-separated is known as the qcqs lemma in Vakil's Foundations of Algebraic geometry, and I like his proof better. Here it is, for comparison:

Cover X by finitely many open affines $\operatorname{Spec} A_i$ (quasi-compactness) and cover their intersections $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ by finitely many open affines $\operatorname{Spec} A_{ijk}$ (quasi-separatedness implies $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ is quasi-compact). Then we have an exact sequence

$$\mathcal{O}_X(X) \to \prod_i A_i \rightrightarrows \prod_{i,j,k} A_{ijk}$$

from the sheaf condition on $\mathcal{O}_X$ which gives sections as equalizers. Localizing at f we get that

$$(\mathcal{O}_X(X))_f \to \left(\prod_i A_i\right)_f \rightrightarrows \left(\prod_{i,j,k} A_{ijk}\right)_f$$

is exact. On the other hand, X_f is covered by $\operatorname{Spec}(A_i)_f$ and the intersections $\operatorname{Spec}(A_i)_f \cap \operatorname{Spec}(A_j)_f$ is covered by $\operatorname{Spec}(A_{ijk})_f$, and considering the same sort of exact sequence as before, we get that

$$\mathcal{O}_X(X_f) \to \prod_i (A_i)_f \rightrightarrows \prod_{i,j,k} (A_{ijk})_f$$

is exact. Using the fact that localization commutes with finite products, we see that $\mathcal{O}_X(X_f)$ and $(\mathcal{O}_X(X))_f$ are both equalizers of the diagram

$$\prod_i (A_i)_f \rightrightarrows \prod_{i,j,k} (A_{ijk})_f$$

and thus they must be isomorphic by the universal property of equalizers.

Exercise II.2.17. *A Criterion for Affineness.*

a. Let $f : X \to Y$ be morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

b. A scheme X is affine if and only if there is a finite set of elements $f_1, \cdots, f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that the open subset X_f are affine, and $f_1, \cdots, f_r$ generate the unit ideal in A. [*Hint*: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Solution.

a. I claim f is a homeomorphism: it's a bijection on points, and it's an open map, as for any $U \subset X$ open, $f(U \cap U_i)$ is open and thus $f(U) = \bigcup_i f(U \cap U_i)$ as open. To show that the map $f^\sharp$ is an isomorphism of sheaves, we look on stalks: we have an induced map of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ which gives us maps on stalks $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Let $f(x) \in U_i$. By the condition that f restricts to an isomorphism on U_i, we have that the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induced by f is an isomorphism, and sine U_i cover X, we see that f induces isomorphisms at every stalk. Thus it's an isomorphism of sheaves, and f is an isomorphism of schemes.

b. Let $Y = \operatorname{Spec} A$, and consider the map $g : X \to Y$ in $\operatorname{Hom}_{Sch}(X, \operatorname{Spec} A)$ corresponding to $id_A \in \operatorname{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X) = A)$ by the bijection from exercise II.2.4. Now we want to show that $g : X \to Y$ is an isomorphism when restricted to $g^{-1}(X_{f_i})$, which by part (a) will let us conclude that $X \cong Y = \operatorname{Spec} A$.

   As the f_i together generate the unit ideal, we have that $D(f_i)$ form an open cover of X, and it's finite by assumption. Next, by the assumption that each $D(f_i)$ is affine, each $D(f_i)$ is quasicompact (ref. exercise II.2.13). Further, each intersection of $D(f_i)$s is quasi-compact: $D(f_i) \cap D(f_j) = D(f_j) \subset \operatorname{Spec} A_{f_i} \cong \operatorname{Spec} A_{f_i f_j}$ which is quasi-compact because it's affine. So we're in the situation of exercise II.2.16(d), and we may apply the result of the exercise to see that $\mathcal{O}_{X_f}(X_f) = \mathcal{O}_X(X_f) \cong A_f$, and so $X_f \cong \operatorname{Spec} A_f$ and we're done.

Exercise II.2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

a. Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

b. Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

c. With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

d. Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective. [*Hint*: Consider $X' = \operatorname{Spec}(A/\ker\varphi)$ and use (b) and (c).]

Solution.

a. The points in $D(f)$ are exactly the prime ideals $\mathfrak{p} \subset A$ not containing f. As the elements belonging to every prime ideal are precisely the nilpotent elements, we have the desired equivalence: if f isn't nilpotent, there's a prime $\mathfrak{p}$ not containing it, and if $D(f)$ is empty, f is in all the prime ideals, and it must be nilpotent.

b. If the induced map on sheaves is injective, then the map on global sections is injective. But φ is exactly this map on global sections, so we have the desired result.

   To show the converse, let $\mathfrak{p} \in X$ be a point at which $f^\sharp_x : \mathcal{O}_{X,\mathfrak{p}} \to \mathcal{O}_{Y,y}$ is not injective. Pick a nonzero element g in the kernel of this morphism, and since $\mathcal{O}_{X,\mathfrak{p}} = A_\mathfrak{p}$, we can write $g = \frac{a}{b}$ for $a \in \mathfrak{p}$ and $b \notin \mathfrak{p}$. Then $a \in \ker\varphi$, and we have the desired result.

   To show that $f(Y) \subset X$ is dense, suppose not: $f(Y)$ dense is equivalent to it intersecting every open subset nontrivially. If it failed to do so, then there would be some nontrivial open subset U which it did not intersect - equivalently, it would be contained in $V(I)$ for I some non-nilpotent ideal. But then this implies our map factors as $\operatorname{Spec} B \to \operatorname{Spec} A/I \to \operatorname{Spec} A$, which after taking global sections means that our map of rings $A \to B$ factors as $A \to A/(g) \to B$, and it's not injective.

c. If φ is surjective, then $B \cong A/\ker\varphi$, and $V(\ker\varphi)$ is exactly the closed subset (in fact, closed subscheme) in question. Bijectivity on points is clear by the correspondence theorem, so we just need to show that our map is either open or closed. We'll pick open: an open set $D(f)$ is exactly sent to $D(f + \ker\varphi)$, so that settles that. For surjectivity of the morphism of sheaves, the map on stalks is given by $A_\mathfrak{p} \mapsto (A/\ker\varphi)_\mathfrak{p}$, which is clearly surjective as localization is exact.

d. If $f^\sharp$ is surjective, it's surjective on each stalk, so for any $b \in B$ and any point $\mathfrak{p}_i \in \operatorname{Spec} A$, we can find a principal affine open neighborhood $D(f_i)$ of $\mathfrak{p}_i$ so that $b_{\mathfrak{p}_i} \in (f_*\mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{p}_i}$ is the image of some $\frac{a}{f_i^{n_i}} \in A_{f_i}$. In particular, this means that $f^{m_i}(a - f_i^{n_i} b) = 0$ in B. Since $\operatorname{Spec} A$ is quasi-compact, we can pick a finite cover of $\operatorname{Spec} A$ by such $D(f_i)$ and select $m \geq m_i$, $n \geq n_i$ for all i. Since $D(f_i) = D(f_i^n)$ and $X = D(f_i)$ is equivalent to f_i generating the unit ideal, the f_i^n generate the unit as well, and we can find $g_i \in A$ so that $1 = \sum g_i f_i^n$. Then $b = \sum g_i f^{n+m} a$ and this is in the image of φ, so φ is surjective.

Exercise II.2.19. Let A be a ring. Show that the following conditions are equivalent:

- $\operatorname{Spec} A$ is disconnected;
- there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents*);

- A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

Solution. If $\operatorname{Spec} A$ is disconnected, it is the union of two disjoint closed sets, say $V(I) \sqcup V(J)$, where we can assume that I, J are radical ideals. As $V(I+J) = V(I) \cap V(J)$, we have that $V(I+J)$ is empty, which means it must contain 1. So I, J are comaximal, and by the Chinese Remainder Theorem, we have that $A \cong A/I \times A/J$. Thus the first item implies the third.

If $A \cong A_1 \times A_2$, then $e_1 = 1_{A_1}$ and $e_2 = 1_{A_2}$ satisfy the relations described in the second item, and so the third item implies the second.

If we have elements e_1, e_2 as in the second item, then $V(e_1) \cap V(e_2) = V(1) = \emptyset$ and so $\operatorname{Spec} A$ is the disjoint union of two nonempty closed subsets, which is exactly the condition that it's disconnected. So (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and the statements are equivalent.

II.3 First Properties of Schemes

This section does a decent job of introducing a lot of basic properties of schemes, but one sore spot for me is exercise II.3.11: this would be best handled using quasi-coherent sheaves, but that material isn't developed until II.5.

Exercise II.3.1. Show that a morphism $f : X \to Y$ is locally of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

Solution. We start with a useful lemma.

Lemma. *If* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are two open affine subsets of a scheme* X*, then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *can be covered by open sets* U *which are simultaneously* $D(f) \subset \operatorname{Spec} A$ *and* $D(g) \subset \operatorname{Spec} B$*.*

Proof. Let $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f \cap \operatorname{Spec} B \subset \operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$, and $\operatorname{Spec} B_g = \operatorname{Spec}(A_f)_{g'}$, so writing $g' = \frac{g''}{f^n}$ for $g'' \in A$, we get that $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$. As $\operatorname{Spec} A_f$ cover $\operatorname{Spec} A$, we have $\operatorname{Spec} A_f \cap \operatorname{Spec} B$ cover $\operatorname{Spec} A \cap \operatorname{Spec} B$, and as $\operatorname{Spec} B_g$ cover $\operatorname{Spec} B$, we have $\operatorname{Spec} B_g$ cover $\operatorname{Spec} A_f \cap \operatorname{Spec} B$, so $\operatorname{Spec} A \cap \operatorname{Spec} B$ really can be covered in the way we claim. ■

Recall Hartshorne's definition of locally of finite type is that such a cover of Y exists, so the only thing we have to do here is show that given an open affine cover of Y by $\operatorname{Spec} B_i$ so that $f^{-1}(\operatorname{Spec} B_i)$ can be covered by open affines $\operatorname{Spec} A_{ij}$ with each A_{ij} a finitely generated B_i-algebra, then for any open affine $\operatorname{Spec} B \subset Y$, we have that $f^{-1}(\operatorname{Spec} B)$ can be covered by open affines $\operatorname{Spec} A$ so that A is a finitely generated B-algebra.

Suppose $\operatorname{Spec} B_i$ is a cover of Y with $f^{-1}(\operatorname{Spec} B_i)$ covered by open affines $\operatorname{Spec} A_{ij}$ such that A_{ij} are finitely-generated B_i-algebras. Let $\operatorname{Spec} B \subset X$ be an arbitrary affine open. Cover $\operatorname{Spec} B \cap \operatorname{Spec} B_i$ by simultaneously-distinguished open affines as per the lemma. Now $f^{-1}(D(g)) \subset \operatorname{Spec} B_i) \cap \operatorname{Spec} A_{ij} = \operatorname{Spec}(A_{ij})_g$, which is a finitely-generated $(B_i)_g$-algebra, and as $(B_i)_g \cong B_{g'}$, we have that $(A_{ij})_g$ is a finitely-generated $B_{g'}$-algebra. As $B_{g'} \cong B[t]/(tg'-1)$, we have that $(A_{ij})_g$ is a finitely-generated B-algebra, and we're done after varying i, j.

Exercise II.3.2. A morphism $f : X \to Y$ is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for *every* open affine subset $V \subset Y$, $f^{-1}(V)$ is quasi-compact.

Solution. Fix V_i a cover of Y by open affines so that $f^{-1}(V_i)$ is quasi-compact, and let $V \subset Y$ be an arbitrary affine open. For any point $v \in V$, pick an $i(v)$ so that $v \in V_{i(v)}$, and then by the fact that distinguished affine opens form a basis for the topology on $V_{i(v)}$, select a distinguished affine open neighborhood $D(g_v) \subset V_{i(v)}$ of v in $V_{i(v)}$ contained entirely inside $V_{i(v)} \cap V$. Now, considering these as subsets of V, we see that they cover V, and by quasi-compactness we may assume this covering is finite. By quasi-compactness, we can write $f^{-1}(V_i)$ as a finite union of affine opens $\operatorname{Spec} A_{ij}$.

But $\operatorname{Spec} A_{ij} \cap f^{-1}(D(g_v)) = \operatorname{Spec}(A_{ij})_{g_v}$, which is quasi-compact because it's affine, and a finite union of these covers $f^{-1}(D(g_v))$, so $f^{-1}(V)$ can be written as a finite union of quasi-compact open subsets and thus it is quasi-compact.

Exercise II.3.3.

a. Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

b. Conclude from this that f is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$, where each A_i is a finitely generated B-algebra.

c. Show also if f is of finite type, then for *every* open affine subset $V = \operatorname{Spec} B \subset Y$, and for *every* open affine subset $U = \operatorname{Spec} A \subset f^{-1}(V)$, A is a finitely generated B-algebra.

Solution.

a. Finite type clearly implies locally of finite type and quasi-compact. On the other hand, by exercises II.3.1 and II.3.2, for any morphism which is locally of finite type and quasi-compact, the preimage of every open affine subset $\operatorname{Spec} B$ can be covered by finitely many affine opens $\operatorname{Spec} A_i$ which are each finitely-generated B-algebras, and thus such a morphism is of finite type.

b. See above.

c. Cover $f^{-1}(\operatorname{Spec} B)$ by finitely many $\operatorname{Spec} A_i$ by part (b). For every point $x \in \operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, select a simultaneously-distinguished open affine neighborhood by the lemma in exercise II.3.1. We can refine this to a finite cover as $\operatorname{Spec} A$ is quasi-compact, so $\operatorname{Spec} A$ is covered by finitely many $D(f_i)$ so that each A_{f_i} is finitely-generated as a B-algebra.

   For each A_{f_i}, pick a finite list of generators for A_{f_i} as a B-algebra and multiply each by an appropriate power of f_i so that it's in the image of the map $A \to A_{f_i}$. Picking a preimage of each such generator in A and taking the union of all of these sets as i varies, we get a finite list of elements $a_1, \cdots, a_n$ of A. Consider the surjection from $B[x_1, \cdots, x_n] \to A$ given by sending the variables x_i to the corresponding elements $a_i \in A$, and let the cokernel of this map of B-modules be denoted C. To show that $B[x_1, \cdots, x_n] \to A$ is surjective and thus A is finitely generated as a B-algebra, it suffices to show that $C = 0$. We already know that $C_{f_i} = 0$ for every i by the assumption, which by the transitivity of localization implies that $C_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subset A$, and so $C = 0$ and we're done.

   (If you need a refresher on the reason that $C_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ implies $C = 0$, here it is: suppose that $x \in C$ is nonzero. Now consider $Ann_A(x) := \{f \in A \mid fx = 0\}$. This is the annihilator of x in A, and it's an ideal. The condition that $C_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ implies that $Ann_A(x)$ isn't contained in any prime ideal, so $Ann_A(x) = A$ and $x = 0$. But this didn't depend on x, so every element of C is equal to zero, and thus $C = 0$.)

Exercise II.3.4. Show that a morphism $f : X \to Y$ is finite if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

Solution. Let $\operatorname{Spec} B_i$ be an open affine cover of Y such that $f^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} A_i$, and let $\operatorname{Spec} B \subset Y$ be an arbitrary affine open. Cover $\operatorname{Spec} B$ by distinguished affine open subsets of $\operatorname{Spec} B$ and $\operatorname{Spec} B_i$, and refine this to a finite subcover since $\operatorname{Spec} B$ is quasi-compact. As $f^{-1}(D(g) \subset \operatorname{Spec} B_i) = \operatorname{Spec}(A_i)_g$, we have that the preimage of every element of our cover is affine, and so by exercise II.2.17, we have that $f^{-1}(\operatorname{Spec} B)$ is affine (after noting that if a finite collection of elements generate the unit ideal, their images under any ring homomorphism do too: just apply the ring homomorphism to 1).

Now we need to show that $\operatorname{Spec} A = f^{-1}(\operatorname{Spec} B)$ is finitely generated as a module. But this is just the same proof as exercise II.3.3(c), substituting 'finitely generated as a module' for 'finitely generated as an algebra'.

Exercise II.3.5. A morphism $f : X \to Y$ is *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

a. Show that a finite morphism is quasi-finite.

b. Show that finite morphism is *closed*, i.e., the image of any closed subset is closed.

c. Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Solution.

a. We immediately reduce to the case $f : \operatorname{Spec} A \to \operatorname{Spec} B$. The fiber of this morphism over a point $\mathfrak{p} \in \operatorname{Spec} B$ is exactly $\operatorname{Spec} A \otimes_B k(\mathfrak{p})$, and $A \otimes_B k(\mathfrak{p})$ is finitely generated as a $k(\mathfrak{p})$-module by definition of the tensor product. Thus $A \otimes_B k(\mathfrak{p})$ is a finite-dimensional $k(\mathfrak{p})$-vector space, so it's Artinian and has finitely many ideals, so it's spectrum has finitely many points.

b. We immediately reduce to the case $f : \operatorname{Spec} B \to \operatorname{Spec} A$ by taking an affine open cover of Y and noting that a set $S \subset Y$ is closed iff there's an open cover $\{U_i\}$ of Y so that $S \cap U_i$ is closed for all i. We also note that it's enough to show that $f(\operatorname{Spec} B)$ is closed in $\operatorname{Spec} A$: if we want to show that $f(V(I))$ is closed for some ideal I, we can just consider the composite $\operatorname{Spec} B/I \to \operatorname{Spec} B \to \operatorname{Spec} A$, which is also a finite map.

   To show that $f(\operatorname{Spec} B)$ is closed in $\operatorname{Spec} A$, we show that for any point $\mathfrak{p} \in \operatorname{Spec} A$ which isn't in the image of $\operatorname{Spec} B$, we can find a function g defined on some neighborhood of $\mathfrak{p}$ so that $V(g)$ contains $f(\operatorname{Spec} B)$ but not $\mathfrak{p}$. For an arbitrary point $\mathfrak{q} \in \operatorname{Spec} A$, the points in $\operatorname{Spec} B$ which map to it are exactly the prime ideals $\mathfrak{r} \in \operatorname{Spec} B$ so that $\mathfrak{r} \cap A = \mathfrak{q}$. As localizating at $\mathfrak{q}$ and then quotienting by $\mathfrak{q}$ preserves this relation, we see that the points in $\operatorname{Spec} B$ which map to $\mathfrak{q}$ are the primes of the ring $B \otimes_A A_{\mathfrak{q}}/\mathfrak{q} \cong B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$. If $\mathfrak{p}$ is not in the image of f, this means that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = 0$. But $B_{\mathfrak{p}}$ is a finitely-generated $A_{\mathfrak{p}}$-module, and the statement that $\mathfrak{p}B_{\mathfrak{p}} = B_{\mathfrak{p}}$ gives that there is some $a \in A_{\mathfrak{p}}$ so that $a = 1$ modulo $\mathfrak{p}$ and $aB_{\mathfrak{p}} = 0$ by Nakayama's lemma. This a is exactly the required function g from earlier.

   (Alternatively, if we know a little more commutative algebra, we may use the Going-Up theorem. Details left to the interested reader, see for instance Stacks tag 01WM.)

c. Consider $\mathbb{A}^1 \sqcup \mathbb{G}_m \to \mathbb{A}^1$ in the obvious fashion, corresponding to $k[x] \to k[x] \times k[x, x^{-1}]$ by $x \mapsto (x, x)$. This isn't finite, as our map isn't closed: the image of $\mathbb{G}_m$ is $\mathbb{A}^1 \setminus \{0\}$, which isn't a closed subset of $\mathbb{A}^1$.

Exercise II.3.6. Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the *function field* of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Solution. We may immediately reduce to the affine case: by exercise II.2.9, the generic point of any affine open subset $\operatorname{Spec} A \subset X$ is the same as the generic point of X, and the stalk of a sheaf at a point can be calculated starting from any open neighborhood of that point. As Hartshorne's definition of an integral scheme is one so that $\mathcal{O}_X(U)$ is an integral domain for all U, we have that any integral affine scheme is the spectrum of an integral domain. Now I claim that the generic point of an integral domain is (0): indeed, (0) is the unique minimal prime ideal, so $V((0))$, the closure of (0), is all of $\operatorname{Spec} A$. By proposition II.2.2, $\mathcal{O}_{\operatorname{Spec} A,(0)} = A_{(0)} = \operatorname{Frac}(A)$, which is a field.

Exercise II.3.7. A morphism $f : X \to Y$, with Y irreducible, is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is *dominant* if $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subset Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite. [*Hint*: First show that the function field of X is a finite field extension of the function field of Y.]

Solution. Let $\operatorname{Spec} A$ be an open affine subscheme of Y and suppose $\operatorname{Spec} B$ is an open affine subscheme of X contained in $f^{-1}(\operatorname{Spec} A)$. As $f(\operatorname{Spec} B) \subset \operatorname{Spec} A/\ker(A \to B)$, we see that if $V(\ker(A \to B))$ is a proper subset of $\operatorname{Spec} A$, f can't be dominant. But this implies that $D(\ker(A \to B)) = \emptyset$, so $\ker(A \to B)$ is a nilpotent ideal and since A is a domain, $\ker(A \to B) = 0$ and $A \to B$ is an injection. This gives an injection on function fields $\operatorname{Frac} A \to \operatorname{Frac} B$. Writing $B = A[x_1, \cdots, x_n]$ where the $x_i \in B$ by the finite-type hypothesis, we see that the fiber over the generic point is exactly the spectrum of $\operatorname{Frac}(A) \otimes_A A[x_1, \cdots, x_n] = \operatorname{Frac}(A)[x_1, \cdots, x_n]$. This is an integral domain with fraction field $\operatorname{Frac}(B)$. If the transcendence degree of $\operatorname{Frac}(A) \subset \operatorname{Frac}(B)$ is positive, then one of the a_i is transcendental and we can find infinitely many prime ideals in $\operatorname{Frac}(A)[x_1, \cdots, x_n]$ by considering ideals of the form $(m_\alpha(x_i))$ as m_α runs over the irreducible polynomials in one variable over $\operatorname{Frac}(A)$ (there are infinitely many of these: any irreducible polynomial gives finitely many roots of $\overline{\operatorname{Frac}(A)}$, and any algebraically closed field is infinite). So $\operatorname{Frac}(X) = \operatorname{Frac}(A) \subset \operatorname{Frac}(B) = \operatorname{Frac}(Y)$ is algebraic and finitely generated as a field extension, so it is a finite extension.

Next I claim that we can find an $a \in A$ so that for any $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$, the map $\operatorname{Spec} B_a \to \operatorname{Spec} A_a$ is finite. Write $B = A[x_1, \cdots, x_n]$ as in the previous paragraph. For any x_i, we can obtain a polynomial with coefficients in A satisfied by x_i by writing the minimal polynomial for x_i over $\operatorname{Frac}(A)$ and clearing denominators. Let l_i be the leading coefficient of each such polynomial. Letting $a = \prod l_i$, we see that $A_a \to B_a$ makes B_a finitely generated as a module over A_a and the claim is proven. By picking a finite cover of $f^{-1}(\operatorname{Spec} A)$ and repeating this process finitely many times, we may assume that our original $\operatorname{Spec} A$ was selected so that it was covered by finitely many open affines $\operatorname{Spec} B \subset X$ each of which are finite over $\operatorname{Spec} A$.

Now I claim that if we have a morphism $X \to \operatorname{Spec} A$ where X and $\operatorname{Spec} A$ are integral and X is covered by finitely many open affines $\operatorname{Spec} B_i$ so that $\operatorname{Spec} B_i \to \operatorname{Spec} A$ is finite for all i, then we can find an element $a \in A$ so that $X_a \to \operatorname{Spec} A$ is an affine morphism. Let $W = \bigcap \operatorname{Spec} B_i$, where the $\operatorname{Spec} B_i$ are the finitely many open affines covering $f^{-1}(\operatorname{Spec} A)$. Since X is irreducible and the intersection is finite, W is nonempty and open. By repeated applications of the lemma from II.3.1, we can find an affine open subset $V \subset W$ which is distinguished in every $\operatorname{Spec} B_i$ (that is, $V = \operatorname{Spec}(B_i)_{b_i}$ for all i). Since $A \to B_i$ is finite for all i, each b_i satisfies a monic polynomial with coefficients in A. Let c_i be the (nonzero) constant terms of these polynomials, and let $a = \prod c_i$. Then $(B_i)_a \cong (B_j)_a$ for all i, j in a manner compatible with the maps from A_a, and so $\operatorname{Spec}(B_i)_a$ is exactly the preimage of $\operatorname{Spec} A_a$ under f. Thus $\operatorname{Spec} A_a$ is a dense affine open subset of Y so that $f^{-1}(\operatorname{Spec} A_a) \to \operatorname{Spec} A_a$ is finite and we're done.

(Intuitively, what's happening here is that behavior at the generic point corresponds to behavior on an open dense set. The tricks in this problem are just figuring out how make this all algebraically precise.)

Exercise II.3.8. *Normalization.* A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec} \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the *normalization* of X. Show also that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Solution. First we show that localization commutes with taking integral closures. Let A be an integral domain, S a multiplicatively closed subset of A, and let $\overline{A}$ denote the integral closure of A in its field of fractions. Then the claim is that $\overline{S^{-1}A} = S^{-1}\overline{A}$.

In one direction, $S^{-1}\overline{A} \subset \overline{S^{-1}A}$: if $z \in \operatorname{Frac} A$ is integral over A, satisfying the monic polynomial $x^n + \sum_{i=0}^{n-1} a_i x^i$ where $a_i \in A$, then $\frac{z}{s}$ is integral over $\overline{S^{-1}A}$ since it satisfies the monic polynomial $x^n + \sum_{i=0}^{n-1} \frac{a_i}{s^{n-i}} x^i$ which has coefficients in $S^{-1}A$. Conversely, if $\frac{f}{g} \in Frac(A)$ is integral over $S^{-1}A$, then it satisfies the monic polynomial $x^n + \sum_{i=0}^{n-1} a_i x^i$ where $a_i \in S^{-1}A$. But then we can find an $s \in S$ so that $s\frac{f}{g}$ satisfies the monic polynomial $x^n + \sum_{i=0}^{n-1} a'_i x^i$ with $a'_i \in A$ by clearing denominators appropriately. So $\overline{S^{-1}A} \subset S^{-1}\overline{A}$, and the claim is proven.

Now suppose that $\{\operatorname{Spec} A_i\}_{i\in I}$ is the collection of all open affine subschemes of X. Since the $\operatorname{Spec} A_i$ glue together to form X, we have a gluing data as per exercise II.2.12. Now cover $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ by simultaneously-distinguished open subsets $\operatorname{Spec}(A_i)_a = \operatorname{Spec}(A_j)_b$ (exercise II.3.1). Since integral closures commute with localizations, we have that the preimage of $\operatorname{Spec}(A_i)_a$ in $\operatorname{Spec} \widetilde{A_i}$ and $\operatorname{Spec}(A_i)_b)$ in $\operatorname{Spec} \widetilde{A_j}$ are isomorphic. This implies that the preimages of $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ in $\operatorname{Spec} \widetilde{A_i}$ and $\operatorname{Spec} \widetilde{A_j}$ are isomorphic, and since we constructed this isomorphism naturally, we have that it forms a gluing data for the $\operatorname{Spec} \widetilde{A_i}$, giving us $\widetilde{X}$. ($\widetilde{X}$ is normal because any point lies in some $\operatorname{Spec} \widetilde{A_i}$, and the stalk at the point is a localization of an integrally closed domain, which is again an integrally closed domain.)

To show the universal property, it's enough to show that this factorization happens affine-locally and glues correctly. The factorization is a consequence of the work we did in exercise I.3.17(e), and the gluing follows from the compatibility of integral closure with localization proven earlier in this problem.

Finiteness of the integral closure in the case that X is finite type over a field follows from theorem I.3.9A: $\widetilde{X} \to X$ is affine-locally $\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ and $\widetilde{A}$ is module-finite A-algebra in this case by the theorem.

Exercise II.3.9. *The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

a. Let k be a field, and let $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

b. Let k be a field, let s and t be indeterminates over k. The $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.

Solution.

a. Since $\mathbb{A}^1_k$ and $\operatorname{Spec} k$ are affine, we can compute the fiber product as the spectrum of $k[x] \otimes_k k[y]$. But $k[x] \otimes_k k[y] \cong k[x, y]$ and this is isomorphic to the coordinate algebra of $\mathbb{A}^2$, so $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$.

   To see that the underlying topological spaces are different (this is what Hartshorne means when he says 'underlying point set'), recall that the points of $\mathbb{A}^1_k$ are given by maximal ideals $\mathfrak{m}_a$ and the generic point $\eta = (0)$. The product as topological spaces therefore consists of the following types of points, with the specified closures:

   - $(\mathfrak{m}_a, \mathfrak{m}_b)$ with closure $(\mathfrak{m}_a, \mathfrak{m}_b)$,
   - $(\mathfrak{m}_a, \eta)$ with closure $\{\mathfrak{m}_a\} \times \mathbb{A}^1_k$,
   - $(\eta, \mathfrak{m}_b)$ with closure $\mathbb{A}^1_k \times \{\mathfrak{m}_b\}$, and
   - (η, η) with closure $\mathbb{A}^1_k \times \mathbb{A}^1_k$ (where we mean product as topological spaces).

   We note that we can't select three one-dimensional closed subsets which all intersect each other in exactly one closed point: the one-dimensional closed subsets are exactly the horizontal and vertical lines, and the intersection of any two lines of the same type is either empty or the whole of the line. On the other hand, we can do this in $\mathbb{A}^2_k$: $V(x)$, $V(y)$, and $V(x+y+1)$ are all one-dimensional closed subsets and the intersection of any two of them is a closed point which isn't on the third. Since such a property is a homeomorphism invariant, we've shown that the topological spaces $\mathbb{A}^1_k \times_{top} \mathbb{A}^1_k$ and $\mathbb{A}^2_k$ are not homeomorphic.

b. The first thing to note is that $k(s) \otimes_k k(t)$ isn't a field: we can describe this as $S^{-1}k[s] \otimes_k T^{-1}k[t] \cong (ST)^{-1}k[s,t]$ where S and T consist of all nonzero polynomials in s and t, respectively, and ST consists of the products of all nonzero polynomials in s and all nonzero polynomials in t. So $st-1$ is an element of $k[s,t]$ which isn't in ST and therefore isn't invertible. Going further, any irreducible polynomial of positive degree $p(x)$ in one variable with nonzero constant term determines a prime ideal of $(ST)^{-1}k[s,t]$ by $(p(st))$. So $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$ has lots of points - it's not a single point like you would expect if you were only looking at the topological fiber product.

To be even clearer about what's happened here, we took $\mathbb{A}^2_k$ and deleted every closed subscheme of the form $X \times_k \mathbb{A}^1_k$ and $\mathbb{A}^1_k \times_k Y$. This gives us a 1-dimensional topological space where the surviving points are exactly the generic points of the curves in $\mathbb{A}^2_k$ which aren't horizontal or vertical lines as well as the generic point of the whole space. (Describe is a pretty fuzzy term, so hopefully this clues you in to what's going on here.)

Exercise II.3.10. *Fibers of a Morphism.*

a. If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\operatorname{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

b. Let $X = \operatorname{Spec} k[s,t]/(s-t^2)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \mapsto s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fiber X_y consists of two points with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fiber X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

Solution.

a. First we prove a lemma and show that open immersions (pg 85) are stable under base change.

Lemma (Stacks 01JR). *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. Let $X \times_S Y$ be the fiber product and p, q the projections to X, Y respectively. If $U \subset S$, $V \subset X$, and $W \subset Y$ are open subschemes with $f(V) \subset U$ and $g(W) \subset U$, then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion identifying $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. If $a : T \to V$ and $b : T \to W$ are morphisms of schemes so that $f \circ a = g \circ b$ as morphisms to U, then they agree as morphisms to S, and define a unique morphism $T \to X \times_S Y$ by the universal property of the fiber product. On the other hand, this morphism's image is contained in the open $p^{-1}(V) \cap q^{-1}(W)$ by assumption - so we have a unique morphism from T to $p^{-1}(V) \cap q^{-1}(W)$ and thus $p^{-1}(V) \cap q^{-1}(W)$ satisfies the same universal property as $V \times_U W$. Since objects satisfying universal properties are unique up to unique isomorphism, this implies that $V \times_U W \cong p^{-1}(V) \cap q^{-1}(W)$ and thus the morphism $V \times_U W \to X \times_S Y$ is an open immersion.

■

We can use this lemma to show that open immersions are stable under base change as follows: given an open immersion $X' \to X$ and a map $Y \to X$, we apply the lemma with $S = U = X$ and $V = X'$ to see that this gives that the natural map $X' \times_S Y \to X \times_X Y \cong Y$ is an open immersion.

Now write $i_y : \operatorname{Spec} k(y) \to Y$ for the inclusion of y in to Y. We'll first reduce to the affine case. To start, we may assume Y is affine: factoring i_y as the composition of $\operatorname{Spec} k(y) \to \operatorname{Spec} B \to Y$ for $\operatorname{Spec} B$ an affine open subset of Y, we see that the fiber product of i_y and f is the same as the fiber product of $i'_y : \operatorname{Spec} k \to \operatorname{Spec} B$ and $\overline{f} : X_{\operatorname{Spec} B} \to \operatorname{Spec} B$. Next, we can cover X by open affines with open immersions $i_A : \operatorname{Spec} A \to Y$. As open immersions are stable under base change, we get that the morphism $\operatorname{Spec} A \times_Y \{y\} \to X_y$ is also an open immersion, and the schemes $\operatorname{Spec} A \times_Y y$ cover X_y. So it's enough to understand the affine situation.

In the affine case, our map $X \to Y$ can be represented as a ring map $f : B \to A$ where a point $\mathfrak{p} \in X$ maps to a point $f^{-1}(\mathfrak{p}) \in Y$. Let $\mathfrak{q}$ be the prime ideal associated to the point y. We will proceed to compute the fiber product by intermediate steps (recalling that for a fiber diagram of affine schemes $\operatorname{Spec} R \to \operatorname{Spec} S$ and $\operatorname{Spec} T \to \operatorname{Spec} S$, the fiber product is the scheme $\operatorname{Spec} R \otimes_S T$ with the obvious natural maps). Consider the following diagram:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X_{\operatorname{Spec} B_{\mathfrak{q}}} \arrow[r] \arrow[d] & X = \operatorname{Spec} A \arrow[d] \\ \{y\} \arrow[r] & \operatorname{Spec} B_{\mathfrak{q}} \arrow[r] & Y = \operatorname{Spec} B \end{tikzcd}$$

Our first step is to localize both A and B at the ideal $\mathfrak{q} \subset B$. Geometrically, this corresponds to taking the fiber product along the map $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} B$ in the above diagram. We then identify $X_{\operatorname{Spec} B_{\mathfrak{q}}} = \operatorname{Spec} A \otimes_B B_{\mathfrak{q}} = \operatorname{Spec} A_{\mathfrak{q}}$. We note that by the description of the ideals of $A_{\mathfrak{q}}$, this preserves the points in the fiber over y: prime ideals of the localization $A_{\mathfrak{q}}$ are exactly those that do not intersect the set we're localizing at, which is equivalent to the preimages of those prime ideals being disjoint from the complement of $\mathfrak{q}$, or equivalently contained in $\mathfrak{q}$.

We are now in the situation of $B_{\mathfrak{q}} \to A \otimes_B B_{\mathfrak{q}} := A_{\mathfrak{q}}$. In order to compute the next fiber product, we recognize the inclusion $\{y\} \to \operatorname{Spec} B_{\mathfrak{q}}$ as the map $\operatorname{Spec} \kappa(y) = \operatorname{Spec} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \to \operatorname{Spec} B_{\mathfrak{q}}$. Therefore $X_y = \operatorname{Spec} A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$, and it remains to identify the prime ideals of this ring. By the correspondence theorem, these are exactly the prime ideals that contain $\mathfrak{q}A_{\mathfrak{q}}$. But these are precisely the prime ideals who's preimage under the map $B_{\mathfrak{q}} \to A_{\mathfrak{q}}$ is exactly $\mathfrak{q}$: that is, the points of $\operatorname{Spec} A_{\mathfrak{q}}$ which are mapped to $\{y\}$. So this step also preserves the points in the fiber over y.

As these two steps preserve the points in the fiber over y and combine to produce the fiber over y, we are done: the scheme-theoretic fiber over y has the same underlying topological space as the topological fiber over y.

For an alternate proof, points of the fiber product $X \times_S Y$ correspond to quadruples $(x, y, s, \mathfrak{p})$ with $f(x) = g(y) = s$ and $\mathfrak{p}$ a prime ideal of $\operatorname{Spec} k(x) \otimes_{k(s)} k(y)$ (see Stacks 01JT). In our case, $k(s) = k(y)$ and so the tensor product is really just $k(x)$, the prime ideal must be zero, and we see the correspondence we're after.

b. Caution: we actually need to assume that $\operatorname{char} k \neq 2$ here in order to have the conclusions Hartshorne states.

By part (a) and the fact that the fiber product of affine schemes can be computed by taking tensor products, this will go pretty quick. In the first case, we're looking at $\operatorname{Spec} k[s,t]/(s - t^2) \otimes_{k[s]} k[s]/(s-a) \cong \operatorname{Spec} k[t]/(t^2 - a)$. As $a \neq 0$, we have $t^2 - a = (t - \sqrt{a})(t + \sqrt{a})$ since k is algebraically closed, and as $\sqrt{a} \neq -\sqrt{a}$ as $\operatorname{char} k \neq 2$, by the Chinese remainder theorem we have $k[t]/(t^2 - a) \cong k \times k$ by $t \mapsto (\sqrt{a}, -\sqrt{a})$ and therefore we have the desired result. In the second case, we're looking at $\operatorname{Spec} k[t]/(t^2)$, which is a nonreduced one-point scheme (this is also what happens when $\operatorname{char} k = 2$: $(t^2 - a) = ((t - \sqrt{a})^2)$, so $\operatorname{Spec} k[t]/(t^2 - a) \cong \operatorname{Spec} k[t]/((t - \sqrt{a})^2))$. In the third case, we're looking at $\operatorname{Spec} k[s,t]/(s - t^2) \otimes_{k[s]} k(s) \cong k(s)[t]/(t^2 - s)$, and $k(s)[t]/(t^2 - s)$ is a degree-two field extension of $k(s)$, the residue field at η.

Exercise II.3.11. *Closed Subschemes.*

a. Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

b. (*) If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subset A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$. [*Hints*: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \emptyset$, if necessary, show that we may assume the $D(f_i)$ cover X. Next show that $f_1, \cdots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subset A$.] *Note*: We will give another proof of this result using sheaves of ideals later (5.10).

c. Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

d. Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the *scheme-theoretic image* of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Solution. There are a couple minor annoyances with this question: it's most natural to do (b) before (a) because the fact that a closed immersion with an affine scheme as a target is of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is so fundamental, and all of this would be easier and more natural to do once we've introduced quasi-coherent sheaves, which Hartshorne has chosen to do in II.5 (which occurs after this!).

a. It's clear that restricting our closed immersion to an open subset $U \subset X$ gives a closed immersion $Y \times_X U = Y \cap U \to U$, and if we have a map $f : Y \to X$ and an open cover $\{U_i\}$ of X so that $Y \times_X U_i \to U_i$ is a closed immersion for all U_i, then f is a closed immersion: checking that f is a homeomorphism on to its image and surjective as a map of sheaves are both local conditions. We can cover $X' = X \times_X X'$ by open affine subschemes given by $\operatorname{Spec} A \times_{\operatorname{Spec} A} \operatorname{Spec} B$ where $\operatorname{Spec} A$ form an affine open cover of X and $\operatorname{Spec} B$ form an affine open cover of X' such that the image of $\operatorname{Spec} B$ lands in $\operatorname{Spec} A$ under the morphism $X' \to X$. Assuming the result of (b), this implies that on an affine open $\operatorname{Spec} A \subset X$, our map $f : Y \to X$ is $\operatorname{Spec} A/I \to \operatorname{Spec} A$, and base-extending along $\operatorname{Spec} B \to \operatorname{Spec} A$ gives us $\operatorname{Spec} A/I \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \operatorname{Spec} B/I \to \operatorname{Spec} A \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \operatorname{Spec} B$, which is again a closed immersion and we're done.

b. First, we note that as a closed subset of a quasi-compact topological space X, Y is also quasi-compact: given any open cover of Y, we can write any element of the cover as the intersection of an open subset of X with Y, and then these opens plus the complement of Y form an open cover of X. Selecting a finite subcover, we have the desired result.

   Now take a finite cover of Y by open affines $\{V_i = \operatorname{Spec} B_i\}$. As $Y \subset X$ has the induced topology, we can find open $U_i \subset X$ so that $U_i \cap Y = V_i$. As the distinguished open sets $D(f)$ for $f \in A$ form a basis for the topology on X, we can write each $U_i = \bigcup_j D(f_{ij})$ for some $f_{ij} \in A$, and in turn this means that Y is covered by $D(f_{ij}) \cap Y$ as i, j vary. In particular, we note that $D(f_{ij}) \cap Y$ is the distinguished open affine $D(\overline{f_{ij}}) \subset V_i$ where the overline represents the image of f_{ij} in B_i under the map $A \to B_i$ induced from $\operatorname{Spec} B_i \hookrightarrow Y \to X = \operatorname{Spec} A$. By quasi-compactness of Y, we can select a finite subcover by $D(\overline{f_{ij}}) \cap Y$, and by the addition of some finite number of $D(g_h)$ for $g_h \in A$, we can assume that the $D(f_{ij})$ and $D(g_h)$ cover X as well. In this case, covering X implies that $(f_{ij}, g_h) = (1)$ inside A, and thus $(\overline{f_{ij}}, \overline{g_h}) = (1)$ inside $\Gamma(Y, \mathcal{O}_Y)$, as $A = \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is a ring homomorphism. But we can now apply exercise II.2.17(b): $\overline{f_{ij}}$ and $\overline{g_h}$ are a finite set of elements which generate the unit ideal inside $\Gamma(Y, \mathcal{O}_Y)$ and $D(\overline{f_{ij}})$ and $D(\overline{g_h}) = \emptyset$ are affine. So Y is affine, and by exercise II.2.18(d), we have that $Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$.

c. The underlying topological spaces of Y and Y' are identical, so it's enough to look on sheaves. On any affine open $\operatorname{Spec} A \subset X$, we have that $Y' \cap \operatorname{Spec} A$ is given by the affine scheme $\operatorname{Spec} A/I$ for some ideal I, and $Y \cap \operatorname{Spec} A$ is given by the affine scheme $\operatorname{Spec} A/\sqrt{I}$. In particular, we have that the induced maps on structure sheaves over $\operatorname{Spec} A$ are given by $A \to A/I$ and $A \to A/\sqrt{I}$. As $A \to A/\sqrt{I}$ factors as $A \to A/I \to A/\sqrt{I}$ and this is compatible with localization (since radicals and localizations commute and localization is exact), we have the desired result.

d. If $f : Z \to X$ factors through a closed subscheme $i : Y \to X$, this means that we can write f as $Z \to Y \to X$, which means that the maps $\mathcal{O}_X \to f_* \mathcal{O}_Z$ and $\mathcal{O}_X \to i_* \mathcal{O}_Y \to f_* \mathcal{O}_Z$ are equal. In particular, letting $\mathcal{I}_Y = \ker \mathcal{O}_X \to i_*\mathcal{O}_Y$, this means that by the first isomorphism theorem, f factors through Y iff the image of $\mathcal{I}_Y \subset \mathcal{O}_X$ is mapped to zero in $f_*\mathcal{O}_Z$. Now by Hartshorne's defintion of a closed immersion, if $\{Y_i\}_{i\in I}$ is a collection of closed subschemes of X, then we can define a closed subscheme $\bigcap_{i\in I} Y_i$ as the topological space which is the intersection of the underlying topological spaces of each Y_i, and equipped with the structure sheaf $\mathcal{O}_X/\mathcal{I}$, where $\mathcal{I}$ is the sum of all the sheaves $\mathcal{I}_{Y_i}$. I claim that if we let $\{Y_i\}_{i\in I}$ be the collection of all closed subschemes through which f factors, then $\bigcap_{i\in I} Y_i$ is the scheme-theoretic image: it's a closed subscheme by construction, and if Y' is any closed subscheme through which f factors, then $\mathcal{I}_{Y'} \subset \mathcal{I}$ and thus by our characterization of 'factors through a closed subscheme' in terms of the ideal sheaves, we have the required claim.

The reason we have to go to all of this trouble is because the scheme-theoretic image can behave in unexpected ways. Some surprising behavior can happen with scheme-theoretic images when $f : Z \to X$ is not quasi-compact and Z is nonreduced. For instance, $\overline{f(Z)}$ is not even the underlying topological space of the scheme-theoretic image in general, and formation of the scheme-theoretic image does not commute with restriction to open subsets without either of these conditions on f and Z As an explicit example, if we take $\coprod_{n\in\mathbb{Z}_{\geq 0}} \operatorname{Spec} k[x]/(x^n) \to \operatorname{Spec} k[x]$ with the obvious map, then the scheme-theoretic image is all of $\operatorname{Spec} k[x]$, because no nonzero polynomial in x is zero modulo every x^n. This is decidedly not the closure of the set-theoretic image, (x), and the base change along $\operatorname{Spec} k[x, x^{-1}] \to \operatorname{Spec} k[x]$ makes the source empty, demonstrating the claim about formation of the scheme-theoretic image not being compatible with restriction to open subsets. Gross!

Now on to showing that $\overline{f(Z)}$ with the reduced induced subscheme structure is the scheme theoretic image of f when X is reduced. First, if we have a subset $S \subset X$ and a sheaf $\mathcal{O}_S$ so that for every open affine subscheme $\operatorname{Spec} A \subset X$ we have $S \cap \operatorname{Spec} A = \operatorname{Spec} A/I \subset \operatorname{Spec} A$ as topological spaces and $\mathcal{O}_S|_{\operatorname{Spec} A\cap S} \cong \mathcal{O}_{\operatorname{Spec} A/I}$, and for every $g \in A$ we have that $S \cap \operatorname{Spec} A_g = \operatorname{Spec} A_g/I_g$ as schemes, then $(S, \mathcal{O}_S)$ is a closed subscheme of Y and the obvious map is a closed immersion. Let $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Z)$. I claim that $(\overline{f(X)}, \mathcal{O}_Y/\mathcal{I})$ fulfills the above property and therefore is a closed subscheme, and further that this is the reduced induced subscheme structure on $\overline{f(X)}$.

In order to show the first part of the claim, it's enough to show that $\mathcal{I}(\operatorname{Spec} A)_g = \mathcal{I}(\operatorname{Spec} A_g)$ as submodules of A_g. Let $\operatorname{Spec} A \subset X$ be an affine open subscheme and cover $f^{-1}(\operatorname{Spec} A)$ by affine open subschemes $\operatorname{Spec} B_j$ as j ranges over an index set J. By the sheaf property, we have that $(f_*\mathcal{O}_Z)(\operatorname{Spec} A) = \mathcal{O}_Z(f^{-1}(\operatorname{Spec} A))$ injects in to $\prod_{j\in J} \mathcal{O}_Z(\operatorname{Spec} B_j) = \prod_{j\in J} B_j$, so the sequence $0 \to \mathcal{I}(\operatorname{Spec} A) \to A \to \prod_{j\in J} B_j$ is exact. Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}(\operatorname{Spec} A) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \prod_{j\in J} B_j \arrow[d] \\
0 \arrow[r] & \mathcal{I}(\operatorname{Spec} A)_g \arrow[r] \arrow[d] & A_g \arrow[r] \arrow[d] & \left(\prod_{j\in J} B_j\right)_g \arrow[d] \\
0 \arrow[r] & \mathcal{I}(\operatorname{Spec} A_g) \arrow[r] & A_g \arrow[r] & \prod_{j\in J} (B_j)_g
\end{tikzcd}$$

where the morphisms from the top row to the bottom row come from the restriction maps, and factor through the middle row by the universal property of localization since g is invertible in each of the entries in the bottom row. To achieve our goal, suppose we have some $a/g^n \in \mathcal{I}(\operatorname{Spec} A_g) \subset A_g$ - this is the same as an element of A_g who's image in $\prod_{j\in J}(B_j)_g$ is zero. We would like to claim that a/g^n maps to zero in $\left(\prod_{j\in J} B_j\right)_g$, which would imply it's in $\mathcal{I}(\operatorname{Spec} A)_g$. In order for this to be true, we need there to exist an m so that $g^m a = 0$ in $\prod_{j\in J} B_j$. We know that for each $j \in J$, there is an m_j so that $g^{m_j} a$ maps to 0 in B_j by the bottom row of the diagram, and so all we have to do is be able to set $m \geq m_j$ for all $j \in J$ and we win.

If Z is reduced, I claim that $m = m_j = 1$ for all j works. If $g^n a$ maps to 0 in B_j, then ga maps to a nilpotent in B_j, as $(ga)^n = g^n a \cdot a^{n-1}$ and the image of the RHS is zero in B_j. But Z is reduced, so B_j is reduced, so ga must actually map to 0, and thus $\mathcal{I}(\operatorname{Spec} A)_g = \mathcal{I}(\operatorname{Spec} A_g)$. (It's not too hard to see that if f is quasi-compact, we can also pick m successfully: J may be chosen to be a finite set, and then m can be set to $\max m_j$. The problem with computing the scheme-theoretic image like this lies with non-quasi-compact maps from nonreduced sources as seen in the example above.)

To show that this means that $S = (\operatorname{Supp} \mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I})$ is $\overline{f(X)}$ with the reduced induced subscheme structure, suppose $\operatorname{Spec} A \subset X$ is an affine open subscheme. For all $g \in A$ so that $f^{-1}(\operatorname{Spec} A_g) = \emptyset$, we have that $\mathcal{I}(\operatorname{Spec} A_g) = A_g$, so $g^n \in \mathcal{I}(\operatorname{Spec} A)$ for some n. Therefore $\overline{f(Z)} \cap \operatorname{Spec} A \supset \bigcap_{g\in\mathcal{I}(\operatorname{Spec} A)} V(g) = S \cap \operatorname{Spec} A$ as sets, so $\overline{f(Z)} \supset S$ as sets. On the other hand, if this is a proper containment, then there exists a point $x \in f(Z) \setminus S$, and taking an affine open neighborhood, we can find a function which vanishes on $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ but not x. Such a function would be in the kernel of $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ but not in the kernel of $\mathcal{O}_X \to f_*\mathcal{O}_Z$, which contradicts the characterization that $f : Z \to X$ should factor through $\operatorname{Im}(f) \to X$ from earlier in the proof. So $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is equal to $\overline{f(Z)}$ as sets.

To show they have the same scheme structure, it suffices to show that $\mathcal{O}_X/\mathcal{I}$ is reduced by the uniqueness of the reduced induced scheme structure (example II.3.2.6). For any affine open $\operatorname{Spec} A \subset X$, we have that $A/\mathcal{I}(\operatorname{Spec} A)$ embeds in the reduced ring $\mathcal{O}_Z(f^{-1}(\operatorname{Spec} A))$ by the first isomorphism theorem and is therefore reduced. Therefore the affine scheme $\overline{f(Z)} \cap \operatorname{Spec} A$ is given by $\operatorname{Spec} A/\mathcal{I}(\operatorname{Spec} A)$, the spectrum of a reduced ring. This suffices to prove that the scheme-theoretic image is reduced: if $y \in \operatorname{Spec} B \subset Y$ where $\operatorname{Spec} B$ is an affine open, then

$\mathcal{O}_{Y,y} \cong \mathcal{O}_{\operatorname{Spec} B,y} \cong B_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset B$, and every localization of a reduced ring is reduced. Thus the scheme-theoretic image is reduced by exercise II.2.3 and we are done.

Exercise II.3.12. *Closed Subschemes of* Proj S.

a. Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to $\operatorname{Proj} T$, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

b. If $I \subset S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as the image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.

   We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

Solution.

a. The open set in question is $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset \varphi(S_+)\}$, which is clearly all homogeneous prime ideals by the fact that φ is surjective and preserves degrees: $\varphi(S_+) = T_+$. On any principal open subset $D(f) \subset \operatorname{Proj} S$ with f homogeneous of positive degree, we have that our map is given by $\operatorname{Spec} T_{(f)} \to \operatorname{Spec} S_{(f)}$, and as $S_{(f)} \to T_{(f)}$ is again surjective since localization is exact, we have that this map is of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$. As these principal affine opens cover $\operatorname{Proj} S$ and the property of being a closed immersion is local on the target, we're finished.

b. We show that for any f homogeneous of positive degree, $(S/I)_{(f)}$ and $(S/I')_{(f)}$ are naturally isomorphic as rings, which gives that their spectra are naturally isomorphic and thus $\operatorname{Proj} S/I \cong \operatorname{Proj} S$ via gluing. For any $f \in S$ homogeneous of positive degree, we may find an $m \geq 0$ so that $m \deg n \geq d_0$. Thus for any element in $(S/I)_{(f)}$ which can be expressed as $\frac{p}{f^n}$ where $\deg p = n \deg f$, we have $\frac{p}{f^n} = \frac{f^m p}{f^{n+m}}$ which is an element of $(S/I')_{(f)}$ and we're done.

Exercise II.3.13. *Properties of Morphisms of Finite Type.*

a. A closed immersion is a morphism of finite type.

b. A quasi-compact open immersion (Ex. 3.2) is of finite type.

c. A composition of two morphisms of finite type is of finite type.

d. Morphisms of finite type are stable under base extension.

e. If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

f. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

g. If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Solution.

a. Affine-locally, a closed immersion looks like $\operatorname{Spec} A/I \to \operatorname{Spec} A$. A/I is generated by 1 as an A-algebra, so this morphism is locally of finite type, and as $\operatorname{Spec} A/I$ is quasi-compact, the morphism is quasi-compact. Thus the claim follows.

b. Let $X \to Y$ be an open immersion. For any open affine $\operatorname{Spec} A \subset Y$, we can cover $f^{-1}(U)$ by distinguished affine opens $\operatorname{Spec} A_a$ as these form a basis for the topology on Y and an open immersion is locally an isomorphism. This shows that the map is locally of finite type, as A_a is generated by $\frac{1}{a}$ as an A-algebra. The assumption that our open immersion is quasi-compact then implies it is of finite type.

c. If $f : X \to Y$ and $g : Y \to Z$ are locally of finite type, then for any affine open $\operatorname{Spec} A \subset Z$, we can cover $g^{-1}(\operatorname{Spec} A)$ by affine opens $\operatorname{Spec} B_i$ which are each finitely generated as A-algebras, and for each $\operatorname{Spec} B_i \subset Y$, we can cover $f^{-1}(\operatorname{Spec} B_i)$ by affine opens $\operatorname{Spec} C_{ij}$ which are each finitely generated as B_i-algebras. But then C_{ij} is finitely generated as an A-algebra, so $g \circ f : X \to Z$ is locally of finite type.

   To check quasi-compactness, pick an affine open in Z. By quasi-compactness, $g^{-1}(Z)$ is a finite union of quasi-compact affine opens, and repeating the procedure on each of those, we see that $(g \circ f)^{-1}(Z)$ is a finite union of affine open subschemes of X, and it is thus quasi-compact.

d. Suppose $f : X \to S$ is our morphism of finite type and $g : Y \to S$ is any morphism. We want to show that $X \times_S Y \to Y$ is of finite type. Given an open affine $\operatorname{Spec} R \subset S$, we can cover $f^{-1}(\operatorname{Spec} R)$ by finitely many open affines $\operatorname{Spec} A_i \subset X$ so that the induced maps $R \to A_i$ make each A_i finitely generated as R-algebras. Now cover $g^{-1}(\operatorname{Spec} R) \subset Y$ by open affines $\operatorname{Spec} B_j$: then the preimage of $\operatorname{Spec} B_j$ in the fiber product $X \times_S Y$ is covered by affine opens of the form $\operatorname{Spec} B_j \times_{\operatorname{Spec} R} \operatorname{Spec} A_i \cong \operatorname{Spec} B_j \otimes_R A_i$. Each of these are finitely-generated as B_j-algebras since A_i is finitely generated as an R-algebra, and there are finitely many of these by assumption.

e. Denote the structure morphisms $f : X \to S$ and $g : Y \to S$. Pick an affine open subset $\operatorname{Spec} R \subset S$, and as f, g are finite type, we may cover $f^{-1}(\operatorname{Spec} R)$ and $g^{-1}(\operatorname{Spec} R)$ by finitely many affine opens $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_j$ respectively so that A_i and B_j are all finitely-generated R-algebras. Then by the construction of the fiber product and the lemma from exercise II.3.10, we have that $f^{-1}(\operatorname{Spec} R) \times_S g^{-1}(\operatorname{Spec} R)$ is covered by finitely many affine opens of the form $\operatorname{Spec} A_i \times_S \operatorname{Spec} B_j \cong \operatorname{Spec} A_i \otimes_R B_j$, each of which is finitely generated as an R-algebra.

f. For any open affines $\operatorname{Spec} A \subset Z$, $\operatorname{Spec} B \subset g^{-1}(\operatorname{Spec} A)$, and $\operatorname{Spec} C \subset f^{-1}(\operatorname{Spec} B) \subset (g \circ f)^{-1}(\operatorname{Spec} A)$ we have that $A \to B \to C$ makes C a finitely generated A-algebra by exercise II.3.3(c). It suffices to prove that if we have a sequence of ring homomorphisms

$A \to B \to C$ so that C is finitely generated as an A-algebra, then C is also finitely generated as a B-algebra. Write $C = A[x_1, \cdots, x_n]/J$ and suppose B is generated as an A-algebra by some collection of elements $\{y_\alpha\}_{\alpha \in A}$. Let $\overline{y_\alpha}$ denote the image of y_α in C. Now I claim that $B[x_1, \cdots, x_n]/(J, y_\alpha - \overline{y_\alpha}) \cong C$, where I mean the ideal generated by the images of all elements of J in B and all elements of the form $y_\alpha - \overline{y_\alpha}$ as α ranges over the index set A. This shows that f is locally of finite type, and as it's quasi-compact by assumption, we have that f is of finite type.

g. Let $\operatorname{Spec} A \subset Y$ be an affine open, and cover $f^{-1}(\operatorname{Spec} A)$ by finitely many affine opens $\operatorname{Spec} B_i$ where B_i are finitely generated A-algebras. Then the B_i are noetherian, as B_i is a quotient of $A[x_1, \cdots, x_n]$ which is noetherian by the Hilbert basis theorem, and noetherianity is preserved under quotients. Covering Y by finitely many affine opens which have preimages covered by finitely many affine opens as above, we see that X is covered by finitely many affine opens which are spectra of noetherian rings, and so X is noetherian by definition (pg 83).

Exercise II.3.14. If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

Solution. By exercise II.3.13(g), as $\operatorname{Spec} k$ is noetherian and $X \to \operatorname{Spec} k$ is of finite type, we have that X is noetherian. In particular, every open subset of X is quasi-compact by exercise II.2.13(a). Now we show that any quasi-compact scheme has a closed point: Let X be a quasi-compact scheme, and take $\{U_i = \operatorname{Spec} A_i\}$ to be a finite open cover of X by open affines. A maximal ideal of A_1 determines a closed point P_1 of U_1, and if this point is closed in X we're finished. If not, then take P_2 to be a point in $\overline{\{P_1\}}$ with $P_2 \neq P_1$. Now P_2 is in some U_i but not U_1. Without loss of generality, it's in U_2. If P_2 is closed, we're done. If not, we can repeat the process to get a P_3 which isn't in either of U_1 or U_2, and without loss of generality it's in U_3. This process can't continue forever, since there are only finitely many U_i, so eventually we must find a closed point.

(One unsettling feature of schemes which aren't quasi-compact is that they don't have to have closed points. Karl Schwede has a nice short expository note about this which should be accessible via Googling 'Karl Schwede scheme with no closed points' if you're interested in the details.)

As for an example where closed points need not be dense for arbitrary schemes, the most accessible example is the spectrum of a DVR, for instance $\operatorname{Spec} k[x]_{(x)}$. This ring has two ideals, (0) and (x), and the only nontrivial closed set in this topology is $\{(x)\}$, so $\{(0)\}$ is an example of an open set containing no closed points.

Exercise II.3.15. Let X be a scheme of finite type over a field k (not necessarily algebraically closed).

a. Show that the following three conditions are equivalent (in which case we say that X is *geometrically irreducible*).

   (i) $X \times_k \overline{k}$ is irreducible, where $\overline{k}$ denotes the algebraic closure of k. (By abuse of notation, we write $X \times_k \overline{k}$ to denote $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$.)

(ii) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.

(iii) $X \times_k K$ is irreducible for every extension field K of k.

b. Show that the following three conditions are equivalent (in which case we say X is *geometrically reduced*).

(i) $X \times_k \overline{k}$ is reduced.

(ii) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.

(iii) $X \times_k K$ is reduced for every extension field K of k.

c. We say that X is *geometrically integral* if $X \times_k \overline{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

Solution. Our proofs in this question don't use the fact that X is of finite type at all, and this is not an oversight: the equivalence of these different characterizations of geometrically irreducible and geometrically reduced are general and do not require any finiteness hypotheses on the k-scheme in question. If you're looking for supplementary material or other takes on these proofs, the equivalence of these characterizations are treated in other texts: Stacks Project tackles them in tags 0364 and 035U, respectively; Vakil treats them in section 9.5 (as of the November 2017 draft), entitled 'Properties not preserved by base change, and how to fix them'; Qing Liu's *Algebraic Geometry and Arithmetic Curves* uses a slightly different approach making use of the finiteness conditions and only working with algebraic extensions in lemma 3.2.6, proposition 3.2.7, and remark 3.2.11 (you may also wish to consult math stack exchange question 3654825 for further details if you have trouble understanding Liu's reduction to the case of a finite extension of fields).

First, a couple preliminary lemmas:

Lemma. *If $X \to S$ is a surjective morphism of schemes and $S' \to S$ is any morphism of schemes, then $X' = X \times_S S' \to S'$ is also surjective. (We say surjectivity is preserved by base-change.)*

Proof. Let $s' \in S'$ with image $s \in S$, and $x \in X$ a point with image $s \in S$. By base-changing our entire diagram along the map $\operatorname{Spec} k(s) \to S$, we may assume that $S = \operatorname{Spec} k(s)$. Then by replacing $S' \to S$ with the composite morphism $\operatorname{Spec} k(s') \to S' \to S$ and $X \to S$ with the composite morphism $\operatorname{Spec} k(x) \to X \to S$, we may assmue that S, S', X are all spectra of fields $k(s), k(s'), k(x)$ with inclusions $k(s) \hookrightarrow k(s')$ and $k(s) \hookrightarrow k(x)$. Now if we can show that $\operatorname{Spec} k(s') \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(x)$ is nonempty, we'll know that X' has a point mapping to s': the canonical projections from the fiber product $\operatorname{Spec} k(s') \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(x)$ give a morphism to X' which when composed with $X' \to S'$ has s' in its image, and the nonempty image of $\operatorname{Spec} k(s') \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(x)$ in X' contains a point in the fiber $X'_{s'}$. Proving that $\operatorname{Spec} k(s') \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(x)$ is nonempty is equivalent to showing that $k(s') \otimes_{k(s)} k(x)$ is not the zero ring, and this is clear: as a vector space, it has dimension $(\dim_{k(s)} k(s'))(\dim_{k(s)} k(x))$, and both these quantities are nonzero. ∎

Lemma (Hilbert's Nullstellensatz, rephrased)**.** *Let k be an algebraically closed field and A a reduced commutative finitely generated k-algebra. Then for any nonzero $a \in A$, there exists a homomorphism $\varphi : A \to k$ with $\varphi(a) \neq 0$.*

Proof. Assume $a \neq 0$ and write $A = k[x_1, \cdots, x_n]/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$. Assume there is no such φ, and note that a must be in $\mathfrak{p}$ by theorem I.1.3A since it evaluates to zero at all maximal ideals containing $\mathfrak{p}$, so $a \in \mathfrak{p}$ or $a = 0 \in A$, contradicting our assumption. ■

Now on to the main events.

a. We'll first show that a scheme S is irreducible if and only if there exists an affine open covering $S = \bigcup_{i \in I} U_i$ so that I is nonempty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. The forward direction is clear: any open subset of an irreducible topological space is irreducible, and the generic point is in every open subset, so any affine open cover suffices. To show the reverse direction, suppose $S = Z_1 \cup Z_2$ is a union of two closed subsets. For every i, we see that $U_i \subset Z_1$ or $U_i \subset Z_2$, so pick some i and assume (up to possibly renumbering the Z) that $U_i \in Z_1$. For any $j \in I$, the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. So $U_j \subset Z_1$ as well, and thus $X = Z_1$.

   This implies we can check any of the definitions of geometric irreducibility of X affine-locally: as X is irreducible, we can cover X by finitely many affine open subschemes U_i which satisfy the conditions of the previous paragraph. Then $(U_i)_K$ are exactly the preimages of the U_i under $X_K \to X$, and we have that they're affine schemes which are all nonempty and have nonempty intersection. Thus if $(U_i)_K$ is irreducible, then the $(U_i)_K$ satisfy the conditions of the previous paragraph and thus X_K is irreducible. Conversely, if X_K is irreducible, then each $(U_i)_K$ is too, as any open subset of an irreducible space is irreducible. So X geometrically irreducible is equivalent to all of the U_i geometrically irreducible.

   Next, we make a topological observation: the image of an irreducible topological space under a surjective mapping is again irreducible. Combining this with the lemma that surjectivity is preserved by base change, we see that if Y is a scheme over F and $F \subset E$ is a field extension, then if Y is reducible Y_E must be as well: $\operatorname{Spec} E \to \operatorname{Spec} F$ is surjective, so $Y_E \to Y$ is surjective, and if Y_E were irreducible, then Y would necessarily be as well by the observation.

   Now we can address the problem at hand. It's clear that (iii) implies (i), and (i) implies (ii) by an application of our topological observation, the lemma that surjectivity is preserved by base change, and the fact that $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k^s$ is surjective.

   To show that (i) implies (iii), we use the Nullstellensatz to reduce the case of a general field extension to an algebraic field extension. By our reductions above, we assume that $X = \operatorname{Spec} A$ is affine and irreducible. If X_K fails to be irreducible, then $X_{\overline{K}}$ cannot be irreducible by our observation. So it is enough to prove that if $X_{\overline{k}}$ is irreducible, then $X_{\overline{K}}$ is irreducible. Next, we may assume that $X_{\overline{k}}$ is reduced: since taking the reduction is an isomorphism on underlying topological spaces, this doesn't affect reducibility. This implies that $A_{\overline{k}}$ is a domain. Now, if A_K is not a domain, that means there are nonzero elements $f, g \subset A_K$ so that $fg = 0$. Now let L be the sub-$\overline{k}$-algebra of K generated by the coefficients of f and g, and select a_f (resp. a_g) to be any nonzero coefficient of f (resp. g). L is reduced, commutative, and finitely generated over $\overline{k}$, so by our rephrasing of the nullstellensatz there exists a homomorphism $\varphi : L \to \overline{k}$ so that $\varphi(a_f a_g) \neq 0$. Then applying φ to the coefficients of

f and g, we get that $\varphi(f), \varphi(g) \neq 0$ in $A_{\overline{k}}$ so that $\varphi(f)\varphi(g) = 0$, contradicting our assumption that $A_{\overline{k}}$ was a domain. So (i) implies (iii).

To check that (ii) implies (i), we may assume that X is reduced, $k = k^s$, and write $A = k[x_1, \cdots, x_n]/I$ for I a prime ideal. Our goal is to show that $A_{\overline{k}} = \overline{k}[x_1, \cdots, x_n]/I_{\overline{k}}$ does not have elements f, g which are non-nilpotent but $fg = 0$. If $A_{\overline{k}}$ had such f, g, then the extension $k \subset F \subset \overline{k}$ generated by the coefficients of f, g is a finite purely inseparable extension, say of degree p^n (where $\operatorname{char} k = p$). Then $f^{p^n}, g^{p^n} \in A \subset A_{\overline{k}}$ and satisfy the same properties, but A was assumed to be an integral domain, contradiction. So (ii) implies (i) and we have shown the equivalence of conditions (i), (ii), and (iii).

b. First, the property of being reduced is affine local on X: X is reduced iff for all open covers of X by affine schemes $\operatorname{Spec} A_i$, all the A_i are reduced, which happens iff there exists a cover of X by affine opens $\operatorname{Spec} A_i$ with A_i reduced. For details, see the final paragraph of the solution of exercise II.3.11(d).

This implies that the property of being geometrically reduced is affine local, too: if $\{U_i\}$ is an affine cover of X and $k \subset K$ a field extension, then $\{(U_i)_K\}$ is an affine cover of X_K, so X_K reduced is equivalent to $(U_i)_K$ reduced for all i by the previous paragraph. Hereafter we assume $X = \operatorname{Spec} A$ is affine.

Now for an algebraic observation: if A is a k-algebra, $f \in A$ is a nonzero nilpotent, and $k \subset K$ is a field extension, then $f \otimes 1 \in A \otimes_k K$ is also nilpotent and nonzero because $k \subset K$ is a flat map of rings and thus $A \to A \otimes_k K$ is an injection. In particular, if A is non-reduced, then A_K cannot be reduced either, so we may assume that X is reduced.

As (iii) trivially implies (i) and (i) implies (ii) by our algebraic observation, we are left to prove that (i) implies (iii) and (ii) implies (i).

To show that (i) implies (iii), we use almost the same argument involving the nullstellensatz from (a). If X_K is nonreduced, then $X_{\overline{K}}$ is nonreduced, so it suffices to show that if $X_{\overline{k}}$ is reduced, then $X_{\overline{K}}$ must be reduced as well. If $X_{\overline{K}}$ is non-reduced, then there is a nonzero solution to $f^n = 0$ for some n as f ranges over elements of $A_{\overline{K}}$. The same argument as in (a) where we choose a homomorphism $\varphi : \overline{K} \to \overline{k}$ which when extended to $A_{\overline{k}} \to A_{\overline{k}}$ does not send f to zero gives a nonzero nilpotent in $A_{\overline{k}}$, contradicting our assumption that $A_{\overline{k}}$ was reduced. Thus (i) implies (iii).

To show that (ii) implies (i), we make further reductions. First, it suffices to consider $X = \operatorname{Spec} A$ irreducible: because A embeds in to $\prod A/\mathfrak{p}$ as $\mathfrak{p}$ runs over the minimal primes of A, $A_{\overline{k}}$ embeds in to $\prod (A/\mathfrak{p})_{\overline{k}}$ by flatness of the extension $k \subset \overline{k}$. So if $A_{\overline{k}}$ is nonreduced, one of $(A/\mathfrak{p})_{\overline{k}}$ must be nonreduced as well. Therefore we may assume that $k = k_p$ and A is a k-algebra which is an integral domain. Next, if there is some element $f \in A_{\overline{k}}$ which is a nonzero nilpotent, then letting $k \subset K$ be the finite separable subextension to which the coefficients of f belong we've reduced to showing the claim for a finite separable extension $k \subset K$. By the theorem of the primitive element, we can write $K = k[x]/(p(x))$ for some $p(x)$ a separable polynomial over k (that is, $p(x)$ has no multiple roots over any field extension of k). As K is a flat k-module, $A \otimes_k K$ embeds in to $\operatorname{Frac}(A) \otimes_k K = \operatorname{Frac}(A)[x]/(p(x))$, which by the

Chinese Remainder Theorem is a finite product of fields (since $p(x)$ factors in to a product of distinct irreducible polynomials over K by the definition of a separable polynomial). So A_K is reduced and we are done.

c. $\operatorname{Spec} \mathbb{R}[x]/(x^2+1)$ is integral, but it's base change to the algebraic closure is $\operatorname{Spec} \mathbb{C}[x]/(x^2+1) \cong \operatorname{Spec} \mathbb{C} \times \mathbb{C} = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$, which is reducible. $\operatorname{Spec} \mathbb{F}_p(t)[x]/(x^p - t)$ is integral, but it's base change to $\mathbb{F}_p(t^{1/p})$ is $\operatorname{Spec} \mathbb{F}_p(t^{1/p})[x]/((x - t^{1/p})^p)$, which is not reduced.

Exercise II.3.16. *Noetherian Induction.* Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

Solution. Let S be the set of all closed subsets V of X so that V is nonempty and does not have property $\mathcal{P}$. If S is nonempty, then by exercise I.1.7 S has a minimal element with respect to inclusion because X is noetherian. But this means that every proper closed subset of our minimal element has property $\mathcal{P}$, so by assumption our minimal element has property $\mathcal{P}$ and thus doesn't belong to S. So $S = \emptyset$ and X has property $\mathcal{P}$.

Exercise II.3.17. *Zariski Spaces.* A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9).

For example, let R be a discrete valuation ring, and let $T = \operatorname{sp}(\operatorname{Spec} R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\emptyset, \{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.

a. Show that if X is a noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.

b. Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points.*

c. Show that a Zariski space X satisfies the axiom T_0: given any two distinct points of X, there is an open set containing one but not the other.

d. If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.

e. If x_0, x_1 are points of a topological space X, and if $x_0 \in \overline{\{x_1\}}$, then we say that x_1 *specializes* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization* of x_1, or that x_1 is a *generization* of x_0. Now let X be a Zariski space. Show that the minimal points, for the particular ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say that closed subsets are *stable under specialization.*) Similarly, open subsets are *stable under generization.*

f. Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

Solution. Many people say 'generalization' instead of 'generization' for the relation described in (e). For instance, this document does.

a. By exercise II.2.9, every nonempty irreducible closed subset has a unique generic point. By exercise II.2.13, $\operatorname{sp}(\operatorname{Spec} R)$ is a noetherian topological space for any noetherian ring R, and any noetherian scheme has a finite cover by open affine noetherian schemes. So it suffices to prove that any topological space X with a finite cover by open noetherian topological spaces $\{U_i\}$ is again noetherian. This is straightforwards: given a descending chain of closed subsets $V_1 \supset V_2 \supset \cdots$, the intersection with each U_i is a descending chain of closed subsets of a noetherian topological space, so it must stabilize eventually at some n_i. Taking the maximum of these finite number of n_i, we see that our chain of closed subsets must stabilize, as two subsets of a topological space are equal iff their intersections with each element of an open cover are.

b. We show that for any closed subset $S \subset X$, S contains a closed point. Since closed subsets of Zariski topological spaces are again Zariski, we might as well replace S by X. Let $s_1 \in X$ be a point. Then $\overline{\{s_1\}}$ is an irreducible closed subset. If $\overline{\{s_1\}} = \{s_1\}$, then we're done. If not, take $s_2 \in \overline{\{s_1\}}$ with $s_2 \neq s_1$, and consider $\overline{\{s_2\}}$. By uniqueness of generic points, $\overline{\{s_2\}}$ is a proper subset of $\overline{\{s_1\}}$. If this process continued forever, then we would have an infinite descending chain of closed subsets of a noetherian topological space which didn't stabilize, a contradiction. So our chain stabilizes at some step, which means we find a point s_i so that $\overline{\{s_i\}} = \{s_i\}$ and we have a closed point.

c. Suppose $x, y \in X$ are distinct points. Consider $\overline{\{x\}}$ and $\overline{\{y\}}$: by the uniqueness of generic points, these are distinct closed subsets of X. If $x \in \overline{\{y\}}$, then the complement of $\overline{\{x\}}$ is an open subset containing y and not x. If $x \notin \overline{\{y\}}$, then the complement of $\overline{\{y\}}$ is an open subset containing x and not y, and we're done.

d. Let η be the generic point of X. Then if $U \subset X$ is a nonempty open subset of X not containing η, we have that U^c is a proper closed subset of X containing η, contradicting the fact that $\overline{\{\eta\}} = X$. So η belongs to every nonempty open subset.

e. If x is minimal with respect to the given order, then x is the only point in $\overline{\{x\}}$, and it is closed. If x is maximal with respect to the given order, then there's no irreducible closed subset properly containing x, as else the generic point of that subset would be a generalization of x. Since $\overline{\{x\}}$ is irreducible, this implies that $\overline{\{x\}}$ is a maximal irreducible closed subset, which is the same thing as an irreducible component by proposition I.1.5.

   To see that closed subsets are stable under specialization, let A be closed with some point $a \in A$. Since A is closed, it must contain $\overline{\{a\}}$, which means it contains any specialization of a. Conversely, if A is open and $a \in A$ has a' as a generalization, then $A \cap \overline{\{a'\}}$ is a nonempty open subset of $\overline{\{a'\}}$. Since $\overline{\{a'\}}$ is a Zariski space, it's generic point must be in this open subset by part (d), so we have that A contains a' and open subsets are stable under generalization.

f. As $t(X)$ is topologized by letting the closed subsets be $t(Y)$ where $Y \subset X$ is closed, we have that any descending chain of closed subsets in $t(X)$ comes from a descending chain of closed subsets of X, and as the latter must stabilize, the former must as well. Further, given an irreducible closed subset $Y \subset X$, we have that $Y \in t(X)$ is the generic point of $t(Y)$, and uniqueness is obvious.

For the final part of the problem, we clarify that the claim to be proven is that for noetherian X, X is Zariski iff $\alpha : X \to t(X)$ is a homeomorphism (it's maybe slightly oddly worded - the main point is that we have to assume X noetherian, else $\mathbb{Z}$ with the left-ray topology is a counterexample). As we've shown above that $t(X)$ is a Zariski space for X noetherian and the property of being Zariski is obviously a homeomorphism invariant, all we have to do is to show that if X is Zariski, then α is a homeomorphism. If X is Zariski, then every point is the generic point of it's closure (which is irreducible), and every irreducible subset has a unqiue generic point, which shows that α is a bijection. As $t(S)$ is defined to be closed for any closed subset $S \subset X$ and α is continuous as proven in proposition II.2.6, we have that α is a homeomorphism.

Exercise II.3.18. *Constructible Sets.* Let X be a Zariski topological space. A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

a. A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

b. Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

c. A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

d. If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

Solution.

a. Any closed set is in $\mathfrak{F}$: it's complement is open, so by condition (3) it belongs to $\mathfrak{F}$. Thus any locally closed subset is a member of $\mathfrak{F}$. To see that $\mathfrak{F}$ is closed under finite unions, write $A \cup B = (A^c \cap B^c)^c$, and as $\mathfrak{F}$ is closed under unions and finite intersections, we have the result. Thus any finite union of locally closed subspaces is constructible.

   Letting $\mathfrak{F}'$ be the collection of finite unions of locally closed subsets, we've shown $\mathfrak{F}' \subset \mathfrak{F}$. So to show that $\mathfrak{F}' = \mathfrak{F}$, it suffices to show that $\mathfrak{F}'$ satisfies conditions (1), (2), and (3), as $\mathfrak{F}$ is defined to be the smallest collection satisfiying these conditions. Clearly every open subset $U \subset X$ can be written as a locally closed subset: write $U = U \cap X$, where U is open and X is closed. Next, any finite intersection of locally closed subsets $\bigcap_{i=1}^n (U_i \cap V_i)$ with U_i open and V_i closed is locally closed: rewriting this as $(\bigcap_{i=1}^n U_i) \cap (\bigcap_{i=1}^n V_i)$, we see that since a finite intersection of open (resp. closed) subsets is again open (resp. closed) we have the desired result. To see that finite unions of locally closed subsets are closed under complements, note that as $(S \cup T)^c = S^c \cap T^c$, it's enough to prove it for the case of a single locally closed subset. For an open set U and a closed set V, we have that $(U \cap V)^c = U^c \cup V^c$, which is a finite union of locally closed subsets and we're done.

b. By (a), write our constructible set as a finite union of locally closed subsets $U_i \cap V_i$ for U_i open and V_i closed. Any V_i is a finite union of irreducible components since it's a closed subset of a noetherian topological space and thus again noetherian, so we may assume that each V_i is irreducible and $U_i \cap V_i \neq \emptyset$. Then as closures commute with finite unions, we have that $\overline{\bigcup (U_i \cap V_i)} = \bigcup V_i$ as any nonempty open subset of an irreducible set is dense. By irreducibility of X, we must have that V_{i_0} is actually X for some i_0.

   Now the conclusions follow easily: by exercise II.3.17(d), the generic point of X is contained in $U_{i_0} \cap V_{i_0}$, and as $V_{i_0} = X$, we have that our constructible set contains U_{i_0} which is open.

c. Closed and open subsets are both locally closed and thus constructible by (a). Closed subsets are stable under specialization and open subsets are stable under generalization by exercise II.3.17(e).

   Now assume S is constructible and stable under specialization. Write S as a finite union of locally closed subsets $U_i \cap V_i$ with each V_i irreducible as in (c), and let v_i be the generic point of V_i for all i. Since S is stable under specialization, S contains every point in $\overline{\{v_i\}}$ which implies that $S \supset \bigcup V_i$. On the other hand, for any point $s \in S$, it's contained in some V_i, so we have $S \subset \bigcup V_i$, and so $S = \bigcup V_i$ is a finite union of closed sets and thus closed.

   To see the result for S constructible and stable under specialization, we note that S^c is constructible and stable under specialization, so S^c is closed and thus S is open.

d. A continuous map is characterized by the fact that the preimage of any open (resp. closed) subset is open (resp. closed). As preimages commute with intersections and unions, we have that the preimage of any finite union of locally closed subsets is again a finite union of locally closed subsets, and by applying (a) twice, we have the desired result.

Exercise II.3.19. The real importance of the notion of constructible subsets derives from the following theorem of Chevalley - see Cartan and Chevalley [1, exposé 7] and see also Matsumura [2, Ch. 2, §6]: let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y. Prove this theorem in the following steps.

a. Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

b. (*) In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subset B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$. [*Hint*: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.]

c. Now use noetherian induction on Y to complete the proof.

d. Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

Solution. We remark that this theorem holds in greater generality than proven here if one is a little more careful about constructible sets. For instance, Stacks 054K gives the result where the morphism $f : X \to Y$ is only required to be quasi-compact and locally of finite presentation, which is much weaker than the hypotheses given here.

a. First, we note that open immersions and closed immersions are stable under base change (proof of exercise II.3.10 and exercise II.3.11). Further, any open or closed subscheme of a noetherian scheme is again noetherian: any open subscheme of a noetherian scheme is locally noetherian (proposition II.3.2) and quasi-compact (exercise I.1.7) and thus noetherian by definition; a closed immersion is of finite type by exercise II.3.13(a) and a map of finite type with noetherian target has noetherian source by exercise II.3.13(g). Finally, any open or closed immersion with a noetherian target is a morphism of finite type by exercise II.3.13(f) as any open immersion with noetherian target is quasi-compact as mentioned above. So any open or closed immersion $S \to X$ will yield a map $S \to Y$ between schemes which will preserve all our hypotheses, and so will any base change along an open or closed immersion.

   Our first step in our reduction is replacing a constructible set by a locally closed subscheme. As any constructible set is a finite union of locally closed subsets and set-theoretic images of morphisms commute with unions, it suffices to prove this for a single locally closed subscheme $U \cap V \subset X$.

   Now we can consider the composition

$$U \cap V \to V \to X \to Y$$

   where the first map is the open immersion of $U \cap V$ in to V and the second map is the closed immersion $V \to X$. By the logic of the first paragraph, the composite $U \cap V \to Y$ is a morphism of finite type of noetherian schemes, so it suffices to show that $f(X)$ is constructible.

   Next, as Y is noetherian, decompose Y as a finite union of irreducible components $\bigcup Y_i$. As set-theoretic images commute with unions, base extension along a closed immersion preserves our hypotheses, and $\bigcup X \times_Y Y_i = X$ as sets, it's enough to show that each induced map $X \times_Y Y_i \to Y_i$ has constructible image. So we can assume Y is irreducible.

Now cover Y by finitely many affine opens $\operatorname{Spec} A_i$ by quasi-compactness. As set-theoretic images commute with unions, base extension along an open immersion preserves our hypotheses, and $\bigcup X \times_Y \operatorname{Spec} A_i = X$ as sets, it's enough to show that each induced map $X \times_Y \operatorname{Spec} A_i \to \operatorname{Spec} A_i$ has constructible image. So we can assume $Y = \operatorname{Spec} A$ is affine.

As X is quasi-compact, so we can cover X by finitely many affine opens $\operatorname{Spec} B_i$, and as set-theoretic images commute with unions and open immersions preserve our hypotheses, we may assume $X = \operatorname{Spec} B$ is affine.

Penultimately, as all we care about is the underlying set, we can replace X with X_{red}, and then by exercise II.2.3 we can replace Y with Y_{red} in order to assume X and Y are reduced. Finally, $\operatorname{Spec} B \to \operatorname{Spec} A$ factors as

$$\operatorname{Spec} B \to \operatorname{Spec} A/\ker(A \to B) \to \operatorname{Spec} A$$

where the final map is a closed immersion and the first map is dominant by exercise II.2.18. The map $\operatorname{Spec} B \to \operatorname{Spec} A/\ker(A \to B)$ is of finite type by exercise II.3.13(f). The scheme $\operatorname{Spec} A/\ker(A \to B)$ is noertherian because any quotient of a noetherian ring is noetherian and any noetherian affine scheme is the spectrum of a noetherian ring (proposition II.3.2). So $\operatorname{Spec} B \to \operatorname{Spec} A/\ker(A \to B)$ is a dominant map of finite type between noetherian schemes, and if we can show that the image of any locally closed subset along the closed immersion $\operatorname{Spec} A/\ker(A \to B) \to \operatorname{Spec} A$ is again locally closed, then we'll have the claim for $\operatorname{Spec} B \to \operatorname{Spec} A$. Now suppose $i : S \to T$ is a closed immersion and $U \cap V$ is a locally closed subset with U open and V closed. Then there's an open $U' \subset T$ so that $U' \cap i(S) = i(U)$ by the subspace topology, and $i(V) = i(S) \cap i(V)$ is closed in T. So $i(U \cap V) = i(U) \cap i(V) = U' \cap i(S) \cap i(V) = U' \cap i(V)$ is locally closed, and therefore we may assume $X \to Y$ dominant.

The work above shows that we've made the necessary reductions: it's enough to show that $f(X)$ is constructible if X and Y are noetherian integral affine schemes with f a dominant morphism of finite type between them. (One should note that frequently these sorts of reductions go by in the blink of an eye. We present the full picture here in the interest of showing you how the sausage is made, and so that one day you too can gloss over most of these steps.)

b. We first show that the commutative algebra result implies that $f(X)$ contains a nonempty open subset of Y. Take $b = 1$ and a as guaranteed by the algebraic result. Since $a \neq 0$ and A is a domain, $D(a)$ is nonempty. Now I claim that $D(a)$ is a nonempty open subset of $f(X)$: pick some prime $\mathfrak{p} \in D(a)$, and let $K = \overline{\operatorname{Frac}(A/\mathfrak{p})}$, the algebraic closure of the residue field at $\mathfrak{p}$. Since the image of a in K is nonzero, we get a homomorphism $\varphi : B \to K$ extending the natural map $A \to K$, and $\mathfrak{q} = \ker \varphi$ is a prime ideal of B with preimage $\mathfrak{p} \subset A$. So $\mathfrak{q} \in \operatorname{Spec} B$ maps to $\mathfrak{p} \in \operatorname{Spec} A$, and as $\mathfrak{p} \in D(a)$ was arbitrary, we see that $D(a)$ is contained in $f(X)$.

To prove the algebraic result, we proceed by induction as per the hint. It's clear that we only need to prove the case when B is generated by a single element over A: if a general B is finitely generated, we can just consider a sequence of extensions adjoining one element each time and use the result from the one-variable case repeatedly. So suppose B is generated by one element x over A.

If x is transcendental over $\mathrm{Frac}(A)$ when considered as an element of $\mathrm{Frac}(B) \supset \mathrm{Frac}(A)$, write $b = \sum_{i=0}^n a_i x^i$ for $a_i \in A$ and $a_n \neq 0$ and let $\varphi : A \to K$ be a homomorphism from A to an algebraically closed field K with $\varphi(a) \neq 0$. Now let $\alpha \in K$ be any element which is not a root of $\sum_{i=0}^n \varphi(a_i) T^i \in K[T]$: then the map $\psi : B \to K$ given by sending $a \mapsto \varphi(a)$ and $x \mapsto \alpha$ is the desired map $B \to K$ with $\psi(b) \neq 0$ extending φ.

Now suppose x is algebraic over $\mathrm{Frac}(A)$ when considered as an element of $\mathrm{Frac}(B) \supset \mathrm{Frac}(A)$. Then $\mathrm{Frac}(B) \supset \mathrm{Frac}(A)$ is an algebraic extension, and so any nonzero $b \in B$ also satisfies a polynomial with coefficients in A, say $\sum_{i=0}^n a_i b^i = 0$ with $a_0, a_n \neq 0$. Let p, q be the constant terms of the polynomials with coefficients from A satisfied by x and b, respectively. I claim that $a = pq$ is the desired element of a. Let $\varphi : A \to K$ be a fixed homomorphism of A to an algebraically closed field with $\varphi(a) \neq 0$. Then by the universal property of localization, this extends to a homomorphism $\varphi_a : A_a \to K$, and it's enough to show that $\varphi_a : A_a \to K$ can be extended to $B_a \to K$, as this extension composed with the natural map $B \to B_a$ will give the required extension of φ to B.

In order to show that $\varphi_a : A_a \to K$ extends to B_a, we first note that $A_a \subset B_a$ is an integral extension of rings. This means that $A_a \subset B_a$ satisfies lying over, and so there's a prime ideal $\mathfrak{q} \subset B_a$ so that $\mathfrak{q} \cap A_a = \ker \varphi_a$ ($\ker \varphi_a$ is prime since φ_a is a homomorphism to a field). Now consider $A_a \to B_a \to B_a/\mathfrak{q}$: this has kernel $\ker \varphi_a$, and so we have an injective homomorphism $f : A_a/\ker \varphi_a \to B_a/\mathfrak{q}$ which makes the latter integral over the former. Taking fraction fields, we see that $\mathrm{Frac}(B_a/\mathfrak{q})$ is algebraic over $\mathrm{Frac}(A_a/\ker \varphi_a)$. But $\mathrm{Frac}(B_a/\mathfrak{q})$ is generated over $\mathrm{Frac}(A_a/\ker \varphi_a)$ by the image of x, and all we need to do to extend a map $\mathrm{Frac}(A_a/\ker \varphi_a) \to K$ to $\mathrm{Frac}(B_a/\mathfrak{q})$ is to decide where the image of x goes. Let $y \in K$ be an element satisfying the polynomial that b satisfies over A (we note that $y \neq 0$ by the fact that the image of p, the constant term of the polynomial in A satisfied by b, is nonzero under $A \to K$). It's clear that defining $\psi : \mathrm{Frac}(B_a/\mathfrak{q}) \to K$ by $\psi(x) = y$ gives us a well-defined extension, and we can check that the composite morphism $B \to B_a \to B_a/\mathfrak{q} \to \mathrm{Frac}(B_a/\mathfrak{q}) \to K$ doesn't send b to zero: if it did, then $q = 0$ and thus $a = 0$, contradicting the fact our morphism was an extension of $A \to K$ which didn't send a to zero.

c. By exercise II.3.16, it suffices to show that if $Y' \subset Y$ is a closed subset with every closed $Y'' \subset Y'$ satisfying $f(X) \cap Y''$ constructible, then $Y' \cap f(X)$ is constructible. It's enough to consider Y' irreducible, as constructible sets are closed under finite unions and any noetherian topological space has finitely many irreducible components.

If $f(X) \cap Y'$ is not dense in Y', then $\overline{f(X) \cap Y'}$ is a closed subset of Y', and by assumption $f(X) \cap Y'$ is constructible in $\overline{f(X) \cap Y'}$. But by the same logic as in our reduction to the dominant case in (a), we have that $f(X) \cap Y'$ is constructible in Y'. On the other hand, if $f(X) \cap Y'$ is dense in Y', then it contains a nonempty open subset U by (b). Suppose V is the closed complement of U: then by assumption $f(X) \cap V$ is closed in V, so we can write

$f(X) \cap Y' = U \sqcup (f(X) \cap V)$, which exhibits $f(X) \cap Y'$ as a constructible set. This finishes the proof by noetherian induction.

d. The canonical example of this is $\mathbb{A}^2_k \to \mathbb{A}^2_k$ by $(x, y) \mapsto (x, xy)$. The image of this is $\mathbb{A}^2$ without the y-axis but with the origin added back in. If the image were open or closed, it's intersection with any subscheme of $\mathbb{A}^2$ would be open or closed in the subspace topology. The intersection of the image with the line $x = 0$ is just the origin, which isn't open in the subspace topology on $\mathbb{A}^1$, so the image isn't open. The intersection of the image with the line $y = 1$ is all of $\mathbb{A}^1$ except the origin, which isn't closed in the subspace topology on $\mathbb{A}^1$, so the image isn't closed.

Exercise II.3.20. *Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

a. For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.

b. Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \operatorname{trdeg} K(X)/k$.

c. If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} \mid P \in Y\}$.

d. If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

e. If U is a nonempty open subset of X, then $\dim U = \dim X$.

f. If $k \subset k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

Solution.

a. Let $\operatorname{Spec} A$ and $\operatorname{Spec} B$ be two affine open subsets of X. I claim that $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$, which implies that $\dim X = \dim \operatorname{Spec} A$ for any affine open subset $\operatorname{Spec} A \subset X$ by exercise I.1.10(b). By exercise II.3.6, we have that $\operatorname{Frac}(A) \cong \operatorname{Frac}(B) \cong \mathcal{O}_{X,\xi}$ where the isomorphism is as k-algebras, and so by theorem I.1.8A we have that $\dim A = \dim B$ and thus $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

   Now let $\operatorname{Spec} A \subset X$ be an affine open subset containing P, which corresponds to a maximal ideal $\mathfrak{p} \subset A$. Then by theorem I.1.8A, we have that $\operatorname{ht} \mathfrak{p} = \dim A$ as $\dim A/\mathfrak{p} = 0$ because $A/\mathfrak{p}$ is a field. But $\operatorname{ht} \mathfrak{p} = \dim A_\mathfrak{p}$ since the primes of $A_\mathfrak{p}$ are the primes contained in $\mathfrak{p}$. As $\mathcal{O}_{X,P} \cong A_\mathfrak{p}$ since stalks don't depend on the neighborhood you take to calculate them, we have that $\dim A_\mathfrak{p} = \dim \mathcal{O}_{X,P}$ and thus $\dim \mathcal{O}_{X,P} = \dim X$, so we have the desired result.

b. This was shown in (a).

c. We first show the result in the case that Y is irreducible and that this infimum is achieved at the unique generic point of Y. This statement plus the fact that Y is noetherian immediately implies the result for reducible Y: we can just take the infimum over the finite collection of generic points of the irreducible components, and as the dimension of Y is the supremum of the dimensions of it's irreducible components, we're done.

To show the result in the irreducible case, we first note that because the claim is purely topological on Y, we may put the reduced induced subscheme structure on Y so that Y is an integral subscheme of X. Now we reduce to the case that X and thus Y are both affine: by the work we did in part (a), if $\operatorname{Spec} A \subset X$ is an affine open subset intersecting Y, then $\dim X = \dim X \cap \operatorname{Spec} A$, $\dim Y = \dim Y \cap \operatorname{Spec} A$, and $\mathcal{O}_{X,P} \cong \mathcal{O}_{\operatorname{Spec} A,P}$ for any point $P \in \operatorname{Spec} A$ as stalks don't depend on the open neighborhood used to calculate them.

In the case that $X = \operatorname{Spec} A$ is affine and Y is integral, Y is cut out by a prime ideal $\mathfrak{q} \subset A$, which is the generic point of Y. Any $P \in Y \subset X$ corresponds to a prime ideal $\mathfrak{p} \subset A$ containing $\mathfrak{q}$. It is therefore immediate that $\operatorname{ht} \mathfrak{q} \leq \operatorname{ht} \mathfrak{p}$, and so $\dim A_{\mathfrak{q}} \leq \dim A_{\mathfrak{p}}$. By an application of theorem I.1.8A, we see that $\operatorname{ht} \mathfrak{q} = \dim A - \dim A/\mathfrak{q} = \dim X - \dim Y = \operatorname{codim}(Y, X)$ and so we're done.

d. If Y is irreducible, this is theorem I.1.8A(b). If Y is not irreducible, then it has finitely many irreducible components because it's noetherian, and we can just pick the irreducible component of largest dimension and work with that as $\dim Y = \sup \dim Z$ where $Z \subset Y$ is closed irreducible.

e. Cover U by affine opens. By the logic of (a) we have that every affine open subset of X is of dimension $\dim X$, and thus by exercise I.1.10(b) we have that $\dim U = \dim X$.

f. It suffices to prove this in the case that X is affine: any covering of X by open affines $\operatorname{Spec} A$ will give a cover of $X_{k'}$ by open affines $\operatorname{Spec} A \otimes_k k'$, and so if we prove that $\operatorname{Spec} A \otimes_k k'$ has pure dimension $\dim X$, we'll be done. So fix $X = \operatorname{Spec} A$, and fix an embedding $X \hookrightarrow \mathbb{A}^n_k$, which is equivalent to choosing generators $R := k[x_1, \cdots, x_n] \to A$, exhibiting A as $k[x_1, \cdots, x_n]/I$ for some prime ideal I and $A \otimes_k K$ as $K[x_1, \cdots, x_n]/I \otimes_k K$.

When $k \subset k'$ is an algebraic extension, let $I' \subset K[x_1, \cdots, x_n]$ be a prime ideal lying above I. As $k[x_1, \cdots, x_n] \subset k'[x_1, \cdots, x_n]$ is an integral extension of normal domains, it satisfies lying over, incomparability, going up, and going down. This means that every prime ideal of $k'[x_1, \cdots, x_n]$ lying over I fits in to a chain of proper inclusions of prime ideals of $k'[x_1, \cdots, x_n]$ of length $\operatorname{ht} I$, and any chain of proper inclusions of prime ideals ending at a prime ideal lying over I restricts to a chain of proper inclusions of prime ideals of $k[x_1, \cdots, x_n]$ ending at I. In particular, every irreducible component of $X_{k'}$ is of the exact same dimension as X and we've shown the claim for $k \subset K$ algebraic.

For a general field extension $k \subset k'$, we may reduce to the case that k is algebraically closed. If $X_{k'}$ has an irreducible component not of dimension $\dim X$, then $X_{\overline{k'}}$ must also have such an irreducible component: letting Z be this component, $Z_{\overline{k'}}$ has the same dimension as Z by our proof of the algebraic case. Since $Z_{\overline{k'}}$ must lie entirely inside a single irreducible component of $X_{\overline{k}}$, we may assume that $X_{\overline{k}}$ is irreducible, and as taking reductions doesn't alter dimensions, we may assume that X is reduced and therefore integral. So if we can show the claim in the case $k = \overline{k}$, this will prove the claim in general.

If $k = \overline{k}$, then X is geometrically integral by exercise II.3.15, so $X_{k'}$ is integral, and in fact if $X = \operatorname{Spec} k[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ then $X_{k'} = \operatorname{Spec} k'[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ and we can talk about the field of fractions of $X_{k'}$. Now I claim that the transcendence degree of $K(X_{k'})$ over k' is the same as the transcendence degree of $K(X)$ over k. First, we show that $\operatorname{trdeg}_{k'} K(X_{k'}) \leq \operatorname{trdeg}_k K(X)$: up to a permutation of the x_i, the set $\{x_1, \cdots, x_d\}$ with $d = \dim X$ is a transcendence basis for $K(X)$, and x_i for $i > d$ is algebraic over $\{x_1, \cdots, x_d\}$ inside $K(X)$, therefore x_i for $i > d$ is also algebraic over $\{x_1, \cdots, x_d\}$ inside $K(X_{k'})$ by the same equation. To show the reverse inequality, suppose there is a nontrivial algebraic dependence relation among $\{x_1, \cdots, x_d\}$ inside $K(X_{k'})$: then by the same technique as in exercise II.3.15 involving the nullstellensatz, we get a nontrivial algebraic dependence relation among $\{x_1, \cdots, x_d\}$ inside $K(X)$, contradicting the assumption that $\{x_1, \cdots, x_d\}$ forms a transcendence basis for $K(X)$. By part (b), this shows that $X_{k'}$ is of dimension $\dim X$, and we are done.

Exercise II.3.21. Let R be a discrete valuation ring containing its residue field k. Let $X = \operatorname{Spec} R[t]$ be the affine line over $\operatorname{Spec} R$. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

Solution. Let $\mathfrak{m} = (u)$ be the maximal ideal of R. Then $\mathfrak{p} = (ut - 1)$ is a maximal ideal of $R[t]$, as $R[t]/(ut-1) = R_u = \operatorname{Frac}(R)$, but it's of height one so $\mathcal{O}_{\operatorname{Spec} R[t], \mathfrak{p}} \cong R[t]_{(ut-1)}$ is of dimension one while $\operatorname{Spec} R[t]$ is of dimension 2, which shows that (a) does not hold. For (d), consider $V(\mathfrak{p})$: this is the spectrum of a field, so it's a single point and has dimension zero, while $\mathfrak{p}$ is an ideal of height one and thus $V(\mathfrak{p})$ has codimension one. For (e), consider $\operatorname{Spec} \operatorname{Frac}(R)[t] \cong \operatorname{Spec} R[t]_u \subset \operatorname{Spec} R[t]$: this is open and 1-dimensional, but X is 2-dimensional.

Exercise II.3.22. (*) *Dimension of the Fibres of a Morphism.* Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.

a. Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$, such that $\eta' \in f(Z)$, and show that $\operatorname{codim}(Z, X) \leq \operatorname{codim}(Y', Y)$.

b. Let $e = \dim X - \dim Y$ be the *relative dimension* of X over Y. For any $y \in f(X)$, show that every irreducible component of the fibre X_y has dimension $\geq e$. [*Hint*: Let $Y' = \overline{y}$, and use (a) and (Ex. 3.20b).]

c. Show that there is a dense open subset $U \subset X$, such that for any $y \in f(U)$, $\dim U_y = e$. [*Hint*: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is a finitely generated B-algebra. Take $t_1, \cdots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \cdots, t_e]$. Then X_1 is isomorphic to affine e-space over Y, and the morphism $X \to X_1$ is generically finite. Now use (Ex. 3.7) above.]

d. Going back to our original morphism $f : X \to Y$, for any integer h, let E_h be the set of points $x \in X$ such that, letting $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geq h$. Show that (1) $E_e = X$ (use (b) above); (2) if $h > e$, then E_h is not dense in X (use (c) above); and (3) E_h is closed, for all h (use induction on $\dim X$).

e. Prove the following theorem of Chevalley - see Cartan and Chevalley [1, exposé 8]. For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains an open dense subset of Y.

Solution. Somehow we haven't yet proven that the base change of a finite morphism is again finite. We'll need this later, so let's prove it now.

Lemma. *If $f : X \to S$ is a finite morphism and $g : S' \to S$ is an arbitrary morphism, then $X' = X \times_S S' \to S'$ is finite as well. (We say that finite morphisms are stable under base change.)*

Proof. Let $U = \operatorname{Spec} A$ be an affine open subset of S, $f^{-1}(U) = \operatorname{Spec} B$ for B a finite A-module, and $U' = \operatorname{Spec} C$ be an affine open subset of S' with $g(U') \subset U$. Then the inverse image of U' in X' is $\operatorname{Spec} B \otimes_A C$, and $B \otimes_A C$ is a finite C-module because it's generated by $b \otimes 1$ as b runs over a generating set for B as an A-module. As U runs over all affine open subsets of S and U' runs over all affine open subsets contained in $g^{-1}(U)$, we obtain a cover of S' so that the induced morphism on each piece of the cover is finite, and we are done.

■

On to the main event.

a. Let us first recall a result from commutative algebra. For a noetherian local ring R with maximal ideal $\mathfrak{m}$ and dimension d, a *system of parameters* is a sequence of d elements $x_1, \cdots, x_d \in \mathfrak{m}$ so that $\sqrt{(x_1, \cdots, x_d)} = \mathfrak{m}$. Every noetherian local ring has such a system of parameters, and there is no sequence $y_1, \cdots, y_r$ with $r < d$ so that $\sqrt{(y_1, \cdots, y_r)} = \mathfrak{m}$. For a proof, see for instance Matsumura's *Commutative Ring Theory*, section 14.

For our problem, let ξ be the generic point of Z, and note that because $\eta' \in f(Z)$, we must have that $f(\xi) = \eta'$. This means that we get a local map of local rings $\varphi : \mathcal{O}_{Y,\eta'} \to \mathcal{O}_{X,\xi}$. Let $I = (x_1, \cdots, x_d) \subset \mathcal{O}_{Y,\eta'}$ be the ideal generated by a system of parameters for $\mathcal{O}_{Y,\eta'}$. Then $\sqrt{I\mathcal{O}_{X,\xi}}$ is the intersection of all prime ideals of $\mathcal{O}_{X,\xi}$ containing $I\mathcal{O}_{X,\xi}$, which is equivalent to the intersection of all prime ideals of $\mathcal{O}_{X,\xi}$ containing $\varphi(I)$, which is equivalent to all of the prime ideals of $\mathcal{O}_{X,\xi}$ who's inverse image under φ contains I. But the only prime ideal containing I in $\mathcal{O}_{Y,\eta'}$ is $\mathfrak{m}_{\eta'}$: such a prime ideal must contain $\sqrt{I} = \mathfrak{m}_{\eta'}$, which is maximal. Since φ is a local map of local rings, this implies that the only prime ideal containing I is $\mathfrak{m}_\xi$. So $\mathfrak{m}_\xi$ is a minimal prime over $(\varphi(x_1), \cdots, \varphi(x_d))$, and by Krull's Height Theorem (theorem I.1.11A), this says that $\operatorname{ht} \mathfrak{m}_\xi \leq d$. So $\dim \mathcal{O}_{X,\xi} \leq \dim \mathcal{O}_{Y,\eta'}$ and thus by exercise II.3.20(c) we have the result.

b. Let $Y' = \overline{\{y\}}$. Let $Z \subset X$, with generic point z, be an irreducible component of $f^{-1}(Y)$ so that $f(z) = y$. Such Z are in bijection with irreducible components Z_y of X_y, the fiber over y: taking the closure of z in the fiber gives an irreducible component, and taking the closure in X of the generic point of an irreducible component of the fiber gives such a Z. By part (a), we have that $\operatorname{codim}(Z,X) \leq \operatorname{codim}(Y',Y)$, and by exercise II.3.20(d) we have that $\operatorname{codim}(Z,X) = \dim X - \dim Z$ and $\operatorname{codim}(Y',Y) = \dim Y - \dim Y'$, so rearranging we get

$$e = \dim X - \dim Y \leq \dim Z - \dim Y'.$$

From $f(z) = y$, we obtain an extension of residue fields $k \subset k(y) \subset k(z)$, and by the properties of transcendence degree, we have that $\operatorname{trdeg} k(z)/k(y) = \operatorname{trdeg} k(z)/k - \operatorname{trdeg} k(y)/k$. By exercise II.3.20(b) we have that $\operatorname{trdeg} k(z)/k - \operatorname{trdeg} k(y)/k$ is exactly $\dim Z - \dim Y$, while $\operatorname{trdeg} k(z)/k(y)$ is the dimension of $Z \cap X_y$ as a scheme over $k(y)$ by the same reasoning. Combining this with the inequality proven in the previous paragraph, we see that every irreducible component of X_y is of dimension at least e.

c. The reduction to the affine case is clear: pick an affine open subset $U \subset Y$ and an affine open subset $V \subset f^{-1}(U) \subset X$, then a dense open subset of V is a dense open subset of X because X is irreducible. Letting $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we have that the corresponding map $B \to A$ is injective because f is dominant, so we may assume that B is a subalgebra of A. Following the hint, choose $t_1, \cdots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \cdots, t_e]$ and let $h : X_1 \to Y$ denote the obvious map. Then X_1 is isomorphic to affine e-space over Y, and the morphism $g : X \to X_1$ is generically finite because the inverse image of the generic point of X_1 is the spectrum of a finite $K(X_1)$-module. We also observe that as $B[t_1, \cdots, t_e] \to A$ is an injection, $g : X \to X_1$ is dominant, so we may apply exercise II.3.7 to see that there is an open subset $V \subset X_1$ so that $g : g^{-1}(V) \to V$ is a finite map. Taking the fiber over $y \in h(V)$, we get $(g^{-1}(V))_y \to \mathbb{A}^e_{k(y)} \to \operatorname{Spec} k(y)$ where the first map is finite because it's the base change of a finite map (cf the lemma we proved at the start).

Next, I claim that a finite map of finite type integral k-schemes $\pi : S \to T$ gives $\dim S \leq \dim T$. We may assume S, T are affine, and that $T = \overline{\pi(S)}$ so that $S \to T$ is dominant, as the inclusion $\overline{\pi(S)} \to T$ clearly maintains the inequality and $S \to \overline{\pi(S)}$ is again finite (it's the base change of $S \to T$ along the closed immersion $\overline{\pi(S)} \to T$). So $k[T] \to k[S]$ is an injective, finite map of integral domains, and thus $k(T) \subset k(S)$ is a finite extension of fields, and thus their transcendence degrees over k are equal, so we have proven the claim by exercise II.3.20(b).

The preceding paragraph gives that any irreducible component of $(g^{-1}(V))_y$ is of dimension at most e. By (b), all the irreducible components of $(g^{-1}(V))_y$ are of dimension at least e, so we have shown the existence of U as requested.

d. The first claim is true by an application of part (b). The second claim follows by an application of part (c): E_h is contained in the complement of the dense open set $U \subset X$ coming from (c), so E_h isn't dense. For the third claim, let X' be the complement of the dense open subset $U \subset X$ coming from part (c). Then $X' = \bigcup X_i$ is a finite union of irreducible subsets, and by putting the reduced induced structure on each X_i as well as $\overline{f(X_i)} \subset Y$, we can get that $f|_{X_i} : X_i \to \overline{f(X_i)}$ is a dominant map between finite type integral schemes over k where $\dim X_i < \dim X$. As the E_h associated to $X \to Y$ is the union of the E_h associated to each

$X_i \to \overline{f(X_i)}$ and the latter E_h are closed by our inductive hypothesis, we see that E_h is indeed closed, being a finite union of closed sets. (The base case of the induction is $\dim X = 0$ which implies that X and Y are both points, and the claim is clear there.)

e. This is an application of Chevalley's theorem on constructible sets to the conclusions obtained in (d). First, note that $f(E_h)$ is exactly the points y of Y where the fiber X_y has dimension at least h. So $C_h = f(E_h) \setminus f(E_{h+1})$, and by exercise II.3.19 $f(E_h)$ is constructible, so C_h is as well.

   To show that C_e contains an open dense subset of Y, it is enough to show that C_e contains the generic point η of Y by exercise II.3.19. As $E_e = X$, we have $\eta \in f(E_e)$ because f is dominant, so we need to show that η is not in $f(E_h)$ for any $h > e$. The generic point of X is the generic point of X_η, so the fiber over the generic point of Y is irreducible. But U from (c) is dense in Y, so $\eta \in f(U)$ and thus $U \cap X_\eta \neq \emptyset$, so $\dim X_\eta = \dim U_\eta = e$ and we are done.

Exercise II.3.23. If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in (I, Ex. 3.15, 3.16), and if t is the functor of (2.6), then $t(V \times W) = t(V) \times_{\operatorname{Spec} k} t(W)$.

Solution. First we show the result for V, W affine with coordinate algebras $k[V], k[W]$ respectively. Then by the proof of proposition II.2.6, we have that $t(V) \cong \operatorname{Spec} k[V]$ and $t(W) \cong \operatorname{Spec} k[W]$, so $t(V) \times_k t(W) \cong \operatorname{Spec} k[V] \otimes_k k[W]$. On the other hand by exercise I.3.15(b), we have that $V \times W$ has coordinate algebra $k[V] \otimes_k k[W]$, so $t(V \times W) \cong \operatorname{Spec} k[V] \otimes_k k[W]$, and so $t(V) \times_k t(W) \cong t(V \times W)$.

To see the result in the general case, we note that the projection maps $V \times W \to V$ and $V \times W \to V$ agree when composed with projection to a point $\mathbb{A}^0_k$, so they give maps $t(V \times W) \to t(V)$ and $t(V \times W) \to t(W)$ which agree with the projection maps from $t(V)$ and $t(W)$ to a point $t(\mathbb{A}^0_k) = \operatorname{Spec} k$. By the universal property of the fiber product, this means we have an induced map $t(V \times W) \to t(V) \times_k t(W)$ and we wish to show that this map is an isomorphism.

As $V \times W$ is covered by open affine subvarieties of the form $V_i \times W_j$ for $V_i \subset V$ open affine and $W_j \subset W$ open affine, we have that $t(V \times W)$ is covered by $t(V_i \times W_j)$. But $t(V_i \times W_j) \cong t(V_i) \times_k t(W_j)$ by our result in the affine case, and by construction $t(V) \times_k t(W)$ is covered by $t(V_i) \times_k t(W_j)$. So $t(V) \times_k t(W)$ is covered by $t(V_i \times W_j)$, which also cover $t(V \times W)$, and the gluing data is the same by the construction of the fiber product. So these schemes are isomorphic, and $t(V \times W) \cong t(V) \times_k t(W)$.

(Alternatively, one may show that $t(V \times W)$ fulfills the same universal property as $t(V) \times_k t(W)$, but in order to make this work one must show that $t(V) \times_k t(W)$ is in the essential image of t, so it's not just a one-liner. This extra step is important: for instance, if we take the fiber product of $V(y)$ and $V(y - x^2)$ over $\mathbb{A}^2$ as varieties in the sense of chapter I, we'll get $V(x, y)$ which gives $\operatorname{Spec} k$ after applying t, while if we take the fiber product in schemes, we'll get $\operatorname{Spec} k[x, y]/(y, x^2)$, which isn't reduced.)

II.4 Separated and Proper Morphisms

Hartshorne's extra noetherian assumptions start to show up in full force in this section. Many results in this section are true without that hypothesis (or under weaker hypotheses, like quasi-separatedness), and sometimes this hypothesis is dropped in the exercises (for instance, the first exercise can be reduced to an algebra exercise relatively quickly with the valuative criteria, except for the fact that the valuative criteria in Hartshorne demands noetherianness, while the exercise doesn't). We'll take the exercises at face value (that is, if it doesn't say noetherian, we won't assume it) and add comments about how noetherian assumptions interact with the task at hand as necessary.

There's also something interesting going on with valuation rings in this section. A general valuation ring is an absolutlely wild thing, and it seems to me that we've lost some facility as a subject with valuation rings - Zariski did a lot with them, but that level of expertise does not really appear present these days. If you want to learn more, it seems like Zariski and Samuel's *Commutative Algebra* is a good source, but be warned that it's not written in the language of schemes and it can be a bit of a challenge to read and connect to how we usually talk about modern algebraic geometry.

Exercise II.4.1. Show that a finite morphism is proper.

Solution. A proper morphism is finite type, separated, and universally closed. We can see that a finite morphism is of finite type by definition. To show separatedness, I claim that any morphism $f : X \to Y$ with $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ for $\operatorname{Spec} B \subset Y$ an affine open subscheme is separated (this is the claim that an affine morphism, which will be introduced in exercise II.5.17, is separated). Cover Y with affine opens $\operatorname{Spec} B$: then $f^{-1}(\operatorname{Spec} B) \times_Y f^{-1}(\operatorname{Spec} B) = f^{-1}(\operatorname{Spec} B) \times_{\operatorname{Spec} B} f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A$ cover $X \times_Y X$. Since a morphism of affine schemes is separated by proposition II.4.1, we have that $\Delta_{X/Y} : X \to X \times_Y X$ is locally a closed immersion on each of these sets, and since a closed immersion can be checked locally, we see that $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion and thus $X \to Y$ is separated.

To show that a finite morphism is universally closed, we note that we've proven a finite morphism is closed in exercise II.3.5(b), so it suffices to prove that finite morphisms are stable under base change. This is direct: assume $f : X \to Y$ is finite, let $g : Y' \to Y$ be any morphism, and denote the natural morphisms $X' = X \times_Y Y' \to Y'$ and $X' \to X$ by f' and g', respectively. Pick an arbitrary affine open $\operatorname{Spec} B \subset Y$ and let $\operatorname{Spec} R \subset g^{-1}(\operatorname{Spec} B) \subset Y'$ be an affine open subscheme. Then $f'^{-1}(\operatorname{Spec} R) = \operatorname{Spec} R \times_Y X = \operatorname{Spec} R \times_{\operatorname{Spec} B} f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} R \times_{\operatorname{Spec} B} \operatorname{Spec} A = \operatorname{Spec} R \otimes_B A$, where we've used the fact that $f^{-1}(\operatorname{Spec} B)$ is an affine scheme $\operatorname{Spec} A$ by the fact $f : X \to Y$ is finite. As $R \otimes_B A$ is finitely generated as an R-module because A is finitely generated as a B-module, we have that $f'^{-1}(\operatorname{Spec} R) = \operatorname{Spec} R \otimes_B A$ is a finite R-module and thus $X' \to Y'$ is finite. So finite morphisms are stable under base change, which means they are universally closed, and thus finite morphisms are proper.

Exercise II.4.2. Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated. [*Hint*: Consider the map $X \to Y \times_S Y$ obtained from f and g.]

Solution. This is a classic of the genre. Consider the following commutative diagram, where $V = X \times_{Y \times_S Y} Y$ is the fiber product:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_{Y/S}"] \\ X \arrow[r, "f \times_S g"] & Y \times_S Y \end{tikzcd}$$

Since Y is separated over S, $\Delta_{Y/S}$ is a closed immersion. As closed immersions are stable under base change, we have that $V \to X$ is a closed immersion. Since f, g agree on an open dense subset $U \subset X$, we get a pair of morphisms $i : U \to X$ and $f \circ i : U \to X \to Y$ which agree when composed with the maps to $Y \times_S Y$, so we have a morphism $U \to V$ such that $U \to V \to X$ is the inclusion i. So $V \to X$ is a closed immersion who's image contains an open dense set, and thus the set-theoretic image of $V \to X$ is all of X. Since X is reduced, this means that $V = X$ as schemes and thus $f = g$.

To find a counterexample when X is nonreduced, let $S = \operatorname{Spec} k$, $X = Y = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ with f, g the maps induced by the identity and the map sending $\varepsilon \mapsto 0$. Then f, g give the same map on topological spaces (the identity map) and the same map on residue fields (the identity map), but $f \neq g$ by the equivalence between the category of affine schemes and commutative rings.

To find a counterexample when Y is nonseparated, let Y be the affine line with a double origin and let X be the affine line. Then there are two distinct inclusions of X in to Y which agree on the complement of the origin, but do not agree on X.

Exercise II.4.3. Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. The $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Solution. $U \times_S V$ is an open affine subset of $X \times_S X$. As the diagonal is a closed subscheme of $X \times_S X$, it's intersection with $U \times_S V$ is a closed subscheme of $U \times_S V$, which by exercise II.3.11 is affine. But by the lemma we proved in exercise II.3.10, $U \times_S V$ is exactly $U \cap V$.

To see a counterexample, let X be an affine plane with a doubled origin given by gluing $\mathbb{A}^2$ and $\mathbb{A}^2$ along $\mathbb{A}^2 \setminus \{(0,0)\}$. Then the two copies of $\mathbb{A}^2$ give two open affine subsets of X which intersect in a copy of $\mathbb{A}^2 \setminus \{(0,0)\}$ which isn't affine by exercise I.3.6.

Exercise II.4.4. Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S. We refer to this result by saying that 'the image of a proper scheme is proper'. [*Hint*: Factor f into the graph morphism $\Gamma_f : X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

Solution. As $Z \to S$ is proper and $Y \to S$ is separated, we have that any S-morphism $Z \to Y$ is proper by corollary II.4.8(e) (we also note this claim is true without any noetherian hypotheses whatsoever, see for instance Stacks 01W6). As proper morphisms are closed, we have that the image of Z inside Y is closed.

Next, $f(Z)$ with it's image subscheme structure is finite type over S: it's a closed subscheme of Y, which means $f(Z) \to Y$ is a closed immersion. As closed immersions are of finite type, $Y \to S$ is finite type by assumption, and compositions of morphisms of finite type are again of finte type, we see $f(Z) \to S$ is of finite type. To see $f(Z)$ is separated over S, we apply corollary II.4.6 parts (a) and (b) (we note that noetherian hypotheses are not needed here - reference Stacks 01L1 and 01KU, respectively). It remains to check that $f(Z) \to S$ is universally closed.

Now we show that if $f : X \to Y$ is a surjective morphism of schemes over S with the structure morphism $p : X \to S$ universally closed, then the structure morphism $q : Y \to S$ is also universally closed. Consider an arbitrary morphism $S' \to S$, and the base change $f' : X_{S'} \to Y_{S'}$ where we denote the structure morphisms $X_{S'} \to S'$ and $Y_{S'} \to S'$ by p' and q', respectively. As surjective morphisms are stable under base extension (via a lemma proven in the solution to II.3.15), we have that $f' : X_{S'} \to Y_{S'}$ is surjective, and $p' : X_{S'} \to S'$ is closed by assumption. For any closed $T \subset Y_{S'}$, we have that $q'(T) = p'(f'^{-1}(T))$, and as f' is continuous and p' is closed, we see that $q'(T)$ is closed and we're done.

Since $Z \to f(Z)$ is a surjective morphism of S-schemes, we have that $f(Z) \to S$ is universally closed and thus $f(Z)$ is proper over S.

Exercise II.4.5. Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has *center* x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.

a. If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.

b. If X is proper over k, then every valuation of K/k has a unique center on X.

c. (*) Prove the converses of (a) and (b). [*Hint*: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.]

d. If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a). [*Hint*: Let $a \in \Gamma(X, \mathcal{O}_X)$ with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in \mathfrak{m}_R$. Then use (b) to get a contradiction.]

Note. If X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

Solution.

a. Let R be the ring of all elements of non-negative valuation. If $x \in X$ is a center for our valuation, then $\mathcal{O}_{X,x} \subset R$ by definition, which gives a map $\operatorname{Spec} R \to X$ sending $\mathfrak{m}_R$ to x and (0) to η, the generic point. By theorem II.4.3 with $Y = \operatorname{Spec} k$, we see there can be at most one such x, so the center of a valuation is unique if it exists.

b. The same logic as (a) except applying theorem II.4.7 instead of theorem II.4.3 gives the desired result.

c. This problem is challenging. There is a proof in EGA II 7.3.10 which uses Chow's lemma which both difficult and a future exercise. We present an alternate proof not using the machinery of Chow's lemma. The main idea is that if $X \to Y$ is a proper dominant morphism of integral schemes over k, then X satisfies the condition on valuations iff Y does. From there, we can replace X by its normalization X' and show that every closed integral subscheme of X' of codimension one satisfies the condition on valuations, implying every closed integral subscheme of X of codimension one satisfies the condition on valuations by the lemma. By downwards induction, this implies that every closed integral subscheme of X satisfies the condition on valuation rings. This shows that X satisfies the valuative criteria, finishing the problem.

On to the proof. First, a preliminary about valuation rings:

Lemma. *If $A \subset K$ is a subring of a field, then A is a valuation ring iff for every nonzero $x \in K$, at least one of x and x^{-1} belongs to A.*

Proof. The forward direction is clear: $x \cdot x^{-1} = 1$, so $v(x) + v(x^{-1}) = 0$ in the value group and therefore at least one of $v(x), v(x^{-1})$ is non-negative. For the reverse direction, if $A \neq K$, then A has a nonzero maximal ideal $\mathfrak{m}$. If there's another maximal ideal $\mathfrak{m}'$, then we can find $x \in \mathfrak{m}$, $y \in \mathfrak{m}'$ with $x \notin \mathfrak{m}'$ and $y \notin \mathfrak{m}$. Then neither x/y or y/x can be in A, contradicting the asumption, so A has a unique maximal ideal. Now suppose A' is a local ring dominating A, and suppose $x \in A'$ - we have to show $x \in A$. If not, then $x^{-1} \in A$ and in fact $x^{-1} \in \mathfrak{m}$, so $x, x^{-1} \in A'$. But this means that x^{-1} goes from being in the maximal ideal of A to being a unit in A', which is impossible because A' dominates A. The claim is proven.

∎

To solve the problem, we'll show that the conditions about centers of valuations on $k(X)/k$ give the appropriate conditions for the valuative criteria for separatedness and properness. We start by explaining how to connect the left side of the diagram in the valuative criteria to valuations of $k(X)/k$. Let R be a valuation ring with field of fractions L and suppose we have the standard commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Let z denote the unique point in the image of $\operatorname{Spec} L \to X$, and let Z be the closure of z equipped with the reduced induced subscheme structure. Then Z is an integral subscheme of X with generic point z and function field $k(Z) = \kappa(z)$, the residue field at z. The map $\operatorname{Spec} L \to X$ gives us an inclusion $\kappa(z) \subset L$, and we let $S = \kappa(z) \cap R$. It is straightforward to check that S is a valuation ring: for any element $a \in \kappa(z) = \operatorname{Frac}(S)$, we have that considering a as an element of L, we have that either $a \in R$ or $a^{-1} \in R$, so either $a \in S$ or $a^{-1} \in S$. So our diagram can be rewritten as follows:

$$\begin{array}{ccccc} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} \kappa(z) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} S & \longrightarrow & \operatorname{Spec} k \end{array}$$

and therefore it's enough to check the valuative criteria on valuation rings R with fields of fractions L the residue field of a point z in X.

In the case where z is the generic point of X, this gives the answer immediately: R is a valuation ring of $k(X)$, so by assumption it has at most one (respectively, a unique) center $x \in X$, which means that there exists at most one (respectively, a unique) lifting $\operatorname{Spec} R \to X$ making the relevant diagram commute by lemma II.4.4. To show the result in general, we will prove that if X is an integral scheme of finite type over a field so that every valuation on $k(X)/k$ has at most one (respectively, a unique) center on X, then the same is true for every integral closed subscheme $Z \subset X$. We can reduce this further to proving that if X satisfies the condition on valuations, then every closed integral subscheme $Z \subset X$ of codimension one does as well by downward induction. To show this, we start with a lemma.

Lemma. *Let $f : X \to Y$ be a proper dominant (equivalently, proper surjective) morphism of integral schemes over k. Every valuation on $k(X)/k$ has at most one (respectively, a unique) center on X iff the same is true for valuations of $k(Y)/k$ and Y.*

Proof. We have four things to prove:

(i) If every valuation on $k(X)/k$ has at most one center, then every valuation on $k(Y)/k$ has at most one center;

(ii) If every valuation on $k(X)/k$ has a center, then every valuation on $k(Y)/k$ has a center;

(iii) If every valuation on $k(Y)/k$ has at most one center, then every valuation on $k(X)/k$ has at most one center;

(iv) If every valuation on $k(Y)/k$ has a center, then every valuation on $k(X)/k$ has a center.

Let R be a valuation ring for $k(Y)/k$. As $f : X \to Y$ is dominant, it maps the generic point of X to the generic point of Y and thus induces an injection of fields $k(Y) \hookrightarrow k(X)$ which we may assume to be an inclusion. Let R' be a valuation ring of $k(X)$ dominating $R \subset k(X)$. (We note that this also implies that $R' \cap k(Y) = R$: if $k(Y) \cap R'$ contained an element e not in R, then e^{-1} is in R, therefore $\mathfrak{m}_{R'} \cap R \neq \mathfrak{m}_R$ which contradicts the fact that R' dominates R.) By lemma II.4.4, a center y for R on Y is equivalent to R dominating $\mathcal{O}_{Y,y}$, which implies R' dominates $\mathcal{O}_{Y,y}$ as subrings of $k(X)$. This means we have a valuative diagram

$$\begin{array}{ccc} \operatorname{Spec} k(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & Y \end{array}$$

and by the valuative criteria for properness, we have a unique lifting $\operatorname{Spec} R' \to X$. Thus for every center $y \in Y$ of R, we get a unique center $x \in X$ of R'. So if there is at most one center in X for all valuation rings R' of $k(X)/k$, then there is at most one center in Y for any valuation ring R on Y, and (i) is proven.

To show that the existence of a center for all valuation rings R' of $k(X)/k$ implies the existence of a center for all valuation rings of $k(Y)/k$, let R and R' be as in the previous paragraph. As R' has a center on X by assumption, we get a map $\operatorname{Spec} R' \to X$ by lemma II.4.4, and composing with the map $X \to Y$, we obtain a map $\operatorname{Spec} R' \to Y$. I claim that $\operatorname{Spec} R' \to Y$ factors through $\operatorname{Spec} R \to Y$. This can be seen from examining the maps on local rings: letting $x \in X$ be the image of the closed point of $\operatorname{Spec} R'$ and $y \in Y$ the image of x, we have a sequence of local maps of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \to R'$ which are all injections because the maps on function fields are. But $\mathcal{O}_{Y,y}$ lands inside $k(Y) \subset k(X)$, which means it lands inside $k(Y) \cap R'$, which is exactly R. Therefore R dominates $\mathcal{O}_{Y,y}$, so y is a center for R and (ii) is proven.

Now suppose R' is a valuation ring for $k(X)/k$ with two centers x, x'. Then by the logic above, the images of x and x' must be centers for $R = R' \cap k(Y)$, and x and x' cannot map to the same point, otherwise this would violate our result that every center for R uniquely determines a center for R'. Therefore we've proven (iii) by contrapositive.

Finally, if R' is a valuation ring for $k(X)/k$, then $R = R' \cap k(Y)$ is a valuation ring for $k(Y)/k$, and so if R has a center, we get a valuative diagram as above. As $X \to Y$ is proper, we get a lifting $\operatorname{Spec} R \to X$, and considering the composition $\operatorname{Spec} R' \to \operatorname{Spec} R \to X$, we see that R' has a center on X by lemma II.4.4, so we've proven (iv) and we're done.

■

Now recall the construction of the normalization from exercise II.3.8: for a X an integral k-scheme of finite type, we let X' denote the normalization, which comes with a natural dominant map $\nu : X' \to X$ which is finite in our case. By exercise II.4.1, a finite map is proper, so the map $\nu : X' \to X$ verifies the conditions of the lemma (we also note ν is surjective).

Let $Z \subset X$ be a codimension one integral closed subvariety. As finite and surjective morphisms are stable under base change (see the solutions for exercise II.3.15 and II.3.22, respectively), we have that $Z \times_X X' \to Z$ is finite and surjective. Since finite morphisms are closed, each irreducible component of $Z \times_X X'$ must map to a closed irreducible subset of Z, and therefore by surjectivity there must be an irreducible component of $Z \times_X X'$ which surjects on to Z. Let $Z' \subset X'$ be such an irreducible component equipped with the reduced induced subscheme structure. As $Z' \to Z \times_X X'$ is a closed immersion, it is finite, so the composite $Z' \to Z$ is a finite surjective morphism, and as finite morphisms are proper by exercise II.4.1, this satisfies the conditions of our lemma. So it suffices to show that if Z is a closed codimension one integral subscheme of a normal integral scheme X of finite type over a field, then the condition about valuations on X implies the condition about valuations on Z.

Let $Z \subset X$ as described in the previous sentence, and let $z \in Z$ be the generic point of Z. Note that $\mathcal{O}_{X,z}$ is a DVR by theorem I.6.2A: it's a noetherian local domain of dimension one which is integrally closed. Suppose $R \subset k(Z) = \kappa(z)$ is a valuation ring trivial on k, and let $q : \mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ be the natural quotient map. Define $S = q^{-1}(R) \subset \mathcal{O}_{X,z}$. I claim S is a valuation ring. Let $e \in k(X)$ be an arbitrary nonzero element. As at least one of e, e^{-1} are in $\mathcal{O}_{X,z}$, we may assume e is actually in $\mathcal{O}_{X,z}$. If $e \in \mathfrak{m}_z$, then $e \in S$. If $e \notin \mathfrak{m}_z$, then $e, e^{-1} \in \mathcal{O}_{X,z}$ and so at least one of $q(e)$ or $q(e^{-1}) = q(e)^{-1}$ is in R, which implies that one of e or e^{-1} is in S, which implies it is a valuation ring.

Suppose $z' \in Z$ is a center for R on Z. Then $\mathcal{O}_{X,z'} = q^{-1}(\mathcal{O}_{Z,z'})$ is dominated by S, so z' is a center for S on X. As $Z \to X$ is injective on underlying sets, this implies that if R has two distinct centers on Z, S must have two distinct centers on X. This shows that if every valuation on $k(X)/k$ has at most one center on X, then every valuation on $k(Z)/k$ has at most one center on Z. Now suppose $z' \in X$ is a center for S: then $\mathcal{O}_{X,z'} \subset S \subset \mathcal{O}_{X,z}$, so $z' \in \overline{z} = Z$ and taking the quotient by the maximal ideal of $\mathcal{O}_{X,z}$ we see that $\mathcal{O}_{Z,z'}$ is dominated by R. We are done.

d. Let $a \in \Gamma(X, \mathcal{O}_X)$ so that $a \notin k$. As k is algebraically closed, we have that a is transcendental over k considered as an element of K, so $k[a^{-1}]$ is a polynomial ring. The localization $k[a^{-1}]_{(a^{-1})}$ is a local ring contained in K, so there exists a valuation ring $R \subset K$ dominating it. As $\mathfrak{m}_R \cap k[a^{-1}]_{(a^{-1})} = (a^{-1})$, we see that $a^{-1} \in \mathfrak{m}_R$.

By considering the diagram in the valuative criteria for properness and taking global sections, we find a unique map $\Gamma(X, \mathcal{O}_X) \to R$ so that $\Gamma(X, \mathcal{O}_X) \to K$ is the same as the composite $\Gamma(X, \mathcal{O}_X) \to R \to K$. But this implies that $a \in R$, while $a^{-1} \in \mathfrak{m}_R$, so we simultaneously have that a has non-negative valuation and a^{-1} has positive valuation, contradiction.

Exercise II.4.6. Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism. [*Hint*: Use (4.11A).]

Solution. This problem can be generalized significantly: the result which is always true no matter what assumptions we have on X and Y is that a morphism being finite is equivalent to being simultaneously affine and proper. It's also worth pointing out (again) that 'affine variety over k' here means an integral scheme of finite type over k - without this assumption, the hint to use II.4.11A is complicated by the fact that non-integral schemes don't have fields of fractions, and one is best served by attacking the problem in an alternate fashion. (See EGA2 6.7.1 or Stacks 01WN for a proof of the claim in this generality.)

Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, with the map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to a ring homomorphism $\varphi : A \to B$. If we can show that an arbitrary element $b \in B$ is integral over A, then we can conlcude that B is finite over A, as a map of rings which is of finite type and integral is module-finite. Now let R be any valuation ring of $K(B)$ containing $\varphi(A)$, and consider the diagram given from taking global sections of the diagram in the valuative criteria for properness:

$$\begin{tikzcd} K(B) & B \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & A \arrow[l] \arrow[u, "\varphi"'] \end{tikzcd}$$

So every $b \in B$ must be in R by commutativity of the diagram, and thus every $b \in B$ is integral over $\varphi(A)$, and so B is integral over A and we're done.

Exercise II.4.7. *Schemes Over* $\mathbb{R}$. For any scheme X_0 over $\mathbb{R}$, let $X = X_0 \times_{\mathbb{R}} \mathbb{C}$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to $\mathbb{C}$. Then X is a scheme over $\mathbb{C}$, and σ is a *semi-linear* automorphism, in the sense that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \mathbb{C} \arrow[r, "\alpha"] & \mathbb{C} \end{tikzcd}$$

Since $\sigma^2 = id$, we call σ an *involution.*

a. Now let X be a separated scheme of finite type over $\mathbb{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over $\mathbb{R}$, such that $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbb{R}} \mathbb{C}$ described above.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbb{R}$, and X, σ will denote the corresponding scheme with involution over $\mathbb{C}$.

b. Show that X_0 is affine if and only if X is.

c. If X_0, Y_0 are two such schemes over $\mathbb{R}$, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

d. If $X \cong \mathbb{A}^1_{\mathbb{C}}$, then $X_0 \cong \mathbb{A}^1_{\mathbb{R}}$.

e. If $X \cong \mathbb{P}^1_{\mathbb{C}}$, then either $X_0 \cong \mathbb{P}^1_{\mathbb{R}}$, or X_0 is isomorphic to the conic in $\mathbb{P}^2_{\mathbb{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

Solution.

a. We tackle the affine case first. If $X = \operatorname{Spec} A$ is an affine $\mathbb{C}$-scheme, then we can check directly that $X^0 = \operatorname{Spec} A^\sigma$ works, by demonstrating that $A^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong A$. To do this, define a map $A \to A^\sigma \otimes_{\mathbb{R}} \mathbb{C}$ by $a \mapsto \frac{a+\sigma(a)}{2} \otimes 1 + \frac{a-\sigma(a)}{2} \otimes i$ and a map $A^\sigma \otimes_{\mathbb{R}} \mathbb{C} \to A$ by $s \otimes (x+iy) = xs + iys$.

It is immediate to see that these are mutually inverse, so $A^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong A$, and this is clearly unique.

For the general case, start by picking a finite σ-invariant affine open cover $\{U_i = \operatorname{Spec} A_i\}$: for any point $x \in X$, we can find an affine open subset U containing x and $\sigma(x)$ by assumption, and then as X is separated we can apply exercise II.4.3 to see that $U \cap \sigma(U)$ is affine and sigma invariant by construction. Then as X is of finite type over $\mathbb{C}$, it's quasi-compact, so we can refine our cover to a finite cover by σ-invariant affine opens. Now the intersections $U_{ij} = \operatorname{Spec} A_{ij}$ of these affine opens are affine again by exercise II.4.3 and σ-invariant by construction, and the immersions $\operatorname{Spec} A_{ij} \to \operatorname{Spec} A_i$ from these intersections are intertwiners for the σ action. We therefore have that the gluing data comes from $\mathbb{R}$-morphisms $A_i^\sigma \to A_{ij}^\sigma$, and by exercise II.2.12, we can use this gluing data to construct X_0. X_0 is finite type over $\mathbb{R}$ because it's covered by finitely many spectra of finitely-generated $\mathbb{R}$-algebras, uniqueness follows from the uniqueness for affines and the uniqueness of gluing, so all that's left to do is to check separatedness.

By corollary II.4.2, it suffices to check that the image of $\Delta_0 : X_0 \to X_0 \times_{\mathbb{R}} X_0$ is closed. As $X \to \mathbb{C}$ is separated, $\Delta : X \to X \times_{\mathbb{C}} X$ is a closed immersion, and it's also the base change of $\Delta_0 : X_0 \to X_0 \times_{\mathbb{R}} X_0$ along $X \times_{\mathbb{C}} X \to X_0 \times_{\mathbb{R}} X_0$. But $X \times_{\mathbb{C}} X \to X_0 \times_{\mathbb{R}} X_0$ is the base change of $X_0 \times_{\mathbb{R}} X_0 \to \operatorname{Spec} \mathbb{R}$ along $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$: a choice of maps from a test scheme T to $X_0 \times_{\mathbb{R}} X_0$ and $\operatorname{Spec} \mathbb{C}$ which agree after composition with the maps to $\operatorname{Spec} \mathbb{R}$ is equivalent to a triple of maps $T \to X_0$, $T \to X_0$, and $T \to \operatorname{Spec} \mathbb{C}$ agreeing after composition with the maps to $\operatorname{Spec} \mathbb{R}$, which is equivalent to a pair of maps $T \to X_0 \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ and $T \to X_0 \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ agreeing after composition with the maps to $\operatorname{Spec} \mathbb{C}$, or a map $T \to X \times_{\mathbb{C}} X$. As $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is finite, $X \times_{\mathbb{C}} X \to X_0 \times_{\mathbb{R}} X_0$ is finite as well, so it's closed. Therefore the image of Δ under the map $X \times_{\mathbb{C}} X \to X_0 \times_{\mathbb{R}} X_0$ is closed, which is exactly the image of Δ_0, because $X \to X_0$ is surjective.

We note that this problem is a baby case of Galois descent, itself a baby case of fpqc descent - one good introduction to this subject in greater generality is Poonen's 'Rational Points on Varieties', available on his website. Specifically, we're proving that 'separated descends along fpqc coverings'.

b. If X_0 is affine, then $X = X_0 \times_{\mathbb{R}} \mathbb{C}$ is clearly affine. If X is affine, then as X_0 is unique and we constructed an affine X_0 in (a), we must have that X_0 is affine too.

c. Any morphism $X_0 \to Y_0$ induces a morphism $X \to Y$ by base change. To show the reverse direction, cover Y by σ-invariant affine opens, and then cover the preimage of each such open by σ-invariant affine opens in X using the procedure we described in (a). Then a map commuting with the involutions gives a collection of ring maps commuting with the involutions and thus a collection of ring maps of the rings of invariants which glue back together to a map $X_0 \to Y_0$.

d. If $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A^\sigma$, and $\mathbb{A}^1_{\mathbb{C}} \cong \operatorname{Spec} \mathbb{C}[t]$, so it remains to show that no matter what the action of σ on t is, we have $\mathbb{C}[t]^\sigma \cong \mathbb{R}[u]$.

Since $\sigma^2 = id$, we must have that $\sigma(t) = at+b$ for some nonzero $a \in \mathbb{C}$, and our goal is to find a $u = xt+y$ with $x \neq 0$ so that $\sigma(u) = u$: then $\mathbb{C}[t] \cong \mathbb{C}[u]$ and the action of σ on $\mathbb{C}[u]$ is the 'standard' complex conjugation on $\mathbb{C}[u]$ fixing u. If $a \neq -1$, then $u = t + \sigma(t) = (1+a)t + b$ works; if $a = -1$, then $it + \sigma(it) = it + (-i)(-t+b) = 2it - ib$ works. So $\mathbb{C}[t]^\sigma \cong \mathbb{R}[u]$ and thus $(\mathbb{A}^1_\mathbb{C})^\sigma \cong \mathbb{A}^1_\mathbb{R}$.

e. The key here is that we can tell which case we're in by whether σ fixes any closed point or not. I claim that if σ fixes a closed point, then $X_0 \cong \mathbb{P}^1_\mathbb{R}$ and if not, $X_0 \cong V(x_0^2 + x_1^2 + x_2^2)$.

   If σ fixes a closed point, then it fixes the complement of that closed point, which is isomorphic to $\mathbb{A}^1_\mathbb{C}$. By part (d), this shows that σ fixes another point in $\mathbb{A}^1_\mathbb{C}$ and by another application of the previous sentence, we have that X can be covered by two σ-invariant copies of $\mathbb{A}^1_\mathbb{C}$. Their intersection is a copy of $\mathbb{A}^1_\mathbb{C} \setminus \{p\}$ which is stable under the action of σ, and up to changes of coordinates we can assume $p = 0$ so that the intersection is $\operatorname{Spec} \mathbb{C}[t, t^{-1}]$ with σ acting by $\sigma(t) = t$. These morphisms are exactly the gluing data of $\mathbb{P}^1_\mathbb{R}$, so $X_0 \cong \mathbb{P}^1_\mathbb{R}$.

   In the case that σ does not fix any closed points, we can pick a point p and via a change of coordinates send $p, \sigma(p)$ to $0, \infty$. Then σ fixes the complement of these points, so σ acts on $\operatorname{Spec} \mathbb{C}[t, t^{-1}]$ fixing no points. As the units of $\mathbb{C}[t, t^{-1}]$ are exactly at^k for $a \neq 0$ and $k \in \mathbb{Z}$ and σ must send t to a unit, we see that $\sigma(t) = at^k$. As $\sigma^2 = id$, we see that $k^2 = 1$. If $k = 1$, then the valuation ring $\mathbb{C}[t]_{(t)} \subset K(\mathbb{P}^1_\mathbb{C})$ would be fixed, implying σ fixes a closed point of $\mathbb{P}^1_\mathbb{C}$, contrary to our assumption. So $k = -1$.

   Now we determine a. As $t \cdot \sigma(t) = a$ is fixed by σ, we see that $a \in \mathbb{R}$. If $a > 0$, then the ideal $(t - \sqrt{a}) = (\sqrt{a} - at^{-1})$ is preserved by σ, as $\sigma(t - \sqrt{a}) = at^{-1} - \sqrt{a}$, which contradicts the assumption of no fixed points. So $a < 0$, and by the change of coordinates $t = \frac{1}{\sqrt{-a}} t$ we can take σ to be $\sigma(t) = -t^{-1}$.

   To compute invariants, we first note that σ stabilizes the $\mathbb{C}$-vector space on t^n and t^{-n}. When $n \neq 0$, this is a 4-real-dimensional vector space with basis $t^n, it^n, t^{-n}, it^{-n}$, and σ acts as

$$\begin{pmatrix} 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & (-1)^{n+1} \\ (-1)^n & 0 & 0 & 0 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}$$

   which after a bit of linear algebra gives that σ stabilizes $it^n + it^{-n}$ and $t^n - t^{-n}$ when n is odd, and stabilizes $t^n + t^{-n}$ and $it^n - it^{-n}$ when n is even. I claim that the invariants are generated as an $\mathbb{R}$-algebra by $t - t^{-1}$ and $it + it^{-1}$. This can be seen by induction: $(t-t^{-1})^n = t^n + (-1)^n t^{-n}$ and $(it+it^{-1})(t-t^{-1})^{n-1} = it^n - (-1)^n it^{-n}$ up to terms with exponent of absolute value less than n which are also invariant under σ. So the algebra of invariants is $\mathbb{R}[t - t^{-1}, it + it^{-1}]$, and the minimal relation between these is that $(t - t^{-1})^2 + (it + it^{-1})^2 + 4 = 0$. Thus up to scaling, our algebra of σ-invariants is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$. Embedding in $\mathbb{P}^2$ and taking the closure, we see that we get the conic $V(X^2 + Y^2 + Z^2) \subset \mathbb{P}^2_\mathbb{R}$ and we've finished.

Exercise II.4.8. Let $\mathscr{P}$ be a property of morphisms of schemes such that:

a. a closed immersion has $\mathscr{P}$;

b. a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

c. $\mathscr{P}$ is stable under base extension.

Then show that:

d. a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

e. if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;

f. if $f : X \to Y$ has $\mathscr{P}$, then $f_{red} : X_{red} \to Y_{red}$ has $\mathscr{P}$.

[*Hint*: For (e), consider the graph morphism $\Gamma_f : X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$.]

Solution. This is a fun exercise in coming up with the right diagrams. One note: any isomorphism is a closed immersion, so every isomorphism has property $\mathcal{P}$.

d. We'll do everything in the relative situation, since it doesn't cost us anything: assume all schemes are over a base scheme S. Suppose $X \to Y$ and $X' \to Y'$ are morphisms of S-schemes each with property $\mathcal{P}$.

$$\begin{tikzcd} X \times_S X' \arrow[r, "\mathcal{P}"] \arrow[d] & X \times_S Y' \arrow[r, "id"] \arrow[d] & X \times_S Y' \arrow[r, "\mathcal{P}"] \arrow[d] & Y \times_S Y' \arrow[d] \\ X' \arrow[r, "\mathcal{P}"] & Y' & X \arrow[r, "\mathcal{P}"] & Y \end{tikzcd}$$

Then $X \times_S Y' \to Y \times_S Y'$ is the base change of $X \to Y$ along $Y \times_S Y' \to Y$, and similarly $X \times_S X' \to X \times_S Y'$ is the base change of $X' \to Y'$ along $X \times_S Y' \to Y'$, so both $X \times_S X' \to X \times_S Y'$ and $X \times_S Y' \to Y \times_S Y'$ have property $\mathcal{P}$. Thus their composition $X \times_S X' \to Y \times_S Y'$ must also have property $\mathcal{P}$.

e. First I claim that $\Gamma_f : X \to X \times_Z Y$ is the base change of $\Delta_{Y/Z} : Y \to Y \times_Z Y$ along $f \times_Z id_Y : X \times_Z Y \to Y \times_Z Y$. We show that (X, f, Γ_f) satisfies the universal property: given morphisms from a test object $q_1 : Q \to Y$ and $q_2 : Q \to X \times_Z Y$, the condition that they agree when composed with the maps to $Y \times_Z Y$ gives us a unique map $Q \to X$ via $Q \overset{q_2}{\to} X \times_Z Y \overset{\pi}{\to} X$.

As g is separated, $\Delta_{Y/Z} : Y \to Y \times_Z Y$ is a closed immersion and thus has property $\mathcal{P}$. Since Γ_f is the base-change of $\Delta_{Y/Z}$, we have that Γ_f has property $\mathcal{P}$. Now as $X \to Z$ and $id_Y : Y \to Y$ have property $\mathcal{P}$, we have by (d) that $X \times_Z Y \to Z \times_Z Y$ has property $\mathcal{P}$ as well. But $Z \times_Z Y$ is naturally isomorphic to Y, and thus as compositions of maps with property $\mathcal{P}$ have property $\mathcal{P}$, we have that $X \to Y$ has property $\mathcal{P}$.

f. $X_{red} \to X$ and $Y_{red} \to Y$ are both closed immersions and thus have property $\mathcal{P}$. Therefore the composite morphism $X_{red} \to X \to Y$ has property $\mathcal{P}$. Since closed immersions are affine morphisms (exercise II.3.11) and affine morphisms are separated (proof of exercise II.4.1), $Y_{red} \to Y$ is separated as well. By exercise II.2.3, the morphism $X_{red} \to X \to Y$ factors as $X_{red} \to Y_{red} \to Y$ since X_{red} is reduced. But as $Y_{red} \to Y$ is separated, we may apply (f) to see that $X_{red} \to Y_{red}$ has property $\mathcal{P}$ and we're done.

Exercise II.4.9. Show that a composition of projective morphisms is projective. [*Hint*: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$.] Conclude that projective morphisms have properties (a)-(f) of (Ex. 4.8) above.

Solution. Assume $X \to Y$ and $Y \to Z$ are projective; that is, $X \to Y$ factors as $X \to \mathbb{P}^n_Y \to Y$ and similarly for $Y \to Z$ (where by $\mathbb{P}^n_Y$ we mean $\mathbb{P}^n \times Y$). Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \mathbb{P}^n_Y \arrow[r] \arrow[d] & \mathbb{P}^n \times \mathbb{P}^m_Z \arrow[d] \\
 & Y \arrow[r] \arrow[dr] & \mathbb{P}^m_Z \arrow[d] \\
 & & Z
\end{tikzcd}$$

The triangles on the left follow from the definition of of projective morphisms. The square in the middle is a fiber square by the logic in exercise II.4.8(d), so $\mathbb{P}^n_Y \to \mathbb{P}^n \times \mathbb{P}^m_Z$ is a closed immersion. As the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ is given by a surjective morphism of graded rings per exercise I.2.14, it is a closed immersion by exercise II.3.12(a), and so its base change $\mathbb{P}^n \times \mathbb{P}^m_Z \cong \mathbb{P}^m \times \mathbb{P}^n \times Z \to \mathbb{P}^{nm+n+m} \times Z \cong \mathbb{P}^{nm+n+m}_Z$ is too. As compositions of closed immersions are closed immersions, we have that $X \to \mathbb{P}^{nm+n+m}_Z$ is a closed immersion and thus $X \to Z$ factors as $X \to \mathbb{P}^{nm+n+m}_Z \to Z$ where the first map is a closed immersion and the second is a projection, and thus $X \to Z$ is projective.

The preceding paragraph shows that projective morphisms satisfy part (b) of exercise II.4.8 A closed immersion $X \to Y$ is a projective morphism because we can write it $X \to \mathbb{P}^0_Z \to Z$, so they satisfy part (a). Further, if $X \to Y$ is projective and we base extend by $Y' \to Y$, then $X \times_Y Y' \to \mathbb{P}^n_Y \times_Y Y' \cong \mathbb{P}^n_{Y'}$ is a closed immersion because closed immersions are stable under base extension, and the map $\mathbb{P}^n_Y \times_Y Y' \to Y \times_Y Y'$ is still the projection, so they satisfy part (c). Thus by exercise II.4.8 projective morphisms have properties (d)-(f) and we're done.

Exercise II.4.10. (*) *Chow's Lemma.* This result says that proper morphisms are fairly close to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subset X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.

a. Reduce to the case X irreducible.

b. Show that X can be covered by a finite number of open subsets U_i, $i = 1, \cdots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.

c. Let $U = \bigcap U_i$, and consider the map

$$f : U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure (Ex. 3.11d) $f(U)$. Let $g : X' \to X$ be the projection on the the first factor, and let $h : X' \to P = P_1 \times_S \cdots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

d. Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

Solution.

a. Since X is finite type over a noetherian scheme, it is again noetherian and thus has finitely many irreducible components X_i. Now I claim that for each X_i we can find a proper S-scheme Y_i and an S-morphism $Y_i \to X_i$ so that $Y_i \to X_i$ is surjective and an isomorphism on $X_i \setminus \cup_{j \neq i} X_j$: let Y_i be the scheme-theoretic image of the open immersion $X_i \setminus \cup_{j \neq i} X_j \to X$. Since X is noetherian, the open subscheme $X_i \setminus \cup_{j \neq i} X_j$ is quasi-compact, and therefore the scheme-theoretic image can be computed affine-locally on the target by our solution to exercise II.3.11(d). This makes it clear that Y_i has underlying set exactly X_i and is isomorphic to X_i over $X_i \setminus \cup_{j \neq i} X_j$. Since Y_i is a closed subscheme of a proper S-scheme X, it is again proper.

   Finally, if we can prove the lemma for each Y_i by producing a projective S-scheme Y_i and a morphism $g_i : Y_i' \to Y_i$ which is an isomorphism on a dense open set of Y_i, then the composition of $\coprod g_i : \coprod Y_i' \to \coprod Y_i$ and $\coprod Y_i \to X$ is a map from a projective S-scheme to X which induces an isomorphism over a dense open subset of X. Therefore we've reduced to the case where X is irreducible.

b. Since S is noetherian, it is quasi-compact and we can cover S by finitely many affine opens S_i. Then by the assumption that $X \to S$ is proper, it is of finite type, so we can cover the preimage of each S_i in X by finitely many affine opens X_{ij}, each of which have a closed immersion in to $\mathbb{A}^{n_{ij}}_{S_i}$. Composing this with the open embeddings $\mathbb{A}^{n_{ij}}_{S_i} \to \mathbb{P}^{n_{ij}}_{S_i}$ and $\mathbb{P}^{n_{ij}}_{S_i} \to \mathbb{P}^{n_{ij}}_{S}$, we have a closed immersion of each X_{ij} into an open subscheme of a projective S-scheme. As X is noetherian, the X_{ij} are noetherian, which means any morphism out of any X_{ij} is quasi-compact; therefore by our solution to exercise II.3.11(d), the scheme theoretic image of $X_{ij} \to \mathbb{P}^{n_{ij}}_{S}$ is a closed subscheme P_{ij} of $\mathbb{P}^{n_{ij}}_{S}$ which has as its underlying set the closure of the set-theoretic image of X_{ij}. Thus X_{ij} admits an open immersion in to P_{ij}, which shows that it is quasi-projective, and we've shown that X can be covered by finitely many U_i, each with an open immersion to a projective S-scheme P_i.

c. One key fact we'll use here is that the scheme-theoretic image of any morphism out of U can be computed locally, and it has as its underlying set the closure of the set-theoretic image of U. This follows from our solution to exercise II.3.11(d): since U is noetherian, every morphism out of it is quasi-compact.

Let us start by fixing some notation as in the following diagram:

$$\begin{tikzcd} & & X \\ U \arrow[r, "j"] & X' \arrow[ur, "g"] \arrow[r, "f'"] \arrow[dr, "h"'] & X \times_S P \arrow[u, "\pi_X"'] \arrow[d, "\pi_P"] \\ & & P \end{tikzcd}$$

where j is the induced map from U to the scheme-theoretic image X', f' is the obvious closed immersion, and π_X and π_P are the projections from the fiber product. Additionally, let $\pi_i : P \to P_i$ denote the projection on to the i^{th} factor.

The first phase of the proof is to construct a convenient open cover of X'. Let $W_i = \pi_i^{-1}(U_i) \subset P$ (for fixed i, this is just the product of U_i and P_j for all $j \neq i$). I claim $h^{-1}(W_i)$ cover X': it suffices to show that $g^{-1}(U_i) \subset h^{-1}(W_i)$, because the open sets $g^{-1}(U_i)$ cover X' as the open sets U_i cover X by assumption. Consider the following diagram:

$$\begin{tikzcd} g^{-1}(U_i) \arrow[r, "h|_{g^{-1}(U_i)}"] \arrow[d, "g|_{g^{-1}(U_i)}"'] & P \arrow[d, "\pi_i"] \\ U_i \arrow[r] & P_i \end{tikzcd}$$

Since $U \subset g^{-1}(U_i) \subset X'$ and U is set-theoretically dense in X' by our key fact, we have that U is set-theoretically dense in $g^{-1}(U_i)$. As our goal is to show the set-theoretic containment $g^{-1}(U_i) \subset \pi_P^{-1}(W_i)$, we can replace $g^{-1}(U_i)$ with its reduction, which doesn't affect the underlying maps of topological spaces and then apply exercise II.4.2 to see that the two ways of traversing the square are the same on the level of topological spaces. Therefore $h^{-1}(W_i) = h^{-1}(\pi_i^{-1}(U_i))$ contains $g^{-1}(U_i)$, and our claim is proven.

The second phase of the proof is to verify that $h|_{h^{-1}(W_i)} : h^{-1}(W_i) \to W_i$ is a closed immersion. Consider the graph morphism $U_i \to X \times_S U_i$: since X is separated, this is a closed immersion. By base changing along the map $W_i \to U_i$, we get a closed immersion $\alpha_i : W_i \to X \times_S W_i$. On the other hand, we see that $f : U \to X \times_S P$ factors through $X \times_S W_i$ by definition and is contained in the closed subscheme $\alpha_i(W_i) \subset X \times_S W_i$. Therefore the scheme-theoretic image of $U \to X \times_S W_i$ must have a closed immersion in to the graph $\alpha_i(W_i)$, and since the scheme theoretic image can be computed locally on $X \times_S P$, we have that $f'|_{h^{-1}(W_i)} : h^{-1}(W_i) \to X \times_S W_i$ factors through W_i. The situation can be seen in the following commutative diagram:

$$\begin{tikzcd}
h^{-1}(W_i) \arrow[r, hook, "cl"] \arrow[rr, bend left, "f'|_{h^{-1}(W_i)}"] \arrow[drr, "h|_{h^{-1}(W_i)}"'] & W_i \arrow[r, hook, "\alpha_i"] \arrow[dr] & X \times_S W_i \arrow[d, "\pi_P|_{X \times_S W_i}"] \\
& & W_i
\end{tikzcd}$$

As the composition $\pi_P|_{X\times_S W_i} \circ \alpha_i$ is the identity on W_i (it's the composition of a graph morphism and the projection to the domain), we have that $h^{-1}(W_i) \to W_i$ which is a closed immersion.

This gives us enough to conclude that h is a closed immersion as follows. Since h is the composition of the proper morphisms f' and π_P, h is proper and therefore closed. By exercise II.3.11, in order to show that h is a closed immersion it suffices to verify that h is a closed immersion locally along each W_i, as $P \setminus h(X')$ together with the W_i form an open cover of P and $\emptyset \to P \setminus h(X')$ is a closed immersion. But this is exactly what we showed in the previous paragraph and therefore h is a closed immersion and X' is projective over S.

d. To check that $h|_{h^{-1}(U)} : h^{-1}(U) \to U$ is an isomorphism, we first note that $f : U \to X \times_S P$ factors as
$$U \to U \times_S P \to X \times_S P,$$
where the first map is the graph of $U \to P$ induced from the open immersions $U \to U_i \to P_i$ for all i and the second map is the open immersion induced by $U \to X$. Since P is separated over S, the graph of the S-morphism $U \to P$ is closed in $U \times_S P$ and is in fact isomorphic to U via the projection $U \times_S P \to U$. Combining this with the observation that the scheme-theoretic image can be computed locally, we get that $f'(X') \cap (U \times_S P)$ is the graph of $U \to P$, or $f(X') \cap (U \times_S P) \cong U$.

Exercise II.4.11. If you are willing to do some harder commutative algebra, and stick to noetherian schemes, then we can express the valuative criteria of separatedness and properness using only *discrete* valuation rings.

a. If $\mathcal{O}, \mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$. Prove this in the following steps. By taking a polynomial ring over $\mathcal{O}$, reduce to the case where L is a *finite* extension field of K. Then show that for a suitable choice of generators $x_1, \cdots, x_n$ of $\mathfrak{m}$, the ideal $\mathfrak{a} = (x_1)$ in $\mathcal{O}' = \mathcal{O}[x_2/x_1, \cdots, x_n/x_1]$ is not equal to the unit ideal. Then let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$, and let $\mathcal{O}'_{\mathfrak{p}}$ be the localization of $\mathcal{O}'$ at $\mathfrak{p}$. This is a noetherian local domain of dimension 1 dominating $\mathcal{O}$. Let $\widetilde{\mathcal{O}'}_{\mathfrak{P}}$ be the integral closure of $\mathcal{O}'_{\mathfrak{p}}$ in L. Use the theorem of Krull-Akizuki (see Nagata [7, p. 115]) to show that $\widetilde{\mathcal{O}'}_{\mathfrak{P}}$ is noetherian of dimension 1. Finally, take R to be a localization of $\widetilde{\mathcal{O}'}_{\mathfrak{P}}$ at one of its maximal ideals.

b. Let $f : X \to Y$ be a morphisms of finite type of noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all *discrete* valuation rings.

Solution. One helpful interpretation of this result is the following: the spectrum of a DVR can be thought of as a little neighborhood of a point in a curve and the spectrum of it's field of fractions can be thought of as that neighborhood minus the point. So separatedness is saying that if you can fill in a punctured section of a curve, you can do so uniquely, and properness says that you can always do so uniquely. This is sort of like saying 'limits along curves are unique/exist and are unique'.

a. We assume that $\mathcal{O}$ is not a field, otherwise this whole thing is a bit silly.

   Suppose $L = K(t_1, \cdots, t_n)$. Then some subset of the t_i is a transcendence basis for L/K, so up to permutation we may assume it's $t_1, \cdots, t_r$. Then $\mathcal{O}[t_1, \cdots, t_r]$ is a polynomial algebra over $\mathcal{O}$, and thus noetherian by Hilbert's basis theorem. Localizing at $(\mathfrak{m}, t_1, \cdots, t_r)$ returns the noetherian local domain $\mathcal{O}[t_1, \cdots, t_r]_{(\mathfrak{m}, t_1, \cdots, t_r)} \subset K(t_1, \cdots, t_r)$ which dominates $\mathcal{O}$, and as $L/K(t_1, \cdots, t_r)$ is a finitely generated algebraic extension, it is in fact finite. So we've reduced to the case where L/K is finite.

   Pick any generating set $x_1, \cdots, x_n$ for $\mathfrak{m} \subset \mathcal{O}$. If the ideal (x_i) were the unit ideal in $\mathcal{O}[\frac{x_1}{x_i}, \cdots, \frac{x_n}{x_i}]$, then we could write $\frac{1}{x_i}$ as a polynomial in $\frac{x_1}{x_i}, \cdots, \frac{x_n}{x_i}$. If this polynomial was of degree d, then clearing denominators by multiplying by x_i^d gives that $x_i^{d-1} \in \mathfrak{m}^d$. If this were the case for all i, then letting D be the maximum d over all i, we would have that $\mathfrak{m}^{D-1} \subset \mathfrak{m}^D$, which implies that $\mathfrak{m} = 0$. If this is the case, then $\mathcal{O}$ was already a field, contradiction. So up to permutation of indicies, we have that (x_1) is not the unit ideal in $\mathcal{O}'$.

   As x_1 is neither a unit nor a zero-divisor, all minimal primes over it have height one by Krull's Height theorem (I.1.11A). Letting $\mathfrak{p}$ be such a minimal prime, we get that $\mathcal{O}'_\mathfrak{p}$ is a noetherian local ring of dimension one containing $\mathcal{O}'$ and thus $\mathcal{O}$. To show that $\mathcal{O}'_\mathfrak{p}$ dominates $\mathcal{O}$, note that $\mathfrak{p}_\mathfrak{p}$ is the maximal ideal of $\mathcal{O}'_\mathfrak{p}$, and $\mathfrak{p}_\mathfrak{p} \cap \mathcal{O}' = \mathfrak{p}$. As $\mathfrak{p} \supset (x_1)$ and $x_i \in (x_1) \subset \mathcal{O}'$ for all i, we see that $\mathfrak{p}_\mathfrak{p} \cap \mathcal{O} \supset \mathfrak{m}$. On the other hand, $\mathfrak{p}_\mathfrak{p} \cap \mathcal{O}$ is a proper prime ideal of $\mathcal{O}$, so since it contains the maximal ideal $\mathfrak{m} \subset \mathcal{O}$, it must be maximal. Thus $\mathcal{O}'_\mathfrak{p}$ dominates $\mathcal{O}$. We quote the Krull-Akizkui theorem from Bourbaki's *Commutative Algebra*, Chapter VII, section 2, number 5, proposition 5:

   Theorem (Krull-Akizuki)**.** *Let A be a reduced noetherian ring of dimension at most one with K its ring of fractions. If B is a subring of a finite extension L of K containing A, then B is a one-dimensional noetherian ring. Furthermore, for every nonzero ideal I of B, the quotient B/I is finite over A.*

   By the Krull-Akizuki theorem, $\widetilde{\mathcal{O}'_\mathfrak{p}}$, the integral closure of $\mathcal{O}'_\mathfrak{p}$ in L, is noetherian of dimension one. As integral closures commute with localization, this impleis that the localization of $\widetilde{\mathcal{O}'_\mathfrak{p}}$ at one of its maximal ideals is an integrally closed noetherian local domain of dimension one, or a DVR.

b. Let's trace through why we had to consider arbitrary valuation rings in the valuative criteria. The key point is that we have a local ring $\mathcal{O} \subset K$, and then by theorem I.6.1A we get a valuation ring $R \subset K$ dominating $\mathcal{O}$. But by the work we did in (a) with $L = K$, we can say that there's a discrete valuation ring R dominating $\mathcal{O}$, so it is enough to consider just DVRs.

Exercise II.4.12. *Examples of Valuation Rings.* Let k be an algebraically closed field.

a. If K is a function field of dimension 1 over k (I, §6), then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of (I, §6).

b. If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose X is a complete nonsingular surface with function field K.

   (1) If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.

   (2) If $f : X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.

   (3) Let x_0 be a closed point. Let $f : X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2 : X_2 \to X_1$ be the blowing up of x_1, an let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A). Show that R is a valuation ring of K/k, and it has center x_0 on X. When is R a discrete valuation ring?

Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

Solution.

a. By theorem I.6.9 and proposition II.4.10, K is the function field of a smooth projective curve C over k. By theorem II.4.9, this means that $C \to \operatorname{Spec} k$ is proper, so by exercise II.4.5(b), we have that any valuation ring R has center on a point $c \in C$. If c is the generic point, then $R = K$. If c is a closed point, then $\mathcal{O}_{C,c}$ is dominated by R. On the other hand, $\mathcal{O}_{C,c}$ is already a DVR: it's a regular local ring of dimension one, hence a noetherian local domain of dimension one which is integrally closed. Since valuation rings are maximal for the relation of domination, this implies that $R = \mathcal{O}_{C,c}$ already, and in particular, is a discrete valuation ring.

b. It's not completely clear what you're actually supposed to do for most of this problem - we'll interpret this as verifying that all of these things are valuation rings (plus that extra bit at the end of (3)).

If x_1 is the generic point of an irreducible curve Y in X, then $\mathcal{O}_{X,x_1}$ is a DVR because it's an integrally closed noetherian local domain of dimension one, and thus a DVR. If we assume that X' is also normal (this appears to be a simple omission on Hartshorne's part) and y is the generic point of Y', the irreducible curve contracted to x_0, then $\mathcal{O}_{X,x_0} \subset \mathcal{O}_{X',y}$ is an inclusion of local rings, and the latter dominates the former: a regular function on X near x_0 vanishing at x_0 implies that its pullback to X' vanishes on Y', which means it is in the maximal ideal of $\mathcal{O}_{X',y}$. So $\mathcal{O}_{X',y}$ is a DVR by the same logic as earlier, and by birationality $\mathcal{O}_{X',y}$ is a valuation ring for $K(X') = K(X)$.

For the third case, the same argument as in the previous paragraph shows that $\mathcal{O}_{X_{i+1},x_{i+1}}$ dominates $\mathcal{O}_{X_i,x_i}$ and thus the union of all of these is in fact a local ring dominating $\mathcal{O}_{X,x_0}$. Taking R to be a valuation ring of $K(X)/k$ dominating R_0 implies that R dominates $\mathcal{O}_{X,x_0}$ by transitivity of dominance, so R has center x_0 on X. If there is a curve $C \subset X_i$ so that the intersection of the exceptional divisor and strict transform of C in the iterated blowup X_j contains x_j, then this is not a discrete valuation, else this is a DVR corresponding to a 'transcendental curve'.

If such a C exists, this is the valuation ring corresponding to the 'composite' valuation constructed from the DVR of the generic point of C in X_i and the DVR of some smooth point c_0 of C mapping down on to x_0 (see the second-to-last paragraph in the solution of II.4.5(c) for why this is a valuation ring). This isn't discrete, because no power of a rational function vanishing only at c_0 will ever be greater (under the valuation ordering) than any rational function vanishing on the whole of C - this shows that the ordering on the valuation group is non-archimedean, and thus the value group isn't $\mathbb{Z}$.

If there is no such curve C, then the idea one should have is that no rational function can vanish after restriction to our 'transcendental curve', so any rational function has a well-defined order of vanishing at x_0. To formalize this, the valuation is defined by the eventually constant order of vanishing of our rational function on the exceptional divisor - that is, given any rational function in $k(X)$, it's pullback along the iterated blowups will eventually give us a constant order of vanishing along the exceptional divisor, and we can use this as our discrete valuation.

II.5 Sheaves of Modules

One common theme of this section is that it's usually better and easier to define a map globally and check locally: this lets one reduce the check to something you already know for modules from a previous abstract-algebra course. There will be times where this isn't the appropriate tack to take, and we'll note those when we come to them.

Hartshorne does readers a disservice here with his definition of coherence. The real, actual definition of coherence is that a sheaf $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is coherent iff every point $x \in X$ has an open neighborhood U with a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ for some integer n (we say that $\mathcal{F}$ is of *finite type*), and for any open set $U \subset X$ and any morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$, the kernel of this morphism is of finite type. This is the same as what Hartshorne defines in the (locally) noetherian situation, see for instance Stacks 01XZ, but is a rather large departure outside of this realm. For instance, there are schemes with non-coherent structure sheaves (take the spectrum of the ring of germs of C^∞ functions $\mathbb{R} \to \mathbb{R}$ at $0 \in \mathbb{R}$, for one example).

Let's also take some time to complain about Hartshorne's treatment of graded rings: there's almost always a 'finitely generated in degree one' assumption hanging around which isn't always required (and sometimes is required for non-obvious reasons!) and can lead to some headaches. To deal with the full consequences of this assumption and when one can remove it, I like the Stacks Project's section dealing with Proj - these sections are tags 01M3, 01MJ, 01MM, etc.

Another headache in this section: Hartshorne defines an immersion as an isomorphism with an open subscheme of a closed subscheme (page 120, just before remark II.5.16.1). This is not the usual definition, which is a closed immersion followed by an open immersion (these are the same except in the case of non-quasi-compact morphisms from non-reduced schemes, see Stacks 01QV and 03DQ). We will complain about this vociferously when appropriate.

Exercise II.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the *dual* of $\mathcal{E}$, denoted $\check{\mathcal{E}}$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

a. Show that $(\check{\mathcal{E}})^\vee \cong \mathcal{E}$.

b. For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

c. For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

d. (*Projection Formula*). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$ module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$.

Solution.

Some slight erata before we solve the problem: the $\mathrm{Hom}_{\mathcal{O}_X}$ that show up in (c) should be $\mathcal{H}om_{\mathcal{O}_X}$ (sheaf Hom).

a. Write $(\check{\mathcal{E}})^\vee$ as $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X)$ and we define a map $\mathcal{E} \to (\check{\mathcal{E}})^\vee$ by sending a section $s \in \mathcal{E}(U)$ to the map which evaluates $s^* \in \check{\mathcal{E}}(U)$ on s. This is clearly compatible with restrictions and thus gives a map of sheaves. To check it is an isomorphism, we may work on stalks by the theory we developed in II.1. So it suffices to check that $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{E}_x, \mathcal{O}_{X,x})\mathcal{O}_{X,x}) \cong \mathcal{E}_x$ when $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module of finite rank via the same map we described above. But this is exactly the same proof that vector spaces are isomorphic to their double duals which should be familiar from your graduate algebra course: pick a basis, prove the claim in the basis using that every element of the dual is given by taking the dot product with some unique element, and verify that this gives an isomorphism.

b. We define a map from the tensor product presheaf $\check{\mathcal{E}} \otimes_{\mathcal{O}_X}^{Psh} \mathcal{F}$ to the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ by $\sum s^* \otimes a \mapsto (s \mapsto a)$ and then check that this gives an isomorphism on stalks. By the theory developed in II.1, this implies we get a map of sheaves $\check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ which has the same maps on stalks and thus our map of sheaves is an isomorphism.

   To check that we have an isomorphism, we restrict to stalks: $\mathcal{E}_x \cong \mathcal{O}_{X,x}^n$ by the assumption that $\mathcal{E}$ is localy free of finite rank, and then it's clear that this is an isomorphism: $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^n, \mathcal{F}_x) \cong \mathcal{F}_x^n \cong \mathcal{O}_{X,x}^n \otimes_{\mathcal{O}_{X,x}} \mathcal{F}$ by our map, and the proposition is proven.

c. This is known as the tensor-hom adjunction, and this is true with no assumptions on the sheaves involved (i.e. we need not require that any of $\mathcal{E}$, $\mathcal{F}$, or $\mathcal{G}$ be locally free of finite rank - this is even true in the setting of just ringed spaces!). We can define this on the level of sections of presheaves and then sheafify everything. A map $\mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \mathcal{G}(U)$ tells us how to send tensors $e \otimes f$ to an element g, while a map $\mathcal{F}(U) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{G}(U))$ requires us to give for each element f of $\mathcal{F}(U)$ a map $\mathcal{E}(U) \to \mathcal{G}(U)$: we do this by plugging in f to the first map. See any graduate algebra text for the verification that this is an isomorphism on the level of modules. This isomorphism of modules gives an isomorphism of presheaves, which after sheafification gives an isomorphism of sheaves.

d. The key observation here is that the pullback of the structure sheaf is the structure sheaf: $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$ naturally, since $R \otimes_R M \cong M$ for any R-module M. Now, if $\mathcal{E}$ is free, then $\mathcal{E} \cong \mathcal{O}_Y^n$, and we can write $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \cong f^*(\mathcal{F}^n)$. As f^* is an additive functor, it commutes with finite direct sums (this is abstract nonsense: a left/right adjoint functor preserves colimits/limits, and as finite products and direct sums are the same in an abelian category, we have that any adjoint functor between abelian categories preserves finite sums/products; or you can prove it directly) and thus $f^*(\mathcal{F}^n) \cong (f^*\mathcal{F})^n \cong f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n \cong f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$. In the case where $\mathcal{E}$ is locally free instead of free, we may restrict to a cover where $\mathcal{E}$ is free and then glue the above result together.

Exercise II.5.2. Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.

a. To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

b. That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Solution.

a. This is just definition-unwinding, for the most part. The open sets in X are X and $\{(0)\}$, and the value of $\mathcal{O}_X$ on these are R and K, respectively. So the data specifying a $\mathcal{O}_X$-module $\mathcal{F}$ is given by a choice of $\mathcal{F}(X)$ and $\mathcal{F}(\{(0)\})$ plus a restriction map: an R-module M, a K-module L, and a homomorphism $f : M \to L$ respectively. To show that f is equivalent to ρ, any f induces $\rho : M \otimes_R K \to L$ as the composite of $f \otimes 1 : M \otimes_R K \to L \otimes_R K$ and $L \otimes_R K \to L$ by $(l, k) \mapsto lk$. Conversely, any ρ gives an f by letting $f(m) = \rho(m \otimes 1)$.

b. Let π be a generator of the maximal ideal of R. Then $\{(0)\} = D(\pi)$ as sets. $\mathcal{F}$ is quasi-coherent iff $\mathcal{F} \cong \widetilde{M}$, and $\widetilde{M}(D(\pi)) \cong M_\pi \cong M \otimes_R R_\pi \cong M \otimes_R K$. So $\mathcal{F}$ is quasi-coherent iff $M \otimes_R K \cong L$ via the restriction map.

Exercise II.5.3. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\widetilde{-}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism

$$\operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}).$$

Solution. Given a map $\varphi : M \to \Gamma(X, \mathcal{F})$ we can get a map of sheaves $\widetilde{M} \to \mathcal{F}$ as follows: on any set of the form $D(f)$, $f \in A$, φ induces a map $M_f \to \Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ by the definition of localization and the fact that $\Gamma(X_f, \mathcal{F})$ is a $\mathcal{O}(X_f) = A_f$-module so the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ factors through $\Gamma(X, \mathcal{F})_f$. By the lemma in the solution to exercise II.2.4 and the fact that the affine open sets $D(f)$ form a basis for the topology on $\operatorname{Spec} A$, this gives a map of sheaves $\widetilde{M} \to \mathcal{F}$.

Given a map $f : \widetilde{M} \to \mathcal{F}$, we can take global sections to get a map $f(X) : \widetilde{M}(X) = M \to \Gamma(X, \mathcal{F})$. Now our goal is to show that these two maps (global sections and localization) are mutually inverse, which will show that the required natural isomorphism.

If we take a map $\varphi : M \to \Gamma(X, \mathcal{F})$, turn it in to a map $\varphi^\sharp : \widetilde{M} \to \mathcal{F}$, and then take global sections, we get φ back by construction. The other direction is similar: given a map $f : \widetilde{M} \to \mathcal{F}$, we get a map $f(X) : M \to \Gamma(X, \mathcal{F})$ by taking global sections, and then reconstruct $\widetilde{M} \to \mathcal{F}$ by the lemma from our solution to exercise II.2.4. To check this really does give the same answer, it suffices to see that we get the same map $\widetilde{M}(D(f)) \to \mathcal{F}(D(f))$. But this is clear by construction: $\widetilde{M}(D(f)) = M_f$, and the map $res_{X,D(f)} : \mathcal{F}(X) \to \mathcal{F}(D(f))$ factors through $\mathcal{F}(X)_f$, so we get the same map of sheaves we started with.

Exercise II.5.4. Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of locally free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent if and only if it locally a cokernel of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

Solution. $\Rightarrow$: Let $x \in X$ be a point with an affine open neighborhood $\operatorname{Spec} A$ so that $\mathcal{F}|_{\operatorname{Spec} A} = \widetilde{M}$ for some A-module M. Then by writing a presentation of M as $A^{\oplus I} \to A^{\oplus J} \to M \to 0$, we can take the associated modules to get that $\mathcal{O}_{\operatorname{Spec} A}^{\oplus I} \to \mathcal{O}_{\operatorname{Spec} A}^{\oplus J} \to \widetilde{M} \to 0$ is also exact, and thus $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free sheaves on U.

$\Leftarrow$: Let $x \in X$ be a point with an open neighborhood U so that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of locally free sheaves on U. By shrinking U, we may assume the sheaves are free and $U = \operatorname{Spec} A$ is affine. Taking global sections, we let M be the cokernel of the morphism of global sections, and so $\widetilde{M}$ is the cokernel of our map of locally free sheaves, showing that our condition is satisfied.

The same logic applies for coherence, except that we need to verify that if A is a noetherian ring and M is a finitely generated A-module, then M can be written as the cokernel of $A^{\oplus m} \to A^{\oplus n}$ for m, n finite. Clearly, if M is finitely generated, then we can take n to be finite, so it remains to prove that m must be finite as well. This is standard (any submodule of a finitely-generated module over a noetherian ring is again finitely-generated).

Exercise II.5.5. Let $f : X \to Y$ be a morphism of schemes.

a. Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

b. Show that a closed immersion is a finite morphism (§3).

c. If f is a finite morphism of noetherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

Solution.

a. Let $X = \mathbb{A}^1_k$ and $Y = \mathbb{A}^0_k$ with f the standard projection and $\mathcal{F} = \mathcal{O}_{\mathbb{A}^1_k}$. Then $f_*\mathcal{F}$ is $k[x]$, which is not a finitely-generated k-module.

b. By the work we did in exercise II.3.11, we have that any closed immersion is an affine morphism which locally looks like $\operatorname{Spec} A/I \to \operatorname{Spec} A$. The A-module A/I is generated by $1 + I$, and thus a closed immersion satisfies the definition for a finite morphism.

c. It's enough to show this when $Y = \operatorname{Spec} A$ as the condition is local on Y. From the definition of a finite morphism, we have that $X = \operatorname{Spec} B$ with the map $X \to Y$ being induced from a map $A \to B$ giving B as a finite A-module, and from the characterization of a coherent sheaf, we may write $\mathcal{F} \cong \widetilde{M}$ for M a finite B-module. Then we have that $f_*\mathcal{F}$ is the sheaf corresponding to M_A, the module given by treating M as an A-module via the restriction of scalars, and our goal is to show that M_A is a finite A-module. This is straightforwards: if the elements b_i generate B as an A-module and the elements m_j generate M as a B-module, then the elements $b_i m_j$ generate M as an A-module, and our claim is proven.

Exercise II.5.6. *Support.* Recall the notions of support of a section of a sheaf, support of sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

a. Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \widetilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the *annihilator* of $m = \{a \in A \mid am = 0\}$.

b. Now suppose that A is noetherian, and M finitely generated. Show $\operatorname{Supp}\mathcal{F} = V(\operatorname{Ann} M)$.

c. The support of a coherent sheaf on a noetherian scheme is closed.

d. For any ideal $\mathfrak{a} \subset A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ by $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathscr{H}_Z^0(\mathcal{F})$, where $Z = V(\mathfrak{a})$ and $\mathcal{F} = \widetilde{M}$. [*Hint*: Use (Ex. 1.20) and (5.8) to show a priori that $\mathscr{H}_Z^0(\mathcal{F})$ is quasi-coherent. Then show that $\Gamma_{\mathfrak{a}}(M) = \Gamma_Z(\mathcal{F})$.]

e. Let X be a noetherian scheme, and let Z be a closed subset. If $\mathcal{F}$ is a quasi-coherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathscr{H}_Z^0(\mathcal{F})$ is also quasi-coherent (respectively, coherent).

Solution.

a. Recall that the support is the set of points p where a section is nonzero in the stalk. For a prime ideal $\mathfrak{p} \subset A$, we have that $m \neq 0$ in $M_{\mathfrak{p}}$ iff there are no elements $a \in A \backslash \mathfrak{p}$ so that $am = 0$, so the condition that $\mathfrak{p} \in \operatorname{Supp} m$ is equivalent to $\operatorname{Ann} m \subset \mathfrak{p}$. Thus $V(\operatorname{Ann} m) = \operatorname{Supp} m$.

b. Let M be finitely generated by $m_1, \cdots, m_n$. Then the condition that $\mathcal{F}_{\mathfrak{p}} = M_{\mathfrak{p}} \neq 0$ is exactly the condition that at least one of the $m_i \neq 0$ in $M_{\mathfrak{p}}$. By the same logic as part (a), this means that $\mathfrak{p} \in \operatorname{Supp} M$ is equivalent to the ideal $\operatorname{Ann} M$ being a subset of $\mathfrak{p}$ and we again have $V(\operatorname{Ann} M) = \operatorname{Supp}\mathcal{F}$. (Noetherianity is not necessary here: just that M be finitely generated.)

c. If $\mathcal{F}$ is coherent, then on any affine open set $\operatorname{Spec} A$ it is of the form $\widetilde{M}$, for M a finitely generated A-module. So sheafifying the result from part b, we have our desired result.

   (One should note that there are a lot of extra hypotheses here: one correct statement with essentially the same proof is that assuming X is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, if every point $x \in X$ has an open neighborhood U so that $\mathcal{F}|_U$ is generated by finitely many sections, then the support of $\mathcal{F}$ is closed. We say that a the support of a finite-type sheaf on a ringed space is closed, see Stacks 01BA.)

d. As quasi-coherence is a local condition, we have that if $\mathcal{F}$ is a quasi-coherent sheaf on $\operatorname{Spec} A$, then $\mathcal{F}|_U$ is a quasi-coherent sheaf on $U = \operatorname{Spec} A \setminus V(\mathfrak{a})$. Next, if $j : U \to \operatorname{Spec} A$ is the open immersion, we have that j_* takes quasi-coherent sheaves to quasi-coherent sheaves by proposition II.5.8. By the exact sequence

$$0 \to \mathscr{H}_Z^0(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U)$$

   from exercise II.1.20, we have that $\mathscr{H}_Z^0(\mathcal{F})$ is the kernel of a map of quasi-coherent sheaves, so by proposition II.5.7 we have that $\mathscr{H}_Z^0(\mathcal{F})$ must also be quasi-coherent. So it's enough to show that the sheaves $\Gamma_{\mathfrak{a}}(M)^{\sim}$ and $\mathscr{H}_Z^0(\mathcal{F})$ have isomorphic global sections. But by exercise II.1.20, we have that $\mathscr{H}_Z^0(\mathcal{F})(X) = \Gamma_Z(\mathcal{F})$, and the conclusion is clear.

e. As every closed subscheme of an affine scheme $\operatorname{Spec} A$ is $V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$, part (d) shows this in the case of an affine noetherian scheme. Since these condition are local and every noetherian scheme can be covered by noetherian affine schemes, we have the result in general.

Exercise II.5.7. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.

a. If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

b. $\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.

c. $\mathcal{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathcal{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

Solution.

a. Since the problem is local, by taking an affine open neighborhood of x we may assume that $X = \operatorname{Spec} A$ is affine and $\mathcal{F} \cong \widetilde{M}$ for M a finite A-module. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. Write x_i for a generating set for M and $\frac{y_i}{a_i}$ with $y_i \in M$, $a_i \in A \setminus \mathfrak{p}$ for the elements of $M_\mathfrak{p}$ corresponding to the standard basis of $A_\mathfrak{p}^n$ under the isomorphism $M_\mathfrak{p} \cong A_\mathfrak{p}^n$. Then we can write $x_i = \sum \frac{b_j}{c_j} \frac{y_j}{a_j}$ for $b_j \in A$ and $c_j \in A \setminus \mathfrak{p}$. Letting $f = (\prod a_i)(\prod c_j)$, we see that these relations hold on $D(f)$, and the $\frac{y_i}{a_i}$ are linearly independent in M_f because they're linearly independent in $M_\mathfrak{p}$, so we have shown that M_f is free, thus $\widetilde{M_f} \cong \mathcal{F}|_{D(f)}$ is free too.

b. The forward direction is trivial: for any point $x \in X$ there is a neighborhood U so that $\mathcal{F}|_U \cong \mathcal{O}_U^n$, and the stalk of $\mathcal{O}_U^n$ at x is clearly $\mathcal{O}_{X,x}^n$. The reverse direction is exactly part (a).

c. Suppose $\mathcal{F}$ is locally free of rank 1. I claim that $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ can be taken as $\mathcal{G}$. By exercise II.5.1b, we have that $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. Clearly $\mathcal{O}_X$ maps in to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ by $s \in \mathcal{O}_X(U)$ mapping to multiplication of sections of $\mathcal{F}$ over U by s, and we want to verify that this map is an isomorphism of sheaves. To show this, it suffices to check on a sufficiently fine open cover: pick an open cover $\{U_i\}$ where $\mathcal{F}$ is free. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})|_{U_i} \cong \mathcal{H}om_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) \cong \mathcal{O}_{U_i}$ and we're done.

Supposing that there is a coherent sheaf $\mathcal{G}$ so that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$, we show that every stalk of $\mathcal{F}$ is free which implies that $\mathcal{F}$ is locally free by part (a). Write $R = \mathcal{O}_{X,x}$, $M = \mathcal{F}_x$, and $N = \mathcal{G}_x$. Our goal is to prove that for a local ring R and two R-modules M, N that $M \otimes_R N \cong R$ implies that $M \cong R$.

Let k denote the residue field of R. We have that

$$k \cong R \otimes_R k \cong (M \otimes_R N) \otimes_R k \cong (M \otimes_R N) \otimes_R k \otimes_R k \cong$$

$$\cong (M \otimes_R k) \otimes_R (N \otimes_R k) \cong (M \otimes_R k) \otimes_k (N \otimes_R k)$$

and so $M \otimes_R k \cong N \otimes_R k \cong k$. By Nakayama's lemma, this means that both of M and N are cyclic R-modules. So $M \cong R/P$ and $N \cong R/Q$ for ideals P, Q. So $R \cong M \otimes_R N \cong R/P \otimes_R R/Q \cong R/(P+Q)$, which is the case iff $P+Q=0$, or $P=Q=0$. We are finished.

Exercise II.5.8. Again let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x),$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

a. The function φ is *upper semi-continuous*, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

b. If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant function.

c. Conversely, if X is reduced, and φ is constant, then $\mathcal{F}$ is locally free.

Solution.

a. We'll prove this two ways: one without using Nakayama's lemma which I think is a bit slicker, and then one using Nakayama's lemma to follow the hint and prepare for part (c).

   This function only depends on the stalks of $\mathcal{F}$, so it suffices to check this in the case that $X = \operatorname{Spec} R$ is affine and $\mathcal{F} = \widetilde{M}$ for M a finite R-module. Write M as the cokernel

   $$R^m \xrightarrow{f} R^n \to M \to 0,$$

   where we represent f as an $n \times m$ matrix acting on the column vectors of R^m by left-multiplication. After after localizing at the prime $\mathfrak{p}$ representing x and tensoring with $k(x)$, this becomes

   $$k(x)^m \xrightarrow{\hat{f}} k(x)^n \to M_\mathfrak{p} \otimes_{R_\mathfrak{p}} k(x) \to 0.$$

   As $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} k(x) \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$, the dimension of the latter is just the corank of $\hat{f}$ as a map of vector spaces $k(x)^m \to k(x)^n$, that is, $\dim \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) = n - \operatorname{rank} \hat{f}$.

   We now analyze $\operatorname{rank} \hat{f}$. Recall the following useful characterization of rank: a matrix is of rank at most r iff all $(r+1) \times (r+1)$ minors have vanishing determinant. Now since the determinant is a polynomial in the entries of the matrix, we have that the determinant of a particular minor of $\hat{f}$ is just the reduction of the determinant of the corresponding minor of f modulo $\mathfrak{p}$, and so any particular minor has vanishing determinant on a closed set. Thus $\hat{f}$ is of rank at most r on a closed set for any r, or $\dim \mathcal{F}_x \otimes k(x)$ is at least $n - r$ on a closed set, and the claim is proven.

   To prove this via Nakayama's lemma as suggested, argue in the same fashion as exercise II.5.7: if $\varphi(x) = n$, then lift a minimal set of n generators for $\mathcal{F}_x \otimes k(x) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ to a minimal generating set for $M_\mathfrak{p}$ also of size n using Nakayama's lemma. By the logic from exercise

II.5.7, these suffice to generate M_f for a well-chosen f, which means that $\varphi(y) < n+1$ for all $y \in D(f)$. This finishes the problem by showing that $\{x \mid \varphi(x) < n+1\}$ is open: its complement is $\{x \mid \varphi(x) \geq n+1\}$, which is therefore closed.

b. If $\mathcal{F}$ is locally free, then φ is locally constant: if $U \subset X$ is an open set so that $\mathcal{F}|_U \cong \mathcal{O}_X|_U^n$, then $\varphi(x) = n$ for any $x \in U$, so the function φ is continuous when we give the integers the discrete topology. Since a locally constant function on a connected space is constant, we have the result.

c. We make the same reduction to the affine case as in (a), and apply the logic of exercise II.5.7 to get that if $\varphi(\mathfrak{p}) = n$, then there exists an $f \in R$ with $\mathfrak{p} \in D(f)$ so that M_f is generated by at most n elements. As $\varphi(\mathfrak{q}) = n$ for any $\mathfrak{q} \in D(f)$, we see that these n generators must be linearly independent in $M_{\mathfrak{q}}$, and thus $M_{\mathfrak{q}}$ is free as well.

Exercise II.5.9. Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.

a. Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$.

b. Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely-generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbb{Z}$ such that for all $d \geq d_0$, $\alpha_d : M_d \to \Gamma(X, \widetilde{M}(d))$ is an isomorphism. [*Hint*: Use the methods of the proof of (5.19).]

c. With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geq d} \cong M'_{\geq d}$. Here $M_{\geq d} = \bigoplus_{n \geq d} M_n$. We will say that a graded S-module is *quasi-finitely generated* if it is equivalent to a finitely generated module. Now show that the functors $\widetilde{-}$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

Solution.

a. Note that by proposition II.5.12(b), we have $\Gamma_*(\widetilde{M}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\widetilde{M}(n)) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\widetilde{M(n)})$. Now I claim that there's a natural homomorphism $M_n \to \Gamma(\widetilde{M(n)})$: simply send $m \in M_n$ to the collection of sections $\frac{m}{1} \in M(n)(D(f)) = (M_n)_{(f)}$ which are clearly compatible on overlaps and thus patch to a section of $\widetilde{M(n)}$ by the sheaf condition. Taking the direct sum over all n, we have our desired result.

b. If you follow the proof of theorem II.5.19 closely, this isn't so difficult, but if you haven't done that, this will probably be pretty hard - it's one of the main theorems of Serre's *Faisceaux algbriques cohrents*. We take the tack of following the proof of theorem II.5.19, after making a reduction to the case of $M = S/\mathfrak{p}$.

As in theorem II.5.19, take a finite filtration

$$0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$$

by graded submodules so that $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for $\mathfrak{p}_i$ a homogeneous prime ideal and n_i an integer. If we know the desired result for $(S/\mathfrak{p}_i)(n_i)$, then we have the result via induction on i in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M^{i-1} \arrow[r] \arrow[d] & M^i \arrow[r] \arrow[d] & (S/\mathfrak{p}_i)(n_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_*(X, \widetilde{M^{i-1}}) \arrow[r] & \Gamma_*(X, \widetilde{M^i}) \arrow[r] & \Gamma_*(X, \widetilde{(S/\mathfrak{p}_i)(n_i)}) &
\end{tikzcd}$$

By induction, there is some d_{i-1} so that the left vertical map is an isomorphism in degrees $\geq d_{i-1}$, and by assumption there is some c_i so that the right vertical map is an isomorphism in degrees $\geq c_i$. By the 5-lemma, the middle map is also an isomorphism and the bottom sequence is exact in degrees $\geq \max(d_{i-1}, c_i) = d_i$. We note that we can also forget about n by absorbing it in to d, so it suffices to prove the claim for $M = S/\mathfrak{p}$. In particular, we can use the identification $\operatorname{Proj} S/\mathfrak{p} = V(\mathfrak{p}) \subset \mathbb{P}^n_A$ to reduce to the special case where S is a graded integral domain, finitely generated by S_1 as an S_0-algebra, where $S_0 = A$ is a finitely generated domain over k.

Continuing as in the proof of theorem II.5.19, we see that $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$ is finitely generated, say by z_i. Then as in the proof of II.5.19, we get that $yz_i \in S_{\geq n}$ for all $y \in S_{\geq n}$ for some n. In particular, if we take $d_0 = \max \deg z_i + n$, then $S'_d \subset S_d$ for all $d \geq d_0$ and we are done.

c. Our work in part (b) gives us that if M is finitely generated, the map $M \to \Gamma_*(\widetilde{M})$ is an isomorphism in high enough degrees. By proposition II.5.15, we have that $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$ for any quasi-coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S$. So it remains to show that if M is a quasi-finitely generated module, then $\widetilde{M}$ is coherent, and if $\mathcal{F}$ is coherent, then $\Gamma_*(\mathcal{F})$ is quasi-finitely generated.

   Suppose M is quasi-finitely generated with N a finitely generated module so that $M \approx N$, and let $f \in S$ be homogeneous of degree one. Then $\widetilde{M}|_{D(f)}$ is $\widetilde{M_{(f)}}$ on $\operatorname{Spec} S_{(f)}$ by construction, and I claim that $\widetilde{M_{(f)}} \cong \widetilde{N_{(f)}}$. To see this, we note that there is some $n > 0$ so that for all $n' \geq n$ we have $M_{n'} \cong N_{n'}$, and so by writing every element $\frac{m}{f^d}$ of $M_{(f)}$ as $\frac{f^n m}{f^{n+d}}$, we see that $M_{(f)} \cong N_{(f)}$ and the claim follows. So on every $D(f)$ we have that $\widetilde{M}|_{D(f)}$ is coherent and thus $\widetilde{M}$ is coherent.

   Suppose $\mathcal{F}$ is coherent. Let n be such that $\mathcal{F}(n)$ is generated by finitely many global sections (by theorem II.5.17) - then for all $n' \geq n$, we have that $\mathcal{F}(n')$ is also generated by finitely many global sections, specifically the images of the multiplication of the global sections of $\mathcal{O}(n'-n)$ and $\mathcal{F}(n)$. Let $M' \subset \Gamma_*(\mathcal{F})$ be the graded module generated by the finitely many global sections which suffice to generate $\mathcal{F}(n)$. The inclusion $M' \subset \Gamma_*(\mathcal{F})$ induces a map $\widetilde{M'} \to \widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$, which after tensoring with $\mathcal{O}(n)$ gives an isomorphism because $\mathcal{F}(n)$ is globally generated. Since $\mathcal{O}(n)$ is invertible, the original map must have been an isomorphism, so $M' \approx \Gamma_*(\widetilde{M'}) \cong \Gamma_*(\mathcal{F})$ and the latter is quasi-finitely generated.

Exercise II.5.10. Let A be a ring, let $S = A[x_0, \cdots, x_r]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

a. For any homogeneous ideal $I \subset S$, we define the *saturation* $\overline{I}$ of I to be $\{s \in S \mid$ for each $i = 0, \cdots, r$ there is an n such that $x_i^n s \in I\}$. We say that I is *saturated* if $I = \overline{I}$. Show that $\overline{I}$ is a homogeneous ideal of S.

b. Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

c. If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathcal{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.

d. There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.

Solution.

a. Let $f, g \in \overline{I}$ be two arbitrary elements. We need to show that $f + g \in \overline{I}$, that $af \in \overline{I}$ for any $a \in A$, and that $\overline{I}$ is homogeneous. Sums: if b_i are the exponents of the x_i so that $x_i^{n_i} f \in I$ and the c_i are the same for g, then letting $d_i = \max(b_i, c_i)$ we have that $x_i^{d_i}(f + g) \in I$ for all i. Products: as I is an ideal, if $x_i^{n_i} f \in I$, then $x_i^{n_i} af = ax_i^{n_i} f \in I$. To show homogeneity, we use the characterization that an ideal is homogeneous iff every element's homogeneous parts belong to the ideal. Let $f = f_0 + f_1 + \cdots + f_t$ be an element of $\overline{I}$, where f_i is homogeneous of degree i. Then $x_i^n f = x_i^n f_0 + \cdots + x_i^n f_t \in I$, and by the definition of a homogeneous ideal, we must have that each $x_i^n f_j$ is in I. But this means that $f_j \in \overline{I}$ and thus $\overline{I}$ is homogeneous.

b. It suffices to check that $\widetilde{I_1} \cong \widetilde{I_2}$, which we can do on each $D(x_i)$. In this case, as $I_{(x_i)} = \overline{I}_{(x_i)}$ they define the same ideal sheaf of $D(x_i)$ and we're done by the correspondence between quasi-coherent sheaves of ideals and closed subschemes.

c. Suppose $f \in \Gamma(\mathcal{O}(n))$ so that $x_i^m f \in \Gamma(\mathcal{I}_Y(n + m))$. On $D(x_i)$, we have that f must vanish, since x_i doesn't (in more formal language, since $\mathcal{O}(m)|_{D(x_i)}$ is trivial, with isomorphism to $\mathcal{O}|_{D(x_i)}$ given by multiplying by x_i^m we have that f is a section of $\mathcal{I}_Y(n)$). So $\Gamma_*(\mathcal{I}_Y)$ is saturated.

d. By proposition II.5.9, every quasi-coherent sheaf of ideals uniquely determines a closed subscheme, and by parts (b) and (c) there is a unique saturated sheaf of ideals cutting out any closed subscheme of $\operatorname{Proj} S$.

Exercise II.5.11. Let S and T be two graded rings with $S_0 = T_0 = A$. We define the *Cartesian product* $S \times_A T$ to be the graded ring $\bigoplus_{d \geq 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) = X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

The Cartesian product of rings is related to the *Segre embedding* of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \cdots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \hookrightarrow \mathbb{P}^r_A$, and if $y_0, \cdots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \hookrightarrow P^s_A$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and hence defines a projective embedding $\operatorname{Proj}(S \times_A T) \hookrightarrow \mathbb{P}^N_A$, with $N = rs + r + s$. This is just the image of $X \times Y \subset \mathbb{P}^r \times \mathbb{P}^s$ in its Segre embedding.

Solution. A couple observations: this is the only place I have ever seen anyone detail this construction; this is true with absolutely no assumptions about S or T being finitely generated or generated in degree one, though those do give a slightly easier proof.

Our strategy is to show the desired isomorphisms by showing that these objects are assembled from the same gluing data (refer to exercise II.1.22 for gluing sheaves and exercise II.2.11 for gluing schemes). Let $f \in S$ and $g \in T$ be homogeneous of positive degree. As $D(f) = D(f^n)$ as sets, we may assume that f and g are of the same homogeneous degree after possibly taking powers. Note that sets of the form $D(f)$ for $f \in S$ homogeneous of positive degree cover $\operatorname{Proj} S$.

We show that $D(f \otimes g) = D(f) \times_A D(g)$, where the former is considered in $\operatorname{Proj} S \times_A T$ and the latter in $\operatorname{Proj} S \times_A \operatorname{Proj} T$. By proposition II.5.11, the former is $\operatorname{Spec}(S \times_A T)_{(f \otimes g)} \cong \operatorname{Spec} S_{(f)} \otimes_A T_{(g)}$, while the latter is $\operatorname{Spec} S_{(f)} \times_{\operatorname{Spec} A} \operatorname{Spec} T_{(g)}$, and these are the same by the construction of the fiber product. To see that the gluing data for both schemes with this collection of covers is the same, it suffices to note that the restriction map on the structure sheaf from $D(f \otimes g)$ to $D(ff' \otimes gg')$ on the first scheme is the same map as the restriction map on the structure sheaf from $D(f) \times_{\operatorname{Spec} A} D(g)$ to $D(ff') \times_{\operatorname{Spec} A} D(gg')$ on the second scheme. Combining this with the observation that the intersection of $D(f)$ and $D(f')$ is $D(ff')$, this enables us to glue correctly. But the restriction maps from sections over $D(f \otimes g)$ to sections over $D(ff' \otimes gg')$ and sections over $D(f) \times_{\operatorname{Spec} A} D(g)$ to sections over $D(ff') \times_{\operatorname{Spec} A} D(gg')$ are the same: they're both the natural inclusion which maps $(s \otimes t)/(f \otimes g)^n$ to $(sf'^n \otimes tg'^n)/(ff' \otimes gg')^n$. Thus the two schemes have the same gluing data for the cover consisting of open sets of the form $D(f \otimes g) = D(f) \times_A D(g)$, and by uniqueness of gluing we have that they are the same scheme.

The first step in checking the isomorphism of sheaves is to compute sections of both over an affine open set of the form $D(f \otimes g)$ where again we make the assumption that $f \in S$ and $g \in T$ are homogeneous of the same degree. By proposition II.5.11, we have that $\mathcal{O}_{\operatorname{Proj} S \times_A T}(1)(D(f \otimes g)) = (S \times_A T)(1)_{(f \otimes g)}$. As $p_1(D(f \otimes g)) = D(f)$ and $p_2(D(f \otimes g)) = D(g)$, we have $p_1^*(\mathcal{O}_X(1))(D(f \otimes g)) = S(1)_{(f)}$ and $p_2^*(\mathcal{O}_Y(1))(D(f \otimes g)) = T(1)_{(g)}$. As $D(f \otimes g)$ is affine, sections commutes with tensor products, so $(p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1)))(D(f \otimes g)) = S(1)_{(f)} \otimes_{(S \times_A T)_{(f \otimes g)}} T(1)_{(g)}$. It is not difficult to see that these modules are isomorphic via the maps $(s \otimes t)/(f \otimes g)^n \mapsto (s/f^n) \otimes (t/g^n)$ and $(s/f^p) \otimes (t/g^q) \mapsto (f^q s \otimes g^p t)/(f \otimes g)^{p+q}$. The restriction maps from sections over $D(f \otimes g)$ to sections over $D(ff' \otimes gg')$ are exactly given by sending $(s \otimes t)/(f \otimes g)^n$ to $(sf'^n \otimes tg'^n)/(ff' \otimes gg')^n$ while the restriction maps from sections over $D(f) \times_A D(g)$ to sections over $D(ff') \times_A D(gg')$ are given by sending $(s/f^n) \otimes (t/g^n) \mapsto (sf'^n/(ff')^n) \otimes (tg'^n/(gg')^n)$, and these maps are compatible with the isomorphisms above. By the same gluing strategy as in the first part, we are finished.

Exercise II.5.12.

a. Let X be a scheme over a scheme Y, and let $\mathcal{L}, \mathcal{M}$ be two very ample invertible sheaves on X. Show that $\mathcal{L} \otimes \mathcal{M}$ is also very ample. [*Hint*: Use a Segre embedding.]

b. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathcal{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathcal{L} \otimes f^*\mathcal{M}$ is a very ample sheaf on X relative to Z.

Solution.

a. **WARNING**: With the usual definitions (where an immersion is a closed immersion followed by an open immersion, which is the opposite order from Hartshorne) this claim is not true without further assumptions (a usual immersion must be quasi-compact or the source of a usual immersion must be reduced in order for it to also be one of Hartshorne's immersions, see Stacks 01QV and 03DQ). There are potentially some technical upgrades to make to Hartshorne's definition of very ample as well - replacing $\mathbb{P}^r$ with relative proj of the symmetric algebra on a quasi-coherent sheaf on Y and insisting that $f : X \to Y$ be quasi-compact is not uncommon. On to the proof of this result using Hartshorne's setup.

   First, a prelimary result: tensor products commute with pullbacks. We will use exercise II.5.1(c), the tensor-hom adjunction, and the fact that if $f : X \to Y$ is a morphism of ringed spaces, then $f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{N}, \mathcal{P}) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, f_*\mathcal{P})$.

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}, \mathcal{P}) &\cong \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{M}, \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{N}, \mathcal{P})) \\
&\cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{N}, \mathcal{P})) \\
&\cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, f_*\mathcal{P})) \\
&\cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}, f_*\mathcal{P}) \\
&\cong \mathcal{H}om_{\mathcal{O}_X}(f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}), \mathcal{P}).
\end{aligned}$$

   Recall that if $f : X \to Y$ is a morphism of schemes and $\mathcal{L}$ is a line bundle on X, we say that $\mathcal{L}$ is very ample if it is $i^*\mathcal{O}_{\mathbb{P}^r_Y}(1)$ for some immersion $i : X \to \mathbb{P}^r_Y$ so that $f = \pi \circ i$ where $\pi : \mathbb{P}^r_Y \to Y$ is the projection. Let $i_1 : X \to \mathbb{P}^r_Y$ and $i_2 : X \to \mathbb{P}^s_Y$ be the two immersions which exhibit $\mathcal{L}$ and $\mathcal{M}$ as $i_1^*\mathcal{O}(1)$ and $i_2^*\mathcal{O}(1)$, respectively. Denote the Segre embedding $\mathbb{P}^r_Y \times \mathbb{P}^s_Y \to \mathbb{P}^{rs+r+s}_Y$ by S. This is a closed immersion, and so we have that $S \circ (i_1, i_2)$ is an immersion of X in to $\mathbb{P}^{rs+r+s}_Y$.

   Now I claim that $(S \circ (i_1, i_2))^*\mathcal{O}_{\mathbb{P}^{rs+r+s}}(1) \cong \mathcal{L} \otimes \mathcal{M}$. We first note that $S^*\mathcal{O}_{\mathbb{P}^{rs+r+s}}(1) \cong \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(1)$. By exercise II.5.11, we have that $\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(1) \cong p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^s}(1))$, so we have that

$$(S \circ (i_1, i_2))^*\mathcal{O}_{\mathbb{P}^{rs+r+s}}(1) \cong (i_1, i_2)^*(p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^s}(1)))$$

which is isomorphic to

$$(i_1, i_2)^* p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes (i_1, i_2)^* p_2^*(\mathcal{O}_{\mathbb{P}^s}(1))$$

since tensor products commute with pullbacks. Next, by functoriality, we have that

$$(i_1, i_2)^* p_j^* = (p_j \circ (i_1, i_2))^*,$$

and as $p_j \circ (i_1, i_2) = i_j$, we have that our tensor product is just $(i_1^* \mathcal{O}_{\mathbb{P}^r}(1)) \otimes (i_2^* \mathcal{O}_{\mathbb{P}^s}(1))$, or $\mathcal{L} \otimes \mathcal{M}$. So $\mathcal{L} \otimes \mathcal{M}$ is very ample and we are done.

b. **WARNING**: This problem is not correct as stated. We'll first offer the intended proof, then explain how it fails, and finish by suggesting ways to fix it. The key issue is that a composition of immersions according to Hartshorne need not be an immersion. To demonstrate a counterexample, let $R = k[x_1, \cdots]$ be the polynomial ring in infinitely many variables, $X = \operatorname{Spec} R$, and let $U = \bigcup_{n=1}^{\infty} D(x_i)$, so that $U \to X$ is an open immersion. Let $I_n = (x_1^n, \cdots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \cdots) \subset R[x_n^{-1}]$, and note that $I_n R[(x_n x_m)^{-1}] = 1$ for any $m \neq n$, so the quasi-coherent sheaves of ideals $\widetilde{I_n}$ glue to a sheaf of ideals on U and determine a closed subscheme $Z \subset U$. I claim that $Z \to X$ cannot factor as a composition of an open immersion $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is a closed immersion. If this were possible, then $\overline{Z}$ would be defined by an ideal $I \subset R$ so that $I_n = IR[x_n^{-1}]$ for all n. But the only element of R which is in all I_n is zero, a contradiction.

The proof Hartshorne intends us to find here is strongly reminiscent of the diagram chase from exercise II.4.9 where we showed that the composition of projective morphisms were projective. Let $i_X : X \to \mathbb{P}^r_Y$ and $i_Y : Y \to \mathbb{P}^s_Z$ be the immersions which give $\mathcal{L} = i_X^* \mathcal{O}_{\mathbb{P}^r}(1)$ and $\mathcal{M} = i_Y^* \mathcal{O}_{\mathbb{P}^s}(1)$ respectively. Let $\pi_Y : \mathbb{P}^r_Y \to Y$ and $\pi_Z : \mathbb{P}^s_Z$ be the standard projections, so that $f = \pi_Y \circ i_X$ and $g = \pi_Z \circ i_Y$. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & \mathbb{P}^r_Y \arrow[r, "j"] \arrow[d, "\pi_Y"] & \mathbb{P}^r_Y \times_Y \mathbb{P}^s_Z \arrow[r, "S"] \arrow[d, "\rho"] & \mathbb{P}^{rs+r+s}_Z \arrow[ddl] \\
 & Y \arrow[r, "i_Y"] \arrow[dr, "g"'] & \mathbb{P}^s_Z \arrow[d, "\pi_Z"] & \\
 & & Z &
\end{tikzcd}$$

Since open and closed immersions are preserved by fiber products, we have that the base change of an immersion is again an immersion, and thus j is an immersion. If we knew that the composition of immersions was an immersion, then $j \circ i_X$ would be an immersion, and just like in part (a), composing this with the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s_Z \to \mathbb{P}^{rs+r+s}_Z$ would again be an immersion. Supposing this is true, the same diagram chase as in part (a) shows that $\mathcal{L} \otimes f^* \mathcal{M}$ is the pullback of $\mathcal{O}_{\mathbb{P}^{rs+r+s}}(1)$ along this immersion.

So how can we ensure that the composition of two immersions is an immersion? Our goal is given a sequence of maps

$$A \xrightarrow{i_1} B \xrightarrow{i_2} C \xrightarrow{i_3} D \xrightarrow{i_4} E$$

where i_1, i_3 are open immersions and i_2, i_4 are closed immersions is to be able to write $i_3 \circ i_2 : B \to D$ as $B \xrightarrow{j_1} C' \xrightarrow{j_2} D$ where j_1 is an open immersion and j_2 is a closed immersion, which will suffice as the composition of two open (resp. closed) immersions is again an open (resp. closed) immersion. In order to do that, we need to guarantee that either B is reduced or $B \to D$ is quasi-compact by StacksProject tags 03DQ or 01QV respectively. One easy way to do this which fits with Hartshorne's philosophy would be to assume that E is locally noetherian: this suffices since a closed immersion is quasi-compact and an open immersion with a locally noetherian target is also quasi-compact (the first claim follows by exercises II.3.2 and II.3.11, for instance, and the second follows by exercise II.3.2 plus the fact that $\operatorname{Spec} A$ is noetherian for a noetherian ring and every subset of a noetherian topological space is quasi-compact).

In our case, it suffices to assume that Z is locally noetherian, which fits well with Hartshorne's philosophy about noetherianity. Then $\mathbb{P}^a_Z$ is locally noetherian for any integer $a \geq 0$, as if Z is locally noetherian it is covered by open affines $\operatorname{Spec} A_i$ with A_i a noetherian ring, and so $\mathbb{P}^a_Z$ is covered by $\operatorname{Spec} A_i[x_1, \cdots, x_a]$ which is again noetherian by the Hilbert basis theorem. By the discussion above, this solves the problem.

Exercise II.5.13. Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geq 0} S_n^{(d)}$ where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.

This construction is related to the d-uple *embedding* (I, Ex. 2.12) in the following way. If $x_0, \cdots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \hookrightarrow \mathbb{P}^r_A$, then the set of monomials of degree d in the x_i is a set of generators for $S_1^{(d)} = S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbb{P}^r_A$.

Solution. If S is generated by $f_i \in S_1$, then $S^{(d)}$ is generated by $f_i^d \in S_1^{(d)}$, and so just as the open subsets $D(f_i)$ cover $\operatorname{Proj} S$, the open subsets $D(f_i^d)$ cover $\operatorname{Proj} S^{(d)}$. As every element of $S_{(f_i)}$ can be written as a fraction of total degree zero with denominator f_i^{nd} for some $n \geq 0$, we see that $S_{(f_i)} = S^{(d)}_{(f_i^d)}$ and so the affine patches $\operatorname{Spec} S_{(f_i)} = D(f_i) \subset \operatorname{Proj} S$ are isomorphic to $\operatorname{Spec} S^{(d)}_{(f_i)} = D(f_i^d) \subset \operatorname{Proj} S^{(d)}$. The gluing data is the same, too: given $D(f)$ and $D(g)$ for f, g homogeneous of degree one, the overlap is $\operatorname{Spec}(S_{(f)})_{\frac{g}{f}} \cong \operatorname{Spec}(S_{(g)})_{\frac{f}{g}}$, and the same isomorphism given by making sure all our denominators are d^{th} powers by multiplying by 1 gives the result. This shows that $\operatorname{Proj} S \cong \operatorname{Proj} S^{(d)}$ by the uniqueness of gluing (exercise II.2.12).

To see the claim about $\mathcal{O}(d)$, recall that $\mathcal{O}(n)$ on $\operatorname{Proj} R$ is the sheaf that corresponds to $\widetilde{f^n R_{(f)}}$ on $D(f)$ where the gluing data $(\mathcal{O}(n)|_{D(f)})|_{D(g)\cap D(f)} \cong (\mathcal{O}(n)|_{D(g)})|_{D(f)\cap D(g)}$ is the map corresponding to multiplication by $\frac{g^n}{f^n}$ between $(f^n R_{(f)})_{g/f}$ and $(g^n R_{(g)})_{f/g}$. Tracing through the definitions, we see that when we consider $\operatorname{Proj} S^{(d)}$ we get $f^d S_{(f)}$ on $D(f)$ and the gluing data corresponding to multiplication by $\frac{g^d}{f^d}$, which is exactly the information specifying $\mathcal{O}(d)$ on $\operatorname{Proj} S$.

Exercise II.5.14. Let A be a ring, and let X be a closed subscheme of $\mathbb{P}^r_A$. We define the *homogeneous coordinate ring* $S(X)$ of X for the given embedding to be $A[x_0, \cdots, x_r]/I$, where I is the ideal $\Gamma_*(\mathcal{I}_X)$ constructed in the proof of (5.16). (Of course if A is a field and X a variety, this coincides with the definition given in (1, §2)!) Recall that a scheme X is *normal* if its local rings are integrally closed domains. A closed subscheme $X \subset \mathbb{P}^r_A$ is *projectively normal* for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}^r_k$. Show that for some $d > 0$, the d-uple embeding of X is projectively normal, as follows.

a. Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure. [*Hint*: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathscr{S} = \bigoplus_{n\geq 0} \mathcal{O}_X(n)$ on X, and show that $\mathscr{S}$ is a sheaf of integrally closed domains.]

b. Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

c. Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding is projectively normal.

d. As a corollary of (a), show that a closed subscheme $X \subset \mathbb{P}^r_A$ is projectively normal if and only if it is normal, and for every $n \geq 0$ the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

Solution. One sort of annoying thing that happens here is that Hartshorne doesn't specialize to the case of $A = k$ an algebraically closed field until the 6th sentence in, which you could miss if you're not paying attention. Indeed, this is only really relevant for part (c) and dealing with the zeroth graded piece of S and S': everything else is general.

a. First, in any normal scheme, irreducible components are connected components: if two irreducible components met at a point, then this would give two distinct minimal primes in the local ring of this point, contradicting the fact it's a domain by normality. So normal and connected implies irreducible, and by the stalk-local description of reducedness, we have that X is integral.

   To show that S is an integral domain, we check that $I = \Gamma_*(\mathcal{I}_X)$ is prime. It suffices to check this for homogeneous elements $f, g \in I$: if $fg \in I$ but $f, g \notin I$ then both f, g have a homogeneous component of least degree not in I and the multiplication of these homogeneous components is in I. So let $f \in \mathcal{O}_X(d)$ and $g \in \mathcal{O}_X(e)$ so that $fg \in \mathcal{I}_X(d+e)$. Assuming $D(x_i) \cap X \neq \emptyset$, we may restrict the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^r_A} \to \mathcal{O}_X \to 0$ to the affine open $D(x_i)$ and after trivializing appropriately, we see that the conditions above are that $(f/x_i^d)(g/x_i^e) = fg/x_i^{d+e}$ is zero in the integral domain $\mathcal{O}_X(D(x_i) \cap X)$. So one of f/x_i^d and

g/x_i^e must be as well, and then for any other j so that $D(x_j) \cap X \neq \emptyset$, we see that after multiplying by the appropriate power of $\frac{x_j}{x_i}$ and noting that the vanishing set of any element of an integral domain is dense iff it is zero, we see that one of f or g is zero when restricted to X. This implies that it must have been in the appropriate twist of $\mathcal{I}_X$ by exactness.

Now we show that we have a sequence of maps $S \hookrightarrow S' \hookrightarrow \mathrm{Frac}(S)$ where the composite is the natural inclusion of S into its field of fractions. First, by taking global sections of

$$0 \to \bigoplus_{n\in\mathbb{Z}} \mathcal{I}_X(n) \to \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^r_A} \to \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n) \to 0,$$

we get the exact sequence

$$0 \to I \to A[x_0, \cdots, x_n] \to S',$$

and since $A[x_0, \cdots, x_n]/I \cong S$, we have a map $S \to S'$. The same logic as above tells us that S' is an integral domain: given two homogeneous elements $f \in \mathcal{O}_X(d)(X)$ and $g \in \mathcal{O}_X(e)(X)$ whose product is zero in $\mathcal{O}_X(d+e)(X)$, we can restrict them to a nonempty $D(x_i) \cap X$ to see that one must be zero, and whichever one is zero must be zero in every nonempty $D(x_j) \cap X$. Now I claim that for any element f of S'_d, there exists an e and a nonzero element s of S_e so that $sf \in S_{d+e}$, which will show our claim about the sequence of maps $S \hookrightarrow S' \hookrightarrow \mathrm{Frac}(S)$. This is the same as saying that the map $A[x_0, \cdots, x_n]_{d+e} \to S'_{d+e}$ is surjective for some $e \gg 0$. To prove this, we apply the result from exercise II.5.9: taking Γ_* of everything, we get that

$$0 \to \Gamma_*(\mathcal{I}_X) \to \Gamma_*(\mathcal{O}_{\mathbb{P}^r_A}) \to \Gamma_*(\mathcal{O}_X) \to 0$$

is exact as a sequence of graded modules up to the modification of finitely many graded pieces. Picking $n_0 \gg 0$ so that all modified graded pieces are in degree less than n_0, we have that for all $n > n_0$ the sequence

$$0 \to \Gamma(\mathcal{I}_X(n)) \to \Gamma(\mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(\mathcal{O}_X(n)) \to 0$$

is honestly exact, and we may take $e = n_0$. (You could also copy the proof of theorem II.5.19, or there's also a cohomological version of this argument - see for instance theorem III.5.2.)

The above work shows that S' is an integral extension of S, as the two are the same in large-enough degrees. It remains to show that S' is actually the integral closure. First we prove that the integral closure of S in $\mathrm{Frac}(S) = K$ consists only of sums of fractions of graded elements of S. Fix a nonzero element $y \in S_1$. Let K_d denote all the elements in K which can be written as fractions s_1/s_2 with $s_1 \in S_{n+d}$ and $s_2 \in S_d$ for some n. Then $K' = \bigoplus_{d\in\mathbb{Z}} K_d$ is a subring of K isomorphic to $K_0[y, y^{-1}]$ and $K = K_0(y)$. As K is the field of fractions of K' and K' is integrally closed, being the localization of a polynomial ring, we see that the integral closure of S must be contained in K', so it consists entirely of sums of homogeneous fractions.

We can easily see that no sum of homogeneous fractions with a fraction of negative total degree can be integral over S: by considering the homogeneous fraction f of highest negative degree, we see that any integral dependence relation has highest negative degree portion only

f^n for some n, or that $f^n = 0$, which is absurd. So it suffices to prove that any homogeneous fraction of nonnegative degree in $\mathrm{Frac}(S)$ is integral over S.

Now suppose we have a homogeneous element $f \in \mathrm{Frac}(S)$ which satisfies some equation with coefficients in S. Since X is normal, any affine open subscheme $\operatorname{Spec} A \subset X$ has A normal (ref exercise I.3.17(c) if you need a reminder). Restricting to each nonempty $X \cap D(x_i)$ and trivializing by dividing by the appropriate power of x_i, we see that $f/x_i^d \in \mathcal{O}_X(D(x_i))$ satisfies an equation with coefficients in $\mathcal{O}_X(D(x_i))$ - but this is an integrally closed domain, so f/x_i^d is actually in $\mathcal{O}_X(D(x_i))$. Undoing our trivialization, we see that f defines a global section of $\mathcal{O}_X(d)$, and so it's an element of S' and S' is integrally closed.

b. We did this in (a).

c. Pick d large enough that $S_{nd} = S'_{nd}$ for all $n > 0$. This works because any integral dependence equation over $S'^{(d)}$ is also an integral dependence relation over S. Now I claim that $S_0 = S'_0$: this is k in the case that X is nonempty, and 0 otherwise. If $X \neq \emptyset$, then $S_0 = k$ because no nonzero global section can vanish on X. In the same scenario we have that any global section of X determines a map $X \to \mathbb{A}^1_k$, and by exercise II.4.4 the image of this is a closed point. Since k is algebraically closed, every closed point of $\mathbb{A}^1_k$ is a copy of $\operatorname{Spec} k$, so the global function on X are just k. If $X = \emptyset$ then both sides are zero.

   This is the one place where we use the fact that k is an algebraically closed field in a key way, and the claim is false without it: consider $\operatorname{Spec}\mathbb{C} \to \mathbb{P}^1_{\mathbb{R}}$ given by the composition of the closed immersion $\operatorname{Spec}\mathbb{C} = \operatorname{Spec}\mathbb{R}[x]/(x^2+1) \to \mathbb{A}^1_{\mathbb{R}}$ with the standard open immersion $\mathbb{A}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$. Then $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}$ has global sections $\mathbb{R}$ while $\mathcal{O}_X$ has global sections $\mathbb{C}$, and the induced map is not surjective, so by (d) this is not projectively normal and no d-uple embedding will be.

d. If X is projectively normal, then S is integrally closed and so $S = S'$ which shows that $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective for all $n \geq 0$. If the aforementioned map is surjective, then $S = S'$ and so X is projectively normal.

Exercise II.5.15. *Extension of Coherent Sheaves.* We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{F}'$ on X such that $\mathcal{F}'|_U \cong \mathcal{F}$.

a. On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathcal{F}$ is the *union* of its subsheaves $\mathcal{F}_\alpha$ if for every open set U, the group $\mathcal{F}(U)$ is the union of the subgroups $\mathcal{F}_\alpha(U)$.

b. Let X be a noetherian affine scheme, U an open subset, and $\mathcal{F}$ coherent on U. Then there exists a coherent sheaf $\mathcal{F}'$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. [*Hint*: Let $i : U \to X$ be the inclusion map. Show that $i_*\mathcal{F}$ is quasi-coherent, then use (a).]

c. With $X, U, \mathcal{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \subset \mathcal{G}|_U$. Show that we can find $\mathcal{F}'$ a coherent subsheaf of $\mathcal{G}$, with $\mathcal{F}'|_U \cong \mathcal{F}$. [*Hint*: Use the same method, but replace $i_*\mathcal{F}$ by $\rho^{-1}(i_*\mathcal{F})$, where ρ is the natural map $\mathcal{G} \to i_*(\mathcal{G}|_U)$.]

d. Now let X be any noetherian scheme, U an open subset, $\mathcal{F}$ a coherent sheaf on U, and $\mathcal{G}$ a quasi-coherent sheaf on X such that $\mathcal{F} \subset \mathcal{G}|_U$. Show that there is a coherent subsheaf $\mathcal{F}' \subset \mathcal{G}$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. Taking $\mathcal{G} = i_*\mathcal{F}$ proves the result announced at the beginning. [*Hint*: Cover X with open affines, and extend over one of them at a time.]

e. As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf is the union of its coherent subsheaves. [*Hint*: If s is a section of $\mathcal{F}$ over an open set U, apply (d) to the subsheaf of $\mathcal{F}|_U$ generated by s.]

Solution.

a. A quasi-coherent sheaf on an affine scheme $\operatorname{Spec} R$ is given by $\widetilde{M}$ for M an R-module. Since we can write every such M as the union of it's finitely-generated submodules, we can apply $\widetilde{-}$ to both sides of this union and get the desired result because a finitely-generated module corresponds to a coherent sheaf on a noetherian affine scheme.

b. Let $X = \operatorname{Spec} A$ be a noetherian affine scheme, U an open subset, $i : U \to X$ the inclusion, and $\mathcal{F}$ coherent on U. Because U is noetherian, we have $i_*\mathcal{F}$ is quasi-coherent by proposition II.5.8(c). As U is noetherian, it is quasi-compact, so we can cover it by finitely many affine opens $\operatorname{Spec} A_{f_i}$. But $(i_*\mathcal{F})|_{\operatorname{Spec} A_{f_i}}$ is just $\mathcal{F}|_{\operatorname{Spec} A_{f_i}}$, which is coherent, so it is generated by finitely many sections. By lemma II.5.3(b), after multiplication by f_i^n, these sections extend to global sections of $i_*\mathcal{F}$, and multiplication by f_i^n doesn't affect their ability to generate $\mathcal{F}|_{\operatorname{Spec} A_{f_i}}$. So there are finitely many global sections of $i_*\mathcal{F}$ which after restriction suffice to generate $\mathcal{F}$, and we may simply take the finitely-generated submodule spanned by these sections as our coherent sheaf $\mathcal{F}'$. (We don't really need to use (a) here, though it is a useful fact to know.)

c. The same proof as in (b) suffices: cover U by finitely many open affines of the form $\operatorname{Spec} A_{f_i}$, construct a finite set of global sections of $\mathcal{G}$ which suffice to generate $\mathcal{F}$, and let $\mathcal{F}'$ be the sheaf generated by these sections. (I know we're ignoring the hints, but sometimes the hints aren't the most helpful thing in the world.)

d. As per the hint, we can cover X by finitely many affine opens $\operatorname{Spec} A_i$. Extend $\mathcal{F}|_{U \cap \operatorname{Spec} A_1}$ to a coherent sheaf $\mathcal{F}_1$ on $\operatorname{Spec} A_i$, and then glue $\mathcal{F}_1$ and $\mathcal{F}$ along the open set $U \cap \operatorname{Spec} A_i$ to a coherent sheaf on $U \cup \operatorname{Spec} A_1$. This works because coherence is a local property. Repeating the above process, the result follows.

e. Let X be our noetherian scheme with open subset U and let $\mathcal{G}$ be the subsheaf of $\mathcal{F}|_U$ generated by s. This is coherent because its restriction to an affine open $\operatorname{Spec} A \subset U$ is the sheafification of the submodule generated by s in $(\mathcal{F}|_U)(\operatorname{Spec} A)$, which is clearly finitely generated. So $\mathcal{G}$ extends to a coherent sheaf on X by (d) and we're done.

Exercise II.5.16. *Tensor Operations on Sheaves.* First we recall the definitions of various tensor operations on a module. Let A be a ring, and let M be an A-module. Let $T^n(M)$ be the tensor product $M \otimes \cdots \otimes M$ of M with itself n times, for $n \geq 1$. For $n = 0$ we put $T^0(M) = A$. Then

$T(M) = \bigoplus_{n\geq 0} T^n(M)$ is a (noncommutative) A-algebra, which we call the *tensor algebra* of M. We define the *symmetric algebra* $S(M) = \bigoplus_{n\geq 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A-algebra. Its component $S^n(M)$ in degree n is called the nth *symmetric product* of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x, y \in M$. As an example, note that if M is a free A-module of rank r, then $S(M) \cong A[x_1, \cdots, x_r]$.

We define the *exterior algebra* $\bigwedge(M) = \bigoplus_{n\geq 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a *skew commutative* graded A-algebra. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$ (he we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$). The nth component $\bigwedge^n(M)$ is called the nth *exterior power* of M.

Now let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *tensor algebra*, *symmetric algebra*, and *exterior algebra* of $\mathcal{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_X$-algebras, and their components in each degree are $\mathcal{O}_X$-modules.

a. Suppose that $\mathcal{F}$ is locally free of rank n. Then $T^r(\mathcal{F})$, $S^r(\mathcal{F})$, and $\bigwedge^r(\mathcal{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

b. Again let $\mathcal{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r \mathcal{F} \otimes \bigwedge^{n-r} \mathcal{F} \to \bigwedge^n \mathcal{F}$ is a perfect pairing for any r, i.e. it induces and isomorphism of $\bigwedge^r \mathcal{F}$ with $(\bigwedge^{n-r} \mathcal{F})^\vee \otimes \bigwedge^n \mathcal{F}$. As a special case, note if $\mathcal{F}$ has rank 2, then $\mathcal{F} \cong \mathcal{F}^\vee \otimes \bigwedge^2 \mathcal{F}$.

c. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathcal{F})$,
$$S^r(\mathcal{F}) = F^0 \supset F^1 \supset \cdots \supset F^r \supset F^{r+1} = 0$$
with quotients
$$F^p/F^{p+1} \cong S^p(\mathcal{F}') \otimes S^{r-p}(\mathcal{F}'')$$
for each p.

d. Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ have ranks n', n, n'' respectively, there is an isomorphism $\bigwedge^n \mathcal{F} \cong \bigwedge^{n'} \mathcal{F}' \otimes \bigwedge n'' \mathcal{F}''$.

e. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be an $\mathcal{O}_Y$-moduile. Then f^* commutes with all the tensor operations on $\mathcal{F}$, i.e., $f^*(S^n(\mathcal{F})) = S^n(f^*\mathcal{F})$ etc.

Solution.

a. It suffices to prove these statements locally, so we might as well asssume $\mathcal{F} \cong \mathcal{O}_X^n$. The statements now follow from the corresponding statements for modules: picking $e_1, \cdots, e_n$ as a basis for $\mathcal{F}$, we get that any sequence of length r of integers between 1 and n gives a unique basis element for the first, any non-descending such sequence gives a unique basis element for the second, and any strictly ascending such sequence gives a unique basis element for the third. The number of such sequences is well known to be n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

b. All the relevant maps are defined globally, so we can check whether this is an isomorphism locally. As in (a), we may assume that $\mathcal{F} \cong \mathcal{O}_X^n$ with basis $e_1, \cdots, e_n$. Then by (a), $\bigwedge^r \mathcal{F}$ is free on $e_{i_1} \wedge \cdots \wedge e_{i_r}$ where $1 \leq i_1 < \cdots < i_r \leq n$. The multiplication map is exactly

$$f(e_{i_1} \wedge \cdots e_{i_r}) \cdot g(e_{j_1} \wedge \cdots \wedge e_{j_{n-r}}) \mapsto fg(e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{n-r}}$$

and it's clear that this induces an isomorphism of $\bigwedge^r \mathcal{F}$ with $\mathcal{H}om_{\mathcal{O}_X}(\bigwedge^{n-r} \mathcal{F}, \bigwedge^n \mathcal{F})$: any such map is determined by where it sends the basis vectors, and given a map sending $e_{j_1} \wedge \cdots \wedge e_{j_{n-r}} \mapsto fe_1 \wedge \cdots \wedge e_n$ and all other basis vectors to zero, this is the map given by multiplying by $\pm fe_{i_1} \wedge \cdots \wedge e_{i_r}$ where the e_i are the complement of the e_j. Adding, we get the result.

c. We define a filtration on $S^r(\mathcal{F})$ so that that F^p consists of tensors with at most p coordinates coming from $\mathcal{F}'$. The way we construct this filtration is that we set it up in a basis-independent way on open subsets $U \subset X$ where each of $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ are free, and then check that it glues in to a filtration of the global sheaves. Fix an arbitrary such U, and choose any splitting $\mathcal{F}|_U \cong \mathcal{F}'|_U \oplus \mathcal{F}|''_U$. This gives us

$$S^r(\mathcal{F}|_U) \cong \bigoplus_{i=0}^{r} S^i(\mathcal{F}'|_U) \otimes S^{r-i}(\mathcal{F}''|_U).$$

Now we construct the filtration by induction - the idea of our strategy is to set F^i to be those tensor products which have at least i entries coming from a basis of $\mathcal{F}'|_U$. To do this, set $F^{r+1} = 0$, and assume we've picked $F^{j+1}, \cdots, F^{r+1}$ satisfying the requested properties. To construct F^j, consider the image of

$$\varphi : S^j(\mathcal{F}'|_U) \otimes S^{r-j}(\mathcal{F}''|_U) \to S^r(\mathcal{F}|_U)/F^{j+1}.$$

I claim the preimage of this under the projection $\pi : S^r(\mathcal{F}|_U) \to S^r(\mathcal{F}|_U)/F^{j+1}$ is independent of the chosen splitting: pick a basis $x_1, \cdots, x_s$ for $\mathcal{F}'|_U$ and a basis $y_1, \cdots, y_t$ for $\mathcal{F}''|_U$, where we identify each with their image inside $\mathcal{F}|_U$. The image of φ is the span of the collection

$$x_{p_1} \otimes \cdots \otimes x_{p_j} \otimes y_{q_1} \otimes \cdots \otimes y_{q_{r-j}}$$

as the p range over $1, \cdots, s$ and the q range over $1, \cdots, t$. Then for any other basis $y_1 + c_1, \cdots, y_t + c_t$ with $c_i \in \mathcal{F}'|_U$, the image is the span of the collection

$$x_{p_1} \otimes \cdots \otimes x_{p_j} \otimes (y_{q_1} + c_{q_1}) \otimes \cdots \otimes (y_{q_{r-j}} + c_{q_{r-j}}).$$

But expanding the tensor product in this sum, we see that it differs from a vector in our original collection by tensor products which have at least $j+1$ entries which are selected from the xs, and thus this difference is in F^{j+1}, so our claim about basis-independence is proven, and we can set F^j to be this preimage.

This means our filtration F^j is basis-independent, and in particular, it's compatible with restriction maps between open subsets where $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are all free since any basis of the sheaves on the larger set is still a basis on the smaller set. Since any scheme is covered by such opens, this means we can patch our filtration together to an honest filtration of sheaves with the specified quotients globally, and we're done.

d. We're going to use the same strategy as part (c): we're going to define a filtration of $\bigwedge^r \mathcal{F}$,

$$\bigwedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

which has successive quotients $F^p/F^{p+1} = \bigwedge^p \mathcal{F}' \otimes \bigwedge^{r-p} \mathcal{F}''$. Considering $r = \operatorname{rank} \mathcal{F}$, this will imply the 'in particular' result, as the filtration will reduce to $\bigwedge^r \mathcal{F} \cong \bigwedge^{rk\mathcal{F}'} \mathcal{F}' \otimes \bigwedge^{rk\mathcal{F}''} \mathcal{F}''$.

The way we construct this filtration is that we set it up in a basis-independent way on open subsets $U \subset X$ where each of $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ are free, and then check that it glues in to a filtration of the global sheaves. Fix an arbitrary such U, and choose any splitting $\mathcal{F}|_U \cong \mathcal{F}'|_U \oplus \mathcal{F}|''_U$. This gives us

$$\bigwedge^r \mathcal{F}|_U \cong \bigoplus_{i=0}^r \left(\bigwedge^i \mathcal{F}'|_U \right) \otimes \left(\bigwedge^{r-i} \mathcal{F}''|_U \right).$$

Now we construct the filtration by induction - the idea of our strategy is to set F^i to be those wedge products which have at least i entries coming from a basis of $\mathcal{F}'|_U$. To do this, set $F^{r+1} = 0$, and assume we've picked $F^{j+1}, \cdots, F^{r+1}$ satisfying the requested properties. To construct F^j, consider the image of

$$\varphi : \bigwedge^j \mathcal{F}'|_U \otimes \bigwedge^{r-j} \mathcal{F}''|_U \to \left(\bigwedge^r \mathcal{F}|_U \right) / F^{j+1}.$$

I claim the preimage of this under the projection $\pi : \bigwedge^r \mathcal{F}|_U \to \left(\bigwedge^r \mathcal{F}|_U\right)/F^{j+1}$ is independent of the chosen splitting: pick a basis $x_1, \cdots, x_s$ for $\mathcal{F}'|_U$ and a basis $y_1, \cdots, y_t$ for $\mathcal{F}''|_U$, where we identify each with their image inside $\mathcal{F}|_U$. The image of φ is the span of the collection

$$x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge y_{q_1} \wedge \cdots \wedge y_{q_{r-j}}$$

as the p range over $1, \cdots, s$ and the q range over $1, \cdots, t$. Then for any other basis $y_1 + c_1, \cdots, y_t + c_t$ with $c_i \in \mathcal{F}'|_U$, the image is the span of the collection

$$x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge (y_{q_1} + c_{q_1}) \wedge \cdots \wedge (y_{q_{r-j}} + c_{q_{r-j}}).$$

But expanding the wedge product in this sum, we see that it differs from a vector in our original collection by wedge products which have at least $j+1$ entries which are selected from the xs, and thus this difference is in F^{j+1}, so our claim about basis-independence is proven, and we can set F^j to be this preimage.

This means our filtration F^j is basis-independent, and in particular, it's compatible with restriction maps between open subsets where $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are all free since any basis of the sheaves on the larger set is still a basis on the smaller set. Since any scheme is covered by such opens, this means we can patch our filtration together to an honest filtration of sheaves with the specified quotients globally, and we're done.

e. See exercise II.5.12(a) for a proof that tensor products commute with pullbacks, which gives us that $f^*T^n(\mathcal{F}) = T^n(\mathcal{F})$. So given the exact sequence $0 \to \mathcal{I} \to T^n(\mathcal{F}) \to S^n(\mathcal{F}) \to 0$ we get the exact sequence $f^*\mathcal{I} \to T^n(f^*\mathcal{F}) \to f^*S^n(\mathcal{F}) \to 0$ after applying the right-exact functor f^*. It suffices to show that $f^*\mathcal{I}$ surjects on to the subsheaf of $T^n(f^*\mathcal{F})$ generated by expressions of the form $x \otimes y - y \otimes x$: once we do that, we win, because $S^n(f^*\mathcal{F})$ and $f^*S^n(\mathcal{F})$ are the same quotient.

Suppose we have a local section of $T^n(f^*\mathcal{F})$ of the form $x \otimes y \otimes \cdots - y \otimes x \otimes \cdots$, and we want to show that it's in the image of $f^*\mathcal{I} \to T^n(f^*\mathcal{F})$. After expanding the elements in $\cdots$ as sums of simple tensors, it suffices to show the claim in the case that $x = \sum a_i \otimes g_i$ and $y = \sum b_i \otimes h_i$ where a_i, b_i, and all entries of $\cdots$ are local sections of $f^{-1}\mathcal{F}$ and g_i, h_i are local sections of $\mathcal{O}_X$. By expanding the tensors, we can write $x \otimes y \otimes \cdots - y \otimes x \otimes \cdots$ as a sum of elements of the form $g_i h_j (a_i \otimes b_j \otimes \cdots - b_j \otimes a_i \otimes \cdots)$ and the claim is clear.

The proof for the exterior product is the same, except that we treat a local section of the form $x \otimes x \otimes \cdots$ where we make the same assumptions as above and expand $x \otimes x \otimes \cdots$ as a sum of terms of the form $g_i^2(a_i \otimes a_i \otimes \cdots)$ and $g_i g_j (a_i \otimes a_j \otimes \cdots + a_j \otimes a_i \otimes \cdots)$.

Exercise II.5.17. *Affine Morphisms*. A morphism $f : X \to Y$ of schemes is *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

a. Show that $f : X \to Y$ is an affine morphism if and only if for *every* open affine $V \subset Y$, $f^{-1}(V)$ is affine. [*Hint*: Reduce to the case Y affine, and use (Ex. 2.17).]

b. An affine morphism is quasi-compact and separated. Any finite morphism is affine.

c. Let Y be a scheme, and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subset Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \hookrightarrow V$ of open affines of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. The scheme X is called $\mathbf{Spec}\,\mathcal{A}$. [*Hint*: Construct X by glueing together the schemes $\operatorname{Spec} \mathcal{A}(V)$, for V open affine in Y.]

d. If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : \mathbf{Spec}\,\mathcal{A} \to Y$ is an affine morphism, and $\mathcal{A} \cong f_*\mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism, then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \mathbf{Spec}\,\mathcal{A}$.

e. Let $f : X \to Y$ be an affine morphism, and let $\mathcal{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{A}$-modules (i.e., quasi-coherent $\mathcal{O}_Y$-modules having a structure of $\mathcal{A}$-module). [*Hint*: For any quasi-coherent $\mathcal{A}$-module $\mathcal{M}$, construct a quasi-coherent $\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$, and show that the functors f_* and $\widetilde{-}$ are inverse to each other.

Solution.

a. Let $\operatorname{Spec} A \subset Y$ be an open affine. Cover $\operatorname{Spec} A \cap V_i$ by affine opens which are simultaneously distinguished in both $\operatorname{Spec} A$ and V_i (see the solution for exercise II.3.1 if you need a reminder on how this works). But the preimage of $D(f) \subset \operatorname{Spec} S$ under an affine morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ corresponding to the ring morphism φ is just $\operatorname{Spec} R_{\varphi(f)}$, so our morphism restricted to $\operatorname{Spec} A$ is affine and it suffices to prove the claim when $Y = \operatorname{Spec} A$.

   Now let $R = \mathcal{O}_X(X)$, which is an A-module via the pullback $f^\sharp : A \to \mathcal{O}_X(X)$. By the same logic from the first paragraph, we may assume that all the V_i are actually distinguished open affines $\operatorname{Spec} A_{f_i} \subset \operatorname{Spec} A$, and since $\operatorname{Spec} A$ is quasi-compact, we may assume that we have a finite cover. Since the $\operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$, we must have that the ideal generated by the f_i is the unit ideal, or that there exist $a_i \in A$ so that $\sum a_i f_i = 1$. But this means that the images $f^\sharp(f_i)$ generate the unit ideal of $\mathcal{O}_X(X)$ as well, by applying $f^\sharp$ to the same equation. By exercise II.2.17, we're now finished: the $f^\sharp(f_i)$ generate the unit ideal of $\mathcal{O}_X(X)$ and each X_{f_i} is affine by assumption.

b. Any quasi-compact set can be covered by finitely many open affines. By (a), the preimage of any affine open under an affine morphism is in fact affine, so the preimage of any quasi-compact set can be covered by finitely many affine opens. As a finite union of quasi-compact spaces is again quasi-compact, we have the desired result.

   Affine morphisms being separated was proven in exercise II.4.1, and finite morphisms are affine by definition.

c. Per the hint, we want to glue the schemes $\operatorname{Spec} \mathcal{A}(V)$ as V runs over the open affine subschemes of Y. To do this, we need to check compatibility on overlaps by the gluing lemma from exercise II.2.12. Supposing $U, V \subset Y$ are affine open subschemes, their intersection can be covered by simultaneously distinguished open affines W_i. By quasi-coherence of $\mathcal{A}$, this gives that $\mathcal{A}(W_i)$ is a simultaneous localization of $\mathcal{A}(U)$ and $\mathcal{A}(V)$ and so its spectrum is a simultaneously distinguished open affine subset of $\operatorname{Spec} \mathcal{A}(U)$ and $\operatorname{Spec} \mathcal{A}(V)$ with the same coordinate ring, $\mathcal{A}(W)$. This gives the pair-wise isomorphisms. To verify compatibility on triple intersections, we use the same construction, except covering the intersection by affine opens simultaneously distinguished in all three pieces we intersect and getting compatibility of the maps from compatibility of restriction maps, since $\mathcal{A}$ is a sheaf.

d. For the first statement, it is clear that f is affine by the construction in (c). To see that $\mathcal{A} \cong f_*\mathcal{O}_X$, we verify that they have the same sections and restriction maps over all affine open subschemes $V \subset Y$: this suffices to show that they are isomorphic, as sheaves can be uniquely reconstructed from the data of their values and restriction morphisms on a basis and the affine opens form a basis. Again, this is clear by construction: $f^{-1}(V) = \operatorname{Spec} \mathcal{A}(V)$ which has sections $\mathcal{A}(V)$, so $\mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$ and similarly for the restriction morphisms.

For the converse, let $V = \operatorname{Spec} B \subset Y$ be an open affine with preimage $U = \operatorname{Spec} A \subset X$. Then $\mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V)) = A$ and so $\mathcal{A}|_V$ is just the sheafification of A viewed as a B-module, so it is quasi-coherent. By uniqueness from part (c), we have the claim that $X \cong \mathbf{Spec}\,\mathcal{A}$: X satisfies all the required properties of $\mathbf{Spec}\,\mathcal{A}$.

e. Given a $\mathcal{A}$-module $\mathcal{M}$, we let $\widetilde{\mathcal{M}}$ be the $\mathcal{O}_X$-module which is the gluing of the sheaves $\widetilde{\mathcal{M}(U)}$ on $\operatorname{Spec} \mathcal{A}(U) \subset X$ as U ranges over all open affine subschemes of Y. We use the same technique as in (d) to prove this works: covering the intersections of two affine open subschemes of Y by simultaneously distinguished affine opens and using the fact that $\mathcal{M}$ is quasi-coherent, we obtain the required sheaf on X by gluing.

To demonstrate the claim that f_* and $\widetilde{-}$ are mutually inverse, we compute the compositions. Over any open affine $U \subset Y$, the sheaf $f_*\widetilde{\mathcal{M}}$ has sections isomorphic to $\mathcal{M}(U)$ by an application of corollary II.5.5 and these isomorphisms agree upon restriction to further affine open subsets, so they glue to define an isomorphism of sheaves. Conversely, if $U \subset Y$ is open, then $(\widetilde{f_*\mathcal{M}})(U) \cong \mathcal{M}|_U$ by an application of corollary II.5.5 and by the fact that this is quasi-coherent and the overlaps of any two open affines in Y can be covered by simultaneously distinguished open affines, we can glue these isomorphisms together to show that $\widetilde{f_*\mathcal{M}} \cong \mathcal{M}$.

Exercise II.5.18. *Vector Bundles.* Let Y be a scheme. *A (geometric) vector bundle* of rank n over Y is a scheme X and a morphism $f : X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$, such that for any i, j, and for any open affine subset $V = \operatorname{Spec} A \subset U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_V = \operatorname{Spec} A[x_1, \cdots, x_n]$ is given by a *linear* automorphism θ of $A[x_1, \cdots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$.

An *isomorphism* $g : (X, f, \{U_i\}, \{\psi_i\}) \to (X', f', \{U_i'\}, \{\psi_i'\})$ of one vector bundle of rank n to another one is an isomorphism $g : X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f, together with the covering of Y consisting of all the U_i and U_i', and the isomorphisms ψ_i and $\psi_i' \circ g$, is also a vector bundle structure on X.

a. Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathcal{E})$ be the symmetric algebra on $\mathcal{E}$, and let $X = \mathbf{Spec}\, S(\mathcal{E})$, with projection morphism $f : X \to Y$. For each open affine subset $U \subset Y$ for which $\mathcal{E}|_U$ is free, choose a basis of $\mathcal{E}$, and let $\psi : f^{-1}(U) \to \mathbb{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathcal{E}(U))$ with $\mathcal{O}(U)[x_1, \cdots, x_n]$. Then $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathcal{E}_U$ chosen. We call it the *geometric vector bundle associated* to $\mathcal{E}$, and denote it by $\mathbb{V}(\mathcal{E})$.

b. For any morphism $f : X \to Y$, a *section* of f over an open set $U \subset Y$ is a morphism $s : U \to X$ such that $f \circ s = id_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{$ set of sections of f over $U\}$ is a sheaf of sets on Y, which we denote by $\mathscr{S}(X,Y)$. Show that if $f : X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathscr{S}(X,Y)$ has a natural structure of $\mathcal{O}_Y$-module, which makes it a locally free $\mathcal{O}_Y$-module of rank n. [*Hint*: It is enough to define the module structure locally, so we can assume $Y = \operatorname{Spec} A$ is affine, and $X = \mathbb{A}^n_Y$. Then a section $s : Y \to X$ comes from an A-algebra homomorphism $\theta : A[x_1, \cdots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \cdots, \theta(x_n)\rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

c. Again let $\mathcal{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbb{V}(\mathcal{E})$, and let $\mathscr{S} = \mathscr{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathscr{S} \cong \mathcal{E}^\vee$, as follows. Given a section $s \in \Gamma(V, \mathcal{E}^\vee)$ over any open set V, we think of s as an element of $\operatorname{Hom}(\mathcal{E}|_V, \mathcal{O}_V)$. So s determines an $\mathcal{O}_V$-algebra homomorphism $S(\mathcal{E}|_V) \to \mathcal{O}_V$. This determines a morphism of spectra $V = \mathbf{Spec}\, \mathcal{O}_V \to \mathbf{Spec}\, S(\mathcal{E}|_V) = f^{-1}(V)$, which is a section of X/Y. Show that this construction gives an isomorphism of $\mathcal{E}^\vee$ to $\mathscr{S}$.

d. Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words 'locally free sheaf' and 'vector bundle' interchangeably, if no confusion seems likely to result.

Solution. This has been written about approximately ∞ times, and Hartshorne tells you exactly what's happening in the exercise statements! If this isn't satisfying to you, go do some searching on your own - StacksProject, Shafarevich's *Basic Algebraic Geometry II*, etc.

a. There's not really much to do here - the problem statement gives everything away. All we have to do is show that the automorphisms we get are linear and don't depend on the choice of basis, and this is easy. For the first condition, if U_1, U_2 are two affine open sets where $\mathcal{E}$ is free, then there's a linear automorphism which takes the restriction of one basis to the other on any affine open subscheme inside $U_1 \cap U_2$. For the second, note that changing the basis of $\mathcal{E}|_U$ can be corrected by applying an automorphism of $\mathbb{A}^n_U$ and we still get the same vector bundle up to isomorphism.

b. The hint solves the problem: assume $Y = \operatorname{Spec} A$ affine, $X = \mathbb{A}^n_Y$, and then a section $Y \to X$ corresponds to an A-algebra homomorphism $A[x_1, \cdots, x_n] \to A$, and these are exactly specified by ordered n-tuples of elements of A: send x_i to the i^{th} element. So sections are in bijection with elements of A^n, and this exactly tells you how to define the module structure: just use the obvious structure on A^n.

c. A section $s \in \mathcal{E}^\vee(V)$ represents a $\mathcal{O}_V$-module homomorphism $\mathcal{E}|_V \to \mathcal{O}_V$, and we can take the symmetric algebra of $\mathcal{E}|_V$ to get a map $S(\mathcal{E}|_V) \to \mathcal{O}_V$. Taking **Spec** of this morphism, we get a map $V \to \mathbb{V}(\mathcal{E}|_V) = f^{-1}(V)$, which is exactly a section of $\mathscr{S}$ over V. Going the other way, we simply reverse all of these steps: section of $\mathscr{S}(V)$ gives a map $V \to f^{-1}(V)$, inducing a map $S(\mathcal{E}|_V) \to \mathcal{O}_V$, from which we can recover a map $\mathcal{E}|_V \to \mathcal{O}_V$, or an element of $\mathcal{E}^\vee(V)$. It is clear that these processes are mutually inverse.

d. Yes, we did this.

II.6 Divisors

Hartshorne refers to condition (*) many times here. For a reminder, this condition is X is a noetherian integral separated scheme which is regular in codimension one. Hartshorne's version of Algebraic Hartogs is proposition II.6.3A, which somehow I spent an hour searching for.

Hartshorne's treatment of curves in this section suffers for not making the assumption that k is algebraically closed more obvious. Basically every problem referring to a complete curve needs to have the reminder that $k = \overline{k}$ attached to it or be modified for the case that k is not algebraically closed. The development of Weil and Cartier divisors is general, though.

Exercise II.6.1. Let X be a scheme satisfying (*). Then $X \times \mathbb{P}^n$ also satisfies (*), and $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbb{Z}$.

Solution. Let's go down the list one by one.

If X is noetherian, it can be covered by finitely many affine opens $\operatorname{Spec} A_i$ where each A_i is a noetherian ring. $\mathbb{P}^n_{\mathbb{Z}}$ can be covered by the $n+1$ copies of $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t_1, \cdots, t_n]$, so $X \times \mathbb{P}^n$ is covered by $\operatorname{Spec} A_i[t_1, \cdots, t_n]$ which is noetherian by Hilbert's basis theorem. $X \times \mathbb{P}^n$ having a finite open cover by noetherian affine schemes implies that $X \times \mathbb{P}^n$ is noetherian.

For integrality, we first note that as X is integral, the A_i are integral domains and therefore so are the $A_i[t_1, \cdots, t_n]$, so the proof of noetherianity above implies that $X \times \mathbb{P}^n$ is reduced. To show that $X \times \mathbb{P}^n$ is irreducible, we apply the criteria for irreducibility from problem II.3.15: a scheme S is irreducible if and only if there exists an affine open covering $S = \bigcup_{i \in I} U_i$ so that I is nonempty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Here our U_i are $\operatorname{Spec} A_i[t_1, \cdots, t_n]$: these are all irreducible affine opens, and their common intersection is exactly $(\bigcap_{i \in I} \operatorname{Spec} A_i) \times (D(t_0 t_1 \cdots t_n))$. As $X \times \mathbb{P}^n$ is reduced and irreducible, it is integral.

For separatedness, we can apply corollary II.4.6(d) to the morphisms $X \to \operatorname{Spec} \mathbb{Z}$ (separated by assumption) and $\mathbb{P}^n \to \operatorname{Spec} \mathbb{Z}$ (separated by theorem II.4.9) to get that $X \times \mathbb{P}^n \to \operatorname{Spec} \mathbb{Z} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$ is separated, and hence $X \times \mathbb{P}^n$ is separated.

Regular in codimension one is taken care of by the proof of proposition II.6.6: every codimension one point is contained in some $X \times \mathbb{A}^n$, and as $X \times \mathbb{A}^n = X \times \mathbb{A}^1 \times \cdots \times \mathbb{A}^1$, we have the result by induction.

To show the final claim about $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbb{Z}$, we use the exact sequence of proposition II.6.5. Let $i : X \times \mathbb{P}^{n-1} \to X \times \mathbb{P}^n$ be the closed immersion with image $X \times V(T_0)$, and $j : X \times \mathbb{A}^n \to X \times \mathbb{P}^n$ be the open immersion with image $X \times D(T_0)$. By proposition II.6.5, we have an exact sequence

$$\mathbb{Z} \cdot (X \times \mathbb{P}^{n-1}) \to \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X \times \mathbb{A}^n) \to 0$$

and the third item is exactly $\mathrm{Cl}(X)$ by repeated applications of proposition II.6.6. It remains to show that the first map is injective and the exact sequence splits.

Let Z denote the image of $i : X \times \mathbb{P}^{n-1} \to X \times \mathbb{P}^n$. If aZ is a principal divisor, then $aZ \cap \mathbb{P}^n_{k(X)}$ must also be a principal divisor: the nonzero function in $k(X \times \mathbb{P}^n)$ cutting out aZ is also a nonzero function in $k(\mathbb{P}^n_{k(X)})$ which cuts out the divisor $aZ \cap \mathbb{P}^n_{k(X)}$. So to show that aZ cannot be principal for any $a \neq 0$, it suffices to show that $aZ \cap \mathbb{P}^n_{k(X)}$ is not principal for any a.

Suppose that $aZ \cap \mathbb{P}^n_{k(X)}$ is principal for some $a \neq 0$, and let $f \in k(X \times \mathbb{P}^n)$ be a rational function with divisor $aZ \cap \mathbb{P}^n_{k(X)}$. Looking at $\mathbb{A}^n_{k(X)} = D(T_0) \subset \mathbb{P}^n_{k(X)}$, we see that since $Z \cap \mathbb{P}^n_{k(X)}$ is disjoint from $D(T_0)$ the valuation of f is zero at every codimension one point and we must have that f and $1/f$ are both in the localization of $k(X)[t_1, \cdots, t_n]$ at every height one prime. Therefore by proposition II.6.3A, f and $1/f$ belong to $k(X)[t_1, \cdots, t_n]$ and are units in that ring. As the only units in a polynomial ring over a field are the elements of the field, we have that $f \in k(X)^\times$, so its valuation in the local ring of $V(T_0)$ is zero. Thus aZ is not a principal divisor for any $a \neq 0$ and the map $\mathbb{Z} \cdot (X \times \mathbb{P}^{n-1}) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ is injective.

We can define a splitting of $\mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X \times \mathbb{A}^n)$ by writing a map $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ given on prime divisor by sending D_i to $D_i \times \mathbb{P}^n$ and composing with the isomorphism $\mathrm{Cl}(X \times \mathbb{A}^n) \to \mathrm{Cl}(X)$ from proposition II.6.6. This map sends a prime divisor $D_i \subset X$ to $D_i \times \mathbb{P}^n$ and then $D_i \times \mathbb{A}^n$ and finally back to D_i by construction, so this is a splitting, and by a general lemma from homological algebra this splitting gives that $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$ and we're finished.

Exercise II.6.2. (*) *Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one (hence satsifies (*)). For any divisor $D = \sum n_i Y_i$ on X, we define the *degree* of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself (I, §7).

a. Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbb{P}^n$ for which $Y_i \cap U_i \neq \emptyset$, and let $n_i = v_{Y_i}(\overline{f_i})$, where $\overline{f_i}$ is the restriction of f_i to $U_i \cap X$. Then we define the *divisor* $V.X$ to be $\sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of $\mathrm{Div}\, \mathbb{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\, X$.

b. If D is a principal divisor on $\mathbb{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus we get a homomorphism $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, X$.

c. Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y)$ defined in (I, §7). Then use the generalized Bezout theorem (I, 7.7) to show that for any divisor D on $\mathbb{P}^n$, none of whose components contain X,

$$\deg(D.X) = (\deg D) \cdot (\deg X).$$

d. If D is a principal divisor on X, show that there is a rational function f on $\mathbb{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism $\deg : \mathrm{Cl}\, X \to \mathbb{Z}$. (This gives another proof of (6.10), since any complete nonsingular curve is projective.) Finally, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Cl}\, \mathbb{P}^n \arrow[r] \arrow[d, "\deg", "\cong"'] & \mathrm{Cl}\, X \arrow[d, "\deg"] \\ \mathbb{Z} \arrow[r, "\cdot(\deg X)"] & \mathbb{Z} \end{tikzcd}$$

and in particular, we see that the map $\operatorname{Cl}\mathbb{P}^n \to \operatorname{Cl}X$ is injective.

Solution. Hartshorne says 'subvariety' here - just to remind ourselves, this means an integral subscheme.

a. To show well-definedness of our map, we note that for two choices U_i, U_i' with corresponding local equations f_i, f_i', the equations must agree up to a unit on the nonempty overlap $U_i \cap U_i'$ (remember, everything in sight is irreducible). This means that they determine the same element of $\mathcal{O}_{X,Y_i}$ up to a unit, so $\nu_{Y_i}(\overline{f_i})$ does not depend on the choice of U_i and f_i. Our map is therefore well-defined and we can extend linearly exactly as requested.

b. This is essentially clear by definition: if $f \in k(\mathbb{P}^n)^*$ is a function with $(f) = D$ and no component of D contains X, then we can find an affine open $U \subset \mathbb{P}^n$ where $U \cap X \neq \emptyset$ and f defines an honest regular function. Then via the composite $k[U] \to k[U \cap X] \to k(X)$, we see that $\overline{f}$ is a regular function defining $D.X$, so $D.X$ is a principal divisor and we're done.

   To be more explicit about the argument, suppose Y_i is a prime divisor intersecting X which has nonzero valuation in D. Then we can find an open neighborhood in $\mathbb{P}^n$ which intersects Y_i and X on which Y_i is defined by a local equation f_i, and up to shrinking our neighborhood we get that $f = u \cdot f_i^{v_{Y_i}(f)}$. Restricting to X, we find that the valuation of $f|_X$ along any $Y_{ij} \subset X \cap Y_i$ is $v_{Y_{ij}}(u \cdot f_i^{v_{Y_i}(f)}) = v_{Y_i}(f) \cdot v_{Y_{ij}}(f_i)$, which is exactly the coefficient of Y_{ij} in the definition of $D.X$.

c. We need to recall (or develop) a couple facts about how the associated sheaf functor $\widetilde{-}$ works with Proj first.

   Lemma. *Suppose S is a noetherian graded ring and M a graded S-module. Then $M \mapsto \widetilde{M}$ is an exact functor.*

   Proof. It's enough to show that $M \mapsto M_{(\mathfrak{p})}$ is exact for $\mathfrak{p}$ a graded prime ideal of S not containing the irrelevant ideal, because the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \operatorname{Proj} S$ is given by $M_{(\mathfrak{p})}$ per proposition II.5.11(a) and exactness of a sequence of sheaves can be checked on stalks.

   First we show that for a homogeneous element $f \in S$, the graded localization functor $M \mapsto M_{(f)}$ is exact. Given an exact sequence of graded S-modules $0 \to M' \to M \to M'' \to 0$, we have that localization at a homogeneous element f gives an exact sequence of graded modules $0 \to M'_f \to M_f \to M''_f \to 0$ because localization is exact and commutes with direct sums: the d^{th} graded piece of M_f is exactly the elements $\frac{m}{f^n}$ where $\deg m - n \deg f = d$. As our maps are maps of graded modules and therefore preserve degree, we can restrict to the elements of any fixed degree to get an exact sequence $0 \to (M'_f)_d \to (M_f)_d \to (M''_f)_d \to 0$, so we have the claim.

   Next, I claim that if $\mathfrak{p} \subset S$ is a graded prime ideal and f is a homogeneous element not in $\mathfrak{p}$, then $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$ where $\mathfrak{p}'$ is $(\mathfrak{p}_{(f)})_0$. This is not hard to see: given an element $\frac{x}{g} \in M_{(\mathfrak{p})}$, we can write it as $(xg^{\deg(f)-1}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)})$ because $\deg(x) = \deg(g)$

and $g^{\deg(f)}/f^{\deg(g)}$ isn't in $\mathfrak{p}'$. Conversely, if we have an element $(x/f^n)/(g/f^m) \in (M_{(f)})_{\mathfrak{p}'}$, we can write it as $(xf^m)/(gf^n)$ and the composition both ways is the identity.

This shows that $M \mapsto M_{(\mathfrak{p})}$ is the composition of the two exact functors $M \mapsto M_{(f)}$ and $M_{(f)} \mapsto (M_{(f)})_{\mathfrak{p}'}$.

■

Let $S = k[t_0, \cdots, t_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$ in the sense of chapter I, $I_X \subset S$ the homogeneous ideal of $X \subset \mathbb{P}^n_k$, and $\hat{f}$ be the homogeneous equation of V. Then by proposition I.7.4, we get a finite filtration of $S/(I_X + \hat{f})$ by graded submodules M_i with subquotients isomorphic to $(S/\mathfrak{p}_i)(n_i)$ for integers n_i and homogeneous prime ideals $\mathfrak{p}_i \subset S$. Chapter I's intersection multiplicity along Y is the number of times $\mathfrak{p}_i = I_Y$ among the subquotients of this filtration. Applying the associated sheaf functor (which is exact), we get a finite filtration of $\mathcal{O}_{X\cap V}$ by sheaves $\widetilde{M_i}$ with subquotients isomorphic to $(j_i)_*\mathcal{O}_{Z_i}(n_i)$ for Z_i integral subschemes of $\mathbb{P}^n_k$ contained in $X \cap V$ with closed immersion $j_i : Z_i \to \mathbb{P}^n_k$. Now we take the stalk at η, the generic point of Y: since $\eta \in Z_i$ iff $Z_i = Y$, we get that the subquotients of the filtration $(\widetilde{M_i})_\eta$ are all either 0 or $\mathcal{O}_{Y,\eta}(n_i) \cong \mathcal{O}_{Y,\eta}$ (this last bit isomorphism relies on the fact that $\mathcal{O}(1)$ is invertible), and so the intersection multiplicity from chapter I is the length of $\mathcal{O}_{X,\eta}/(f)$. But $\mathcal{O}_{X,\eta}$ is a DVR, so $(f) = \mathfrak{m}_\eta^{\nu_Y(f)}$, and $\mathcal{O}_{X,\eta}/(f)$ has length $\nu_Y(f)$: it's filtered by

$$0 \subset \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta \subset \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^2 \subset \cdots \subset \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^{\nu_Y(f)} = \mathcal{O}_{X,\eta}/(f).$$

Thus the two quantities are equal, and writing $n_i = \deg(D)$ we see that by theorem I.7.7 we have that $\deg(D.X) = (\deg D) \cdot (\deg X)$.

d. Let $\hat{f} \in k(X)$ be a nonzero rational function demonstrating that D is principal, that is $(f) = D$, and let ξ be the generic point of X. Then as $k(X) = \mathcal{O}_{X,\xi}$ is a quotient of $\mathcal{O}_{\mathbb{P}^n_k,\xi}$ by the fact that the maps on stalks of a closed immersion are surjective, we can lift $\hat{f}$ to an element $f \in \mathcal{O}_{\mathbb{P}^n_k,\xi}$, and via the inclusion $\mathcal{O}_{\mathbb{P}^n_k,\xi} \hookrightarrow k(\mathbb{P}^n_k)$, we get the required rational function f. It is clear that this is the right thing to do by construction: to carry out the procedure in part (a) we restrict several times, and this fixes any choices we ever made. This shows that the homomorphism $\operatorname{Div} X \to \mathbb{Z}$ given by $\sum n_i Y_i \mapsto \sum n_i \deg Y_i$ descends to the quotient $\operatorname{Div} X/\operatorname{PDiv} X \cong \operatorname{Cl} X$. The content of (c) exactly shows that the diagram is commutative.

Exercise II.6.3. (*) *Cones.* In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex. 2.10). So let V be a projective variety in $\mathbb{P}^n$, which is of dimension ≥ 1 and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in $\mathbb{A}^{n+1}$, and let $\overline{X}$ be its projective closure in $\mathbb{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

a. Let $\pi : \overline{X} \setminus P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ for each i, and then show as in (6.6) that $\pi^* \operatorname{Cl} V \to \operatorname{Cl}(\overline{X} \setminus P)$ is an isomorphism. Since $\operatorname{Cl} \overline{X} \cong \operatorname{Cl}(\overline{X} \setminus P)$, we have also $\operatorname{Cl} V \cong \operatorname{Cl} \overline{X}$.

b. We have $V \subset \overline{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\operatorname{Cl}\overline{X}$ is equal to π^*(class of $V.H$) where H is any hyperplane of $\mathbb{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Cl} V \to \operatorname{Cl} X \to 0,$$

where the first arrow sends $1 \mapsto V.H$, and the second is π^* followed by the restriction to $X \setminus P$ and inclusion in X. (The injectivity of the first arrow follows from the previous exercise.)

c. Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine cooridnate ring of X). Show that $S(V)$ is a unique factorization domain if and only if (1) V is projectively normal (Ex. 5.14), and (2) $\operatorname{Cl} V \cong \mathbb{Z}$ and is generated by the class of $V.H$.

d. Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\operatorname{Cl} X \to \operatorname{Cl}(\operatorname{Spec} \mathcal{O}_P)$.

Solution. We assume we're working over a field here, but make no other assumptions.

a. The fibers of $X \setminus \{P\}$ over $D(x_i) \cap V$ in $\mathbb{A}^{n+1}$ are copies of $\mathbb{A}^1$ minus the origin, and taking the projective closure adds the point at infinity on this line, making the fiber of π a copy of $\mathbb{A}^1$. The proof that the class groups of V and $\overline{X} \setminus \{P\}$ are isomorphic is exactly the same as proposition II.6.6, so we omit it. The isomorphism $\operatorname{Cl} V \cong \operatorname{Cl} X \setminus \{P\} \cong \operatorname{Cl}\overline{X}$ comes from the fact that P is a point of codimension at least two - since class groups are defined based on behavior in codimension one and this alteration doesn't touch any of that data, we have an isomorphism.

b. $V \subset \overline{X}$ as the hyperplane section at infinity follows from the following chain of facts: $V \subset \mathbb{P}^n$ is cut out by a homogeneous prime ideal I in $k[x_0, \cdots, x_n]$; the affine cone $X \subset \mathbb{A}^{n+1}$ is cut out by I; the projective closure is given by homogenizing with respect to x_{n+1} (which doesn't do anything since I is already homogeneous); and finally taking the intersection with the hyperplane at infinity is modding out by x_{n+1} on the coordinate algebra side.

Suppose H is a hyperplane in $\mathbb{P}^n$ not containing V. Then H is the intersection of a hyperplane $H' \subset \mathbb{P}^{n+1}$ with $\mathbb{P}^n_\infty$, the hyperplane at infinity, and in fact H' is linearly equivalent to $\mathbb{P}^n_\infty$ because all hyperplanes in projective space are linearly equivalent. This means that under $\pi^* : \operatorname{Cl} V \to \operatorname{Cl}\overline{X}$, the class of $V.H$ has image $\pi^*(V \cap H) = \overline{X} \cap H' \sim \overline{X} \cap \mathbb{P}^n_\infty = V$ where $\sim$ denotes linear equivalence. By proposition II.6.5, we have an exact sequence

$$\mathbb{Z} \to \operatorname{Cl}\overline{X} \to \operatorname{Cl} X \to 0$$

where the first map sends $1 \mapsto V.H$ and the second map is restriction. Part (d) of exercise II.6.2 shows that the first map is injective, and then the isomorphism $\operatorname{Cl}\overline{X} \cong \operatorname{Cl} V$ constructed earlier lets us rewrite the middle term and alter the second map as desired.

c. Our goal here is to apply proposition II.6.2: a noetherian domain A is a UFD iff $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl} X = 0$. Normality of $S(V) = \mathcal{O}_X(X)$ is equivalent to projective normality of V by exercise II.5.14, and $\operatorname{Cl} V \cong Z$ generated by the class of $V.H$ iff $\operatorname{Cl} X = 0$, so $S(V)$ is a UFD iff these two statements hold by proposition II.6.2.

d. First we describe the scheme $\operatorname{Spec} \mathcal{O}_P$: it consists of all the points in X which are generalizations of P, and we have a natural map from divisors on X to divisors on $\operatorname{Spec} \mathcal{O}_P$ by intersecting. This clearly descends to a homomorphism of class groups: the intersection of the divisor of some rational function $f \in K(X)$ with $\operatorname{Spec} \mathcal{O}_P$ is just the divisor of f inside $\operatorname{Spec} \mathcal{O}_P$ because the fraction field of $\mathcal{O}_P$ is $K(X)$. This map is clearly surjective - we need to show injectivity to get an isomorphism. This amounts to showing that every divisor on X not supported on P is principal.

Let's translate this to algebra. Let $A = S(V) = k[x_0, \cdots, x_n]/I$ for a homogeneous ideal I, let $\mathfrak{m} = (x_0, \cdots, x_n)$, and let $\mathfrak{p} \subset A$ be the height one ideal corresponding to the generic point of a prime divisor D not passing through the cone point. By the assumption D does not pass through P, we have $\mathfrak{p} \not\subset \mathfrak{m}$. As $\mathfrak{p} \not\subset \mathfrak{m}$, we can select $f \in \mathfrak{p}$ so that $f = 1$ modulo $\mathfrak{m}$. Now pick any i so that $D(x_i) \cap X \neq \emptyset$ (equivalently, $x_i \notin I$), and consider the following chain of localizations:

$$A \hookrightarrow A_{x_i} = A_{(x_i)}[x_i, x_i^{-1}] \hookrightarrow K[x_i, x_i^{-1}] \subset \operatorname{Frac} A$$

where $K = \operatorname{Frac} A_{(x_i)} = K(V)$. (Geometrically, what's going on here is that $\mathfrak{p}$ represents the generic point of D, and in analogy with proposition II.6.6, we're showing that $\mathfrak{p}$ is a codimension-one point of type II for the map $X \setminus P \to V$.)

Since x_i is not in $\mathfrak{p}$, we have that $\mathfrak{p}A_{(x_i)}[x_i, x_i^{-1}]$ does not intersect $A_{(x_i)}$, and thus $\mathfrak{p}K[x_i, x_i^{-1}]$ is a proper prime ideal of $K[x_i, x_i^{-1}]$. As $K[x_i, x_i^{-1}]$ is a PID, we can find a single generator f of $\mathfrak{p}K[x_i, x_i^{-1}]$, and after multiplying by an appropriate power of x_i and scaling by an element of k, we may assume that $f = 1 + \sum_{q=1}^{r} a_q x_i^q$. This generator f must be irreducible since it generates a prime ideal, and therefore f has valuation 1 in the local ring of the generic point of D. Note that this immediately implies that the valuation of f in the local ring of any other type two divisor is zero: irreducible elements of a UFD can only be in one height one prime ideal, since height one primes are principal in UFDs.

It remains to check that f has valuation 0 in the local ring associated to any type one divisor, so suppose D' is a type one prime divisor with generic point lying in $D(x_i)$ and associated valuation $v_{D'}$ at the generic point of D'. Next, write $h = fg$ for $g = 1 + \sum_{q=1}^{t} b_q x_i^q \in K[x_i, x_i^{-1}]$. We see from the properties of valuations that $v_{D'}(f)$ and $v_{D'}(g)$ are both nonpositive: they're equal to the minimum of the valuation of the nonzero coefficients of f and g, and both of these quantities are bounded above by $v_{D'}(1) = 0$. Next, as h is not in any homogeneous prime ideal, $v_{D'}(h) = 0$, and therefore $v_{D'}(h) = v_{D'}(fg) = v_{D'}(f) + v_{D'}(g) = 0$ and thus $v_{D'}(f) = 0$. So f has valuation 0 at all prime divisors with generic point in $D(x_i) \cap X$.

All that remains is to check the valuation of f for prime divisors that lie in $D(x_j)$ and not $D(x_i)$ for $j \neq i$. Fix $j \neq i$, and rewrite f, g as $f = 1 + \sum_{q=1}^{r} (a_q \frac{x_i^r}{x_j^r}) x_j^q$ and $g = 1 + \sum_{q=1}^{t} (b_q \frac{x_i^r}{x_j^r}) x_j^q$. The previous argument holds with these new coefficients, so by varying j we get that f has valuation 0 at all prime divisors of type one, and thus D is the principal divisor associated to f.

Exercise II.6.4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \cdots, x_n]$ be a *square-free* nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \cdots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring. [*Hint*: The quotient field K of A is just $k(x_1, \cdots, x_n)[z]/(z^2 - f)$. It is a Galois extension of $k(x_1, \cdots, x_n)$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g, h \in k(x_1, \cdots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Now show that α is integral over $k[x_1, \cdots, x_n]$ if and only if $g, h \in k[x_1, \cdots, x_n]$. Conclude that A is the integral closure of $k[x_1, \cdots, x_n]$ in K.]

Solution. We do not need any assumptions on k here besides $\operatorname{char} k \neq 2$.

We've already pretty much done this in exercise I.3.17 (but without the $\operatorname{char} k \neq 2$ assumption). We see that $K = \operatorname{Frac} A$ is a degree-two extension of $k(x_1, \cdots, x_n)$, so we can write any element of K as $u + vz$ for $u, v \in k(x_1, \cdots, x_n)$. Now $u + vz$ satisfies $P(T) = T^2 - 2uT - (v^2 f - u^2)$, and I claim that if $u + vz$ is integral over A, the coefficients of this polynomial are in $k[x_1, \cdots, x_n]$. If $v = 0$, the minimal polynomial for $u \in k(x_1, \cdots, x_n)$ is $T - u$, which must divide the polynomial which witnesses $u = u + vz$ as integral over A. So u is a root of a monic polynomial with coefficients in $k[x_1, \cdots, x_n]$ and is thus integral over $k[x_1, \cdots, x_n]$ and is in $k[x_1, \cdots, x_n]$. If $v \neq 0$, then $P(T)$ is the minimal polynomial of $u + vz$ as an element of $K/k(x_1, \cdots, x_n)$, so it must divide the polynomial which shows that $u + vz$ is integral over A. But this means that the roots of $P(T)$ are integral over $k[x_1, \cdots, x_n]$, so the coefficients of this polynomial must also be integral over $k[x_1, \cdots, x_n]$ by Vieta's formulas. But the elements of $k(x_1, \cdots, x_n)$ integral over $k[x_1, \cdots, x_n]$ are exactly the elements of $k[x_1, \cdots, x_n]$ because $k[x_1, \cdots, x_n]$ is integrally closed in its field of fractions, and the claim is proven. (This generalizes to the statement that if $R \subset F$ is the inclusion of an integrally closed domain in to its field of fractions and $F \subset E$ is a finite field extension, then any element in E which is integral over R has minimal polynomial in $R[T]$, which always feels like a bit of magic every time I use it even though it's not a particularly difficult proof.)

As $\operatorname{char} k \neq 2$, we are quickly done: $2u \in k[x, y]$ implies $u \in k[x, y]$, and thus $v^2 f$ must also be in $k[x, y]$ because f is squarefree, so $k[x, y, z]/(xy - z^2)$ is integrally closed.

Exercise II.6.5. (*) *Quadric Hypersurfaces.* Let $\operatorname{char} k \neq 2$, and let X be the affine quadric hypersurface $\operatorname{Spec} k[x_0, \cdots, x_n]/(x_0^2 + x_1^2 + \cdots + x_r^2)$ - cf. (I, Ex. 5.12).

a. Show that X is normal if $r \geq 2$ (use (Ex. 6.4)).

b. Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \cdots + x_r^2$. Now imitate the method of (6.5.2) to show that:

   (1) If $r = 2$, then $\operatorname{Cl} X \cong \mathbb{Z}/2\mathbb{Z}$;

   (2) If $r = 3$, then $\operatorname{Cl} X \cong \mathbb{Z}$ (use (6.6.1) and (Ex. 6.3) above);

   (3) If $r \geq 4$ then $\operatorname{Cl} X = 0$.

c. Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equation. Show that:

   (1) If $r = 2$, then $\operatorname{Cl} Q \cong \mathbb{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;

(2) If $r = 3$, $\operatorname{Cl} Q \cong \mathbb{Z} \oplus \mathbb{Z}$;

(3) If $r \geq 4$ then $\operatorname{Cl} Q \cong \mathbb{Z}$, generated by $Q.H$.

d. Prove Klein's theorem, which says that if $r \geq 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subset \mathbb{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection. (First show that for $r \geq 4$, the homogeneous cooridnate ring $S(Q) = k[x_0, \cdots, x_n]/(x_0^2 + \cdots + x_r^2)$ is a UFD.)

Solution. We do not need any hypotheses here besides $\operatorname{char} k \neq 2$ and $x^2 + 1$ admits a solution in k.

a. If $r \geq 2$, then we can write $x_0^2 + x_1^2 + \cdots + x_r^2$ as $x_0^2 - (-x_1^2 - \cdots - x_r^2)$, and the term in the parentheses is squarefree: if $\sqrt{-1} \in k$ and $r = 2$, it's a product of linear terms, otherwise $r > 2$ and we can check irreducibility by noting that the singular locus of $V(x_1^2 + \cdots + x_r^2) \subset \mathbb{A}^n_k$ is codimension r, whereas if f^2 divided $x_1^2 + \cdots + x_r^2$ the Jacobian matrix of $V(x_1^2 + \cdots + x_r^2)$ would vanish along $V(f)$, which is of codimension one. Applying exercise II.6.4, we have that X is normal.

b. First, we note that by repeated applications of proposition II.6.6, it suffices to consider the case when $r = n$. As $k[x_0, \cdots, x_n]/(x_0^2 + \cdots + x_r^2) \cong k[x_0, \cdots, x_r]/(x_0^2 + \cdots + x_r^2) \otimes_k k[x_{r+1}, \cdots, x_n]$, we have that $X \cong \operatorname{Spec} k[x_0, \cdots, x_r]/(x_0^2 + \cdots + x_r^2) \times_k \mathbb{A}^{n-r}_k$.

To achieve the desired change of coordinates, we need to assume that $i = \sqrt{-1} \in k$. Then $x_0^2 + x_1^2 = (x_0 + ix_1)(x_0 - ix_1)$ and it's clear what linear change of coordinates to apply.

The method of example II.6.5.2 shows that the class group of a quadric hypersurface is a homomorphic image of $\mathbb{Z}$: letting $\overline{x_0}$ be the image of x_0 in the coordinate algebra of X, we can consider $V(\overline{x_0}) = V(x_0, x_2^2 + \cdots + x_r^2) \subset X$ and the standard exact sequence

$$\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl}(X \setminus V(x_0, x_2^2 + \cdots + x_r^2)) \to 0.$$

Next, we note that $X \setminus V(\overline{x_0})$ is affine: it's the spectrum of $k[x_0^{\pm 1}, x_1, \cdots, x_r]/(x_0 x_1 = x_1^2 + \cdots + x_r^2)$. Further, it's coordinate algebra is a UFD, because $k[x_0^{\pm 1}, x_1, \cdots, x_r]/(x_0 x_1 = x_1^2 + \cdots + x_r^2) \cong k[x_0^{\pm 1}, x_2, \cdots, x_r]$ by the homomorphism sending $x_1 \mapsto x_0^{-1}(x_1^2 + \cdots + x_r^2)$. By proposition II.6.2, this means that the class group of $X \setminus V(x_1, x_2^2 + \cdots + x_r^2)$ is zero and thus $\mathbb{Z} \to \operatorname{Cl} X$ is a surjection.

Case 1: $r = 2$. The case with $n = 2$ is exactly what's proven in example II.6.5.2.

Case 2: $r = 3$. We can apply the same linear transformation to x_1, x_2 as at the start of this problem to see that we're looking to compute the class group of $\operatorname{Spec} k[x, y, z, w]/(xy - zw)$. This is the cone on the quadric hypersurface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ discussed in example II.6.6.1, where it's determined it has class group $\mathbb{Z}^2$. From exercise II.6.3, we see that the class group of X fits in to an exact sequence $0 \to \mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \to \operatorname{Cl} X \to 0$, which means it is of the form $\mathbb{Z} \oplus \mathbb{Z}/n$ for some n. On the other hand, we showed that it was the surjective image of $\mathbb{Z}$ earlier. The only way for both of these things to be true is if $n = 1$, and $\operatorname{Cl} X \cong \mathbb{Z}$.

Case 3: $r \geq 4$. I claim $V(\overline{x_0})$ is a principal prime divisor. Principal divisor is clear - to check primality (ie $V(\overline{x_0})$ isn't a multiple of another prime divisor or a sum of prime divisors) it suffices to check that the coordinate algebra of $V(\overline{x_0}) \subset X$ is a domain. We compute:

$$k[x_0, \cdots, x_r]/(x_0, x_0x_1 = x_2^2 + \cdots + x_r^2) \cong k[x_1, \cdots, x_r]/(x_2^2 + \cdots + x_r^2)$$

which is a domain by the logic in part (a). So the image of $V(\overline{x_0})$ in $\operatorname{Cl} X$ is zero, and thus $\operatorname{Cl} X = 0$.

c. The exact sequence of exercise II.6.3(b) gives

$$0 \to \mathbb{Z} \to \operatorname{Cl} Q \to \operatorname{Cl} X \to 0$$

where the first map sends $1 \mapsto Q.H$, and we'll use this plus some light homological algebra to reach our conclusions.

In the case where $r = 2$, we have that $\operatorname{Cl} X = \mathbb{Z}/2$, so after tensoring our exact sequence with $\mathbb{Q}$, we get that $\mathbb{Q}$ surjects on to $(\operatorname{Cl} Q) \otimes_{\mathbb{Z}} \mathbb{Q}$, which means that $\operatorname{Cl} Q$ is of the form $\mathbb{Z} \oplus T$ where T is a torsion abelian group. Tensoring with $\mathbb{Z}/p$ for p an odd prime gives that $\mathbb{Z}/p \to \mathbb{Z}/p \otimes_{\mathbb{Z}} \operatorname{Cl} Q$ is surjective; tensoring with $\mathbb{Z}/2$ gives that $(\operatorname{Cl} Q) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to \mathbb{Z}/2$ is an isomorphism. Thus $T = 0$, and so $\operatorname{Cl} Q \cong \mathbb{Z}$ and the map $1 \mapsto V.H$ sends $1 \mapsto 2$.

When $r = 3$, this is exactly example II.6.6.1 after applying II.6.3(a) repeatedly, because $\operatorname{Proj} k[x_0, \cdots, x_r]/(x_0x_1 = x_2x_3)$ is the $r-3$ fold cone on $\mathbb{P}^1 \times \mathbb{P}^1$.

When $r \geq 4$, the exact sequence reduces to $0 \to \mathbb{Z} \to \operatorname{Cl} Q \to 0 \to 0$ and the claim is obvious.

d. By part (a), $S(Q)$ is normal, and by part (b), it has $\operatorname{Cl} \operatorname{Spec} S(Q) = 0$. So by proposition II.6.2 it is a UFD. Thus every height one prime in $S(Q)$ is principal. Thus I_Y is generated by a single element, and any lift of this to $k[x_0, \cdots, x_n]$ suffices to generate the ideal of a hypersurface which when intersected with Q gives Y.

Exercise II.6.6. Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$ of (6.10.2).

a. Show that three points P, Q, R of X are collinear if and only if $P + Q + R = 0$ in the group law on X. (Note that the point $P_0 = (0, 1, 0)$ is the zero element in the group structure on X.)

b. A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

c. A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point. (An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity ≥ 3 with the curve at P.)

d. Let $k = \mathbb{C}$. Show that the points of X with coordinates in Q form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

Solution. We assume $k = \overline{k}$ here. (The version where k is not algebraically closed is more interesting - consult a text on elliptic curves.) Note that the fact we're working with this specific elliptic curve is only relevant in the final portion of part (d) where we calculate the subgroup structure of $X(\mathbb{Q})$ - everything else is general and works for any elliptic curve over a field.

a. Suppose P, Q, R are collinear and lie on ℓ. Then as $\ell \sim V(z)$ we have $P + Q + R \sim 3P_0$, so $(P - P_0) + (Q - P_0) + (R - P_0) \sim 0$ and therefore $P + Q + R = 0$ in the group law. Conversely, if $P + Q + R = 0$ in the group law, consider the line ℓ through P, Q. By Bezout, this intersects X in a third point T. According to the previous logic, we have $P + Q + T = 0$ in the group law. But this implies that $T = R$ by uniqueness of inverses.

b. If the tangent line ℓ to P passes through P_0, then $2P + P_0 \sim 3P_0$ in the class group because $\ell \sim V(z)$. But then in the class group this is the same as $2P = 0$ by the same method as (a). The converse is proven using the same logic as (a): let ℓ be the tangent line to P. By Bezout, its intersection with X is P with order > 1, so it's intersection is $2P + Q$ for Q some point on X (possibly P). But then $2P = Q$ in the group law, so $Q = 0$.

c. Same logic as (b), except in this case we really do have $Q = P$ - we know that the tangent line to P has intersection multiplicity exactly three with X at P by Bezout (this gives ≤ 3) and the assumption that P is an inflection point (this gives ≥ 3).

d. We need to check that the inverse of any point with rational coordinates has rational coordinates, and that the sum of two points with rational coordinates has rational coordinates. The big idea is that if $n - 1$ points of intersection of a line and a degree n curve C in $\mathbb{P}^2_k$ are defined over some subfield $F \subset k$, then the last point of intersection is too. This can be seen by Vieta's formulas: restricting to some $\mathbb{A}^2_k \subset \mathbb{P}^2_k$ containing all the points of intersection, we see that the restriction of the dehomogenization of the defining equation of C is a degree n polynomial in $k[x]$, the coordinate algebra of $\mathbb{A}^1_k = \ell \cap \mathbb{A}^2_k$. Then by Vieta's formulas, if $n - 1$ roots of this polynomial are defined over F, the final root must be as well. Backsubstituting in to the equation for ℓ and using the fact ℓ was defined over F, we see that the final intersection point has coordinates in F in $\mathbb{A}^2$, and the last coordinate we add is $z = 1$.

   From the characterization of the inverse of $P \neq (0, 1, 0)$ as the third point of intersection of the line through P and $(0, 1, 0)$ with X, we see that the inverse has rational coordinates, and from the characterization of $P + Q$ as the inverse of the third point of intersection of the line through P and Q with X, it follows that the points of an elliptic curve defined over $F \subset k$ with coordinates in F form a group.

   Now we attack the problem of determining the group structure of $X(\mathbb{Q})$. There are four obvious rational points: $(0, 1, 0), (-1, 0, 1), (0, 0, 1), (1, 0, 1)$, and each of the non-identity points are of order two by (b) because they have vertical tangents. This subgroup is isomorphic to the Klein four-group, and I claim that this is all of $X(\mathbb{Q})$.

   To check this, suppose $x_0 = a/b \neq 0$ is rational so that there is a y_0 with $(x_0, y_0, 1)$ a rational point on X. By considering the p-adic valuation of the two sides of the equation $y^2 = x^3 - x$, we see that $v_p(x)$ is even for all p: the left hand side is $v_p(y^2) = 2v_p(y)$, and the right hand side is $v_p(x^3-x) = v_p(x^2-1) + v_p(x)$, which is $v_p(x)$ if $v_p(x) > 0$ and $3v_p(x)$ if $v_p(x) < 0$. So x is plus or minus the square of a rational number, and therefore $x^2 - 1$ must be as well. Writing $x = \frac{u^2}{v^2}$, this gives that the integer $|v^4(x^2-1)| = |u^4 - v^4|$ is a square. I claim that $u^4 - v^4 = \pm w^2$ has no solutions in integers unless one of u, v, w is zero, which implies the points we've found so far are the only rational points.

We will prove something stronger, originally due to Fermat: no right triangle with integer sides can have two sides each with length equal to a square or twice a square. This implies our result as our equation rearranges to $u^4 = w^2 + v^4$ or $u^4 + w^2 = v^4$, and any rational solution with u, v, w nonzero gives such a triangle. So suppose we have such a triangle. By scaling all the sides we may assume they are pairwise coprime and thus we have a primitive Pythagorean triple with two sides each a square or twice a square. According to the well-known parametrization of primitive Pythagorean triples, there are coprime integers p, q so that our triangle has sides $p^2 - q^2$, $2pq$, and $p^2 + q^2$. If $p^2 \pm q^2$ were both squares, then the triangle with sides $\sqrt{(p^2+q^2)(p^2-q^2)} = \sqrt{p^4 - q^4}$, q^2, and p^2 would be another triangle satisfying the condition. Else, just one of $p^2 + q^2$ and $p^2 - q^2$ is a square, and $2pq$ is a square or twice a square, implying that p and q are squares or twice squares. But then either the triangle with sides q, $\sqrt{p^2 - q^2}$, p or q, p, $\sqrt{p^2+q^2}$ is another triangle satisfying the conditions. In any case, the triangle we've proven has strictly smaller hypotenuse, which is impossible: this implies that there is an infinite descending sequence of positive integers. So no such triangle exists and the equation $u^4 - v^4 = \pm w^2$ has no nontrivial solutions in integers.

Exercise II.6.7. (*) Let X be the nodal cubic curve $y^2 z = x^3 + x^2 z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, $\operatorname{CaCl}^\circ X$, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.

Solution. We assume that $k = \overline{k}$.

The proof of example II.6.11.4 goes through exactly as written until one needs to figure out the group structure on the set of smooth closed points of X. Here we should note that we're in characteristic not 2: if $\operatorname{char} k = 2$, then the change of coordinates $y = y + x$ turns our curve in to $(y^2 + x^2)z = x^3 + x^2 z$, or $y^2 z = x^3$, the cuspidal cubic - so we can't show that the group structure on $X \setminus Z$ is $\mathbb{G}_m$, because it's $\mathbb{G}_a$, and those aren't isomorphic over any field. We should also note that Hartshorne is implicitly assuming that we're working over an algebraically closed field - otherwise the appropriate notion of degree of a divisor on a curve $\sum n_i P_i$ is $\sum [\kappa(P_i) : k] n_i$, and sending a divisor P to $P - P_0$ doesn't even land in $\operatorname{CaCl}^\circ$. All of this is fixable with some more rigor and attention to what exactly a group variety is, but we will not discuss now.

First, make the change of coordinates $y = y + x$ which changes our equation to $(y^2 + 2xy + x^2)z = x^3 + x^2 z$, or $y^2 z + 2xyz = x^3$. Next, make the changes of coordinates $x = 4x - 4y$, $y = 8y$ so that our equation becomes $64y^2 z + 64(x-y)yz = 64(x-y)^3$, and as $2 \neq 0$, this reduces to $y^2 z + (x-y)yz = (x-y)^3$, or $xyz = (x-y)^3$. We note that this chain of coordinate changes has moved our singular point to $[0:0:1]$, and there are no other points of our curve on $y = 0$. So $D(y) \cap V(xyz = (x-y)^3)$ is the nonsingular locus, with coordinate algebra $k[x,z]/(xz - (x-1)^3)$ which is isomorphic to $k[\xi^{\pm 1}]$ via $x \mapsto \xi, z \mapsto (\xi - 1)^3/\xi$ and $\xi \mapsto x$ (note that x is invertible in

$k[x,z](xz-(x-1)^3)$ with inverse $z+x^2-3x+3$). Thus $X\setminus Z\cong \mathbb{G}_m$, and it remains to show that the group operations are compatible.

For this it suffices to note that any line which meets $V(xz-(x-1)^3)$ in three points counted with multiplicity has the product of the x-coordinates equal to one: $-P$ is the third intersection of the line through P and the point $(1,0)$ with X, and $P+Q$ is the inverse of the third intersection of the line through P,Q with X, so if the claim holds then we have that our isomorphism $\mathbb{G}_m\cong X\setminus Z$ respects multiplication. As any line meeting $V(xz-(x-1)^3)$ in three points is not of the form $x=c$, we can solve the equation of the line for z, plug in, and get a polynomial in x with constant term 1, which finishes the problem by Vieta.

Exercise II.6.8.

a. Let $f:X\to Y$ be a morphism of schemes. Show that $\mathcal{L}\mapsto f^*\mathcal{L}$ induces a homomorphism of Picard groups, $f^*:\operatorname{Pic}Y\to\operatorname{Pic}X$.

b. If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^*:\operatorname{Cl}Y\to\operatorname{Cl}X$ defined in the text, via the isomorphisms of (6.16).

c. If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f:X\to\mathbb{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

Solution. Part (a) is general, part (b) and (c) both should assume $k=\overline{k}$.

a. Two facts combine to prove the claim: the pullback of a locally free sheaf of rank r is again a locally free sheaf of rank r, and tensor products commute with pullbacks. The first fact boils down to the pullback of the structure sheaf being the structure sheaf, which is immediate from the definition of $f^*\mathcal{F}=f^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X$: by taking an appropriate open cover $\{Y_i\}$ of Y so that $\mathcal{L}|_{Y_i}$ is free of rank one, we have that $f^*\mathcal{L}$ is free of rank one on $f^{-1}(Y_i)$ and thus $f^*\mathcal{L}$ is invertible. For the second fact, refer to the solution to exercise II.5.12 to see that tensor product commutes with pullbacks.

b. By propositions II.6.11 and II.6.15, we have the following diagram, where the horizontal arrows are isomorphisms:

$$\begin{tikzcd}
\operatorname{Cl}Y \arrow[r, "\alpha_Y"] \arrow[d, "f^*"] & \operatorname{CaCl}Y \arrow[r, "\beta_Y"] & \operatorname{Pic}Y \arrow[d, "f^*"] \\
\operatorname{Cl}X \arrow[r, "\alpha_X"] & \operatorname{CaCl}X \arrow[r, "\beta_Y"] & \operatorname{Pic}X
\end{tikzcd}$$

I claim this diagram commutes. We first define a map $f^*:\operatorname{CaCl}Y\to\operatorname{CaCl}X$ induced by f^* on line bundles. Given a line bundle, there is an associated Cartier divisor divisor D by proposition II.6.15, so suppose $\mathcal{L}$ is a line bundle on Y with associated Cartier divisor $\{(V_i,g_i)\}$. Then $f^*\mathcal{L}$ is a line bundle with associated Cartier divisor $\{(f^{-1}(V_i),f^*g_i)\}=f^{-1}(D)$, and this map is a group homomorphism because it makes the square commute.

By linearity, I claim it is enough to check that $f^* : \mathrm{CaCl}\, Y \to \mathrm{CaCl}\, X$ agrees with $f^* : \mathrm{Cl}\, Y \to \mathrm{Cl}\, X$ for a point $P \in Y$. So what's the Cartier divisor associated to a point $P \in Y$? Let π be a uniformizer of the DVR $\mathcal{O}_{Y,P}$, defined on an open subscheme $U \subset Y$ and having no zeroes on U besides P (we can always do this by shrinking U). Then $\{(U, \pi), (Y \setminus \{P\}, 1)\}$ is the Cartier divisor associated to P, and its pullback to X is $\{(f^{-1}(U), f^*\pi), (X \setminus f^{-1}(P), 1)\}$. The Weil divisor associated to this is exactly $\sum_{P_i \in f^{-1}(P)} v_{P_i}(f^*\pi)$ by the proof of proposition II.6.11, which matches the definition of f^*P as a Weil divisor as $\sum_{Q \in f^{-1}(P)} v_Q(f^*\pi)$ in the definition on the bottom of page 137.

c. The same diagram and argument as above shows that it suffices to consider $f^* : \mathrm{CaCl}\, \mathbb{P}^n \to \mathrm{CaCl}\, X$ instead of $f^* : \mathrm{Pic}\, \mathbb{P}^n \to \mathrm{Pic}\, X$. We also see that it is enough to check the claim with a hyperplane $H \subset \mathbb{P}^n$ which does not contain X: since $\mathrm{Pic}\, \mathbb{P}^n \cong \mathrm{CaCl}\, \mathbb{P}^n \cong \mathrm{Cl}\, \mathbb{P}^n \cong \mathbb{Z}$ is generated by the class of a hyperplane, all we need to do to verify commutativity of the diagram is to check that the image of $[H] \in \mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, X$ is the same as the image of $[H] \in \mathrm{CaCl}\, \mathbb{P}^n \to \mathrm{CaCl}\, X$.

   We run the same construction as in (b): let ℓ be a linear form cutting out H, a hyperplane containing X so that the Cartier divisor associated to H is $\{\cdots, (D(x_i), \ell/x_i), \cdots, (D(\ell), 1)\}$. Then the pullback of this Cartier divisor to X is $\{\cdots, (X \cap D(x_i), f^*(\ell/x_i)), \cdots, (X \setminus H, 1)\}$, and the Weil divisor associated to this is just $\sum_Y v_Y(f^*(\ell/x_i))$ where Y is a prime divisor and i is chosen so that $Y \cap D(x_i) \neq \emptyset$. But this is the exact same thing as the image of H under the map defined in exercise II.6.2.

Exercise II.6.9. (*) *Singular Curves.* Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\widetilde{X}$ be its normalization, and let $\pi : \widetilde{X} \to X$ be the projection map (Ex. 3.8). For each point $P \in X$, let $\mathcal{O}_P$ be its local ring, and let $\widetilde{\mathcal{O}_P}$ be the integral closure of $\mathcal{O}_P$. We use a * to denote the group of units in a ring.

a. Show that there is an exact sequence

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^* / \mathcal{O}_P^* \to \mathrm{Pic}\, X \xrightarrow{\pi^*} \mathrm{Pic}\, \widetilde{X} \to 0.$$

   [*Hint*: Represent $\mathrm{Pic}\, X$ and $\mathrm{Pic}\, \widetilde{X}$ as the groups of Cartier divisors modulo principal divisors, and use the exact sequence of sheaves on X

$$0 \to \pi_* \mathcal{O}_{\widetilde{X}}^* / \mathcal{O}_X^* \to \mathcal{K}^* / \mathcal{O}_X^* \to \mathcal{K}^* / \pi_* \mathcal{O}_{\widetilde{X}}^* \to 0.]$$

b. Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Pic}\, X \to \mathbb{Z} \to 0,$$

   and if X is a plane nodal cubic curve, there is an exact sequence

$$0 \to \mathbb{G}_m \to \mathrm{Pic}\, X \to \mathbb{Z} \to 0.$$

Solution. We assume $k = \overline{k}$ here. This is important, because the statement of part (a) is not true without extra hypotheses if $k \neq \overline{k}$: there's a step where we need to argue that no element of the first term of the exact sequence determines a trivial Cartier divisor on $\widetilde{X}$, which is equivalent to the global sections of $\mathcal{O}_X$ and $\mathcal{O}_{\widetilde{X}}$ being the same field. Our assumption that $k = \overline{k}$ is the easiest way out of this, though assuming that X is geometrically integral (exercise II.3.15(c)) will also suffice (brief proof outline: show that $\mathcal{O}_X(X) \otimes_k \overline{k} \cong \mathcal{O}_{X \times_k \overline{k}}(X \times_k \overline{k})$, show that $\dim_k \mathcal{O}_X(X)$ is the number of connected components of $X \times_k \overline{k}$ assuming X is geometrically reduced using the first statement, and $\dim_k \mathcal{O}_{\widetilde{X}}(\widetilde{X})$ is the number of irreducible components of X, as normalization turns irreducible components in to irreducible components, so X geometrically integral implies $\mathcal{O}_X(X) = \mathcal{O}_{\widetilde{X}}(\widetilde{X}) = k$).

Before beginning the main proof, we show that a proper connected reduced scheme over a field has global sections a finite extension of that field.

Lemma (Zariski's Lemma)**.** *Suppose k is a field and K is a field extension of k which is finitely generated as a k-algebra. Then K is a finite field extension of k.*

Proof. See Atiyah-Macdonald or Google. ■

Lemma. *If X is a proper connected reduced scheme over a field k, then $\mathcal{O}_X(X)$ is a field which is a finite extension of k.*

Proof. Any global sections of X is equivalent to a morphism to $\mathbb{A}^1_k$, as we get compatible maps $k[t] \to A$ for any open affine $\operatorname{Spec} A \subset X$ by sending t to the global section. By exercise II.4.4, this means the image of X in $\mathbb{A}^1_k$ is a connected subscheme which is proper over k, or a closed point. This means that the map factors as $X \to P \to \mathbb{A}^1_k$ where P is a closed point of $\mathbb{A}^1_k$. Such closed points are $\operatorname{Spec} F$ for a field $F = k[x]/\mathfrak{m}$, so by considering the induced map on global sections, we have that every nonzero global section has an inverse and therefore the global sections form a field.

Now let $x \in X$ be a closed point inside the affine open $\operatorname{Spec} A \subset X$, represented by the maximal ideal $\mathfrak{m}_x$. As $\mathfrak{m}_x \cap \mathcal{O}_X(X) = 0$, we see that $\mathcal{O}_X(X)$ embeds inside the residue field $k(x) = A/\mathfrak{m}_x$, which is a finite extension of k by Zariski's lemma, so $\mathcal{O}_X(X)$ must also be a finite extension of k. ■

a. On any affine open $\operatorname{Spec} A \subset X$, we have that $(\pi_* \mathcal{O}^*_{\widetilde{X}}) = \widetilde{A}$, the normalization of A, so $\mathcal{O}^*_X \subset \pi_* \mathcal{O}^*_{\widetilde{X}}$ as subsheaves of $\mathcal{K}^*$. We get an exact sequence of sheaves $0 \to \pi_* \mathcal{O}^*_{\widetilde{X}}/\mathcal{O}^*_X \to \mathcal{K}^*/\mathcal{O}^*_X \to \mathcal{K}^*/\pi_*\mathcal{O}^*_{\widetilde{X}} \to 0$ on X by the third isomorphism theorem.

   As normalization is birational, we have that $\pi_*\mathcal{O}^*_{\widetilde{X}}/\mathcal{O}^*_X$ is supported on a proper closed subset of X, which is a finite set of points since we're working with a curve. Since a sheaf on a finite discrete space is equal to the direct sum of the skyscraper sheaves of the stalks at each point, we have that $\pi_*\mathcal{O}^*_{\widetilde{X}}/\mathcal{O}^*_X \cong \bigoplus_{P \in X} i_P((\pi_*\mathcal{O}^*_{\widetilde{X}})_P/\mathcal{O}^*_{X,P})$. Next, by definition of $\mathcal{O}_{\widetilde{X}}$ and the fact that integral closure commutes with localization, we have that $(\pi_*\mathcal{O}_{\widetilde{X}})_P \cong \widetilde{\mathcal{O}_{X,P}}$, and thus $\bigoplus_{P \in X} i_P((\pi_*\mathcal{O}^*_{\widetilde{X}})_P/\mathcal{O}^*_{X,P}) \cong \bigoplus_{P\in X} i_P(\widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}^*_{X,P})$, so our exact sequence of sheaves stands at

$$0 \to \bigoplus_{P\in X} i_P(\widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}_{X,P}^*) \to \mathcal{K}_X^*/\mathcal{O}_X^* \to \mathcal{K}_X^*/\pi_*\mathcal{O}_{\widetilde{X}}^* \to 0.$$

The next thing to do is to take global sections. As $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \operatorname{CaDiv} X$, we have that the middle term is isomorphic to $\operatorname{CaDiv} X$. As a direct sum of skyscraper sheaves is flasque, we have that taking global sections of this sequence remains exact by exercise II.1.16(a). Next, we argue that $\Gamma(X, \mathcal{K}_X^*/\pi_*\mathcal{O}_{\widetilde{X}}^*) \cong \operatorname{CaDiv}(\widetilde{X})$: given a global section $\{(U_i, f_i)\}_{i\in I}$, we get that $\{(\pi^{-1}(U_i), f_i)\}_{i\in I}$ is a global section of $\mathcal{K}_{\widetilde{X}}^*/\mathcal{O}_{\widetilde{X}}^*$, so we have an injection $\Gamma(X, \mathcal{K}_X^*/\pi_*\mathcal{O}_{\widetilde{X}}^*) \to \Gamma(\widetilde{X}, \mathcal{K}_{\widetilde{X}}^*/\mathcal{O}_{\widetilde{X}}^*) \cong \operatorname{CaDiv}(\widetilde{X})$.

To see this map is surjective, we know that by proposition II.6.11 the Weil and Cartier divisors on $\widetilde{X}$ are the same, so it suffices to show that we can represent any prime Weil divisor on $\widetilde{X}$ as a global section of $\mathcal{K}_{\widetilde{X}}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*$. Let $Q \in X$ be an arbitrary closed point and let $U \subset X$ be an affine open neighborhood of Q. Consider the closed subscheme

$$Y = \bigsqcup_{P\mapsto Q} (\{P\}, \mathcal{O}_{\widetilde{X},P}/(u_P^2)) \subset \pi^{-1}(U)$$

where u_P is a generator of the maximal ideal of $\mathcal{O}_{\widetilde{X},P}$. Since π is affine, $\pi^{-1}(U)$ is affine and therefore the map $\mathcal{O}_{\widetilde{X}}(\pi^{-1}(U)) \to \mathcal{O}_Y(U)$ is surjective. Let $s_P \in \mathcal{O}_{\widetilde{X}}(\pi^{-1}(U))$ be a section which maps to $u_{P'} \in \mathcal{O}_{\widetilde{X},P'}/(u_{P'}^2)$ if $P = P'$ and $1_{P'} \in \mathcal{O}_{\widetilde{X},P'}/(u_{P'}^2)$ otherwise. This section might have other zeroes on $\pi^{-1}(U)$, but none of them are at any point mapping to Q. Because π is finite and therefore closed, we may find a smaller open subset $U' \subset U$ which does not contain any point of $\pi(V(s_P))$ except for Q. Then $\{(X \setminus Q, 1), (U', s_P)\}$ is a global section of $\mathcal{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*$ which has P as its divisor, and we've shown that our map is indeed surjective. So our exact sequence stands at

$$0 \to \bigoplus_{P\in X} \widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}_{X,P}^* \to \operatorname{CaDiv} X \to \operatorname{CaDiv} \widetilde{X} \to 0.$$

To get from here to the requested exact sequence, our goal is to mod out the group of principal Cartier divisors in $\operatorname{CaDiv} X$ and $\operatorname{CaDiv} \widetilde{X}$ without touching the first term in our exact sequence. If we can do that, we're done by proposition II.6.15: X and $\widetilde{X}$ are both integral, implying $\operatorname{CaCl} X \cong \operatorname{Pic} X$. To be precise, we need to show that the image of our first term only has trivial intersection with the image $\Gamma(X, \mathcal{K}^*) = K(X)^*$ in $\operatorname{CaDiv} X$, or that the map $\operatorname{CaDiv} X \to \operatorname{CaDiv} \widetilde{X}$ doesn't send any nontrivial principal divisor to 0.

If some element of $\bigoplus_{P\in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^*$ has image the principal Cartier divisor associated to f on X, then as $K(X) = K(\widetilde{X})$ and the map $\operatorname{CaDiv} X \to \operatorname{CaDiv} \widetilde{X}$ sends the principal Cartier divisor associated to $f \in K(X)$ on X to the principal Cartier divisor associated to $f \in K(\widetilde{X})$ on $\widetilde{X}$, we need to show that any f which determines a trivial Cartier divisor on $\widetilde{X}$ also determines a trivial Cartier divisor on X. By the second lemma above, we have that $\mathcal{O}_X(X)$ and $\mathcal{O}_{\widetilde{X}}(\widetilde{X})$ are both finite extensions of k, and since $k = \overline{k}$, we have that they're both k.

As any element f of $K(\widetilde{X})^*$ determining the trivial Cartier divisor must be in $\mathcal{O}^*_{\widetilde{X},x}$ for every $x \in X$, we have by proposition II.6.3 that f is in A for every affine open $\operatorname{Spec} A \subset \widetilde{X}$ and thus $f \in k^*$, which clearly determines the trivial Cartier divisor on X too. So quotienting out by principal Cartier divisors gives an exact sequence

$$0 \to \bigoplus_{P\in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^* \to \operatorname{Pic} X \to \xrightarrow{\pi^*} \operatorname{Pic} \widetilde{X} \to 0.$$

(Counterexample in the case that $\mathcal{O}_X(X) \neq \mathcal{O}_{\widetilde{X}}(\widetilde{X})$: consider $X = V(x^2+y^2) \subset \mathbb{P}^2_{\mathbb{Q}}$ and the rational function x/y. Then $x/y \in \widetilde{\mathcal{O}_{X,[0:0:1]}}^*$, $x/y \notin \mathcal{O}^*_{X,[0:0:1]}$, and x/y is a constant global section of the normalization.)

b. Let $X = \operatorname{Proj} k[x,y,z]/(y^2z - x^3)$ be the cuspidal cubic and $Y = \operatorname{Proj} k[x,y,z]/(y^2 - x^3 - x^2)$ be the nodal cubic. We assume characteristic not 2 for the nodal cubic - if you want to do this in characteristic 2, replace Y by $\operatorname{Proj} k[x,y,z]/((y-x)yz - x^3)$ (this can be obtained from the form we wrote down in exercise II.6.7 by the change of coordinates sending $y \mapsto -y$ followed by the change of coordinates sending $x \mapsto x - y$; for a proof that all nodal cubics with the nodal directions defined over k are projectively equivalent to $xyz - (x-y)^3$, see Silverman's *Arithmetic of Elliptic Curves* page 61 proposition 2.5; the fun bit with the units changes from computing the integral closure of $k[t^2-1, t(t^2-1)]$ to $k[t^2-t, t^3-t^2]$ and the rest is left to you). We'll show that the normalization of either scheme is a copy of $\mathbb{P}^1_k$ and check that the quotients of groups of units are as described, which will finish the problem by part (a).

We'll start by computing the normalization of X. Note that $D(y)$ and $D(z)$ cover X, and $D(y)\cap X = \operatorname{Spec} k[x,z]/(z-x^3) \cong \operatorname{Spec} k[x]$ which is normal, while $D(z)\cap X = \operatorname{Spec} k[x,y]/(y^2-x^3)$ which isn't normal as y/x squares to x. As $k[x,y]/(y^2-x^3) \cong k[t^2,t^3]$ via $x \mapsto t^2$ and $y \mapsto t^3$, the normalization $D(z)\cap X$ is $\operatorname{Spec} k[t]$, which glues to produce a copy of $\mathbb{P}^1$ mapping to X where the map of local rings at the non-normal point is $k[t^2,t^3]_{(t^2,t^3)} \to k[t]_{(t)}$. The units of the former are, up to scaling, $\frac{1+t^2p(t)}{1+t^2q(t)}$, and the units of the latter are, up to scaling, $\frac{1+tr(t)}{1+ts(t)}$. Multiplying both the numerator and denominator of $\frac{1+tr(t)}{1+ts(t)}$ by $1-ts(t)$, we see that $\frac{1+tr(t)}{1+ts(t)} = \frac{1+tu(t)}{1-t^2s(t)^2}$, so every unit is equivalent up the action of $k[t^2,t^3]^\times_{(t^2,t^3)}$ to one of the form $1+ct$ plus higher-order terms in t. Further, I claim that any two such units with identical c are equivalent up to the action of $k[t^2,t^3]^\times_{(t^2,t^3)}$: after multiplying their quotient $\frac{1+ct+t^2p(t)}{1+ct+t^2q(t)}$by $1 = \frac{1-ct-t^2q(t)}{1-ct-t^2q(t)}$ we see that neither the numerator nor denominator have a t term. So the quotient group is exactly $1+ct$ as c runs over k, which is precisely $\mathbb{G}_a$ and therefore we have the exact sequence that was requested.

To compute the normalization of Y, we note $D(x)\cap Y = \operatorname{Spec} k[y,z]/(y^2z-2) \cong \operatorname{Spec} k[y,y^{-1}]$ and $D(y)\cap Y = \operatorname{Spec} k[x,z]/(z-x^3-x^2) \cong \operatorname{Spec} k[x]$ are normal while $D(z)\cap Y = \operatorname{Spec} k[x,y]/(y^2-x^3-x^2)$ is not, as y/x squares to $x+1$. As $k[x,y]/(y^2-x^3-x^2)$ is isomorphic to $k[t^2-1, t^3-t]$ via $x \mapsto t^2-1$ and $y \mapsto t^3-t$, we see that the normalization of $D(z) \cap Y$ is a copy of $\mathbb{A}^1 = \operatorname{Spec} k[t]$ which glues with the other two opens in our cover to form a $\mathbb{P}^1$. The units issues here are a bit less explicit: the units of the integral closure $k[t^2-1, t^3-t]_{(t^2-1,t^3-t)}[t]$ are those rational functions in $k(t)$ which are defined and do not vanish at $t = \pm 1$, while the units of $k[t^2-1, t^3-t]_{(t^2-1,t^3-t)}$ are rational functions in $k(t)$ which are defined and take the same non-vanishing value at $t = \pm 1$. So the nonzero ratio $u(1)/u(-1)$ is an invariant of units of the integral closure which is equal to one exactly when the unit is in $k[t^2-1, t^3-t]_{(t^2-1,t^3-t)}$, and all values in $k^\times$ are achieved as $1 + \frac{a-1}{2}(t+1)$ evaluates to a at $t = 1$ and 1 at -1. Therefore the first term in the exact sequence is $\mathbb{G}_m$ and we have the requested exact sequence.

Exercise II.6.10. *The Grothendieck Group $K(X)$*. Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves on X. If $\mathcal{F}$ is a coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.

a. If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

b. If X is any integral scheme, and $\mathcal{F}$ a coherent sheaf, we define the *rank* of $\mathcal{F}$ to be $\dim_K \mathcal{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism rank : $K(X) \to \mathbb{Z}$.

c. If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X \setminus Y) \to 0,$$

where the first map is extension by zero, and the second map is restriction. [*Hint*: For exactness in the middle, show that if $\mathcal{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_n = 0$, such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use (Ex. 5.15).]

For further information about $K(X)$, and its applications to the generalized Riemann-Roch theorem, see Borel-Serre [1], Manin [1], and Appendix A.

Solution.

a. As the global sections functor gives an equivalence of the category of coherent sheaves on $\mathbb{A}^1_k$ with finitely-generated modules over $k[t]$, we can work in the latter setting. By the classification of finitely generated modules over a PID, we have that every finitely generated module over $k[t]$ is isomorphic to $k[t]^n \bigoplus_{i\in I} k[t]/(p_i^{n_i})$ for $p_i \in k[t]$ irreducible, n, n_i non-negative integers, and I finite. Now I claim that any module of the form $k[t]/(p_i^{n_i})$ is equal to zero in the Grothendieck group: it fits in to an exact sequence of the form $0 \to k[t] \xrightarrow{\cdot p_i^{n_i}} k[t] \to k[t]/(p_i^{n_i}) \to 0$, so $k[t] - k[t] + k[t]/(p_i^{n_i})$ is in the group we modded out by. We also know that $k[t]^m + k[t]^n = k[t]^{n+m}$ from the obvious exact sequence. So the Grothendieck group is the free abelian group on a single generator $k[t]$, or $\mathbb{Z}$.

b. As taking stalks is exact, if we have an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, then $0 \to \mathcal{F}'_\xi \to \mathcal{F}_\xi \to \mathcal{F}''_\xi \to 0$ is an exact sequence of $\mathcal{O}_{X,\xi} = K$-modules. As every exact sequence of vector spaces splits, we have $\dim_K \mathcal{F}_\xi = \dim_K \mathcal{F}'_\xi + \dim_K \mathcal{F}''_\xi$ or $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{F}' + \operatorname{rank} \mathcal{F}''$. So the group homomorphism rank which sends $\sum n_i \mathcal{F}_i$ in the free abelian group of coherent sheaves to $\sum n_i \operatorname{rank}(\mathcal{F}_i)$ descends to a group homomorphism $K(X) \to \mathbb{Z}$ by the first isomorphism theorem. It is clearly surjective by considering $n\gamma(\mathcal{O}_x)$.

c. By exercise II.5.5 parts (b) and (c), the pushforward of a coherent sheaf along the closed immersion $i : Y \to X$ is again coherent. By exercise II.1.19, $(i_*\mathcal{F})_x$ is $\mathcal{F}_x$ if $x \in Y$ and 0 else, so the pushforward of an exact sequence of coherent sheaves on Y is still exact on X and we get a homomorphism $K(Y) \to K(X)$. As restriction to an open subscheme is exact because it preserves stalks, any exact sequence of sheaves on X is still exact when restricted to $X \setminus Y$, so we get a homomorphism $K(X) \to K(X \setminus Y)$. Surjectivity of $K(X) \to K(X \setminus Y)$ follows directly from the statement of exercise II.5.15, which says that for a noetherian scheme X with open subscheme U and a coherent sheaf $\mathcal{F}$ on U, then there is a coherent sheaf $\mathcal{F}'$ on X so that $\mathcal{F}'|_U \cong \mathcal{F}$.

All that's left is exactness in the middle, and this is slightly more interesting. Clearly the direct image of any coherent sheaf on Y is sent to 0 when restricting to $X \setminus Y$, so it will suffice to show that the class of any coherent sheaf on X which is supported on Y is in the image of $K(Y) \to K(X)$. We'll proceed as in the hint and find a finite filtration $\mathcal{F} = \mathcal{F}_0 \supset \cdots \supset \mathcal{F}_n = 0$ so that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is a $\mathcal{O}_Y$-module. To see that this solves the problem, consider the exact sequences $0 \to \mathcal{F}_{i+1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i+1} \to 0$: this implies that $\gamma(\mathcal{F}_i) - \gamma(\mathcal{F}_{i+1}) = \gamma(\mathcal{F}_i/\mathcal{F}_{i+1})$, so summing over all i we have that $\gamma(\mathcal{F}) = \sum \gamma(\mathcal{F}_i/\mathcal{F}_{i+1})$, and each $\gamma(\mathcal{F}_i/\mathcal{F}_{i+1})$ is in the image of $K(Y)$.

To construct the filtration, consider the natural morphism $\mathcal{F} \to i_*i^*\mathcal{F}$ coming from the adjunction between i^* and i_* (cf exercise II.1.18 and page 110). If $\mathcal{F}$ is coherent, then so is $i_*i^*\mathcal{F}$: $i^*\mathcal{F}$ is coherent by the description of coherent sheaves on a noetherian scheme as those which are affine-locally $\widetilde{M}$ and proposition II.5.2(e); and i_* of a coherent sheaf is coherent as mentioned in the first paragraph. I claim that $\mathcal{F} \to i_*i^*\mathcal{F}$ is surjective, which we can check affine-locally: working on an affine open $\operatorname{Spec} A \subset X$, we have that $X \to Y$ corresponds to $A \to A/I$ for some ideal $I \subset A$, and $\mathcal{F} \to i_*i^*\mathcal{F}$ corresponds to $M \to M/IM$, with kernel IM. Set $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_{i+1} = \ker \mathcal{F}_i \to i_*i^*\mathcal{F}_i$, which gives that $\mathcal{F}_i/\mathcal{F}_{i+1} \cong i_*i^*\mathcal{F}_i$, a coherent $\mathcal{O}_Y$-module.

It remains to explain why the filtration we obtain is finite. We work affine locally on $\operatorname{Spec} A \subset X$, where $Y \cap \operatorname{Spec} A = \operatorname{Spec} A/I$, and $\mathcal{F}|_{\operatorname{Spec} A} = \widetilde{M}$ for a noetherian ring A, ideal $I \subset A$, and M a finitely-generated A-module. By exercise II.5.6(b), we have $\operatorname{Supp}(\mathcal{F}) = V(\operatorname{Ann} M)$, and this is set-theoretically contained in $Y = V(I)$ by assumption. As $V(\operatorname{Ann} M) \subset V(I)$ implies that $\sqrt{\operatorname{Ann} M} \supset \sqrt{I}$, we have that $\sqrt{\operatorname{Ann} M} \supset I$ and every element $f \in I$ has some power f^n in $\operatorname{Ann} M$. Since A is noetherian, I is finitely generated, say by $g_1, \cdots, g_m$, and we for any $1 \leq j \leq m$ we have $g_j^{n_j} \in \operatorname{Ann} M$ for some integer n_j. Letting N be the maximum of the n_j, we get that $I^N \subset \operatorname{Ann} M$, so $I^N M = 0$. As the n^{th} step of our filtration is $I^n M$, we have that our filtration is zero on $\operatorname{Spec} A$ by step N. Since X is noetherian, it is quasi-compact, and may be covered by finitely many open affines $\operatorname{Spec} A_i$ so the filtration is zero after step N_i on $\operatorname{Spec} A_i$. Taking $N = \max_i N_i$, we see that our filtration is zero at step N on X, and we're done.

Exercise II.6.11. (*) *The Grothendieck Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$, in several steps.

a. For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that for any D, $\psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.

b. For any coherent sheaf $\mathcal{F}$ on X, show that there exist locally free sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ and an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$. Let $r_0 = \operatorname{rank} \mathcal{E}_0$, $r_1 = \operatorname{rank} \mathcal{E}_1$, and define $\det \mathcal{F} = (\bigwedge^{r_0} \mathcal{E}_0) \otimes (\bigwedge^{r_1} \mathcal{E}_1)^{-1} \in \operatorname{Pic} X$. Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathcal{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathcal{L}(D)$.

c. If $\mathcal{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathcal{L}(D)^{\oplus r} \to \mathcal{F} \to \mathcal{T} \to 0$, where $\mathcal{T}$ is a torsion sheaf. Conclude that if $\mathcal{F}$ is a sheaf of rank r, then $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$.

d. Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Solution.

a. If $D = D_1 + D_2$ where D_1 and D_2 are effective divisors with no points in common, then we have $\psi(D) = \psi(D_1) + \psi(D_2)$ by definition and $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_{D_1}) + \gamma(\mathcal{O}_{D_2})$ as $\mathcal{O}_D \cong \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}$, so it suffices to treat the case of $D = nP$. Write f for the function which locally cuts out D in a neighborhood of P. Then $\mathcal{O}_D$ is the skyscraper sheaf with value $\mathcal{O}_{X,P}/f$, which is a vector space of dimension $v_P(f) = n$ by the proof of proposition II.6.11. Recall that $\mathcal{O}_{X,P}$ is a DVR, so up to multiplying f by a unit in $\mathcal{O}_{X,P}$, we may assume that $f = \pi^n$ for π a uniformizer of $\mathcal{O}_{X,P}$. Then $\mathcal{O}_{X,P}/f$ has a filtration by submodules $\pi^j \mathcal{O}_{X,P}/f$, so $\mathcal{O}_D$ has a filtration by subsheaves $i_P(\pi^j \mathcal{O}_{X,P}/f)$, the skyscraper sheaves with value $\pi^j \mathcal{O}_{X,P}/f$ at P. Just as in exercise II.6.10(c), this lets us write $\gamma(\mathcal{O}_D) = \sum \gamma(i_P(\pi^j \mathcal{O}_{X,P}/f)/i_P(\pi^{j+1}\mathcal{O}_{X,P}/f))$, and as each of these successive quotients are isomorphic to $k(P)$, we have that $\psi(D) = \gamma(\mathcal{O}_D)$.

To check that this only depends on the linear equivalence class of D, we note that from the exact sequence $0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, we have $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X) - \gamma(\mathcal{I}_D)$, while by proposition II.6.18 $\mathcal{I}_D \cong \mathcal{L}(-D)$ and by proposition II.6.13 $\mathcal{L}(-D) \cong \mathcal{L}(-D')$ iff $D \sim D'$. As isomorphisms give equalities in the Grothendieck group, we have that $\gamma(\mathcal{O}_D)$ only depends on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$ once we note that any divisor can be written as a difference of effective divisors.

b. By proposition II.6.7, a complete curve over an algebraically closed field is projective, so by corollary II.5.18, any coherent sheaf $\mathcal{F}$ on X can be written as a quotient of a direct sum of twists of the structure sheaf. As such a sheaf is locally free, we have $\mathcal{E}_0$ as requested.

Let $\mathcal{E}_1$ be the kernel of $\mathcal{E}_0 \to \mathcal{F}$. I claim this is locally free, too: every stalk of $\mathcal{E}_1$ is a submodule of the stalk of $\mathcal{E}_0$, which is free, so as $\mathcal{O}_{X,x}$ is a PID for every $x \in X$, every stalk of $\mathcal{E}_1$ is again free. By exercise II.5.7, this shows the requested exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$.

To show that $\det \mathcal{F}$ is independent of the resolution, first we define the determinant bundle for a locally free sheaf: if $\mathcal{E}$ is a locally free sheaf of rank e, define $\det \mathcal{E} = \bigwedge^e \mathcal{E}$. From exercise II.5.16(d), we see that if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves, $\det \mathcal{E} \cong (\det \mathcal{E}') \otimes (\det \mathcal{E}'')$. Now suppose we have two resolutions $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ and $0 \to \mathcal{G}_1 \to \mathcal{G}_0 \to \mathcal{F} \to 0$ and consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d] & \mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_0 \oplus \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{K}$ is the kernel of $\mathcal{E}_0 \oplus \mathcal{G}_0 \to \mathcal{F}$ by the direct sum of the maps $\mathcal{E}_0 \to \mathcal{F}$ and $\mathcal{G}_0 \to \mathcal{F}$, which is locally free for the same reason as $\mathcal{E}_1$ was above. By the nine lemma, the left column is exact. We can also consider the same diagram with the roles of the $\mathcal{E}$ and $\mathcal{G}$ reversed - note that this doesn't change $\mathcal{K}$, so $\det \mathcal{K} \cong (\det \mathcal{E}_1) \otimes (\det \mathcal{G}_0)$ and $\det \mathcal{K} \cong (\det \mathcal{G}_1) \otimes (\det \mathcal{E}_0)$. Now we push some determinants around:

$$\begin{aligned}
(\det \mathcal{E}_0) \otimes (\det \mathcal{E}_1)^{-1} &\cong (\det \mathcal{E}_0) \otimes (\det \mathcal{E}_1)^{-1} \otimes (\det \mathcal{G}_0)^{-1} \otimes (\det \mathcal{G}_0) \\
&\cong (\det \mathcal{E}_0) \otimes (\det \mathcal{K})^{-1} \otimes (\det \mathcal{G}_0) \\
&\cong (\det \mathcal{E}_0) \otimes (\det \mathcal{G}_1)^{-1} \otimes (\det \mathcal{E}_0)^{-1} (\det \mathcal{G}_0) \\
&\cong (\det \mathcal{G}_1)^{-1} \otimes (\det \mathcal{G}_0)
\end{aligned}$$

and we see that $\det \mathcal{F}$ is independent of the chosen resolution.

To check that $\det \mathcal{F}$ defines a homomorphism from $K(X)$, we need to show that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, then $\det \mathcal{F} \cong (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$. So suppose $0 \to \mathcal{E}_1' \to \mathcal{E}_0' \to \mathcal{F}' \to 0$ is a presentation of $\mathcal{F}'$ as a quotient of locally free sheaves, and similarly for $0 \to \mathcal{E}_1'' \to \mathcal{E}_0'' \to \mathcal{F}'' \to 0$. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_1' \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_0' \arrow[r] \arrow[d] & \mathcal{E}_0' \oplus \mathcal{E}_0'' \arrow[r] \arrow[d] & \mathcal{E}_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathcal{K}$ is the kernel of the map $\mathcal{E}_0' \oplus \mathcal{E}_0'' \to \mathcal{F}$, which is defined locally and glued together. We use the same determinant calculus as above:

$$\begin{aligned}
\det \mathcal{F} &\cong (\det(\mathcal{E}_0' \oplus \mathcal{E}_0'')) \otimes (\det \mathcal{K})^{-1} \\
&\cong (\det \mathcal{E}_0') \otimes (\det \mathcal{E}_0'') \otimes (\det \mathcal{E}_1')^{-1} \otimes (\det \mathcal{E}_1'')^{-1} \\
&\cong (\det \mathcal{E}_0') \otimes (\det \mathcal{E}_1')^{-1} \otimes (\det \mathcal{E}_0'') \otimes (\det \mathcal{E}_1'')^{-1} \\
&\cong (\det \mathcal{F}') \otimes (\det \mathcal{F}'')
\end{aligned}$$

and so we indeed have a homomorphism $\det : K(X) \to \operatorname{Pic} X$.

Writing $D = D_1 - D_2$ for D_1, D_2 effective divisors and using that det and ψ are homomorphisms, it suffices to show the claim for D effective. In that case consider $0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ - as both $\mathcal{I}_D$ and $\mathcal{O}_X$ are locally free of rank one, we have that $\det \mathcal{O}_D \cong \mathcal{O}_X \otimes \mathcal{I}_D^{-1} \cong \mathcal{L}(-D)^{-1} \cong \mathcal{L}(D)$ by propositions II.6.18 and II.6.13.

c. Let ξ be the generic point of X. As $\mathcal{F}_\xi \cong k(X)^r$ is a free $k(X)$-module, we have by exercise II.5.7(a) that $\mathcal{F}$ is free in an open neighborhood U of ξ, which implies that we have an isomorphism $\mathcal{O}_X^r|_U \to \mathcal{F}|_U$ on U. Suppose $X \setminus U = \{P_1, \cdots, P_n\}$, and let $D = \sum P_i$. As $\mathcal{L}(eD)|_U$ is trivial for any e, we have $\mathcal{F}|_U \cong (\mathcal{F} \otimes \mathcal{L}(eD))|_U$. I claim that for a sufficiently large integer e, the composite $\mathcal{O}_X^{\oplus r}|_U \to (\mathcal{F} \otimes \mathcal{L}(eD))|_U$ lifts to a map $\mathcal{O}_X^{\oplus r} \to \mathcal{F} \otimes \mathcal{L}(eD)$, which after tensoring by $\mathcal{L}(-eD)$ gives an injection $\mathcal{L}(-eD) \to \mathcal{F}$ which is surjective on U, and thus the quotient $\mathcal{F}/\mathcal{L}(-eD)$ is supported on the finite set $X \setminus U$ and is torsion.

To show this, let $\{(U_i, f_i), (U, 1)\}$ be the Cartier divisor associated to D, where we assume that each U_i is actually $\operatorname{Spec} A_i$ after an appropriate shrinking and subdivision and the number of U_i is finite by quasi-compactness. Now the map $\mathcal{O}_X^{\oplus r}|U \to \mathcal{F}|_U$ is given by an r-tuple of sections in $\mathcal{F}(U)$, and each section of $\mathcal{F}(U)$ gives us a section of $\mathcal{F}$ over $D(f_i) \subset U_i$. By lemma II.5.3, for each i and each section over $D(f_i)$ there is an integer n so that f^n times that section extends to a section over U_i. Taking the maximum of these n across all finitely many i and the r sections over each $D(f_i)$, we obtain e so that $\mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ extends to

$\mathcal{O}_X^{\oplus r} \to \mathcal{F} \otimes \mathcal{L}(eD)$. The kernel of this map is a subsheaf of $\mathcal{O}_X^{\oplus r}$ and is thus locally free by the logic from (b), plus it's zero at the generic point by definition, so $\mathcal{O}_X^{\oplus r} \to \mathcal{F} \otimes \mathcal{L}(eD)$ is injective. As tensoring with an invertible sheaf is an isomorphism, we get an injection $\mathcal{L}(-eD)^{\oplus r} \to \mathcal{F}$ with cokernel a torsion sheaf $\mathcal{T}$ as requested.

From the exact sequence $0 \to \mathcal{L}(-eD)^{\oplus r} \to \mathcal{F} \to \mathcal{T} \to 0$, we get that $\gamma(\mathcal{F}) = r\gamma(\mathcal{L}(-eD)) + \gamma(\mathcal{T})$ in $K(X)$. As $\mathcal{L}(-eD) \cong \mathcal{I}_{eD}$ fits in to the exact sequence $0 \to \mathcal{L}(-eD) \to \mathcal{O}_X \to \mathcal{O}_{eD} \to 0$, we have that $\gamma(\mathcal{L}(-eD)) = \gamma(\mathcal{O}_X) - \gamma(\mathcal{O}_{eD})$, so $\gamma(\mathcal{F}) = r(\gamma(\mathcal{O}_X) - \gamma(\mathcal{O}_{eD})) + \gamma(\mathcal{T})$ and therefore $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X) = r\gamma(\mathcal{O}_{eD}) + \gamma(\mathcal{T})$. As any coherent sheaf supported on a finite set of points of X can be filtered as in (a), we see that $\gamma(\mathcal{T})$ is in the image of ψ, so $r\gamma(\mathcal{O}_{eD}) + \gamma(\mathcal{T})$ is in the image of ψ and therefore $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X)$ is.

d. Here's the diagram:

$$\begin{tikzcd}
\operatorname{Pic} X \arrow[r, bend left, "\psi"] & K(X) \arrow[l, bend left, "\det"] \arrow[r, bend left, "\mathrm{rank}"] & \mathbb{Z} \arrow[l, bend left, "1 \mapsto \gamma(\mathcal{O}_X)"]
\end{tikzcd}$$

The composition $\det \circ \psi$ is the identity by (b), so det is surjective and we have a splitting $K(X) \cong \operatorname{Pic} X \oplus (\ker \det)$ by the splitting lemma applied to

$$0 \to \ker \det \to K(X) \overset{\det}{\to} \operatorname{Pic} X \to 0.$$

Meanwhile, $\mathcal{O}_X$ is in the kernel and by (c) we have that $K(X) = \mathbb{Z} \cdot \gamma(\mathcal{O}_X) + \operatorname{Im} \psi$, so $\gamma(\mathcal{O}_X)$ generates the kernel. As $\operatorname{rank} \gamma(\mathcal{O}_X) = 1$ is non-torsion, we see that $\gamma(\mathcal{O}_X)$ is non-torsion, so $\ker \det = \mathbb{Z} \cdot \gamma(\mathcal{O}_X)$ and we have the desired splitting.

Exercise II.6.12. Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathcal{F} \in \mathbb{Z}$, such that:

1. If D is a divisor, $\deg \mathcal{L}(D) = \deg D$;
2. If $\mathcal{T}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathcal{F} = \sum_{P \in X} \operatorname{length}(\mathcal{F}_P)$; and
3. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, then $\deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$.

Solution. Defining deg as a homomorphism $K(X) \to \mathbb{Z}$ will make it automatically satisfy the third condition. The first condition says that $\deg \gamma(\mathcal{O}_X) = 0$, while the second condition says that $\deg \mathcal{O}_{X,P} = 1$, as any torsion coherent sheaf admits a finite filtration with subquotients given by the structure sheaves of points by part (a) of exercise II.6.11. By part (d) of exercise II.6.11, we have $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z} \cdot \gamma(\mathcal{O}_X)$, so defining $\deg : K(X) \to \mathbb{Z}$ by sending $D \mapsto \deg D$ and $\gamma(\mathcal{O}_X) \mapsto 0$ completely characterizes deg.

II.7 Projective Morphisms

This section isn't used a ton in the rest of chapter II or chapter III, but once you reach chapter IV you'll start to see its importance.

Exercise II.7.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. [*Hint*: Reduce to a question of modules over a local ring by looking at stalks.]

Solution. We start with a lemma from abstract algebra:

Lemma. *Let R be a commutative ring. Then any surjective R-module endomorphism of the regular module R is an isomorphism.*

Proof. Any R-linear endomorphism f of the regular module is multiplication by an element of R: if $r \in R$ is arbitrary, then $f(r) = rf(1)$. Now let u denote an element satisfying $f(u) = 1$. Then $uf(1) = 1$, so $f(1)$ is a unit with inverse u, and therefore the map $g(r) = ru$ is a two-sided inverse to f.

■

By our previous results in sheaf theory, a map of sheaves is surjective (respectively, an isomorphism) iff it is on stalks. As invertible sheaves are locally free of rank one, every stalk is isomorphic to $\mathcal{O}_{X,x}$, so the induced map on stalks is a surjective $\mathcal{O}_{X,x}$-endomorphism of the regular module $\mathcal{O}_{X,x}$. By the lemma, this is an isomorphism, so f is an isomorphism.

Exercise II.7.2. Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \cdots, s_n\}$ and $\{t_0, \cdots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subset \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphisms $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m \setminus L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m-n-1$.

Solution. This is mostly linear algebra. Suppose $\dim V = l+1$ - then there are linearly independent subsets of the s_i and t_j of size $l+1$ so that these subsets span V and generate the sheaf at every point (the second condition is implied by the first by passing to the stalk, which is a k-linear map and thus preserves linear dependence relations). Without loss of generality, we may assume that these are $\{s_0, \cdots, s_l\}$ and $\{t_0, \cdots, t_l\}$. As these are bases, for every $i \leq r$ we can write $t_i = \sum_{j=0}^r a_{ij} s_j$ uniquely. For $r < i \leq n$, write $t_i = s_i + \sum_{j=0}^r b_{ij} s_j$, where the coefficients b_{ij} are uniquely determined. Thus the matrix

$$M = \begin{pmatrix} A & 0 & 0 \\ B & I_{r-n} & 0 \end{pmatrix}$$

where $A = (a_{ij})_{0 \leq i,j \leq r}$, $B = (b_{ij})_{r < i \leq n, 0 \leq j \leq r}$, and I_{r-n} is the identity matrix of size $r-n$ is the matrix for an injective map $k\langle s_0, \cdots, s_n\rangle \to k\langle t_0, \cdots, t_m\rangle$ which commutes with the surjections $k\langle s_0, \cdots, s_n\rangle \to V$ and $k\langle t_0, \cdots, t_m\rangle \to V$.

This map induces an injection of graded rings $k[s_0, \cdots, s_n] \to k[t_1, \cdots, t_m]$, and by the construction of theorem II.7.1 we get the following commutative triangle of schemes:

$$\begin{tikzcd}
\operatorname{Proj} k[s_0, \cdots, s_n] & \operatorname{Proj} k[t_0, \cdots, t_m] \arrow[l] \\
& X \arrow[u] \arrow[ul]
\end{tikzcd}$$

Up to a linear automorphism of $k\langle t_0, \cdots, t_m\rangle$ inducing an automorphism of $\operatorname{Proj} k[t_0, \cdots, t_m]$, we can transform our matrix M to be the matrix $\begin{pmatrix} I_n & 0 \end{pmatrix}$, so that our map of vector spaces is the canonical injection $s_i \mapsto t_i$, corresponding to the injection $k[s_0, \cdots, s_n] \to k[t_0, \cdots, t_m]$ sending $s_i \mapsto t_i$. Setting $L = V(t_{n+1}, \cdots, t_m)$, this is exactly the projection from $\operatorname{Proj} k[t_0, \cdots, t_m] \setminus L = \mathbb{P}^m \setminus L$ to $\operatorname{Proj} k[s_0, \cdots, s_n]$.

Exercise II.7.3. Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

a. either $\varphi(\mathbb{P}^n) = pt$ or $m \geq n$ and $\dim \varphi(\mathbb{P}^n) = n$;

b. in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \geq 1$, (2) a linear projection $\mathbb{P}^N \setminus L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m$. Also, φ has finite fibers.

Solution. Let's recall some facts: by corollary II.6.17, the invertible sheaves on $\mathbb{P}^n_k$ are exactly $\mathcal{O}(n)$ for $n \in \mathbb{Z}$. By proposition II.5.13, we have that the global sections of $\mathcal{O}_{\mathbb{P}^n_A}(d)$ is exactly the d-graded piece of $A[x_0, \cdots, x_n]$, which is a free A-module of rank $\binom{n+d}{n}$.

a. By theorem II.7.1, a map from a scheme X to $\mathbb{P}^m_k$ is equivalent to the data of a line bundle $\mathcal{L}$ on X plus a choice of $m+1$ global sections $s_0, \cdots, s_m$ of $\mathcal{L}$ which generate the sheaf at each point. If we pick $\mathcal{O}(d)$ on $X = \mathbb{P}^n$ for $d < 0$, this has no nonzero global sections and therefore cannot determine a morphism to $\mathbb{P}^m$. If we pick $\mathcal{O}(d)$ for $d = 0$, then the global sections are the constants and this gives a constant map to $\mathbb{P}^m$. If we pick $\mathcal{O}(d)$ for $d > 0$, then the locus where our $m+1$ sections fail to generate the sheaf is exactly $V(s_0, \cdots, s_m)$, which is of codimension at most $m+1$ by Krull's height theorem. Since we need this locus to be empty, we need $n - (m+1) < 0$, which is equivalent to $n \leq m$.

   To show the claim about dimension, we fix n and induct on m. Suppose we have a map $\varphi : \mathbb{P}^n \to \mathbb{P}^m$. If $n > m$, it's image is a point by the above. If $n \leq m$, either it's surjective, or it's not.

   In the first case, $\dim(\varphi(\mathbb{P}^n)) = m \geq n$ and I claim that this implies $m = n$. Since the generic point of $\mathbb{P}^n$ must map to the generic point of $\mathbb{P}^m$, we see that we get an induced map on function fields $k(\mathbb{P}^m) \to k(\mathbb{P}^n)$, which implies $m \leq n$ by the properties of transcendence degree and exercise II.3.20.

   In the second case, $\varphi(\mathbb{P}^n)$ is a proper closed subset of $\mathbb{P}^m$ by exercise II.4.4, so there is a closed point $P \in \mathbb{P}^m$ which is missed by φ by exercise II.3.14. Let π be the projection $\mathbb{P}^m \setminus \{P\} \to \mathbb{P}^{m-1}$, and consider $\varphi' := \pi \circ \varphi : \mathbb{P}^n \to \mathbb{P}^{m-1}$. By the inductive hypothesis, we have that either $\varphi'(\mathbb{P}^n)$ is of dimension n or a point. If $\dim \varphi'(\mathbb{P}^n) = n$, then we have $\dim \varphi(\mathbb{P}^n) = n$ by the same logic as when φ was surjective as $\varphi(\mathbb{P}^n)$ surjects on to $\varphi'(\mathbb{P}^n)$.

On the other hand, if $\varphi'(\mathbb{P}^n)$ is a point, then $\varphi(\mathbb{P}^n)$ lies in the fiber over that point in $\mathbb{P}^m \setminus P$. This is a copy of $\mathbb{A}^1$, and by exercise II.4.4, the image of φ in this fiber (and thus $\mathbb{P}^m$) must be a point.

b. By the classification of line bundles on $\mathbb{P}^n$, we see that d is determined by $\mathcal{L} \cong \mathcal{O}(d)$. Let $\{s_0, \cdots, s_p\}$ denote the global sections of $\mathcal{O}(d)$ over $\mathbb{P}^n$ which determine the morphism $\mathbb{P}^n \to \mathbb{P}^m$, and then add sections $s_{p+1}, \cdots, s_r$ so that $\{s_0, \cdots, s_r\}$ spans the global sections of $\mathcal{O}(d)$. On the other hand, we can consider the sections $\{t_0, \cdots, t_{\binom{n+d}{d}-1}, 0, \cdots, 0\}$ where the t_i are the standard basis of $k[x_0, \cdots, x_n]_d$ and we add enough zeroes to get at least r sections - then by problem II.7.2, we have the claim about the projection.

   To check that the fibers are finite, let π be the projection. The linear space L is of dimension $N - m - 1$, and the fiber of π over a point is exactly a copy of of $\mathbb{P}^{N-m}$ over the residue field of that point. As our copy of $\mathbb{P}^n$ doesn't intersect L, it's intersection with the fiber must be 0-dimensional by the projective dimension theorem (I.7.2).

Exercise II.7.4.

a. Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

b. Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate $\operatorname{Pic} X$, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

Solution.

a. First, recall that closed immersions are separated because they are affine: this is exercise II.3.12 plus the proof of exercise II.4.1. Next, I claim the open immersions are separated. Suppose $U \to Z$ is an open immersion, and consider the map $U \to U \times_Z U$: $U \times_Z U \cong U \times_U U \cong U$, and the map is an isomorphism which is a closed immersion.

   Now let $\mathcal{L}$ be the ample sheaf on X. By theorem II.7.6, $\mathcal{L}^m$ is ample for some $m > 0$, so there is an immersion $i : X \to \mathbb{P}^n_A$. Then i is the composition of an open immersion and closed immersion, both of which are separated morphisms. By theorem II.4.9, projective morphisms are proper and thus separated, so the map $X \to \operatorname{Spec} A$ factors as a composition of separated morphisms and is therefore separated.

b. A line bundle $\mathcal{L}$ on X is the data of a line bundle $\mathcal{L}_1, \mathcal{L}_2$ on each copy of $\mathbb{A}^1$ which agrees on the intersection $\mathbb{A}^1 \setminus \{0\}$. Since all of these are affine with trivial Picard group, we can assume that the line bundles on each of these are just a copy of the structure sheaf. By the gluing lemma, to classify line bundles on X, it is enough to classify the isomorphisms $\mathcal{L}_1|_{X\setminus\{0_1,0_2\}} \cong \mathcal{L}_2|_{X\setminus\{0_1,0_2\}}$, which by the previous sentence amounts to classifying automorphisms of $k[x, x^{-1}]$ as $X \setminus \{0_1, 0_2\}$ is affine. All such automorphisms are of the form cx^n for $n \in \mathbb{Z}$ and $c \in k^\times$, and it is not difficult to see that we can always assume the automorphism is of the form x^n by the automorphism of one of the $\mathcal{O}_{\mathbb{A}^1_k}$ which is scaling by c.

Denote the line bundle associated to the automorphism x^n by $\mathcal{L}_n$. To compute global sections, we can use the two obvious copies of $\mathbb{A}^1$ and get an exact sequence

$$0 \to \mathcal{L}_n(X) \to k[x_1] \oplus k[x_2] \to k[x, x^{-1}]$$

by the sheaf condition. Up to a module automorphism of $k[x, x^{-1}]$ we may assume that $x_1 \mapsto 1$ and $x_2 \mapsto x^n$, so that the global sections are pairs $(f(x_1), x_2^n f(x_2))$ or $(x_1^n f(x_1), f(x_2))$ depending on the sign of n. It is clear that if $n \neq 0$, these global sections cannot generate the stalk at one of the two origins, so the only possible ample sheaf is the one with $n = 0$, the structure sheaf of X. But this is clearly not ample, because $\mathcal{L}_1 \otimes \mathcal{L}_0^{\otimes t} \cong \mathcal{L}_1$, contradicting the definition of ample (pg 153). Thus there is no ample invertible sheaf on X.

Exercise II.7.5. Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

a. If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, the $\mathscr{L} \otimes \mathscr{M}$ is ample.

b. If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.

c. If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.

d. If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.

e. If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geq n_0$.

Solution. We begin with a lemma:

Lemma. *If two sheaves $\mathcal{F}$ and $\mathcal{G}$ on a scheme X are generated by global sections, then $\mathcal{F} \otimes \mathcal{G}$ is too.*

Proof. Generated by global sections means that there exist surjections $\mathcal{O}_X^{\oplus I} \to \mathcal{F}$ and $\mathcal{O}_X^{\oplus J} \to \mathcal{G}$ for sets I, J. Since tensor products are right-exact, we get a surjection $\mathcal{O}_X^{\oplus I} \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$, and as $\mathcal{O}_X^{\oplus I} \otimes \mathcal{G} \cong \mathcal{G}^{\oplus I}$ admits a surjection from $\mathcal{O}_X^{\oplus(I \times J)}$, we see that there is a surjection $\mathcal{O}_X^{\oplus(I \times J)} \to \mathcal{F} \otimes \mathcal{G}$ by composition.

■

Now on to the main proof.

a. If $\mathcal{L}$ is ample, then for any sheaf $\mathcal{F}$ there is an $n_0 > 0$ so that for all $n \geq n_0$ we have $\mathcal{F} \otimes \mathcal{L}^n$ globally generated. The same n works for $\mathcal{L} \otimes \mathcal{M}$: $\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^n \cong \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{M}^n$, which is a tensor product of the globally generated sheaves $\mathcal{F} \otimes \mathcal{L}^n$ and $\mathcal{M}^n$ and thus globally generated by our lemma.

b. Pick $n > 0$ so that $\mathcal{M} \otimes \mathcal{L}^n$ is globally generated, and then note that $\mathcal{L} \otimes (\mathcal{L}^n \otimes \mathcal{M})$ is ample by (a).

c. Find an $n > 0$ so that $\mathcal{L} \otimes \mathcal{M}^n$ is globally generated by the definition of ample. Then $(\mathcal{L} \otimes \mathcal{M})^n \cong \mathcal{L}^{n-1} \otimes (\mathcal{L} \otimes \mathcal{M}^n)$, which is ample by (a). Now, apply proposition II.7.5, which says that a sheaf is ample iff some positive tensor power of it is.

d. Let $s_0, \cdots, s_m$ be global sections generating $\mathcal{L}$ and $t_0, \cdots, t_m$ be global sections generating $\mathcal{M}$, with corresponding morphisms $i : X \to \mathbb{P}^m_A$ and $j : X \to \mathbb{P}^n_A$. Then the global sections $s_i \otimes t_j$ generate $\mathcal{L} \otimes \mathcal{M}$, and the corresponding morphism $X \to \mathbb{P}^{mn+m+n}$ is the composition of $(i,j) : X \to \mathbb{P}^m \times \mathbb{P}^n$ with the Segre embedding $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$. By exercise II.5.11, we have that $\sigma^* \mathcal{O}_{\mathbb{P}^{mn+m+n}}(1) \cong p^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes q^* \mathcal{O}_{\mathbb{P}^n}(1)$ where p, q are the projections from the product. As $p \circ (i,j) = i$ and $q \circ (i,j) = j$ while tensor products commute with pullbacks, we note that $(i,j)^*(p^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes q^* \mathcal{O}_{\mathbb{P}^n}(1)) \cong i^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes j^* \mathcal{O}_{\mathbb{P}^n}(1)$. Next, $i^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \mathcal{L}$ by the definition of very ample, and $j^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{M}$ by the definition of the morphism associated to a set of sections and the fact that the sections t_j generate $\mathcal{M}$, so the pullback of $\mathcal{O}_{\mathbb{P}^{mn+m+n}}(1)$ to X along our map is precisely $\mathcal{L} \otimes \mathcal{M}$.

To check that our morphism is a closed immersion, we write it as the composition $X \to X \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$. The first morphism is a closed immersion because it's the graph Γ_j and X is separated over A (see the solution to exercise II.4.8(e) for the graph morphism), the second morphism is a closed immersion by exercise II.4.8(d) applied to the property 'is a closed immersion', and the final morphism is the Segre embedding. Therefore we've verified the definition of $\mathcal{L} \otimes \mathcal{M}$ being very ample.

e. Pick $p > 0$ so that $\mathcal{L}^p$ is very ample by theorem II.7.6. Pick $q_0 > 0$ so that $\mathcal{L} \otimes \mathcal{L}^q$ is generated by global sections for all $q > q_0$ by the definition of ample. Now set $n_0 = p + q + 1$: if $n > n_0$, then $\mathcal{L}^n \cong \mathcal{L}^m \otimes (\mathcal{L} \otimes \mathcal{L}^q)$ with $q > q_0$, so $\mathcal{L} \otimes \mathcal{L}^q$ is generated by global sections and $\mathcal{L}^m$ is ample. By (d) we are done.

Exercise II.7.6. *The Riemann-Roch Problem.* Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behavior for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathscr{L}^n)$ as a function of n.

a. Show that if D is very ample, and if $X \hookrightarrow \mathbb{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the *Hilber polynomial* of X (I, §7). Thus in this case $\dim |nD|$ is a polynomial function of n, for n large.

b. If D corresponds to a torsion element of $\operatorname{Pic} X$, of order r, then $\dim |nD| = 0$ if $r | n$, -1 otherwise. In this case the function is periodic of period r.

It follows from the general Riemann-Roch thereom that $\dim |nD|$ is a polynomial function for n large, whenever D is an *ample* divisor. See (IV, 1.3.2), (V, 1.6), and Appendix A. In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials $P_1, \cdots, P_r$, such that for all n sufficiently large, $\dim |nD| = P_{i(n)}(n)$, where $i(n) \in \{1, 2, \cdots, r\}$ is a function of n.

Solution.

a. Let $i : X \hookrightarrow \mathbb{P}^n_k$ denote the immersion coming from D. Then $i^*\mathcal{O}(1) \cong \mathcal{L}$, and as pullback commutes with tensor products, we have that $\mathcal{L}^n \cong i^*\mathcal{O}(n)$, and so $\Gamma(X, \mathcal{L}^n) \cong \mathcal{O}_X(n)(X)$. By exercise II.5.14(b), this is equal to $\dim_k S(X)_n$ for $n \gg 0$, which is exactly the definition of the Hilbert polynomial from chapter I section 7. Remembering to subtract 1 for the translation to linear systems, we have the result.

b. If $r|n$, then $\mathcal{L}^n \cong \mathcal{O}_X$ which has global sections k. To show that $\dim |nD| = -1$ when r does not divide n it suffices to show that $\dim_k H^0(\mathcal{L}^n) = 0$ in this case. So suppose $\mathcal{L}^n$ has a nonzero global section s. By proposition II.7.7(a), such a section gives an effective divisor D' on X linearly equivalent to nD. Now multiply both sides of $nD \sim D'$ by r: we get that $nrD \sim rD'$, but nrD is the trivial divisor, and so rD' is trivial. But this is impossible unless D' was already trivial: for any representation of D' as $\{(U_i, f_i)\}$, we have that rD' can be represented by $\{(U_i, f_i^r)\}$, and if f_i^r is a unit then so is f_i.

Exercise II.7.7. *Some Rational Surfaces.* Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$ where x, y, z are the homogeneous coordinates of X.

a. The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface (I, Ex. 2.13).

b. Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbb{P}^4$. The image is called the *Veronese surface* in $\mathbb{P}^4$. Cf. (IV, Ex. 3.11).

c. Let $\mathfrak{d} \subset |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X \setminus P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\widetilde{X}$, then this map extends to give a closed immersion of $\widetilde{X}$ in $\mathbb{P}^4$. Show that $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and the lines in X through P are transformed into straight lines in $\widetilde{X}$ which do not meet. $\widetilde{X}$ is the union of all of these lines, so we say $\widetilde{X}$ is a *ruled surface* (V, 2.19.1).

Solution.

a. We'll apply proposition II.7.2 to see that the map coming from these sections is a closed embedding. Clearly the nonvanishing set of each of these sections is affine: it's Spec $k[x,y,z]_{(s_i)})$ by the definition of Proj in II.2. It remains to check that $k[y_0, \cdots, y_5] \to \Gamma((\mathbb{P}^1_k)_{s_i}, \mathcal{O}_{(\mathbb{P}^1_k)_{s_i}})$ is surjective.

   By symmetry, it suffices to consider the cases $s_i = z^2$ and $s_i = yz$. In the first case, $\Gamma(X_i, \mathcal{O}_{X_i}) \cong k[x/z, y/z]$ and we see that $xz/z^2 = x/z$ and $yz/z^2 = y/z$ are in the image of $k[y_0, \cdots, y_5] \to \Gamma(X_i, \mathcal{O}_{X_i})$, so the map is surjective. In the second case, $\Gamma(X_i, \mathcal{O}_{X_i}) \cong k[x/z, y/z, (y/z)^{-1}]$ and we see that $xy/yz = x/z$, $y^2/yz = y/z$, and $z^2/yz = (y/z)^{-1}$ are in the image of $k[y_0, \cdots, y_5] \to \Gamma(X_i, \mathcal{O}_{X_i})$, so the map is surjective.

To see that this is indeed the Veronese from chapter 1, we can compute in affine charts. Unraveling the definitions on a k-rational point which we may assume is in $D(x)$, we see that $[1 : a : b]$ maps to $[1 : a^2 : b^2 : a : b : ab]$, which is exactly what the Veronese does.

b. This question needs the additional assumption that $\operatorname{char} k \neq 2$. We proceed as in part (a). The same reasoning shows that the nonvanishing set of each of the sections are affine, so we are to show surjectivity of the ring morphisms.

We again break things in to the case where $s_i = x^2,\ y^2,\ z^2$, or $s_i = y(x-z),\ (x-y)z$. In the first case, by symmetry, we need only consider $s_i = z^2$ with $\Gamma(X_i, \mathcal{O}_{X_i}) = k[x/z, y/z]$. Writing

$$\frac{(x-y)z - y(x-z)}{z^2} + \frac{x^2z^2 + y^2z^2 - ((x-y)z)^2}{2z^4} = \frac{xz - xy}{z^2} + \frac{2xyz^2}{2z^4} = \frac{x}{z}$$

and

$$\frac{-y(x-z)}{z^2} + \frac{x^2z^2 + y^2z^2 - ((x-y)z)^2}{2z^4} = \frac{yz - xy}{z^2} + \frac{2xyz^2}{2z^4} = \frac{y}{z},$$

we see that all of the generators of $\Gamma(X_i, \mathcal{O}_{X_i})$ can be written as polynomials in $\frac{s_j}{s_i}$. In the second case, we also need only consider $s_i = y(x-z)$. Here, $\Gamma(X_i, \mathcal{O}_{X_i}) = k[x, y, z]_{(y(x-z))}$, which is generated as a k-algebra by $\frac{p}{y(x-z)}$ for p among $\{x^2, y^2, z^2, xy, xz, yz\}$. Clearly the first three options are in the image of $k[s_j/s_i] \to \Gamma(X_i, \mathcal{O}_{X_i})$, and by cyclic symmetry, recognizing that $x(y-z) = y(x-z) - (x-y)z$, it suffices to show that $\frac{xy}{y(x-z)}$ is in the image as well. The expression

$$\frac{x^2y^2 - y^2z^2 + (y(x-z))^2}{2y^2(x-z)^2} = \frac{x^2y^2 - xy^2z}{y^2(x-z)^2} = \frac{xy}{y(x-z)}$$

verifies this, so we're done.

To show that the statement of this exercise is false in characteristic 2, I claim that taking $s_i = z^2$, there is no way to write $\frac{x}{z} \in k[x/z, y/z]$ as the image of some element of $\Gamma(X_i, \mathcal{O}_{X_i})$. Clearing denominators, we're looking to show that there is no integer k and polynomial p for which the equation

$$xz^{2k+1} = p(x^2, y^2, z^2, y(x-z), z(x-y))$$

holds. As we're in characteristic 2, we have that $(y(x-z))^2 = x^2y^2 - y^2z^2$, so we may assume that p is of degree at most one in $y(x-z)$ and $z(x-y)$, or that our equation is of the form

$$xz^{2k+1} = q_0(x^2, y^2, z^2) + y(x-z)q_1(x^2, y^2, z^2) + z(x-y)q_2(x^2, y^2, z^2).$$

We may assume each of the q_i are homogeneous of the appropriate degree. Now, writing everything out in the standard basis $x^\alpha y^\beta z^\gamma$ of the vector space of polynomials of degree $2k+2$, we see that the only terms which can be involved in this equation are $z^{2k}y(x-z)$ and $z^{2k}z(x-y)$, which immediately produces a contradiction by evaluating at $x = y = z = 1$.

(Geometrically, what's happening here is that we're projecting the Veronese in $\mathbb{P}^5$ to a copy of $\mathbb{P}^4$ from $p = [0 : 0 : 0 : 1 : 1 : 1]$, and such a projection restricts to an isomorphism on a variety $V \subset \mathbb{P}^5$ iff p on isn't on any secants or tangents to V. This point is on such a line when $\operatorname{char} k = 2$, but isn't when $\operatorname{char} k \neq 2$.)

c. We'll assume k is algebraically closed here (Hartshorne doesn't mention it, but it's almost certainly what's intended). Up to a change of coordinates, we may assume that $P = [0:0:1]$, which gives $\mathfrak{d}$ as the subspace spanned by $\{x^2, y^2, xy, xz, yz\}$ inside the polynomials of degree two in x, y, z. These generate $\mathcal{O}(2)$ at every point which isn't P, and generate the maximal ideal $\mathfrak{m}_P\mathcal{O}(2)$, so we have an immersion $U \to \mathbb{P}^4$ which extends to a morphism $\widetilde{X} \to \mathbb{P}^4$ after blowing up P by example II.7.17.3.

To check this map is a closed immersion, we note that in order to obtain $\widetilde{X}$ we're blowing up $\mathcal{I}_P(2)$, which admits a surjective map from $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ by $(s_1, s_2) \mapsto xs_1 + ys_2$. This gives a closed immersion $\widetilde{X} \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \cong \mathbb{P}^2 \times \mathbb{P}^1$, which admits a closed immersion to $\mathbb{P}^5$ via the Segre embedding. Tracing the maps of the global sections, we see that the global sections $x_0, x_1, x_2, x_3, x_4, x_5$ of $\mathcal{O}_{\mathbb{P}^5}(1)$ pull back to the global sections $(x,0), (y,0), (z,0), (0,x), (0,y), (0,z)$ of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2})}(1)$ which pull back to the global sections x^2, xy, xz, yx, y^2, yz of $\mathcal{I}_P(2) = \mathcal{O}_{\widetilde{X}}(1)$. So the closed immersion $\widetilde{X} \to \mathbb{P}^5$ factors as the composition of $\widetilde{X} \to \mathbb{P}^4$ with a linear inclusion $\mathbb{P}^4 \to \mathbb{P}^5$, so our original map $\widetilde{X} \to \mathbb{P}^4$ must have been a closed immersion as well.

To show that this embedding gives us a surface of degree 3, we may intersect our surface with two well-chosen hyperplanes and measure the degree of the resulting intersection. This is equivalent to taking the intersection of two well-chosen conics in X which meet at P and throwing away P as a point of intersection: it is easy to see that the result is $2 \cdot 2 - 1 = 3$, so our surface is of degree three. (For an explicit example of two conics to choose, take $x^2 - yz$ and $y^2 - xz$: these intersect at $[0:0:1]$, $[1:1:1]$, $[\omega:\omega^2:1]$, and $[\omega^2:\omega:1]$ where ω is a nontrivial cubic root of 1, and the fact that this intersection at $[0:0:1]$ is transverse means it's eliminated by the blowup.) To check that lines through P are sent to lines in $\mathbb{P}^4$, taking the intersection of the image of our line with a hyperplane in $\mathbb{P}^4$ gives the intersection of our line with a conic through P, not counting one intersection at P. This is a single point, so the image of our line has degree one which implies it's again a line.

Distinct lines in X through P have strict transforms which do not meet in the blowup by exercise II.7.12 (yes, that's after this one, but it doesn't use this exercise in its proof at all). Again, by the equivalence of taking intersections in $\mathbb{P}^4$ and on X throwing away a copy of P, we see that these lines are all disjoint, so $\widetilde{X}$ is in fact the union of all of them and thus a ruled surface.

Exercise II.7.8. Let X be a noetherian scheme, let $\mathcal{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between *sections* of π (i.e., morphism $\sigma : X \to \mathbb{P}(\mathcal{E})$ such that $\pi \circ \sigma = id_X$) and quotient invertible sheaves $\mathcal{E} \to \mathcal{L} \to 0$ of $\mathcal{E}$.

Solution. Apply proposition II.7.12 with $Y = X$ and $g = id_X$.

Exercise II.7.9. Let X be a regular noetherian scheme, and $\mathcal{E}$ a locally free coherent sheaf of rank ≥ 2 on X.

a. Show that $\operatorname{Pic} \mathbb{P}(\mathcal{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

b. If $\mathcal{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(E')$ (over X) if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Solution.

a. First, note that we should require that X is connected: else we can have something silly like $X = \operatorname{Spec} k \sqcup \operatorname{Spec} k$ with trivial Picard group and $X \times \mathbb{P}^1 \cong \mathbb{P}^1_k \sqcup \mathbb{P}^1_k$ with Picard group $\mathbb{Z} \oplus \mathbb{Z}$.

   Let π denote the map $\mathbb{P}(\mathcal{E}) \to X$. There's an obvious map $\operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathcal{E})$: $(\mathcal{L}, n) \mapsto \pi^*\mathcal{L} \otimes \mathcal{O}(n)$. We'll show it's injective and surjective.

   For injectivity, suppose $\pi^*\mathcal{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ for some $\mathcal{L}$ a line bundle on X and $n \in \mathbb{Z}$. Then by proposition II.7.11, we have that $\pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$, and so by the projection formula we have $\mathcal{L} \otimes \pi_*\mathcal{O}(n) \cong \mathcal{O}_X$, so $n = 0$ by proposition II.7.11 and $\mathcal{L} \cong \mathcal{O}_X$.

   For surjectivity, we first note that as X is regular and connected, it is irreducible, and as a regular scheme is reduced, X is integral. Now let $\{U_i\}$ be an open affine cover of X so that $\mathcal{E}$ is free of rank r on each U_i. Then each U_i is a noetherian integral regular affine scheme, and in particular satisfies condition (*) from section 6 of chapter II. Over each U_i, we have that $\mathbb{P}(\mathcal{E})|_{U_i} \cong U_i \times \mathbb{P}^n$, and by exercise II.6.1 plus corollary II.6.16, we see that

$$\operatorname{Pic}(U_i \times \mathbb{P}^n) \cong \operatorname{Cl}(U_i \times \mathbb{P}^n) \cong \operatorname{Cl}(U_i) \times \mathbb{Z} \cong \operatorname{Pic}(U_i) \times \mathbb{Z},$$

   where the isomorphism sends $\mathcal{O}(1)$ to $(0,1) \in \operatorname{Pic}(U_i) \times \mathbb{Z}$.

   Now suppose $\mathcal{L}$ is a line bundle on $\mathbb{P}(\mathcal{E})$ and pick two elements U_i, U_j of our cover. By the isomorphism of Picard groups, we see that the restriction of $\mathcal{L}$ to $\mathbb{P}(\mathcal{E})|_{U_i} \cong U_i \times \mathbb{P}^n$ is isomorphic to $\pi_i^*\mathcal{L}_i \otimes \mathcal{O}(n_i)$ and the restriction of $\mathcal{L}$ to $\mathbb{P}(\mathcal{E})|_{U_j} \cong U_j \times \mathbb{P}^n$ is isomorphic to $\pi_j^*\mathcal{L}_j \otimes \mathcal{O}(n_j)$. Since the restriction of each of these to $(U_i \cap U_j) \times \mathbb{P}^n$ is the same, we see from our proof of injectivity that $n_i = n_j$, and as all the U_i meet because X is irreducible, we have that all the n_i are the same. Now I claim that $\mathcal{L} \otimes \mathcal{O}(-n_i)$ is the pullback of a line bundle on X: applying the projection formula and proposition II.7.11 to $(\pi_i)_*(\pi_i^*\mathcal{L}_i \otimes \mathcal{O}(-n_i))$, we see that this is exactly $\mathcal{L}_i$, and by considering the restrictions of $\mathcal{L}$ to $(U_i \times \mathbb{P}^n) \cap (U_j \times \mathbb{P}^n)$ inside $\mathbb{P}(\mathcal{E})$ we obtain a gluing data for the $(\pi_i)_*(\pi_i^*\mathcal{L}_i \otimes \mathcal{O}(-n_i))$ which shows that $\pi_*(\mathcal{L} \otimes \mathcal{O}(-n_i))$ is a line bundle on X. So $\mathcal{L} \cong \pi^*\pi_*(\mathcal{L} \otimes \mathcal{O}(-n_i)) \otimes \mathcal{O}(n_i)$, showing surjectivity, and thus $\operatorname{Pic}(\mathbb{P}(\mathcal{E})) \cong \operatorname{Pic}(X) \times \mathbb{Z}$.

b. If there is such an $\mathcal{L}$, we have the desired result by lemma II.7.9.

   Conversely, suppose $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}')$ is an isomorphism over X with inverse g, and let the projection maps $\mathbb{P}(\mathcal{E}) \to X$ and $\mathbb{P}(\mathcal{E}') \to X$ be denoted by π and π', respectively. Consider $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$ and $g^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: by the results of part (a), these must be $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)$ and $\pi'^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(k')$ for $\mathcal{L}, \mathcal{L}'$ line bundles on X and k, k' integers. Now we investigate $g^*f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$: on the one hand, as $g \circ f = id$, we have that it is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$, while on the other hand, we have the following chain of isomorphisms:

$$\begin{aligned} g^*f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) &\cong g^*(\pi^*\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)) \\ &\cong g^*\pi^*\mathcal{L}\otimes g^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)) \\ &\cong \pi'^*\mathcal{L}\otimes(g^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{\otimes k} \\ &\cong \pi'^*\mathcal{L}\otimes(\pi'^*\mathcal{L}'\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(k'))^{\otimes k} \\ &\cong \pi'^*\mathcal{L}\otimes\pi'^*\mathcal{L}'^{\otimes k}\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(kk') \\ &\cong \pi'^*(\mathcal{L}\otimes\mathcal{L}'^{\otimes k})\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(kk') \end{aligned}$$

where we've used that pullback commutes with tensor product several times. By the injectivity in part (a), we see that $kk'=1$, so $k=k'=\pm 1$. By proposition II.7.11 and the fact that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$ have sections over $U\times_X\mathbb{P}(\mathcal{E})\cong U\times_X\mathbb{P}(\mathcal{E}')$ where $U\subset X$ is an open affine subset on which $\mathcal{E}$ and $\mathcal{E}'$ are free, we see we must have $k=k'=1$.

Now consider $\pi_*f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$. On one hand, as $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)\cong\pi^*\mathcal{L}'\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, we can apply the projection formula and proposition II.7.11 to see that $\pi_*f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)\cong\mathcal{L}'\otimes\mathcal{E}$. On the other hand, since f is an isomorphism over X, we have $\pi_*f^*=\pi'_*$, so again by proposition II.7.11 we get that $\pi_*f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)\cong\pi'_*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)\cong\mathcal{E}'$, so $\mathcal{E}'\cong\mathcal{L}'\otimes\mathcal{E}$ and we're done.

Exercise II.7.10. $\mathbb{P}^n$-*Bundles Over a Scheme.* Let X be a noetherian scheme.

a. By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a *projective n-space bundle* over X, as a scheme P with a morphism $\pi: P\to X$ such that P is locally isomorphic to $U\times\mathbb{P}^n$, $U\subset X$ open, and the transition automorphisms on $\operatorname{Spec} A\times\mathbb{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0,\cdots,x_n]$ (e.g., $x_i'=\sum a_{ij}x_j$, $a_{ij}\in A$).

b. If $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^n$-bundle over X.

c. (*) Assume that X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on X. [*Hint*: Let $U\subset X$ be an open set such that $\pi^{-1}(U)\cong U\times\mathbb{P}^n$, and let $\mathcal{L}_0$ be the invertible sheaf $\mathcal{O}(1)$ on $U\times\mathbb{P}^n$. Show that $\mathcal{L}_0$ extends to an invertible sheaf $\mathcal{L}$ on P. Then show that $\pi_*\mathcal{L}=\mathcal{E}$ is a locally free sheaf on X and that $P\cong\mathbb{P}(\mathcal{E})$.] Can you weaken the hypothesis 'X regular'?

d. Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbb{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathcal{E}$ of rank $n+1$ under the equivalence relation $\mathcal{E}'\cong\mathcal{E}$ if and only if $\mathcal{E}'\cong\mathcal{E}\otimes\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on X.

Solution.

a. Yup.

b. Cover X by affine open neighborhoods $U_i=\operatorname{Spec} A_i$ where $\mathcal{E}$ is free on each U_i. Then $\mathbb{P}(\mathcal{E})$ over U_i is given by $\operatorname{Proj} A_i[x_0,\cdots,x_n]\cong\mathbb{P}^n_{A_i}$, which satisfies the first condition of the definition. To check the second part, we note that if $\operatorname{Spec} A$ is an affine open lying in both U_i and U_j, we get an isomorphism $\mathcal{O}_{\operatorname{Spec} A}^{n+1} \cong (\mathcal{E}|_{U_i})|_{\operatorname{Spec} A} \cong (\mathcal{E}|_{U_j})|_{\operatorname{Spec} A} \cong \mathcal{O}_{\operatorname{Spec} A}^{n+1}$ which is given by an invertible $(n+1) \times (n+1)$ matrix with values in A. This induces the required linear automorphism of $A[x_0, \cdots, x_n]$, and these glue to form the required bundle.

c. Cover X by affine open subschemes $\{U_i = \operatorname{Spec} A_i\}_{0 \le i \le m}$ so that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^n$. Let $H_0 = V(x_0) \subset U_i \times \mathbb{P}^n$, and let $H = \overline{H_0} \subset P$. Define an invertible sheaf $\mathcal{L}$ on P by gluing the invertible sheaves $\mathcal{O}_{U_i \times \mathbb{P}^n}(H \cap (U_i \times \mathbb{P}^n))$ along their overlaps. Then $\mathcal{L}$ is an invertible sheaf extending $\mathcal{O}_{U_0 \times \mathbb{P}^n}(1)$. For each i, let $p_i : U_i \times \mathbb{P}^n \to U_i$ and $q_i : U_i \times \mathbb{P}^n \to \mathbb{P}^n$ denote the projection maps. By the argument from II.7.9(a), $\mathcal{L}|_{U_i \times \mathbb{P}^n} \cong p_i^* \mathcal{M}_i \otimes q_i^* \mathcal{O}_{\mathbb{P}^n}(n_i)$ for all i where $\mathcal{M}_i$ is an invertible sheaf on U_i and $n_i \in \mathbb{Z}$. From the same argument, we have that $n_i = n_j$ for all i, j, which implies $n_i = n_0 = 1$ for all i. By the projection formula and proposition II.7.11, $(\pi|_{U_i \times \mathbb{P}^n})_*(\mathcal{L}|_{U_i \times \mathbb{P}^n}) \cong ((p_i)_* p_i^* \mathcal{M}_i) \otimes ((p_i)_* q_i^* \mathcal{O}(1)) \cong \mathcal{M}_i^{\oplus n+1}$, which is locally free.

This implies that $\pi_* \mathcal{L}$ is locally free, and the map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is surjective because the map $p_i^* \mathcal{M}_i^{\oplus n+1} \to p_i^* \mathcal{M}_i \otimes q_i^* \mathcal{O}(1)$ is. Therefore by proposition II.7.12, we have a map $f : P \to \mathbb{P}(\pi_* \mathcal{L})$ over X coming from the surjective map of sheaves $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$. We can check that this is an isomorphism locally on X: picking an affine open set $U \subset X$ where $\mathcal{M}$ is free and $\pi^{-1}(U) \cong U \times \mathbb{P}^n$ is trivial, we see the map is just $\pi^* \mathcal{O}_U^{\oplus n+1} \to \mathcal{O}_{U \times \mathbb{P}^n}(1)$ given by sending the i^{th} coordinate to x_i, which clearly induces $\pi^{-1}(U) \cong \mathbb{P}_U^n$.

We cannot remove the hypothesis X regular, as the following example related by MO/MSE user Sasha shows (link). Let $X = (\mathbb{P}_k^1 \times \mathbb{P}_k^1)/(\mathbb{Z}/2)$, with $\mathbb{Z}/2$ acting digaonally and nontrivially on each component. Then X is a surface with four ordinary double points. Let $\pi : Y \to X$ be the blowup of one of these points, and $E \subset Y$ the exceptional curve. Then $E^2 = -2$ and

$$\operatorname{Ext}^1(\mathcal{O}_Y, \mathcal{O}_Y(E)) = H^1(Y, \mathcal{O}_Y(E)) \cong H^1(Y, \mathcal{O}_E(E)) \cong k.$$

Let F be the vector bundle on Y which is the nontrivial extension

$$0 \to \mathcal{O}_Y(E) \to F \to \mathcal{O}_Y \to 0.$$

Then by construction we have $F|_E \cong \mathcal{O}_E(-1)^{\oplus 2}$.

Consider the projective bundle $\mathbb{P}_Y(F)$, the surface $S = \mathbb{P}_E(F|_E) \subset \mathbb{P}_Y(F)$, and the composition

$$\mathbb{P}_Y(F) \to Y \to X.$$

This can also be factorized as

$$\mathbb{P}_Y(F) \to Z \to X,$$

where the first morphism contracts S and the second morphism is a $\mathbb{P}^1$-bundle, which is not isomorphic to the projectivization of a vector bundle. To see this, let $x_0 = \pi(E)$ and let $C \subset Z$ be the fiber of $Z \to X$ over x_0. If $Z \to X$ is a projectivization, then there is a line bundle L on Z so that $L|_C \cong \mathcal{O}(1)$. The pullback of L to $\mathbb{P}_Y(F)$ restricts to $S \cong E \times \mathbb{P}^1$ as $\mathcal{O}(0,1)$. On the other hand, $\mathcal{O}_{\mathbb{P}_Y(F)}(1)$ restricts to S as $\mathcal{O}(1,1)$. So there must be a line bundle on Y that restricts to E as $\mathcal{O}(1)$, but this is impossible because $\operatorname{Pic}(Y)$ is generated by $\pi^* \operatorname{Pic}(X)$, which restricts to E trivially and by $\mathcal{O}_Y(E)$, which restricts to E as $\mathcal{O}(-2)$.

d. By (c), every $\mathbb{P}^n$-bundle on X is of the form $\mathbb{P}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ on X. By exercise II.7.9, two such bundles $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic iff $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on X, which proves the claim.

Exercise II.7.11. On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

a. If $\mathcal{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathcal{I}^d$ for any $d \geq 1$ gives a scheme isomorphic to the blowing up of $\mathcal{I}$ (cf. Ex. 5.13).

b. If $\mathcal{I}$ is any coherent sheaf of ideals, and if $\mathcal{J}$ is an invertible sheaf of ideals, then $\mathcal{I}$ and $\mathcal{I} \cdot \mathcal{J}$ give isomorphic blowings-up.

c. If X is regular, show that (7.17) can be strengthened as follows. Let $U \subset X$ be the largest open set such that $f : f^{-1}U \to U$ is an isomorphism. Then $\mathcal{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X \setminus U$.

Solution.

a. Consider the map of quasi-coherent sheaves of algebras $\bigoplus_{n\geq 0} \mathcal{I}^{dn} \to \bigoplus_{n \geq 0} \mathcal{I}^n$ given by sending $\mathcal{I}^{dn} \mapsto \mathcal{I}^{dn}$. By exercise II.5.13, this induces a map $\operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathcal{I}^{dn}\right) \to \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathcal{I}^{n}\right)$ which is locally an isomorphism, therefore this map is globally an isomorphism.

b. This is a direct application of lemma II.7.9 once we show that the natural map $\mathcal{I}^d \otimes \mathcal{J}^{\otimes d} \to \mathcal{I}^d \cdot \mathcal{J}^d$ is an isomorphism. To prove this statement, we work locally. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset on which $\mathcal{J}|_U$ is free, that is, there is an isomorphism $\mathcal{O}_U \to \mathcal{J}|_U$. On the one hand, this gives us that $\mathcal{I}^d \otimes \mathcal{J}^{\otimes d} \cong \mathcal{I}^d$ over U by repeated applications of $\mathcal{O}_U \cong \mathcal{J}|_U$ and $\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{O}_U \cong \mathcal{F}$. On the other hand, taking sections over U, we see that the isomorphism $\mathcal{O}_U \to \mathcal{J}|_U$corresponds to an isomorphism $A \to \mathcal{J}(U)$ given by $1 \mapsto s \in \mathcal{J}(U)$. By repeated applications, we can check that $A \cong \mathcal{J}^d(U)$ by $1 \mapsto s^d$, and restriction of this to the map to $\mathcal{I}^d(U) \subset A$ gives that $\mathcal{I}^d \cong \mathcal{I}^d \cdot \mathcal{J}^d$ over U. Composing these isomorphisms, we see that they give the natural map described above and we have the desired result.

c. Write $f : Z \to X$ as the blowup of X along some coherent sheaf of ideals $\mathcal{I}$ by proposition II.7.17. Our goal will be to produce a sheaf of ideals $\mathcal{J}$ with $\mathcal{J}|_U = \mathcal{O}_U$ and an invertible sheaf of ideals $\mathcal{O}_X(D)$ so that $\mathcal{I} = \mathcal{J} \cdot \mathcal{O}_X(D)$: by part (b), this will show that $Bl_{\mathcal{I}}X \cong Bl_{\mathcal{J}}X$.

Assume U is nonempty and X is connected, thus X is irreducible and integral by regularity. Noting that $\pi : \pi^{-1}(U) \to U$ is the identity, we see that $(\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Z)|_{\pi^{-1}(U)} \cong \mathcal{I}|_U$ is an invertible sheaf by the universal property of blowing up. $\mathcal{I}|_U$ corresponds to an effective Cartier divisor, which by regularity corresponds to an effective Weil divisor $D' \subset U$, and the closure $D \subset X$ is again an effective Weil divisor on X, giving us an effective Cartier divisor on X with corresponding line bundle $\mathcal{O}_X(-D)$ having the property that $\mathcal{O}_X(-D)|_U = \mathcal{I}|_U$.

Now I claim that $\mathcal{I} \subset \mathcal{O}_X(-D)$. We'll check this on stalks - it's true more or less by definition for $x \in U$, so we focus our attention on $x \in X \setminus U$. Write $D = \sum n_i D_i$ for D_i prime divisors with generic points η_i. Let $f_{i,x}$ be the equation of D_i in $\mathcal{O}_{X,x}$ (this is 1 if $x \notin D_i$). As the localization map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\eta_i}$ is injective and the image of $\mathcal{I}_x$ lands inside $\mathcal{I}_{\eta_i}$, we have that

$$\mathcal{I}_x \subset \bigcap f_{i,x}^{n_i} \mathcal{O}_{X,\eta_i}$$

where the intersection is considered inside $\operatorname{Frac} \mathcal{O}_{X,x}$. Combining this with the relatively anodyne statement that $\mathcal{I}_x \subset \mathcal{O}_{X,x}$, we get that

$$\mathcal{I}_x \subset \left(\bigcap f_{i,x}^{n_i} \mathcal{O}_{X,\eta_i}\right) \cap \mathcal{O}_{X,x} = \bigcap f_{i,x}^{n_i} \mathcal{O}_{X,x} = \left(\prod f_{i,x}^{n_i}\right) \mathcal{O}_{X,x}$$

where the final equality comes from the fact that X is locally factorial and hence $\mathcal{O}_{X,x}$ is a UFD. This is exactly the stalk of $\mathcal{O}_X(D)$, so we've proven that $\mathcal{I} \subset \mathcal{O}_X(D)$.

Now we define $\mathcal{J}$: let $\{U_j = \operatorname{Spec} A_j\}_{j \in J}$ be an affine open covering of X with $\mathcal{O}_X(-D)$ the sheafification of the principal ideal $(s_j) \subset A_j$, and define $I_j = \mathcal{I}(U_j)$. Letting $J_j = \{a \in A_j \mid s_j a \in I_j\}$, we see that J_j is an ideal of A_j, finitely generated by noetherianness of X, and it is easy to see that they are compatible on overlaps so they glue to a coherent sheaf of ideals $\mathcal{J}$.

By construction, we see that $\mathcal{J} \cdot \mathcal{O}_X(-D) = \mathcal{I}$, so we have that $Bl_{\mathcal{I}} X \cong Bl_{\mathcal{J}} X$. To verify that $\mathcal{J}$ is supported on $X \setminus U$, recall that the support of a coherent sheaf of is closed (exercise II.5.6) and note that for any open subset $U' \subset X$ with $\mathcal{J}|_{U'} = \mathcal{O}_{U'}$ we have $U \times_X Bl_{\mathcal{J}} X \cong U$, so if $\mathcal{J}$ were supported on a proper closed subset of $X \setminus U$, we would have that the complement were a larger open set where π was an isomorphism, contradicting our assumption that U was the largest such subset of X.

Exercise II.7.12. Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\widetilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathcal{I}_Y + \mathcal{I}_Z$). Show that the strict transforms $\widetilde{Y}$ and $\widetilde{Z}$ of Y and Z in $\widetilde{X}$ do not meet.

Solution. Before we begin, we should note that the condition should be 'no irreducible component of Y containing an irreducible component of Z or vice-versa'.

The question is affine-local on X, so we may assume $X = \operatorname{Spec} A$ is affine and $\mathcal{I}_Y, \mathcal{I}_Z$ are the sheaves corresponding to $I_Y, I_Z \subset A$ with sum $I = I_Y + I_Z$. So $\widetilde{X} = \operatorname{Proj} \bigoplus_{d \geq 0} I^d$, with the closed immersions of $\widetilde{Y}$ and $\widetilde{Z}$ coming from Proj of the surjective ring homomorphisms $\bigoplus_{d\geq 0} I^d \to \bigoplus_{d \geq 0} I^d/(I^d \cap I_Y)$ and $\bigoplus_{d \geq 0} I^d \to \bigoplus_{d\geq 0} I^d/(I^d \cap I_Z)$ by corollary II.7.15. A point of intersection of $\widetilde{Y}$ and $\widetilde{Z}$ is a homogeneous prime ideal containing both $I^d \cap I_Y$ and $I^d \cap I_Z$ for all $d > 1$ but not $\bigoplus_{d \geq 1} I^d$. This is clearly impossible by the definition of I as $I_Y + I_Z$, so $\widetilde{Y} \cap \widetilde{Z} = \emptyset$.

Exercise II.7.13. (*) *A Complete Nonprojective Variety.* Let k be an algebraically closed field of char $\neq 2$. Let $C \subset \mathbb{P}^2_k$ be the nodal cubic curve $y^2 z = x^3 + x^2 z$. If $P_0 = (0, 0, 1)$ is the singular point, then $c \setminus P_0$ is isomorphic to the multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ (Ex. 6.7). For each $a \in k$, $a \neq 0$, consider the translation of $\mathbb{G}_m$ given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a.

Now consider $C \times (\mathbb{P}^1 \setminus \{0\})$ and $C \times (\mathbb{P}^1 \setminus \{\infty\})$. We glue their open subsets $C \times (\mathbb{P}^1 \setminus \{0, \infty\})$ by the isomorphism $\varphi : \langle P, u\rangle \mapsto \langle \varphi_u(P), u\rangle$ for $P \in C$, $u \in \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$. Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi : X \to \mathbb{P}^1$.

a. Show that π is a proper morphism, and hence that X is a complete variety over k.

b. Use the method of (Ex. 6.9) to show that $\operatorname{Pic}(C \times \mathbb{A}^1) \cong \mathbb{G}_m \times \mathbb{Z}$ and $\operatorname{Pic}(C \times (\mathbb{A}^1 \setminus \{0\})) \cong \mathbb{G}_m \times \mathbb{Z} \times \mathbb{Z}$. [*Hint*: If A is a domain and if * denotes the group of units, then $(A[u])^* \cong A^*$ and $(A[u, u^{-1}])^* \cong A^* \times \mathbb{Z}$.]

c. Now show that the restriction map $\operatorname{Pic}(C \times \mathbb{A}^1) \to \operatorname{Pic}(C \times (\mathbb{A}^1 \setminus \{0\}))$ is of the form $\langle t, n\rangle \mapsto \langle t, 0, n\rangle$ and the automorphism φ of $C \times (\mathbb{A}^1 \setminus \{0\})$ induces a map of the form $\langle t, d, n\rangle \mapsto \langle t, d + n, n\rangle$ on its Picard group.

d. Conclude that the image of the restriction map $\operatorname{Pic} X \to \operatorname{Pic}(C \times \{0\})$ consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

Solution.

a. $C \times (\mathbb{P}^1 \setminus \{0\}) \to (\mathbb{P}^1 \setminus \{0\})$ and $C \times (\mathbb{P}^1 \setminus \{\infty\}) \to (\mathbb{P}^1 \setminus \{\infty\})$ are both proper, as they're base changes of $C \to \operatorname{Spec} k$ along $(\mathbb{P}^1 \setminus \{0\}) \to \operatorname{Spec} k$ and $(\mathbb{P}^1 \setminus \{\infty\}) \to \operatorname{Spec} k$. Since properness is local on the base (corollary II.4.8(f)) and the sets $(\mathbb{P}^1 \setminus \{0\})$ and $(\mathbb{P}^1 \setminus \{\infty\})$ form an open cover of $\mathbb{P}^1_k$, we have the result.

b. To ease notation, let $V = \mathbb{A}^1_k$ or $\mathbb{A}^1_k \setminus \{0\}$, and let $Y = C \times V$. The normalization of Y is then $\widetilde{Y} = \mathbb{P}^1_k \times V \xrightarrow{\nu} Y$. Just as in exercise II.6.9, we get an exact sequence of sheaves

$$0 \to \nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y \to \mathcal{K}^*_Y / \mathcal{O}^*_Y \to \mathcal{K}^*_Y / \pi_* \mathcal{O}^*_{\widetilde{Y}} \to 0$$

on Y, which after taking global sections gives us the exact sequence

$$0 \to \Gamma(Y, \nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y) \to \operatorname{CaDiv}(Y) \to \operatorname{CaDiv}(\widetilde{Y}).$$

Our task now is to determine the first term.

Letting Y^{sm} denote the smooth locus of Y, we observe that $\nu : \nu^{-1}(Y^{sm}) \to Y^{sm}$ is an isomorphism, and so the sheaf $\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y$ is supported on $\{[0:0:1]\} \times V$, a copy of $\mathbb{A}^1_k \subset Y$. This means that for any open sets $U, U' \subset Y$, if $U \cap (\{[0:0:1]\} \times V) = U' \cap (\{[0:0:1]\} \times V)$, then $\Gamma(U, \nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y) = \Gamma(U', \nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y)$: sections over U are equivalent a section on $U \cap U'$ and a section on $U \cap (Y \setminus \{[0:0:1]\} \times V)$ agreeing on the overlap $U \cap U' \cap (Y \setminus \{[0:0:1]\} \times V)$, but the sections of the sheaf in question are zero over the latter two open sets. Combining this observation with the facts that global sections of $\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y$ are given by glueable sections of the quotient presheaf along some open cover, the open sets of V are all of the form $D(f)$ for some $f \in \mathcal{O}_V(V)$, and that Y is quasi-compact, we see that every global section of $\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y$ comes from a compatible family of sections on a collection of open sets of the form

$\{C' \times D(f_1), \cdots, C' \times D(f_n)\}$ where $C' = C \cap D(z)$ and $f_1, \cdots, f_n$ generate the unit ideal in the coordinate ring of V.

Let $R = k[t^2-1, t(t^2-1)]$ be the coordinate ring of C'. Then the coordinate ring of $C \times D(f)$ is $R[x]_f$ (or $R[x, x^{-1}]_f$), which has elements which look like $p(t^2-1, t(t^2-1))/f^n$ where p is a polynomial with coefficients in $k[x]$ (or $k[x, x^{-1}]$). Now suppose $p(t^2-1, t(t^2-1))/f^n$ is invertible, with inverse $q(t^2-1, t(t^2-1))/f^m$. Then working in $k[t, x]$ (or $k[t, x, x^{-1}]$), we see that $pq = f^{nm}$ which implies that neither p nor q can have any term of positive degree in t: if they did, f^{nm} must also have such a term, which is absurd. So the units of of $R[x]_f$ (resp. $R[x, x^{-1}]_f$) are exactly products of units of R and units of $k[x]_f$ (resp units of R and units of $k[x, x^{-1}]_f$). (Note that the same proof works when we replace R by $k[t]$, too.) The upshot is that we can pick representatives of our sections of $\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y$ on $C' \times D(f_i)$ which are all constant in the V direction, as any unit in $k[x]_f$ (resp. $k[x, x^{-1}]_f$) is a section of $\mathcal{O}^*_Y$. Therefore a global section of $\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y$ is honestly an element of $\nu_* \mathcal{O}^*_{\widetilde{Y}}(C' \times V) / \mathcal{O}^*_Y(C' \times V)$, or $k[t, x]^* / k[t^2-1, t(t^2-1), x]^*$ (resp. $k[t, x, x^{-1}]^* / k[t^2-1, t(t^2-1), x, x^{-1}]^*$).

By the hint, which can be proven via exactly the same argument involving units in the previous paragraph, we get that these quotients are precisely $k[t]^*/k[t^2-1, t(t^2-1)]^* \cong k^*$ when $V = \mathbb{A}^1_k$ and $(k[t]^*/k[t^2-1, t(t^2-1)]^*) \times \mathbb{Z} \cong k^* \times \mathbb{Z}$ when $V = \mathbb{A}^1_k \setminus \{0\}$ per the argument in exercise II.6.9(b), with $f = 1 + \frac{t-1}{2}(\frac{y}{x} + 1)$ representing $t \in k^*$ in the former and $f = 1 + \frac{tu^d - 1}{2}(\frac{y}{x} + 1)$ representing $(t, d) \in k^* \times \mathbb{Z}$. This gives us the two exact sequences

$$0 \to k^* \to \mathrm{CaDiv}(C \times \mathbb{A}^1_k) \to \mathrm{CaDiv}(\mathbb{P}^1_k \times \mathbb{A}^1_k)$$

when $V = \mathbb{A}^1_k$ and

$$0 \to k^* \times \mathbb{Z} \to \mathrm{CaDiv}(C \times (\mathbb{A}^1_k \setminus \{0\})) \to \mathrm{CaDiv}(\mathbb{P}^1_k \times (\mathbb{A}^1_k \setminus \{0\}))$$

when $V = \mathbb{A}^1_k \setminus \{0\}$. The same logic as exercise II.6.9 lets us upgrade CaDiv to CaCl in both cases. I claim that these maps between Cartier class groups are surjective: any Cartier divisor on $\widetilde{Y}$ can be moved via linear equivalence to be supported entirely on the locus where $\nu : \widetilde{Y} \to Y$ is an isomorphism, where it's clearly in the image of the pullback map. This gives us the following exact sequences:

$$0 \to k^* \to \mathrm{CaCl}(C \times \mathbb{A}^1_k) \to \mathrm{CaCl}(\mathbb{P}^1_k \times \mathbb{A}^1_k) \to 0,$$

$$0 \to k^* \times \mathbb{Z} \to \mathrm{CaCl}(C \times (\mathbb{A}^1_k \setminus \{0\})) \to \mathrm{CaCl}(\mathbb{P}^1_k \times (\mathbb{A}^1_k \setminus \{0\})) \to 0.$$

Further, CaCl is the same as Pic in all of these cases by proposition II.6.15. By a smorgasbord of results from section II.6 (proposition 5, 6, corollary 16, and exercise 1) we see that $\mathrm{Pic}(C \times V) = \mathbb{Z}$ in both cases, and thus these exact sequences are split by the well-known fact from homological algebra that any exact sequence $0 \to M \to N \to P \to 0$ in an abelian category with P projective splits. Therefore we have that $\mathrm{Pic}(C \times \mathbb{A}^1) \cong \mathbb{G}_m \times \mathbb{Z}$ and $\mathrm{Pic}(C \times (\mathbb{A}^1 \setminus \{0\})) \cong \mathbb{G}_m \times \mathbb{Z} \times \mathbb{Z}$.

c. Fixing $Y = C \times \mathbb{A}^1_k$ and $Y' = C \times (\mathbb{A}^1 \setminus \{0\})$, restriction from Y to Y' applied to the short exact sequence of sheaves

$$0 \to \nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y \to \mathcal{K}^*_Y / \mathcal{O}^*_Y \to \mathcal{K}^*_Y / \pi_* \mathcal{O}^*_{\widetilde{Y}} \to 0$$

gives a morphism of exact sequences of sections

$$\begin{tikzcd}
0 \arrow[r] & (\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y)(Y) \arrow[r] \arrow[d] & (\mathcal{K}^*_Y / \mathcal{O}^*_Y)(Y) \arrow[r] \arrow[d] & (\mathcal{K}^*_Y / \pi_* \mathcal{O}^*_{\widetilde{Y}})(Y) \arrow[d] \\
0 \arrow[r] & (\nu_* \mathcal{O}^*_{\widetilde{Y}} / \mathcal{O}^*_Y)(Y') \arrow[r] & (\mathcal{K}^*_Y / \mathcal{O}^*_Y)(Y') \arrow[r] & (\mathcal{K}^*_Y / \pi_* \mathcal{O}^*_{\widetilde{Y}})(Y')
\end{tikzcd}$$

which is compatible with the calculations we did in part (b) and thus gives a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & k^* \arrow[r] \arrow[d] & \operatorname{Pic}(C \times \mathbb{A}^1) \arrow[r] \arrow[d] & \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{A}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k^* \times \mathbb{Z} \arrow[r] & \operatorname{Pic}(C \times (\mathbb{A}^1 \setminus \{0\})) \arrow[r] & \operatorname{Pic}(\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})) \arrow[r] & 0
\end{tikzcd}$$

where the first map sends an element t of k^* to $(t, 0) \in k^* \times \mathbb{Z}$, and the final map sends $p^*\mathcal{O}_{\mathbb{P}^1}(d) \otimes q^*\mathcal{O}_{\mathbb{A}^1}$ to $p^*\mathcal{O}_{\mathbb{P}^1}(d) \otimes q^*\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$. This shows that the middle map is of the form $\langle t, n \rangle \mapsto \langle t, 0, n \rangle$ as the exact sequences are split.

We will now make a coordinate change on C so our computations relating to the automorphism φ are easier. Change coordinates by $x \mapsto (x - y)$ and $y \mapsto (x + y)$ so that C has an affine model by $\operatorname{Spec} k[x, y]/((x + y)^2 = (x - y)^3 + (x - y)^2)$: keeping the same geometric description of the normalization map, we see that the normalization map $\nu : \mathbb{P}^1 \to C$ is given by $\operatorname{Frac} k[x, y]/((x + y)^2 = (x - y)^3 + (x - y)^2) \to k(t)$ by $x \mapsto \frac{4t}{(t-1)^3}$, $y \mapsto \frac{4t^2}{(t-1)^3}$, and the isomorphism between $\mathbb{P}^1 \setminus \{0, \infty\}$ and $C \setminus P_0$ now respects the multiplication on both (see exercise II.6.7 for more details). The rational function $f_{a,d}$ on Y which witnesses the element $(a, d) \in k^* \times \mathbb{Z}$ is taken to $\frac{au^d x - y}{x - y}$ under this change of coordinates, and this pulls back to the rational function $\frac{t - au^d}{t - 1}$ on $\widetilde{Y} = \mathbb{P}^1_{k[u^{\pm 1}]}$, where t is a coordinate on $\mathbb{P}^1$. The automorphism φ_u pulls back to an automorphism of $\widetilde{Y}$ by sending $t \mapsto ut$ and preserving u. Applying the automorphism to $f_{a,d}$, we see it transforms in to $\frac{ut - au^d}{ut - 1}$, which also represents (a, d): setting $t = 0$, we get au^d, and setting $t = \infty$, we get 1. Thus φ acts as the identity on $k^* \times \mathbb{Z} \times \{0\} \subset k^* \times \mathbb{Z} \times \mathbb{Z} = \operatorname{Pic} Y$.

To determine what happens to the last factor of $\mathbb{Z}$ in $\operatorname{Pic} Y$, we can pick a representative for $(1, 0, 1)$, apply the automorphism, and see what happens: the Cartier divisor which is

$$\{(C' \times (\mathbb{A}^1 \setminus \{0\}), 1), ((C \setminus P_0) \times (\mathbb{A}^1 \setminus \{0\}), t - 1)\}$$

represents $(1,0,1) \in \operatorname{Pic} Y$. As the automorphism φ is given by the spectrum of the ring automorphism $k[t^{\pm 1}, u^{\pm 1}]$ by $t \mapsto tu$, $u \mapsto u$ on $(C \setminus P_0) \times (\mathbb{A}^1 \setminus \{0\}) \cong \operatorname{Spec} k[t^{\pm 1}, u^{\pm 1}]$, our Cartier divisor is taken to

$$\{(C' \times (\mathbb{A}^1 \setminus \{0\}), 1), (\varphi^{-1}((C \setminus P_0) \times (\mathbb{A}^1 \setminus \{0\})), ut-1)\}$$

by pullback along φ, which is linearly equivalent to

$$\{(C' \times (\mathbb{A}^1 \setminus \{0\}), \frac{t-1}{ut-1}), (\varphi^{-1}((C \setminus P_0) \times (\mathbb{A}^1 \setminus \{0\})), t-1)\}$$

after multiplication by $\frac{t-1}{ut-1}$. This divisor pulls back to $\mathcal{O}(1)$ on $\widetilde{Y}$, and the ratio of the evaluation at 0 and ∞ of the pullback of $\frac{t-1}{ut-1}$ to $\widetilde{Y}$ is $1/(1/u) = u$, so this Cartier divisor is $(1,1,1) \in \operatorname{Pic} Y$. This shows that φ acts on $\operatorname{Pic} Y$ by $(t,d,n) \mapsto (t, d+n, n)$.

d. A line bundle on X is a choice of line bundles on $C \times (\mathbb{P}^1 \setminus \{0\})$ and $C \times (\mathbb{P}^1 \setminus \{\infty\})$ agreeing on the overlap $C \times (\mathbb{P}^1 \setminus \{0, \infty\})$. What this means on Picard groups is that $\operatorname{Pic} X$ is the equalizer of $k^* \times \mathbb{Z} \rightrightarrows k^* \times \mathbb{Z} \times \mathbb{Z}$, where one map is $(t,n) \mapsto (t,0,n)$ and the other is $(t,n) \mapsto (t,n,n)$ by our computations in part (c). If $(t,0,n) = (t,n,n)$, then $n = 0$ and thus $\operatorname{Pic} X = k^*$, so the image of the restriction map $\operatorname{Pic} X \to \operatorname{Pic}(C \times \{0\})$ consists entirely of divisors of degree zero.

This implies that given any line bundle $\mathcal{L}$ on X and choice of global sections $s_0, \cdots, s_n$, the morphism to $\mathbb{P}^n_k$ must be constant on $C \times \{0\}$ since those global sections restrict to global sections of our line bundle restricted to $C \times \{0\}$. In particular, there cannot be any closed immersion from X to $\mathbb{P}^n_k$: such a morphism is injective on underlying topological spaces. Thus X is not projective over k and π is not a projective morphism.

Exercise II.7.14.

a. Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathcal{E}$, such that the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ is *not* very ample relative to X.

b. Let $f : X \to Y$ be a morphism of finite type, let $\mathcal{L}$ be an ample invertible sheaf on X, and let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let $P = \mathbf{Proj}\, \mathcal{S}$, let $\pi : P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n \gg 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n$ is very ample on P relative to Y. [*Hint*: Use (7.10) and (Ex. 5.12).]

Solution.

a. Let $X = \mathbb{P}^1_k$ and $\mathcal{E} = \mathcal{O}_X(-1)$. Then $\mathbb{P}(\mathcal{E}) = \mathbb{P}^1$ with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E} = \mathcal{O}_X(-1)$ by lemmas II.7.9 and II.7.11. If $\mathcal{E}$ were relatively very ample, we would have a closed immersion in to $\mathbb{P}^n_{\mathbb{P}^1_k} = \mathbb{P}^n \times \mathbb{P}^1_k$ for some n. Composing this with the Segre embedding $\mathbb{P}^n \times \mathbb{P}^1 \to \mathbb{P}^N$, we would have a closed immersion $i : \mathbb{P}^1_k \to \mathbb{P}^N_k$ with $i^* \mathcal{O}_{\mathbb{P}^N_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(-1)$. But this is absurd: $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ has no global sections, while $\mathcal{O}_{\mathbb{P}^N_k}(1)$ does.

b. This argument is quick once we prove that $\mathcal{L}$ ample implies some tensor power or $\mathcal{L}$ is very ample relative to f (this is a relative version of one direction of theorem II.7.6).

   We don't need to do much to adapt the proof of theorem II.7.6 to our situation: cover Y by affine opens $\operatorname{Spec} A_i$, and take $x \in X$ to be in $f^{-1}(\operatorname{Spec} A_i)$ for some i, and assume our open affine neighborhood U of x where $\mathcal{L}$ is free is entirely contained in one particular A_i. Proceeding as in the proof of theorem II.7.6, we end up with a finite list of sections s_j with each $X_j = X_{s_j} = \operatorname{Spec} B_j$ contained entirely inside $f^{-1}(\operatorname{Spec} A_i)$. Then the induced map $\operatorname{Spec} B_j \to \operatorname{Spec} A_i$ makes B_j in to a finitely generated A_i-algebra, say by b_{jk}. Then $s_j^n b_{jk}$ extends to a global section by lemma II.5.14, and we can again take one value of n to work for all j, k because we have finitely many sections. As the s_j^n generate $\mathcal{L}^n$ at every point, we can define a map $X \to \mathbb{P}^N_Y$ by the sections s_j^n and $s_j^n b_{jk}$. We can check that this is an immersion affine-locally, and indeed this is easy to verify on the open subscheme $\mathbb{P}^N_{A_i} \subset \mathbb{P}^N_Y$ by the unupgraded version of theorem II.7.6. It should be noted that this proof also now means that the conclusion of exercise II.7.5(e) holds in our situation.

   By proposition II.7.10, $\mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^l$ is very ample relative to π for some $l > 0$. Next, a direct application of exercise II.5.12(b) to the π-relatively very ample sheaf $\mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^l$ and the f-relatively ample sheaf $\mathcal{L}^m$ for any $m \geq m_0$ (via the upgraded version of exercise II.7.5(e)) shows that $\mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^l \otimes \pi^*\mathcal{L}^m \cong \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^{l+m}$ is very ample (all of the cautions regarding exercise II.5.12 are moot because the (†) condition assumes X noetherian, which implies P is also noetherian, being finite type over a noetherian scheme by part (a) of proposition II.7.10). Thus $\mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^n$ is very ample for all $n > l + m$.

II.8 Differentials

Differentials are great fun! This section could be a bit more explicit with the connection between differentials and smoothness (of arbitrary morphisms of schemes), but Hartshorne doesn't introduce us to the concept of a smooth morphism until chapter III section 10 - that feels perhaps a little late. Given how useful smoothness is, it feels like a little bit of a missed opportunity.

A brief reminder: when Hartshorne says 'variety', he means an integral separated scheme of finite type over an algebraically closed field.

Exercise II.8.1. Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

a. Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),
$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$
is exact on the left also. [*Hint*: In copying the proof of (8.7), first pass to $B/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $B/\mathfrak{m}^2$.]

b. Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \operatorname{trdeg} k(B)/k$.

c. Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{X,x}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

d. Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is a regular local ring}\}$ is an open dense subset of X.

Solution.

a. Our goal is to show that $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B)$ is injective. Equivalently, we want to show that the dual map
$$\delta^* : \operatorname{Hom}_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) \to \operatorname{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$
is surjective. The left term is isomorphic to $\operatorname{Hom}_B(\Omega_{B/k}, k(B)) \cong \operatorname{Der}_k(B, k(B))$ by the tensor-hom adjunction and the universal property of Kähler differentials. δ^* take a derivation $d : B \to k(B)$ and restricts it to $\mathfrak{m} \subset B$; this descends to a map $\mathfrak{m}/\mathfrak{m}^2 \to k(B)$ as $d(fg) = fdg + (df)g$, and if both $f, g \in \mathfrak{m}$, then they're sent to $0 \in k(B)$ under evaluation.

To show surjectivity of δ^*, let $h \in \operatorname{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$ be arbitrary. For any $b \in B$, write $b = c + \lambda$ uniquely with $\lambda \in k(B)$, $c \in \mathfrak{m}$ via the section $k(B) \to B \to k(B)$ from theorem II.8.25A. Declaring $d(b) = h(\bar{c})$ for $\bar{c}$ the residue class of $c \in \mathfrak{m}/\mathfrak{m}^2$, we see that d is a $k(B)$-linear derivation with $\delta^*(d) = h$, and we're done.

b. If $\Omega_{B/k}$ is free of rank $\dim B + \operatorname{trdeg} k(B)/k$, then by the exact sequence of vector spaces from part (a) we have that $\dim B + \operatorname{trdeg} k(B)/k = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \Omega_{k(B)/k}$. By theorem II.8.6A, $\dim \Omega_{k(B)/k} = \operatorname{trdeg} k(B)/k$, as k is perfect, so we deduce that $\dim B = \dim \mathfrak{m}/\mathfrak{m}^2$ and therefore B is a regular local ring.

   Conversely, suppose B is a regular local ring which is $A_\mathfrak{p}$ for some finitely-generated k-algebra A with prime ideal $\mathfrak{p} \subset A$. Let $K = \operatorname{Frac} B$, which is a separably generated extension of k by theorem I.4.8A. By proposition II.8.2A, we have $\Omega_{B/k} \otimes_B K \cong \Omega_{K/k}$, which is a k-vector space of dimension $\operatorname{trdeg} K/k$ by theorem II.8.6A. Thus $\dim_K \Omega_{B/k} \otimes_B K = \operatorname{trdeg} K/k = \dim A = \operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p}$ where the last two equalities are by theorem I.1.8A. As $\operatorname{ht} \mathfrak{p} = \dim B$ and $\dim A/\mathfrak{p} = \operatorname{trdeg} \operatorname{Frac}(A/\mathfrak{p})/k = \operatorname{trdeg} k(B)/k$, we have that

$$\dim_K \Omega_{B/k} \otimes_B K = \dim B + \operatorname{trdeg} k(B)/k.$$

   On the other hand, by the exact sequence of (a), we have that

$$\dim_{k(B)} \Omega_{B/k} \otimes_B k(B) = \dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 + \dim_{k(B)} \Omega_{k(B)/k} = \dim B + \operatorname{trdeg} k(B)/k.$$

   Therefore $\dim B + \operatorname{trdeg} k(B)/k = \dim_k \Omega_{B/k} \otimes_B k = \dim_K \Omega_{B/k} \otimes_B K$ and by lemma II.8.9, $\Omega_{B/k}$ is free of rank $\dim B + \operatorname{trdeg} k(B)/k$.

c. Let $x \in X$ with $\operatorname{Spec} A$ an affine open neighborhood of x. As X is of finite type over k, A is of finite type over k by exercise II.3.1, and $\mathcal{O}_{X,x} = A_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset A$ by quasi-coherence of $\mathcal{O}_X$. By part (b), we see that $\mathcal{O}_{X,x}$ is a regular local ring iff $\Omega_{\mathcal{O}_{X,x}/k}$ is free of rank $\dim \mathcal{O}_{X,x} + \operatorname{trdeg} k(x)/k$, which is exactly $\dim A$ by the argument in (b). Since $\dim A = \dim \operatorname{Spec} A = \dim X$ where the last equality follows by exercise II.3.20(e), we're done.

d. The same proof works verbatim, substituting a reference to part (c) instead of theorem II.8.15 in the final sentence of the proof.

Exercise II.8.2. Let X be a variety of dimension n over k. Let $\mathcal{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subset \Gamma(X, \mathcal{E})$ be a vector space of global sections which generate $\mathcal{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathcal{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathcal{E}$ giving rise to an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{E}' \to 0$$

where $\mathcal{E}'$ is also locally free. [*Hint*: Use a method similar to the proof of Bertini's theorem (8.18).]

Solution. We first reduce to the case where V is finite-dimensional. Let $r = \operatorname{rank} \mathcal{E}$ and let $\eta \in X$ be the generic point. There are some finite number of elements $\{v_i\}_{i \in I} \subset V$ whose image in $\mathcal{E}_\eta$ generate $\mathcal{E}_\eta$ as a $\mathcal{O}_{X,\eta}$-module. Let $\mathcal{E}'$ be the quotient of $\mathcal{E}$ by the image of the morphism $\mathcal{O}_X^{\oplus I} \to \mathcal{E}$ given by sending $e_i \mapsto v_i$. This is a quasi-coherent sheaf which is zero at the generic point by construction, so it's supported on some closed lower-dimensional set X', which is a finite union of irreducible components by X being noetherian. Now do the same thing for all the generic points of the irreducible components of X': we add a finite list of elements of our vector space V which generate the stalk of $\mathcal{E}$ at each such generic point to form a collection $\{v_i\}_{i\in I'}$ and let $\mathcal{E}''$ be the quotient of $\mathcal{E}$ by the image of the morphism $\mathcal{O}_X^{\oplus I'} \to \mathcal{E}$ given by sending $e_i \mapsto v_i$. Eventually, by induction, the process terminates with a finite number of elements in our collection $\{v_i\}$ whose images generate $\mathcal{E}_x$ for all $x \in X$. Therefore we may take V finite dimensional.

Now consider the map of locally free sheaves $V \otimes \mathcal{O}_X \to \mathcal{E}$ given by sending $v \mapsto v$. This induces a map of geometric vector bundles $X \times_k V \to \mathbb{V}(\mathcal{E})$ per the construction in exercise II.5.18. On an open affine subset $\operatorname{Spec} A \subset X$ where $\mathcal{E}$ is trivial, the restriction of this map is isomorphic to an A-linear morphism $\mathbb{A}_A^{\dim V} \to \mathbb{A}_A^r$ given in coordinates by some $r \times (\dim V)$ matrix $M = \{m_{ij}\}$ with entries in A. Taking coordinates $x_1, \cdots, x_{\dim V}$ on $\mathbb{A}_A^{\dim V}$, the locus $\{(x,s) \mid s|_x = 0\}$ is then cut out inside $\mathbb{A}_A^{\dim V}$ by the r linear equations corresponding to the n entries of the vector $M(x_1, \cdots, x_{\dim V})$, and this description is compatible with overlaps by the fact that the transition functions on $\mathbb{V}(\mathcal{E})$ are linear. So $Y = \{(x,s) \mid s|_x = 0\} \subset X \times_k V$ is a closed subscheme, and we see that its fiber over any closed point $x \in X$ is of dimension $\dim V - \operatorname{rank} \mathcal{E}$ by the rank nullity theorem and the fact that we chose V to generate $\mathcal{E}$.

By dimension counting, we see that $\dim Y \leq \dim X + \dim V - \operatorname{rank} \mathcal{E}$, which is strictly less than $\dim V$ by the assumption that $\operatorname{rank} \mathcal{E} > n = \dim X$. This implies that the projection $Y \hookrightarrow X \times_k V \to V$ cannot contain the generic point of V by looking at transcendence degrees: the transcendence degree over k of the residue field for a point $y \in Y$ is bounded above by $\dim Y$, so it cannot be an extension of the residue field of the generic point of V which has transcendence degree $\dim V > \dim Y$. By exercise II.3.19, we have that the image of Y is contained in a proper closed subset. As the k-rational points of $\mathbb{A}_k^n$ are dense, we have that there is a k-rational point in V not in the image of Y, which exactly gives us a section $v \in V$ which does not vanish at any point $x \in X$ (we only need k infinite for this, not the full power of algebraically closed). By exercise II.5.7(b), we see that the cokernel $\mathcal{E}'$ of $\mathcal{O}_X \to \mathcal{E}$ by $1 \mapsto v$ is again locally free, and we're done.

(One thing to note about Hartshorne's original proof of Bertini's theorem in II.8.18 is that he glosses over the proof of why B is closed. In that proof the correct justification is that the closed set in question is given by the simultaneous vanishing of the determinant of all the minors of the appropriate size of the projective Jacobian.)

Exercise II.8.3. *Product Schemes.*

a. Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X\times Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

b. If X and Y are nonsingular varieties over a field k, show that $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

c. Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

Solution.

a. Considering the fiber diagram, we obtain the exact sequences

$$p_1^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/X} \to 0$$

and

$$p_2^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y} \to 0$$

by applying proposition II.8.11 to each of the two ways to go around the square. By applying proposition II.8.10 we see that the third terms in the exact sequences are isomorphic to $p_2^*\Omega_{Y/S}$ and $p_1^*\Omega_{X/S}$, respectively, giving us

$$p_1^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to p_2^*\Omega_{Y/S} \to 0$$

and

$$p_2^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S} \to p_1^*\Omega_{X/S} \to 0.$$

Now I claim that the composition of the first map in the first sequence with the second map in the second sequence is the identity. It's enough to verify this affine-locally, so suppose $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} R$ are all affine. Our exact sequence becomes

$$\Omega_{A/R} \otimes_A (A \otimes_R B) \to \Omega_{A\otimes_R B/R} \to \Omega_{B/R} \otimes_B (A \otimes_R B) \to 0,$$

and simplifying the tensor products on the outside terms it becomes

$$\Omega_{A/R} \otimes_R B \to \Omega_{A\otimes_R B/R} \to \Omega_{B/R} \otimes_R A \to 0.$$

By construction of the maps involved in the exact sequence, we see that the first map sends $d(a)\otimes b$ to $(1\otimes b)d(a\otimes 1)$ while the second map sends $d(a\otimes b) = (1\otimes b)d(a\otimes 1)+(a\otimes 1)d(1\otimes b)$ to $ad(b)$ and all the claims are immediately verified.

By symmetry, this implies that both sequences are exact on the left and by the splitting lemma both split, so we have $\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

b. By part (a), we have $\Omega_{X\times Y} \cong p_1^*\Omega_{X/k} \oplus p_w^*\Omega_{Y/k}$. The definition of $\omega_{X\times Y}$ is $\bigwedge^{n+m} \Omega_{X\times Y}$, which by exercise II.5.16 parts (d) and (e) gives

$$\bigwedge^{n+m} \Omega_{X\times Y} \cong \bigwedge^{n} p_1^*\Omega_{X/k} \otimes \bigwedge^{m} p_2^*\Omega_{Y/k} \cong p_1^*\omega_{X/k} \otimes p_2^*\omega_{Y/k}.$$

c. Hartshorne is assuming that k is algebraically closed here - this is the only place in the problem we need this.

By example II.8.20.3, $\omega_Y \cong \mathcal{O}_Y$, so $\omega_{Y\times_Y} \cong \mathcal{O}_{Y\times Y}$ which has global sections k by exercise II.4.5(d), so $p_g(Y\times Y) = 1$. On the other hand, by the result of exercise I.7.2(e), we have that $p_a(Y\times Y) = p_a(Y)^2 - 2p_a(Y) = -1$.

Exercise II.8.4. *Complete Intersections in* $\mathbb{P}^n$. A closed subscheme Y of $\mathbb{P}^n_k$ is call a (*strict, global*) *complete intersection* if the homogeneous ideal I of Y in $S = k[x_0, \cdots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n)$ elements (I, Ex. 2.17).

a. Let Y be a closed subscheme of codimension r in $\mathbb{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \cdots, H_r$ such that $Y = H_1 \cap \cdots \cap H_r$ *as schemes*, i.e. $\mathcal{I}_Y = \mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}$. [*Hint*: Use the fact that the unmixedness theorem holds in S (Matsumura [2, p. 107]).]

b. If Y is a complete intersection of dimension ≥ 1 in $\mathbb{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14). [*Hint*: Apply (8.23) to the affine cone over Y.]

c. With the same hypotheses as (b), conclude that for all $l \geq 0$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to \Gamma(Y, \mathcal{O}_Y(l))$ is surjective. In particular, taking $l = 0$, show that Y is connected.

d. Now suppose given integers $d_1, \cdots, d_r \geq 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \cdots, H_r$ in $\mathbb{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \cdots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbb{P}^n$.

e. If Y is a nonsingular complete intersection as in (d), show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

f. If Y is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

g. If Y is a nonsingular curve in $\mathbb{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

Solution.

a. The first thing to do is to show that if a homogeneous ideal $I \subset k[x_0, \cdots, x_n]$ can be generated by r elements, it can be generated by r homogeneous elements (this is perhaps an unintentional omission by Hartshorne, but it's comforting to know it's true). Let $\{f_s\}_{s\in S}$ be a minimal system of homogeneous generators for I. Any relation $\sum g_s f_s = 0$ among the f_s must have $g_s \in (x_0, \cdots, x_n)$: if some g_{s_0} has a nonzero constant term, then looking at the $\deg f_{s_0}$ component of the equation, we see that f_{s_0} is in the ideal generated by the other f_s involved. Now define $I \to k^{\oplus S}$ by $\sum g_s f_s = (g_s(0))_{s\in S}$: this is surjective and does not depend on the representation of an element as $\sum g_s f_s$ by the previous sentence. This implies that $k^{\oplus S}$ can be generated by r elements, or $r \geq |S|$. Since every generating set of I must have at least $\operatorname{codim} Y$ elements, we see that I can indeed be generated by r homogeneous elements.

Now assume that Y is a complete intersection. Writing $I = (f_1, \cdots, f_r)$ for f_i homogeneous and $H_i = V(f_i)$, we have that $\mathcal{I}_Y = \mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}$ which shows that $Y = H_1 \cap \cdots \cap H_r$ as schemes.

Conversely, suppose that Y is the intersection of r hypersurfaces. The first thing to check is that the homogeneous ideal $I_H \subset k[x_0, \cdots, x_n]$ corresponding to a hypersurface H is principal. Since $\mathcal{I}_H$ is locally principal, it's of the form $\mathcal{O}_{\mathbb{P}^n}(-d)$ for some $d \geq 0$. As $H \neq \emptyset$, $1 \notin \mathcal{I}_H(\mathbb{P}^n)$, so $d \geq 1$, $\mathcal{I}_H(\mathbb{P}^n) = 0$, and $I_H = \Gamma_*(\mathcal{I}_H) \cong k[x_0, \cdots, x_n](-d)$ by the results of section II.5. This means that I_H is principal, generated by the element corresponding to $1 \in k[x_0, \cdots, x_n](-d)$.

Writing f_i for the generator of I_{H_i}, we have that I_Y is the saturation of $(f_1, \cdots, f_r)$ by exercise II.5.10. As $I = (f_1, \cdots, f_r)$ has height r, the unmixedness theorem applied to $k[x_0, \cdots, x_n]$ gives that I is unmixed, or equivalently, $k[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$ has no embedded primes. Thus $(x_0, \cdots, x_n) \subset k[x_0, \cdots, x_n]$ can't be an associated prime of $(f_1, \cdots, f_r)$, so if $x_i y \in I$ for all i we must have $y \in I$. But this implies that I is saturated: supposing y is in the saturation of I, we may examine monomials of the form $(\prod x_i^{n_i})y$ and find that there must be one so that multiplication by any x_i lands in I - so our monomial was actually in I, and repeating we find that $y \in I$. Thus $I = I_Y$.

b. By (a), $I_Y = I_{C(Y)} \subset k[\mathbb{A}^{n+1}]$ is generated by r elements and thus $C(Y)$ is also a complete intersection. As Y is a complete intersection and normal, it is regular in codimension one by proposition II.8.23(b). As the singular set of $C(Y)$ is the cone on the singular set of Y, we have that it is a closed set of codimension at least two, so another application of proposition II.8.23(b) gives that $C(Y)$ is normal, or that $k[x_0, \cdots, x_n]/I_Y$ is integrally closed, which is exactly the condition that Y is projectively normal.

c. This is a verbatim application of exercise II.5.14(d).

d. We add the hypothesis that k is algebraically closed in order to apply Bertini's theorem. This can be done by induction: suppose we've obtained an irreducible, nonsingular Y which is the intersection of nonsingular hypersurfaces $H_1, \cdots, H_{s-1}$ for some $0 \leq s \leq r$ (we take the convention that $\mathbb{P}^n_k$ is the empty intersection). Taking the d_s-uple embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$, we see that taking the intersection of $Y \subset \mathbb{P}^N$ with a hyperplane $H \subset \mathbb{P}^N$ is equivalent to taking the intersection of $Y \subset \mathbb{P}^n$ with a hypersurface H_{d_s} of degree d_s in $\mathbb{P}^n$. By Bertini's theorem we can pick an H so that both $Y \cap H = Y \cap H_{d_s}$ and $H_{d_s} = \mathbb{P}^n \cap H \subset \mathbb{P}^N$ are nonsingular and the former is of dimension $\dim Y - 1$; as both $Y \cap H_{d_s}$ and H_{d_s} are connected (part (c) and the fact that every hypersurface in $\mathbb{P}^n$ for $n > 1$ is connected, respectively) we have that they must be irreducible.

e. We proceed inductively as in (d). Note that $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ by example II.8.20.1, so the base case is proven. Now suppose we know the result for $Y = H_1 \cap \cdots \cap H_{s-1}$ with $s \leq r$. Then $Y \cap H_s$ is a nonsingular subvariety of codimension one of the nonsingular variety Y, so $\omega_{Y \cap H_s} \cong \omega_Y \otimes \mathcal{I}_{H_s}^\vee \otimes \mathcal{O}_{Y \cap H}$ by proposition II.8.20. But $\mathcal{I}_{H_s}$ is the restriction of $\mathcal{O}(-d_s)$ (H_s is irreducible and does not contain Y), so it's dual is $\mathcal{O}(d_s)$, giving $\omega_{Y \cap H_s} \cong \omega_Y(d_s) \otimes \mathcal{O}_{Y \cap H_s} \cong \mathcal{O}_{Y \cap H}(\sum d_i - n - 1)$.

f. Twisting the standard exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$ by $(-n-1+d)$ and taking global sections gives the exact sequence

$$0 \to \Gamma(\mathbb{P}^n, \mathcal{I}_Y(-n-1+d)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1+d)) \to \Gamma(Y, \mathcal{O}_Y(-n-1+d))$$

which is exact on the right by part (c). As $\mathcal{I}_Y \cong \mathcal{O}(-d)$, we get the exact sequence

$$0 \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1+d)) \to \Gamma(Y, \mathcal{O}_Y(-n-1+d)) \to 0.$$

The dimensions of the first two spaces are 0 and $\binom{d-1}{n}$ by proposition II.5.13 and the fact that the dimension of the q-graded piece of $k[x_0, \cdots, x_n]$ is $\binom{q+n}{n}$, while the dimension of the last space is exactly $p_g(Y)$ by part (e). So $p_g(Y) = \binom{d-1}{n}$ and the rest of the conclusions follow.

g. The same procedure as in (f) gives us the exact sequence

$$0 \to \Gamma(\mathbb{P}^3, \mathcal{I}_Y(d+e-4)) \to \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) \to \Gamma(Y, \mathcal{O}_Y(d+e-4)) \to 0,$$

implying $p_g(Y) = \dim \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) - \dim \Gamma(\mathbb{P}^3, \mathcal{I}_Y(d+e-4))$. The dimension of the middle term is $\binom{d+e-4+3}{3}$, while the dimension of the first term is $\binom{e-4+3}{3} + \binom{d-4+3}{4}$ as the homogeneous ideal I of Y is generated two coprime irreducible polynomials f, g of degree d and e which means that the degree $d+e-4$ polynomials are of the form $af + bg$ with $\deg a = e-4$ and $\deg b = d-4$. After some light algebra, we get the result:

$$\binom{d+e-1}{3} - \binom{e-1}{3} - \binom{d-1}{3} = \frac{1}{2}de(d+e-4) + 1.$$

Exercise II.8.5. *Blowing up a Nonsingular Subvariety.* As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \widetilde{X} \to X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

a. Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$, and $\mathbb{Z} \to \operatorname{Pic} X$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

b. Show that $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathcal{L}((r-1)Y')$. [*Hint*: By (a) we can write in any case $\omega_{\widetilde{X}} \cong f^*\mathcal{M} \otimes \mathcal{L}(qY')$ for some invertible sheaf $\mathcal{M}$ on X, and some integer q. By restricting to $\widetilde{X} \setminus Y' \cong X \setminus Y$, show that $\mathcal{M} \cong \omega_X$. To determine q, proceed as follows. First show that $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y. Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbb{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r-1$.]

Solution.

a. By proposition II.6.5, we have an exact sequence

$$\mathbb{Z} \to \operatorname{Cl} \widetilde{X} \to \operatorname{Cl}(\widetilde{X} \setminus Y') \to 0.$$

Since $\widetilde{X} \setminus Y' \cong X \setminus Y$ and Y is of codimension at least two, we have that the third term is isomorphic to $\operatorname{Cl} X$. Pullback along π gives a section $\operatorname{Cl} X \to \operatorname{Cl} \widetilde{X}$, so it remains to show that $\mathbb{Z} \to \operatorname{Cl} \widetilde{X}$ is injective, and then we'll have the desired splitting.

If $nY' \sim 0$, then there's some rational function f which vanishes exactly on $Y' \subset \widetilde{X}$, which is equivalent to f vanishing exactly on $Y \subset X$ and having no poles. This is absurd: rational functions on smooth varieties vanish along sets of codimension one, and Y has codimension two. (Alternately, consult the proof given in exercise II.6.1: the rational functions f and $1/f$ are nonvanishing at every codimension one point, which means f restricts to a unit on every affine open as smooth varieties are normal because regular local rings are UFDs.)

b. Following the hint, we can write $\omega_{\widetilde{X}} \cong \pi^*\mathcal{M} \otimes \mathcal{L}(qY')$ for some $\mathcal{M}$ a line bundle on X and $q \in \mathbb{Z}$. Since $(\pi^*\mathcal{M})|_{\widetilde{X}\setminus Y'} \cong \mathcal{M}|_{X\setminus Y}$ and $\operatorname{Pic} X \setminus Y \cong \operatorname{Pic} X$ (corollary II.6.16 plus the corresponding statement on class groups from before), we see that $\mathcal{M}$ is determined by its behavior on $\widetilde{X} \setminus Y' \cong X \setminus Y$, and therefore $\mathcal{M} \cong \omega_X$.

Continuing with the hint, we have that $\omega_{Y'} \cong \omega_{\widetilde{X}} \otimes \mathcal{L}(Y') \otimes \mathcal{O}_{Y'}$ by proposition II.8.20. By our assumption that $\omega_{\widetilde{X}} \cong \pi^*\omega_X \otimes \mathcal{L}(qY')$, we see that $\omega_{Y'} \cong \pi^*\omega_X \otimes \mathcal{L}((q+1)Y') \otimes \mathcal{O}_{Y'}$. As $\mathcal{L}(Y') \cong \mathcal{I}_{Y'}^\vee$ by proposition II.6.18 and $\mathcal{I}_{Y'} \cong \mathcal{O}_{\widetilde{X}}(1)$ by the proof of proposition II.7.13, we obtain $\omega_{Y'} \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(-q-1) \otimes \mathcal{O}_{Y'}$.

Taking a closed point $y \in Y$ and letting $Z = \{y\} \times_Y Y' \cong \mathbb{P}^{r-1}$, we see that $\omega_Z \cong \omega_{Y'} \otimes \bigwedge^r \mathcal{N}_{Z/Y'}$ by proposition II.8.20. This gives us that

$$\omega_Z \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(-q-1) \otimes \mathcal{O}_{Y'} \otimes \bigwedge^r \mathcal{N}_{Z/Y'}.$$

Since $\mathcal{N}_{Z/Y'} \cong \pi^*\mathcal{N}_{\{y\}/Y}$ and exterior powers commute with pullback, our expression simplifies to

$$\omega_Z \cong \pi^*(\omega_X \otimes \bigwedge^r \mathcal{N}_{\{y\}/Y}) \otimes \mathcal{O}_{\widetilde{X}}(-q-1),$$

which by another application of proposition II.8.20 is just

$$\omega_Z \cong \pi^*\omega_{\{y\}} \otimes \mathcal{O}_{\widetilde{X}}(-q-1).$$

As $\omega_{\{y\}} \cong \mathcal{O}_{\{k\}}$ and $\pi^*\omega_{\{k\}} \cong \mathcal{O}_Z$, we see that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. On the other hand, $Z \cong \mathbb{P}^{r-1}$ so $\omega_Z \cong \mathcal{O}_Z(-r)$ and therefore $q = r-1$.

Exercise II.8.6. *The Infinitesimal Lifting Property.* The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism making a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
 & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[ur, dashed, "g"] & B \arrow[d] \\
 & 0
\end{tikzcd}$$

We call this result the *infinitesimal lifting property* for A. We prove this result in several steps.

a. First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about $\operatorname{Spec} A$ being nonsingular.)

b. Now let $P = k[x_1, \cdots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ J & & I \\ \downarrow & & \downarrow \\ P & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

and show that h induces an A-linear map $\overline{h} : J/J^2 \to I$.

c. Now use the hypothesis $\operatorname{Spec} A$ nonsingular and (8.17) to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0.$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

Solution.

a. If g and g' are two k-algebra homomorphisms lifting f, then $g(c) = g'(c)$ for $c \in k$ by the definition of a k-algebra homomorphism. To verify the Leibniz rule for θ, we first note that any two liftings of f give the same A-module structure on I: $g(a) \cdot i = (g'(a) + i') \cdot i = g'(a) \cdot i + i'i = g'(a)$ because $g(a)$ and $g'(a)$ differ by $i' \in I$ and $I^2 = 0$. Then

$$\begin{aligned}\theta(ab) &= g(a)g(b) - g'(a)g'(b) \\ &= g(a)g(b) - g(a)g'(b) + g(a)g'(b) - g'(a)g'(b) \\ &= g(a)\theta(b) + \theta(a)g'(b),\end{aligned}$$

so θ satisfies the Leibniz rule and is a k-linear derivation.

Conversely, if $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, then $g+\theta$ certainly is a set-theoretic lift of f since θ lands in I. To check it's a homomorphism of k-algebras, we note that g is k-linear and θ vanishes on $k \subset A$; that θ and g distribute over addition; and as $\theta(a)\theta(b) = 0$ (since $I^2 = 0$), $g+\theta$ is multiplicative:

$$\begin{aligned}g(ab) + \theta(ab) &= g(a)g(b) + g(a)\theta(b) + \theta(b)g(a) \\ &= g(a)g(b) + g(a)\theta(b) + \theta(b)g(a) + \theta(a)\theta(b) \\ &= (g(a) + \theta(a))(g(b) + \theta(b)).\end{aligned}$$

b. Let $q : P \to A$ be the quotient map. Then by the universal property of polynomial rings, we can define a map $h : P \to B'$ lifting f by sending x_i to any element of B' which maps to $f(q(x_i))$, and this is exactly what we need to make the diagram commute.

For the subsequent assertion about a homomorphism $\overline{h}$, J/J^2 is an A-module for the same reason I is an A-module: if p, p' are two elements of P mapping to $a \in A$, their difference is in J, so $pj - p'j \in J^2$ for $j \in J$ and therefore we can define an A-module structure on J/J^2 where a acts as any lift of a to P. To define $\overline{h}$, note that $h(J) \subset I$ by commutativity of the diagram, so $h(J^2) = 0$ in B'. By the first isomorphism theorem, this descends to a map of k-modules $J/J^2 \to I$, and by the description of the A-action on both J/J^2 and I we can upgrade this to an A-morphism $\overline{h} : J/J^2 \to I$.

c. By the hypothesis that $\operatorname{Spec} A$ is nonsingular and theorem II.8.17, we have an exact sequence of sheaves $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathbb{A}^n_k/k} \otimes \mathcal{O}_{\operatorname{Spec} A} \to \Omega_{\operatorname{Spec} A/k} \to 0$. Taking global sections and noting that $\Omega_{P/k} \otimes A \to \Omega_{A/k}$ is surjective by proposition II.8.3A combined with the observation that $R \to S$ surjective implies $\Omega_{S/R} = 0$, we have the desired exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Applying $\mathrm{Hom}_A(-, I)$ and using the tensor-hom adjunction to conclude $\mathrm{Hom}_A(\Omega_{P/k} \otimes_P A, I) \cong \mathrm{Hom}_P(\Omega_{P/k}, I)$, we obtain the long exact sequence

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to \mathrm{Ext}^1_A(\Omega_{A/k}, I) \to \cdots.$$

As $\operatorname{Spec} A$ is nonsingular over k, $\Omega_{\operatorname{Spec} A/k}$ is locally free by theorem II.8.17, and by the well-known equivalence of finite locally free modules with finite projective modules (see Bourbaki's *Commutative Algebra*, chapter II, 5.2; Serre's *Modules projectifs et espaces fibres a fibre vectorielle*; Stacks 00NV or many others), we see that $\Omega_{A/k}$ is projective, which implies $\mathrm{Ext}^1_A(\Omega_{A/k}, -) = 0$, giving us our desired exact sequence.

Finally, let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$ by the work we did in part (b). θ gives a k-derivation $P \to B'$ by the composition

$$P \xrightarrow{d} \Omega_{P/k} \xrightarrow{\theta} I \to B'$$

to get a k-derivation $P \to B'$. By the construction of θ, we see that $\theta(b) = h(b)$ for any $b \in J$, so $h' = h - \theta$ descends to a homomorphism $A \to B'$ lifting f.

Exercise II.8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathcal{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathcal{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. Such a pair $X', \mathcal{I}$ we call an *infinitesimal extension* of the scheme X by the sheaf $\mathcal{F}$. One such extension, the *trivial* one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathcal{F}$.

The general problem of classifying extensions of X by $\mathcal{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathcal{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

Solution. We'll tackle the case where X and X' are both affine first. Let $X \cong \operatorname{Spec} A$ be a nonsingular affine variety over an algebraically closed field k, with $\mathcal{F} \cong \widetilde{M}$ a coherent sheaf on X, and suppose $X' \cong \operatorname{Spec} A'$ for an A-algebra A' with ideal $I \subset A'$ satisfying $I^2 = 0$, $A'/I \cong A$, and $I \cong M$ as A-modules. By the infinitesimal lifting property, we have a ring homomorphism $A \to A'$ lifting the isomorphism $A'/I \cong A$. Via the splitting lemma, this gives a direct sum decomposition $A' \cong A \oplus I \cong A \oplus M$ as k-vector spaces, and since our map $A \to A'$ is a ring homomorphism it's immediate to verify that this can be upgraded to an isomorphism of A-modules by $a(0, m) = (0, am)$. Therefore $A' \cong A \oplus I$ as rings with the multiplication on the RHS given by $(a, m)(a', m') = (aa', am' + a'm)$.

To handle the general case, it suffices to show that if $X \cong (X')_{red}$ is affine, then X' must also be affine. This is exercise III.3.1, which doesn't rely on the material we're using here and it's much nicer to follow the exposition rather than try and reinvent that particular wheel without saying the word 'cohomology'.

Exercise II.8.8. Let X be a projective nonsingular variety over k. For any $n > 0$ we define the nth *plurigenus* of X to be $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$. Thus in particular $P_1 = p_g$. Also, for any q, $0 \leq q \leq \dim X$ we define an integer $h^{q,0} = \dim_k \Gamma(X, \Omega^q_{X/k})$ where $\Omega^q_{X/k} = \bigwedge^q \Omega_{X/k}$ is the sheaf of regular q-forms on X. In particular, for $q = \dim X$, we recover the geometric genus again. The integers $h^{q,0}$ are called *Hodge numbers*.

Using the method of (8.19), show that P_n and $h^{q,0}$ are *birational* invariants of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

Solution. Here is the necessary alteration: in the third sentence, $f^*\omega_X \to \omega_Y$ induces $f^*\omega_X^{\otimes n} \to \omega_Y^{\otimes n}$ inductively by composing the maps $f^*\omega_X^{\otimes(n-1)} \otimes f^*\omega_X \to \omega_Y^{\otimes(n-1)} \otimes f^*\omega_X$ and $\omega_Y^{\otimes(n-1)} \otimes f^*\omega_X \to$

$\omega_Y^{\otimes(n-1)} \otimes \omega_Y$ given by tensoring $f^*\omega_X^{\otimes(n-1)} \to \omega_Y^{\otimes(n-1)}$ with $f^*\omega_X$ and $f^*\omega_X \to \omega_Y$ with $\omega_Y^{\otimes n-1}$, respectively. As the tensor powers of ω_X and ω_Y remain line bundles, the rest of the proof goes through with no changes.

II.9 Formal Schemes

This is a weird section. Nothing serious happens, we don't see much about how to actually work with formal schemes, and it's not clear why we should care. This is widely agreed to be the most skippable section of this book.

If you're actually interested in formal schemes, you will find a much more substantial discussion in EGA: see EGA I section 10 for a more substantial introduction, and then EGA III for the big fun. And the theorem on formal functions is indeed quite wonderful.

Exercise II.9.1. Let X be a noetherian scheme, Y a closed subscheme, and $\hat{X}$ the completion of X along Y. We call the ring $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the ring of *formal-regular functions* on X along Y. In this excercise we show that if Y is a connected, nonsingular, positive-dimensional subvariety of $X = \mathbb{P}^n_k$ over an algebraically closed field k, then $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

a. Let $\mathcal{I}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on Y, $\mathcal{I}/\mathcal{I}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.

b. Show that for any $r \geq 1$, $\Gamma(Y, \mathcal{I}^r/\mathcal{I}^{r+1}) = 0$.

c. Use the exact sequences

$$0 \to \mathcal{I}^r/\mathcal{I}^{r+1} \to \mathcal{O}_X/\mathcal{I}^{r+1} \to \mathcal{O}_X/\mathcal{I}^r \to 0$$

and induction on r to show that $\Gamma(Y, \mathcal{O}_X/\mathcal{I}^r) = k$ for all $r \geq 1$. (Use 8.21Ae.)

d. Conclude that $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$. (Actually, the same result holds without the hypothesis Y nonsingular, but the proof is more difficult - see Hartshorne [3, (7.3)].)

Solution.

a. Theorem II.8.17 gives an injection $\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^n_k/k} \otimes \mathcal{O}_Y$, while theorem II.8.13 gives an injection $\Omega_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}$ as part of an exact sequence

$$0 \to \Omega_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0.$$

To check that $\Omega_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}$ stays an injection after tensoring with $\mathcal{O}_Y$, it suffices to show that after passing to stalks and tensoring with $\mathcal{O}_{Y,x}$, the resulting sequence is exact. But $\mathcal{O}_{X,x}$ is flat over itself, so $\operatorname{Tor}_1^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ and thus $\Omega_{\mathbb{P}^n_k/k} \otimes \mathcal{O}_Y \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \otimes \mathcal{O}_Y$ is an injection because it is on stalks. (Really, what we're showing is that the sheaf Tor is zero, but we don't exactly have the technology to define that yet - wait until chapter III.) Composing the two maps, we have the result.

b. As Y is nonsingular, $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf on Y by theorem II.8.17. Applying Sym^n to the inclusion $\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_Y(-1)$, we get a morphism $\operatorname{Sym}^n(\mathcal{I}/\mathcal{I}^2) \to \operatorname{Sym}^n(\mathcal{O}_Y(-1)^{r+1})$. First, by an application of theorem II.8.21A(e), we see that $\operatorname{Sym}^n(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{I}^n/\mathcal{I}^{n+1}$. Next, we observe that $\mathrm{Sym}^n(\mathcal{O}_Y(-1)^{r+1}) \cong \mathcal{O}_Y(-n)^N$, where N is some large number we don't care about. So we have a morphism $\mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_Y(-n)^N$: if we can show this is injective, we're done, since the latter has no global sections.

To show injectivity, we exploit exercise II.5.16(e): applying that statement to the morphism of ringed spaces $(\{*\}, \mathcal{O}_{X,x}) \to X$, we see that Sym^n commutes with passage to stalks. As the maps found in part (a) all locally split (on any affine open, a finite locally free sheaf is a projective module as mentioned in the solution to II.8.6), we see that the map on stalks $(\mathcal{I}^n/\mathcal{I}^{n+1})_y \to \mathcal{O}_{Y,y}(-n)^N$ exhibits the source as a direct summand of the target. Since direct sums commute with tensor products, this map is injective, and therefore we're done.

c. Taking global sections, we obtain

$$0 \to \Gamma(Y, \mathcal{I}^r/\mathcal{I}^{r+1}) \to \Gamma(Y, \mathcal{O}_X/\mathcal{I}^{r+1}) \to \Gamma(\mathcal{O}_X/\mathcal{I}^r).$$

By (b), the first term is always zero, so the second map is always an injection. When $r = 1$, the third term is exactly k by exercise II.4.5(d). For any $r > 0$, we always have that $k \subset \Gamma(Y, \mathcal{O}_X/\mathcal{I}^{r+1})$, so we see that $\Gamma(Y, \mathcal{O}_X/\mathcal{I}^2) = k$, and then inductively we have that $\Gamma(Y, \mathcal{O}_X/\mathcal{I}^{r+1}) = k$ for all r.

d. By proposition II.9.2 we have the desired conclusion.

Exercise II.9.2. Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subset X = \mathbb{P}^n_k$ be as above, and let $f : X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

Solution. Let Y_n denote the closed subscheme $(Y, \mathcal{O}_X/\mathcal{I}_Y^n) \subset X$, and let $\operatorname{Spec} B \subset Z$ be an affine open neighborhood of P. Since Y_n is quasi-compact, formation of the scheme-theoretic image commutes with restriction to opens the target by our solution to exercise II.3.11(d), so the map $Y_n \to Z$ factors through $Y_n \to \operatorname{Spec} B$. By an application of exercise II.2.4, our map $Y_n \to \operatorname{Spec} B$ comes from a ring map $B \to \mathcal{O}_{Y_n}(Y_n)$: as $\mathcal{O}_{Y_n}(Y_n) = k$ and our map is a k-morphism, we see that this map factors through $B/\mathfrak{m}_P \cong k$, where $\mathfrak{m}_P$ is the maximal ideal of B corresponding to P.

Now let $\operatorname{Spec} A \subset X$ be an affine open subscheme contained in $f^{-1}(\operatorname{Spec} B)$ and intersecting Y. Consider the following diagram of schemes:

$$\begin{array}{ccc} Y_n \cap \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} B/\mathfrak{m}_P & \longrightarrow & \operatorname{Spec} B \end{array}$$

Writing $I = \mathcal{I}_Y(\operatorname{Spec} A)$, we have that $Y_n \cap \operatorname{Spec} A = \operatorname{Spec} A/I^n$ and therefore we get the following commutative diagram of rings:

$$\begin{array}{ccc} A/I^n & \longleftarrow & A \\ \uparrow & & \uparrow \\ B/\mathfrak{m}_P & \longleftarrow & B \end{array}$$

The upshot is that $\mathfrak{m}_P \subset B$ must land in $I^n \subset A$ for all n. But this means $\mathfrak{m}_P$ lands in $\bigcap_{n=1}^{\infty} I^n$, which is zero by the Krull intersection theorem. So $\operatorname{Spec} A$ maps to P, which implies X maps to P.

Exercise II.9.3. Prove the analogue of (5.6) for formal schemes, which says, if $\mathfrak{X}$ is an affine formal scheme, and if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, and if $\mathfrak{F}'$ is coherent, then the sequence of global sections

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is exact. For the proof, proceed in the following steps.

a. Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$, and for each $n > 0$ consider the exact sequence

$$0 \to \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'' \to 0.$$

Use (5.6), slightly modified, to show that for every open affine subset $\mathfrak{U} \subset \mathfrak{X}$, the sequence

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$

is exact.

b. Now pass to the limit, using (9.1), (9.2), and (9.6). Conclude that $\mathfrak{F} \cong \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'$ and that the sequence of global sections above is exact.

Solution.

a. The sequence

$$0 \to \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'' \to 0$$

is evidently exact. By proposition II.9.6, $\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}'$ is coherent on the affine scheme Y_n, where affineness of Y_n follows from proposition II.9.4. Tracing the proof of proposition II.5.6, we see that the only place we use quasi-coherence is that the first sheaf in our exact sequence should be quasi-coherent: the entire proof goes through with the assumptions that we are working over an affine scheme, that the left sheaf is quasi-coherent and that the middle and right terms need only be abelian sheaves. So we have an exact sequence of global sections

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$

for any n and any affine open $\mathfrak{U} \subset \mathfrak{X}$.

b. Since the maps $\Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^{n+1}\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}')$ are surjective for any n, we have that the inverse system $(\Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}'))$ satisfies the Mittag-Leffler condition. By an application of propositions II.9.1(b) and II.9.2, passing to the inverse limit along n gives us the exact sequence

$$0 \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$

on any affine open $\mathfrak{U} \subset \mathfrak{X}$. By proposition II.9.6(a), we get that the first term of this exact sequence is $\Gamma(\mathfrak{U}, \mathfrak{F}')$, giving us

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}') \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0.$$

As $\Gamma(\mathfrak{U}, \mathfrak{F}') \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')$ factors through $\Gamma(\mathfrak{U}, \mathfrak{F})$, we can write down the following commutative diagram of abelian groups (n.b.: this is **not** an exact sequence):

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}) \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0.$$

Since the composite $\Gamma(\mathfrak{U}, \mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}) \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')$ is injective and the composite $\Gamma(\mathfrak{U}, \mathfrak{F}) \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'')$ is surjective, we must have that $\Gamma(\mathfrak{U}, \mathfrak{F}) \to \Gamma(\mathfrak{U}, \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}')$ is an isomorphism. Therefore

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is exact.

Exercise II.9.4. Use (Ex. 9.3) to prove that if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $\mathfrak{X}$, and if $\mathfrak{F}', \mathfrak{F}''$ are coherent, then $\mathfrak{F}$ is coherent also.

Solution. (A brief word on coherence: since the completion of a noetherian ring along an ideal is again noetherian by Atiyah-MacDonald theorem 10.26, we are able to use Hartshorne's definition of coherence here without worry.) Since coherence is local, it suffices to show the claim when $\mathfrak{X}$ is affine: we assume $X = \operatorname{Spec} A$ and $\mathfrak{X} = \widehat{X}$ as in the situation of theorem II.9.7. By exercise II.9.3, we have the exact sequence

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0,$$

and by theorem II.9.7, the outside two terms are finitely generated A-modules. By the standard argument considering the images of the generators of $\Gamma(\mathfrak{X}, \mathfrak{F}')$ and the preimages of the generators of $\Gamma(\mathfrak{X}, \mathfrak{F}'')$, we see that $\Gamma(\mathfrak{X}, \mathfrak{F})$ must be finitely generated as well. Applying the functor $(-)^{\triangle}$ to this sequence of global sections, by theorem II.9.7 we get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathfrak{X}, \mathfrak{F}')^{\triangle} \arrow[r] \arrow[d] & \Gamma(\mathfrak{X}, \mathfrak{F})^{\triangle} \arrow[r] \arrow[d] & \Gamma(\mathfrak{X}, \mathfrak{F}'')^{\triangle} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{F}' \arrow[r] & \mathfrak{F} \arrow[r] & \mathfrak{F}'' \arrow[r] & 0
\end{tikzcd}$$

where the outside arrows are isomorphisms. By the 5-lemma, the middle arrow must be an isomorphism as well, so $\mathfrak{F}$ is coherent.

Exercise II.9.5. If $\mathfrak{F}$ is a coherent sheaf on a noetherian formal scheme $\mathfrak{X}$, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

Solution. Pick a finite cover of $\mathfrak{X}$ by affine noetherian formal schemes $\mathfrak{U}_i$. On each $\mathfrak{U}_i$, the coherent sheaf $\mathfrak{F}|_{\mathfrak{U}_i}$ is generated by a finite number of sections $\{s_{ij}\}$ by theorem II.9.7. To finish, pick a finite number of global sections which generate all of the $\{s_{ij}\}$.

Exercise II.9.6. Let $\mathfrak{X}$ be a noetherian formal scheme, let $\mathfrak{I}$ be an ideal of definition, and for each n, let Y_n be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n)$. Assume that the inverse system of groups $(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies the Mittag-Leffler condition. The prove that $\operatorname{Pic}\mathfrak{X} = \varprojlim \operatorname{Pic} Y_n$. As in the case of a scheme, we define $\operatorname{Pic}\mathfrak{X}$ to be the group of locally free $\mathcal{O}_{\mathfrak{X}}$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

a. Use the fact that $\ker(\Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}}) \to \Gamma(Y_n, \mathcal{O}_{Y_n}))$ is a nilpotent ideal to show that the inverse system $(\Gamma(Y_n, \mathcal{O}^*_{Y_n}))$ of units in the respective rings also satisfies (ML).

b. Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, and assume that for each n, there is some isomorphism $\varphi_n : \mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathcal{O}_{Y_n}$. Then show that there is an isomorphism $\mathfrak{F} \cong \mathcal{O}_{\mathfrak{X}}$. Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})$ and $(\mathcal{O}_{Y_n})$! Conclude that the natural map $\operatorname{Pic}\mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is injective.

c. Given an invertible sheaf $\mathcal{L}_n$ on Y_n for each n, and given isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{Y_n} \cong \mathcal{L}_n$, construct maps $\mathcal{L}_{n'} \to \mathcal{L}_n$ for each $n' \geq n$ so as to make an inverse system, and show that $\mathfrak{L} = \varprojlim \mathcal{L}_n$ is a coherent sheaf on $\mathfrak{X}$. Then show that $\mathfrak{L}$ is locally free of rank 1, and thus conclude that the map $\operatorname{Pic}\mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective. Again be careful, because even though each $\mathcal{L}_n$ is locally free of rank 1, the open set needed to make them free might get smaller and smaller with n.

d. Show that the hypothesis '$(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies (ML)' is satisfied if either $\mathfrak{X}$ is affine, or each Y_n is projective over a field k.

   Note: See (III, Ex. 11.5-11.7) for further examples and applications.

Solution.

a. First we observe that $\Gamma(Y_n, \mathcal{O}^*_{Y_n}) = \Gamma(Y_n, \mathcal{O}_{Y_n})^*$. Next, if $f \in \Gamma(Y_n, \mathcal{O}_{Y_n})$ is a unit, any lift $\overline{f} \in \Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}})^*$ is also a unit: supposing $fg = 1$, then $1 - \overline{f}\overline{g} \in \ker(\Gamma(Y_n, \mathcal{O}_{Y_n}) \to \Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}}))$, which is a nilpotent ideal. So $1 - \overline{f}\overline{g} = 1 - s$ for some nilpotent s, which has inverse $1 + s + s^2 + \cdots$ which is a finite sum since eventually $s^a = 0$. Therefore the inverse system $(\Gamma(Y_n, \mathcal{O}^*_{Y_n}))$ also satisfies the Mittag-Leffler condition.

b. φ_{n+1} descends to an isomorphism $\overline{\varphi_{n+1}}$ of $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}$ and $\mathcal{O}_{Y_{n+}}/\mathfrak{I}^n \cong \mathcal{O}_{Y_n}$ after modding out by $\mathfrak{I}^n(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})$ and $\mathfrak{I}^n\mathcal{O}_{Y_{n+1}}$, respectively. Thus $\varphi_n \circ \overline{\varphi_{n+1}}^{-1}$ is a module isomorphism of $\mathcal{O}_{Y_n}$ with itself. Morphisms $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_n}$ are in bijection with global sections, and invertible morphisms are in bijection with invertible global sections. Therefore by our observation that any unit in

$\Gamma(Y_n, \mathcal{O}_{Y_n})$ lifts to a unit in $\Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}})$, we can adjust φ_{n+1} by multiplication by a unit of $\mathcal{O}_{Y_{n+1}}$ so that $\overline{\varphi_{n+1}} = \varphi_n$.

Applying this procedure inductively, we can produce a compatible system of isomorphisms $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \to \mathcal{O}_{Y_n}$ which have as their limit an isomorphism $\mathfrak{F} \to \mathcal{O}_{\mathfrak{X}}$. Therefore any invertible sheaf $\mathfrak{F}$ on $\mathfrak{X}$ which pulls back to $\mathcal{O}_{Y_n}$ on each Y_n is actually isomorphic to $\mathcal{O}_{\mathfrak{X}}$ and thus the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is an injection.

c. Consider the exact sequence

$$0 \to \mathfrak{I}^n \to \mathcal{O}_{Y_{n+k}} \to \mathcal{O}_{Y_n} \to 0.$$

After tensoring by the line bundle $\mathcal{L}_{n+k}$ and repeatedly applying the isomorphisms $\mathcal{L}_{n+k} \otimes \mathcal{O}_{Y_{n+k-1}} \cong \mathcal{L}_{n+k-1}$, we get the exact sequence

$$0 \to \mathfrak{I}^n\mathcal{L}_{n+k} \to \mathcal{L}_{n+k} \to \mathcal{L}_n \to 0,$$

and the maps $\mathcal{L}_{n+k} \to \mathcal{L}_n$ are all compatible because the isomorphisms $\mathcal{O}_{Y_{n+k}}/\mathfrak{I}^n \cong \mathcal{O}_{Y_n}$ are all compatible. By an application of proposition II.9.6(b), these assemble in to a coherent sheaf $\mathfrak{L}$ on $\mathfrak{X}$.

Now suppose we know that there is an affine open set $\mathfrak{U} \subset \mathfrak{X}$ so that $\mathcal{L}_n$ is free on $\mathfrak{U}$, where the isomorphism $\mathcal{O}_{Y_n}|_{\mathfrak{U}} \to \mathcal{L}_n|_{\mathfrak{U}}$ is given by $1 \mapsto \ell \in \mathcal{L}_n(\mathfrak{U})$. Let $\overline{\ell} \in \mathcal{L}_{n+1}(\mathfrak{U})$ be any lift of ℓ. I claim that $\mathcal{O}_{Y_{n+1}}|_{\mathfrak{U}} \to \mathcal{L}_n|_{\mathfrak{U}}$ by $1 \mapsto \overline{\ell}$ is an isomorphism. Let's check on stalks: at any point $u \in \mathfrak{U}$, we have that $\mathcal{L}_{n,u} \cong \mathcal{O}_{Y_n,u}$ and $\mathcal{L}_{n+1,u} \cong \mathcal{O}_{Y_{n+1},u}$, so the map $1 \mapsto \ell$ and $1 \mapsto \overline{\ell}$ induce the following commutative diagram of local rings:

$$\begin{array}{ccc} \mathcal{O}_{Y_{n+1},u} & \xrightarrow{\overline{\ell}} & \mathcal{O}_{Y_{n+1},u} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_n,u} & \xrightarrow{\ell} & \mathcal{O}_{Y_n,u} \end{array}$$

where the vertical maps are the reduction and we slightly abuse notation in using ℓ and $\overline{\ell}$ to refer to their images in the stalk. Since the lower map is an isomorphism, $\ell \in \mathcal{O}_{Y_n,u}$ is a unit, and by the logic of part (a), $\overline{\ell}$ must also be a unit. Thus the top map is an isomorphism for all u and therefore $\mathcal{L}_{n+1}|_{\mathfrak{U}}$ is free. Picking an affine open cover $\{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ where $\mathcal{L}_1$ is free on each $\mathfrak{U}_i$, we see that $\mathfrak{L}$ is locally free of rank 1. This shows that any collection of compatible line bundles on each Y_n gives a line bundle on $\mathfrak{X}$, and thus the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective.

d. If $\mathfrak{X}$ is affine, then $Y_n = \operatorname{Spec} A/I^n$ for some ring A and ideal I, and the maps in $(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ are just given by $A/I^{n+k} \to A/I$, which are clearly surjective and thus trivially satisfy the Mittag-Leffler condition. If each Y_n is projective over a field k, then each $\Gamma(Y_n, \mathcal{O}_{Y_n})$ is a finite-dimensional vector space over k by theorem II.5.19, where any ascending or descending chain of subspaces stabilizes.

Chapter III

Cohomology

III.1 Derived Functors

There are no problems here! Maybe I should write something about derived functors in general.

III.2 Cohomology of Sheaves

Cohomology is a wonderful tool which linearizes problems, and it's only natural that algebraic geometry would want in on the action. This section is very general in comparison to the sorts of cohomological material we'll cover later: we'll later specialize from sheaves of abelian groups on arbitrary topological spaces to quasi-coherent sheaves on schemes.

Exercise III.2.1.

a. Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X \setminus \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.

b. (*) More generally, let $Y \subset X = \mathbb{A}^n_k$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X \setminus Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

Solution.

a. Consider the exact sequence of sheaves

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_{\{P,Q\}} \to 0.$$

Taking the long exact sequence in cohomology, we get

$$0 \to H^0(X, \mathbb{Z}_U) \to \mathbb{Z} \to \mathbb{Z}^2 \to H^1(X, \mathbb{Z}_U) \to \cdots$$

which shows that $H^1(X, \mathbb{Z}_U) \neq 0$ as otherwise $\mathbb{Z} \to \mathbb{Z}^2$ is a surjection, which is obviously impossible.

b. Let $Y_{n,m}$ be a collection of m distinct hyperplanes in $\mathbb{A}^n_k$ and let $U_{n,m}$ be its open complement. We first prove that $H^i(\mathbb{A}^n_k, Y_{n,m}) = 0$ for all $i \geq \min(1, m-1)$ if $m \leq n$ by induction on m. For the base case $m = 1$, consider the long exact sequence in cohomology associated to the exact sequence of sheaves

$$0 \to \mathbb{Z}_{U_{n,1}} \to \mathbb{Z}_X \to \mathbb{Z}_{Y_{n,1}} \to 0.$$

As $\mathbb{Z}_X$ and $\mathbb{Z}_{Y_{n,1}}$ are constant sheaves on the irreducible spaces $X \cong \mathbb{A}^n_k$ and $Y_{n,1} \cong \mathbb{A}^{n-1}_k$, they are flasque, so their cohomology vanishes in degrees above zero. By considering the portion of the long exact sequence

$$\cdots \to H^{i-1}(X, \mathbb{Z}_{Y_{n,1}}) = 0 \to H^i(X, \mathbb{Z}_{U_{n,1}}) \to H^i(X, \mathbb{Z}_X) = 0 \to \cdots$$

for $i > 1$, we see that $H^i(X, \mathbb{Z}_{U_{n,1}}) = 0$ for $i > 1$. This leaves us with the long exact sequence

$$0 \to H^0(X, \mathbb{Z}_{U_{n,1}}) \to H^0(X, \mathbb{Z}_X) \to H^0(X, \mathbb{Z}_{Y_{n,1}}) \to H^1(X, \mathbb{Z}_{U_{n,1}}) \to 0.$$

We now observe that the restriction map $\mathbb{Z}_X \to \mathbb{Z}_{Y_{n,1}}$ is an isomorphism on global sections, so $H^i(X, \mathbb{Z}_{U_{n,1}})$ is actually zero for all i.

Now assume we've proven the claim for some fixed m with $m < n$. Consider the long exact sequence in cohomology associated to the short exact sequence of sheaves

$$0 \to \mathbb{Z}_{U_{n,m}} \to \mathbb{Z}_{U_{n,m+1}} \to \mathbb{Z}_{U_{n,m+1}}|_{H_{m+1}} \to 0.$$

Looking near $H^i(X, \mathbb{Z}_{U_{n,m+1}})$, we see it has the form

$$\cdots \to H^i(X, \mathbb{Z}_{U_{n,m}}) \to H^i(X, \mathbb{Z}_{U_{n,m+1}}) \to H^i(X, \mathbb{Z}_{U_{n,m+1}}|_{H_{m+1}}) \to \cdots .$$

Since $\mathbb{Z}_{U_{n,m+1}}|_{H_{m+1}}$ is the pushforward of $\mathbb{Z}_{U_{n-1,m}}$ along the closed immersion $H_{m+1} \cong \mathbb{A}^{n-1}_k \to \mathbb{A}^n_k$, we can write $H^i(X, \mathbb{Z}_{U_{n,m+1}}|_{H_{m+1}}) \cong H^i(\mathbb{A}^{n-1}_k, \mathbb{Z}_{U_{n-1,m}})$. But then for $i \geq \min(1, m-1)$ both terms surrounding $H^i(X, \mathbb{Z}_{U_{n,m+1}})$ vanish by our inductive hypothesis.

Suppose now that $H_1, \cdots, H_{n+1}$ are $n+1$ hyperplanes in X in suitably general position (all we'll need is that $H_1 \cap H_{n+1}, \cdots, H_n \cap H_{n+1}$ is a collection of hyperplanes in $\mathbb{A}^{n-1}_k$ which are also in suitably general position, and if $n = 1$ then we have two distinct points). We proceed by induction on n: the case $n = 1$ was handled in part (a). Consider the exact sequence of sheaves

$$0 \to \mathbb{Z}_{U_{n,n+1}} \to \mathbb{Z}_{U_{n,n}} \to \mathbb{Z}_{U_{n,n}}|_{H_{n+1}} \to 0$$

and take the long exact sequence in cohomology. By the above, we have that $H^i(X, \mathbb{Z}_{U_{n,n}})$ vanishes for $i = n-1, n$, so the tail end of our long exact sequence on cohomology looks as follows:

$$\cdots \to H^{n-1}(X, \mathbb{Z}_{U_{n,n}}) = 0 \to H^{n-1}(X, \mathbb{Z}_{U_{n,n}}|_{H_{n+1}}) \to H^n(X, \mathbb{Z}_{U_{n,n+1}}) \to H^n(X, \mathbb{Z}_{U_{n,n}}) = 0 \to \cdots$$

Therefore $H^{n-1}(X, \mathbb{Z}_{U_{n,n}}|_{H_{n+1}}) \cong H^n(X, \mathbb{Z}_{U_{n,n+1}})$. But $\mathbb{Z}_{U_{n,n}}|_{H_{n+1}}$ is exactly $\mathbb{Z}_{U_{n-1,n}}$ for the hyperplanes which are $H_i \cap H_{n+1}$ for $i < n+1$ inside H_{n+1}. Since we assumed that these smaller-dimensional hyperplanes were in suitably general position, we have that

$$H^{n-1}(X, \mathbb{Z}_{U_{n,n}}|_{H_{n+1}}) \cong H^{n-1}(\mathbb{A}^{n-1}_k, \mathbb{Z}_{U_{n-1,n}}) \neq 0,$$

and so $H^n(X, \mathbb{Z}_{U_{n,n+1}})$ doesn't vanish either.

Exercise III.2.2. Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Solution. The sections of $\mathcal{K}/\mathcal{O}$ on an open set U are just $\bigoplus_{P \in U} I_P$, and the restriction map between $U \supset U'$ is just the projection, which is clearly surjective. Thus $\mathcal{K}/\mathcal{O}$ is flasque, so the only possible nontrivial cohomology of $\mathcal{O}$ is the cokernel of $H^0(X, \mathcal{K}) \to H^0(X, \mathcal{K}/\mathcal{O})$. By the computation in exercise II.1.21(e), this is zero.

Exercise III.2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

a. Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H^i_Y(X,\cdot)$. They are the *cohomology groups* of X *with supports* in Y, and coefficients in a given sheaf.

b. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathcal{F}') \to \Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{F}'') \to 0$$

is exact.

c. Show that if $\mathcal{F}$ is flasque, then $H^i_Y(X,\mathcal{F}) = 0$ for all $i > 0$.

d. If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(X-Y,\mathcal{F}) \to 0$$

is exact.

e. Let $U = X \setminus Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0_Y(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F}|_U) \to \\ &\to H^1_Y(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(U,\mathcal{F}|_U) \to \\ &\to H^2_Y(X,\mathcal{F}) \to \cdots . \end{aligned}$$

f. *Excision.* Let V be an open subset of X containin Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H^i_Y(X,\mathcal{F}) \cong H^i_Y(V,\mathcal{F}|_V).$$

Solution.

a. Let $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$ be an exact sequence of sheaves of abelian groups on X. It's clear that the image of $\Gamma_Y(X,\mathcal{F})$ lands in $\Gamma_Y(X,\mathcal{G})$ under the map $\varphi(X)$: being supported on Y means that $s_x = 0$ for all $x \notin Y$, and as $\varphi(X)(s)_x = \varphi_x(s_x)$, we see that $\varphi(X)(s)$ is also supported on Y. The same logic shows that $\Gamma_Y(X,\mathcal{G})$ lands in $\Gamma_Y(X,\mathcal{H})$ under $\psi(X)$.

As the restriction of an injective map is again injective, we have that $\Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{G})$ is injective. Since $\Gamma(X,-)$ is left exact, the composition $\psi(X) \circ \varphi(X)$ is zero, and thus it's restriction to $\Gamma_Y(X,\mathcal{F}) \subset \Gamma(X,\mathcal{F})$ is as well. Therefore $0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{G}) \to \Gamma_Y(X,\mathcal{H})$ is exact, and $\Gamma_Y(X,-)$ is a left exact functor.

b. Note that for any sheaf $\mathcal{F}$, $\Gamma_Y(X,\mathcal{F})$ is the kernel of the restriction map $\mathcal{F}(X) \to \mathcal{F}(X \setminus Y)$. By two applications of exercise II.1.16(b), we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}''(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}'(X \setminus Y) \arrow[r] & \mathcal{F}(X \setminus Y) \arrow[r] & \mathcal{F}''(X \setminus Y) \arrow[r] & 0
\end{tikzcd}$$

and applying the snake lemma plus the identification $\Gamma_Y(X, \mathcal{F}) \cong \ker(\mathcal{F}(X) \to \mathcal{F}(X \setminus Y))$ we obtain the exact sequence

$$0 \to \Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'') \to \operatorname{coker}(\mathcal{F}'(X) \to \mathcal{F}'(X \setminus Y)).$$

As $\mathcal{F}'$ is flasque, the cokernel vanishes, so we get that

$$0 \to \Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'') \to 0$$

is exact.

c. The same proof as in propostition III.2.5 applies, slightly modified: embed $\mathcal{F}$ in to an injective sheaf $\mathcal{I}$ with quotient $\mathcal{G}$. By lemma III.2.4, $\mathcal{I}$ is flasque, so $\mathcal{G}$ is flasque by exercise II.1.16(c). From (b) we have the exact sequence

$$0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{I}) \to \Gamma_Y(X, \mathcal{G}) \to 0.$$

On the other hand, as $\Gamma_Y(X, -)$ is a covariant left exact functor by (a), we can apply theorem III.1.1A(e) to see that $H^i_Y(X, \mathcal{I}) = R^i\Gamma_Y(X, \mathcal{I}) = 0$ for $i > 0$ as $\mathcal{I}$ is injective. By the long exact sequence in cohomology, we have that $H^1_Y(X, \mathcal{F}) = 0$ and $H^i_Y(X, \mathcal{F}) \cong H^{i-1}_Y(X, \mathcal{G})$ for $i \geq 2$. But $\mathcal{G}$ is flasque, so by induction on i we get the result.

d. Left exactness follows from the observation that $\Gamma_Y(X, \mathcal{F})$ is the kernel of $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus Y, \mathcal{F})$ made in (b), while surjectivity follows from the fact that $\mathcal{F}$ is flasque.

e. Take a flasque resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$. By (d), we get an exact sequence of chain complexes

$$0 \to \Gamma_Y(X, \mathcal{I}^\bullet) \to \Gamma(X, \mathcal{I}^\bullet) \to \Gamma(X \setminus Y, \mathcal{I}^\bullet) \to 0,$$

which as its homology exactly the given long exact sequence in cohomology.

f. Since $H^i_Y(X, -)$ and $H^i_Y(V, (-)|_V)$ are both universal δ-functors, it suffices to show a natural isomorphism between $\Gamma_Y(X, -)$ and $\Gamma_Y(V, (-)|_V)$ by corollary III.1.4. There's a natural restriction map $\Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(V, \mathcal{F}|_V)$ for any $\mathcal{F}$, and I claim this is an isomorphism. It is injective: the kernel is the sections s satisfying $\operatorname{Supp}(s) \subset X \setminus V$ and $\operatorname{Supp}(s) \subset Y$, but $X \setminus V$ does not meet Y and thus any such section must be zero. It is surjective: given any section $s \in \Gamma_Y(V, \mathcal{F}|_V)$, it is zero on the open set $V \setminus Y$, so it glues with the section 0 on $X \setminus Y$ to form a section of $\mathcal{F}$ over X supported on Y which restricts to s on V.

Exercise III.2.4. *Mayer-Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\cdots \to H^i_{Y_1\cap Y_2}(X,\mathcal{F}) \to H^i_{Y_1}(X,\mathcal{F}) \oplus H^i_{Y_2}(X,\mathcal{F}) \to H^i_{Y_1\cup Y_2}(X,\mathcal{F}) \to$$
$$\to H^{i+1}_{Y_1\cap Y_2}(X,\mathcal{F}) \to \cdots.$$

Solution. From the proof of proposition III.2.2, we may choose a resolution of $\mathcal{F}$ by injective sheaves $\mathcal{I}^\bullet$ each of which are isomorphic to a direct product of sheaves of the form $j_*(I_x)$, where $j : \{x\} \to X$ is the inclusion of a point and I_x is an injective $\mathcal{O}_{X,x}$-module. I claim that

$$0 \to \Gamma_{Y_1\cap Y_2}(X, I^i) \to \Gamma_{Y_1}(X, I^i) \oplus \Gamma_{Y_2}(X, I^i) \to \Gamma_{Y_1\cup Y_2}(X, I^i) \to 0$$

is an exact sequence, where the first map is the inclusion in to each factor and the second map is the difference. This follows immediately from the description of I^i as a product of sheaves each supported on a single point: the given sequence is exact for any sheaf $j_*(I_x)$, and a product of exact sequences of modules over a ring is again exact. This gives us a short exact sequence of chain complexes inducing the desired long exact sequence in cohomology.

Exercise III.2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \overline{\{Q\}}$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j : X_P \to X$ be the inclusion, and for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_P = j^*\mathcal{F}$. Show that for all i, $\mathcal{F}$, we have

$$H^i_P(X,\mathcal{F}) = H^i_P(X_P,\mathcal{F}_P).$$

Solution. (Minor notational gripe: j^* means j^{-1} here.) After some preliminary investigation of j^*, we will prove that $H^i_P(X,-)$ and $H^i_P(X_P,\mathcal{F}_P)$ are universal δ-functors and that $\Gamma_P(X,-) \cong \Gamma_P(X_P, j^*(-))$ which shows the claim in view of corollary III.1.4.

First, I claim that the presheaf $\mathcal{P}$ on X_P defined by $U \subset X_P \mapsto \varinjlim_{V\supset U} \mathcal{F}(V)$ where $V \subset X$ is open is already a sheaf. To check locality, assume we have two sections $s, t \in \mathcal{P}(U)$ with representatives s', t' defined on W_s, W_t open subsets of X respectively. If s and t are equal in every $\mathcal{P}(U_i)$, this means that for each i there is an open subset W_i of X containing U_i and contained in $W_s \cap W_t$ where $s'|_{W_i} = t'|_{W_i}$, implying that $s = t$ when restricted to $\bigcup W_i$. But $\bigcup W_i$ contains U, so $s = t$ in the limit.

To check gluing, we explain how to glue when our cover has two open sets: that is, $U \subset X_P$ is covered by $U_1, U_2 \subset X_P$. This will suffice to prove the general case because X is topologically noetherian, so every subspace of X is quasi-compact and we need only consider finite covers. Suppose we have a section of $\mathcal{P}(U_i)$ represented by $s_i \in \mathcal{F}(V_i)$ for $V_i \subset X$ an open set with $V_i \cap X_P = U_i$ for $i = 1, 2$. The statement that these are compatible on the overlap means that there's a third section represented by $s_{12} \in \mathcal{F}(V_{12})$ for $V_{12} \subset X$ an open set with $V_{12} \cap X_P = U_{12} = U_1 \cap U_2$ and $V_{12} \subset V_1 \cap V_2$ so that $s_1|_{V_{12}} = s_2|_{V_{12}}$. We'd love to glue s_1 and s_2 to a section of $V_1 \cup V_2$ by applying the sheaf axiom for $\mathcal{F}$, but there's a problem: it may be the case that $V_1 \cap V_2$ is strictly larger than V_{12}.

To rectify this issue, I claim that we can shrink V_1 and V_2 so that their intersection is contained in V_{12}: doing this will let us glue s_1 and s_2 to a section of $\mathcal{F}(V_1 \cup V_2)$ which will then provide a gluing of our two sections of $\mathcal{P}$. Let $Z = X \setminus V_{12}$, which is a closed subset of X and thus a finite union of irreducible components Z_j, each of which has a unique generic point η_j. Note that in order for V_1

or V_2 to avoid Z_j, it suffices to avoid that they avoid η_j. As $\eta_j \notin V_{12}$, $\eta_j \notin U_1 \cap U_2$ as a subset of X, so without loss of generality, $\eta_j \notin U_1$. In fact, since U_1 is closed under generalization as a subset of X_P, it's closed under generalization as a subset of X since X_P is closed under generalizations; thus $\eta_j \notin U_i$ means that $\overline{\{\eta_j\}} \cap U_i = \emptyset$. So $V_1 \setminus \overline{\{\eta_j\}}$ is an open subset of X containing U_1, and letting this be our new V_1, we've thrown Z_j out of $V_1 \cap V_2$. After repeating this finitely many times, we've arranged that $V_1 \cap V_2 \subset V_{12}$ and our proof that $\mathcal{P}$ is a sheaf is finished.

The upshot of knowing that $\mathcal{P}$ is already a sheaf is that j^* sends flasque sheaves to flasque sheaves: given two open subsets $U_1 \subset U_2 \subset X_P$ with lifts $V_1 \subset V_2 \subset X$ and a section $s \in (j^*\mathcal{F})(U_1)$, we can pick a representative $s' \in \mathcal{F}(V_1)$ and by flasqueness of $\mathcal{F}$ find a section $s'' \in \mathcal{F}(V_2)$ so that $s''|_{V_2} = s'$, and the image of $s'' \in \varinjlim_{V \supset U_2} \mathcal{F}(V)$ is exactly s. Therefore $H^i_P(X_P, j^*(-))$ is effaceable for all $i > 0$. Combining this with the fact that j^* is exact and $H^i_P(X_P, -)$ is a δ-functor, we see that $H^i_P(X_P, j^*(-))$ is a universal δ-functor from the category of sheaves on X to abelian groups by theorem III.1.3A. As $H^i_P(X, -)$ is also a universal δ-functor between the same two categories, it is enough to show that $\Gamma_P(X, -)$ is isomorphic to $\Gamma_P(X_P, j^*(-))$ in order to get an isomorphism between their derived functors $H^i_P(X, -)$ and $H^i_P(X_P, j^*(-))$ by corollary III.1.4.

Suppose $\mathcal{F}$ is a sheaf on X, and let $s \in \mathcal{F}(X)$ be a global section supported on P. Then the image of s in $(j^*\mathcal{F})(X_P) = \varinjlim_{V \supset X_P} \mathcal{F}(V)$ is also supported at P, so we have a map $\Gamma_P(X, \mathcal{F}) \to \Gamma_P(X_P, j^*\mathcal{F})$. On the other hand, given a global section s' of $j^*\mathcal{F}$ supported at P, it lifts to a section s of $\mathcal{F}$ defined on some open set of X containing P, and s is supported on P. So we can glue s and the section which is zero on $X \setminus P$ to a global section of $\mathcal{F}$ which is supported on P, and this process gives a map $\Gamma_P(X_P, j^*\mathcal{F}) \to \Gamma_P(X, \mathcal{F})$. It's clear that these two maps are mutually inverse, so the two functors are isomorphic and we're done.

Exercise III.2.6. Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf is injective if and only if for every open set $U \subset X$, and for every subsheaf $\mathscr{R} \subset \mathbb{Z}_U$, and for every map $f : \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map $\mathbb{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

Solution. Warning: the finite generation condition in the hint is at best imprecise. Even if you interpret the statement to mean locally finitely generated (in the sense that for any $x \in X$ there is an open neighborhood U of x and a surjection $\mathbb{Z}^n_U \to \mathcal{R}|_U$), then a subsheaf of the form $i_!\mathbb{Z}_U$ may fail to be finitely generated for some $U \subset X$. Consider $X = \mathbb{A}^1_k$ and U the complement of the origin, with $i : U \to X$ the inclusion map. Then $i_!\mathbb{Z}_U$ is certainly a subsheaf of $\mathbb{Z}_X$, but it's not locally finitely generated: for any neighborhood of the origin, $i_!\mathbb{Z}_U$ has no nonzero sections and therefore cannot receive a nonzero morphism from any constant sheaf. We'll still show that $\mathcal{R} \to \mathcal{I}$ factors through some $\mathcal{I}_\alpha$, but our method of proof will be slightly different.

We begin by showing the first claim from the hint: $\mathcal{I}$ is injective iff for all open $U \subset X$, all subsheaves $\mathcal{R} \subset \mathbb{Z}_U$, and all maps $\mathcal{R} \to \mathcal{I}$, there exists an extension $\mathbb{Z}_U \to \mathcal{I}$. Letting $i : U \to X$ denote the inclusion, the forward direction is just the definition of an injective object applied to the inclusion $i_!\mathcal{R} \to i_!\mathbb{Z}_U$ and the morphism $i_!\mathcal{R} \to \mathcal{I}$. The reverse direction is the interesting part: this is an analogue of Baer's criterion in the case of injective modules over a ring, and our method of proof will be similar.

Let $\mathcal{F} \to \mathcal{G}$ be an injective map of sheaves on X, and suppose $f : \mathcal{F} \to \mathcal{I}$ is a map from $\mathcal{F}$ to a sheaf $\mathcal{I}$ satisfying the property from the hint. Consider the poset whose elements are pairs $(\mathcal{F}', f')$ where $\mathcal{F}'$ is a submodule of $\mathcal{G}$ containing $\mathcal{F}$ and $f' : \mathcal{F}' \to \mathcal{I}$ is an extension of f, and the ordering is given by $(\mathcal{F}', f') \leq (\mathcal{F}'', f'')$ when $\mathcal{F}' \subset \mathcal{F}''$ and f'' extends f'. By Zorn's lemma, this poset has maximal elements, say $(\mathcal{F}', f')$. Suppose $\mathcal{F}' \neq \mathcal{G}$: then we can find an open set $U \subset X$ and an element $g \in \mathcal{G}(U) \setminus \mathcal{F}'(U)$. Let $\mathcal{K} = \mathcal{F}' \times_{\mathcal{G}} i_!\mathbb{Z}_U$, where $i : U \to X$ is the inclusion and $i_!\mathbb{Z}_U$ maps to $\mathcal{G}$ by $1 \mapsto g$. By the construction of $\mathcal{K}$ as the kernel of $\mathcal{F}' \oplus i_!\mathbb{Z}_U \to \mathcal{G}$ by $(-f', 1 \mapsto g)$, we see $\mathcal{K}$ is a subsheaf of $i_!\mathbb{Z}_U$. Applying the property of $\mathcal{I}$ from the hint, we obtain a map $i_!\mathbb{Z}_U \to \mathcal{I}$ compatible with $\mathcal{F}' \to \mathcal{G}$, and thus we can extend our map $\mathcal{F}' \to \mathcal{I}$ to a map $\operatorname{Im}(\mathcal{F}' \oplus i_!\mathbb{Z}_U \to \mathcal{G}) \to \mathcal{I}$. This image sheaf contains the section $g \in \mathcal{G}(U)$ by construction, so it's strictly larger than $\mathcal{F}'$, contradicting our choice of $\mathcal{F}'$ as maximal. Therefore $\mathcal{F}'$ must actually be $\mathcal{G}$, and we've shown that $\mathcal{I}$ is injective.

In order to show that $\mathcal{R} \to \mathcal{I}$ factors through some $\mathcal{I}_\alpha$, we begin with a preliminary about noetherian topological spaces: if X is noetherian, X is locally connected. To prove the statement, pick $x \in X$ and let E be an open neighborhood of x. Since noetherianness is hereditary (exercise I.1.7(c)), E is again noetherian and decomposes in to finitely many irreducible components $Y_1, \cdots, Y_n$. Then $\left(\bigcup_{x \in Y_i} Y_i\right) \setminus \left(\bigcup_{x \notin Y_i} Y_i\right)$ is an open connected neighborhood of x.

Now define the set $X_d = \{x \in X \mid d \in \mathcal{R}_x\}$. I claim this is open: if $d \in \mathcal{R}_x$, then there is some open connected neighborhood U of x so that $d \in \mathcal{R}(U)$. As the restriction maps $\mathcal{R}(U) \to \mathcal{R}_y$ for U connected and $y \in U$ are injective, we see that any $y \in U$ must also be in X_d and therefore X_d is a union of open sets. Next, we observe that $\operatorname{Supp} \mathcal{R} = \bigcup_{d>0} X_d$. As $\operatorname{Supp} \mathcal{R}$ is noetherian, we can select a finite subcover and write $\operatorname{Supp} \mathcal{R} = \bigcup_{d=1}^{D} X_d$ for some integer D. Now we see that $\mathcal{R} = +_{d=1}^{D} i_!(d\mathbb{Z}_{X_d})$, so to specify a morphism out of $\mathcal{R}$ it suffices to define where the generating sections of each constant sheaf go. As X_d is noetherian, it has finitely many connected components, so each $\mathbb{Z}_{X_d}$ has finitely many generating sections over X_d, and thus the amount of data needed to specify a morphism out of $\mathcal{R}$ is finite (for each generating section, pick a representative for it's image in some $\mathcal{I}_\alpha$, and then take a common upper bound for all α we find this way by the definition of a directed set). Therefore any map $\mathcal{R} \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ factors through some $\mathcal{I}_\alpha$.

To finish the proof, suppose we have an injection $\mathcal{R} \to i_!\mathbb{Z}_U$ and a map $\mathcal{R} \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$. Factoring $\mathcal{R} \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ through some $\mathcal{I}_\alpha$ and applying the extension property, we see that $\mathcal{R} \to \mathcal{I}_\alpha$ extends to a morphism $i_!\mathbb{Z}_U \to \mathcal{I}_\alpha$, which when composed with the natural map $\mathcal{I}_\alpha \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ gives an extension of $\mathcal{R} \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ to $i_!\mathbb{Z}_U \to \varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ and therefore $\varinjlim_{\alpha \in A} \mathcal{I}_\alpha$ is injective.

Exercise III.2.7. Let S^1 be the circle (with its usual topology), and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.

a. Show that $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$, using our definition of cohomology.

b. Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.

Solution.

a. Let $\mathcal{F}$ be the flasque sheaf which assigns to each $U \subset S^1$ the set of all functions $U \to \mathbb{R}$. Embed $\mathbb{Z} \to \mathcal{F}$ in the obvious way and let $\mathcal{Q}$ be the quotient. Taking the long exact sequence in cohomology arising from
$$0 \to \mathbb{Z} \to \mathcal{F} \to \mathcal{Q} \to 0,$$
we note that $H^1(S^1, \mathcal{F}) = 0$ implying $H^1(S^1, \mathbb{Z}) = \operatorname{coker}(\Gamma(S^1, \mathcal{F}) \to \Gamma(S^1, \mathcal{Q}))$. By exercise II.1.3, any section $s \in \Gamma(S^1, \mathcal{Q})$ is the image of a family $\{(s_i, U_i)\}_{i \in I}$ with $s_i \in \mathcal{F}(U_i)$ where U_i form an open cover of S^1 and $(s_i - s_j)|_{U_i \cap U_j}$ is a section of $\mathbb{Z}_{U_i \cap U_j}$. Since S^1 is compact, we may assume I is finite; after subdividing, throwing away redundant elements, and reordering we may assume that our cover consists of connected open subsets so that U_i only intersects U_{i-1} and U_{i+1} with indices interpreted modulo $|I|$.

Now I claim that it suffices to consider $|I| = 3$. Let n_{i+1} be the value of $s_{i+1} - s_i$ on $U_i \cap U_{i+1}$. Replacing s_{i+1} with $s_{i+1} - n_{i+1}$, which does not change the image of s_{i+1} in $\mathcal{Q}(U_{i+1})$, we see that $s_i = s_{i+1}$ on $U_i \cap U_{i+1}$. Therefore we can glue s_i and s_{i+1} to form a section of $\mathcal{F}$ over $U_i \cup U_{i+1}$ without changing its image in $\mathcal{Q}$. Repeating this process for $i = 3, \cdots, |I| - 1$, we see that we can glue the sections s_i on $U_3 \cup \cdots \cup U_{|I|}$ so that we're only looking at $\{(s_1, U_1), (s_2, U_2), (s_3, U_3 \cup \cdots \cup U_{|I|})\}$.

If we have a section $\{(s_1, U_1), (s_2, U_2), (s_3, U_3)\}$, by the same logic we may assume that $s_1 = s_2$ on $U_1 \cap U_2$ and $s_2 = s_3$ on $U_2 \cap U_3$. Therefore up to adding a global section of $\mathcal{F}$, the global sections of $\mathcal{Q}$ are exactly those of the form $\{(0, U_1), (0, U_2), (n, U_3)\}$ for $n \in \mathbb{Z}$ and opens U_i satisfying our ordering and intersection assumptions. Since any two such sections are equivalent up to an element of $\Gamma(S^1, \mathcal{F})$ iff their ns match, we see that the cokernel is exactly $\mathbb{Z}$.

b. The same argument from the first paragraph of (a) applies verbatim. Now I claim that we can assemble the $\{(s_i, U_i)\}$ in to a global section of $\mathcal{F}$ which has image s. Let Φ_i be a partition of unity on S^1 relative to the cover $\{U_i\}$: each Φ_i is a continuous function $S^1 \to [0, 1]$ with support contained in U_i and $\sum \Phi_i = 1$. Let r_{i+1} be the function on U_{i+1} which takes the value 0 over $U_{i+1} \setminus U_i$ and $s_{i+1} - s_i$ on $U_i \cap U_{i+1}$. Now consider $\Phi_i r_{i+1}$ and $\Phi_{i+1} r_{i+1}$: the former extends by zero to a continuous function on U_{i+1} while the latter extends by zero to a continuous function on U_i, so we can replace s_i with $s_i - \Phi_{i+1} r_{i+1}$ and s_{i+1} with $s_{i+1} + \Phi_i r_{i+1}$ without changing their images in $\mathcal{Q}(U_i)$ and $\mathcal{Q}(U_{i+1}$, respectively. Further, $(s_{i+1} + \Phi_i r_{i+1}) - (s_i - \Phi_{i+1} r_{i+1}) = s_{i+1} - s_i - r_{i+1} = 0$ on $U_i \cap U_{i+1}$, so these glue to a section of $\mathcal{F}$ over $U_i \cup U_{i+1}$. In fact we can do this for all i, so $\Gamma(S^1, \mathcal{F}) \to \Gamma(S^1, \mathcal{Q})$ is surjective and $H^1(S^1, \mathcal{R}) = 0$.

III.3 Cohomology of a Noetherian Affine Scheme

One very nice result which has been proven since the publication of this book is that the category of quasi-coherent sheaves on *any* scheme has enough injectives (this generalizes corollary III.3.6). This result is due to Gabber and is well-documented at tag 077K of the Stacks Project.

Exercise III.3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [*Hint*: Use (3.7), and for any coherent sheaf $\mathcal{F}$ on X, consider the filtration $\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots$, where $\mathcal{N}$ is the shaef of nilpotent elements on X.]

Solution. X_{red} is a closed subscheme of X, so if X is affine, then X_{red} is as well by exercise II.3.11 or corollary II.5.10.

For the other direction, let $\mathcal{F}$ be a quasi-coherent sheaf and consider the filtration $\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots$ as in the hint. This gives exact sequences

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to (\mathcal{N}^i\mathcal{F})/(\mathcal{N}^{i+1}\mathcal{F}) \to 0$$

for all $i \geq 0$. As $(\mathcal{N}^i\mathcal{F})/(\mathcal{N}^{i+1}\mathcal{F}) \cong \mathcal{N}^i\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{N}$ is a quasi-coherent sheaf on X_{red}, we have that

$$H^j(X, (\mathcal{N}^i\mathcal{F})/(\mathcal{N}^{i+1}\mathcal{F})) \cong H^j(X_{red}, (\mathcal{N}^i\mathcal{F})/(\mathcal{N}^{i+1}\mathcal{F})) = 0$$

for $j > 0$ by theorem III.3.7. Since X is noetherian, $\mathcal{N}^n = 0$ for some $n > 0$, so $\mathcal{N}^n\mathcal{F} = 0$ and $H^j(X, \mathcal{N}^n\mathcal{F}) = 0$ for all j. Looking at the long exact sequence in cohomology, we see that $H^j(X, \mathcal{N}^i\mathcal{F})$ fits between $H^j(X, \mathcal{N}^{i+1}\mathcal{F})$ and $H^j(X_{red}, (\mathcal{N}^i\mathcal{F})/(\mathcal{N}^{i+1}\mathcal{F}))$. By induction, this shows that $H^j(X, \mathcal{N}^i\mathcal{F}) = 0$ for all $j > 0$ and all i; therefore by theorem III.3.7 X is also affine.

Exercise III.3.2. Let X be a reduced affine noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

Solution. Minor gripe: affine is a descriptor of a scheme, not a set or topological space - $\mathbb{P}^1_k$ and $\mathbb{A}^1_k$ are homeomorphic but one is affine and the other isn't. Really, we should say 'each irreducible component equipped with the reduced induced subscheme structure'.

If X is affine, equipping each irreducible component with the reduced induced subscheme structure, we see that they're closed subschemes of X and therefore affine by exercise II.3.11 or corollary II.5.10.

Conversely, let $X_1, \cdots, X_n$ be the irreducible components of X equipped with the reduced induced subscheme structure, and let $\mathcal{I}_i$ be the ideal sheaf of $X_1 \cup \cdots \cup X_i$ (we take $\mathcal{I}_0 = \mathcal{O}_X$). Given a quasi-coherent sheaf $\mathcal{F}$ on X, we can filter $\mathcal{F}$ as

$$\mathcal{F} \supset \mathcal{I}_1\mathcal{F} \supset \cdots \supset \mathcal{I}_n\mathcal{F} = 0.$$

Consider the exact sequence $0 \to \mathcal{I}_{i+1}\mathcal{F} \to \mathcal{I}_i\mathcal{F} \to \mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F} \to 0$: the third entry is a quasi-coherent sheaf on X_{i+1}, so

$$H^j(X, \mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}) \cong H^j(X_{i+1}, \mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}) = 0$$

because X_{i+1} is affine. Looking at the long exact sequence in cohomology, we see that $H^j(X, \mathcal{I}_i\mathcal{F})$ fits between $H^j(X, \mathcal{I}_{i+1}\mathcal{F})$ and $H^j(X, \mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F})$. By induction, this shows that $H^j(X, \mathcal{I}_i\mathcal{F}) = 0$ for all $j > 0$ and all i; therefore by theorem III.3.7 X is also affine.

Exercise III.3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

a. Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdots)$.

b. Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M}),$$

where $H^i_Y(X, \cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

c. For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

Solution.

a. Let $\varphi : M \to N$ be an R-module homomorphism. If $m \in \Gamma_{\mathfrak{a}}(M)$, then $\varphi(m) \in \Gamma_{\mathfrak{a}}(N)$ because $\varphi(\mathfrak{a}^n m) = \mathfrak{a}^n \varphi(m)$. Therefore if $0 \to M \to N \to P \to 0$ is an exact sequence of R-modules, we get induced maps $\Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N) \to \Gamma_{\mathfrak{a}}(P)$, and it's immediate that the first map is injective and the composition is zero from each of these being the restriction of the maps in our original exact sequence. Therefore $\Gamma_{\mathfrak{a}}(-)$ is a left-exact functor from R-modules to R-modules.

b. By our work in exercise II.5.6, we have that $\Gamma_{\mathfrak{a}}(M) = \Gamma_{V(\mathfrak{a})}(X, \widetilde{M})$. Taking an injective resolution $0 \to M \to I^\bullet$ of M, after sheafifying we see that $\widetilde{I^\bullet}$ is a complex of flasque sheaves resolving $\widetilde{M}$, and since flasque sheaves are acyclic for cohomology with supports by exercise III.2.3, we have that $H^i_{V(\mathfrak{a})}(X, \widetilde{M})$ is the i^{th} cohomology of $\Gamma_{V(\mathfrak{a})}(X, \widetilde{I^\bullet})$. On the other hand, this complex is simultaneously $\Gamma_{\mathfrak{a}}(M)(I^\bullet)$, so it computes the cohomology groups $H^i_{\mathfrak{a}}(M)$ and we see the two are isomorphic.

c. $H^i_{\mathfrak{a}}(M)$ is a quotient of $\Gamma_{\mathfrak{a}}(I)$ for some injective R-module I, thus every element of $H^i_{\mathfrak{a}}(M)$ is annihilated by some power of $\mathfrak{a}$.

Exercise III.3.4. *Cohomological Interpretation of Depth.*

If A is a ring, $\mathfrak{a}$ an ideal, and M an A-module, then $depth_{\mathfrak{a}} M$ is the maximum length of an M-regular sequence $x_1, \cdots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

a. Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

b. Show inductively, for M finitely generated, that for any $n \geq 0$, the following conditions are equivalent:

   (i) $\operatorname{depth}_{\mathfrak{a}} M \geq n$;

(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details and related results, see Grothendieck [7].

Solution.

a. The forward direction is just the definition of depth: if $a \in A$ is the first entry in an M-regular sequence, then a is not a zero divisor on M, and thus a^n isn't either for any $n > 0$. So there are no elements $m \in M$ which satisfy $\mathfrak{a}^n m = 0$ for any n, and $\Gamma_{\mathfrak{a}}(M) = 0$.

   Conversely, suppose $\Gamma_{\mathfrak{a}}(M) = 0$ and M finitely generated. $\Gamma_{\mathfrak{a}}(M) = 0$ means that for any $m \in M$ and $n \geq 0$, there is some $x \in \mathfrak{a}^n$ with $xm \neq 0$, or that $\mathfrak{a}$ is not contained in any associated prime of M. Since M is a finitely generated module over a noetherian ring, it has finitely many associated primes, and therefore we may apply prime avoidance to see that $\mathfrak{a}$ is not a subset of the union of all the associated primes. Therefore there is some $a \in \mathfrak{a}$ which isn't in any associated prime and thus is not a zero divisor on M. So the $\mathfrak{a}$-depth of M is at least one.

b. $n = 1$ was (a); we'll proceed by induction.

   Suppose M has $\mathfrak{a}$-depth at least $n+1$ with $x_1, \cdots, x_{n+1} \in \mathfrak{a}$ an M-regular sequence. Consider the exact sequence
   $$0 \to M \xrightarrow{\cdot x_1} M \to M/x_1M \to 0$$
   which induces the long exact sequence in cohomology
   $$\cdots \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/x_1M) \to \cdots.$$
   Now we make some observations: the induced map $H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M)$ is multiplication by x_1, and as both M and M/x_1M have depth at least n, we see that $H^i_{\mathfrak{a}}(M/x_1M) = H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$. So the nontrivial portion of our exact sequence begins
   $$0 \to H^n_{\mathfrak{a}}(M) \xrightarrow{\cdot x_1} H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M/x_1M) \to \cdots.$$
   By exercise III.3.3(c), $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, so multiplication by x_1 cannot be an injective automorphism unless $H^n_{\mathfrak{a}}(M) = 0$.

   Conversely, suppose $H^i_{\mathfrak{a}}(M) = 0$ for $i < n+1$. Then M has depth at least $n > 0$ by the inductive hypothesis, so we can find an x_1 in $\mathfrak{a}$ which begins an M-regular sequence. By considering the same short exact sequence as in the previous paragraph, we get the long exact sequence
   $$\cdots \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M/x_1M) \to \cdots.$$
   By hypothesis, $H^{n-1}_{\mathfrak{a}}(M) = H^n_{\mathfrak{a}}(M)$, so $H^{n-1}_{\mathfrak{a}}(M/x_1M) = 0$, and therefore M/x_1M is of depth at least n, so M is of depth at least $n+1$.

Exercise III.3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i). depth $\mathcal{O}_P \geq 2$;

(ii). if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U \setminus P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20) in view of (II, 8.22A).

Solution. The second condition is equivalent to the restriction map $H^0(U, \mathcal{O}_U) \to H^0(U \setminus P, \mathcal{O}_U)$ being bijective. This is equivalent to $H^0_P(U, \mathcal{O}_U) = H^1_P(U, \mathcal{O}_U) = 0$ by the long exact sequence in exercise III.2.3(e). Next, $H^i_P(U, \mathcal{O}_U) = H^i_P(U_P, (\mathcal{O}_U)_P)$ by exercise III.2.5 and $U_P = \operatorname{Spec} \mathcal{O}_{X,P}$ while $(\mathcal{O}_U)_P = \mathcal{O}_{X,P}$; therefore by exercise III.3.3(b), we have that $H^i_P(U, \mathcal{O}_U) = H^i_{\mathfrak{m}}(\mathcal{O}_{X,P})$. By exercise III.3.4, the vanishing of $H^0_{\mathfrak{m}}(\mathcal{O}_{X,P})$ and $H^1_{\mathfrak{m}}(\mathcal{O}_{X,P})$ is equivalent to $\mathcal{O}_{X,P}$ having depth at least two and we're done.

Exercise III.3.6. Let X be a noetherian scheme.

a. Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

b. (*) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathscr{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subset X$ is an open subset, then $\mathscr{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines ...]

c. Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

Solution.

a. Let $\{\operatorname{Spec} A_i\}$ be the finite affine open cover of X constructed in corollary III.3.6, and let I_i be the injective A_i-module in which $\mathcal{F}(\operatorname{Spec} A_i)$ embeds. Now if $\mathcal{F}'$ is a sheaf receiving an injection from $\mathcal{F}$, we have injections $\mathcal{F}(\operatorname{Spec} A_i) \to \mathcal{F}'(\operatorname{Spec} A_i)$ and maps $\mathcal{F}'(\operatorname{Spec} A_i) \to I_i$ extending $\mathcal{F}(\operatorname{Spec} A_i) \to I_i$ by the universal property of injective modules. Applying the associated sheaf functor, which is an equivalence of categories, we obtain a map $f^*\mathcal{F}' \to \widetilde{I_i}$ extending $f^*\mathcal{F} \to \widetilde{I_i}$, and by the pushforward-pullback adjunction, we get a map $\mathcal{F}' \to f_*\widetilde{I_i}$. Taking the direct sum, we get a map $\mathcal{F}' \to \mathcal{G}$ which extends $\mathcal{F} \to \mathcal{G}$ and thus $\mathcal{G}$ is injective.

b. Let $\mathcal{I}$ be an injective quasi-coherent sheaf. By corollary III.3.6, embed $\mathcal{I}$ in a flasque quasi-coherent sheaf $\mathcal{F}$, and let $\mathcal{Q}$ be the quotient. Then because $\mathcal{I}$ is injective, the exact sequence

$$0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{Q} \to 0$$

splits, or $\mathcal{F} \cong \mathcal{I} \oplus \mathcal{Q}$. But the restriction maps of $\mathcal{I} \oplus \mathcal{Q}$ are just the direct sum of the restriction maps of $\mathcal{I}$ and $\mathcal{Q}$, so they're surjective as well. Therefore $\mathcal{I}$ is flasque.

c. By (b), an injective resolution in $\mathfrak{Qco}(X)$ is a flasque resolution in $\mathfrak{Ab}(X)$, so it computes abelian sheaf cohomology by proposition III.2.5.

Exercise III.3.7. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subset A$ be an ideal, and let $U \subset X$ be the open set $X \setminus V(\mathfrak{a})$.

a. For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

b. Apply this in the case of an injective A-module I, to give another proof of (3.4).

Solution.

a. This is proven in EGA I prop. 6.9.17 (make sure you check the published book, not the pdf on Numdam which is the earlier version and does not have the proof). We'll do something slightly different.

First, I claim that $\varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M)$ only depends on the radical of $\mathfrak{a}$. Consider the directed system $\mathcal{D}$ of ideals of A with radical $\sqrt{\mathfrak{a}}$, ordered by inclusion: I claim that powers of $\mathfrak{a}$ are coinitial. If $\mathfrak{b} \in \mathcal{D}$, then $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}$ and by the noetherian hypothesis we have $\mathfrak{a}^n \subset \mathfrak{b}$ for some $n > 0$, showing the claim. This implies that the entries $\operatorname{Hom}_A(\mathfrak{a}^n, M)$ are cofinal among the directed system $\operatorname{Hom}_A(\mathfrak{b}, M)$ indexed over $\mathcal{D}$, so the limit depends only on $V(\mathfrak{a})$.

Now define a presheaf $\mathcal{P}$ on $\operatorname{Spec} A$ which assigns to each open set $U = X \setminus V(\mathfrak{a})$ the module $\varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, M)$ with the restriction map corresponding to the inclusion $X \setminus V(\mathfrak{a}) = V \subset U = X \setminus V(\mathfrak{b})$ the induced map $\varinjlim \operatorname{Hom}_A(\mathfrak{b}^n, M)$ to $\varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, M)$ coming from the inclusion $\mathfrak{a}^n \subset \mathfrak{b}$ (as $V(\mathfrak{a}) \supset V(\mathfrak{b})$ implies $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$ and thus $\mathfrak{a}^n \subset \mathfrak{b}$ for some n). I claim $\mathcal{P}$ is actually a sheaf. To check locality, suppose $U = \bigcup U_i$ (where our cover is finite as $\operatorname{Spec} A$ is noetherian) and φ is an element in $\mathcal{P}(U)$ which is zero on each U_i. Write $U_i = X \setminus V(\mathfrak{a}_i)$ - representing φ as a map $\mathfrak{a}_i \to M$, $\varphi = 0$ if and only if it's restriction to $\mathfrak{a}_i^n$ is zero for some n, and by replacing $\mathfrak{a}_i$ with $\mathfrak{a}_i^n$, we may assume $\varphi = 0$ on $\mathfrak{a}_i$ for all i. Then writing $U_i = X \setminus V(\mathfrak{a}_i)$, we have $U = X \setminus V(\sum \mathfrak{a}_i)$ and it's clear that φ acts as zero on $\sum \mathfrak{a}_i$. To check gluing, let $\varphi, \psi \in \mathcal{P}(U)$ so that they agree on each U_i. Fix an ideal $\mathfrak{a}_i$ for each U_i, and let n_i be an integer so that $\varphi = \psi$ as maps $\mathfrak{a}_i^{n_i} \to M$. Then $\varphi = \psi$ as maps $\sum \mathfrak{a}_i^{\max n_i}$, which means they're equal in $\mathcal{P}(U)$.

As sheaves are determined by their values on a basis, it suffices to show that $\mathcal{P}(D(a)) = \widetilde{M}(D(a)) = M_a$ and the restriction map $\mathcal{P}(D(a)) \to \mathcal{P}(D(b))$ is the standard map $M_a \to M_b$ in order to prove Deligne's formula. Let $\varphi \in \varinjlim \operatorname{Hom}_A((a^n), M)$: then picking a representative $\varphi_i \in \operatorname{Hom}_A((a^i), M)$, we can define an element of M_a by $\frac{\varphi_i(a^i)}{a^i}$ and this is clearly compatible with the limit. Conversely, given an element $\frac{m}{a^i} \in M_a$ we can define a map $\varphi_i : (a^i) \to M$ by $\varphi(a^i) = m$ and extending by the A-action; this is also clearly compatible with any other way to write $\frac{m}{a^i}$ in M_a. These two maps are mutually inverse, so $\mathcal{P}(D(a)) = M_a$. Now let's check the restriction maps. If $D(b) \subset D(a)$, then $V(b) \supset V(a)$ or $\sqrt{(b)} \subset \sqrt{(a)}$, so $(b) \subset \sqrt{(b)} \subset \sqrt{(a)}$ and therefore $b^n = ca$. The restriction map $M_a \to M_b$ is given by sending $\frac{m}{a^i} \mapsto \frac{c^i m}{c^i a^i} = \frac{c^i m}{b^{ni}}$, while the restriction map $\operatorname{Hom}_A((a^i), M) \to \operatorname{Hom}_A((b^{ni}), M)$ is given by restricting φ to (b^{ni}), and it sends b^{ni} to $c^i\varphi(a^i)$. So restriction maps are compatible with our isomorphisms and we're done.

b. Suppose $V \subset U$ is an inclusion of open sets in X with $V = X \setminus V(\mathfrak{b})$ and $U = X \setminus V(\mathfrak{a})$ for $\mathfrak{a} \supset \mathfrak{b}$ radical ideals. Then by (a), the restriction map is given by

$$\varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, I) \to \varinjlim \operatorname{Hom}_A(\mathfrak{b}^n, I)$$

induced by $\mathfrak{b}^n \to \mathfrak{a}^n$. Since I is injective, the map $\operatorname{Hom}_A(\mathfrak{a}^n, I) \to \operatorname{Hom}_A(\mathfrak{b}^n, I)$ is a surjection, and since colimits commute with cokernels, the induced map in the directed limit is a surjection. Applying (a), we see that $\Gamma(U, \widetilde{I}) \to \Gamma(V, \widetilde{I})$ is a surjection, so $\widetilde{I}$ is flasque.

Exercise III.3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, \cdots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \cdots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

Solution. Suppose $i \in I$ maps to $\frac{1}{x_0} \in I_{x_0}$. Then $\frac{i}{1} = \frac{1}{x_0}$ in I_{x_0}, or $ix_0 = 1$. Multiplying by a high-enough power of x_0, we get that $ix_0^n = 0$, so $0 = x_0^n$ in I which is nonsense. Therefore $\frac{1}{x_0}$ is not in the image of $I \to I_{x_0}$.

III.4 Čech Cohomology

You can actually calculate things with Čech cohomology, which is pretty rad.

Exercise III.4.1. Let $f : X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.16). Show that for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

[*Hint*: Use (II, 5.8).]

Solution. By proposition II.5.8, $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. Let $\mathfrak{U} = \{U_i\}$ be an affine open cover of Y: then the complex $C^\bullet(\mathfrak{U}, f_*\mathcal{F})$ computes the cohomology $H^p(Y, f_*\mathcal{F})$ by theorem III.4.5. On the other hand, this is the same complex as $C^\bullet(f^{-1}\mathfrak{U}, \mathcal{F})$, which computes the cohomology $H^p(X, \mathcal{F})$. So the two cohomologies are isomorphic.

(This generalizes beyond what we've proven here - the Leray spectral sequence with $E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}))$ converges to $H^{p+q}(X, \mathcal{F})$, but the higher direct images of an affine morphism vanish and thus the spectral sequence degenerates on the E_2 page and we get $H^p(X, \mathcal{F}) = H^p(Y, \mathcal{F})$ for any affine morphism.)

Exercise III.4.2. Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

a. Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathcal{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

b. For any coherent sheaf $\mathcal{F}$ on Y, show that there is a coherent sheaf $\mathcal{G}$ on X, and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^r$ which is an isomorphism at the generic point of Y. [*Hint*: Apply $\mathcal{H}om(\cdot, \mathcal{F})$ to α and use (II, Ex. 5.17e).]

c. Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider $\ker \beta$ and $\operatorname{coker} \beta$, and use noetherian induction on Y.

Solution.

a. First, a preliminary about finite surjective morphisms. Let X have generic point η_X and Y have generic point η_Y. Then $f^{-1}(\eta_Y) = \eta_X$: clearly $\eta_X \in f^{-1}(\eta_Y)$, and any other point in the fiber must generalize to η_X because X is integral. But $f^{-1}(\eta_Y) = \operatorname{Spec} k(\eta_Y) \times_Y X$ is a finite scheme over $\operatorname{Spec} k(\eta)$, so it is discrete and has no nontrivial generalizations. Thus for any sheaf $\mathcal{M}$ on X, the stalk of $f_*\mathcal{M}$ at η_Y is just the stalk of $\mathcal{M}$ at η_X.

   $K(X)$ is finite over $K(Y)$ by our solution to exercise II.3.7, so let $x_1, \cdots, x_n$ be a basis. Define $\mathcal{M}$ to be the $\mathcal{O}_X$-submodule of $K(X)$ generated by $\{x_1, \cdots, x_n\}$. $\mathcal{M}$ is coherent: over an affine open $\operatorname{Spec} A \subset X$, it is the sheafification of the A-submodule of $K(X) = \operatorname{Frac}(A)$ generated by the x_i. The x_i are global sections of $f_*\mathcal{M}$, and defining α to be the morphism $\mathcal{O}_Y^r \to f_*\mathcal{M}$ by $e_i \mapsto x_i$, we see that $\mathcal{O}_{Y,\eta_Y}^r \to (f_*\mathcal{M})_{\eta_Y} = K(X)$ is an isomorphism.

b. Applying $\mathcal{H}om_Y(-,\mathcal{F})$ to $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M}$, we get a map $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F}) \to \mathcal{H}om_Y(\mathcal{O}_Y^r,\mathcal{F}) \cong \mathcal{F}^r$. Now I claim that $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F})$ is quasi-coherent: we may apply $\mathcal{H}om_U(-,\mathcal{F})$ to the locally exact sequence $\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to f_*\mathcal{M} \to 0$ given by coherence of $f_*\mathcal{M}$ to represent $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F})$ as locally the kernel of a morphism of quasi-coherent sheaves $\mathcal{F}^n \to \mathcal{F}^m$ and apply proposition II.5.7. Further, as $\mathcal{F}$ is coherent and any submodule of a finitely-generated module over a noetherian ring is again finitely generated, we see that $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F})$ is in fact a coherent sheaf. Since $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F})$ has the structure of a $f_*\mathcal{O}_X$-module via the pushforward of the structure of $\mathcal{M}$ as a $\mathcal{O}_X$-module, we see by exercise II.5.17(e) that it is $f_*\mathcal{G}$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$, and it's not too hard to see that coherence is preserved in the proof given in that exercise.

To see that our map $\mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F}) \to \mathcal{H}om_Y(\mathcal{O}_Y^r,\mathcal{F})$ is an isomorphism at the generic point, we will show that $\mathcal{H}om_Y(\mathcal{A},\mathcal{B})_y \cong \operatorname{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{A}_y,\mathcal{B}_y)$ when ther exists an exact sequence $\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to \mathcal{A} \to 0$ in a neighborhood U of y (we say '$\mathcal{A}$ is locally finitely presented at y'). Without loss of generality, we may assume $U = Y$. Applying $\mathcal{H}om_Y(-,\mathcal{B})$ to the exact sequence $\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to \mathcal{A} \to 0$ gives

$$0 \to \mathcal{H}om_Y(\mathcal{A},\mathcal{B}) \to \mathcal{H}om_Y(\mathcal{O}_Y^n,\mathcal{B}) \to \mathcal{H}om_Y(\mathcal{O}_Y^m,\mathcal{B}),$$

and taking stalks shows that $\mathcal{H}om_Y(\mathcal{A},\mathcal{B})_y \cong \ker(\mathcal{B}^n \to \mathcal{B}^m)_y$. On the other hand, taking stalks first, we have that $\mathcal{O}_{Y,y}^m \to \mathcal{O}_{Y,y}^n \to \mathcal{A}_y \to 0$ is again exact and applying $\operatorname{Hom}_{\mathcal{O}_{Y,y}}(-,\mathcal{B}_y)$ we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{A}_y,\mathcal{B}_y) \to \operatorname{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}^n,\mathcal{B}_y) \to \operatorname{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}^m,\mathcal{B}_y)$$

which this time shows that $\operatorname{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{A}_y,\mathcal{B}_y) \cong \ker(\mathcal{B}_y^n \to \mathcal{B}_y^m)$. As $\ker(\mathcal{B}_y^n \to \mathcal{B}_y^m) \cong \ker(\mathcal{B}^n \to \mathcal{B}^m)_y$, we have the desired result. It's now straightforward to see that since $\mathcal{O}_Y^r \to f_*\mathcal{M}$ is an isomorphism at the generic point, $f_*\mathcal{G} \cong \mathcal{H}om_Y(f_*\mathcal{M},\mathcal{F}) \to \mathcal{H}om_Y(\mathcal{O}_Y^r,\mathcal{F}) \cong \mathcal{F}^r$ is also an isomorphism at the generic point.

c. Write $Y = \bigcup Y_i$ where the Y_i are irreducible components of Y, and equip each Y_i with the reduced induced subscheme structure. In order to prove that Y is affine, it is enough to show that each Y_i is by exercise III.3.2. Now consider the base change diagram:

$$\begin{tikzcd} X \times_Y Y_i \arrow[r] \arrow[d] & X \arrow[d] \\ Y_i \arrow[r] & Y \end{tikzcd}$$

As $Y_i \to Y$ is a closed immersion, $X \times_Y Y_i \to X$ is as well, so $X \times_Y Y_i$ is affine by exercise II.3.11. As finite morphisms and surjective morphisms are preserved by base change (proofs of exercise II.3.22 and II.3.15, respectively) $X \times_Y Y_i \to Y_i$ is again finite surjective. Since finite morphisms are closed, each irreducible component of $X \times_Y Y_i$ must map to a closed irreducible subset of Y_i, and by surjectivity there must be an irreducible component $X' \subset X \times_Y Y_i$ which surjects on to Y_i. As closed immersions are finite, the composition of the closed immersion of this irreducible component equipped with its reduced induced structure to $X \times_Y Y_i$ with the map $X \times_Y Y_i \to Y_i$ produces a finite surjective map from an affine integral X' to an integral Y_i. Finally, noting that closed immersions are separated and preserve noetherianness, we see that X' and Y_i are integral noetherian separated schemes, and we've achieved the desired reduction.

We aim to use Serre's criterion (theorem III.3.7). Let $\mathcal{I}$ be a coherent sheaf of ideals on Y, let $\beta : f_*\mathcal{G} \to \mathcal{I}$ be the morphism constructed in (b), and consider the two exact sequences

$$0 \to \ker\beta \to f_*\mathcal{G} \to \operatorname{Im}\beta \to 0$$

and

$$0 \to \operatorname{Im}\beta \to \mathcal{I} \to \operatorname{coker}\beta \to 0$$

which give rise to the following long exact sequences in cohomology:

$$\cdots \to H^i(Y, \ker\beta) \to H^i(Y, f_*\mathcal{G}) \to H^i(Y, \operatorname{Im}\beta) \to \cdots,$$
$$\cdots \to H^i(Y, \operatorname{Im}\beta) \to H^i(Y, \mathcal{I}) \to H^i(Y, \operatorname{coker}\beta) \to \cdots.$$

By exercise III.4.1, $H^i(Y, f_*\mathcal{G}) \cong H^i(X, \mathcal{G})$ which vanishes for all $i > 0$ by theorem III.3.7, so $H^i(Y, \operatorname{Im}\beta) = H^{i+1}(Y, \ker\beta)$ for all $i > 0$. As Y is noetherian, $\ker\beta$ and $\operatorname{coker}\beta$ are again coherent (any submodule of a finitely generated module over a noetherian ring is finitely generated, and any quasi-coherent sheaf which is locally finitely generated is coherent), we have that $Y_1 = \operatorname{Supp}\ker\beta$ and $Y_2 = \operatorname{Supp}\operatorname{coker}\beta$ are closed by exercise II.5.6. I claim that we can upgrade these Y_i to closed subschemes: since annihilators commute with localization for finitely-generated modules, we can patch the closed subschemes given by $V(\operatorname{Ann}(\ker\beta)(\operatorname{Spec} A))$ for $\operatorname{Spec} A \subset Y$ an affine open subshcheme to a closed subscheme of Y. Since $f_*\mathcal{G} \to \mathcal{I}$ is an isomorphism at the generic point, Y_1 and Y_2 are proper closed subschemes of Y, and by our construction of Y_i, we have that $\ker\beta = (i_1)_* i_1^* \ker\beta$ and $\operatorname{coker}\beta = (i_2)_* i_2^* \operatorname{coker}\beta$. Applying exercise III.4.1, it's enough to show that Y_1, Y_2 are affine in order to show that Y is affine. Now we can conclude that Y is affine by noetherian induction: repeating our argument up to this point, we eventually reduce to the case in which Y is a point - but a one-point scheme is affine by the definition of a scheme.

Exercise III.4.3. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and let $U = X \setminus \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine - cf. (I, Ex. 3.6).)

Solution. Cover U by $D(x)$ and $D(y)$. Then the Čech complex corresponding to this cover is

$$k[x, x^{-1}, y] \times k[x, y, y^{-1}] \to k[x, x^{-1}, y, y^{-1}],$$

and the cokernel at the second step computes $H^1(U, \mathcal{O}_U)$ by theorem III.4.5. As the image of the differential is the sum of the images of $k[x, x^{-1}, y] \to k[x, x^{-1}, y, y^{-1}]$ and $k[x, y, y^{-1}] \to k[x, x^{-1}, y, y^{-1}]$, it's enough to observe that the image of the first map is $\operatorname{Span}\{x^i y^j \mid i \geq 0\}$ and the image of the second map is $\operatorname{Span}\{x^i y^j \mid j \geq 0\}$. So the image of the map in our Čech complex is $\{x^i y^j \mid i, j \not< 0\}$ and we're done after computing the quotient.

Exercise III.4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathcal{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

a. Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{B} = (V_j)_{j\in J}$, together with a map $\lambda : J \to I$ of index sets, such that for each $j \in J$, $V_j \subset U_{\lambda(j)}$. If $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathcal{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathcal{F}) \to \check{H}^i(\mathfrak{B}, \mathcal{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathcal{F}).$$

b. For any abelian sheaf $\mathcal{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$$

are compatible with the refinement maps above.

c. Now prove the following theorem. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$$

is an isomorphism. [*Hint*: Embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$, and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so that we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0.$$

Define a complex $D^\bullet(\mathfrak{U})$ by

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to D^\bullet(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes $D^\bullet(\mathfrak{U}) \to C^\bullet(\mathfrak{U}, \mathcal{R})$ and see what happens under refinement.]

Solution.

a. If $V_j \subset U_{\lambda(j)}$, then for any $J' \subset J$ we have $\bigcap_{j\in J'} V_j \subset \bigcap_{j\in J'} U_{\lambda(j)}$, and this respects the differential because the differential is a sum of restriction maps. So we get an induced map of chain complexes $C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{B}, \mathcal{F})$, and considering the induced map on homology we get the desired map on cohomology.

b. We construct a map of complexes $\alpha : \mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}) \to \mathcal{C}^\bullet(\mathfrak{B},\mathcal{F})$ by declaring the map on any open set $V \subset U$ to be the map $C^\bullet(\mathfrak{U}\cap V,\mathcal{F}) \to C^\bullet(\mathfrak{B}\cap V,\mathcal{F})$ from part (a). Let $0\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution; as $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F})$ and $\mathcal{C}^\bullet(\mathfrak{B},\mathcal{F})$ are both resolutions of $\mathcal{F}|_U$ and α induces the identity on $\mathcal{F}|_U$, we obsevre that the composite of α with the map $\mathcal{C}^\bullet(\mathfrak{B},\mathcal{F}) \to \mathcal{I}^\bullet$ inducing the identity on $\mathcal{F}|_U$ from lemma III.4.4 gives a map $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}) \to \mathcal{I}^\bullet$ which induces the identity on $\mathcal{F}|_U$. But then this map must be homotopic to the map $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}) \to \mathcal{I}^\bullet$ of lemma III.4.4, so the composite $\check{H}^i(\mathfrak{U},\mathcal{F}) \to \check{H}^i(\mathfrak{B},\mathcal{F}) \to H^i(U,\mathcal{F})$ is equal to $\check{H}^i(\mathfrak{U},\mathcal{F}) \to H^i(U,\mathcal{F})$, and we've proven the claim.

c. Following the hint, embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$ and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$ be the quotient. Then we have an exact sequence
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$$
and a long exact sequence in homology
$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{R}) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) = 0$$
where the final entry is zero since $\mathcal{G}$ is flasque. Define $D^\bullet(\mathfrak{U})$ to fit in the exact sequence
$$0 \to C^\bullet(\mathfrak{U},\mathcal{F}) \to C^\bullet(\mathfrak{U},\mathcal{G}) \to D^\bullet(\mathfrak{U}) \to 0.$$
Taking homology gives a long exact sequence
$$0 \to \check{H}^0(\mathfrak{U},\mathcal{F}) \to \check{H}^0(\mathfrak{U},\mathcal{G}) \to H^0(D^\bullet(\mathfrak{U})) \to \check{H}^1(\mathfrak{U},\mathcal{F}) \to \check{H}^1(\mathfrak{U},\mathcal{G}) = 0,$$
where the final entry is zero by an application of proposition III.4.3. Noting $H^0(X,\mathcal{F}) \cong \check{H}^0(\mathfrak{U},\mathcal{F})$ and similarly for $\mathcal{G}$ while $H^0(D^\bullet(\mathfrak{U}))$ maps to $H^0(X,\mathcal{R})$, we obtain the following diagram where the first two maps are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathfrak{U},\mathcal{G}) \arrow[r] \arrow[d] & H^0(D^\bullet(\mathfrak{U})) \arrow[r] \arrow[d] & \check{H}^1(\mathfrak{U},\mathcal{F}) \arrow[r] & 0 \\
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

This induces a map $\alpha : \check{H}^1(\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F})$. By the 5-lemma, we can show that α is an isomorphism by showing that $\varinjlim \check{H}^0(D^\bullet(\mathfrak{U})) \to H^0(X,\mathcal{R})$ is an isomorphism.

This is straightforward: an element of $\check{H}^0(D^\bullet(\mathfrak{U}))$ is a collection $\{(s_i \in \mathcal{G}(U_i)/\mathcal{F}(U_i)) \mid U_i \in \mathfrak{U}\}$ so that the sections agree on overlaps, which is the exact same description as an element of the quotient sheaf once we take the limit along all open coverings.

Exercise III.4.5. For any ringed space $(X,\mathcal{O}_X)$, let $\operatorname{Pic} X$ denote the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X,\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U,\mathcal{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathcal{L}$ on X, cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i\cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U},\mathcal{O}_X^*)$. Now use (Ex. 4.4).]

Solution. Follow the hint: given an invertible sheaf $\mathcal{L}$ on X, we can cover X by opens U_i where $\mathcal{L}|_{U_i}$ is free for each i and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$; then $\varphi_i^{-1} \circ \varphi_j$ is an automorphism of $\mathcal{O}_{U_i \cap U_j}$, or a section of $\mathcal{O}_X^*$ over $U_i \cap U_j$. These automorphisms satisfy the cocycle condition, which means they define an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. It is not difficult to see that if $\mathcal{L} \cong \mathcal{M}$, then the transition automorphisms differ by the isomorphism $\mathcal{L}|_{U_i} \cong \mathcal{M}|_{U_i}$, which is a coboundary, and given an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ we can construct an invertible sheaf on X trivialized on $\mathfrak{U}$ which maps to our element. So $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ is precisely the set of invertible sheaves on X up to isomorphism which can be trivialized on $\mathfrak{U}$. Applying exercise III.4.4, we see the result.

Exercise III.4.6. Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively $\mathcal{O}_{X_0}$); the map $\mathcal{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X, \mathcal{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathcal{I}) \to \ldots.$$

Solution. We can check the exact sequence on stalks, where the claim reduces to showing that for a local ring A with ideal $I \subset A$ such that $I^2 = 0$, then $0 \to I \xrightarrow{\varphi} A^* \xrightarrow{\psi} (A/I)^* \to 0$ is also exact. Now we have some checking to do: φ is well-defined, since $\varphi(a - a) = \varphi(a) \cdot \varphi(-a) = 1 - a^2 = 0$ since $I^2 = 0$; it's injective, since if $1 + a = 1$, then $a = 0$; exactness in the middle is clear; and the latter map is surjective by the argument about lifting units along nilpotent reductions from exercise II.9.6 applied to $A = A/I^2 \to A/I$. Taking cohomology, we obtain the requested exact sequence.

Exercise III.4.7. Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation $f(x_0, x_1, x_2)$ of degree d. (Do not assume f is irreducible.) Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1,$$
$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

Solution. We adopt the convention x_{ij} for x_i/x_j in order to ease notation. Since $(1, 0, 0)$ is the only point not in $D(x_1) \cup D(x_2) \subset \mathbb{P}^2_k$, it's clear that $X \cap D(x_1)$ and $X \cap D(x_2)$ cover X as $(1, 0, 0) \notin X$, and that both $X \cap D(x_1)$ and $X \cap D(x_2)$ are affine. Further, $D(x_1) \cap D(x_2) = D(x_1 x_2)$ is affine, and we see that the Čech complex for this cover is

$$k[x_{02}, x_{12}]/(f(x_{02}, x_{12}, 1)) \oplus k[x_{01}, x_{21}]/(f(x_{01}, 1, x_{21})) \to k[x_{02}, x_{12}, x_{21}]/(f(x_{02}, x_{12}, 1)),$$

with the differential defined on generators by $(x_{02}, 0) \mapsto x_{02}$, $(x_{12}, 0) \mapsto x_{12}$, $(0, x_{01}) \mapsto -x_{02}x_{21}$, and $(0, x_{21}) \mapsto -x_{21}$. We observe that $(1, 0, 0) \notin V(f)$ implies that f is (up to multiplication by a unit) monic as a polynomial in x_0, so we can write elements of $k[x_{02}, x_{12}]/(f(x_{02}, x_{12}, 1))$ uniquely as $\sum_{i=0}^{d-1} x_{02}^i p_i(x_{12})$; similarly, we can write elements of $k[x_{01}, x_{21}]/(f(x_{01}, 1, x_{21}))$ uniquely as $\sum_{i=0}^{d-1} x_{01}^i q_i(x_{21})$ and elements of $k[x_{02}, x_{21}, x_{12}]/(f(x_{02}, x_{12}, 1))$ uniquely as $\sum_{i=0}^{d-1} x_{02}^i r_i(x_{12}, x_{21})$.

Now we compute. Starting with

$$\left(\sum_{i=0}^{d-1} x_{02}^i p_i(x_{12}), \sum_{i=0}^{d-1} x_{01} q_i(x_{21})\right) \mapsto \sum_{i=0}^{d-1} x_{02}(p_i(x_{12}) - x_{12}^{-i} q_i(x_{12}^{-1}))$$

we see that in order to determine H^0, it's enough to determine when $p_i(x_{12}) - x_{12}^{-i} q_i(x_{12}^{-1}) = 0$. For $i > 0$, this means that $q_i = 0$ and $p_i = 0$, while for $i = 0$ we get that $p_i = q_i$ is a constant, and therefore H^0 is one-dimensional. To determine H^1, it's enough to observe that the monomials $x_{02}^i x_{12}^j$ are in the image of the differential exactly when $j \geq 0$ or $j \leq -i$. Therefore the cokernel is isomorphic to the span of $x_{02}^i x_{12}^j$ for $0 \leq i < d$ and $-i < j < 0$, which is $\frac{1}{2}(d-1)(d-2)$ dimensional by a counting argument.

Exercise III.4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\operatorname{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\operatorname{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\operatorname{cd}(X) \leq \dim X$.

a. In the definition of $\operatorname{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

b. If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

c. Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\operatorname{cd}(X) \leq r$.

d. (*) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude (independently of (2.7)) that $\operatorname{cd}(X) \leq \dim X$.

e. Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbb{P}_k^n$. Show that $\operatorname{cd}(X \setminus Y) \leq r - 1$.

Solution. a. Exercise II.5.15 shows that every quasi-coherent sheaf is the direct limit of its coherent subsheaves, while proposition III.2.9 says that direct limits commute with cohomology on noetherian schemes. Therefore if $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and all coherent sheaves $\mathcal{F}$, we have that $H^i(X, \mathcal{G}) = 0$ for all $i > n$ and all quasi-coherent sheaves $\mathcal{G}$, so it suffices to consider coherent sheaves.

b. By corollary II.5.18, any coherent sheaf on a projective scheme over a noetherian ring is a quotient of a finite direct sum of locally free sheaves. If $i : X \hookrightarrow P$ is an open immersion of X in to a projective scheme over k, then for any coherent sheaf $\mathcal{F}$ on X, there is a coherent sheaf $\mathcal{F}'$ on P with $\mathcal{F}'|_X \cong \mathcal{F}$ by exercise II.5.15, and restricting to X we see that the result of corollary II.5.18 holds for any coherent sheaf on a quasi-projective scheme over a noetherian ring.

From here, construct an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is a locally free coherent sheaf surjecting on to $\mathcal{F}$ by the above paragraph. Then, examining the long exact sequence on homology, we see that if $H^i(\mathcal{E}) = 0$ for all $i > n$, then $H^i(X, \mathcal{F}) \cong H^{i+1}(X, \mathcal{K})$ for all $i > n$. Repeating this process with $\mathcal{K}$ in the role of $\mathcal{F}$, we can see that for any $j > i$, there is a coherent sheaf $\mathcal{K}_j$ with $H^i(X, \mathcal{F}) \cong H^j(X, \mathcal{K}_j)$. Since X is noetherian, it is finite-dimensional, so picking $j > \dim X$ we see that $H^j(X, \mathcal{K}_j) = 0$ by Grothendieck vanishing. Therefore if there exists an n so that $H^i(X, \mathcal{E}) = 0$ for all $i > n$ and all locally free coherent sheaves $\mathcal{E}$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and all coherent sheaves $\mathcal{F}$.

c. The Čech complex for a covering of $r+1$ sets is zero in degrees $\geq r+1$. Therefore $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > r$, so $H^i(X, \mathcal{F}) = 0$ for $i > r$ by theorem III.4.5.

d. We first solve the problem in the case that X is projective and k is algebraically closed. Suppose $X \subset \mathbb{P}^n_k$ is projective of dimension r. We observe that there's a linear subspace L of dimension $n-r-1$ which is disjoint from X as follows: while $r < n$, we can project from some point not in X on to a hyperplane, and eventually we project the r-dimensional subvariety $\pi(X) \subset \mathbb{P}^{r+1}_k$ on to $\mathbb{P}^r_k$ from some point P. Taking the preimage of P in $\mathbb{P}^n_k$, we see that the preimage L is of dimension $n-r-1$ and does not intersect X by construction. Now L is cut out by $r+1$ linear equations $\{l_0, \cdots, l_r\}$, which means $\mathbb{P}^n_k \setminus L$ is covered by the $r+1$ affine opens $D(l_i)$, so $D(l_i) \cap X$ are $r+1$ affine opens inside X which cover X.

Now we expand to the case where X is quasi-projective but k is still algebraically closed. Let $X \hookrightarrow P$ be an open immersion where P is projective over k, and let Z be the closed complement with the reduced induced subscheme structure. Let $\pi : Bl_Z P \to P$ be the blowup of P along Z, and let $E = \pi^{-1}(Z)$ be the exceptional divisor (we know that this is pure codimension one because its sheaf of ideals is invertible, per proposition II.7.13). Repeating the strategy of the first paragraph with $Bl_Z P \subset \mathbb{P}^N_k$ in the role of X, we find that there's a projection ρ from a linear subspace L of dimension $N-r-1$ to $\mathbb{P}^{\dim Bl_Z P}$, and we observe that $\rho(E)$ is can be cut out (set-theoretically) by one equation since it's a hypersurface. Letting f be an equation cutting out $\rho(E)$ and $\{l_0, \cdots, l_r\}$ a choice of linear equations cutting out L, we see that the affine open sets $D(l_i \cdot f)$ cover $X \subset Bl_Z P$, and we've found our affine open cover consisting of $r+1$ sets.

Finally, to expand to the case when k is not algebraically closed, run the proof above for $X_{\overline{k}}$: we get $r+1$ homogeneous polynomials f_i in $\overline{k}[x_0, \cdots, x_n]$ so that $D(f_i) \cap X_{\overline{k}}$ cover $X_{\overline{k}}$. First, I claim that we can arrange it so that all the coefficients of f_i are in the separable closure of k: if $k \subset \overline{k}$ is a separable extension, we're done; else we can decompose it as $k \subset k_{sep} \subset \overline{k}$, where the final extension is purely inseparable. Since every element of a purely inseparable extension has minimal polynomial $x^{p^n} - \alpha$, $f_i^{p^n}$ has coefficients in k_{sep} for some n, and $D(f_i^{p^n}) = D(f_i)$. Now, since X is stable under the Galois action, $D(f_i) \cap X_{\overline{k}} = D(\sigma(f_i)) \cap X_{\overline{k}}$ for $\sigma \in Gal(k_{sep}/k)$. Let f_i' be the product of the Galois conjugates of f_i. We have that $D(f_i') \cap X_{\overline{k}} = D(f_i) \cap X_{\overline{k}}$, and that f_i' is a polynomial in $k[x_0, \cdots, x_N] \subset \overline{k}[x_0, \cdots, x_N]$ since it's stable under the Galois action. So $D(f_i') \subset \mathbb{P}^N_k$ cover X since the $D(f_i') \subset \mathbb{P}^N_{\overline{k}}$ do, and we've finished.

e. Write $I_Y = (f_1, \cdots, f_r)$. Then $X \setminus Y = \bigcup D(f_i)$, each $D(f_i)$ is affine, and $X \setminus Y$ is noetherian and separated since it's an open subscheme of $\mathbb{P}^n_k$. Therefore we have the desired result by part (c).

Exercise III.4.9. Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbb{P}^4_k$ is also not a set-theoretic complete intersection. [*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X \setminus Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X \setminus P, \mathcal{O}_X) \neq 0$.]

Solution. The same proof of exercise III.4.8(e) shows that the complement of a set-theoretic complete intersection of codimension r in $\mathbb{A}^n$ has cohomological dimension at most $r-1$. Since Y is of codimension 2, it suffices to show that $H^2(X \setminus Y, \mathcal{O}_X) \neq 0$ in order to show that Y is not a set-theoretic complete intersection.

Recall the long exact sequence of exercise III.2.3:

$$\cdots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(X \setminus Z, \mathcal{F}) \to \cdots .$$

When X is affine and $\mathcal{F}$ is quasi-coherent, $H^i(X, \mathcal{F})$ vanishes for all $i > 0$ and we see that $H^i(X \setminus Z, \mathcal{F}) \cong H^{i+1}_Z(X, \mathcal{F})$ for $i > 0$. As the Y_i are complete intersections of dimension 2, our first paragraph shows that $H^i(X \setminus Y_i, \mathcal{F}) = 0$ for any $i > 1$ and quasi-coherent sheaf $\mathcal{F}$. This lets us rewrite the relevant portion of the Mayer-Vietoris exact sequence of exercise III.2.4 for our problem as

$$\cdots \to 0 \to H^2(X \setminus Y, \mathcal{O}_X) \to H^3(X \setminus P, \mathcal{O}_X) \to 0 \to \cdots ,$$

so $H^2(X \setminus Y, \mathcal{O}_X) \cong H^3(X \setminus P, \mathcal{O}_X)$.

To compute $H^3(X \setminus P, \mathcal{O}_X)$, cover $X \setminus P$ by $D(x_1)$, $D(x_2)$, $D(x_3)$, and $D(x_4)$ which gives that $H^3(X \setminus P, \mathcal{O}_X) \cong \check{H}^3(X \setminus P, \mathcal{O}_X)$ is isomorphic to the cokernel of

$$\mathcal{O}_X(D(x_1x_2x_3)) \oplus \mathcal{O}_X(D(x_1x_2x_4)) \oplus \mathcal{O}_X(D(x_1x_3x_4)) \oplus \mathcal{O}_X(D(x_2x_3x_4)) \to k[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}].$$

The same logic as our proof in exercise III.4.3 shows that the cokernel of this map is isomorphic to $\operatorname{Span}\{x_1^a x_2^b x_3^c x_4^d \mid a, b, c, d < 0\}$, so $H^3(X \setminus P, \mathcal{O}_X) \neq 0$ and we're finished.

Exercise III.4.10. (*) Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II, §8). [*Hint*: Use (II, Ex. 8.6) and (4.5).]

Solution. We imitate exercise III.4.5 to show that $\mathcal{O}_{X'}$ is determined by cocycles. Let X' be an infinitesimal extension of X by $\mathcal{F}$, and let $\mathfrak{U} = \{U_i = \operatorname{Spec} A_i\}$ be an affine open covering of X. By exercise II.8.7, we get an isomorphism $\varphi_i : \mathcal{O}_{X'}|_{U_i} \to \mathcal{O}_X|_{U_i} \oplus \mathcal{F}|_{U_i}$, giving an automorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{X'}|_{U_i \cap U_j} \cong \mathcal{O}_X|_{U_i \cap U_j} \oplus \mathcal{F}|_{U_i \cap U_j}$. By the same logic of as in exercise III.4.5, these cocycles suffice to determine $\mathcal{O}_{X'}$.

Our goal is now to interpret these cocycles as elements of $\check{H}^1(\mathfrak{U}, \mathcal{T} \otimes \mathcal{F})$. Since X is separated, $U_i \cap U_j$ is again affine and our infinitesimal extension is trivial on $U_i \cap U_j$. So our goal is to understand the automorphisms of $A \oplus F$ with the ring structure $(a, f)(a', f') = (aa', a'f + af')$ are. By the results of exercise II.8.6, these are in bijection with elements of $\operatorname{Hom}_A(\Omega_{A/k}, F)$. As

$$\operatorname{Hom}_A(\Omega_{A/k}, F) \cong \operatorname{Hom}_{\mathcal{O}_{U_i \cap U_j}}(\Omega_{X/k}|_{U_i \cap U_j}, \mathcal{F}|_{U_i \cap U_j})$$

by exercise II.5.3, we can apply the fact that $\Omega_{X/k} = \mathcal{T}_X^\vee$ to get that

$$\operatorname{Hom}_{\mathcal{O}_{U_i \cap U_j}}(\Omega_{X/k}|_{U_i \cap U_j}, \mathcal{F}|_{U_i \cap U_j}) \cong \operatorname{Hom}_{\mathcal{O}_{U_i \cap U_j}}(\mathcal{O}_{U_i \cap U_j}, (\mathcal{T}_X \otimes \mathcal{F})|_{U_i \cap U_j}) \cong \Gamma(U_i \cap U_j, \mathcal{T}_X \otimes \mathcal{F})$$

by exercise II.5.1. So our cocycles are Čech 1-cocycles for $\mathcal{T}_X \otimes \mathcal{F}$, and therefore infinitesimal extensions are parametrized by the cohomology group $\check{H}^1(\mathfrak{U}, \mathcal{T}_X \otimes \mathcal{F})$. By theorem III.4.5, we have that $\check{H}^1(\mathfrak{U}, \mathcal{T}_X \otimes \mathcal{F}) \cong H^1(X, \mathcal{T}_X \otimes \mathcal{F})$ and the result is proven.

Exercise III.4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathcal{F}|_V) = 0$. Then prove that for all $p \geq 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

Solution. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ in the category of $\mathcal{O}_X$-modules. Let $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ be the Čech resolution. We may construct the following double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathcal{I}^2 \arrow[r] \arrow[u] & \mathcal{C}^0(\mathfrak{U},\mathcal{I}^2) \arrow[r] \arrow[u] & \mathcal{C}^1(\mathfrak{U},\mathcal{I}^2) \arrow[r] \arrow[u] & \mathcal{C}^2(\mathfrak{U},\mathcal{I}^2) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathcal{I}^1 \arrow[r] \arrow[u] & \mathcal{C}^0(\mathfrak{U},\mathcal{I}^1) \arrow[r] \arrow[u] & \mathcal{C}^1(\mathfrak{U},\mathcal{I}^1) \arrow[r] \arrow[u] & \mathcal{C}^2(\mathfrak{U},\mathcal{I}^1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r] \arrow[u] & \mathcal{C}^0(\mathfrak{U},\mathcal{I}^0) \arrow[r] \arrow[u] & \mathcal{C}^1(\mathfrak{U},\mathcal{I}^0) \arrow[r] \arrow[u] & \mathcal{C}^2(\mathfrak{U},\mathcal{I}^0) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{C}^0(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{C}^1(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{C}^2(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Applying $\Gamma(X, -)$ to everything and deleting the bottom row and left column, we see that the rows are exact except at the first entry by an application of proposition III.4.3 since they're the Čech complexes associated to injective objects. We see that the columns are exact except at the first entry since they compute products of $H^\bullet(V, \mathcal{F}|_V)$ for V a finite intersection of sets in $\mathfrak{U}$, and all such cohomology groups are assumed to be zero. By a classical result in homological algebra (see for instance Osborne's *Basic Homological Algebra*, proposition 3.9), this implies that the cohomology groups of $\Gamma(X, \mathcal{I}^\bullet)$ and $\Gamma(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}))$ are isomorphic.

This recovers theorem III.4.5 easily: in a separated scheme, the intersection of open affine schemes is again affine (exercise II.4.3), and by Serre vanishing (theorem III.3.7), $H^i(U, \mathcal{F}|_U) = 0$ for U affine, $\mathcal{F}$ quasi-coherent, and $i > 0$.

III.5 The Cohomology of Projective Space

Computing the cohomology of projective space is the first big calculation you can do - it's a little like computing $\pi_1(S^1)$ in your first algebraic topology class. Just like in topology, there are some decent applications you can get to just from knowing this. Let's dig in!

Exercise III.5.1. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Solution. By theorem III.5.2, $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space for any coherent $\mathcal{F}$ on X, and by theorem III.3.7, $H^i(X, \mathcal{F}) = 0$ for all $i > \dim X$. So the long exact sequence on cohomology associated to our exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a finite length exact sequence of finite dimensional vector spaces. Since every vector space is free, we can split this exact sequence as the direct sum of shifts of the exact sequence $0 \to k^m \stackrel{id}{\to} k^m \to 0$ and therefore the alternating sum of dimensions of the vector spaces in our exact sequence is zero. But that alternating sum is exactly $\chi(\mathcal{F}') - \chi(\mathcal{F}) + \chi(\mathcal{F}'')$, and we're done.

Exercise III.5.2.

a. Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the *Hilbert polynomial* of $\mathcal{F}$ with respect to the sheaf $\mathcal{O}_X(1)$. [*Hints*: Use induction on $\dim \operatorname{Supp} \mathcal{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathcal{R} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{Q} \to 0.]$$

b. Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathcal{F})$, considered as a graded $S = k[x_0, \cdots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathcal{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

Solution.

a. We prove the claim by induction on $\dim \operatorname{Supp} \mathcal{F}$. If $\mathcal{F}$ is a coherent sheaf on a projective scheme X over k with zero-dimensional support, then $\mathcal{F}(X) \cong \prod_{x \in \operatorname{Supp} \mathcal{F}} \mathcal{F}_x$ as $\operatorname{Supp} \mathcal{F}$ is discrete, and $\mathcal{F}_x \cong \mathcal{F}(1)_x$ since $\mathcal{O}_X(1)$ is locally free. So $\mathcal{F}(X) \cong \mathcal{F}(1)(X)$, and therefore the Hilbert polynomial of a coherent sheaf with zero-dimensional support is constant.

Now suppose we know the result for coherent sheaves with support of dimension n or less, and let $\mathcal{F}$ be a coherent sheaf with support of dimension $n+1$. Embed $X \hookrightarrow \mathbb{P}^n_k$ via $\mathcal{O}_X(1)$, and pick a hyperplane H cut out by h which does not contain any $n+1$ dimensional component of $\operatorname{Supp}\mathcal{F}$. There's a morphism $\mathcal{O}_X(-1) \to \mathcal{O}_X$ given by multiplication by h, and tensoring with $\mathcal{F}$ we get a morphism $\mathcal{F}(-1) \stackrel{\cdot h}{\to} \mathcal{F}$. As $\mathcal{F}(-1) \to \mathcal{F}$ is an isomorphism on $X \setminus H$, we see that $\mathcal{R} = \ker(\mathcal{F}(-1) \to \mathcal{F})$ and $\mathcal{Q} = \operatorname{coker}(\mathcal{F}(-1) \to \mathcal{F})$ have support contained in $\operatorname{Supp}\mathcal{F} \cap H$, which is of dimension at most n. Since $\mathcal{O}_X(1)$ is a line bundle, the functor $- \otimes \mathcal{O}_X(1)$ is exact, and so we get a family of exact sequences

$$0 \to \mathcal{R}(r) \to \mathcal{F}(r-1) \to \mathcal{F}(r) \to \mathcal{Q}(r) \to 0.$$

We may split this exact sequence to obtain the two exact sequences

$$0 \to \mathcal{R}(r) \to \mathcal{F}(r-1) \to \mathcal{I}(r) \to 0 \text{ and } 0 \to \mathcal{I}(r) \to \mathcal{F}(r) \to \mathcal{Q}(r) \to 0.$$

By exercise III.5.1, we have that $\chi(\mathcal{F}(r-1)) = \chi(\mathcal{R}(r)) + \chi(\mathcal{I}(r))$ and $\chi(\mathcal{F}(r)) = \chi(\mathcal{I}(r)) + \chi(\mathcal{Q}(r))$, or $\chi(\mathcal{F}(r)) - \chi(\mathcal{F}(r-1)) = \chi(\mathcal{Q}(r)) - \chi(\mathcal{R}(r))$. As $\chi(\mathcal{Q}(r))$ and $\chi(\mathcal{R}(r))$ are numerical polynomials by our inductive assumption, we have that $\chi(\mathcal{F}(r))$ is a numerical polynomial by proposition I.7.3.

b. By theorem III.5.2(b), we have that for $n \gg 0$, $H^i(X, \mathcal{F}(n)) = 0$ for all $i > 0$. Thus $\chi(\mathcal{F}(n)) = \dim_k H^0(X, \mathcal{F}(n)) = \dim_k M_n$, so for $n \gg 0$ this agrees with the definition of the Hilbert polynomial from chapter I and we're finished.

Exercise III.5.3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

Note it depends only on X, not any projective embedding.

a. If X is integral, and k is algebraically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

In particular, if X is a curve, we have

$$p_a = \dim_k H^1(X, \mathcal{O}_X).$$

[*Hint*: Use (I, 3.4).]

b. If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

c. If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of dergree $d \geq 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used geometric genus.)

Solution. a. We proved the claim that an integral proper scheme over an algebraically closed field k has global sections k in problem II.6.9. (Alternatively, the same fact can be deduced via proposition II.2.6 and theorem I.3.4.) By theorem III.2.7, $H^i(X, \mathcal{O}_X) = 0$ for $i > r$. Hence

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1) = (-1)^r \sum_{i=1}^{r} (-1)^i \dim_k H^i(X, \mathcal{O}_X)$$

and substituting $i = r - i$, we have that

$$p_a(X) = (-1)^r \sum_{i=0}^{r-1} (-1)^{r-i} \dim_k H^{r-i}(X, \mathcal{O}_X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

b. The definition from exercise I.7.2 is $p_a(Y) = (-1)^r(P_Y(0) - 1)$, where P_Y is the Hilbert polynomial of Y. From exercise III.5.2(b), we have that $P_Y(0) = \chi(\mathcal{O}_Y)$, so $p_a(Y) = (-1)^r(\chi(\mathcal{O}_Y) - 1)$ and we're done.

c. The quantification here is a little messy - Hartshorne asks us to show that among nonsingular projective curves over a fixed algebraically closed field k, two birational curves have the same arithmetic genus. By the proof of theorem II.8.19, if $f : X \dashrightarrow Y$ is a birational equivalence of two nonsingular projective varieties over a field and $V \subset X$ is the maximal domain of definition of f, then $X \setminus V$ has codimension at least two. But this implies that if X is a curve, then f is an honest morphism, and further by exercise II.4.2 that f is an isomorphism (let $g : Y \to X$ be birational inverse which also extends to a morphism, look at fg and gf: they're morphisms from reduced schemes to separated schemes which agree with the identity on an open dense subset, so they must be the identity, and thus $g = f^{-1}$). Since $\chi(\mathcal{O}_X)$ is clearly an isomorphism invariant, we have that p_a is a birational invariant among nonsingular projective curves.

The nonrationality of a nonsingular plane curve of degree $d \geq 3$ follows from the computation of the arithmetic genus as $\frac{1}{2}(d-1)(d-2)$ by the degree-genus formula (exercise I.7.2(b)).

Exercise III.5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group of a noetherian scheme X.

a. Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

b. Now let $X = \mathbb{P}^r_k$. For each $i = 0, 1, \cdots, r$ let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) \mid i = 0, \cdots, r\}$, and
(2) the map $P : K(X) \to \mathbb{Q}[z]$ is injective.

[*Hint*: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

Solution.

a. By exercise III.5.1, the Euler characteristic is additive over exact sequences, which implies the Hilbert polynomial is additive over exact sequences as well. This implies that the morphism from the free abelian group of coherent sheaves on X to $\mathbb{Q}[z]$ given by sending $\mathcal{F} \mapsto P(\mathcal{F})$ factors (uniquely) through the quotient by the group generated by expressions of the form $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, which is exactly the Grothendieck group.

b. We first observe that $P(\mathcal{O}_{L_i})$ is equal to the Hilbert polynomial of $\mathbb{P}^i_k$, so $P(\mathcal{O}_{L_i})$ is a nonzero polynomial of degree i. Therefore if (1) holds, (2) will hold since there are no nontrivial $\mathbb{Q}$-linear dependence relations among a set of nonzero polynomials in $\mathbb{Q}[z]$ with distinct different degrees.

We will now show that (1) holds by induction. When $r = 0$, $\mathbb{P}^0_k = \operatorname{Spec} k$ and coherent sheaves are precisely vector spaces. In this case, $K(X) \cong \mathbb{Z}$, the free abelian group on $\mathcal{O}_X$. Now suppose we've shown the assertion for r: by considering the closed immersion $i : \mathbb{P}^r_k \to \mathbb{P}^{r+1}_k$ with image $V(x_{r+1})$, via exercise II.6.10(c) we get an exact sequence

$$K(\mathbb{P}^r_k) \to K(\mathbb{P}^{r+1}_k) \to K(\mathbb{A}^{r+1}_k) \to 0.$$

First I claim that $K(\mathbb{P}^r_k) \to K(\mathbb{P}^{r+1}_k)$ by $\gamma(\mathcal{F}) \mapsto \gamma(i_*\mathcal{F})$ is an injection. Let $s \in K(\mathbb{P}^r_k)$ be in the kernel: then $P(i_*s) = 0$. But as a result of exercise III.4.1, we have that $H^p(\mathbb{P}^r_k, \mathcal{F}) = H^p(\mathbb{P}^{r+1}_k, i_*\mathcal{F})$, so $P(i_*s) = P(s)$, and by our inductive hypothesis, we see that $s = 0$.

Next, I claim that $K(\mathbb{A}^{r+1}_k) \cong \mathbb{Z}$: by splitting a long exact sequence in to short exact sequences, it's enough to show that any coherent sheaf on $\mathbb{A}^{r+1}_k$ admits a free resolution of finite length. Via the equivalence between coherent sheaves on $\mathbb{A}^{r+1}_k$ and finitely-generated modules over $k[x_1, \cdots, x_{r+1}]$, it suffices to show that any finitely generated modules over a polynomial ring over a field has a finite length free resolution. But this is exactly the content of Hilbert's syzygy theorem.

Since $\mathbb{Z}$ is free, the exact sequence

$$0 \to K(\mathbb{P}^r_k) \to K(\mathbb{P}^{r+1}_k) \to K(\mathbb{A}^{r+1}_k) \to 0$$

splits, and $K(\mathbb{P}^{r+1}_k)$ is isomorphic to a free abelian group on the classes of the linear subspaces L_i.

Exercise III.5.5. Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection (II, Ex. 8.4). Then:

a. for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

b. Y is connected;

c. $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

d. $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

Solution. Follow the hint: we induct on the codimension of Y in X. The base case is $Y = X$, where the map of (a) is the identity, (b) is trivial since $\mathbb{P}^n_k$ is irreducible, (c) follows by theorem III.5.1, and (d) is a consequence of parts (b) and (c).

Now suppose we've proven the result for complete intersections of codimension $r < n-1$. Let Z be a complete intersection of codimension r and f a homogeneous polynomial of degree d so that $Y = Z \cap V(f)$ is a complete intersection. This means there's an exact sequence

$$0 \to \mathcal{O}_Z(-d) \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$$

which after twisting by n has as it's associated long exact sequence on cohomology

$$0 \to H^0(Z, \mathcal{O}_Z(n-d)) \to H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n)) \to \cdots.$$

Since $H^1(Z, \mathcal{O}_Z(n-d)) = 0$ by assumption, $H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective and therefore the composite $H^0(X, \mathcal{O}_X(n)) \to H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is also surjective, so we've shown (a).

Setting $n = 0$, we see that the vector space $H^0(Y, \mathcal{O}_Y)$ is at most one-dimensional. As every k-scheme Y has $k \subset \Gamma(Y, \mathcal{O}_Y)$, we see this $H^0(Y, \mathcal{O}_Y)$ is in fact one-dimensional, so Y is connected and we've shown (b).

Next, we see that $H^i(Y, \mathcal{O}(n))$ lies between $H^i(X, \mathcal{O}_X(n))$ and $H^{i+1}(X, \mathcal{O}_X(n-d))$, which by assumption are both zero for all $n \in \mathbb{Z}$ and all $0 < i < q-1$. So $H^i(Y, \mathcal{O}_Y(n)) = 0$ for all $0 < i < q-1$ and all $n \in \mathbb{Z}$, proving (c).

Finally, (d) follows from (b), (c), and the definition of the arithmetic genus, so we're done.

Exercise III.5.6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbb{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z}$, so we can talk about the *type* (a, b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbb{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.

a. Use the special cases $(q,0)$ and $(0,q)$, with $q>0$, when Y is a disjoint union of q lines $\mathbb{P}^1$ in Q, to show

(1) if $|a-b|\leq 1$, then $H^1(Q,\mathcal{O}_Q(a,b))=0$;
(2) if $a,b<0$, then $H^1(Q,\mathcal{O}_Q(a,b))=0$;
(3) if $a\leq -2$, then $H^1(Q,\mathcal{O}_Q(a,0))\neq 0$.

b. Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a,b) with $a,b>0$, then Y is connected;
(2) now assume k algebraically closed. Then for any $a,b>0$, there exists an irreducible nonsingular cuvrve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).
(3) an irreducible nonsingular curve Y of type (a,b), $a,b>0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a-b|\leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type $(1,3)$, which is just the rational quartic curve (I, Ex. 3.18).

c. If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y)=ab-a-b+1$. [*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbb{P}^1$. See (V, 1.5.2) for another method.]

Solution.

a. Consider the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^3}(-2)\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_Q$$

where the first map is multiplication by the equation defining Q. Twisting by a and taking the long exact sequence in cohomology, we get

$$\cdots H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(a))\to H^1(Q,\mathcal{O}_Q(a))\to H^2(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(a-2))\to\cdots$$

and the outer two terms are zero by theorem III.5.1, so $H^1(Q,\mathcal{O}_Q(a))=H^1(Q,\mathcal{O}_Q(a,a))=0$. Now let $Y\subset Q$ be a copy of $\mathbb{P}^1$ of type $(0,1)$. We have the exact sequence

$$0\to\mathcal{O}_Q(0,-1)\to\mathcal{O}_Q\to\mathcal{O}_Y\to 0,$$

then twisting by $a=(a,a)$ and taking the long exact sequence on cohomology, we obtain

$$\cdots\to H^0(Q,\mathcal{O}_Q(a))\to H^0(Y,\mathcal{O}_Y(a))\to H^1(Q,\mathcal{O}_Q(a,a-1))\to H^1(Q,\mathcal{O}_Q(a))\to\cdots.$$

By our first paragraph, $H^1(Q,\mathcal{O}_Q(a))=0$, so it suffices to show that $H^0(Q,\mathcal{O}_Q(a))\to H^0(Y,\mathcal{O}_Y(a))$ is surjective in order to show that $H^1(Q,\mathcal{O}_Q(a,a-1))=0$. $H^0(Q,\mathcal{O}(a))$ is just the space of bihomogeneous polynomials in x, y, z, w of bidegree (a, a), which surjects on to the space of homogeneous polynomials on x, y of degree a by plugging in for z, w. So we've dealt with (1).

For (2), let Y be a union of $n > 1$ copies of $\mathbb{P}^1$, and consider the exact sequence

$$0 \to \mathcal{O}_Q(0, -n) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by a and taking the long exact sequence on cohomology, we get

$$\cdots \to H^0(Y, \mathcal{O}_Y(a)) \to H^1(Q, \mathcal{O}_Q(a, a-n)) \to H^1(Q, \mathcal{O}_Q(a)) \to \cdots.$$

The last term vaishes by (1), and when $a < 0$, the first term vanishes by theorem III.5.1, so the middle term must also vanish.

Finally, in the case of (3), we may argue with the same long exact sequence as in (2): we have

$$\cdots \to H^0(Q, \mathcal{O}_Q) \to H^0(Y, \mathcal{O}_Y) \to H^1(Q, \mathcal{O}_Q(-a, 0)) \to H^1(Q, \mathcal{O}_Q) \to \cdots.$$

By (1), the last term vanishes; by exercise III.5.5(b), the first term is k, and as $Y = \bigsqcup_a \mathbb{P}^1_k$, we have $H^0(Y, \mathcal{O}_Y) \cong \bigoplus_a H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^n_k}) = k^a$. So $\dim H^1(Q, \mathcal{O}_Q(-a, 0)) \geq a - 1$, which means it's nonzero.

b. Consider the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

When $a, b > 0$, we have that $H^1(Q, \mathcal{O}_Q(-a, -b)) = 0$ by (a)(2), so $H^0(Y, \mathcal{O}_Y)$ admits a surjection from $H^0(Q, \mathcal{O}_Q) = k$. Therefore $H^0(Y, \mathcal{O}_Y) = k$ and Y is connected.

By example II.7.6.2, $\mathcal{O}_Q(a, b)$ is a very ample invertible sheaf on Q when $a, b > 0$. Applying theorem II.8.18 when k is algebraically closed, we see that there exists a nonsingular hyperplane section of Q embedded by $\mathcal{O}_Q(a, b)$. But this is the same thing as a divisor of type (a, b), so the statement of (2) is proven.

To attack (3), we'll use the criteria of exercise II.5.14(d): a closed subscheme $Y \subset \mathbb{P}^r_A$ is projectively normal iff Y is normal and $H^0(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective. As the map $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \to H^0(Y, \mathcal{O}_Y(n))$ factors through $H^0(Q, \mathcal{O}_Q(n))$ and $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \to H^0(Q, \mathcal{O}_Q(n))$ is surjective by exercise III.5.5(a), it suffices to check whether $H^0(Q, \mathcal{O}_Q(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective. Considering the exact sequence

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0,$$

twisting by n, and taking the long exact sequence in cohomology, we see that if $H^1(Q, \mathcal{O}_Q(n + a, n + b)) = 0$, then $H^0(Q, \mathcal{O}_Q(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective. So by (a)(1), if $|a - b| \leq 1$, then $H^0(Q, \mathcal{O}_Q(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective.

Conversely, suppose $a = b + n$ where $n > 1$. Then after twisting the standard exact sequence

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

by b, we get the exact sequence

$$0 \to \mathcal{O}_Q(-n,0) \to \mathcal{O}_Q(b) \to \mathcal{O}_Y(b) \to 0.$$

Taking the long exact sequence on cohomology, we get

$$\cdots H^0(Q,\mathcal{O}_Q(b)) \to H^0(Y,\mathcal{O}_Y(b)) \to H^1(Q,\mathcal{O}_Q(-n,0)) \to H^1(Q,\mathcal{O}_Q(b)) \to \cdots.$$

By (a)(1), the final term vanishes, while by (a)(3), the second to last term doesn't. So $H^0(Q,\mathcal{O}_Q(b)) \to H^0(Y,\mathcal{O}_Y(b))$ cannot be surjective, and thus Y cannot be projectively normal.

c. Let $Z \cong \mathbb{P}^1$ be a divisor of type $(0,1)$ with closed immersion $i : Z \to Q$ and consider the exact sequence

$$0 \to \mathcal{O}_Q(0,-1) \to \mathcal{O}_Q \to i_*\mathcal{O}_Z \to 0.$$

Twisting by $\mathcal{O}_Q(x,y)$, we get

$$0 \to \mathcal{O}_Q(x,y-1) \to \mathcal{O}_Q(x,y) \to i_*\mathcal{O}_Z \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(x,y) \to 0.$$

As $\mathcal{O}_Q(x,y) \cong p_1^*\mathcal{O}(x) \otimes p_2^*\mathcal{O}(y)$ and tensor products commute with pullback, we can apply the projection formula (exercise II.5.1(d)) to see that

$$\mathcal{O}_Q(x,y) \otimes_{\mathcal{O}_Q} i_*\mathcal{O}_Z \cong i_*(i^*p_1^*\mathcal{O}_{\mathbb{P}^1}(x) \otimes i^*p_2^*\mathcal{O}_{\mathbb{P}^1}(y) \otimes \mathcal{O}_Z).$$

Since $p_2 \circ i$ is a constant map which factors through an open set where $\mathcal{O}_{\mathbb{P}^1}(y)$ is trivial, $i^*p_1^*\mathcal{O}_{\mathbb{P}^1}(y) \cong i^*p_1^*\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_Z$, while $p_1 \circ i$ is the identity, so $i^*p_1^*\mathcal{O}(x) \cong \mathcal{O}(x)$. Therefore

$$i_*(i^*p_1^*\mathcal{O}_{\mathbb{P}^1}(x) \otimes i^*p_2^*\mathcal{O}_{\mathbb{P}^1}(y) \otimes \mathcal{O}_Z) \cong i_*\mathcal{O}_Z(x),$$

so by the additivity of Euler characteristic across exact sequences (exercise III.5.1), we have that $\chi(\mathcal{O}_Q(x,y)) - \chi(\mathcal{O}_Q(x,y-1)) = \chi(\mathcal{O}_{\mathbb{P}^1}(x))$.

On the other hand, examining the defining exact sequence of a divisor Y of type (a,b), we see that $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q) - \chi(\mathcal{O}_Q(-a,-b))$. Writing $\chi(\mathcal{O}_Q) - \chi(\mathcal{O}_Q(-a,-b))$ as the telescoping sum

$$\sum_{i=0}^{b-1} \left(\chi(\mathcal{O}_Q(0,-i)) - \chi(\mathcal{O}(0,-i-1))\right) + \sum_{i=0}^{a-1} \left(\chi(\mathcal{O}_Q(-i,-b)) - \chi(\mathcal{O}(-i-1,-b))\right),$$

or

$$\sum_{i=0}^{b-1} \chi(\mathcal{O}_{\mathbb{P}^1}(0)) + \sum_{i=0}^{a-1} \chi(\mathcal{O}_{\mathbb{P}^1}(-b)).$$

As $\chi(\mathcal{O}_{\mathbb{P}^1}(x)) = 1 + x$, this sum is equal to $b + a(1-b) = -ab + a + b$, and plugging in to the formula for p_a, we get $p_a = (-1)(-ab + a + b - 1) = ab - a - b + 1$ as desired.

Exercise III.5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathcal{L}$ an invertible sheaf.

a. If $\mathcal{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathcal{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

b. $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{red} = \mathcal{L} \otimes \mathcal{O}_{X_{red}}$ is ample on X_{red}.

c. Suppose X is reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i for each irreducible component X_i of X.

d. Let $f : X \to Y$ be a finite surjective morphism, and let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample on Y if and only if $f^*\mathcal{L}$ is ample on X. [*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. 1. §4] for more details.]

Solution.

a. By theorem II.7.6, $\mathcal{L}$ is ample iff $\mathcal{L}^m$ is very ample for $X \to \operatorname{Spec} A$ for some $m > 0$. Let $j : X \to \mathbb{P}^n_A$ be the closed immersion for which $\mathcal{L}^m \cong j^*\mathcal{O}(1)$; then $j \circ i : Y \to \mathbb{P}^n_A$ is also a closed immersion, and $i^*j^*\mathcal{O}(1) \cong i^*\mathcal{L}^m \cong (i^*\mathcal{L})^m$ as tensor products commute with pullbacks. Again by theorem II.7.6, $i^*\mathcal{L}$ is ample as some tensor power is very ample.

b. With $Y = X_{red}$, the forward direction is an immediate application of (a).

   For the reverse direction, we will use proposition III.5.3 and a strategy similar to that of exercise III.3.1. Let $\mathcal{N}$ be the sheaf of nilpotent elements of $\mathcal{O}_X$ (aka the ideal sheaf of X_{red}), and let $i : X_{red} \to X$ be the closed immersion. As X is noetherian, $\mathcal{N}^r = 0$ for some $r > 0$, and for any coherent sheaf $\mathcal{F}$ on X we have that $\mathcal{F} \supset \mathcal{N}\mathcal{F} \supset \mathcal{N}^2\mathcal{F} \supset \cdots \supset 0$ is a finite filtration of $\mathcal{F}$. Now consider the exact sequence

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{L}^n \to \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n \to (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n \to 0.$$

   As $\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \cong \mathcal{N}^i\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{N}$, we have that $(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n \cong \mathcal{O}_X/\mathcal{N} \otimes_{\mathcal{O}_X} (\mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n)$, which is in turn isomorphic to $i^*(\mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) \cong i^*(\mathcal{N}^i\mathcal{F}) \otimes_{\mathcal{O}_{X_{red}}} (i^*\mathcal{L})^n$. By proposition III.5.3, there exists some n_i so that for $n \geq n_i$, the higher cohomology of $i^*(\mathcal{N}^i\mathcal{F}) \otimes (i^*\mathcal{L})^n$ vanishes since $i^*\mathcal{L}$ is ample. Taking the long exact sequence in cohomology, we get

$$\cdots H^p(X, \mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{L}^n) \to H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) \to H^p(X_{red}, i^*(\mathcal{N}^i\mathcal{F}) \otimes (i^*\mathcal{L})^n) \to \cdots,$$

   and by induction we see that $H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n)$ vanishes for all $p > 0$, all i, and all $n \geq \max(n_i)$, so by proposition III.5.3 $\mathcal{L}$ is ample.

c. The forward direction is again an immediate application of (a) as Y ranges over the irreducible components of X equipped with the reduced induced subscheme structure.

The reverse direction uses proposition III.5.3 and a strategy similar to that of exercise III.3.2. Let $X_1, \cdots, X_r$ be the irreducible components of X equipped with the reduced induced subscheme structure, and let $\mathcal{I}_i$ be the ideal sheaf of $X_1 \cup \cdots \cup X_i$. Then for any coherent sheaf $\mathcal{F}$ on X, we can filter $\mathcal{F}$ as

$$\mathcal{F} \supset \mathcal{I}_1\mathcal{F} \supset \cdots \supset \mathcal{I}_n\mathcal{F} = 0.$$

Now consider the exact sequence

$$0 \to \mathcal{I}_{i+1}\mathcal{F} \otimes \mathcal{L}^n \to \mathcal{I}_i\mathcal{F} \otimes \mathcal{L}^n \to (\mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}) \otimes \mathcal{L}^n \to 0.$$

Noting that $\mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}$ is a quasi-coherent sheaf on X_i, the long exact sequence on cohomology associated to our short exact sequence gives that $H^p(X, \mathcal{I}_i\mathcal{F} \otimes \mathcal{L}^n)$ fits between $H^p(X, \mathcal{I}_{i+1}\mathcal{F} \otimes \mathcal{L}^n)$ and $H^p(X_i, \otimes(\mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}) \otimes (\mathcal{L}|_{X_i})^n)$. So by proposition III.5.3, for all $p > 0$ and all n greater than some n_i, we have that $H^p(X_i, \otimes(\mathcal{I}_i\mathcal{F}/\mathcal{I}_{i+1}\mathcal{F}) \otimes (\mathcal{L}|_{X_i})^n)$ vanishes. Arguing by induction, we have that $H^p(X, \mathcal{I}_i\mathcal{F} \otimes \mathcal{L}^n)$ vanishes for all p and all $n \geq \max(n_i)$, therefore $\mathcal{L}$ is ample by proposition III.5.3.

d. First, we observe that for $\mathcal{F}$ coherent on X, then $f_*\mathcal{F}$ is coherent on Y: $f_*\mathcal{F}$ is quasi-coherent by proposition II.5.8, and locally over any affine open $\operatorname{Spec} A \subset Y$ with preimage $\operatorname{Spec} B \subset X$, we have surjections $A^n \to B$ and $B^m \to \mathcal{F}(\operatorname{Spec} B) = (f_*\mathcal{F})(\operatorname{Spec} A)$, and by noetherianity of Y, the kernel of $A^{mn} \to (f_*\mathcal{F})(\operatorname{Spec} A)$ is also finitely generated.

   Now suppose $\mathcal{L}$ is ample on Y. Then $H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) \cong H^i(Y, f_*(\mathcal{F} \otimes f^*\mathcal{L}^n))$ by exercise III.4.1. As $f_*(\mathcal{F} \otimes f^*\mathcal{L}^n) \cong f_*\mathcal{F} \otimes \mathcal{L}^n$ by the projection formula (exercise II.5.1)and $f_*\mathcal{F}$ is coherent, by proposition III.5.3 we have that $H^i(Y, f_*(\mathcal{F} \otimes f^*\mathcal{L}^n)) \cong H^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n))$ vanishes for all $i > 0$ and all $n \gg 0$. Therefore $H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n)$ does as well, and $f^*\mathcal{L}$ is ample by proposition III.5.3.

   Conversely, suppose $f^*\mathcal{L}$ is ample on X. Let $\mathcal{F}$ be a coherent sheaf on Y. By the logic of exercise III.4.2, we may reduce to the case that X and Y are integral and induct on the dimension of Y. Again by the logic of exercise III.4.2, we may find a coherent sheaf $\mathcal{G}$ on X and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^{\oplus r}$ which is an isomorphism at the generic point of Y. Let $\mathcal{Q} = \ker \beta$, $\mathcal{I} = \operatorname{Im} \beta$, and $\mathcal{R} = \operatorname{coker} \beta$: this gives us the two exact sequences

$$0 \to \mathcal{Q} \otimes \mathcal{L}^n \to f_*\mathcal{G} \otimes \mathcal{L}^n \to \mathcal{I} \otimes \mathcal{L}^n \to 0 \text{ and } 0 \to \mathcal{I} \otimes \mathcal{L}^n \to \mathcal{F}^{\oplus r} \otimes \mathcal{L}^n \to \mathcal{R} \otimes \mathcal{L}^n \to 0.$$

   The long exact sequence in cohomology for the first reads

$$\cdots \to H^p(Y, \mathcal{Q} \otimes \mathcal{L}^n) \to H^p(Y, f_*\mathcal{G} \otimes \mathcal{L}^n) \to H^p(Y, \mathcal{I} \otimes \mathcal{L}^n) \to \cdots.$$

   As $f^*\mathcal{L}$ is ample on X, $(f^*\mathcal{L})|_{f^{-1}(\operatorname{Supp} \mathcal{Q})}$ is ample on $f^{-1}(\operatorname{Supp} \mathcal{Q})$ by (a), and by induction, this implies that $\mathcal{L}|_{\operatorname{Supp} \mathcal{Q}}$ is ample on $\operatorname{Supp} \mathcal{Q}$. Since $H^p(Y, \mathcal{Q} \otimes \mathcal{L}^n) \cong H^p(\operatorname{Supp} \mathcal{Q}, \mathcal{Q} \otimes (\mathcal{L}|_{\operatorname{Supp} \mathcal{Q}})^n)$, there is some $n_{\mathcal{Q}}$ so that for all $n \geq n_{\mathcal{Q}}$, we have that $H^p(Y, \mathcal{Q} \otimes \mathcal{L}^n) = 0$.The long exact sequence in homology associated to the other short exact sequence of sheaves reads

$$\cdots \to H^p(Y, \mathcal{I} \otimes \mathcal{L}^n) \to H^p(Y, \mathcal{F}^{\oplus r} \otimes \mathcal{L}^n) \to H^p(Y, \mathcal{R} \otimes \mathcal{L}^n) \to \cdots,$$

and the same argument with $\mathcal{R}$ instead of $\mathcal{Q}$ shows that there is an integer $n_{\mathcal{R}}$ so that for all $p > 0$ and all $n \geq n_{\mathcal{R}}$, $H^p(Y, \mathcal{R} \otimes \mathcal{L}^n) = 0$.

As in the forward direction, the same argument with the projection formula and exercise III.4.1 shows that $H^p(Y, f_*\mathcal{G} \otimes \mathcal{L}^n) \cong H^p(X, \mathcal{G} \otimes f^*\mathcal{L}^n)$, so by ampleness of $f^*\mathcal{L}$, we see that this cohomology group must vanish for all $p > 0$ and all $n \geq n_{\mathcal{G}}$. Taking n_0 to be the maximum of $n_{\mathcal{Q}}$, $n_{\mathcal{G}}$, and $n_{\mathcal{R}}$, we see that for all $p > 0$ and $n \geq n_0$, the cohomology groups $H^p(Y, \mathcal{Q} \otimes \mathcal{L}^n)$, $H^p(Y, f_*\mathcal{G} \otimes \mathcal{L}^n)$, and $H^p(Y, \mathcal{R} \otimes \mathcal{L}^n)$ vanish. This implies that $H^p(Y, \mathcal{F}^{\oplus r} \otimes \mathcal{L}^n)$ vanishes for all $p > 0$ and all $n \geq n_0$, which in turn implies that $H^p(Y, \mathcal{F} \otimes \mathcal{L}^n)$ vanishes under the same conditions as cohomology and tensor products commute with direct sums. Thus $\mathcal{L}$ is ample by proposition III.5.3.

Exercise III.5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

a. If X is irreducible and nonsingular, then X is projective by (II, 6.7).

b. If X is integral, let $\widetilde{X}$ be its normalization (II, Ex. 3.8). Show that $\widetilde{X}$ is complete and nonsingular, hence projective by (a). Let $f : \widetilde{X} \to X$ be the projection. Let $\mathcal{L}$ be a very ample invertible sheaf on $\widetilde{X}$. Show that there is an effective divisor $D = \sum P_i$ on $\widetilde{X}$ with $\mathcal{L}(D) \cong \mathcal{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathcal{L}_0$ on X with $f^*\mathcal{L}_0 \cong \mathcal{L}$. Then use (Ex. 5.7d), (II, 7.6), and (II, 5.16.1) to show that X is projective.

c. If X is reduced, but not necessarily irreducible, let $X_1, \cdots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

d. Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{red}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

Solution.

a. Yes.

b. The normalization map of an integral scheme of finite type over a field is a finite map by exercise II.3.8. So by exercise II.4.1, $\widetilde{X} \to X$ is proper, and therefore $\widetilde{X} \to \operatorname{Spec} k$ is proper (aka $\widetilde{X}$ is complete) as a composition of proper morphisms is proper. $\widetilde{X}$ is normal by definition of the normalization, and it's of dimension one since the normalization map is an isomorphism at the generic point. By proposition II.8.23 the singular set of $\widetilde{X}$ is of codimension at least two, and so as $\widetilde{X}$ is of dimension one, we get that $\widetilde{X}$ is nonsingular and hence projective by (a).

Let $\mathcal{L}$ be a very ample invertible sheaf on $\widetilde{X}$, and consider $\widetilde{X} \subset \mathbb{P}^n_k$ embedded by $\mathcal{L}$. f is an isomorphism away from finitely many points, and as k is algebraically closed and thus infinite, there is a hyperplane in $\mathbb{P}^n_k$ that misses all of those points in $\widetilde{X}$. Letting the intersection of this hyperplane with X counted with the appropriate multiplicites be the divisor D, we see that $\mathcal{L}(D) \cong \mathcal{L}$ and D is supported on the locus where f is an isomorphism. Therefore $\mathcal{L}$ is the pullback of $\mathcal{L}(f(D))$, and by exercise III.5.7(d), $\mathcal{L}(f(D))$ is ample. As X admits an ample line bundle, theorem II.7.6 shows that X admits a very ample line bundle and by remark II.5.16.1, X is projective.

c. Consider the map $\mathcal{O}_X \to \bigoplus_{i=1}^r \mathcal{O}_{X_i}$. The kernel consists of $\bigcap \mathcal{I}_i$, which is zero as X is reduced, so the map is injective. The cokernel is supported on the locus of points which belong to more than one irreducible component, and this set is finite. We may conclude the $\mathcal{O}_X^* \to \bigoplus_{i=1}^r \mathcal{O}_{X_i}^*$ is also an injection with cokernel supported on a finite discrete set. Considering the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_X^* \to \bigoplus_{i=1} \mathcal{O}_{X_i}^* \to \mathcal{Q} \to 0$$

we get that $H^1(X, \mathcal{O}_X^*) \to H^1(X, \bigoplus \mathcal{O}_{X_i}^*)$ is surjective by exercise III.4.1 and Grothendieck vanishing applied to $H^1(X, \mathcal{Q}) = H^1(\operatorname{Supp} \mathcal{Q}, \mathcal{Q})$. As $H^1(X, \bigoplus \mathcal{O}_X^*) \cong \bigoplus H^1(X, \mathcal{O}_{X_i}^*) \cong \bigoplus H^1(X_i, \mathcal{O}_{X_i}^*)$ Therefore by exercise III.4.5, $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective.

Picking ample line bundles $\mathcal{L}_i$ on X_i by (b), we can find a line bundle $\mathcal{L}$ on X which restricts to $\mathcal{L}_i$ on each X_i. By exercise III.5.7(c), this shows that $\mathcal{L}$ is ample and therefore X is projective.

d. Let $\mathcal{N} \subset \mathcal{O}_X$ be the sheaf of nilpotent elements, and consider the exact sequences

$$0 \to \mathcal{N}^{2^i}/\mathcal{N}^{2^{i+1}} \to \mathcal{O}_X/\mathcal{N}^{2^{i+1}} \to \mathcal{O}_X/\mathcal{N}^{2^i} \to 0.$$

Taking the associated long exact sequence in cohomology and applying theorem III.2.7, we see that $H^2(X, \mathcal{N}^{2^i}/\mathcal{N}^{2^{i+1}}) = 0$ and therefore $H^1(X, \mathcal{O}_X/\mathcal{N}^{2^{i+1}}) \to H^1(X, \mathcal{O}_X/\mathcal{N}^{2^i})$ is surjective. As X is noetherian, we have that $\mathcal{N}^n = 0$ for some $n > 0$, so inductively we may apply exercise III.4.6 to see that $H^1(X, \mathcal{O}_X)$ surjects on to $H^1(X, \mathcal{O}_X/\mathcal{N})$. As $H^1(X, \mathcal{O}_X/\mathcal{N}) \cong H^1(X_{red}, \mathcal{O}_{X_{red}})$, exercise III.4.5 shows that $\operatorname{Pic} X$ surjects on to $\operatorname{Pic} X_{red}$.

We finish in the same way as (c): picking an ample line bundle $\mathcal{L}$ on X_{red} by (c), we can find an ample line bundle $\mathcal{M}$ on X with $\mathcal{M}|_{X_{red}} \cong \mathcal{L}$. By exercise III.5.7(b), $\mathcal{L}$ is ample iff $\mathcal{M}$ is, so X is projective iff X_{red} is. Therefore we've shown every proper one-dimensional scheme over an algebraically closed field is projective. (The result extends to any field without the hypothesis of being algebraically closed; see Stacks Project tag 0A22, for instance.)

Exercise III.5.9. *A Nonprojective Scheme.* We show that the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbb{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathcal{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since $\dim X = 2$, we have $\omega \otimes \mathcal{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\cdots \to H^1(X, \omega) \to \operatorname{Pic} X' \to PicX \xrightarrow{\delta} H^2(X, \omega) \to \cdots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X,\omega) \cong k$. Since $\operatorname{char} k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X,\omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k. Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X,\Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show tha $\delta(\mathcal{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathcal{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

Solution. We'll use Čech cohomology to compute here. By the logic of exercise III.4.6 with affine opens instead of stalks, we see that the exact sequence $0 \to \omega \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ is exact on each affine open. Therefore using the standard affine open cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ we have an exact sequence of Čech complexes

$$0 \to C^\bullet(\mathfrak{U},\omega) \to C^\bullet(\mathfrak{U},\mathcal{O}_{X'}^*) \to C^\bullet(\mathfrak{U},\mathcal{O}_X) \to 0$$

giving rise to the long exact sequence on cohomology

$$\cdots \to \check{H}^i(\mathfrak{U},\omega) \to \check{H}^i(\mathfrak{U},\mathcal{O}_{X'}^*) \to \check{H}^i(\mathfrak{U},\mathcal{O}_X^*) \to \cdots .$$

I claim that we can apply exercise III.4.11 to see that the Čech cohomology of each sheaf is the honest cohomology of each sheaf: for ω, this follows from theorem III.3.7 as it is quasi-coherent and any intersection of elements of $\mathfrak{U}$ is affine. For $\mathcal{O}_X^*$, this follows from the fact that the Picard group of $\mathbb{A}^2$ and $D(f) \subset \mathbb{A}^2$ both vanish (ref. corollary II.6.16, example II.6.3.1, and proposition II.6.5). Finally, for $\mathcal{O}_{X'}^*$, this follows from the long exact sequence on cohomology: if $H^p(U,\omega)$ and $H^p(U,\mathcal{O}_X^*)$ both vanish, then $H^p(U,\mathcal{O}_{X'}^*)$ must vanish as well. So we can do all our computations with the long exact sequence on cohomology arising from the exact sequences of Čech complexes.

We'll need to take some extra care around $C^\bullet(\mathfrak{U},\mathcal{O}_{X'}^*)$: even though we have that locally on any set in our cover, $\mathcal{O}_{X'} \cong \mathcal{O}_X \oplus \omega$, keeping track of the isomorphisms is important. Let $\psi_i : \mathcal{O}_X|_{U_i} \oplus \omega|_{U_i} \to \mathcal{O}_{X'}|_{U_i}$ be the isomorphism identifying the restrictions of $\mathcal{O}_X \oplus \omega$ and $\mathcal{O}_{X'}$ over U_i. Then the construction of $\mathcal{O}_{X'}$ from the cocycle ξ in exercise III.4.10 shows that $\psi_i^{-1} \circ \psi_j$ is the endomorphism of $\mathcal{O}_X|_{U_{ij}} \oplus \omega|_{U_{ij}}$ given by sending $(f,\zeta) \mapsto (f, \zeta + df \wedge \xi_{ij})$.

To compute $\delta(1)$, we follow the proof of the snake lemma. The element

$$\left\{ \left(U_{01}, \frac{x_0}{x_1}\right), \left(U_{02}, \frac{x_0}{x_2}\right), \left(U_{12}, \frac{x_1}{x_2}\right) \right\} \in C^1(\mathfrak{U},\mathcal{O}_X^*)$$

represents $\mathcal{O}_X(1)$. This lifts to the element

$$\left\{ \left(U_{01}, \psi_0(\frac{x_0}{x_1})|_{U_{01}}\right), \left(U_{02}, \psi_0(\frac{x_0}{x_2})|_{U_{02}}\right), \left(U_{12}, \psi_1(\frac{x_1}{x_2})|_{U_{12}}\right) \right\} \in C^1(\mathfrak{U},\mathcal{O}_{X'}^*)$$

which maps down to the element

$$\left\{\left(U_{012}, \psi_0(\frac{x_0}{x_1})|_{U_{012}} \cdot \psi_0(\frac{x_0}{x_2})|_{U_{012}}^{-1} \cdot \psi_1(\frac{x_1}{x_2})|_{U_{012}}\right)\right\} \in C^2(\mathfrak{U}, \mathcal{O}_{X'}).$$

To see what element of $(\mathcal{O}_X(U_{012}) \oplus \omega(U_{012}))^*$ this corresponds to, apply ψ_0^{-1}, which gives

$$\psi_0^{-1}\left(\psi_0\left(\frac{x_0}{x_1}\right)|_{U_{012}} \cdot \psi_0\left(\frac{x_0}{x_2}\right)|_{U_{012}}^{-1} \cdot \psi_1\left(\frac{x_1}{x_2}\right)|_{U_{012}}\right) = \frac{x_0}{x_1} \cdot \frac{x_2}{x_0} \cdot \left(\frac{x_1}{x_2} + d\frac{x_1}{x_2} \wedge \frac{x_1}{x_0} d\frac{x_0}{x_1}\right)$$

$$= 1 + \frac{x_2}{x_0} d\frac{x_1}{x_2} \wedge d\frac{x_0}{x_1} = -\frac{dx_0 \wedge dx_1}{x_0 x_1} + \frac{dx_0 \wedge dx_2}{x_0 x_2} - \frac{dx_1 \wedge dx_2}{x_1 x_2}.$$

To check that $e = -\frac{dx_0 \wedge dx_1}{x_0 x_1} + \frac{dx_0 \wedge dx_2}{x_0 x_2} - \frac{dx_1 \wedge dx_2}{x_1 x_2}$ represents a nonzero class in $\check{H}^2(\mathfrak{U}, \omega)$, we compute the image of the differential and show it misses e. Defining a module homomorphism by $dx_0 \wedge dx_1 \mapsto \frac{1}{x_2}$, $dx_0 \wedge dx_2 \mapsto \frac{1}{x_1}$, and $dx_1 \wedge dx_1 \mapsto \frac{1}{x_0}$, we see that the Čech complex for ω turns in to the Čech complex for $\mathcal{O}_X(-3)$ and e is equal to the negative of the generator of H^2 by the calculation in theorem III.5.1. So $\delta(\mathcal{O}(1))$ indeed represents a nonzero class in $H^2(\omega)$ and $\operatorname{Pic} X'$ is trivial. Therefore X' has no ample invertible sheaves and cannot be projective.

Exercise III.5.10. Let X be a projective scheme over a noetherian ring A, and let $\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \to \mathcal{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geq n_0$, the sequence of global sections

$$\Gamma(X, \mathcal{F}^1(n)) \to \Gamma(X, \mathcal{F}^2(n)) \to \cdots \to \Gamma(X, \mathcal{F}^r(n))$$

is exact.

Solution. The condition that our sequence is exact means that $\operatorname{Im}(\mathcal{F}^i \to \mathcal{F}^{i+1}) = \ker(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2})$. This means that if $\mathcal{G}^i = \ker(\mathcal{F}^i \to \mathcal{F}^{i+1})$ for $0 < i < r-1$ and $\mathcal{G}^r = \operatorname{coker}(\mathcal{F}^{r-1} \to \mathcal{F}^r)$, then $0 \to \mathcal{G}^i \to \mathcal{F}^i \to \mathcal{G}^{i+1} \to 0$ is exact for all i. Let n_i be the integer guaranteed by theorem III.5.2(b) for which $H^p(X, \mathcal{G}(n)) = 0$ for all $n \geq n_i$ and $p > 0$. Set $n_0 = \max(n_i)$. For $n \geq n_0$, we see that $H^1(X, \mathcal{G}^i(n)) = 0$ for all i, or that $0 \to \Gamma(X, \mathcal{G}^i(n)) \to \Gamma(X, \mathcal{F}^i(n)) \to \Gamma(X, \mathcal{G}^{i+1}(n)) \to 0$ is exact for all i. Combining all of these exact sequences, we see that

$$\Gamma(X, \mathcal{F}^1(n)) \to \cdots \to \Gamma(X, \mathcal{F}^r(n))$$

is exact for all $n \geq \max(n_i)$, and we're done.

III.6 Ext Groups and Sheaves

Homological algebra is fun! When Hartshorne defines 'enough locally frees', he omits (or is at least unclear about) the condition that the surjection be from a locally free sheaf of *finite rank*. We'll assume what he meant, not what he said.

Exercise III.6.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}', \mathcal{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathcal{F}''$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathcal{F}'$ and $\mathcal{F}''$. Given an extension as above consider the long exact sequence arising from $\mathrm{Hom}(\mathcal{F}'', \cdot)$, in particular the map

$$\delta : \mathrm{Hom}(\mathcal{F}'', \mathcal{F}'') \to \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}'),$$

and let $\xi \in \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}')$ be $\delta(1_{\mathcal{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathcal{F}''$ by $\mathcal{F}'$, and elements of the group $\mathrm{Ext}(\mathcal{F}'', \mathcal{F}')$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

Solution. This is a completely general fact in any abelian category which has either enough projectives or enough injectives (sheaves over ringed spaces - this is us). We start with a couple preliminaries about abelian categories. First, recall that abelian categories have binary pullbacks and pushouts: given $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} C$, the pullback is the kernel of $A \oplus B \xrightarrow{\alpha-\beta} C$, while given $X \xrightarrow{\gamma} Y$ and $X \xrightarrow{\delta} Z$, the pushout is the cokernel of $X \xrightarrow{\gamma \oplus \delta} Y \oplus Z$.

Lemma. *Suppose the following diagram is a pullback diagram in an abelian category:*

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\alpha_2"] \\ C_1 \arrow[r, "\alpha_1"'] & C \end{tikzcd}$$

Then if α_1 is a monomorphism or epimorphism, so is π_2.

Proof. Monomorphisms first: suppose $f : T \to P$ is a morphism so that $\pi_2 \circ f = 0$:

$$\begin{tikzcd} T \arrow[r, "f"] & P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\alpha_2"] \\ & C_1 \arrow[r, "\alpha_1"'] & C \end{tikzcd}$$

Then $\alpha_2 \circ \pi_2 \circ f = 0$, and so $\alpha_1 \circ \pi_1 \circ f$ is also zero by commutativity of the diagram. Since α_1 is a monomorphism, we see that $\pi_1 \circ f = 0$, and as morphisms $T \to P$ correspond uniquely to pairs of morphisms $T \to C_1$ and $T \to C_2$ which agree after composition with α_1 and α_2, we see that $f = 0$ and π_2 is a monomorphism.

Now suppose that α_1 is an epimorphism. As $\operatorname{coker}(C_1 \oplus C_2 \overset{\alpha_1 \oplus \alpha_2}{\to} C)$ is the pushout of $C \to \operatorname{coker}\alpha_1$ and $C \to \operatorname{coker}\alpha_2$, we see that if α_1 is an epimorphism, then $\operatorname{coker}\alpha_1 = 0$ and so $\operatorname{coker}(C_1 \oplus C_2 \overset{\alpha_1 \oplus \alpha_2}{\to} C)$ must also vanish. Therefore our square is both a pullback square and a pushout square, so the sequence

$$0 \to P \to C_1 \times C_2 \overset{\alpha_1 - \alpha_2}{\to} C \to 0$$

is exact. Next, let $f : C_2 \to T$ be a morphism so that $f \circ \pi_2 = 0$:

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[r, "f"] \arrow[d, "\alpha_2"] & T \\ C_1 \arrow[r, "\alpha_1"'] & C & \end{tikzcd}$$

Then $0 : C_1 \to T$ gives morphisms $C_1 \to T$ and $C_2 \to T$ which agree after precomposition with π_1 and π_2, so we get a unique induced morphism $g : C \to T$ by the universal property of the pushout. As $g \circ \alpha_1 = 0$, by α_1 being an epimorphism we see $g = 0$. Thus $f = g \circ \alpha_2 = 0$ and π_2 is an epimorphism.

■

We'll change notation by letting $B = \mathcal{F}'$ and $A = \mathcal{F}''$. Given an element $\xi \in \operatorname{Ext}^1(A, B)$ our goal is to find an extension $0 \to B \to X \to A \to 0$ so that after applying $\operatorname{Hom}(A, -)$, the element $1 \in \operatorname{Hom}(A, A)$ maps to ξ under the long exact sequence. Embed B in to an injective I and let N be the cokernel. Applying $\operatorname{Hom}(A, -)$ to the exact sequence

$$0 \to B \to I \overset{\pi}{\to} N \to 0$$

we see that as $\operatorname{Ext}^1(A, I)$ vanishes because I is injective, the portion of the exact sequence

$$\cdots \to \operatorname{Hom}(A, N) \to \operatorname{Ext}^1(A, B) \to \operatorname{Ext}^1(A, I) = 0 \to \cdots$$

tells us that $\operatorname{Hom}(A, N) \to \operatorname{Ext}^1(A, B)$ is surjective. Let $\gamma \in \operatorname{Hom}(A, N)$ be a lift of ξ, and let X be the pullback of $I \to N$ along $\gamma : A \to N$. Considering $0 : B \to A$ and $B \to I$, the universal property of the pullback gives us the following commutative diagram where $B \to B$ is the identity:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & B \arrow[r] & I \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

I claim that the top row is an exact sequence. First, $B \to X$ is a monomorphism as $B \to I$ factors through X. Next, as π is an epimorphism, our lemma gives that $X \to A$ is also an epimorphism. The composite $B \to X \to A$ is equal to zero by the universal property of the pullback, so the image of B in X lies in the kernel of $X \to A$. Further, as $\ker(X \to A)$ is mapped to $\ker(I \overset{\pi}{\to} N) = B$ by the vertical map, we see that $\ker(X \to A)$ really is just B, and the top row is a short exact sequence.

After applying $\operatorname{Hom}(A, -)$ and taking the long exact sequence associated to each short exact sequence, we obtain a morphism of long exact sequences and concentrate on the following portion:

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,B) \arrow[d] \\
\mathrm{Hom}(A,N) \arrow[r] & \mathrm{Ext}^1(A,B)
\end{tikzcd}$$

By functoriality, the first vertical morphism is postcomposition with $\gamma : A \to N$, and the second vertical morphism is the identity. Going south and then east, we have that id_A maps to γ which then maps to ξ by construction, so the extension $0 \to B \to X \to A \to 0$ has $\delta(1) = \xi$. This shows that the map from isomorphism classes of extensions to $\mathrm{Ext}^1(A,B)$ is surjective.

To see the map from isomorphism classes of extensions to $\mathrm{Ext}^1(A,B)$ is injective, note that if X and X' are two extensions given by choices of γ, γ' lifting ξ, then $\gamma - \gamma' \in \mathrm{Hom}(A,N)$ is in the image of some $\zeta \in \mathrm{Hom}(A,I)$. Then as $A \oplus I \stackrel{\gamma-\pi}{\to} N$ is isomorphic to $A \oplus I \stackrel{\gamma'-\pi}{\to} N$ by the automorphism of $A \oplus I$ given by $\begin{pmatrix} id_A & \zeta \\ 0 & id_I \end{pmatrix}$, we have that $X \cong X'$ by the universal property of the pullback, and the maps we used to induce the exact sequence $0 \to B \to X \to A \to 0$ don't see this change. Thus the two extensions $0 \to B \to X \to A \to 0$ and $0 \to B \to X' \to A \to 0$ are isomorphic, and the map from isomorphism classes of extensions to $\mathrm{Ext}^1(A,B)$ is injective.

Exercise III.6.2. Let $X = \mathbb{P}^1_k$, with k an infinite field.

a. Show that there does not exist a projective object $\mathcal{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathcal{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j : V \to X$ is the inclusion.]

b. Show that there does not exist a projective object $\mathcal{P}$ in $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathcal{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathcal{L} \to \mathcal{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathcal{L}$ is an invertible sheaf on X.]

Solution.

a. We can prove something much more general here.

Lemma. *Suppose X is a locally connected topological space with a point x so that any connected neighborhood of x contains a strictly smaller neighborhood. Then the category of sheaves of abelian groups on X does not have enough projectives.*

Proof. Let U denote a connected neighborhood of x, and V a smaller neighborhood. As $\{V, X \setminus \{x\}\}$ covers X, there is a surjection

$$\mathbb{Z}_V \oplus \mathbb{Z}_{X\setminus\{x\}} \to \mathbb{Z}_X \to 0$$

(where $\mathbb{Z}_V = i_!\mathbb{Z}_V$ for $i : V \to X$ the open inclusion). If there are enough projectives, then there is a surjective map $\mathcal{P} \to \mathbb{Z}_X$ for $\mathcal{P}$ projective, and by the definition of a projective object, this map must factor through $\mathbb{Z}_V \oplus \mathbb{Z}_{X\setminus\{x\}}$. But this means that $\mathcal{P}(U) \to \mathbb{Z}_X(U)$ is the zero map, so by our assumption on X, this is true for all connected neighborhoods of x. Since such neighborhoods are cofinal by the assumption that X is locally connected, we see that $\mathcal{P}_x \to \mathbb{Z}_{X,x}$ is the zero map, contradicting the assumption that $\mathcal{P} \to \mathbb{Z}_X$ is surjective.

■

To apply this to our situation, note that $\mathbb{P}^n_k$ is noetherian and thus locally connected by the argument given in the solution to exercise III.2.6. As $\mathbb{P}^1_k$ is an infinite set with the cofinite topology (this is true without any assumption on k!), it satisfies the assumptions of our claim and therefore $\mathfrak{Mod}(X)$ does not have enough projectives.

b. Suppose $\mathcal{P}$ is a quasi-coherent projective module and $\varphi : \mathcal{P} \to \mathcal{O}_X$ is a surjection. Then by exercise II.5.15(d), $\mathcal{P}$ is the union of it's coherent subsheaves $\mathcal{P}_i$, and as $\mathcal{O}_X = \bigcup \operatorname{Im}(\varphi|_{\mathcal{P}_i})$ we see that there is a coherent sheaf $\mathcal{P}_i$ surjecting on to $\mathcal{O}_X$.

   Now we may assume $\mathcal{P}$ is a coherent sheaf surjecting on to $\mathcal{O}_X$. Let $\mathcal{K}$ be the kernel of the surjection $\mathcal{P} \to \mathcal{O}_X$: by theorem III.5.2, we may find an n so that $0 \to \Gamma(X, \mathcal{K}(n)) \to \Gamma(\mathcal{P}(n), X) \to \Gamma(X, \mathcal{O}_X(n)) \to 0$ is surjective. We note that there exists a surjection $\mathcal{O}_X(r) \to k(x)$ for any r given by twisting the obvious surjection $\mathcal{O}_X \to k(x)$. Therefore the surjection $\mathcal{P} \to k(x)$ obtained by composing $\mathcal{P} \to \mathcal{O}_X$ and $\mathcal{O}_X \to k(x)$ must factor through the surjection $\mathcal{O}_X(-n-1) \to k(x)$. Twisting by n and taking global sections, this means that $\Gamma(X, \mathcal{P}(n)) \to \Gamma(X, k(x))$ is surjective on global sections, while factoring through $\Gamma(X, \mathcal{O}_X(-1)) = 0$, a contradiction. So neither $\mathfrak{Qco}(X)$ nor $\mathfrak{Coh}(X)$ have enough projectives (again without any assumptions on k).

Exercise III.6.3. Let X be a noetherian scheme, and let $\mathcal{F}, \mathcal{G} \in \mathfrak{Mod}(X)$.

a. If $\mathcal{F}, \mathcal{G}$ are both coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is coherent, for all $i \geq 0$.

b. If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is quasi-coherent, for all $i \geq 0$.

Solution.

a. As the conditions and criteria are all local, we may immediately reduce to the case where $X = \operatorname{Spec} A$ and $\mathcal{F} \cong \widetilde{M}$, $\mathcal{G} \cong \widetilde{N}$ for A-modules M, N. Then by exercise III.6.7 (don't worry, there's no circular logic here, that problem doesn't depend on this one) we have that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is the sheafification of $\operatorname{Ext}^i_A(M, N)$. Resolving M by finitely generated free modules (this is where we use noetherianity), we see that $\operatorname{Ext}^i_A(M, N)$ is a subquotient of $\operatorname{Hom}_A(A^{n_i}, N) \cong N^{n_i}$, and is therefore finitely generated as a module (we use noetherianity again here).

b. We've already proven this in our argument for (a).

Exercise III.6.4. Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees*. Then for any $\mathcal{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathcal{E}xt^i(\cdot, \mathcal{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathcal{E}xt^i(\cdot, \mathcal{G})$ is coeffaceable (§1) for $i > 0$.]

Solution. While not explicitly stated in the text, the result of theorem III.1.3A is valid in the contravariant setting as well, so it suffices to show that $\mathcal{E}xt^i(-, \mathcal{G})$ is coeffaceable for $i > 0$ in order to prove it is universal.

Suppose $\mathcal{F}$ is a coherent sheaf admitting a surjection $\mathcal{H} \to \mathcal{F}$ for $\mathcal{H}$ locally free of finite rank. Applying $\mathcal{E}xt^i(-, \mathcal{G})$, we get a map $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^i(\mathcal{H}, \mathcal{G})$. But by proposition III.6.7, this is just $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G} \otimes \mathcal{H}^\vee)$, which vanishes for $i > 0$ by proposition III.6.3 and therefore $\mathcal{E}xt^i(-, \mathcal{G})$ is coeffaceable.

Exercise III.6.5. Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathcal{F}$ we define the *homological dimension* of $\mathcal{F}$, denoted $\operatorname{hd}(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ (or $+\infty$ if there is no finite one). Show:

a. $\mathcal{F}$ is locally free $\Leftrightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathfrak{Mod}(X)$;

b. $\operatorname{hd}(\mathcal{F}) \leq n \Leftrightarrow \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G} \in \mathfrak{Mod}(X)$;

c. $\operatorname{hd}(\mathcal{F}) = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathcal{F}_x$.

Solution.

a. As in the previous exercise, $\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \cong \mathcal{E}xt^1(\mathcal{O}_X, \mathcal{G} \otimes \mathcal{F}^\vee) = 0$ by propositions III.6.7 and III.6.3, which proves the forward direction.

   Conversely, if $\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathfrak{Mod}(X)$, then by proposition III.6.8 we have that $\operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) = 0$ for all x. By proposition III.6.10A(a), this shows that $\mathcal{F}_x$ is a projective $\mathcal{O}_{X,x}$-module, hence a free $\mathcal{O}_{X,x}$-module because all finite-rank projective modules over local rings are free. By exercise II.5.7, $\mathcal{F}$ is locally free.

b. By proposition III.6.5, if $\mathcal{L}_\bullet \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ by locally free sheaves, $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \cong h^i(\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}))$. Therefore if there exists a resolution $\mathcal{L}_\bullet$ of length at most n, we have that all the terms of this complex vanish in degrees above n and therefore the homology above degree n also vanishes.

   Conversely, if $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G}$, consider a resolution

$$0 \to \mathcal{H} \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

   where $\mathcal{L}_i$ are locally free and $\mathcal{H}$ is coherent. Splitting the resolution in to short exact sequences and taking the associated long exact sequences in $\mathcal{E}xt$, we see that $\mathcal{E}xt^1(\mathcal{H}, \mathcal{G}) \cong \mathcal{E}xt^{n+1}(\mathcal{F}, \mathcal{G})$. Therefore by (a), $\mathcal{H}$ is locally free and $\mathcal{F}$ has homological dimension at most n.

c. Clearly if $\mathcal{L}_\bullet \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ by locally free modules of finite rank, then after taking stalks we have that $(\mathcal{L}_x)_\bullet$ is a resolution of $\mathcal{F}_x$ by free modules of finite rank, so $\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x \leq \operatorname{hd}(\mathcal{F})$ for all x.

   To show equality, it's enough to find a $\mathcal{O}_X$-module $\mathcal{G}$ so that $\mathcal{E}xt^{\operatorname{hd}(\mathcal{F})}(\mathcal{F},\mathcal{G}) \neq 0$ by (b) (if $\operatorname{hd}(\mathcal{F}) = \infty$, for each $n > 0$ we find a $\mathcal{G}$ so that $\mathcal{E}xt^n(\mathcal{F},\mathcal{G}) \neq 0$). Let $x \in X$ be a point where the supremum of $\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is achieved, which exists as X is quasi-compact by assumption. As $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})_x \cong \operatorname{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$ by proposition III.6.8, we can apply proposition III.6.10A(b) to find an $\mathcal{O}_{X,x}$-module N so that $\operatorname{Ext}^{\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, N) \neq 0$ and then let $\mathcal{G}$ be the skyscraper sheaf with value N at the point X. This proves the claim.

Exercise III.6.6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

a. M is projective if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > 0$. [*Hint*: Use (6.11A) and descending induction on i to show that $\operatorname{Ext}^i(M,N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct sumand of a free A-module (Matsumura [2, p. 129]).]

b. Use (a) to show that for any n, $\operatorname{pd} M \leq n$ if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > n$.

Solution.

a. The forward direction is an application of proposition III.6.10A(a). For the reverse direction, suppose N is a finitely generated A-module and choose a surjection $A^r \to N$. Completing this to an exact sequence $0 \to K \to A^r \to N \to 0$ and applying $\operatorname{Ext}^\bullet(M,-)$ we get the long exact sequence

$$\cdots \to \operatorname{Ext}^i(M,K) \to \operatorname{Ext}^i(M,A^r) \to \operatorname{Ext}^i(M,N) \to \cdots$$

   and as $\operatorname{Ext}^i(M,A^r) = 0$, we see that $\operatorname{Ext}^i(M,N) \cong \operatorname{Ext}^{i+1}(M,K)$ for all $i > 0$. By induction, $\operatorname{Ext}^i(M,N)$ is isomorphic to the $(\dim A + 1)^{th}$ ext group of M with some finitely generated A-module and by proposition III.6.11A this vanishes. So $\operatorname{Ext}^1(M,N)$ vanishes for all finitely generated A-modules N. In particular, if we write $0 \to R \to A^n \to M \to 0$ and apply $\operatorname{Ext}^\bullet(M,-)$, then we get that

$$0 \to \operatorname{Hom}(M,R) \to \operatorname{Hom}(M,A^n) \to \operatorname{Hom}(M,M) \to 0$$

   is exact. Choosing $\gamma \in \operatorname{Hom}(M,A^n)$ lifting id_M, we see that γ splits the exact sequence $0 \to R \to A^n \to M \to 0$ and thus M is a direct summand of A^n and hence projective.

b. The forward direction follows from computing $\operatorname{Ext}^i(M,A)$ via a projective resolution of M of length at most n.

   For the reverse direction, we induct on $\operatorname{pd} M$. The case of $\operatorname{pd} M = 0$ is (a). Suppose we've proven the claim for modules of projective dimension at most n, and suppose M is a module of projective dimension $n+1$. Construct the same exact sequence $0 \to K \to A^r \to M \to 0$ as in (a) and apply $\operatorname{Ext}^\bullet(-, A)$ to it: we get that $\operatorname{Ext}^{i+1}(M, A) \cong \operatorname{Ext}^i(K, A)$ for $i > 0$. If $\operatorname{Ext}^i(M, A)$ vanishes for all $i > n+1$, then $\operatorname{Ext}^i(K, A)$ vanishes for $i > n$ and by our induction hypothesis, K has a projective resolution $P_\bullet$ of length at most n. Composing $P_\bullet \to K$ with $K \to A^r \to M$, we get that $P_\bullet \to A^r \to M \to 0$ is a projective resolution of M of length at most $n+1$, so $\operatorname{pd} M \le n+1$ and we're done.

Exercise III.6.7. Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$

and

$$\mathcal{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)^\sim.$$

Solution. As M is finitely generated, there exists a free resolution $F_\bullet \to M \to 0$ by free A-modules F_j. Sheafifying, we get an exact sequence $\widetilde{F}_\bullet \to \widetilde{M} \to 0$ where $\widetilde{F}_j$ is a free sheaf of finite rank n_j. Applying proposition III.6.5, we get that $\mathcal{E}xt^i(\widetilde{M}, \widetilde{N}) \cong h^i(\mathcal{H}om(\widetilde{F}_\bullet, \widetilde{N}))$. As $\widetilde{F}_j \cong \mathcal{O}_X^{n_j}$, we have that $\mathcal{H}om(\widetilde{F}_j, \widetilde{N}) \cong \widetilde{N}^{n_j}$, so the chain complex $\mathcal{H}om(\widetilde{F}_\bullet, \widetilde{N})$ consists of quasi-coherent sheaves. Since kernels, images, and quotients of quasi-coherent sheaves are quasi-coherent (proposition II.5.7), the homology sheaves of this complex are quasi-coherent, proving that $\mathcal{E}xt^i_X(\widetilde{M}, \widetilde{N})$ is the sheafification of $\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N})$.

Since global sections is an exact equivalence of categories from quasi-coherent sheaves on X to A-modules, we see that it commutes with taking homology of the chain complex. In one order, we get that $\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N})$, while the other order gives $\operatorname{Ext}^i_A(M, N)$, so these are isomorphic and we've finished.

Exercise III.6.8. Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

a. First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathcal{L})$, and various invertible sheaves $\mathcal{L}$ on X, form a base for the topology of X. [*Hint*: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is a an $\mathcal{L}, s$ such that $x \in X_s \subset U$, first reduce to the case that $Z = X \setminus U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}_x$, $f \notin \mathcal{O}_\zeta$. Let $D = (f_\infty)$, and let $\mathcal{L} = \mathcal{L}(D)$, $s \in \Gamma(X, \mathcal{L}(D))$ correspond to D (II, §6).]

b. Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathcal{L}_i^{n_i}$ for various invertible sheaves $\mathcal{L}_i$ and various integers n_i.

Solution.

a. We're going to proceed slightly differently than in the hint. We begin with a lemma:

Lemma. *If X is an integral noetherian normal integral separated scheme, then the complement of any proper nonempty affine open subscheme $U \subset X$ is of pure codimension one.*

Proof. Let $W \subset X$ be an arbitrary affine open subscheme. Then $W \cap U$ is again affine as X is separated, and the map $W \cap U \to W$ is completely determined by the restriction map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap U)$. If $W \setminus U$ is of codimension two or more, then $W \setminus U$ contains all the height one points of W, and by Hartog's (proposition II.6.3A), we have that $\mathcal{O}_X(W) = \mathcal{O}_X(W \cap U)$. But this means that $W = W \cap U$. Taking W to be an affine open in X intersecting exactly one irreducible component of $X \setminus U$, we see that this irreducible component must be codimension one and the claim is proven.

■

In order to write an arbitrary open subset $U \subset X$ as a union of sets of the form X_s for $s \in \Gamma(X, \mathcal{L})$, it suffices to show that for any $x \in U$ we can find a line bundle $\mathcal{L}$ and a global section s so that $x \in X_s \subset U$. Let $x \in U$ be arbitrary, and let U_x be an affine open neighborhood of x contained in U, which we can find as the affine opens form a basis for the topology. Since a locally factorial scheme is normal, we may apply the lemma to see that $Z = X \setminus U_x$ is of pure codimension one and therefore an effective Weil divisor. As X satisfies the assumptions of proposition II.6.11, Z corresponds to an effective Cartier divisor $\{(U_i, f_i)\}$ which has a canonical global section (the gluing of the f_i) which is nonvanishing exactly on U_x, and we are done.

b. After choosing a finite affine open cover U_i, by coherence we can find finitely many sections $f_{ij} \in \mathcal{F}(U_i)$ which generate $\mathcal{F}|_{U_i}$. Up to refinement, we may assume each U_i is actually of the form X_{s_i} for s_i a global section of some line bundle $\mathcal{L}_i$ on X. Then by lemma II.5.14(b), for each i we can find a n_i so that each $s_i^{n_i} f_{ij}$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}_i^{n_i}$. Defining $\mathcal{O}_X^{\oplus J_i} \to \mathcal{F} \otimes \mathcal{L}_i^{n_i}$ by sending the basis vector e_j to $s_i^{n_i} f_{ij}$, we see that this map is surjective on X_{s_i}. As twisting by $\mathcal{L}_i^{-n_i}$ preserves this property, we obtain a map $\bigoplus_{J_i} \mathcal{L}_i^{-n_i} \to \mathcal{F}$ which is surjective over U_i. Taking the sum over all i we find that $\mathcal{F}$ admits a surjective map from a direct sum of line bundles.

Exercise III.6.9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is *regular* if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathcal{E} - \mathcal{E}' - \mathcal{E}''$, whenever $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre, [1, §4]) as follows.

a. Given a coherent sheaf $\mathcal{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathcal{E}. \to \mathcal{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

b. For each $\mathcal{F}$, choose a finite locally free resolution $\mathcal{E}. \to \mathcal{F} \to 0$, and let $\delta(\mathcal{F}) = \sum(-1)^i\gamma(\mathcal{E}_i)$ in $K_1(X)$. Show that $\delta(\mathcal{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

Solution.

a. Let $\mathcal{F}$ be a coherent sheaf on X. As a regular local ring is a UFD by remark II.6.11.1A, we may apply exercise III.6.8 to find a surjection $\mathcal{E}_0 \to \mathcal{F}$. Letting $\mathcal{K}_0$ be the kernel of $\mathcal{E}_0 \to \mathcal{F}$, we can again find a surjection $\mathcal{E}_1 \to \mathcal{K}_0$. Continuing in this fashion, we find a resolution $\mathcal{E}_\bullet \to \mathcal{F} \to 0$ of $\mathcal{F}$ by locally free sheaves of finite rank $\mathcal{E}_\bullet$.

   By exercise III.6.5, $\operatorname{hd}(\mathcal{F}) = \sup_x \operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x$, while by proposition III.6.11A, $\operatorname{pd}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \leq \dim \mathcal{O}_{X,x}$, which is bounded above by $\dim X$. Thus $\operatorname{hd}(\mathcal{F})$ is finite and so $\mathcal{F}$ has a finite length resolution by locally free sheaves of finite rank.

b. If the objects of $\mathcal{E}_\bullet$ were projective, we would be done by the standard proof that a projective resolution is unique up to homotopy. But they're not, and the key property one would like to use (lifting maps) isn't available here, so we need to construct an auxiliary resolution to compare resolutions of $\mathcal{F}$.

   Lemma. *Suppose we have surjective morphisms $A_1 \to A_0$, $B_1 \to B_0$, $C_0 \to A_0$ and $C_0 \to B_0$ in an abelian category. Let C_1 be the limit of this diagram, as shown below:*

$$\begin{tikzcd} A_1 \arrow[d, two heads] & C_1 \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & B_1 \arrow[d, two heads] \\ A_0 & C_0 \arrow[l, two heads] \arrow[r, two heads] & B_0 \end{tikzcd}$$

   Then all three dotted arrows originating from C_1 are surjective.

   Proof. Note that C_1 is the pullback of $P = A_1 \times_{A_0} C_0$ and $Q = B_1 \times_{B_0} C_0$ over C_0, as seen in the following diagram:

$$\begin{tikzcd} & & C_1 \arrow[dl] \arrow[dr] & & \\ A_1 \arrow[d, two heads] & P \arrow[l] \arrow[dr] & & Q \arrow[r] \arrow[dl] & B_1 \arrow[d, two heads] \\ A_0 & & C_0 \arrow[ll, two heads] \arrow[rr, two heads] & & B_0 \end{tikzcd}$$

   (It suffices to observe that C_1 constructed as the limit of the first diagram satisfies the same universal property as the pullback of P and Q over C_0 by talking through the universal property of the pullback/limit.) Now several applications of the surjectivity portion of the lemma from exercise III.6.1 show that all the arrows in our diagram are surjective.

   ■

Now we prove that this lemma enables us to compare resolutions of $\mathcal{F}$. More precisely, we show that for any two resolutions $\mathcal{E}_\bullet$ and $\mathcal{E}'_\bullet$ of $\mathcal{F}$ by locally free sheaves, we can construct a third resolution $\mathcal{E}''_\bullet$ of $\mathcal{F}$ by locally free sheaves with morphisms $\mathcal{E}''_\bullet \to \mathcal{E}_\bullet$ and $\mathcal{E}'_\bullet$ which induce the identity morphism on $\mathcal{F}$. In order to produce such a $\mathcal{E}''_\bullet$, let $\mathcal{D}_0$ be the limit of the solid diagram below (where the vertical maps come from the resolutions $\mathcal{E}_\bullet \to \mathcal{F}$ and $\mathcal{E}'_\bullet \to \mathcal{F}$) and apply the above lemma to see that the dashed arrows are surjections:

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[d, two heads] & \mathcal{D}_0 \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E}'_0 \arrow[d, two heads] \\
\mathcal{F} & \mathcal{F} \arrow[l, two heads, "id"'] \arrow[r, two heads, "id"] & \mathcal{F}
\end{tikzcd}$$

Letting $\mathcal{E}''_0$ be a locally free sheaf surjecting on to $\mathcal{D}_0$ by the resolution property, we have the first step of our claimed resolution. We proceed inductively by applying the lemma to the following diagram, again letting $\mathcal{E}''_{i+1}$ be a locally free sheaf surjecting on to $\mathcal{D}_{i+1}$:

$$\begin{tikzcd}
\mathcal{E}_{i+1} \arrow[d, two heads] & \mathcal{D}_{i+1} \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E}'_{i+1} \arrow[d, two heads] \\
\ker(\mathcal{E}_i \to \mathcal{E}_{i-1}) & \ker(\mathcal{E}''_i \to \mathcal{E}''_{i-1}) \arrow[l, two heads] \arrow[r, two heads] & \ker(\mathcal{E}'_i \to \mathcal{E}'_{i-1})
\end{tikzcd}$$

This gives us the resolution $\mathcal{E}''_\bullet$ with surjections to $\mathcal{E}_\bullet$ and $\mathcal{E}''_\bullet$ inducing the identity on $\mathcal{F}$.

Now I claim that $\gamma(\mathcal{E}_\bullet) = \gamma(\mathcal{E}''_\bullet)$ in the Grothendieck group. To see this, let $\mathcal{C}_\bullet$ be the mapping cone of $\varphi : \mathcal{E}''_\bullet \to \mathcal{E}_\bullet$, which is a chain complex having $\mathcal{E}''_i \oplus \mathcal{E}_{i-1}$ in degree i and differential $\begin{pmatrix} -d^i_{\mathcal{E}''_\bullet} & 0 \\ \varphi & d^{i-1}_{\mathcal{E}_\bullet} \end{pmatrix}$. There is a long exact sequence on homology sheaves

$$\cdots \to h_i(\mathcal{E}''_\bullet) \xrightarrow{\varphi} h_i(\mathcal{E}_\bullet) \to h_i(\mathcal{C}_\bullet) \to \cdots .$$

As $h_0(\mathcal{E}''_\bullet) \xrightarrow{\varphi} h_0(\mathcal{E}_\bullet)$ is the identity map $\mathcal{F} \to \mathcal{F}$ by construction and $h_i(\mathcal{E}''_\bullet) = h_i(\mathcal{E}_\bullet) = 0$ for $i > 0$ by the definition of a resolution, we see that $\mathcal{C}_\bullet$ is exact. By splitting this exact chain complex in to short exact sequences, we see that $\sum(-1)^i\gamma(\mathcal{E}_i \oplus \mathcal{E}''_{i+1}) = 0$, or $\gamma(\mathcal{E}_\bullet) = \sum(-1)^i\gamma(\mathcal{E}_i) = \sum(-1)^i\gamma(\mathcal{E}''_i) = \gamma(\mathcal{E}''_\bullet)$, and $\delta(\mathcal{F})$ is therefore independent of the resolution.

To check that δ is a homomorphism it suffices to check that for an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have that $\delta(\mathcal{F}) = \delta(\mathcal{F}') + \delta(\mathcal{F}'')$. Let $\mathcal{E}''_\bullet \to \mathcal{F}$ be a finite length resolution of $\mathcal{F}$ consisting of locally free sheaves of finite rank. We can build a resolution $\mathcal{E}_\bullet \to \mathcal{F}$ satisfying the same properties by letting $\mathcal{E}_0$ be a locally free sheaf of finite rank surjecting on to $\mathcal{F} \times_{\mathcal{F}''} \mathcal{E}''_0$ and then inductively defining $\mathcal{E}_{i+1}$ to be a locally free sheaf of finite rank surjecting on to $\ker(\mathcal{E}_i \to \mathcal{E}_{i-1}) \times_{\ker(\mathcal{E}''_i \to \mathcal{E}''_{i-1})} \mathcal{E}''_{i+1}$. Now $\mathcal{E}'_\bullet$ defined as the kernel of $\mathcal{E}_\bullet \to \mathcal{E}''_\bullet$ is a resolution of $\mathcal{F}'$, and furthermore it consists of locally free sheaves: taking stalks at any point $x \in X$, we find that $(\mathcal{E}'_i)_x$ fits in to a short exact sequence

$$0 \to (\mathcal{E}'_i)_x \to (\mathcal{E}_i)_x \to (\mathcal{E}''_i)_x \to 0$$

and as $(\mathcal{E}_i'')_x$ is free, this sequence splits. Therefore $(\mathcal{E}_i')_x$ is projective, being a direct summand of a free module, and in fact free as every finite rank projective module over a local ring is free. Thus $\mathcal{E}_i'$ is locally free by exercise II.5.7. The exact sequence of complexes $0 \to \mathcal{E}_\bullet' \to \mathcal{E}_\bullet \to \mathcal{E}_\bullet'' \to 0$ gives that $\gamma(\mathcal{E}_i) = \gamma(\mathcal{E}_i') + \gamma(\mathcal{E}_i'')$ for all i and therefore δ respects exact sequences.

It is immediate to see that ε and δ are mutually inverse, and we're finished.

Exercise III.6.10. *Duality for a Finite Flat Morphism.*

a. Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$ module $\mathcal{G}$, $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathcal{G}$ (II, Ex. 5.17e).

b. Show that for any coherent $\mathcal{F}$ on X and any quasi-coherent $\mathcal{G}$ on Y, there is a natural isomorphism
$$f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}).$$

c. For each $i \geq 0$, there is a natural map
$$\varphi_i : \operatorname{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \operatorname{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$
[*Hint*: First construct a map
$$\operatorname{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \operatorname{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G}).$$
Then compose with a suitable map from $f_*f^!\mathcal{G}$ to $\mathcal{G}$.]

d. Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat - see §9). Show that φ_i is an isomorphism for all i, all $\mathcal{F}$ coherent on X, and all $\mathcal{G}$ quasi-coherent on Y. [*Hints*: First do $i = 0$. Then do $\mathcal{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathcal{F}$ locally free. Do the general case by induction on i, writing $\mathcal{F}$ as a quotient of a locally free sheaf.]

Solution.

a. As f is finite, $f_*\mathcal{O}_X$ is coherent: locally, on $\operatorname{Spec} A \subset Y$ with preimage $\operatorname{Spec} B \subset X$, we have that $f_*\mathcal{O}_X$ is the sheafification of B_A (B viewed as an A-module) which is coherent exactly because B is a finite A-module. Therefore by exercise III.6.3, $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $\mathcal{O}_Y$-module, and clearly has the structure of a $f_*\mathcal{O}_X$-algebra via acting on the first coordinate. By exercise II.5.17(e), this corresponds to a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

b. As the question is local on Y, we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $f : X \to Y$ is the spectrum of a ring map $\varphi : A \to B$. Writing $\mathcal{F} = \widetilde{F}$ for F a B-module and $\mathcal{G} = \widetilde{G}$ for G an A-module and using the result of exercise III.6.3 on quasi-coherence of $\mathcal{H}om$, we see that we're being asked to prove that there is a natural isomorphism
$$\operatorname{Hom}_B(F, \operatorname{Hom}_A(B_A, G))_A \to \operatorname{Hom}_A(F_A, G).$$
But this is exactly the tensor-hom adjunction, given by evaluating at $1 \in B_A$ on the inside Hom.

c. We follow the hint. In order to construct a natural map $\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \mathrm{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G})$, we construct a morphism $\mathrm{Ext}^i_X(\mathcal{A}, -) \to \mathrm{Ext}^i_Y(f_*\mathcal{A}, f_*(-))$ of functors $\mathfrak{Qco}(X) \to \mathfrak{Ab}$ for a fixed coherent $\mathcal{O}_X$-module $\mathcal{A}$ and plug in $f^!\mathcal{G}$. As f_* is exact because f is affine, we see that $\mathrm{Ext}^i_Y(f_*\mathcal{A}, f_*(-))$ is a δ-functor. By exercise III.3.6(a), $\mathfrak{Qco}(X)$ has enough injectives, so $\mathrm{Ext}^i_X(\mathcal{A}, -)$ is effaceable and therefore by theorem III.1.3A we have that it is universal. So it suffices to construct a morphism $\mathrm{Hom}_X(\mathcal{A}, -) \to \mathrm{Hom}_Y(f_*\mathcal{A}, f_*(-))$, which we can do by taking global sections of the natural morphism $f_*\mathcal{H}om_X(\mathcal{A}, -) \to \mathcal{H}om_Y(f_*\mathcal{A}, f_*(-))$. Thus we've constructed a map $\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \mathrm{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G})$.

We finish by defining a map $f_*f^!\mathcal{G} \to \mathcal{G}$. It suffices to define compatible maps on any affine open subset of Y, and as both sides are quasi-coherent, this means we're looking for a map $(\mathrm{Hom}_A(B_A, G))_A \to G$. We use the evaluation at 1 map, which sends $\varphi \in \mathrm{Hom}_A(B_A, G)$ to $\varphi(1)$, and it's immediate to see that this does everything we would want. Taking $\mathrm{Ext}^i_Y(f_*\mathcal{F}, -)$ of this map we obtain the requested morphisms $\varphi_i : \mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G})$.

d. We discard the hint and proceed in a slicker fashion. First, we note that when $f_*\mathcal{O}_X$ is locally free, we may apply exercise II.5.1 to see that $f^!$ is exact, as $\mathcal{H}om_Y(f_*\mathcal{O}_X, -) \cong (f_*\mathcal{O}_X)^\vee \otimes -$ is an exact functor, as is $\widetilde{-}$ which provides the equivalence of quasi-coherent $f_*\mathcal{O}_X$-modules and $\mathcal{O}_X$-modules (exercise II.5.17(e)). Therefore we may consider $\mathrm{Ext}^i_X(\mathcal{F}, f^!(-))$ and $\mathrm{Ext}^i_Y(f_*\mathcal{F}, -)$ as δ-functors $\mathfrak{Qco}(Y) \to \mathfrak{Ab}$ for fixed $\mathcal{F}$. By our definition of φ_i, they are morphisms of δ-functors. Now I claim that both sides are effaceable: the right side is clear from the general theory of Ext. The left side comes from the fact that $\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G})$ by (b), which lets us apply the same logic as lemma III.6.6 as f_* is exact for affine morphisms. Now, finally, since φ^0 is an isomorphism by (b), we have that φ^i is an isomorphism for all i by theorem III.1.3A.

III.7 The Serre Duality Theorem

Duality is fun! If you're looking for more material on it, check Hartshorne's *Residues and Duality* and Brian Conrad's Duality book.

Exercise III.7.1. Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathcal{L}$ be an ample invertible sheaf on X. The $H^0(X, \mathcal{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

Solution. Suppose not: let $s \in H^0(X, \mathcal{L}^{-1})$ be a nonzero section. Then we can define a morphism $\mathcal{O}_X \to \mathcal{L}^{-1}$ by $1 \mapsto s$, and examining the map on stalks shows that this is injective. Since tensoring with $\mathcal{L}$ preserves injectivity, we obtain a sequence of injective maps

$$\mathcal{L}^n \to \mathcal{L}^{n-1} \to \cdots \to \mathcal{L} \to \mathcal{O}_X.$$

Let φ_n be the map $\mathcal{L}^n \to \mathcal{O}_X$ and note that φ is injective on global sections. Since $\mathcal{O}_X(X)$ is a field by the lemma we proved in exercise II.6.9, we see that for any $x \in X$ and any nonzero element $t \in \mathcal{O}_X(X)$, t_x is invertible in $\mathcal{O}_{X,x}$ and thus $t_x \notin \mathfrak{m}_x$. Therefore if u is a nonzero global section of $\mathcal{L}^n$, u_x cannot be in $\mathfrak{m}_x\mathcal{L}^n_x$, as $\varphi_x(u_x) = (\varphi(X)(u))_x$, which isn't in $\mathfrak{m}_x$. This implies that for any $n \geq 0$ and any nonzero $u \in \mathcal{L}^n(X)$, we have $X_u = X$.

On the other hand in the proof of theorem II.7.6 it is shown that if $\mathcal{L}$ is ample, then for any $x \in X$ there is an $n > 0$ and a section $s \in \mathcal{L}^n(X)$ so that X_s is an affine open neighborhood of x. Therefore X is both affine and projective over k, which implies it's finite by exercise II.4.6 contradicting our assumption that X was of positive dimension.

Exercise III.7.2. Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

a. Show that $f^!\omega_Y^\circ$ is a dualizing sheaf for X, where $f^!$ is defined as in (Ex. 6.10).

b. If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_*\omega_X \to \omega_Y$.

Solution. Secretly what's happening here is that the dualizing complex can be expressed as $\pi_Y^!\omega_{\operatorname{Spec} k}$ for $\pi_Y : Y \to \operatorname{Spec} k$ the structure map while viewing $\pi_Y^!$ as a derived functor. As $\pi_Y \circ f = \pi_X$, we get that $\pi_X^! = \pi_Y^! \circ f^!$ as derived functors, and then we're in a land where all of these are honest functors and the the dualizing complex is concentrated in a single degree. (Generally, $f^!$ only exists in the derived setting - this stuff is lots of fun and I'd encourage you to check out Hartshorne's *Residues and Duality* and/or Brian Conrad's Duality book for more.) On to the problem:

a. Let $\mathcal{F}$ be a coherent sheaf on X. As f is finite, $f_*\mathcal{F}$ is a coherent sheaf on Y, so by duality on Y we have $\operatorname{Hom}_Y(f_*\mathcal{F}, \omega_Y) \cong H^n(Y, f_*\mathcal{F})'$. By exercise III.4.1, we have a natural isomorphism $H^n(Y, f_*\mathcal{F}) \cong H^n(X, \mathcal{F})$ and after taking global sections of the natural isomorphism

$f_*\mathcal{H}om_X(\mathcal{F}, f^!\omega_Y) \cong \mathcal{H}om_Y(f_*\mathcal{F}, \omega_Y)$ from exercise III.6.10(b), we have a natural isomorphism $\operatorname{Hom}_X(\mathcal{F}, f^!\omega_Y) \cong \operatorname{Hom}_Y(f_*\mathcal{F}, \omega_Y)$. Combining these three isomorphisms, we get an isomorphism $\operatorname{Hom}_X(\mathcal{F}, f^!\omega_Y) \cong H^n(X, \mathcal{F})'$.

It remains to check that we can define a trace morphism $H^n(X, f^!\omega_Y) \to k$ so that our isomorphism is induced by the pairing $\operatorname{Hom}_X(\mathcal{F}, f^!\omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, f^!\omega_X)$ followed by the trace. We claim that t_X, the image of $id \in \operatorname{Hom}_X(f^!\omega_Y, f^!\omega_Y)$ under the map $\operatorname{Hom}_X(f^!\omega_Y, f^!\omega_Y) \to H^n(X, f^!\omega_Y)'$, suffices. Consider the following diagram:

$$\begin{array}{ccc} \operatorname{Hom}_X(f^!\omega_Y, f^!\omega_Y) & \longrightarrow & H^n(X, f^!\omega_Y)' \\ \downarrow & & \downarrow \\ \operatorname{Hom}_X(\mathcal{F}, f^!\omega_Y) & \longrightarrow & H^n(X, \mathcal{F})' \end{array}$$

If the left vertical map is given by precomposing with a morphism $\mathcal{F} \to f^!\omega_Y$, then by naturality the right vertical map is the dual of the induced map $H^n(X, \mathcal{F}) \to H^n(X, f^!\omega_Y)$. Therefore the image of id from the top left is mapped to $H^n(X, \mathcal{F}) \to H^n(X, f^!\omega_Y)' \xrightarrow{t_X} k$ in the bottom right, and thus t_X induces the isomorphism we were looking for.

b. By proposition III.7.2, the dualizing sheaf is unique, so we have an isomorphism $\omega_X \to f^!\omega_Y$. After using the adjuction between f_* and $f^!$, this gives a trace map $f_*\omega_X \to \omega_Y$.

Exercise III.7.3. Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q \leq n$.

Solution. We'll need two main ingredients for this exercise: the Euler sequence

$$0 \to \Omega^1_X \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

from theorem II.8.13, and the result of exercise II.5.16(d), which says that for an exact sequence of locally free sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have that $\bigwedge^r \mathcal{F}$ admits a filtration

$$\bigwedge^r \mathcal{F} = F^0 \supset F^1 \supset \cdots \supset F^r \supset F^{r+1} = 0$$

with subquotients $F^p/F^{p+1} \cong (\bigwedge^p \mathcal{F}') \otimes \left(\bigwedge^{r-p} \mathcal{F}''\right)$.

Apply the filtration to a $\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1}$: as $\bigwedge^s \mathcal{O}_X = 0$ for $s > 1$ and $\bigwedge^s \mathcal{O}_X = \mathcal{O}_X$ for $s = 0, 1$, we see that $F^p/F^{p+1} = 0$, or $F^p = F^{p+1}$ for $p < r-2$. Therefore the filtration collapses to $\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \supset \bigwedge^r \Omega_X \supset 0$, and the top subquotient is $\bigwedge^{r-1} \Omega_X$. In other words, we have a short exact sequence

$$0 \to \Omega^r_X \to \bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Omega^{r-1}_X \to 0.$$

Now take cohomology: we get a long exact sequence

$$\cdots \to H^i(X, \Omega^r_X) \to H^i(X, \bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1}) \to H^i(X, \Omega^{r-1}_X) \to \cdots .$$

As $\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \cong \mathcal{O}_X(-r)^N$, we see that the cohomology of the middle term vanishes for all $0 < i < n$ by theorem III.5.1 and $i = 0$ by proposition II.5.13. This implies that $H^i(X, \Omega^r_X) \cong H^{i-1}(X, \Omega^{r-1}_X)$ for all $i > 0$. By an application of corollary III.7.13, we may assume that when calculating $H^q(X, \Omega^p_X)$ that we have $p \geq q$; therefore $H^q(X, \Omega^p_X) \cong H^0(X, \Omega^{p-q}_X)$. The same long exact sequence shows that $H^0(X, \Omega^s_X) = 0$ for $s > 0$, while for $s = 0$ we have $\Omega^0_X \cong \mathcal{O}_X$ and therefore $H^0(X, \Omega^0_X) = k$. This proves the result.

Exercise III.7.4. (*) *The Cohomology Class of a Subvariety.* Let X be a nonsinular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n-p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega^{n-p}_X \to \Omega^{n-p}_Y$. This induces a map on cohomology $H^{n-p}(X, \Omega^{n-p}_X) \to H^{n-p}(Y, \Omega^{n-p}_Y)$. Now $\Omega^{n-p}_Y = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y : H^{n-p}(Y, \Omega^{n-p}_Y) \to k$. Composing, we obtain a linear map $H^{n-p}(X, \Omega^{n-p}_X) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X, \Omega^p_X)$ which we call the *cohomology class* of Y.

a. If $P \in X$ is a closed point, show that $\tau_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X, \Omega^n_X)$ and t_X is the trace map.

b. If $X = \mathbb{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its *degree* as a projective variety (I, §7). [*Hint*: Cut with a hyperplane $H \subset X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

c. For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log : \mathcal{O}^*_X \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathcal{O}^*_X) \to H^1(X, \Omega_X)$ which we denote by c - see (Ex. 4.5).

d. Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathcal{L}(Y))$, where $\mathcal{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

Solution. Unfortunately, part (a) is not correct and this kind of ruins most of the rest of the problem. My verdict: skip it!

a. There's fair bit of unwinding here, and ultimately things don't work out - the fact that the duality map for a point is only defined up to an isomorphism $k \cong k$ mucks everything up.

   Step 1: $n = p$, so $n - p = 0$ and as the empty tensor product is the base ring, we have that the map $\Omega^{n-p}_X \to \Omega^{n-p}_P$ is the natural quotient map $\mathcal{O}_X \to \mathcal{O}_P$. This induces a map on cohomology $H^0(X, \mathcal{O}_X) \cong H^{n-p}(X, \Omega^{n-p}_X) \to H^{n-p}(P, \Omega^{n-p}_P) \cong H^0(P, \mathcal{O}_P)$ which is the restriction of global sections from X to P. These global sections are both just k and the restriction map is the identity. Composing the restriction map $H^0(X, \mathcal{O}_X) \to H^0(P, \mathcal{O}_P)$ with the trace map $t_P : H^0(P, \mathcal{O}_P) \to k$, we get a map $\alpha : H^0(X, \mathcal{O}_X) \cong k \to k$ which can be described by multiplication by a constant which we will also refer to as α.

Step 2: The map $\alpha : H^0(X, \mathcal{O}_X) \cong k \to k$ corresponds by III.7.13 to an element $\eta(P) \in H^n(X, \Omega_X^n) \cong k$. Unwinding the duality statements, we find that the correspondence is as follows: by duality, we have a k-bilinear map

$$f : H^n(X, \omega_X) \times H^0(X, \mathcal{O}_X) \to H^n(X, \omega_X)$$

so that when followed by the k-linear map

$$t_X : H^n(X, \omega_X) \to k,$$

f induces a duality between $H^n(X, \omega_X)$ and $H^0(X, \mathcal{O}_X)$. (f is actually the multiplication of a cohomology class by an element of k, which will be important in the next step.) The way we find the element $\eta(P)$ is as the thing we plug in to the first slot of f so that $t_X(f(\eta(P), -))$ is the same map as $\alpha : H^0(X, \mathcal{O}_X) \to k$ from earlier.

Step 3: Now let's actually figure out what $\eta(P)$ is. The natural map f is just the multiplication of an element of $H^n(X, \omega_X)$ by a global section of $\mathcal{O}_X$, and as $H^n(X, \omega_X)$ is one-dimensional, $t_X(-)$ can be written as multiplication by some $c \in k^*$. As $H^0(X, \mathcal{O}_X) = k$, we can test for agreement between α and $t_X(f(\eta(P), -))$ by plugging in 1 in both maps: $\alpha(1) = \alpha$, while $t_X(f(\eta(P), 1)) = c\eta(P)$. So $\eta(P) = \alpha/c$, and evaluating t_X on it we find that $t_X(\alpha/c) = c\alpha/c = \alpha$. Thus if we chose α to be the identity then we would have the desired equality, but this is not the definition: the trace morphism is only specified up to a constant in the definition on page 241.

b. This problem is wrong because part (a) is wrong - if the hint works as intended, you can indeed reduce to Y a finite set of $\deg Y$ smooth points, and then you want to claim that $\eta(Y)$ is the sum over all the points of $\eta(P) = 1$ by (a). But that last bit's not true!

c. There's not really much to verify here: fg gets sent to

$$(fg)^{-1} d(fg) = (fg)^{-1} f dg + (fg)^{-1} g df = f^{-1} df + g^{-1} df,$$

and this is compatible with restriction; as H^1 is a functor, we get the map c.

d. This is again ruined by the fact that (a) is wrong. If it were the case that (a) were correct (and (b), too) then the point would be to show that if $\deg Y = d$ and therefore $\mathcal{L}(Y) \cong \mathcal{O}(d)$, then $c(\mathcal{O}(d)) = d$ in $H^1(X, \Omega_X)$. To verify this, it's enough to show that $\mathcal{O}(1)$ maps to $1 \in H^1(X, \Omega_X)$ under c, which can be checked by working in Čech cohomology: $\mathcal{O}(1)$ is represented by $\{(U_{ij}, \frac{x_i}{x_j})\}$ in $C^1(\mathfrak{U}, \mathcal{O}_X^*)$ which maps to $\{(U_{ij}, \frac{dx_i}{x_i} - \frac{dx_j}{x_j})\}$ in $C^1(\mathfrak{U}, \Omega_X)$, which is a perfectly reasonable choice of 1 for $H^1(X, \Omega_X)$.

III.8 Higher Direct Images of Sheaves

Don't worry if you feel like you could enjoy these problems more - we'll have more fun with higher direct images later.

Exercise III.8.1. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$. Show that there are natural isomorphism, for each $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

(This is a degenerate case of the Leray spectral sequence - see Godement [1, II, 4.17.1].)

Solution. Embed $\mathcal{F}$ in to an injective sheaf $\mathcal{I}$ and let $\mathcal{Q}$ be the cokernel. Then applying $R^\bullet f_*$ to this exact sequence, we get a long exact sequence of sheaves on Y:

$$0 \to R^0 f_*\mathcal{F} \to R^0 f_*\mathcal{I} \to R^0 f_*\mathcal{Q} \to R^1 f_*\mathcal{F} \to \cdots.$$

As $R^i f_*\mathcal{F}$ and $R^i f_*\mathcal{I}$ vanish for all $i > 0$ (by assumption and by injectivity, respectively) we have that $R^i f_*\mathcal{Q} = 0$ for all $i > 0$ as well. Therefore if $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, we see that $0 \to f_*\mathcal{F} \to f_*\mathcal{I}^\bullet$ is also exact. Because an injective sheaf is flasque and the pushforward of a flasque sheaf is again flasque, we see that $\Gamma(Y, f_*\mathcal{I}^\bullet)$ computes cohomology of $f_*\mathcal{F}$. But this is just the complex $\Gamma(X, \mathcal{I}^\bullet)$, which computes cohomology of $\mathcal{F}$, so the cohomologies are the same.

Exercise III.8.2. Let $f : X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for each $i \geq 0$. (This gives another proof of (Ex. 4.1).)

Solution. Let $\operatorname{Spec} A \subset X$ be an affine open subscheme. Then $f^{-1}(\operatorname{Spec} A) \subset Y$ is again affine, say $\operatorname{Spec} B$. By theorem III.3.5, we have that $H^i(\operatorname{Spec} B, \mathcal{F}|_{\operatorname{Spec} B}) = 0$ for all $i > 0$, and thus the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ assigns every affine open 0 for $i > 0$. Since every open set in X can be covered by affine opens, we see that the sheafification of this presheaf is the zero sheaf for $i > 0$. Since this is precisely the definition of $R^i f_*\mathcal{F}$, we have the desired result.

Exercise III.8.3. Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the *projection formula* (cf. (II, Ex. 5.1))

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Solution. Since tensoring with a locally free sheaves of finite rank is an exact functor, both sides are δ-functors from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules. By exercise II.5.1(d) these functors agree for $i = 0$, so it suffices to show that they're coeffaceable by theorem III.1.3A.

The right hand side is coeffaceable because $R^i f_*$ is. To show that the left hand side is also coeffaceable, suppose $U \subset Y$ is an open subset where $\mathcal{E}$ is free. Then for any $\mathcal{O}_X$-module $\mathcal{G}$, we have $R^i f_*(\mathcal{G} \otimes f^*\mathcal{E})|_U \cong R^i f_*(\mathcal{G} \otimes f^*(\mathcal{E}|_U)) \cong R^i f_*(\mathcal{G}^{\oplus n})$, so plugging in $\mathcal{I}$ for $\mathcal{G}$, we see that these vanish for all $i > 0$, therefore $R^i f_*(\mathcal{I} \otimes f^*\mathcal{E})$ vanishes for all $i > 0$. Taking an injection from $\mathcal{F}$ to an injective object $\mathcal{I}$, we see that the right hand side is also coeffaceable and we're done.

Exercise III.8.4. Let Y be a noetherian scheme, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geq 1$. Let $X = \mathbb{P}(\mathcal{E})$ (II, §7), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

a. Then $\pi_*(\mathcal{O}(l)) \cong S^l(\mathcal{E})$ for $l \geq 0$, $\pi_*(\mathcal{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.

b. Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathcal{E})(-1) \to \mathcal{O} \to 0,$$

cf. (II, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathcal{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ (cf (7.1.1)).

c. Now show, for any $l \in \mathbb{Z}$, that

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathcal{E})^\vee.$$

d. Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

e. In particular, if Y is a nonsingular projective curve of genus g, and $\mathcal{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).

Solution. This problem is basically checking that everything we defined in III.5 is natural enough to glue. It's both miserable and obvious.

a. The first two statements are found in proposition II.7.11. The latter two statements can be deduced from the vanishing of $H^i(\mathbb{P}^n_A, \mathcal{O}(l))$ found in theorem III.5.1 and the argument that the vanishing of $H^i(f^{-1}(\operatorname{Spec} A), \mathcal{F}|_{f^{-1}(\operatorname{Spec} A)})$ for all $\operatorname{Spec} A \subset Y$ implies the vanishing of $R^i f_*\mathcal{F}$ from the previous exercise.

b. By proposition II.7.11(b), we have a natural surjective morphism $\pi^*\mathcal{E} \to \mathcal{O}(1)$, which after twisting by -1 gives a surjective morphism $(\pi^*\mathcal{E})(-1) \to \mathcal{O}$. Locally, on the inverse image of any affine open subscheme of Y where $\mathcal{E}$ is free, we have that this recovers the exact sequence of theorem II.8.13, or that $\ker((\pi^*\mathcal{E})(-1) \to \mathcal{O})$ is locally isomorphic to $\Omega_{\mathbb{P}^n_A/\operatorname{Spec} A}$. Further, this isomorphism is independent of the choice of basis of $\mathcal{E}|_{\operatorname{Spec} A}$ by the proof of theorem II.8.13.By the computation of the map $\Omega_{\mathbb{P}^n_A/\operatorname{Spec} A} \to \mathcal{E}(-1)|_{\mathbb{P}^n_A}$ from theorem II.8.13, we see that restriction of the map is the same as the map we obtain over $\mathbb{P}^n_{A_a}$, so it glues as we expect and we obtain the desired exact sequence.

By exercise II.5.16(d), we have that

$$\bigwedge^{n+1} (\pi^*\mathcal{E})(-1) \cong (\bigwedge^{n} \Omega_{X/Y}) \otimes (\bigwedge^{1} \mathcal{O}),$$

or

$$\bigwedge^{n+1}(\pi^*\mathcal{E})(-1) \cong \bigwedge^{n}\Omega_{X/Y} \cong \omega_{X/Y}.$$

There is a canonical homomorphism $(\pi^*\mathcal{E}\otimes\mathcal{O}(-1))^{\otimes n+1} \to (\bigwedge^{n+1}\pi^*\mathcal{E})\otimes(\mathcal{O}(-1)^{\otimes n+1})$ inducing $\bigwedge^{n+1}((\pi^*\mathcal{E})(-1)) \to (\bigwedge^{n+1}\pi^*\mathcal{E})(-n-1)$ which is locally an isomorphism as $\mathcal{O}(-1)$ is locally free. This proves the assertion that $\omega_{X/Y} \cong (\pi^*\mathcal{E})(-n-1)$.

To obtain the final isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$, we can compute $R^n\pi_*(\omega_{X/Y})$ locally on Y and show that the result glues correctly. Over any affine open $\operatorname{Spec} A \subset Y$ where $\mathcal{E}$ is trivial, we recover that $\Gamma(\operatorname{Spec} A, R^n\pi_*(\omega_{X/Y})) = H^n(\mathbb{P}^n_A, \omega_{X/Y}) = A$. By the computations of exercise III.5.9, we have that $\omega_{X/Y}|_{\mathbb{P}^n_A} \cong \mathcal{O}(-n-1)$ and linear changes of coordinates on $\mathbb{P}^n_A$ preserve the generator of $H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) = A\langle\frac{1}{x_0\cdots x_n}\rangle$ by the Čech cohomology computation of theorem III.5.1. Since that computation doesn't depend on A, it behaves correctly under localization of A, and therefore we can patch the local isomorphisms $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ together to a natural global isomorphism.

c. On any open subscheme $U \subset Y$, there's a natural pairing

$$\operatorname{Hom}_{\pi^{-1}(U)}(\mathcal{O}(l)|_{\pi^{-1}(U)}, \omega_{X/Y}|_{\pi^{-1}(U)}) \times H^n(\pi^{-1}(U), \mathcal{O}(l)) \to H^n(\pi^{-1}(U), \omega_{X/Y})$$

which induces a pairing of sheaves

$$\pi_*\mathcal{H}om_X(\mathcal{O}(l), \omega_{X/Y}) \times R^n\pi_*\mathcal{O}(l) \to R^n\pi_*\omega_{X/Y}.$$

This gives a map

$$R^n\pi_*\mathcal{O}(l) \to \mathcal{H}om_Y(\pi_*\mathcal{H}om_X(\mathcal{O}(l), \omega_{X/Y}), R^n\pi_*\omega_{X/Y})$$

which locally over an affine open subscheme $\operatorname{Spec} A \subset Y$ where $\mathcal{E}$ is free is an isomorphism by the calculation of theorem III.5.1; therefore this map is an isomorphism.

By our computation from (b), we have that $R^n\pi_*\omega_{X/Y} \cong \mathcal{O}_Y$, so

$$R^n\pi_*\mathcal{O}(l) \cong \mathcal{H}om_Y(\pi_*\mathcal{H}om_X(\mathcal{O}(l), \omega_{X/Y}), \mathcal{O}_Y) \cong \pi_*\mathcal{H}om_X(\mathcal{O}(l), \omega_{X/Y})^\vee.$$

Applying exercise II.5.1, we get $\mathcal{H}om_X(\mathcal{O}(l), \omega_{X/Y}) \cong \omega_{X/Y}(-l)$, so

$$R^n\pi_*\mathcal{O}(l) \cong (\pi_*\omega_{X/Y}(-l))^\vee.$$

By part (b), we have that $\omega_{X/Y} \cong (\pi^*\bigwedge^{n+1}\mathcal{E})(-n-1)$, so

$$R^n\pi_*\mathcal{O}(l) \cong (\pi_*(\pi^*\bigwedge^{n+1}\mathcal{E})(-l-n-1))^\vee,$$

which is $(\bigwedge^{n+1}\mathcal{E})^\vee \otimes \pi_*(\mathcal{O}(-l-n-1))$ by the projection formula.

d. Here Hartshorne is assuming we're working with projective schemes over a field because that's the context he's defined these invariants in, but he does not specify this.

   By part (a), we have $R^i\pi_*\mathcal{O}_X \cong \mathcal{O}_Y$ for $i = 0$ and 0 otherwise, so the assumptions of exercise III.8.1 hold for the sheaf $\mathcal{O}_X$, and we have $H^i(X, \mathcal{O}_X) \cong H^i(Y, \mathcal{O}_Y)$ for all i. As $\dim X = \dim Y + n$, the result $p_a(X) = (-1)^n p_a(Y)$ follows (see exercise III.5.3 if you need a reminder about the definition).

   Recall that Hartshorne defines the geometric genus of a nonsingular variety over k to be $\dim_k \Gamma(X, \omega_X)$, so in order to compute it here, we should assume Y is nonsingular. By proposition II.8.11, we have an exact sequence

$$\pi^*\Omega_{Y/\operatorname{Spec} k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

   By theorem II.8.15, $\Omega_{Y/\operatorname{Spec} k}$ is locally free, and as X is locally isomorphic to $\mathbb{P}^n_k \times_k Y$ we may apply exercise II.8.3 to see that our sequence is locally split, which implies that it's actually a short exact sequence. Thus we may apply exercise II.5.16(d) to see that $\omega_{X/\operatorname{Spec} k} \cong \omega_{X/Y} \otimes \pi^*\omega_{Y/\operatorname{Spec} k}$. Therefore $\omega_{X/\operatorname{Spec} k} \cong \pi^*(\omega_{Y/\operatorname{Spec} k} \otimes \bigwedge^{n+1} \mathcal{E})(-n-1)$, which can be seen to have no global sections as follows: let $U \subset Y$ be an affine open where the line bundle $\omega_{Y/\operatorname{Spec} k} \otimes \bigwedge^{n+1} \mathcal{E}$ is trivial; then $\omega_{X/\operatorname{Spec} k}|_{\pi^{-1}(U)} \cong \mathcal{O}_{\mathbb{P}^n_U}(-n-1)$ which has no global sections. Thus $p_g(X) = 0$.

e. Yes, exactly.

III.9 Flat Morphisms

Flatness is a concept that is famously difficult to gather geometric intuition for. The biggest insights I have are that it's the condition that sheaves/modules restrict 'the way they should' to closed subschemes; it's equivalent to openness in the presence of even mild finiteness conditions; and that it's wonderful.

Exercise III.9.1. A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open, i.e., for every open subset $U \subset X$, $f(U)$ is open in Y. [*Hint*: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

Solution. By exercise II.3.19, $f(X) \subset Y$ is constructible; by exercise II.3.18, it suffices to show that $f(X)$ is stable under generalization to show that it's open. As a set is open iff it's intersection with every element of an open cover is open, we may reduce to the case when $Y = \operatorname{Spec} R$ is affine. Next, covering U by affine open subschemes, it suffices to prove the claim when $X = \operatorname{Spec} A$ is affine because the open immersion $U \to X$ is of finite type (this relies on X noetherian).

Now suppose $\mathfrak{p}_1 \in \operatorname{Spec} R$ is a point in $f(X)$, and let $\mathfrak{p}_2 \subset \mathfrak{p}_1$ be a generalization. Our assumption that $\mathfrak{p}_1 \in f(X)$ means there is a prime $\mathfrak{q}_1 \subset A$ with inverse image $\mathfrak{p}_1$ under our map $R \to A$, and we're looking for a prime ideal $\mathfrak{q}_2 \subset \mathfrak{q}_1$ with preimage $\mathfrak{p}_2$. This is exactly Going Down, which holds because $A \to B$ is flat.

(Let me also remark that this theorem is true in greater generality: if $f : X \to Y$ is a morphism of schemes, then locally of finite presentation + flat = universally open, see Stacks Project tag 00F3.)

Exercise III.9.2. Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

Solution. The setup from exercise I.3.14 is that we're projecting the twisted cubic

$$V(x_1^2 - x_0x_2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2)$$

from the point $[0 : 0 : 1 : 0]$ to the cuspidal cubic $y^3 - x^2z$. What happens in example III.9.8.4 is we scale the coordinate we're projecting from, which in this case is x_2. Thus

$$I(X_a) = (x_1^2 - a^{-1}x_0x_2, x_1x_3 - a^{-2}x_2^2, x_0x_3 - a^{-1}x_1x_2).$$

The next step is to take the scheme-theoretic image of this under the inclusion

$$\mathbb{P}^3_{k[a,a^{-1}]} \to \mathbb{P}^3_{k[a]},$$

which is exactly Proj of the kernel of

$$k[a][x_0, x_1, x_2, x_3] \to k[a, a^{-1}][x_0, x_1, x_2, x_3]/(x_1^2 - a^{-1}x_0x_2, x_1x_3 - a^{-2}x_2^2, x_0x_3 - a^{-1}x_1x_2).$$

This kernel consists of all the elements of the form $a^i f$ which are in $k[a][x_0, x_1, x_2, x_3]$ for $i \in \mathbb{Z}$ and $f \in I(X_a)$, so clearly $(ax_1^2 - x_0x_2, a^2x_1x_3 - x_2^2, ax_0x_3 - x_1x_2)$ is in this kernel. I claim that all we need to do get the full kernel is to add $x_0^2x_3 - x_1^3$: it's not too much trouble to verify that $a(x_0^2x_3 - x_1^3) = -x_1(ax_1^2 - x_0x_2) + x_0(ax_0x_3 - x_1x_1)$ is in our ideal, therefore $x_0^2x_3 - x_1^3$ is in the kernel, and the same calculation as in exercise I.2.9 where we send $x_0 \mapsto t^3$, $x_1 \mapsto t^2u$, $x_2 \mapsto atu^2$, and $x_3 \mapsto u^3$ will show that this is enough to form a generating set.

Taking the fiber over $a = 0$, we see we're looking at $\operatorname{Proj} k[x_0, x_1, x_2, x_3]/(x_0x_2, x_1x_2, x_2^2, x_0^2x_3 - x_1^3)$ which is the cuspidal cubic with a nonreduced origin.

Exercise III.9.3. Some examples of flatness and nonflatness.

a. If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field, then f is flat.

b. Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and $X = \operatorname{Spec} k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.

c. Again let $Y = \operatorname{Spec} k[x, y]$ but take $X = \operatorname{Spec} k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{red} \cong Y$, X has no embedded points, but that f is not flat.

Solution.

a. This is a baby case of exercise III.10.9, which usually goes by the name 'miracle flatness'. On the other hand, the fact that the fiber dimension is zero doesn't actually make the argument any shorter - you're advised to just read the proof of exercise III.10.9.

b. The map $k[x, y] \to k[x, y, z, w]/(z, w) \cap (x + z, y + w)$ by $x \mapsto x$ and $y \mapsto y$ is a finite map of rings: $k[x, y, z, w]/(z, w) \cap (x + z, y + w)$ is spanned by $1, z, w$ as a $k[x, y]$-module since $z^2 = -xz$ since $z(x + z) \in (z, w) \cap (x + z, y + w)$ and similarly for w^2. We can embed X in to $\mathbb{A}^2_Y$ via the obvious surjection from $(k[x, y])[z, w]$, and then embed this in to $\mathbb{P}^2_Y$ in the obvious fashion. As X is finite over Y, it is proper, and this map is actually a closed immersion. So we can view $X \subset \mathbb{P}^n_Y$ and apply theorem III.9.9 to see that if $X \to Y$ were flat, the Hilbert polynomial of the fibers shouldn't depend on the fiber. We'll show that this is not the case - since the fibers are finite, the Hilbert polynomial will just be the dimension over k of their coordinate rings.

   We start by writing $(z, w) \cap (x+z, y+w)$ as $(z^2+xz, yz+wz, wx+wz, yw+w^2)$. Computing the coordinate ring of the fiber over $x = a$, $y = b$ is equivalent to to computing $k[x, y, z, w]/(z^2 + xz, yz + wz, wx + wz, yw + w^2, x - a, y - b)$. Over the point $x = 1$, $y = 0$, the fiber is the spectrum of $k[z, w]/(z^2 + z, wz, w + wz, w^2) \cong k[z, w]/(z^2 + z, w)$, a two-dimensional vector space. Over the point $x = 0$, $y = 0$, the fiber is the spectrum of $k[z, w]/(z^2, wz, wz, w^2) \cong k[z, w]/(z^2, wz, w^2)$, a three-dimensional vector space. Thus the map is not flat.

c. The map $k[x, y] \to k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$ by $x \mapsto x$ and $y \mapsto y$ is also a finite map of rings (the target has spanning set $1, z, w$ as a $k[x, y]$-module, for instance), and we'll use the same logic as in part (b) to show that it is not flat. The coordinate ring of the fiber over

$x = 1$, $y = 0$ is $k[z, w]/(z^2, zw, w^2, z) \cong k[w]/(w^2)$, which is two dimensional. The coordinate ring of the fiber over $x = 0$, $y = 0$ is $k[z, w]/(z^2, zw, w^2)$, which is three dimensional, and thus $X \to Y$ is not flat.

To check that X has no embedded points, let's first remember what an embedded point is - Hartshorne doesn't do a great job of defining them. For an affine scheme $\operatorname{Spec} A$ and a quasi-coherent sheaf $\widetilde{M}$, an associated point is an associated prime of M, or a prime ideal $\mathfrak{p} \subset A$ so that M has an element m with $\operatorname{Ann}(m) = \mathfrak{p}$. An embedded point is a non-minimal associated prime of the structure sheaf, which in this case is a non-minimal associated prime of $k[X] \cong k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$.

After modding out by the nilradical (z, w) we see that this ring is a domain, so (z, w) is the unique minimal prime of $k[X]$. We'll show that the annihilators of nonzero elements of $k[X]$ are precisely (z, w) or 0. First I claim that any element of (z, w) has annihilator exactly (z, w): clearly z and w annihilate any element of (z, w), which shows that the annihilator contains (z, w). To show the annihilator is contained in (z, w), we can write any element $f \in k[X]$ uniquely as $f = p(x, y) + zq(y) + wr(x, y)$ by applying the relation $xz = yw$. Now suppose $f \in (z, w)$, or $p = 0$, while $f' = p'(x, y) + zq'(y) + wr'(x, y)$ is such that $ff' = 0$: this gives that $zq(y)p'(x, y) + wr(x, y)p'(x, y) = 0$. Writing $p'(x, y) = \sum_{i=0}^{d} a_i(y)x^i$ for d minimal, this expands to

$$zq(y)a_0(y) + w(r(x, y)p'(x, y) + \sum_{i=1}^{d} ya_i(y)x^{i-1}).$$

If this is zero, by looking at the coefficient of wx^d we see that $d = 0$, and then that $a_0 = 0$, or $p' = 0$ and $f' \in (z, w)$. So no element of (z, w) has annihilator which is a non-minimal associated prime.

Next I claim that the annihilator of all elements outside of (z, w) is zero: the above computation shows that no element of (z, w) can annihilate an element outside (z, w), while if $f, g \notin (z, w)$, then $fg \notin (z, w)$ because (z, w) is prime, and in particular $fg \neq 0$. So the only annihilators of nonzero elements of $k[X]$ are 0 and (z, w), neither of which is a non-minimal prime of $k[X]$, so X has no embedded points.

Exercise III.9.4. *Open Nature of Flatness.* Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X \mid f \text{ is flat at } x\}$ is an open subset of X (possibly empty) - see Grothendieck [EGA IV3, 11.1.1].

Solution. This is not a particularly enjoyable problem to prove yourself, and it's covered well enough in the literature you should probably just cite/read it if you can help it. Besides the EGA reference in the problem statment, one can also consult Stacks Project tag 0398 or Matsumura's *Commutative Ring Theory*, theorem 24.3.

We'll need some preliminary results, some taken from the Stacks Project's exposition and one from Matsumura. Here they are:

Lemma (Better Generic Finiteness)**.** *Let R be a domain, and suppose $R \to S$ is a finite type extension of domains. Then there exists an integer d and a factorization of $R \to S$ as $R \to$*

$R[x_1, \cdots, x_d] \to S' \to S$ so that S' is finite over $R[x_1, \cdots, x_d]$ and $S_f \cong S'_f$ for some nonzero $f \in R$.

Proof. This is a slightly upgraded version of exercise II.3.7 (or a worse version of tag 07NA in the Stacks Project).

Pick generators $x_1, \cdots, x_n \in S$ for S as an R-algebra. Denote $K = \mathrm{Frac}(R)$ and $S_K = S \otimes_R K$. Then S_K is a domain which is finitely generated over K, and $\mathrm{Frac}(S_K) = K(x_1, \cdots, x_n)$. Up to reordering, we may assume $x_1, \cdots, x_d$ form a transcendence basis for $\mathrm{Frac}(S_K)/K$ (where $d = \dim \operatorname{Spec} S \otimes_R K$) so that $\mathrm{Frac}(S_K)$ is finite over $K(x_1, \cdots, x_d)$ and generated by $x_{d+1}, \cdots, x_n$. (We can generalize this to S not necessarily a domain at the price of making this step a bit more involved, see Stacks Project 07NA.)

Now let $p_i \in K[x_1, \cdots, x_d][T]$ be a monic polynomial vanishing on x_i for $i > d$. Recalling that $K = \mathrm{Frac}(R)$ and taking the product of the denominators present in each p_i, we can find $f \in R$ so that $fp_i \in R[x_1, \cdots, x_d][T]$ for all i so that $fp_i(x_i) = 0$ in S_K. Therefore after possibly replacing f by fr for r another nonzero element in R, we may assume that $fp_i(x_i) = 0$ in S. Setting $x'_i = fx_i$, let $S' \subset S$ be the sub-R-algebra generated by $x_1, \cdots, x_d$ and x'_i for $d < i \leq n$. Since $f^{\deg_T p_i} p_i(T/f)$ vanishes at $T = x'_i$, we have that each x'_i is integral over $R[x_1, \cdots, x_d]$ and therefore S' is an integral and thus finite extension of $R[x_1, \cdots, x_d]$. Since $S' \subset S$, we have that $S'_f \subset S$, but on the other hand, $x_i = x'_i/f \in S'_f$ generate S, so $S'_f \supset S_f$ and we're done.

■

Lemma (Generic Freeness). *Suppose R is a domain, A is an algebra of finite type over R, and M is a finitely-generated A-module. Then there exists some nonzero $f \in R$ so that M_f is a free R_f-module.*

Proof. This is a slightly upgraded version of exercise II.5.7 and covered as tag 051R in the Stacks Project.

First we can reduce to the case where $M = A$: by the ungraded version of proposition I.7.4, we can find a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of M so that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i \subset A$. If we can show that for each M_i/M_{i-1} there is some f_i so that $(M_i/M_{i-1})_{f_i}$ is a free R_{f_i}-module, we're done: letting $f = \prod f_i$, we have that the subquotients of our filtration are free R_f-modules, and any extension of free modules is free. So it suffices to treat the case when A is a domain and $M = A$.

Let $K = \mathrm{Frac}(R)$ and $A_K = A \otimes_R K$. We will argue by induction on $\dim \operatorname{Spec} A_K$, which is finite type because A_K is of finite type over a field. If $A_K = 0$, then $R \to A$ has nontrivial kernel and taking f to be any nonzero element of this kernel, we have $A_f = 0$.

Now we assume $R \to A$ is an extension of domains. Applying our upgraded version of generic finiteness, we may replace R by R_f as in the conclusion of the previous lemma to get a factorization $R \subset R[x_1, \cdots, x_d] \subset A$ where the second extension is finite. Choose $y_1, \cdots, y_r \in A$ which form a basis for $\mathrm{Frac}(A)$ over $\mathrm{Frac}(R[x_1, \cdots, x_d])$, and let Q be the quotient of A by the image of $R[x_1, \cdots, x_d]^{\oplus r} \to A$ given by sending the i^{th} basis vector to y_i. Then Q is a finitely-generated module not supported at the generic point of $\operatorname{Spec} R[x_1, \cdots, x_d]$, so there is $g \in R[x_1, \cdots, x_d]$ with Q a finite $R[x_1, \cdots, x_d]/(g)$-module. As the fiber of $R[x_1, \cdots, x_d]/(g)$ over the generic point is of dimension less than d, the inductive hypothesis gives that there is some $f \in R$ so that Q_f is a free R_f-module and we're done.

■

Lemma (Local Criteria for Flatness)**.** *Suppose $R \to S$ is a local homomorphism of noetherian local rings, $I \subset R$ is a proper ideal, and M is a finite S-module. M is flat over R iff M/IM is flat over R/I and $\operatorname{Tor}_1^R(R/I, M) = 0$.*

Proof. This is (1) $\Leftrightarrow$ (3') in theorem 22.3 of Matsumura's *Commutative Ring Theory.*

■

On to the problem. We may immediately reduce to the case when $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$ are affine. Further, as X is of finite type over B, we may embed $X \hookrightarrow \mathbb{A}^n_Y$ and reduce to proving that for a finitely-generated module M over $A = R[x_1, \cdots, x_n]$, the locus of points where $\widetilde{M}$ is flat over Y is open.

We'll show that the set is constructible and stable under generalization, which is enough to show openness by exercise II.3.17. To be more precise, we'll show that the flat locus is constructible by showing that if $\mathfrak{p}$ is in the flat locus, then $V(\mathfrak{p})$ contains an open which belongs to the flat locus (this is essentially the argument we used in exercise II.3.19 to show that the set-theoretic image of a morphism of finite type of noetherian schemes is constructible).

Stable under generalization is straightforward: suppose $\mathfrak{p}_1 \subset \mathfrak{p}_2$ are primes in $R[x_1, \cdots, x_n]$ with $\mathfrak{q}_i = R \cap \mathfrak{p}_i$ and $M_{\mathfrak{p}_2}$ a flat $R_{\mathfrak{q}_2}$-module. Since $M_{\mathfrak{p}_1}$ is the localization of the flat $R_{\mathfrak{p}_1}$-module $M_{\mathfrak{p}_2} \otimes_{R_{\mathfrak{p}_2}} R_{\mathfrak{p}_1}$, and localization preserves flatness, we have that $M_{\mathfrak{p}_1}$ is flat over $A_{\mathfrak{q}_1}$.

The second condition is where we use all the preliminary results. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset R[x_1, \cdots, x_n]$ be prime ideals with $\mathfrak{q}_i = \mathfrak{p}_i \cap R$. Suppose M is flat over R at $\mathfrak{p}_1$, which means $M_{\mathfrak{p}_1}$ is a flat $R_{\mathfrak{q}_1}$-module. To show that M is flat over R at $\mathfrak{p}_2$ we need to show that $M_{\mathfrak{p}_2}$ is flat over $R_{\mathfrak{q}_2}$, which by the local criteria is equivalent to showing that $M_{\mathfrak{p}_2}/\mathfrak{q}_1 M_{\mathfrak{p}_2}$ is flat over $R_{\mathfrak{q}_2}/\mathfrak{q}_1 R_{\mathfrak{q}_2}$ and $\operatorname{Tor}_1^{R_{\mathfrak{q}_2}}(R_{\mathfrak{q}_2}/\mathfrak{q}_1 R_{\mathfrak{q}_2}, M_{\mathfrak{q}_2}) = 0$.

To show that the first part of the local criteria holds for an open set of $\mathfrak{p}_2 \in V(\mathfrak{p}_1)$, we apply generic freeness to $M/\mathfrak{q}_1 M$ over $R/\mathfrak{q}_1$ as a finite $(R/\mathfrak{q}_1)[x_1, \cdots, x_n]$-module. We obtain a nonzero $f \in R/\mathfrak{q}_1$ with $(M/\mathfrak{q}_1 M)_f$ a free $(R/\mathfrak{q}_1)_f$-module. Lifting $f \in R/\mathfrak{q}_1$ to $\overline{f} \in R \setminus \mathfrak{q}_1$, this gives that $(M/\mathfrak{q}_1 M)_f \cong (M/\mathfrak{q}_1 M)_{\overline{f}}$ is a free $(R/\mathfrak{q}_1)_{\overline{f}}$-module, and thus for any $\mathfrak{p}_2$ not containing $\overline{f}$, we have that $(M/\mathfrak{q}_1 M)_{\mathfrak{p}_2}$ is a free $(R/\mathfrak{q}_1)_{\mathfrak{p}_2} = (R/\mathfrak{q}_1)_{\mathfrak{q}_2}$-module, and we're done with the first part of the local criteria.

To attack the statment involving Tor, we first note that $\operatorname{Tor}_1^R(R/\mathfrak{q}_1, M)$ is a finitely generated $R[x_1, \cdots, x_n]$-module because $R/\mathfrak{q}_1$ admits a resolution by finite free R-modules. This implies that the support of its associated sheaf on $\operatorname{Spec} R[x_1, \cdots, x_n]$ is closed. Now by the assumption that $M_{\mathfrak{p}_1}$ is flat over $R_{\mathfrak{q}_1}$, we may apply the local criteria to see that $\operatorname{Tor}_1^{R_{\mathfrak{q}_1}}(R_{\mathfrak{q}_1}/\mathfrak{q}_1 R_{\mathfrak{q}_1}, M_{\mathfrak{p}_1}) = 0$. Writing this as $\operatorname{Tor}_1^R(R/\mathfrak{q}_1, M) \otimes_{R[x_1, \cdots, x_n]} R[x_1, \cdots, x_n]_{\mathfrak{p}_1}$, we see that this means that the sheaf associated to $\operatorname{Tor}_1^R(R/\mathfrak{q}_1, M)$ has vanishing stalk at $\mathfrak{p}_1$, so this sheaf vanishes on a dense open subset of $V(\mathfrak{p})$, which gives the second part of the local criteria from the same localization at the start of this sentence. We're finished.

Exercise III.9.5. *Very Flat Families.* For any closed subscheme $X \subset \mathbb{P}^n$, we denote by $C(X) \subset \mathbb{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subset k[x_0, \cdots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generate by I in $k[x_0, \cdots, x_{n+1}]$.

a. Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbb{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbb{P}^{n+1}$.

b. To remedy this situation, we make the following definition. Let $X \subset \mathbb{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subset S_t = k(t)[x_0, \cdots, x_n]$ be the homogeneous ideal of X_t in $\mathbb{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is *very flat* if for all $d \geq 0$,
$$\dim_{k(t)}(S_t/I_t)_d$$
is independent of t. Here $(\ \ )_d$ means the homogeneous part of degree d.

c. If $\{X_t\}$ is a very flat family in $\mathbb{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbb{P}^{n+1}$, and hence flat.

d. If $\{X_{(t)}\}$ is an algebraic family of projectively normal varieties in $\mathbb{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_{(t)}\}$ is a very flat family of schemes.

Solution.

a. Before we propose a counterexample, let's explore why something could go wrong by comparing the Hilbert polynomial of a subscheme $X \subset \mathbb{P}^n_k$ and it's cone $C(X) \subset \mathbb{P}^{n+1}_k$. If $I \subset k[x_0, \cdots, x_n]$ is the homogeneous ideal of X, then the homogeneous ideal of $C(X)$ is the ideal I' of $k[x_0, \cdots, x_{n+1}]$ generated by I. This means that $\dim_k I'_m = \sum_{i=0}^m \dim_k I_i$ (equivalently, $P_{C(X)}(m) = \sum_{i=0}^m \dim_k(k[x_0, \cdots, x_n]/I)_i$) which could be a problem: if we can find a family X_t so that $\dim_{k(t)}(I_{X_t})_m$ is independent of t for $m \geq m_0$ but $\sum_{i=0}^{m_0} \dim_{k(t)}(I_{X_t})_i$ depends on t, then $P(C(X_t))$ depends on t, which implies $C(X_t)$ isn't a flat family by theorem III.9.9.

If you've played with Hilbert polynomials and Hilbert functions before, this should give you a good idea of where to look and what example we're going to pick: three (reduced, closed) points in $\mathbb{P}^2$ have Hilbert function
$$\begin{cases} 0 & n < 1 \\ 2 & n = 1 \\ 3 & n > 1 \end{cases}$$
if they're collinear, while they have Hilbert function
$$\begin{cases} 0 & n < 1 \\ 3 & n \geq 1 \end{cases}$$

if they're not collinear. The proof is straightforward: define a map $k[x_0, x_1, x_2]_m \to k^3$ by evaluation at representatives for our three points. For $n > 1$, the map is surjective: to find the preimage of the i^{th} basis vector, consider an appropriate scaling of $x_j^{n-2} p_k$, where x_j is a monomial not vanishing on the i^{th} point and p_k is a quadratic polynomial vanishing on both points which aren't the i^{th} point. Therefore $\dim_k(k[x_0, x_1, x_2]/I)_n = 3$ for $n > 1$. When $n = 1$, $\dim_k k[x_0, x_1, x_2]_1 = 3$, and there is a linear form vanishing on all three points iff they're in a line, so our claim about the Hilbert functions is proven. Now we can use the family $\{[1:0:0], [0:1:0], [1:1:t]\} \subset \mathbb{P}^2_{k[t]}$, which is flat by our computation and theorem III.9.9, as our counterexample: these points are collinear iff $t = 0$, so the Hilbert function behaves as described in the first paragraph.

b. Sure.

c. If $\dim_{k(t)}(S_t/I_t)_d$ is independent of t, then the Hilbert polynomial of each fiber, which equals this quantity for $d \gg 0$, does not depend on the fiber. Hence by theorem III.9.9, $\{X_t\}$ is a flat family.

To show that $\{C(X_t)\}$ is also a very flat family, we need to show that we can interchange the operations of taking the cone and taking the fiber. The first step is defining the cone on a closed subscheme $X \subset \mathbb{P}^n_T$. Let π denote the natural map $\mathbb{P}^n_T \to T$. If $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n_T}$ is the sheaf of ideals defining X, then we can write

$$X = \mathbf{Proj}(\mathcal{O}_T[x_0, \cdots, x_n]/\bigoplus_{d\in\mathbb{Z}} \pi_* \mathcal{I}_X(d))$$

and define

$$C(X) = \mathbf{Proj}(\mathcal{O}_T[x_0, \cdots, x_{n+1}]/(\bigoplus_{d\in\mathbb{Z}} \pi_* \mathcal{I}_X(d)))$$

which is a closed subscheme of $\mathbf{Proj}(\mathcal{O}_T[x_0, \cdots, x_{n+1}]) = \mathbb{P}^{n+1}_T$. Let $t \in T$ be a point, let $i_t : \operatorname{Spec} k(t) \to T$ be the obvious morphism, let $i_{\mathbb{P}^n} : \mathbb{P}^n_{k(t)} \to \mathbb{P}^n_T$ be the base change of i_t along $\pi : \mathbb{P}^n_T \to T$, and let π' be the other projection $\mathbb{P}^n_{k(t)} \to \operatorname{Spec} k(t)$. We'll show that $C(X)_t$ is exactly $C(X_t)$, which will prove that $C(X)_t$ is a very flat family by the formula for the Hilbert polynomial of a cone from (a) and theorem III.9.9.

First, we observe that relative Proj is compatible with pullbacks: that is, if $\varphi : S' \to S$ is a morphism of schemes, $\mathcal{F}$ is a graded sheaf of $\mathcal{O}_S$-algebras, and $X = \mathbf{Proj}(\mathcal{F})$, then $X \times_S S' \cong \mathbf{Proj}(\varphi^* \mathcal{F})$. This lets us write

$$X_t \cong \mathbf{Proj}\left(i_t^* \left(\mathcal{O}_T[x_0, \cdots, x_n]/\bigoplus_{d\in\mathbb{Z}} \pi_* \mathcal{I}_X(d) \right)\right)$$

and

$$C(X)_t = \mathbf{Proj}\left(i_t^* \left(\mathcal{O}_T[x_0, \cdots, x_{n+1}]/\left(\bigoplus_{d\in\mathbb{Z}} \pi_* \mathcal{I}_X(d)\right)\right)\right),$$

while $C(X_t)$ is

$$\operatorname{Proj}\left(k(t)[x_0,\cdots,x_{n+1}]/\left(\Gamma_*(\mathcal{I}_{X_t})\right)\right) \cong \mathbf{Proj}\left(\mathcal{O}_{k(t)}[x_0,\cdots,x_{n+1}]/\left(\bigoplus_{d\in\mathbb{Z}}\pi'_*\mathcal{I}_{X_t}(d)\right)\right).$$

Now we have several compatibilities to prove: first, i_t^* commutes with the quotients in the definitions of X_t and $C(X)_t$; second, $i_t^*\pi_*\mathcal{F} \cong \pi'_*i_{\mathbb{P}^n}^*\mathcal{F}$ for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_T$; and finally $i_{\mathbb{P}^n}^*\mathcal{I}_X(d) \cong \mathcal{I}_{X_t}(d)$. Once we've proven all three of those, we will have that $C(X)_t \cong C(X_t)$ as claimed.

Let us start by proving $i_{\mathbb{P}^n}^*\mathcal{I}_X(d) \cong \mathcal{I}_{X_t}(d)$. We begin with a lemma:

Lemma. *Suppose X is a scheme over S, and $S' \to S$ is a morphism of schemes. Let $\varphi : X' = X \times_S S' \to X$ be the projection from the definition of the fiber product. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasi-coherent sheaves on X and $\mathcal{H}$ is flat over S, then*

$$0 \to \varphi^*\mathcal{F} \to \varphi^*\mathcal{G} \to \varphi^*\mathcal{H} \to 0$$

is exact.

Proof. We may immediately reduce to the case where $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, and $X' = \operatorname{Spec} A' = \operatorname{Spec} A \otimes_R R'$ are affine and $\mathcal{F} \cong \widetilde{F}$, $\mathcal{G} \cong \widetilde{G}$, $\mathcal{H} \cong \widetilde{H}$ for A-modules F, G, and H. The exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

corresponds to the exact sequence

$$0 \to F \to G \to H \to 0,$$

and its pullback

$$\varphi^*\mathcal{F} \to \varphi^*\mathcal{G} \to \varphi^*\mathcal{H} \to 0$$

corresponds to

$$F \otimes_A A' \to G \otimes_A A' \to H \otimes_A A' \to 0.$$

As $F \otimes_A A' \cong F \otimes_A A \otimes_R R' \cong F \otimes_R R'$, this exact sequence may also be written as

$$F \otimes_R R' \to G \otimes_R R' \to H \otimes_R R' \to 0,$$

where the failure of exactness on the left is measured by $\operatorname{Tor}_1^R(H, R')$. As H is R-flat, this Tor vanishes, and the sequence is exact.

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Applying this to the exact sequence $0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^n_T}(d) \to \mathcal{O}_X(d) \to 0$ on $\mathbb{P}^n_T$, we obtain the exact sequence

$$0 \to i_{\mathbb{P}^n}^*\mathcal{I}_X(d) \to i_{\mathbb{P}^n}^*\mathcal{O}_{\mathbb{P}^n_T}(d) \cong \mathcal{O}_{\mathbb{P}^n_{k(t)}}(d) \to i_{\mathbb{P}^n}^*\mathcal{O}_X(d) \cong \mathcal{O}_{X_t}(d) \to 0.$$

But $\mathcal{I}_{X_t}(d)$ is the kernel of the second map, so $i_{\mathbb{P}^n}^*\mathcal{I}_X(d) \cong \mathcal{I}_{X_t}(d)$ and the first compatibility is taken care of.

Next, interchanging π_* and i^* follows from corollary III.9.4 and proposition III.8.5 after restricting to an affine open neighborhood of t.

Finally, we show that i_t^* commutes with the quotients. By writing

$$\mathcal{O}_T[x_0, \cdots, x_{n+1}]/(\bigoplus_{d\in\mathbb{Z}} \pi_*\mathcal{I}_X(d)) \cong \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \left[\mathcal{O}_T[x_0,\cdots,x_n]/\bigoplus_d \pi_*\mathcal{I}_X(d)\right] x_{n+1}^i$$

it suffices to prove the claim for X. As pullbacks commute with direct sums, we may further reduce the claim to showing that

$$i_t^*(\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d)) \cong (i_t^*\mathcal{O}_T[x_0,\cdots,x_n]_d)/(i_t^*\pi_*\mathcal{I}_X(d))$$

for all d. Just like the first compatibility we proved, we have that $i_t^*(\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d))$ is isomorphic to the quotient of $i_t^*\mathcal{O}_T[x_0,\cdots,x_n]_d$ by the image of $i^*\pi_*\mathcal{I}_X(d)$, and if we can show that $\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d)$ is flat over T, then we may apply our lemma to see that the map $i_t^*\pi_*\mathcal{I}_X(d) \to i_t^*\mathcal{O}_T[x_0,\cdots,x_n]_d$ is injective and we'll have the result we're after. By an application of theorem III.5.2(a), we have that $\pi_*\mathcal{I}_X(d)$ is a coherent sheaf on T; as $\mathcal{O}_T[x_0,\cdots,x_n]_d$ is also coherent, the quotient is coherent, so by proposition III.9.2(e) $\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d)$ is flat iff it is locally free.

To show that $\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d)$ is locally free, we'll apply the criteria of exercise II.5.8(c): if a coherent sheaf $\mathcal{F}$ on a reduced noetherian scheme S satisfies the property that $\dim_{k(s)} \mathcal{F}\otimes k(s)$ does not depend on $s \in S$, then it is locally free. But

$$k(t)\otimes(\mathcal{O}_T[x_0,\cdots,x_n]_d/\pi_*\mathcal{I}_X(d)) \cong k(t)[x_0,\cdots,x_n]_d/i_t^*\pi_*\mathcal{I}_X(d),$$

and $i_t^*\pi_*\mathcal{I}_X(d) \cong \pi'_*i_{\mathbb{P}^n}^*\mathcal{I}_X(d) \cong \pi'_*\mathcal{I}_{X_t}(d)$, so this quotient is precisely S_t/I_t, and its dimension doesn't depend on t by our assumption that X_t is a very flat family! This finishes the proof that $C(X_t) \cong C(X)_t$.

d. It suffices to work affine-locally on $T = \operatorname{Spec} R$. I claim that $X \subset \mathbb{P}^n_R$ being a family of projectively normal subschemes of $\mathbb{P}^n_R$ is equivalent to X being projectively normal as a subscheme of $\mathbb{P}^n_R$. Recall that from exercise II.5.14(d), a closed subscheme $X \subset \mathbb{P}^n_R$ is projectively normal iff $\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) \to \Gamma(X, \mathcal{O}_X(d))$ is surjective for all $d \geq 0$. By corollary III.9.4, $\Gamma(\mathbb{P}^n_{k(t)}, \mathcal{O}_{\mathbb{P}^n_{k(t)}}(d)) \cong \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) \otimes k(t)$ and $\Gamma(X_t, \mathcal{O}_{X_t}(d)) \cong \Gamma(X, \mathcal{O}_X(d)) \otimes k(t)$, and by theorem III.5.2(a), $\Gamma(X, \mathcal{O}_X)$ is a finitely-generated R-module. Now we may conclude that $\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) \to \Gamma(X, \mathcal{O}_X(d))$ is surjective for all $d \geq 0$: a map of R-modules $M \to N$ with N finitely generated that is surjective after tensoring with $k(\mathfrak{p})$ for all $\mathfrak{p} \subset R$ is surjective, as we can lift surjectivity of $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \to N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}$ to surjectivity of $M_\mathfrak{p} \to N_\mathfrak{p}$ by Nakayama's lemma (take generators of $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}$, pick lifts in $M_\mathfrak{p}$, note these lifts map to elements of $N_\mathfrak{p}$ which form a generating set by Nakayama). The converse is just the statement that the restriction map $H^0(\mathbb{P}^n_{k(t)}, \mathcal{O}_{\mathbb{P}^n_{k(t)}}(d)) \to H^0(X_t, \mathcal{O}_{X_t}(d))$ is obtained from the restriction map $H^0(\mathbb{P}^n_T, \mathcal{O}_{\mathbb{P}^n_T}(d)) \to H^0(X, \mathcal{O}_X(d))$ by tensoring with $k(t)$ by corollary III.9.4.

Next, I claim that for a projectively normal subscheme $X \subset \mathbb{P}^n_R$, the cone $C(X) \subset \mathbb{P}^{n+1}_R$ is also projectively normal: the condition that X is projectively normal is equivalent to S_X, the projective coordinate ring being normal, and the projective coordinate ring of the projective cone is just $S_X[x_{n+1}]$, which is clearly normal. As a projectively normal subscheme is normal, we may apply theorem III.9.11 to see that when T is a nonsingular curve over an algebraically closed field and $X \subset \mathbb{P}^n_T$ is a family of projectively normal subschemes, X is a flat family and $C(X) \subset \mathbb{P}^n_T$ is also a flat family.

Penultimately, we claim that $C(X)_t \cong C(X_t)$ as subschemes of $\mathbb{P}^{n+1}_{k(t)}$. It is enough to show that these agree set-theoretically and are both reduced by the uniqueness of the reduced induced subscheme structure. For reducedness, $C(X)_t$ is reduced by the proof of theorem III.9.11, while as X_t is projectively normal, the cone on it is Proj of a domain and by proposition II.2.5 identifying the stalks of the structure sheaf plus the fact that a subring of a domain is a domain, we see that $C(X_t)$ is reduced. Set theoretic agreement is equivalent to showing that the two homogeneous ideals cutting out these subschemes have the same radical. Let $I \subset R[x_0, \cdots, x_n]$ denote the ideal of X, and let $J = I \otimes_R k(t) \subset k(t)[x_0, \cdots, x_n]$ denote the ideal of X_t. Then $C(X_t)$ is cut out by $J^{sat}[x_{n+1}]$ while $C(X)_t$ is cut out by $J[x_{n+1}]$. Clearly $J[x_{n+1}] \subset J^{sat}[x_{n+1}]$ and thus $\sqrt{J[x_{n+1}]} \subset \sqrt{J^{sat}[x_{n+1}]}$, so we need to show that $\sqrt{J^{sat}[x_{n+1}]} \subset \sqrt{J[x_{n+1}]}$. Suppose $j^a \in J^{sat}[x_{n+1}]$: for each $0 \leq i \leq n$, picking a large enough d_i, we may assume $x_i^{d_i a} j^a \in J[x_{n+1}]$ which implies $x_i^{d_i} j \in \sqrt{J[x_{n+1}]}$. But the radical of a homogeneous ideal is saturated, so $j \in \sqrt{J[x_{n+1}]}$. Thus we've shown that that $C(X)_t$ and $C(X_t)$ agree set theoretically, finishing our proof that $C(X)_t \cong C(X_t)$.

Finally, the following lemma will finish our proof that X is very flat:

Lemma. *Given a sequence of real numbers a_n, let $a_n^{(0)} = a_n$ and define*

$$a_n^{(p)} = \sum_{i=0}^{n} a_i^{(p-1)}$$

for p a positive integer. Define an equivalence relation $\sim$ on sequences by $a_n \sim b_n$ iff there exists an n_0 so that for all $n \geq n_0$ we have $a_n = b_n$. If a_n and b_n are two sequences so that $a_n^{(p)} \sim b_n^{(p)}$ for all $p \geq 0$, then $a_n = b_n$ for all n.

We apply this by taking $a_i = \dim_{k(t_1)}(S_{t_1}/I_{t_1})_i$ and $b_i = \dim_{k(t_2)}(S_{t_2}/I_{t_2})_i$ for two points $t_1, t_2 \in T$: flatness of X over T and flatness of the p-fold iterated projective cone on X over T gives the hypotheses of the lemma, while the conclusion is exactly the claim that X_t is a very flat family.

Proof. First, we note that if $a_n = b_n$ for all $n \geq n_0$, then $a_n^{(1)} = b_n^{(1)}$ for all $n \geq n_0$ as well: by the condition that $a_n^{(1)} \sim b_n^{(1)}$, there exists some j_0 so that we have $\sum_{i=0}^{j} a_i = \sum_{i=0}^{j} b_i$ for all $j \geq j_0$, and then subtracting off $a_n = b_n$ for $n \geq n_0$, we obtain the same relation for all $j \geq n_0$. Inductively, this shows that if $a_n = b_n$ for all $n \geq n_0$, then $a_n^{(p)} = b_n^{(p)}$ for all p and all $n \geq n_0$.

Next, we claim that

$$a_n^{(p+1)} = \sum_{i=0}^{n} \binom{p+n-i}{p} a_i$$

for all $p \geq 0$ by induction. As $a_n^{(1)} = \sum_{i=0}^n a_i$, the relation holds for $p = 0$ and all n. Now suppose the relation holds for all pairs (p, n) with $p < p'$, or with $p = p'$ and $n < n'$. Then since

$$a_{n'}^{(p'+1)} = a_{n'}^{(p')} + a_{n'-1}^{(p'+1)},$$

the coefficient of a_i in this sum is

$$\binom{p'-1+n'-i}{p'-1} + \binom{p'+n'-1-i}{p'} = \binom{p'+n'-i}{p'},$$

and the claim is proven.

By the definition of $\sim$ and our first observation, we have that for all p and for all j greater than some fixed j_0, $a_j^{(p+1)} = b_j^{(p+1)}$, or

$$\sum_{i=0}^{j} \binom{p+j-i}{p} a_i = \sum_{i=0}^{j} \binom{p+j-i}{p} b_i \Leftrightarrow \sum_{i=0}^{j} \binom{p+j-i}{p} (a_i - b_i) = 0.$$

Now assume there is some index n so that $a_n \neq b_n$, and fix integers n_0 and n_1 so that $a_n = b_n$ for all $n < n_0$ and $n > n_1$ and n_0 is the maximum such n_0. This equation simplifies to

$$\sum_{i=n_0}^{n_1} \binom{p+j-i}{p} (a_i - b_i) = 0$$

which implies

$$\binom{p+j-n_0}{p} = \sum_{i=n_0+1}^{n_1} \binom{p+j-i}{p} \left| \frac{a_i - b_i}{a_{n_0} - b_{n_0}} \right|$$

for all p and all $j > n_1$ independent of p.

On the other hand, I claim that given $c > 0$ and a fixed integer j, I can always choose integers p so that

$$\binom{p+j-n_0}{p} > c \sum_{i=n_0+1}^{n_1} \binom{p+j-i}{p},$$

contradicting the above. As $\sum_{i=n_0+1}^{j} \binom{p+j-i}{p} \geq \sum_{i=n_0+1}^{n_1} \binom{p+j-i}{p}$, while $\sum_{i=n_0+1}^{j} \binom{p+j-i}{p} = \binom{p+j-n_0}{p+1}$ by the hockeystick identity, it's enough to show that there exists a p so that $\binom{p+j-n_0}{p} \geq c\binom{p+j-n_0}{p+1}$ for any fixed choice of j, c, and n_0. But $\binom{p+j-n_0}{p} = \frac{p+1}{j-n_0}\binom{p+j-n_0}{p+1}$, and we're done. ■

Exercise III.9.6. Let $Y \subset \mathbb{P}^n$ be a nonsingular variety of dimension ≥ 2 over an algebraically closed field k. Suppose $\mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^{n-1}$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbb{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbb{P}^n$ if and only if Y' is a complete intersection in $\mathbb{P}^{n-1}$. [*Hint*: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

Solution. The forward direction is fairly direct: assume $Y = \bigcap H_i$ is a complete intersection. Then Y' can be written as $\bigcap(H_i \cap \mathbb{P}^{n-1})$, and each $H_i \cap \mathbb{P}^{n-1}$ is a hypersurface in $\mathbb{P}^{n-1}$: if it wasn't, then $\dim Y' \not< \dim Y$ by theorem I.7.2.

The reverse direction is where it gets interesting. Suppose $Y' \subset \mathbb{P}^{n-1}$ is a complete intersection, and $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is given by $V(x_n)$. First I claim that Y is projectively normal and $I_{C(Y')} = I_{C(Y)} + (x_n)$ (all ideals will be considered in $k[x_0, \cdots, x_n]$). By the Jacobian criteria it's clear that as Y is smooth, $C(Y)$ is smooth (and therefore normal) away from the origin, so all we need to do to verify that Y is projectively normal is to check out what happens at the origin. Let's look at Y': since this is smooth, it is normal, so by exercise II.8.4(b) we have that $C(Y')$ is normal and therefore an integral scheme. There is an obvious closed immersion $C(Y') \hookrightarrow C(Y)$ which has two relevant properties: it is set-theoretically cut out by x_n (as it's scheme-theoretically cut out by x_n on the complement of the cone point inside $C(Y)$) and as $C(Y')$ is integral, the ideal cutting out $C(Y')$ inside $k[C(Y)]$ is prime. This exactly sets us up to apply lemma III.9.12, which tells us that $C(Y)$ is normal at the cone point (hence normal) and that $C(Y')$ is the closed subscheme of $C(Y)$ cut out by (x_n). This implies $I_{C(Y')} = I_{C(Y)} + (x_n)$.

From the assumption that $Y' \subset \mathbb{P}^{n-1}$ is a complete intersection, we can write $I_{C(Y')} = (f_1, \cdots, f_r, x_n)$ for f_i homogeneous. I claim that writing $f_i = g_i + x_n h_i$ where $g_i \in I_{C(Y)}$ is homogeneous, we have $I_{C(Y)} = (g_1, \cdots, g_r)$. Writing $V(g_1, \cdots, g_r) = Z_1 \cup \cdots \cup Z_s$ for irreducible components $Z_i \subset \mathbb{A}^{n+1}$, we have that $C(Y') = V(g_1, \cdots, g_r) \cap V(x_n) = (Z_1 \cap V(x_n)) \cup \cdots \cup (Z_s \cap V(x_n))$. By Krull's height theorem, each Z_i is of codimension at most r in $\mathbb{A}^{n+1}$, so all the irreducible components of each $Z_i \cap V(x_n)$ are of codimension at most $r+1$: this means each Z_i must contain $C(Y')$. On the other hand, as $C(Y')$ is smooth away from the origin, the rank of the Jacobian matrix $\frac{\partial(g_1, \cdots, g_r, x_n)}{\partial(x_0, \cdots, x_n)}$ is $r+1$ away from the origin, so the rank of the Jacobian matrix $\frac{\partial(g_1, \cdots, g_r)}{\partial(x_0, \cdots, x_n)}$ is r away from the origin and $V(g_1, \cdots, g_r) \subset \mathbb{A}^{n+1}$ is smooth away from the origin. As any point on multiple irreducible components is singular, we see that $V(g_1, \cdots, g_r)$ is irreducible and in fact $I_{C(Y)}$ is the minimal prime lying over $(g_1, \cdots, g_r)$.

As $k[x_0, \cdots, x_n]$ is Cohen-Macaulay, the unmixedness theorem applied to $(g_1, \cdots, g_r)$, an ideal of height r generated by r elements, shows that $V(g_1, \cdots, g_r)$ has no embedded associated primes. This implies that $V(g_1, \cdots, g_r)$ is reduced: as $(g_1, \cdots, g_r)$ is smooth away from the origin, it is reduced away from the origin, so any nilpotents must be concentrated at the origin, which would give that $V(g_1, \cdots, g_r)$ has an embedded point at the origin. Thus $C(Y) \subset V(g_1, \cdots, g_r)$ is an inclusion of closed integral subschemes of $\mathbb{A}^{n+1}$ with the same underlying topological space. By the uniqueness of the reduced induced structure, $C(Y) = V(g_1, \cdots, g_r)$ and therefore Y is a complete intersection.

Exercise III.9.7. Let $Y \subset X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an *infinitesimal deformation* of

Y as a closed subscheme of X, to be a closed subscheme $Y' \subset X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathcal{N}_{Y/X})$, where

$$\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

Solution. Let's investigate the affine case first: suppose $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$ for A a finitely generated k-algebra with $I \subset A$ an ideal. Then a deformation $Y' \subset X' = \operatorname{Spec} A[t]/(t^2)$ is given by an ideal $I' \subset A[t]/(t^2)$ so that $I'/tI' = I$ so that $(A[t]/(t^2))/I'$ is flat over $k[t]/(t^2)$. Applying proposition III.9.1A(b), we see that as A is flat over k, $A[t]/(t)^2$ is flat over $k[t]/(t^2)$; as Y' is flat over $k[t]/(t^2)$, we see that by (e) we must have that I' is flat over $k[t]/(t^2)$; finally, by (a), we have that $(t) \otimes I' \to I'$ must be injective.

We can get a fair amount of mileage out of this statement. Let's start by investigating what $(t) \otimes I'$ is: our first observation is that elements of the form $t \otimes ti$ for $i \in I$ are zero, since we can pull the t across the tensor product. So $t \otimes (a + bt) = t \otimes (a + bt + it)$ for any $a + bt \in I'$ and $i \in I$. Now the condition that $(t) \otimes I' \to I'$ should be injective shows that if $a + bt$ and $a + b't$ are both in I, then $b - b'$ should be in I: else $a + bt$ and $a + b't$ both map to at under $(t) \otimes I' \to I'$, but $(b - b')t \neq 0$ so we've violated injectivity. Therefore I' is of the form $\{a + (f(a) + b)t \mid a, b \in I\}$ for some function $I \to A/I$.

I claim that this function is actually A-linear: by injectivity, $(a + f(a)t + bt) + (a' + f(a')t + b't)$ and $(a + a') + f(a + a')t + (b + b')t$ must map to the same place, so $f(a) + f(a')$ and $f(a + a')$ are the same up to an element of I; again by injectivity $a'(a + f(a)t + bt) = a'a + a'f(a)t + a'bt$ and $a'a + f(a'a)t + a'bt$ must map to the same place, so $a'f(a)$ and $f(a'a)$ are the same up to an element of I. Finally, by A-linearity, we see that this descends to an A-module map $I/I^2 \to A/I$. Thus infinitesmial deformations over an affine scheme are in bijection with maps $I/I^2 \to A/I$, and after a gluing argument we obtain the desired global result.

Exercise III.9.8. (*) Let A be a finitely-generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\operatorname{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\operatorname{Hom}_A(\Omega_{P/k} \otimes A, A) \to \operatorname{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infintesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

Solution. From the previous exercise, we know that the middle term $\mathrm{Hom}_A(J/J^2, A)$ classifies deformations of $\operatorname{Spec} A$ as a closed subscheme of affine space. We have two tasks: show that every abstract deformation of $\operatorname{Spec} A$ gives a deformation of $\operatorname{Spec} A$ as a closed subscheme of $\operatorname{Spec} P$, and show that any two deformations of $\operatorname{Spec} A \subset \operatorname{Spec} P$ are abstractly isomorphic iff they differ by an element of $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A)$.

We'll begin by showing all abstract deformations extend to deformations of $\operatorname{Spec} A \subset \operatorname{Spec} P$. Let A' be an abstract infinitesimal deformation of A, that is, a flat D-algebra along with an isomorphism $\varphi : A'/tA' \to A$. Since $P = k[x_1, \cdots, x_n]$ surjects on to A, the images of x_i generate A - let $\overline{x_i}$ denote arbitrary lifts of $x_i \in A$ to A'. By construction, $\overline{x_i}$ generate A' as a D-algebra up to some D-submodule contained in tA'. On the other hand, by an application of III.9.1A(a), we have that the map $(t) \otimes A' \to A'$ must be injective - this gives that the obvious map $A'/tA' \to tA'$ must be an isomorphism, so we have that the $\overline{x_i}$ actually generate A', so the map $P \otimes_k D \to A'$ by $x_i \otimes 1 \mapsto \overline{x_i}$ shows that A' may be written as an infinitesimal deformation of $\operatorname{Spec} A$ as a closed subscheme of $\operatorname{Spec} P$.

Now we check that if two deformations of $\operatorname{Spec} A \subset \operatorname{Spec} P$ differ by an element of $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A)$, they are are abstractly isomorphic. Write $P = k[x_1, \cdots, x_n]$ so that $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) = A\langle \partial_1, \cdots, \partial_n \rangle$ where ∂_i is the element so that $\partial_i(dx_j) = \delta_{ij}$, and by construction of the map in lemma II.8.4A we have that the map $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A)$ is given by

$$\sum a_i \partial_i \mapsto \left(p \mapsto \sum a_i \frac{\partial p}{\partial x_i} \right).$$

Now consider the D-algebra automorphism of $P \otimes_k D$ given by sending $p + tq \mapsto p + t(q + \sum \overline{a_i} \frac{\partial p}{\partial x_i})$ where $\overline{a_i} \in P$ is a lift of $a_i \in A$: this takes the ideal $\{a + (f(a) + b)t \mid a, b \in J\}$ isomorphically on to $\{a + (f(a) + b + \sum \overline{a_i} \frac{\partial a}{\partial x_i})t \mid a, b \in J\}$, so this D-algebra automorphism induces an isomorphism between the two abstract deformations given by quotienting by these ideals.

It remains to show the converse. Suppose $A' = P \otimes_k D/I'$ and $A'' = P \otimes_k D/I''$ are two deformations of $\operatorname{Spec} A \subset \operatorname{Spec} P$ which are abstractly isomorphic via an A-isomorphism $\varphi : A' \to A''$. Consider $\overline{x_i} \in (P \otimes_k D)/I'$, the image of $x_i \otimes 1 \in P \otimes_k D$ under the obvious surjection. From the condition that φ is an isomorphism over A, we can pick a representative of $\varphi(\overline{x_i}) \in A''$ of the form $\overline{x_i + a_i t}$ for constants $a_i \in A$ (a priori, they're only in P, but any two choices differing by an element of J give the same deformation). These a_i determine φ, and as $t^2 = 0$, we can check that by expanding the product $\prod x_i^{c_i}$ we get that $\varphi(\prod \overline{x_i}^{c_i}) = (\prod \overline{x_i}^{c_i}) + (\sum a_i \frac{\partial \prod \overline{x_j}^{c_j}}{\partial x_i})t$. Therefore the difference of the two functions f' and f'' comes from $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A)$ and we're done.

Exercise III.9.9. A k-algebra A is said to be *rigid* if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x, y, z, w]/(x, y) \cap (z, w)$, and show that A is rigid. This corresponds to two planes in $\mathbb{A}^4$ which meet at a point.

Solution. We'll show that given any map $J/J^2 \to A$, it factors as $J/J^2 \to \Omega_{P/k} \otimes_P A \to A$: this shows that $T^1(A) = 0$. In the notation of the previous exercise, we may take $P = k[x, y, z, w]$ and $J = (x, y) \cap (z, w) = (xz, xw, yz, yw)$. As $\Omega_{P/k}$ is the free P-module on 4 generators dx, dy, dz, dw, we have that $\Omega_{P/k} \otimes_P A$ is the free A-module on 4 generators dx, dy, dz, dw and an A-module map $\Omega_{P/k}\otimes_P A\to A$ is uniquely specified by the images of these generators. A map $J/J^2\to A$ is uniquely specified by the images a_{xy}, ax_{xw}, a_{yz}, and a_{yw} of xy, xw, yz, and yw respectively, subject to the four relations $wa_{xz}=za_{xw}$, $ya_{xz}=xa_{yz}$, $ya_{xw}=xa_{yw}$, and $wa_{yz}=za_{yw}$.

By the definition of the exact sequence from proposition II.8.4A, the map $J/J^2\to\Omega_{P/k}\otimes_P A$ is given by sending $b\in J/J^2$ to $d\overline{b}\otimes 1$ for $\overline{b}\in J$ a lift of b, so $xz\mapsto xdz\otimes 1+zdx\otimes 1$ and similarly down the line. Therefore if we want to factor $J/J^2\to A$ as $J/J^2\to\Omega_{P/k}\otimes_P A\to A$, we need to pick where dx, dy, dz, and dw go so that $xdz\otimes 1+zdx\otimes 1$ maps to a_{xz} and similarly.

To show that this is possible, let's reinterpret A as $(f,g)\in k[x,y]\times k[z,w]$ so that f and g have the same constant term. Writing $a_{xz}=(f_{xz}(x,y),g_{xz}(z,w))$ and so on, we see that the relations $ya_{xz}=xa_{yz}$ become $yf_{xz}(x,y)=xf_{yz}(x,y)$ and so on, which is perfect for us: if $yf_{xz}=xf_{yz}$ in $k[x,y]$, then by unique factorization we can write $f_{xz}=xp_z$ and $f_{yz}=yp_z$ for some $p_z\in k[x,y]$. Repeating this argument for each variable, we obtain polynomials $f_z,f_w\in k[x,y]$ and $g_x,g_y\in k[z,w]$ with zero constant term so that $a_{ij}=if_j+jg_i$. Therefore the map sending $dx\mapsto(0,g_x)$, $dy\mapsto(0,g_y)$, $dz\mapsto(f_z,0)$ and $dw\mapsto(f_w,0)$ is a map $\Omega_{P/k}\otimes_P A\to A$ which gives our desired factorization.

Exercise III.9.10. A scheme X_0 over a field is *rigid* if it has no deformations.

a. Show that $\mathbb{P}^1_k$ is rigid, using (9.13.2).

b. One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f:X\to\mathbb{A}^2$ over k algebraically closed, so that $X_0\cong\mathbb{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbb{A}^2$ for which $f^{-1}(U)\cong U\times\mathbb{P}^1$.

c. (*) Show, however, that one can trivialize a global deformation of $\mathbb{P}^1$ after a flat base extension, in the following sense: let $f:X\to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t\in T$ such that $X_t\cong\mathbb{P}^1_k$. Then there exists a nonsingular curve T', and a flat morphism $g:T'\to T$, whose image contains t, such that if $X'=X\times_T T'$ is the base extension, the new family $f':X'\to T'$ is isomorphic to $\mathbb{P}^1_{T'}\to T'$.

Solution.

a. By example III.9.13.2, infinitesimal deformations of $\mathbb{P}^1_k$ are classified by $H^1(\mathbb{P}^1_k,\mathcal{T})$. The tangent sheaf of $\mathbb{P}^1_k$ is $\Omega^\vee_{\mathbb{P}^1_k}\cong\mathcal{O}_{\mathbb{P}^1_k}(-1-1)^\vee\cong\mathcal{O}_{\mathbb{P}^1_k}(2)$, which has vanishing first cohomology by theorem III.5.1.

b. Let k be algebraically closed of characteristic not 2, let $\mathbb{A}^2_k$ have coordinates a,b, and let $\mathbb{P}^2_k$ have coordinates x, y, z. I claim that $X=V(x^2+(1-a)y^2+(1-b)z^2)\subset\mathbb{P}^2\times\mathbb{A}^2$ with the natural projection $X\to\mathbb{P}^2\times\mathbb{A}^2\to\mathbb{A}^2$ suffices. Properness and flatness are both relatively straightforward: $X\to\mathbb{P}^2\times\mathbb{A}^2$ is a closed immersion and thus proper, $\mathbb{P}^2\times\mathbb{A}^2\to\mathbb{A}^2$ is proper, and compositions of proper maps are proper; flatness can be checked via noticing that $V(x^2+(1+a)y^2+(1+b)z^2)\subset\mathbb{P}^2_{k(p)}$ is a quadratic hypersurface for any $p\in\mathbb{A}^2$

and every hypersurface of degree d in a projective space has same Hilbert polynomial by considering $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$. It is also not difficult to see that $X_0 \cong \mathbb{P}^1_k$: $X_0 = V(x^2+y^2+z^2) \subset \mathbb{P}^2_k$, which is exactly a copy of $\mathbb{P}^1$.

The more interesting part of the problem is showing that there is no open $U \subset \mathbb{A}^2$ containing 0 where $f^{-1}(U) \cong U \times \mathbb{P}^1$. If this was true, there would necessarily be a section $s : U \to f^{-1}(U)$ - we'll show this is impossible. If such an s existed, then by examining what happens at the generic point of U, we would have a $k(a,b)$-rational point on $V(x^2+(1-a)y^2+(1-b)z^2) \subset \mathbb{P}^2_{k(a,b)}$. I claim no such point exists, or that you cannot solve $x^2+(1-a)y^2+(1-b)z^2$ in rational functions $x, y, z \in k(a,b)$.

By clearing denominators intelligently, we may assume that $x, y, z \in k[a,b]$ solve $x^2 + (1 - a)y^2 + (1-b)z^2$ and have no common divisor. Now apply the homomorphism $k[a,b] \to k[b]$ by setting $a = 1$: this gives that $x^2|_{a=1} + (1-b)z^2|_{a=1} = 0$, which by examining the valuation of $x^2|_{a=1}$ and $(1-b)z^2|_{a=1}$ with respect to $1-b$ shows that $x^2|_{a=1} = z^2|_{a=1} = 0$, or that $1-a$ divides both x and z. Therefore $1-a$ must also divide y, contradiction.

c. The only obstruction is the one we encountered in part (b): if there exists a section $s : T \to X$, then we can find an open neighborhood U of t so that $X \times_T U \to U$ is isomorphic to $\mathbb{P}^1_U \to U$. The first step in finding a section is to find a rational point on the fiber of X over the generic point. So let η be the generic point of T, and consider X_η: this is a scheme of finite type over $k(\eta)$, so it has a closed point. By Zariski's lemma, such a closed point is defined over a finite extension F of $k(\eta)$. By the equivalence of categories between regular projective curves over k with surjective maps and field extensions of k of transcendence degree one with injective maps of fields, we obtain a map from a smooth projective curve T' to a projective completion of T, and then shrinking T' to be the preimage of T, we obtain a smooth curve T' mapping surjectively to T so that $X_{T'}$ has a $k(t')$-point in every fiber for every $t' \in T'$. Further, $T' \to T$ is flat by proposition III.9.7. We also observe that for any t' mapping to t we have that $(X \times_T T')_{t'} \cong \mathbb{P}^1_k$ and that $X_{T'} \to T'$ is again flat and projective since these properties are preserved by base change. So by replacing T by T' and X by $X_{T'}$, we may assume that X_η has a $k(\eta)$-rational point.

Now we use the valuative criteria for properness to construct a local section near t. As T is regular, every local ring of a closed point is a DVR, so we may apply the valuative criteria to obtain a map $\operatorname{Spec} \mathcal{O}_{T,t} \to X$ which after composing with $X \to T$ is equal to the obvious map $\operatorname{Spec} \mathcal{O}_{T,t} \to T$. Up to shrinking T by taking an affine open neighborhood of t, this defines a section $s : T \to X$ which is a closed immersion by exercise II.4.8 (as $X \to T$ is projective, therefore separated).

Next, I claim that up to shrinking T around t we may assume X is regular of dimension two. Letting $f \in \mathcal{O}_{T,t}$ be a uniformizer and $x \in X_t$ be any closed point mapping to x, we have that $\mathcal{O}_{X,x}/(f) \cong \mathcal{O}_{\mathbb{P}^1,x}$, so X is regular at x. As the non-regular locus is closed by the Jacobian criteria and $X \to T$ is proper, we have that the image of the non-regular locus of X is a closed subset of T not containing t. We also have that X is irreducible: since $X \to T$ is proper, flat, and of finite type, we have that it is both closed and open. Thus every irreducible component of X must surject on to T, and as $X_t \cong \mathbb{P}^1$ is connected, we must have that any two irreducible components meet in X_t which would imply that some $x \in X_t$ singular as a point of X, which cannot happen. Therefore our claim is proven and we may assume that X is integral and regular of dimension two.

This implies that the sheaf $\mathcal{O}_X(s(T))$ associated to the divisor $s(T) \subset X$ is a line bundle, and by construction, the pullback to $X_t \cong \mathbb{P}^1_k$ is $\mathcal{O}(1)$. Up to possibly shrinking T around t so it is affine, we may apply the proof of corollary III.9.4 to find that $H^i(X_t, \mathcal{O}_X(s(T))_t) \cong H^i(X, \mathcal{O}_X(s(T))) \otimes k(t) \cong H^i(T, f_*\mathcal{O}_X(s(T))) \otimes k(t)$ for all $i \geq 0$. As $\mathcal{O}_X(s(T))_t \cong \mathcal{O}_{\mathbb{P}^1}(1)$, we have that $H^0(X_t, \mathcal{O}_X(s(T))) \cong k^2$ and all other cohomologies are zero. Because $\mathcal{O}_X(s(T))$ is locally free, it is flat over T and therefore the Euler characteristic of the restriction to the fiber is independent of the fiber (combine proposition III.9.9 with the fact that the Euler characteristic can be expressed in terms of the Hilbert polynomial). Since the fibers of $X \to T$ are of dimension one, the Euler characteristic is just $\dim H^0 - \dim H^1$, and as $R^1 f_*\mathcal{O}_X(s(T))$ is coherent by theorem III.5.2 and has rank 0 at t by our previous work, we may shrink T so that $R^1 f_*\mathcal{O}_X(s(T)) = 0$. Therefore $f_*\mathcal{O}_X(s(T))$ is of rank two and hence by exercise II.5.8(c), locally free. We can play the same game with $f_*\mathcal{O}_X$ to show that up to shrinking T, every fiber of $X \to T$ is connected: $\dim H^0(X_t, \mathcal{O}_{X_t})$ is the number of connected components. (If you've read ahead to section III.12, just say 'by semicontinuity' to turn this paragraph in to a one-liner.)

Now consider the sections $x_0, x_1 \in H^0(X_t, \mathcal{O}(1)) \cong H^0(T, f_*\mathcal{O}_X(s(T))) \otimes k(t)$: lifting these to two sections of $f_*\mathcal{O}_X(s(T))$ over some open neighborhood U of t, we find sections x_0', x_1' of $\mathcal{O}_X(s(T))$ on $f^{-1}(U)$, and as the locus where these sections fail to be linearly independent when restricted to the fibers is a closed subset not containing X_t, its image under f is a closed subset not containing t. Thus we may shrink U so that x_0', x_1' are linearly independent in every fiber over $f^{-1}(U)$, and therefore are define a map $\varphi : f^{-1}(U) \to \mathbb{P}^1_U$. It remains to prove that φ is an isomorphism.

We'll show that φ is a finite surjective map of degree one, which proves it is an isomorphism. First, since φ is a section of the projection $\mathbb{P}^1_U \to U$ and $\mathbb{P}^1$ is separated, we may apply exercise II.4.8(e) to see that φ is projective. Next, φ is quasi-finite: if not, then we must have that the $\mathbb{P}^1$ lying over some $u \in U$ is contracted to a point, which is impossible: the sections x_0' and x_1' are linearly independent when restricted to any fiber over U. Therefore by exercise III.11.2 φ is finite. We may also use the same argument as in quasi-finiteness to conclude that φ is surjective, which we check fiberwise. Since each fiber $f^{-1}(U)_u$ is connected by our construction of U, it has connected image in $\mathbb{P}^1_u$ under φ_u. But the connected subsets of $\mathbb{P}^1_u$ are singletons and the whole of $\mathbb{P}^1_u$, and we know the image of $f^{-1}(U)_u$ cannot be the former. Finally, as $f^{-1}(U)$ and $\mathbb{P}^1_U$ are nonsingular, exercise III.9.3(a) shows that φ is flat. Therefore φ is a finite flat surjective map.

This implies that $\varphi_*\mathcal{O}_{f^{-1}(U)}$ is a flat coherent sheaf on $\mathbb{P}^1_U$, which by proposition III.9.2(e) must be locally free. But since the pullback to $\mathbb{P}^1_t$ is the structure sheaf, we see that $\varphi_*\mathcal{O}_{f^{-1}(U)}$ is locally free of rank 1, proving our claim.

Exercise III.9.11. Let Y be a nonsingular projective curve of degree d in $\mathbb{P}^n_k$, over an algebraically closed field k. Show that

$$0 \leq p_a(Y) \leq \frac{1}{2}(d-1)(d-2).$$

[*Hint*: Compare Y to a suitable projection of Y into $\mathbb{P}^2$, as in (9.8.3) and (9.8.4).]

Solution. The case $n = 1$ is trivial, and when $n = 2$ the claim follows from the calculation in exercise II.8.4(f). For $n > 2$, the strategy is to show that we can find a projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ which has $\deg Y = \deg \pi(Y)$ and $p_a(Y) \leq p_a(\pi(Y))$. Applying this inductively, this gives that we can project Y to a plane curve $Y' \subset \mathbb{P}^2$ of degree d and arithmetic genus $\frac{1}{2}(d-1)(d-2) \geq p_a(Y)$ - once we note that $p_a(Y) \geq 0$ by exercise III.5.3, we're finished.

The first step to show the existence of such a projection is given by exercise I.4.9: let $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ be a projection with Y birational to $\pi(Y)$. Up to a change of coordinates, we may assume that the projection is simply the projection from $[0 : \cdots : 0 : 1]$ to $V(x_n)$. Then the condition that $\pi : Y \to \pi(Y)$ is birational means that $\deg Y = \deg \pi(Y)$: by picking a hyperplane H in $\mathbb{P}^{n-1}$ which doesn't intersect the locus where $Y \to \pi(Y)$ fails to be an isomorphism, the result follows by generalized Bezout (theorem I.7.7) after comparing $H \cap \pi(Y)$ with $\langle H, [0 : \cdots : 0 : 1] \rangle \cap Y$ where $\langle H, [0 : \cdots : 0 : 1] \rangle$ denotes the hyperplane in $\mathbb{P}^n$ spanned by H and $[0 : \cdots : 0 : 1]$.

On the other hand, by the technique of example III.9.8.3, the projection $\pi(Y)$ is the reduction of the flat limit Y_0 of Y_a over 0. By corollary III.9.10, $\deg Y_0 = \deg Y_1 = \deg Y$. So we have a surjective map of projective coordinate algebras $k[Y_0] \to k[\pi(Y)]$ and both have Hilbert polynomials with the same leading coefficient. This means that the kernel of this map has constant Hilbert polynomial, and that $H_{Y_0}(z) = H_{\pi(Y)}(z) + c$ for some non-negative integer c. As $p_a(C) = 1 - P_C(0)$ for a curve C, this shows $p_a(Y) \leq p_a(\pi(Y))$.

III.10 Smooth Morphisms

Smoothness is a fundamental tool in algebraic geometry, and it would be nice to have a bit more time to explore this.

Exercise III.10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subset \mathbb{A}^2_k$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

Solution. First we show that X is regular. We deal with the characteristic two case on it's own: since $y^2 - x^2 = (x+y)^2$, we have that $k[x,y]/(y^2 - x^2 + t) \cong k[x,y]/(y^2+t) \cong k(t^{1/2})[x]$ which is clearly a regular ring. When $p \neq 2$, we have slightly more work to do. As $y^2 - x^p + t$ is irreducible, the local ring of the generic point of X is a field and therefore a regular local ring. Next, for a closed point, let $\mathfrak{m} \subset k[x,y]$ be a maximal ideal containing $(y^2 - x^p + t)$. Then the assertion that X is regular at $\mathfrak{m}$ is that

$$\frac{\mathfrak{m}/(y^2 - x^p + t)}{\mathfrak{m}^2/(y^2 - x^p + t) \cap \mathfrak{m}^2} = \frac{\mathfrak{m}}{\mathfrak{m}^2 + (y^2 - x^p + t)}$$

is one-dimensional, or that $y^2 - x^p + t$ is not in $\mathfrak{m}^2$.

When $y \notin \mathfrak{m}$, we can evaluate $\frac{\partial}{\partial y}(y^2 - x^p + t)$ at $\mathfrak{m}$: if $y^2 - x^p + t \in \mathfrak{m}^2$, this would vanish, but it gives $2y \notin \mathfrak{m}$, so $y^2 - x^p + t \notin \mathfrak{m}^2$. When $y \in \mathfrak{m}$ and therefore $\mathfrak{m} = (x^p - t, y)$, we can see that $y^2 - x^p + t \notin \mathfrak{m}^2$: if it were, $x^p - t$ would be in $\mathfrak{m}^2 = (y^2, y(x^p - t), x^{2p} - 2x^p t + t^2)$ too, but by degree considerations this is impossible. Thus X is regular.

On the other hand, to show that X is not smooth let $\overline{k}$ be an algebraic closure of k and consider $X_{\overline{k}}$: if X was smooth over k, this would be regular by example III.10.0.3, but $X_{\overline{k}}$ is not regular at (x,y) because the Jacobian $\begin{pmatrix} px^{p-1} & 2y \end{pmatrix}$ vanishes there.

Exercise III.10.2. Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

Solution. First, f is open by exercise III.9.1 and closed by the definition of proper, so it's image is a clopen subset of Y. Since Y is integral, this means that $f(X) = Y$ and therefore the generic point of X maps to the generic point of Y and we get an extension of fields $k(Y) \subset k(X)$. Since transcendence degree adds over towers of extensions, we see that $k(X)$ has transcendence degree $n = \dim X - \dim Y$ over $k(Y)$, and hence $\Omega_{X/Y}$ has rank at least $\dim X - \dim Y$ at the generic point of X by the compatibility of $\Omega_{X/Y}$ with localization (proposition II.8.2A). Thus by exercise II.5.8(a), we have that the closed set of points $x \in X$ where $\Omega_{X/Y}$ is of rank at least n is all of X because it contains the generic point.

Next, by corollary III.9.6, we have that every irreducible component of X_y is of dimension $\dim X - \dim Y$ for any y, so by the definition of a smooth morphism we have that $\Omega_{X_y/\{y\}}$ is of rank n for every $x \in X_y$. Now I claim that the rank of $\Omega_{X/Y}$ is exactly n for every $x \in X$ mapping to y: writing $\Omega_{X/Y} \otimes k(x)$ as the pullback of $\Omega_{X/Y}$ along the map $\operatorname{Spec} k(x) \to X$, we can factor this map as $\operatorname{Spec} k(x) \to X_y \to X$, and as $\Omega_{X_y/\{y\}}$ is the pullback of $\Omega_{X/Y}$ along $X_y \to X$ by proposition II.8.10, we have that the rank of $\Omega_{X/Y}$ at x is the same as the rank of $\Omega_{X_y/\{y\}}$ at x. Therefore by exercise II.5.8(a), the set Z of points where $\Omega_{X/Y}$ has rank $> n$ is closed and does not contain X_y. So $f(Z)$ is a closed subset of Y not containing y, and letting $U = Y \setminus f(Z)$ we have found our U: condition (1) in the definition of 'smooth of relative dimension n' holds as flatness is stable under base change, condition (2) holds by our previous discussion involving corollary III.9.6, and condition (3) holds by construction of U.

Exercise III.10.3. A morphism $f : X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i). f is étale;

(ii). f is flat, and $\Omega_{X/Y} = 0$;

(iii). f is flat and unramified.

Solution. We need a few preliminaries about $\Omega_{B/A}$ before we begin.

Lemma. *Let $\varphi : A \to B$ be a ring map, and let $R \subset A$ be a multiplicatively closed subset. If R maps to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/R^{-1}A}$.*

Proof. It's enough to show that $d(\varphi(r)^{-1}) = 0$. Consider what happens to $1 = \varphi(r)\varphi(r)^{-1}$ under d:

$$d(1) = d(\varphi(r)\varphi(r)^{-1})$$
$$0 = \varphi(r)d(\varphi(r)^{-1}) + \varphi^{-1}d(\varphi(r))$$
$$0 = \varphi(r)d(\varphi(r)^{-1})$$

and therefore $d(\varphi(r)^{-1}) = 0$.

■

Lemma. *Let $\varphi : A \to B$ be a ring map. If φ is surjective, then $\Omega_{B/A}$ is zero.*

Proof. For any $b \in B$, write $b = \varphi(a)$ and then note that $db = d(\varphi(a)) = 0$.

■

Lemma. *Let $\varphi : A \to B$ be a ring map. If $\Omega_{B/A} = 0$, then the induced map on reductions $A_{red} \to B_{red}$ also has $\Omega_{B_{red}/A_{red}} = 0$.*

Proof. Apply proposition II.8.3A to the sequence of rings maps $A \to A_{red} \to B_{red}$ and $A \to B \to B_{red}$: the first sequence gives that $\Omega_{B_{red}/A} \cong \Omega_{B_{red}/A_{red}}$ by noting that $\Omega_{A_{red}/A} = 0$ by surjectivity, while the second gives that $0 = \Omega_{B/A} \otimes_B B_{red}$ surjects on to $\Omega_{B_{red}/A} \cong \Omega_{B_{red}/A_{red}}$ as $\Omega_{B_{red}/B} = 0$.
■

(i) $\Rightarrow$ (ii): As f smooth of relative dimension zero includes the statement that f is flat, all we need to do is to show that $\Omega_{X/Y} = 0$. I claim that $\dim_{k(x)} \Omega_{X/Y} \otimes k(x) = 0$ implies $(\Omega_{X/Y})_x = 0$. By corollary II.8.5, $\Omega_{X/Y}$ is a coherent sheaf on X, so we may apply Nakayama to see that $\Omega_{X/Y} \otimes k(x) = 0$ implies $(\Omega_{X/Y})_x = 0$ for all x, and therefore $\Omega_{X/Y} = 0$.

(ii) $\Rightarrow$ (i): by exercise III.9.1, f is open, so it remains to check the condition about dimensions of irreducible components from the definition of a smooth morphism of a particular relative dimension. Let Y' be an irreducible component of Y and let X' be an irreducible component of X mapping in to Y'. Let $Y'' \subset Y'$ be an affine open subset which is also open in Y, and let $X'' \subset X'$ an affine open subset which is also open in X and maps in to Y''. By openness of f, the image of X'' in Y'' is open and thus dense, so the generic point of X'' maps to the generic point of Y''.

By our final preliminary, we may replace X'' and Y'' by their reductions while maintaining their dimensions and the condition that $\Omega_{X''/Y''} = 0$. Now look at the map on generic points: by proposition II.8.2A and our first preliminary, the module of differentials associated to the map of function fields $k(Y'') \to k(X'')$ vanishes, and by proposition II.8.6A we have that $k(Y'')$ and $k(X'')$ are of the same transcendence degree over k. Thus X'' and Y'' have the same dimension, so f is smooth of relative dimension zero, and we've shown that (ii) implies (i).

We may reduce the equivalence of (ii) and (iii) to an algebra problem: if we have a local ring map $A \to B$ so that $\Omega_{B/A}$ is a finitely-generated B-module, then $A \to B$ is unramified iff $\Omega_{B/A} = 0$. To see this, let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$: then by proposition II.8.2A, we have that $(\Omega_{B/A})_{\mathfrak{p}} = \Omega_{B_{\mathfrak{p}}/A}$, which by one of our preliminary results is then $\Omega_{B_{\mathfrak{p}}/A_{\mathfrak{q}}}$, where $\mathfrak{q}$ is the preimage of $\mathfrak{p}$ along $A \to B$. Therefore $\Omega_{X/Y} = 0$ iff $\Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}} = 0$ for all $x \in X$, and we've finished our reduction.

(iii) $\Rightarrow$ (ii): Let A (resp. B) be a local ring with maximal ideal $\mathfrak{m}$ (resp. $\mathfrak{n}$) and residue field E (resp. F). Let $A \to B$ be an unramified map of local rings. Letting $A' = E$ and $B' = B \otimes_A E$, we may apply proposition II.8.2A to see that $\Omega_{(B/\mathfrak{m}B)/E} \cong \Omega_{B/A} \otimes_B (B \otimes_A E) \cong \Omega_{B/A} \otimes_A E$. But $\mathfrak{m}B = \mathfrak{n}$ by assumption, so $B/\mathfrak{m}B \cong F$, and as $E \subset F$ is assumed to be separable algebraic extension, we have that $\Omega_{F/E} = 0$ by theorem II.8.6A. Therefore by Nakayama, $\Omega_{B/A} = 0$.

(ii) $\Rightarrow$ (iii): We show that if either the conditions $\mathfrak{m}B = \mathfrak{n}$ or 'F is a separable extension of E' is violated, then $\Omega_{B/A} \neq 0$. Let's start with the extension of fields. Writing $E \to B/\mathfrak{m}N \to F$ and applying proposition II.8.3A, we get the following exact sequence:

$$\Omega_{(B/\mathfrak{m}B)/E} \otimes_{B/\mathfrak{m}B} F \to \Omega_{F/E} \to \Omega_{(B/\mathfrak{m}B)/F} \to 0.$$

The final term vanishes because $B/\mathfrak{m}B \to B/\mathfrak{n} = F$ is surjective, so the first map must be a surjection. In particular, if $E \subset F$ is not a separable algebraic extension, then by theorem II.8.6A we have that the middle term is nonzero, therefore the left term must be nonzero as well. As $\Omega_{(B/\mathfrak{m}B)/E} \cong \Omega_{B/A} \otimes_A E$, we see that $\Omega_{B/A} \neq 0$ by Nakayama.

There are two ways we can fail to have $\mathfrak{m}B = \mathfrak{n}$ - either $\sqrt{\mathfrak{m}B}$ is equal to $\mathfrak{n}$ or it is not. If they're not equal, proposition II.8.3A applied to $E \to B/\mathfrak{m}B \to B/\sqrt{\mathfrak{m}B}$ combined with the observation that $B/\sqrt{\mathfrak{m}B}$ has a fraction field which is of positive transcendence degree over E gives that $\Omega_{B/A} \neq 0$ similarly to the above conclusion.

In the other case, $\sqrt{\mathfrak{m}B} = \mathfrak{n}$ but $\mathfrak{m}B \neq \mathfrak{n}$, we have that $B/\mathfrak{m}B$ is Artinian and thus finite-dimensional as an E-vector space. Let K be an algebraic closure of $A/\mathfrak{m}$. Then $B/\mathfrak{m}B \otimes_E K$ is again Artinian and a finite product of Artinian local rings with K as their residue field. As $B/\mathfrak{m}B$ has a nilpotent element, $(B/\mathfrak{m}B) \otimes_E K$ has a nilpotent element and there exists some local ring factor $B' \subset K \otimes_E B/\mathfrak{m}B$ with nonzero nilpotent maximal ideal $\mathfrak{q}$. By proposition II.8.7, we have that $\mathfrak{q}/\mathfrak{q}^2 \cong \Omega_{B'/K} \otimes_{B'} K$ and thus the RHS is nonzero, and so $\Omega_{B'/K}$ is nonzero by Nakayama. As $\Omega_{B'/K}$ is a localization of $\Omega_{((B/\mathfrak{m}B)\otimes_E K)/K} \cong \Omega_{(B/\mathfrak{m}B)/E} \otimes_E K$ at a maximal ideal, $\Omega_{(B/\mathfrak{m}B)/E}$ is nonzero and we are done.

Exercise III.10.4. Show that a morphism $f : X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisified: for each $x \in X$, let $y = f(x)$. Let $\widehat{\mathcal{O}}_x$ and $\widehat{\mathcal{O}}_y$ be the completions of the local rings of x and y. Choose fields of representatives (II, 8.25A) $k(x) \subset \widehat{\mathcal{O}}_x$ and $k(y) \subset \widehat{\mathcal{O}}_y$ so that $k(y) \subset k(x)$ via the natural map $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$. Then our condition is that for every $x \in X$, $k(x)$ is a separable algebraic extension of $k(y)$, and the natural map

$$\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \widehat{\mathcal{O}}_x$$

is an isomorphism.

Solution. Hartshorne kind of elides details about *why* you can choose fields of representatives like this, but I'm not going to make a fuss about it.

Before we begin, we develop a little material on faithful flatness. An R-module M is *faithfully flat* whenever a sequence of R-modules $0 \to A \to B \to C \to 0$ is exact iff the sequence $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is.

I claim that for M a flat R-module, faithful flatness is equivalent to $M \otimes_R \kappa(r) \neq 0$ for all closed points $r \in \operatorname{Spec} R$: the forward direction is immediate from considering the action of $M \otimes_R -$ on $id : \kappa(r) \to \kappa(r)$. For the reverse direction, consider a complex $A \to B \to C$, let H be the homology at B, and assume $A \otimes_R M \to B \otimes_R M \to C \otimes_R M$ is exact. By flatness of M, we have that $H \otimes_R M = 0$. Let $h \in H$ and declare $I = \operatorname{Ann}(h)$; since $R/I \subset H$, we get that $M/IM \subset H \otimes_R M = 0$ by flatness. If $I \neq R$, then we can find a maximal ideal containing I, contradiction.

The upshot of this is that if $R \to S$ is a flat ring map, then $R \to S$ is faithfully flat iff every closed point of $\operatorname{Spec} R$ is in the image of $\operatorname{Spec} S \to \operatorname{Spec} R$. This last condition is equivalent to $\operatorname{Spec} S \to \operatorname{Spec} R$ being surjective by the combination of the fact that the image of a flat map is stable under generalization by our proof of exercise III.9.1 and the fact that every point in a quasi-compact scheme is a generalization of closed point: given an arbitrary point, take the closure and then apply the result of exercise II.3.14 that every quasi-compact topological space has a closed point.

One very nice consequence for us is that if $A \to B$ is a ring homomorphism and $A \to A'$ is a faithfully flat ring homomorphism, then $A \to B$ is flat iff $A \otimes_A A' \to B \otimes_A A'$ is: starting with an exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, by faithful flatness

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$$

is exact iff

$$0 \to M_1 \otimes_A B \otimes_A A' \to M_2 \otimes_A B \otimes_A A' \to M_3 \otimes_A B \otimes_A A' \to 0$$

is exact, but $M_i \otimes_A B \otimes_A A' \cong (M_i \otimes_A A') \otimes_{A'} (A' \otimes_A B)$ and thus these two conditions are precisely that $A \to B$ is flat and that $A' \to B'$ are flat.

The way faithful flatness enters our picture here is that the completion map $\mathcal{O}_x \to \widehat{\mathcal{O}}_x$ is faithfully flat. To see flatness, by proposition III.9.1A(a) it suffices to check flatness on finitely generated ideals, which is guaranteed by theorem II.9.3A(c). As the completion map is a local homomorphism of local rings, the unique closed point is in the image and we have shown faithful flatness of $\mathcal{O}_x \to \widehat{\mathcal{O}}_x$.

We'll also need a statement about when tensor products commute with inverse limits. Working over a ring A, if (M_n) is an inverse system where each module is finitely generated and N is finitely presented, then $(\varinjlim M_n) \otimes_A N \cong \varinjlim (M_n \otimes_A N)$. To see this, let $A^r \to A^s \to N \to 0$ be a presentation of N and take $\varinjlim$ of the morphism of inverse systems $M_n^r \to M_n^s \to M_n \otimes_N \to 0$ obtained by tensoring our presentation of N with M_n: by proposition II.9.1, this is exact, so we have the desired conclusion.

Now back to the problem at hand. Suppose $k(x)/k(y)$ is separable algebraic and $\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \cong \widehat{\mathcal{O}}_x$. Writing

$$k(x) \cong k(y) \otimes_{k(y)} k(x) \cong k(y) \otimes_{\widehat{\mathcal{O}}_y} \widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \cong k(y) \otimes_{\widehat{\mathcal{O}}_y} \widehat{\mathcal{O}}_x$$

and recalling that $k(y) \cong \widehat{\mathcal{O}}_y/\widehat{\mathfrak{m}}_y$, we have that $k(x) \cong \widehat{\mathcal{O}}_x/\widehat{\mathfrak{m}}_y \widehat{\mathcal{O}}_x$, or that $\widehat{\mathfrak{m}}_y \widehat{\mathcal{O}}_x = \widehat{\mathfrak{m}}_x$. Since $\widehat{M}/\widehat{\mathfrak{m}}^n M \cong M/\mathfrak{m}^n M$ for any module M over a local ring R with $\widehat{M}$ its completion, we see that this implies that the map $(\mathfrak{m}_y \mathcal{O}_x)/\mathfrak{m}_x(\mathfrak{m}_y \mathcal{O}_x) \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective, and therefore $\mathfrak{m}_y \mathcal{O}_x \to \mathfrak{m}_x$ is surjective by Nakayama. Since $\mathfrak{m}_y \mathcal{O}_x \to \mathfrak{m}_x$ is injective, this gives that $\mathfrak{m}_y \mathcal{O}_x = \mathfrak{m}_x$. This shows that the ring map $\mathcal{O}_y \to \mathcal{O}_x$ is unramified, so per exercise III.10.3 all that remains to prove etaleness is to show that this ring map is flat.

By our previous work with faithful flatness, it suffices to show that $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$ is flat. First, we note that $\mathcal{O}_x$ is a finitely-generated $\mathcal{O}_y$-module: since $k(x)$ is finitely generated over k, it is finitely generated over $k(y)$, and as this is a separable algebraic extension, we have that $k(x)/k(y)$ is finite. Combining this with the fact that $\mathfrak{m}_y \mathcal{O}_x = \mathfrak{m}_x$, we have the claim. Writing

$$\widehat{\mathcal{O}}_x = \varprojlim \mathcal{O}_x/\mathfrak{m}_x^n = \varprojlim \mathcal{O}_x/\mathfrak{m}_y^n \mathcal{O}_x = \varprojlim \mathcal{O}_x \otimes_{\mathcal{O}_y} \mathcal{O}_y/\mathfrak{m}_y^n,$$

we may conclude that $\widehat{\mathcal{O}}_x \cong \mathcal{O}_x \otimes_{\mathcal{O}_y} \widehat{\mathcal{O}}_y$ by our work on commuting inverse limits and tensor products since $\mathcal{O}_x$ is module-finite over $\mathcal{O}_y$. Thus we're only on the hook for showing that $\widehat{\mathcal{O}}_x$ is flat over $\widehat{\mathcal{O}}_y$. But this is immediate from our assumptions: $\widehat{\mathcal{O}}_x \cong \widehat{\mathcal{O}}_y \otimes_{k(y)} k(x)$ is the tensor product of the $k(y)$-module $\widehat{\mathcal{O}}_y$ along the map $k(y) \to k(x)$, which is flat because $k(y)$ is a field. Therefore we've shown half of the equivalence.

For the other direction, f etale is equivalent to f flat and unramified by exercise III.10.3, which immediately gives that $k(x)$ is a separable algebraic extension of $k(y)$. To show that $\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \cong \widehat{\mathcal{O}}_x$, it suffices to show that we can commute the inverse limit and the tensor product, that is, that $\varinjlim (\mathcal{O}_y/\mathfrak{m}_y^n \otimes_{k(y)} k(x)) \cong (\varinjlim \mathcal{O}_y/\mathfrak{m}_y^n) \otimes_{k(y)} k(x)$ and that the inverse systems $(\mathcal{O}_y/\mathfrak{m}_y^n \otimes_{k(y)} k(x))$ and $(\mathcal{O}_x/\mathfrak{m}_x^n)$ are isomorphic. As $k(x)$ is a finite extension of $k(y)$ from our previous work, we may apply our earlier work on commuting tensor products and inverse limits to show the first claim.

We finish by demonstrating that $\mathcal{O}_y/\mathfrak{m}_y^n \otimes_{k(y)} k(x) \cong \mathcal{O}_x/\mathfrak{m}_x^n$. As $\mathcal{O}_y \to \mathcal{O}_x$ is flat, we can base change along $\mathcal{O}_y \to \mathcal{O}_y/\mathfrak{m}_y^n$ to see that $\mathcal{O}_y/\mathfrak{m}_y \to \mathcal{O}_x/\mathfrak{m}_y^n \mathcal{O}_x \cong \mathcal{O}_x/\mathfrak{m}_x^n$ is a flat ring extension. Since the latter is an artinian $k(x)$-algebra, it is a finite-dimensional $k(x)$-algebra, and as $k(x)$ is finite over $k(y)$ we have that it's a finitely generated $\mathcal{O}_y/\mathfrak{m}_y^n$-module and therefore free by proposition III.9.1A. Therefore the natural map $\mathcal{O}_y/\mathfrak{m}_y^n \otimes_{k(y)} k(x) \to \mathcal{O}_x/\mathfrak{m}_x^n$ is a map of finite free $\mathcal{O}_y/\mathfrak{m}_y^n$ modules, and we can check whether it's an isomorphism after modding out the maximal ideal by Nakayama. This is the end of our journey: after modding out the maximal ideal, we just have that identity map $k(x) \to k(x)$.

Exercise III.10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathcal{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathcal{F}$ is a free $\mathcal{O}_U$-module, then $\mathcal{F}$ is locally free on X.

Solution. It suffices to show that $\mathcal{F}$ is flat over X by a combination of proposition III.9.1A(e) and exercise II.5.7. But $\mathcal{O}_{U,u}$ is a faithfully flat $\mathcal{O}_{X,x}$-module, and by our material on faithful flatness in the solution to exercise III.10.4, we have that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$ if and only if $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{U,u}$ where $u \mapsto x$ under $U \to X$. Since $(f^*\mathcal{F})_u \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{U,u}$ and any locally free module is flat, we have the result.

Exercise III.10.6. Let Y be the plane nodal cubic curve $y^2 = x^2(x+1)$. Show that Y has a finite étale covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Fig. 12).

Figure coming soon.

Solution. This is visually clear from looking at the picture, but we have to prove it. We make the standard assumption that $\operatorname{char} k \neq 2$ here (else we have to change the form of the nodal cubic, you can do this if you want, it's not so bad).

From looking at the picture, we see that we want X to be two copies of $\mathbb{A}^1_k$ intersecting at two points and that the 'direction' of the parametrizations needs to be opposite - that is, if we let $X' = \operatorname{Spec} k[t] \times k[u]$ be a disjoint union of the two lines before we enforce the intersection, the map $X' \to Y$ should correspond to

$$k[x,y]/(y^2 = x^3 + x^2) \to k[t] \times k[u]$$

$$x \mapsto (t^2 - 1, u^2 - 1), y \mapsto (t^3 - t, -u^3 + u).$$

Now I claim that we can glue $t = \pm 1$ to $u = \pm 1$ to write X as two lines which meet at two points. One straightforward way to do this is to write $X = V(b = a^2 - 1) \cup V(b = -a^2 + 1) = \operatorname{Spec} k[a,b]/(b^2 - (a^2-1)^2) \subset \operatorname{Spec} k[a,b]$ where the map $X' \to X$ is the spectrum of $a \mapsto (t,u)$ and $b \mapsto (t^2 - 1, -u^2 + 1)$: this gives that $k[x,y]/(y^2 = x^3 + x^2) \to k[t] \times k[u]$ factors through $k[a,b]/(b^2 - (a^2-1)^2)$, giving $X \to Y$ as the spectrum of

$$k[x,y]/(y^2 = x^3 + x^2) \to k[a,b]/(b^2 = (a^2-1)^2),$$

$$x \mapsto a^2 - 1, y \mapsto ab.$$

Now we can check that this is indeed etale. Restricting the map $X \to Y$ to $f^{-1}(Y^{sm}) \to Y^{sm}$, we see that this map becomes a disjoint union of isomorphisms from each component on to Y^{sm}, which is etale. It remains to check what happens over the singular point, where we verify the criteria of exercise III.10.4. Both points mapping to the singular point of Y are k-rational, so both have residue field k and it remains to demonstrate that the map $X \to Y$ induces isomorphism on the completed local rings. We'll look at $(1,0) \mapsto (0,0)$, with the other map being exactly the same up to swapping 1 and -1. The map on completed local rings is $k[[x,y]]/(y^2-x^3-x^2) \to k[[(a-1),b]]/(b^2-(a^2-1)^2)$ by $x \mapsto a^2-1$ and $y \mapsto ab$, which has inverse $(a-1) \mapsto -1+\sqrt{x+1}$, $b \mapsto \frac{y}{\sqrt{x+1}}$ where by $\sqrt{x+1}$ we mean the formal power series for $\sqrt{x+1}$, $\sum_{n=0}^{\infty} \binom{1/2}{n} x^n$ which is defined in our field because $2 \neq 0$.

Exercise III.10.7. (*Serre*). *A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbb{P}^2_k$ be the seven points of the projective plane over the prime field $\mathbb{F}_2 \subset k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

a. $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X \setminus \{P_i\}$ to $\mathbb{P}^2$.

b. Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a $1-1$ correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. Thus the singular points of elements of $\mathfrak{d}$ move all over.

Solution.

a. The first claim follows from verifying that the 7 constraints on the 10-dimensional vector space of degre-three homogeneous forms coming from evaluating at each $P_1, \cdots, P_7$ are linearly independent: after projectivizing, this gives the desired result. This can be seen somewhat quickly: writing the equations in a matrix with the columns labeled in order by $\{x^3, y^3, z^3, x^2y, x^2z, y^2z, xyz, xy^2, xz^2, yz^2\}$ and rows labeled in order by $\{[1:0:0],[0:1:0],[0:0:1],[1:1:0],[1:0:1],[0:1:1],[1:1:1]\}$, one observes that the top left 7×7 minor is lower triangular with 1s on the diagonal, giving that $\mathfrak{d} = \operatorname{Span}(x^2y - xy^2, x^2z - xz^2, y^2z - yz^2)$.

   Before we can address the second claim, we need to explain what an inseparable morphism is (Hartshorne forgot to define this before using it!). We'll take as our definition for an inseparable morphism that the map $X \to Y$ is dominant and induces a purely inseparable field extension $K(Y) \subset K(X)$.

   First we check that the map actually is dominant and therefore does give a function field extension. It suffices to show that there is no nonzero polynomial $p(a,b,c) \in k[a,b,c]$ vanishing when a,b,c are set to $xy(x+y), xz(x+z), yz(y+z)$ respectively. Suppose we have such a polynomial p: as the polynomials we evaluate at are all homogeneous of degree three, we may assume p is homogeneous of degree n. Suppose i is the largest integer so that a term divisible by a^i appears in p with nonzero coefficient. Then we can write

$$p(a,b,c) = \left(\sum_{j=0}^{n-i} \xi_j a^i b^j c^{n-i-j} \right) + q(a,b,c)$$

where every monomial in q has degree less than i in a. This means that every term found in $q(a,b,c)$ after evaluating is of degree strictly greater than $n-i$ in z. On the other hand, evaluating the term inside the parentheses we find that the terms of degree exactly $n-i$ in z are

$$\xi_j x^{i+2j} y^{2n-2j-i} (x+y)^i z^{n-i}$$

as j runs from 0 to $n-i$. We see that the only term contributing a monomial of the form $x^i y^{2n} z^{n-i}$ is the one labeled by ξ_0, so $\xi_0 = 0$, and continuing by considering monomials of the form $x^{i+2j} y^{2n-2j} z^{n-i}$ as j goes from 1 to $n-i$, we see that in fact all ξ are zero and $p = 0$.

Now we want to investigate the field extension $K(Y) \subset K(X)$. We may identify $K(X)$ the subfield of degree-zero elements of $k(x,y,z)$ and $K(Y)$ with the subfield of degree-zero elements of $k(xy(x+y), xz(x+z), yz(y+z))$. Playing around, we find

$$\frac{(x^2z + xz^2)(x^2y + xy^2 + x^2z + xz^2)}{(y^2z + yz^2)(x^2y + xy^2 + y^2z + yz^2)} = \frac{x^2(x^2yz + xy^2z + x^2z^2 + xyz^2 + y^2z^2 + z^4)}{y^2(x^2yz + xy^2z + y^2z^2 + x^2z^2 + xyz^2 + z^4)} = \frac{x^2}{y^2},$$

and the LHS is not a square. Therefore $K(X)(\frac{x}{y})$ is a purely inseparable field extension, and I claim that it is actually $K(Y)$. It suffices to show that $\frac{x}{z}$ is in this extension, so after some more tinkering we happen upon

$$\frac{x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2}{y^2z + yz^2} \cdot (1 + \frac{x}{y})^{-1} = \frac{x^2}{yz} + \frac{x}{y}$$

which demonstrates that $\frac{x}{z}$ is in our extension too, and we're done.

b. Since k is algebraically closed of characteristic two, we can write any element of k uniquely as a square. So let $\alpha^2, \beta^2, \gamma^2 \in k$ and consider the curve given by $\alpha^2(x^2y + xy^2) + \beta^2(x^2z + xz^2) + \gamma^2(y^2z + yz^2)$ with Jacobian

$$\begin{pmatrix} \alpha^2y^2 + \beta^2z^2 & \alpha^2x^2 + \gamma^2z^2 & \beta^2x^2 + \gamma^2y^2 \end{pmatrix}.$$

The vanishing of these three equations is equivalent to the vanishing of the three equations $\alpha y + \beta z$, $\alpha x + \gamma z$, and $\beta x + \gamma y$. These have a unique common solution: $[\gamma : \beta : \alpha]$, which lies on the curve (plug in!) and therefore we have shown the correspondence between $C \in \mathfrak{d}$ and points of $\mathbb{P}^2$ via the unique singular point of C.

Unfortunately the remainder of the classification of members of $\mathfrak{d}$ is not correct - there are members of $\mathfrak{d}$ which are conics with a tangent line in addition to three lines meeting at a P_i and cuspidal cubics. This is it, though - these are the only three types of curves in this linear system.

We'll being with investigating when a member of $\mathfrak{d}$ is irreducible by looking for a factorization

$$\alpha^2(x^2y+xy^2)+\beta^2(x^2z+xz^2)+\gamma^2(y^2z+yz^2) = (px+qy+rz)(ax^2+by^2+cz^2+dxy+exz+fyz).$$

We start with casework on how many of p, q, r are zero:

(i) If none are, then $a = b = c = 0$ by considering the x^3, y^3, and z^3 terms, which gives us that our product is $pdx^2y+qdxy^2+pex^2z+rexz^2+qfy^2z+rfyz^2+(pf+qe+rd)xyz$. Thus $p = q = r$, which we may assume is 1, and $d+e+f = 0$ with $d = \alpha^2$, $e = \beta^2$, $f = \gamma^2$. Without loss of generality, assume $\alpha^2 = 1$: this makes the Jacobian of the quadractic factor $(y+\beta^2 z, x+(1+\beta^2)z, \beta^2 x+(1+\beta^2)y)$ which uniquely vanishes at $[(1+\beta^2) : \beta^2 : 1]$. Plugging this in to our quadratic, we get $\beta^2(1+\beta^2)$ which is zero iff $\beta \in \mathbb{F}_2$. When $\beta \in \mathbb{F}_2$, this gives that C decomposes in to a product of three lines through $[1+\beta^2 : \beta^2 : 1] \in \mathbb{P}^2(\mathbb{F}_2)$. On the other hand, when $\beta \notin \mathbb{F}_2$, we see that the tangent line to the conic at the point of intersection between the conic and the line has direction $(\beta+\beta^2, \beta+\beta^2, \beta+\beta^2)$, parallel to $V(x+y+z)$, and so we have a smooth conic with a tangent line.

(ii) If one is, say $r = 0$, then $a = b = 0$ by considering the x^3 and y^3 terms, which gives us that our product is $pdx^2y+qdxy^2+pex^2z+pcxz^2+qfy^2z+qcyz^2+(pf+qe)(xyz)$. Thus $p = q$, which we may assume is 1, and $c = e = f$ with $d = \alpha^2$, $c = \beta^2 = \gamma^2$. Now split in to sub-cases: if $\alpha^2 = 0$, then we may scale so that $\beta^2 = 1$ and we have $(x+y)xy$, which is a product of three lines meeting at $[0 : 0 : 1]$. If $\alpha^2 \neq 0$, scale so $\alpha^2 = 1$ and then we have $(x+y)(\beta^2 z^2+xy+\beta^2 xz+\beta^2 yz)$: this makes the Jacobian of the quadratic factor $(y+\beta^2 z, x+\beta^2 z, \beta^2(x+y))$ which uniquely vanishes at $[\beta^2 : \beta^2 : 1]$. Plugging this in to our quadratic, we get $\beta^2+\beta^4 = \beta^2(1+\beta^2)$, which is zero iff $b \in \mathbb{F}_2$. When $\beta \in \mathbb{F}_2$, this gives that C decomposes in to a union of three lines through $[\beta^2 : \beta^2 : 1] \in \mathbb{P}^2(\mathbb{F}_2)$. On the other hand, when $\beta \notin \mathbb{F}_2$, we see that the tangent line to the conic at the point of intersection between the conic and the line has direction $(\beta+\beta^2, \beta+\beta^2, 0)$, parallel to $V(x+y)$, and so we have a smooth conic with a tangent line.

(iii) If two are, say $q = r = 0$, then $a = 0$ by considering the x^3 term, which gives us that our product is $pdx^2y+pbxy^2+pex^2z+pcxz^2+pfxyz$. Thus we may take $p = 1$ and find that $b = d = \alpha^2$, $c = e = \beta^2$, and $f = 0$. Now split in to sub-cases: if $\alpha^2 = 0$, then we may scale so that $\beta^2 = 1$ and we have $xz(x+z)$, which is a product of three lines meeting at $[0 : 1 : 0]$. If $\alpha^2 \neq 0$, scale so $\alpha^2 = 1$ and then we have $x(y^2+\beta^2 z^2+xy+\beta^2 xz)$: this makes the Jacobian of the quadratic factor $(y+\beta^2 z, x, \beta^2 x)$ which uniquely vanishes at $[0 : \beta^2 : 1]$. Plugging this in to our quadratic, we get $\beta^4+\beta^2 = \beta^2(1+\beta^2)$, which is zero iff $\beta \in \mathbb{F}_2$. When $\beta \in \mathbb{F}^2$, this gives that C demcomposes in to a union of three lines through $[0 : \beta^2 : 1]$. On the other hand, when $\beta \notin \mathbb{F}_2$, we see that the tangent line to the conic at the point of intersection between the conic and the line has direction $(\beta+\beta^2, 0, 0)$, parallel to $V(x)$, and so we have a smooth conic with a tangent line.

Therefore C is reducible exactly when the set $\{\alpha^2, \beta^2, \gamma^2\}$ (equivalently, $\{\alpha, \beta, \gamma\}$) is linearly dependent over $\mathbb{F}_2$. When the $\mathbb{F}_2$-span of $\{\alpha, \beta, \gamma\}$ has dimension one, then $[\gamma : \beta : \alpha] \in \mathbb{P}^2(\mathbb{F}_2)$

and C is three lines meeting at this point. Otherwise, the $\mathbb{F}_2$-span is two-dimensional, and C is a smooth conic passing through $[\gamma : \beta : \alpha]$ and it's tangent line through that point.

When C is irreducible, we demonstrate that C is projectively equivalent to $y^2 z = x^3$, the cuspidal cubic. Since the set $\{\alpha, \beta, \gamma\}$ is linearly independent over $\mathbb{F}_2$, we may scale so that $\alpha = 1$. Now apply the projective automorphism $x \mapsto x + \gamma z$, $y \mapsto y + \beta z$ to put the singular point at $[0 : 0 : 1]$. Our equation after this automorphism is $x^2 y + xy^2 + \beta(1+\beta)x^2 z + \gamma(1+\gamma)y^2 z$. As β and γ aren't in $\mathbb{F}_2$, $\beta(1+\beta)$ and $\gamma(1+\gamma)$ are nonzero, so we can set them as B^2 and C^2 for some $B, C \in k^\times$. Then setting $x = Bx + Cy$, our equation turns in to $x^2 z + B^2 x^2 y + Bxy^2 + (C + C^2)y^3$. Substituting $z \mapsto \frac{B^3}{C^4+C^2}x + (B^2 + \frac{B^2}{(C^2+C)^{2/3}})y$, we find our equation is of the form $x^2 z + (\frac{B}{(C^2+C)^{2/3}}x + (C^2 + C)^{1/3}y)^3$, a cuspidal cubic.

Exercise III.10.8. *A linear system with moving singularities contained in the base locus (any characteristic).* In affine 3-space with coordinates x, y, z, let C be the conic $(x-1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0, 0, t)$ on the z-axis. Let Y_t be the closure in $\mathbb{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

Solution. Hartshorne omits an assumption that $\operatorname{char} k \neq 2$ here, which is necessary.

The first step is to determine the equations of Y_t. Take the standard cone $x^2 + y^2 = z^2$, tilt the axis by applying the transformation $x \mapsto x + z$ to get $(x+z)^2 + y^2 = z^2$, translate the cone so the vertex is at P by $z \mapsto z - t$ to get $(x + z - t)^2 + y^2 = (z-t)^2$, and shrink the section at $t = 0$ by applying the transformation $x \mapsto tx$, $y \mapsto ty$ to get $(tx + z - t)^2 + (ty)^2 = (z-t)^2$ which when evaluated at $z = 0$ gives $t^2(x-1)^2 + t^2 y^2 = t^2$, or $(x-1)^2 + y^2 = 1$ (as long as $t \neq 0$). This equation simplifies to $t^2 x^2 - 2t^2 x + t^2 y^2 + 2txz$, and after dividing out the extra factor of t we have $t(x^2 + 2x + y^2) + 2xz$. Writing $t = a/b$ and homogenizing the equation, we get that Y_t is cut out by $a(x^2 + 2xw + y^2) + b(2xz)$ inside $\mathbb{P}^3$, which is the linear system spanned by $x^2 + 2xw + y^2$ and $2xz$ inside $\mathcal{O}_{\mathbb{P}^3}(2)$.

To check the singularities, we compute the Jacobian to be $2(aw + ax + bz, ay, bx, ax)$. As at least one of a and b is nonzero, we find the singular points must lie on the line $V(x, aw + bz)$. When $a \neq 0$, the only intersection of this with our surface is $[0 : 0 : a : b]$, which is exactly P, while when $a = 0$, the singular locus is $V(x, z)$, the y-axis, so it contains P. To verify that the base locus is as claimed, we note it is exactly $V(x^2 + 2xw + y^2, 2xz) = V(x^2 + 2xw + y^2, x) \cup V(x^2 + 2xw + y^2, z) = C \cup V(y^2, x)$ which is set-theoretically C and the z-axis.

Exercise III.10.9. Let $f : X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen-Macaulay, and that every fiber of f has dimension equal to $\dim X - \dim Y$. Then f is flat. [*Hint*: Imitate the proof of (10.4), using (II, 8.21A).]

Solution. By the same logic as the proof of proposition III.10.4, it is enough to show that $\mathcal{O}_x$ is flat over $\mathcal{O}_y$ for every closed point x with $y = f(x)$. We prove this by induction on $\dim \mathcal{O}_y$. When $\dim \mathcal{O}_y = 0$, we have that $\mathcal{O}_y = k$ by the regularity assumption, and any k-algebra is flat.

Now suppose $\dim \mathcal{O}_y = n > 0$ and we've shown the result for all smaller dimensions. Since $\mathcal{O}_y$ is a regular local ring, we can find a generating set for $\mathfrak{m}$ of size n, say $\alpha_1, \cdots, \alpha_n$. Then

$\mathcal{O}_x/(\alpha_1, \cdots, \alpha_n) \cong \mathcal{O}_{X_y,x}$, which by assumption is of dimension $\dim X - \dim Y = \dim X - n$. By theorem II.8.21A(c), this shows that $\alpha_1, \cdots, \alpha_n$ forms a regular sequence in $\mathcal{O}_x$, and therefore α_1 is not a zero divisor in $\mathcal{O}_x$. Applying the local criteria for flatness, lemma III.10.3A, we see that $\mathcal{O}_y \to \mathcal{O}_x$ is flat iff $\mathcal{O}_y/(\alpha_1) \to \mathcal{O}_x/\alpha_1\mathcal{O}_x$ is flat. But $\mathcal{O}_y/(\alpha_1)$ is a regular local ring of dimension $\dim Y - 1$, $\mathcal{O}_x/\alpha_1\mathcal{O}_x$ is a Cohen-Macaulay local ring of dimension $\dim X - 1$ by theorem II.8.21A(d), and we didn't change the fiber. So the inductive hypothesis applies, and we're finished.

III.11 The Theorem on Formal Functions

The theorem on formal functions is really quite powerful. It's a real treat to use!

Exercise III.11.1. Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbb{A}^n_k$, let $P = (0, \dots, 0)$, let $U = X \setminus P$, and let $f : U \to X$ be the inclusion. Then the fibers of f all have dimension 0, but $R^{n-1} f_* \mathcal{O}_U \neq 0$.

Solution. By proposition II.8.5, we have that $R^{n-1} f_* \mathcal{O}_U$ is isomorphic to the sheaf associated to $H^{n-1}(U, \mathcal{O}_U)$. We may compute this using Čech cohomology, and by the same reasoning as exercise III.4.3 we see that this cohomology group is nonzero.

Exercise III.11.2. Show that a projective morphism with finite fibers (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

Solution. Suppose $f : X \to Y$ is a projective morphism of noetherian schemes which is quasi-finite (Hartshorne omits the noetherian assumption by mistake). It suffices to show that f is affine, as affine plus proper equals finite by exercise II.4.6.

We'll show that if Y is affine, then X is too. First, we note that any fiber of f is of dimension zero: such a fiber is a finite type scheme over a field, which must have infinitely many points if it has positive dimension (the map from the base change to the algebraic closure is finite-to-one, and then apply exercise I.4.8). This means that by corollary III.11.2, we have that $R^1 f_* \mathcal{F} = 0$ for all coherent $\mathcal{F}$ on X. By an application of theorem III.3.7, we are done.

Exercise III.11.3. Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is *not composite with a pencil*, which means that if $f : X \to \mathbb{P}^n_k$ is the morphism determined by $\mathfrak{d}$, then $\dim f(X) \geq 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1). [*Hints*: Use (11.5), (Ex. 5.7), and (7.9).]

Solution. When $f : X \to \mathbb{P}^n_k$ is finite, we have that X is proper over k. We also have that $f^* \mathcal{O}(1)$ is ample by exercise III.5.7(a) and (d), so X is projective by remark II.5.16.1, and we may apply corollary III.7.9 to see that $f^{-1}(H)$ is connected for any hyperplane H. We will reduce to this from the general case by an application of Stein factorization.

Factor $f : X \to \mathbb{P}^n_k$ as $X \xrightarrow{g} X' \xrightarrow{h} \mathbb{P}^n_k$ where g has connected fibers and h is finite by Stein factorization. As the surjective image of an irreducible space is irreducible, X' is irreducible. Since X is reduced we can factor $X \to X'$ through X'_{red}, and therefore we may assume $X' = X'_{red}$ since $X'_{red} \to X'$ is finite. Therefore we may assume that X' is also integral. If $f^{-1}(H)$ is disconnected for some $H \subset \mathbb{P}^n$, then it must be the case that $h^{-1}(H)$ is disconnected, else some fiber of g is disconnected. Finally, since finite surjective morphisms preserve dimension, $\dim X' = \dim f(X) \geq 2$. Thus all we need to do is to show that X' is normal and then we can apply our first paragraph to finish the problem.

By our previous work, X' is integral, so we need only verify that it is normal. This is not so bad: suppose $\varphi \in k(X')$ is an element which satisfies an integral dependence relation $p(z)$ over some affine open subscheme $U \subset X'$. Then φ may also be regarded as an element of $k(X)$ via the injection $k(X') \to k(X)$, and the integral dependence relation $p(z)$ pulls back to an integral dependence relation over $g^{-1}(U)$. Since X is normal, we find that (the image of) φ belongs to $\mathcal{O}_X(g^{-1}(U))$ and hence $\varphi \in \mathcal{O}_{X'}(U)$, proving that for any affine open $U \subset X'$, $\mathcal{O}_{X'}(U)$ is normal. Thus X' is normal and we've won.

Exercise III.11.4. *Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbb{P}^n_k$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subset T$, such that for all closed points $t \in U$, X_t is connected. Then prove that X_t is connected for all $t \in T$.

Solution. This one's much worse than it looks at first glance. You'd love to reduce this by Stein factorization to the point where you could say that $f_*\mathcal{O}_X$ is a finite flat (hence locally free) $\mathcal{O}_T$-module, and then claim that connected fibers implies rank 1. Unfortunately this last claim is false - you might have fibers which look like finite extensions of fields, or fibers that look like $k \to k[x]/(x^n)$, so one has to be a bit more careful. (If you assume geometrically connected fibers instead of just connected fibers, perhaps by taking $k = \overline{k}$, you can skip ahead to this point, but that's not in the text of the question.)

We make a series of reductions to the case where $f : X \to T$ is a finite morphism curves over k with T regular and affine. First, we may assume T is reduced: base change $f : X \to T$ along $T_{red} \to T$ preserves the fibers and flatness. Next, we may assume T is normal (and thus regular): base change along the normalization map $\nu : T' \to T$ is an isomorphism on a dense open subset, so the condition on the connected fibers of $X_{T'} \to T'$ is preserved and $X_{T'}$ is again flat over T'.

Now I claim that we may also assume that X is finite over T: letting $X \xrightarrow{f'} X' \xrightarrow{g} T$ be a Stein factorization of $X \to T$, we see by the same logic as the previous exercise that the fibers of $X \to T$ over $t \in T$ is connected iff the fiber of $X' \to T$ over t is connected. To show X' is flat over T, it is enough to show $g_*\mathcal{O}_{X'} = g_*f'_*\mathcal{O}_{X'} = f_*\mathcal{O}_X$ is flat over T. Let $\operatorname{Spec} A \subset T$ be an affine open subset, and cover $f^{-1}(\operatorname{Spec} A)$ by finitely many affine opens $\operatorname{Spec} B_i \subset T$. Note that $f^{-1}(\operatorname{Spec} A) \subset \mathbb{P}^n_A$ is projective, so it is separated, and the intersection of any two affine opens contained in it is again affine. As the sections of $\mathcal{O}_X$ over any affine open in $f^{-1}(\operatorname{Spec} A)$ are flat over A, we may exhibit $\mathcal{O}_X(f^{-1}(\operatorname{Spec} A))$ as the kernel of $\prod_i B_i \to \prod_{i,j} \mathcal{O}_X(\operatorname{Spec} B_i \cap \operatorname{Spec} B_j)$. Since there are only finitely many B_i, these products are actually sums, and so by proposition III.9.1A(e), $\mathcal{O}_X(f^{-1}(\operatorname{Spec} A))$ is flat over A.

But wait, there's more reduction! We may also assume that X is reduced: by proposition III.9.7, $X \to T$ flat is equivalent to all associated points of X mapping to the generic point of T, and this condition is preserved by reduction - reduction also doesn't change the underlying topological spaces, so the condition on the fibers is preserved as well. I claim X may be taken to be irreducible as well: first, every irreducible component of X must dominate T, and as $X \to T$ is closed, this means that every irreducible component has set-theoretic image T. But the locus of points of X on multiple irreducible components is finite, so if X has multiple irreducible components, then over a dense open subset of T the fiber of f must have at least as many points as there are irreducible components. As f was assumed to have connected fibers over a dense open subset, we see that X is irreducible, hence integral. Finally, it suffices to verify this condition on each affine open of T, so we may as well assume T is affine itself.

Now we're in the situation we claimed at the beginning: let $T = \operatorname{Spec} R$ for R a normal domain over k and let $X = \operatorname{Spec} A$ with A an integral domain which is finitely generated as an R-module. By proposition III.9.1A(f), we have that A is a locally free R-module, so passing to an open cover of T we may assume that $A \cong R^{\oplus n}$ for some n. We can go even further: by considering successive extensions $R \subset R[\alpha_1] \subset R[\alpha_1, \alpha_2] \subset \cdots \subset A$ where the field extension $\operatorname{Frac}(R[\alpha_1, \cdots, \alpha_m]) \subset \operatorname{Frac}(R[\alpha_1, \cdots, \alpha_{m+1}])$ is a simple extension and applying our reductions to each inclusion in turn, we may assume that $A = R[x]/(\xi(x))$ for a monic irreducible polynomial $\xi \in R[x]$ of degree at least two.If $\operatorname{Frac}(R) \subset \operatorname{Frac}(A)$ is purely inseparable then ξ is of the form $x^{\operatorname{char} k} - r$ for $r \in R$, and it is easy to see that the fiber over $\operatorname{Spec} R/\mathfrak{m} \subset \operatorname{Spec} R$ is either a purely inseparable or nilpotent extension of $R/\mathfrak{m}$, both of which have only one point. As the fiber over the generic point of $\operatorname{Spec} R$ is the generic point of $\operatorname{Spec} A$, we have our desired conclusion.

In the case where $\operatorname{Frac}(R) \subset \operatorname{Frac}(A)$ is separable, I claim that for infinitely many closed points $z \in \operatorname{Spec} R$, there is a point $z' \in \operatorname{Spec} A$ mapping to z with $\kappa(z) = \kappa(z')$. This is enough to finish the problem, because it implies that the fibers are non-reduced on a dense set (take the intersection of the set where the fibers are connected and thus singletons with the set satisfying the property with residue fields), or that $\Omega_{A/R}$ is nonzero on a dense set, while the condition that $\operatorname{Frac}(R) \subset \operatorname{Frac}(A)$ is separable implies the stalk of $\Omega_{A/R}$ at the generic point is zero. The combination of two results suffices: first, if $\operatorname{Spec} A \to \operatorname{Spec} R$ is a finite separable morphism of curves with $p \in \operatorname{Spec} R$, then there are only finitely many points p which have a $p' \in \operatorname{Spec} A$ mapping to p with $\kappa(p) \subset \kappa(p')$ inseparable; and second, if $X \to Y$ is a dominant morphism of curves over a field, then there exists a closed point $x \in X$ such that $\kappa(f(x)) \subset \kappa(x)$ is purely inseparable. If we know both of these results, then we can restrict to the locus where $\operatorname{Spec} A \to \operatorname{Spec} R$ has only separable residue field extensions, apply the second result to find a purely inseparable field extension which must be an isomorphism, and then run the proof again after throwing away the point we just found.

Now let us prove these claims. First, recall $A = R[x]/(\xi(x))$ for ξ separable as a polynomial in $\operatorname{Frac}(R)[x]$. This means that we have that $(\xi(x), \xi'(x)) \subset \operatorname{Frac}(R)[x]$ is the unit ideal, or that we can write $\alpha\xi + \beta\xi' = 1$ for $\alpha, \beta \in \operatorname{Frac}(R)[x]$. Therefore for any closed point $p \in \operatorname{Spec} R$ where α and β are defined on $\operatorname{Spec} A$, we have that the equation $\alpha\xi + \beta\xi' = 1$ holds after evaluation of α, β, and the coefficients of ξ in the residue field at p, so $\xi(x)$ in $\kappa(p)[x]$ is separable for an open dense subset of $p \in \operatorname{Spec} R$, and at any such p the fiber is the spectrum $\kappa(p)[x]/(\xi(x))$ which decomposese as a product of separable extensions of $\kappa(p)$ and we have our first claim.

The second claim, that if $f : X \to Y$ is a dominant morphism of curves (pure dimension one schemes of finite type over a field) then there is a point $x \in X$ so that $\kappa(f(x)) \subset \kappa(x)$ is purely inseparable, is best and most elegantly handled by a result due to Poonen in *Points having the same residue field as their image under a morphism*. We'll follow his proof here, though there are some simplifications due to our assumptions. First, we note that the claim can be reduced to the affine case: pick some affine open in Y intersecting the one-dimensional image of X, and then pick some affine open in X contained in the preimage. Next, we note that if we have a composition $X' \to X \to Y \to Y'$, proving the result for $X' \to Y'$ is enough to prove it for $X \to Y$: if $x' \mapsto x \mapsto y \mapsto y'$, then we have a chain of field extensions $\kappa(y') \subset \kappa(y) \subset \kappa(x) \subset \kappa(x')$, and if the outer extension $\kappa(y') \subset \kappa(x')$ is purely inseparable, then the inner extension $\kappa(y) \subset \kappa(x)$ must be as well. Therefore we may replace X by X_{red} and then by some irreducible component which dominates Y; we may also reduce to the case where $Y = \mathbb{A}^1_k$ by taking a morphism corresponding an injection $k[t] \to \mathcal{O}_Y(Y)$ (which exists because $\operatorname{Frac}(\mathcal{O}_Y(Y))$ is a field extension of k of transcendence degree one).

Now consider the extension of fraction fields $k(Y) = k(t) \subset k(X)$. We may factor this as $k(t) \subset L_0 \subset L_1 \subset \cdots \subset L_r = L$ with L_0 separable over $k(t)$ and $L_{i+1} = L_i(u_i)$ purely inseparable of degree $p = \operatorname{char} k$ over L_i. By the theorem of the primitive element, we can write $L_0 = k(t)(z)$, and multiplying z by some nonzero element of $k[t]$ we may assume that the characteristic polynomial $P(T)$ of z in L_0 over $k(t)$ has coefficients in $B = k[t]$. Let $A_i = B[z, u_1, \cdots, u_i]$ so that $(k[t] \setminus \{0\})^{-1} A_i = L_i$. Pick $q \in B$ nonzero so that $u_{i+1}^p \in A_i[q^{-1}]$ for each i and $A_r[q^{-1}] = A[q^{-1}]$.

The claim is that for some nonzero $b \in B$, $b - z \notin A_0[q^{-1}]^\times$. It's enough to find a $b \in B$ with $P(b) = \operatorname{Norm}_{L_0/k(t)}(b - z)$ is not a unit in $B[q^{-1}]$. Let T_n be the set of poylnomials in t of degree exactly n with coefficients chosen from $\{0, 1\}$ so that $|T_n| = 2^n$, and let $d = \deg P$. Then $\{P(b) \mid b \in T_n\}$ consists of at least $2^n/d$ distinct polynomials, each monic of degree nd as long as n is greater than the biggest degree of the coefficients of P with respect to t. On the other hand, factoring q over $\overline{k}$ gives that there are at most $O((nd)^{\deg q})$ monic polynomials of degree nd in $B[q^{-1}]^\times$ as $n \to \infty$. Therefore by picking n large enough, we can find $b \in T_n$ so that $P(b) \notin B[q^{-1}]^\times$, and therefore $b - z \notin A_0[q^{-1}]^\times$. Picking a maximal ideal $\mathfrak{n}_0$ in $A_0[q^{-1}]$ containing $b - z$, we see that since $A_{i+1}[q^{-1}] \cong A_i[q^{-1}][U]/(U^p - \alpha_i)$ for some $\alpha_i \in A_i[q^{-1}]$, there is a unique maximal ideal $\mathfrak{n}$ of $A_r[q^{-1}] = A[q^{-1}]$ over $\mathfrak{n}_0$. Therefore the field extension of the corresponding closed points is purely inseparable, and we're done.

Exercise III.11.5. (*) Let Y be a hypersurface in $X = \mathbb{P}^N_k$ with $N \geq 4$. Let $\widehat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic} \widehat{X} \to \operatorname{Pic} Y$ is an isomorphism. [*Hint*: Use (II, Ex. 9.6), and then study the maps $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ for each n using (Ex. 4.6) and (Ex. 5.5).]

Solution. Suppose $Y = V(f)$ for f of degree d, and let $Y_n = V(f^n)$. By exercise II.9.6, as each Y_n is projective over a field, we have that $\operatorname{Pic} \widehat{X} \cong \varprojlim \operatorname{Pic} Y_n$. Therefore it's enough for us to show that the projection $\varprojlim \operatorname{Pic} Y_n \to \operatorname{Pic} Y$ is an isomorphism.

Let's look at the closed immersion $Y_n \subset Y_{n+1}$. The kernel of this is $\mathcal{O}_Y(-dn)$, which has vanishing cohomology in degrees 1 and 2 by exercise III.5.5(c). Thus the long exact sequence from exercise III.4.6

$$\cdots \to H^1(Y_{n+1}, \mathcal{O}_Y(-dn)) \to \operatorname{Pic} Y_{n+1} \to \operatorname{Pic} Y_n \to H^2(Y, \mathcal{O}_Y(-dn)) \to \cdots$$

gives that $\operatorname{Pic} Y_{n+1} \to \operatorname{Pic} Y_n$ is an isomorphism and so the projection $\varprojlim \operatorname{Pic} Y_n \to \operatorname{Pic} Y$ is an isomorphism.

Exercise III.11.6. Again let Y be a hypersurface in $X = \mathbb{P}^N_k$, this time with $N \geq 2$.

a. If $\mathcal{F}$ is a locally free sheaf on X, show that the natural map

$$H^0(X, \mathcal{F}) \to H^0(\widehat{X}, \widehat{\mathcal{F}})$$

is an isomorphism.

b. Show that the following conditions are equivalent:

   (i) for each locally free sheaf $\mathfrak{F}$ on $\widehat{X}$, there exists a coherent sheaf $\mathcal{F}$ on X such that $\mathfrak{F} \cong \widehat{\mathcal{F}}$ (i.e., $\mathfrak{F}$ is *algebraizable*);

   (ii) for each locally free sheaf $\mathfrak{F}$ on $\widehat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geq n_0$.

   [*Hint*: For (ii) $\Rightarrow$ (i), show that one can find sheaves $\mathcal{E}_0, \mathcal{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\widehat{\mathcal{E}_1} \to \widehat{\mathcal{E}_0} \to \mathfrak{F} \to 0$ on $\widehat{X}$. Then apply (a) to the sheaf $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_0)$.]

c. Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \widehat{X}$ is an isomorphism.

   Note. In fact, (i) and (ii) always hold if $N \geq 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [SGA 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbb{P}^N_k$ with $N \geq 4$, then $\operatorname{Pic} Y \cong \mathbb{Z}$, and it is generated by $\mathcal{O}_Y(1)$. See Hartshorne [5, Ch. IV] for more details.

Solution. We use the same language as the previous problem: suppose $Y = V(f)$ for f of degree d, and let $Y_n = V(f^n)$.

a. Consider the short exact sequence $0 \to \mathcal{O}_X(-dn) \to \mathcal{O}_X \to \mathcal{O}_{Y_n} \to 0$. As tensoring with a locally free sheaf is exact, we may tensor with $\mathcal{F}$ to get an exact sequence $0 \to \mathcal{F}(-dn) \to \mathcal{F} \to \mathcal{F}|_{Y_n} \to 0$ Taking global sections, we get

$$0 \to H^0(X, \mathcal{F}(-dn)) \to H^0(X, \mathcal{F}) \to H^0(Y, \mathcal{F}|_{Y_n}) \to H^1(X, \mathcal{F}(-dn)) \to \cdots .$$

   By Serre duality, $H^0(X, \mathcal{F}(-dn))$ and $H^1(X, \mathcal{F}(-dn))$ are dual to $H^N(X, \mathcal{F}^\vee(dn - N - 1))$ and $H^{N-1}(X, \mathcal{F}^\vee(dn - N - 1))$, which both vanish for $n \gg 0$ by theorem III.5.2. Therefore for all large enough n, we have that the map $H^0(X, \mathcal{F}) \to H^0(Y, \mathcal{F}|_{Y_n})$ is an isomorphism and therefore $H^0(X, \mathcal{F}) \to \varprojlim H^0(Y, \mathcal{F}|_{Y_n}) \cong H^0(\widehat{X}, \widehat{\mathcal{F}})$ is an isomorphism.

b. (i) $\Rightarrow$ (ii): Suppose $\mathfrak{F} \cong \widehat{\mathcal{F}}$ for some coherent sheaf $\mathcal{F}$ on X. By theorem II.5.17 there is some integer n_0 such that for all $n \geq n_0$ we have that $\mathcal{F}(n)$ is generated by global sections, so we have a surjection $\mathcal{O}_X^{\oplus I_n} \to \mathcal{F}(n)$. Applying the functor $- \otimes_{\mathcal{O}_X} \mathcal{O}_X^\wedge$, we obtain a surjective map $(\mathcal{O}_X^\wedge)^{\oplus I_n} \to \mathcal{F}(n)^\wedge$, and as $\mathcal{F}(n)^\wedge \cong \widehat{\mathcal{F}}(n)$, we have the implication.

   (ii) $\Rightarrow$ (i): The goal is to write $\mathfrak{F}$ as the cokernel of a map of direct sums of twists of $\mathcal{O}_{\widehat{X}}$, and then we can apply (a) to see that such a map necessarily comes from a map of direct sums of twists of $\mathcal{O}_X$.

   So suppose $\mathfrak{F}$ is a locally free sheaf with n_0 an integer such that for all $n \geq n_0$ we have that $\mathfrak{F}(n)$ is generated by global sections. Picking one such n and untwisting, we obtain a surjection $\mathcal{E}_0^\wedge = \mathcal{O}_{\widehat{X}}^{\oplus I_n}(-n) \to \mathfrak{F}$. I claim the kernel $\mathfrak{K}$ is locally free: as this is a local statement, we can work in the affine setting, say over the formal scheme associated to $I \subset A$ for I an ideal and

A an I-adically complete noetherian ring, and assume $\mathfrak{F}$ is free. By exercise II.9.3, we see that $\Gamma(\widehat{X}, \mathfrak{K})$ is a finitely generated projective A-module, therefore from the standard proof that any finitely generated projective module is locally free there exist finitely many $a_1, \cdots, a_m$ so that $\Gamma(\widehat{X}, \mathfrak{K})_{a_i}$ is free over A_{a_i} and $(a_1, \cdots, a_m) = 1$. But by theorem II.9.7 this shows that $(\Gamma(\widehat{X}, \mathfrak{K})_{a_i})^{\Delta} \cong \mathfrak{K}|_{D(a_i)}$ is free on $\operatorname{Spf} A_{a_i}$ and thus $\mathfrak{K}$ is locally free. Therefore by our assumption, we may choose a surjection $\mathcal{E}_1^\wedge = \mathcal{O}_{\widehat{X}}^{\oplus I_m}(-m) \to \mathfrak{K}$ which gives $\mathfrak{F}$ as the cokernel of $p^\wedge : \mathcal{E}_1^\wedge \to \mathcal{E}_0^\wedge$.

Now regard $p^\wedge$ as a global section of $\mathcal{H}om_{\widehat{X}}(\mathcal{E}_1^\wedge, \mathcal{E}_0^\wedge)$ on $\widehat{X}$. As $\mathcal{H}om_{\widehat{X}}(\mathcal{E}_1^\wedge, \mathcal{E}_0^\wedge)$ is locally free on $\widehat{X}$, we may apply part (a) to find a map $p : \mathcal{E}_1 \to \mathcal{E}_0$ such that $p^\wedge$ is the completion of p. But by exactness of completion (corollary II.9.8), $\operatorname{coker}(\mathcal{E}_1 \to \mathcal{E}_0)^\wedge$ is also the cokernel of $\mathcal{E}_1^\wedge \to \mathcal{E}_0^\wedge$, therefore $\mathfrak{F}$ is algebraizable.

c. To check that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \widehat{X}$ is injective, it suffices to show that the composite map $\operatorname{Pic} X \to \operatorname{Pic} \widehat{X} \to \operatorname{Pic} Y$ is injective, or that $\mathcal{O}_X(e)|_Y \cong \mathcal{O}_Y(e) \not\cong \mathcal{O}_Y$ for any $e < 0$. But this is clear from looking at the global sections: $h^0(Y, \mathcal{O}_Y) = 1$, while $H^0(Y, \mathcal{O}_Y(e))$ sits between $H^0(X, \mathcal{O}_X(e))$ and $H^1(X, \mathcal{I}_Y(e)) \cong H^1(X, \mathcal{O}(d-e))$ in the long exact sequence on cohomology associated to $0 \to \mathcal{I}_Y(e) \to \mathcal{O}_X(e) \to \mathcal{O}_Y(e) \to 0$, and as both of those cohomology groups vanish we have that $H^0(Y, \mathcal{O}_Y(e)) = 0$ as well.

In order to check surjectivity, we know that any line bundle $\mathfrak{F}$ on $\widehat{X}$ is the completion of some coherent sheaf $\mathcal{F}$ on X and we claim that $\mathcal{F}$ must be a line bundle in a neighborhood of Y. We can verify this locally, so let's work on an affine open $\operatorname{Spec} A \subset X$ where $Y \cap \operatorname{Spec} A$ is cut out by f, $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for an A-module M, and $\mathfrak{F}$ is free of rank 1. Taking an element $m \in M$ so that $m + fM \in M/fM$ corresponds to $1 + fA$ under the isomorphism $M/fM \cong A/fA$ induced from the isomorphism $\varprojlim M/f^nM \cong \varprojlim A/f^nA$, we get a map $A \to M$ by $1 \mapsto m$ which becomes an isomorphism after modding out by any power of f. This means that the kernel and cokernel are both stable under multiplication by f, which means they are supported away from $V(f) \subset \operatorname{Spec} A$. Therefore on some open neighborhood of Y we have $\mathcal{F}$ is a line bundle. The complement of this open neighborhood must be a collection of points: any positive-dimensional subset must meet Y for dimension reasons. Finally, by proposition II.6.5 as $N \geq 2$ there's a line bundle $\mathcal{F}'$ on X which agrees with $\mathcal{F}$ on this open neighborhood and therefore has completion isomorphic to $\mathfrak{F}$.

Exercise III.11.7. Now let Y be a curve in $X = \mathbb{P}^2_k$.

a. Use the method of (Ex. 11.5) to show that $\operatorname{Pic} \widehat{X} \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.

b. Conclude that there is an invertible sheaf $\mathfrak{L}$ on $\widehat{X}$ which is not algebraizable.

c. Conclude that there is a locally free sheaf $\mathfrak{F}$ on $\widehat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections. Cf. (II, 9.9.1)

Solution.

a. We reuse most of exercise III.11.5: suppose $Y = V(f)$ for f of degree d, and let $Y_n = V(f^n)$. By exercise II.9.6, as each Y_n is projective over a field, we have that $\operatorname{Pic}\widehat{X} \cong \varprojlim \operatorname{Pic} Y_n$. Further, the kernel of the natural projection from this limit is the product of the kernels of each projection step. To investigate those kernels, we recall the long exact sequence from exercise III.4.6:
$$\cdots \to H^0(Y_{n+1}, \mathcal{O}^*_{Y_{n+1}}) \to H^0(Y_n, \mathcal{O}^*_{Y_n}) \to$$
$$\to H^1(Y_{n+1}, \mathcal{O}_Y(-dn)) \to \operatorname{Pic} Y_{n+1} \to \operatorname{Pic} Y_n \to H^2(Y, \mathcal{O}_Y(-dn)) \to \cdots .$$
As the Y_i are projective curves over k, the first two terms are just k^* and the map between them is the identity. By Grothendeick vanishing, the final term is zero. Therefore the kernel of $\operatorname{Pic} Y_{n+1} \to \operatorname{Pic} Y_n$ is precisely $H^1(Y_{n+1}, \mathcal{O}_{Y_{n+1}}(-dn))$. We can determine this by Serre duality: per proposition III.7.5, $\omega_{Y_{n+1}} \cong \mathcal{E}xt^1(\mathcal{O}_{Y_{n+1}}, \omega_{\mathbb{P}^2})$ which is $\mathcal{O}_{Y_{n+1}}(d(n+1)-3)$ by applying $\mathcal{H}om(-, \omega_{\mathbb{P}^2})$ to the locally free resolution $\mathcal{O}_{\mathbb{P}^2}(-d(n+1)) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{Y_{n+1}}$ and recalling that $\omega_{\mathbb{P}^N} \cong \mathcal{O}(-N-1)$. Therefore $H^1(Y_{n+1}, \mathcal{O}_{Y_{n+1}}(-dn)) \cong H^0(Y_{n+1}, \mathcal{O}_{Y_{n+1}}(2dn + d - 3)$, which has plenty of sections for all big enough n.

b. This is the contrapositive of the implication proven in exercise III.11.6(c).

c. Same as (b).

Exercise III.11.8. Let $f : X \to Y$ be a projective morphism, let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathcal{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathcal{F})$ is zero in a neighborhood of y.

Solution. (We should and do assume Y locally noetherian here.) By theorem III.11.1, $R^i f_*(\mathcal{F})^\wedge_y \cong \varprojlim H^i(X_n, \mathcal{F}_n)$. Now consider the exact sequence $0 \to \mathfrak{m}_y^{n-1}/\mathfrak{m}_y^n \to \mathcal{O}_Y/\mathfrak{m}_y^n \to \mathcal{O}_Y/\mathfrak{m}_y^{n-1} \to 0$: as $\mathcal{F}$ is flat, we may tensor by it to get an exact sequence
$$0 \to \mathcal{F}_y^{\oplus r} \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to 0.$$
By examining the long exact sequence on homology as n varies, we see that $H^i(X_n, \mathcal{F}_n) = 0$ for all n and therefore $R^i f_*(\mathcal{F})^\wedge_y = 0$, and by the same logic as the proof of corollary III.11.2 we have that $R^i f_* \mathcal{F}$ is zero in a neighborhood of y.

III.12 The Semicontinuity Theorem

Semicontinuity is a blast! Basically any invariant which you can think of which behaves in a semicontinuous way (and there are *tons* of these) really comes from the material we develop in this section.

Exercise III.12.1. Let Y be a scheme of finite type over an algebraically closed field k. Show that the function
$$\varphi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$$
is upper semicontinuous on the closed points of Y.

Solution. It suffices to solve this in the case that Y is affine, so suppose $Y \subset \mathbb{A}_k^n$ is cut out by an ideal $I \subset k[x_1, \cdots, x_n]$. Pick a generating set for I and form the Jacobian matrix J with respect to this generating set. By the proof of theorem I.5.1, we have that $\varphi(y) = n - \operatorname{rk} J(y)$. Now note that $R_{\leq c}$, the locus where $J(y)$ has rank at most c, is the common vanishing locus of the determinants of the $(c+1)\times(c+1)$ minors of J. Thus $R_{\leq c} = \varphi^{-1}([n-c, \infty))$ is a closed subscheme of Y containing $R_{\leq c-1}$, and we see immediately that φ is upper semicontinuous.

(Alternatively, one can check that $\varphi(y)$ is the rank of $\Omega_{Y/k}$ at the point y via proposition II.8.2A and exercise II.8.1(a), which is upper-semicontinuous by example III.12.7.2.)

Exercise III.12.2. Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbb{P}_k^n$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

Solution. Consider the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{X_t} \to 0$. Taking cohomology, we get a long exact sequence
$$\cdots \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to H^i(X_t, \mathcal{O}_{X_t}) \to \cdots.$$
The higher cohomology of $\mathcal{O}_{\mathbb{P}^n}$ vanishes, and we find that when $i > 0$ we get $H^i(X_t, \mathcal{O}_{X_t}) \to H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d))$ is an isomorphism and thus $h^i(X_t, \mathcal{O}_{X_t})$ is independent of t. When $i = 0$, we see that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) = 0$ and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$, so we find that $H^0(X_t, \mathcal{O}_{X_t})$ is of dimension $h^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) - 1$, which is again independent of t.

Exercise III.12.3. Let $X_1 \subset \mathbb{P}_k^4$ be the *rational normal quartic curve* (which is the 4-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$). Let $X_0 \subset \mathbb{P}_k^3$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbb{P}^4$, parametrized by $T = \mathbb{A}^1$, with the given fibers X_1 and X_0 for $t = 1$ and $t = 0$.

Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^4 \times T}$ be the ideal sheaf of the total family $X \subset \mathbb{P}^4 \times T$. Show that $\mathcal{I}$ is flat over T. Then show that
$$h^0(t, \mathcal{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$
and also
$$h^1(t, \mathcal{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$
This gives another example of cohomology groups jumping at a special point.

Solution. This question is incorrect as stated: $h^i(t,\mathcal{I}) = 0$ for all t and all i. I'm not sure what was supposed to happen here - at the end I give a little evidence that we were supposed to consider $\mathcal{I}(1)$ instead of $\mathcal{I}$, but the fix is not as obvious as some other situations where there's been a slip-up.

Assuming we have such a flat family $X \subset \mathbb{P}^4 \times T$, it is true that $\mathcal{I}$ is flat over T: as $\mathcal{O}_X$ is flat over T and $\mathcal{O}_{\mathbb{P}^4\times T}$ is flat over T, an application of proposition III.9.1A(e) gives that $\mathcal{I}$ is flat over T. This means that $\chi(t,\mathcal{I})$ is independent of t by corollary III.9.9 and the observation that $\chi(t,\mathcal{I}) = P_{\mathcal{I}_t}(0)$. Now we may consider the long exact sequence in cohomology associated to $0 \to \mathcal{I}_t \to \mathcal{O}_{\mathbb{P}^4_t} \to \mathcal{O}_{X_t} \to 0$. As X_t is a closed subscheme of $\mathbb{P}^4_t$ for any t, we get that the map $k = H^0(\mathcal{O}_{\mathbb{P}^4_t}) \to H^0(\mathcal{O}_{X_t})$ is injective, so $H^0(\mathcal{I}) = 0$. As the higher cohomologies of $\mathcal{O}_{\mathbb{P}^4_t}$ vanish and the cohomologies of $\mathcal{O}_{X_t}$ vanish above degree 1 by Grothendieck vanishing, the only places we could get a nonzero $h^i(\mathcal{I}_t)$ are in degree 1 and 2, where $h^1(t,\mathcal{I}) = h^0(\mathcal{O}_{X_t}) - 1$ and $h^2(t,\mathcal{I}) = h^1(\mathcal{O}_{X_t})$ from the long exact sequence in cohomology.

Taking the flat family which is the (scheme-theoretic) image of $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^4 \times \mathbb{A}^1$ by $([u : v], t) \mapsto ([u^4 : u^3v : tu^2v^2 : uv^3 : v^4], t)$, we may compute by a procedure similar to determining the ideal of the twisted cubic in REFERENCE (or by plugging in to our favorite computer algebra system) that the ideal defining the flat family is

$$(x_1x_3 - x_0x_4, x_1x_4t - x_2x_3, x_3^2t - x_2x_4, x_0x_3t - x_1x_2, x_1^2t - x_0x_2, x_3^3 - x_1x_4^2, x_0x_3^2 - x_1^2x_4,$$

$$x_1^3 - x_0^2x_3, x_0x_4t^2 - x_2^2, x_0x_4^2t - x_2x_3^2, x_0^2x_4t - x_1^2x_2).$$

Since cohomology commutes with flat base change, we may assume that $k = \overline{k}$. This means $X_t \cong X_1$ when $t \neq 0$, so we just need to investigate the fibers at $t = 1$ and $t = 0$.

When $t = 1$, we get back the ideal of the rational normal quartic, which is $\mathbb{P}^1$ after a closed embedding. This has $h^0 = 1$ and $h^1 = 0$. When $t = 0$, we get that X_0 is cut out by

$$(x_1x_3 - x_0x_4, x_3^3 - x_1x_4^2, x_0x_3^2 - x_1^2x_4, x_1^3 - x_0^2x_3, x_2x_0, x_2x_1, x_2^2, x_2x_3, x_2x_4),$$

which has saturation

$$(x_1x_3 - x_0x_4, x_3^3 - x_1x_4^2, x_0x_3^2 - x_1^2x_4, x_1^3 - x_0^2x_3),$$

which is again the ideal of a $\mathbb{P}^1$ after a closed embedding. So $h^i(t,\mathcal{I}) = 0$ and nothing interesting happenend here. (If you change this to $\mathcal{I}(1)$, then something interesting *does* happen here - the same proofs about flatness go through, but we do have that $h^0(t,\mathcal{I})$ jumps at $t = 0$ because X_0 is contained in a plane.)

Exercise III.12.4. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : X \to Y$ be a flat projective morphism whose fibers are all integral schemes. Let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathcal{L}_y \cong \mathcal{M}_y$ on the fiber X_y. Then show that there is an invertible sheaf $\mathcal{N}$ on Y such that $\mathcal{L} \cong \mathcal{M} \otimes f^*\mathcal{N}$. [*Hint*: Use the results of this section to show that $f_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is locally free of rank 1 on Y.]

Solution. By considering $\mathcal{L} \otimes \mathcal{M}^\vee$, it suffices to show that every invertible sheaf on X which is trivial on the fibers is the pullback of an invertible sheaf on Y. Let $\mathcal{F}$ be such an invertible sheaf.

In this scenario we have that for all closed points y, X_y is an integral projective scheme over an algebraically closed field k, so $H^0(y, \mathcal{F}) \cong H^0(X_y, \mathcal{O}_{X_y}) \cong k$. Therefore $h^0(y, \mathcal{F})$ is constant: the set where $h^0(y, \mathcal{F}) > 1$ is closed by semicontinuity and therefore must contain a closed point if it is not empty because we're working over a scheme finite type over a field. We may apply corollary III.12.9 to see that $f_*\mathcal{F}$ is locally free of rank 1.

Now we want to show that $f^*f_*\mathcal{F}$ is isomorphic to $\mathcal{F}$. As this is a local condition, we can check it on a cover of X by open subsets of the form $f^{-1}(U)$ where $U \subset Y$ is open and $f_*\mathcal{F}|_U$ is free. So it suffices to treat the case where $Y = \operatorname{Spec} R$ is affine and $f_*\mathcal{F} \cong \mathcal{O}_Y$.

From the adjunction between f_* and f^*, we have a natural map $f^*f_*\mathcal{F} \to \mathcal{F}$ and we will show that this is an isomorphism. This natural map is the image of $id_{f_*\mathcal{F}} \in \operatorname{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, f_*\mathcal{F})$ under the isomorphism of $\mathcal{O}_Y(Y) = R$-modules given by the adjunction

$$\operatorname{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, f_*\mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_X}(f^*f_*\mathcal{F}, \mathcal{F}).$$

But since $f_*\mathcal{F} = \mathcal{O}_Y$, we have $\operatorname{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, f_*\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{O}_Y(Y) = R$, and $id_{f_*\mathcal{F}} = id_{\mathcal{O}_Y} = 1 \in R$. On the other hand, the pullback of the structure sheaf is the structure sheaf, so $\operatorname{Hom}_{\mathcal{O}_X}(f^*f_*\mathcal{F}, \mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$, so $\mathcal{F}(X) = R$. Since the adjunction is an isomorphism of R-modules, we see that $id_{f_*\mathcal{F}}$ must be sent to a unit in $\operatorname{Hom}_{\mathcal{O}_X}(f^*f_*\mathcal{F}, \mathcal{F})$. Because $\operatorname{Hom}_{\mathcal{O}_X}(f^*f_*\mathcal{F}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}(X)$ as R-modules, we see that our map $\mathcal{O}_X \to \mathcal{F}$ is given by multiplication by an invertible global section. Therefore it's an isomorphism and we're done.

Exercise III.12.5. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathcal{E}$ be a locally free sheaf on Y, and let $X = \mathbb{P}(\mathcal{E})$ - see (II, §7). Then show that $\operatorname{Pic} X \cong (\operatorname{Pic} Y) \times \mathbb{Z}$. This strengthens (II, Ex. 7.9).

Solution. Let $\operatorname{rk} \mathcal{E} = r + 1$, where we should assume $r \geq 1$ here. Let $\pi : X \to Y$ be the natural projection. There's a natural map $(\operatorname{Pic} Y) \times \mathbb{Z} \to \operatorname{Pic} X$ by $(\mathcal{L}, n) \mapsto \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$, and we'll show that it's injective and surjective.

For injectivity, the same reasoning from exercise II.7.9 works. Suppose we have $\mathcal{L}$ a line bundle on Y and $n \in \mathbb{Z}$ so that $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}$. Then by proposition II.7.11, we have that $\pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) \cong \mathcal{O}_Y$, and so by the projection formula we have $\mathcal{L} \otimes \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \cong \mathcal{O}_Y$, so $n = 0$ by proposition II.7.11 and $\mathcal{L} \cong \mathcal{O}_Y$.

Checking surjectivity is where we get to use our new tools introduced in this section. Let $\mathcal{M}$ be an invertible sheaf on X. Now note that $h^0(y, \mathcal{M}(n))$ encodes the degree of $\mathcal{M}_y$ on $X_y \cong \mathbb{P}^r_{k(y)}$, and by theorem III.12.8 it is upper semicontinuous. On the other hand, $h^0(y, \mathcal{M}^\vee(n))$ encodes the degree of $\mathcal{M}_y$ on the fibers too - but $\deg(\mathcal{M}^\vee)_y = -\deg \mathcal{M}$, so the degree of $\mathcal{M}_y$ must in fact be a continuous function of y. As Y is irreducible, it is connected, so $\mathcal{M}_y \cong \mathcal{O}_{\mathbb{P}^r_{k(y)}}(n)$ for some fixed n independent of y. By twisting by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-n)$, we may assume that $\mathcal{M}_y \cong \mathcal{O}_{\mathbb{P}^r_{k(y)}}$ for all y. Now we may apply corollary III.12.9 to see that $\pi_*\mathcal{M}$ is a line bundle on Y and the same work from the end of exercise III.12.4 shows that $\mathcal{M} \cong \pi^*\pi_*\mathcal{M}$, finishing the proof.

Exercise III.12.6. (*) Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.

a. If $\mathcal{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathcal{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.

b. Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$. (Do *not* assume that T is reduced!) Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\operatorname{Pic}(X \times T) \neq \operatorname{Pic} X \times \operatorname{Pic} T$. [*Hint*: Apply (12.11) with $i = 0, 1$ for suitable invertible sheaves on $X \times T$.]

Solution.

a. Let $\pi : X \times T \to T$ and $\psi : X \times T \to X$ be the projection maps. Note that it suffices to show the claim for $\mathcal{L} \otimes \psi^*\mathcal{M}$ for $\mathcal{M}$ an invertible sheaf on X: the restriction of $\mathcal{L} \otimes \psi^*\mathcal{M}$ to X_t is $\mathcal{L}_t \otimes \mathcal{M}$ for any t. So let $t \in T$ be an arbitrary closed point and let us replace $\mathcal{L}$ by $\mathcal{L} \otimes \psi^*(\mathcal{L}_t)^{-1}$ which restricts to $\mathcal{O}_{X_t} \cong \mathcal{O}_X$ on the fiber over t.

   By our assumption that $H^1(X, \mathcal{O}_X) = 0$, this means that $h^1(t, \mathcal{L}) = 0$ so after an application of theorem III.12.11(a), we have that $R^1\pi_*\mathcal{L}$ vanishes in a neighborhood of t. Next, part (b) of that same theorem says we can apply part (a) again to $i = 0$, which gives that $\pi_*\mathcal{L}$ is locally free of rank 1 in a neighborhood of t as $h^0(t, \mathcal{L}) = h^0(X, \mathcal{O}_X) = 1$. Let $U \subset T$ be a neighborhood of t where $\pi_*\mathcal{L} \cong \mathcal{O}_U$. By the logic of exercise III.12.4, we have that $\mathcal{L}|_{\pi^{-1}(U)} \cong \pi|_U^*(\pi_*\mathcal{L})|_U \cong \mathcal{O}_{\pi^{-1}(U)}$, and so $\mathcal{L}_t \cong \mathcal{L}_{t'}$ for all $t' \in U$.

   Now it remains to show that this implies that $\mathcal{L}_t$ is independent of t up to isomorphism. If T is irreducible, this is clear: pick any closed point t, take the U corresponding to this, pick any U' corresponding to any $t' \notin U$, and note that U and U' must intersect and their intersection contains a closed point. Now suppose T has multiple irreducible components, say $T_1, \cdots, T_n$ (there are finitely many because T is of finite type over a field). If T_i and T_j intersect, then their intersection contains a closed point and we may take an open neighborhood of this closed point to show that the restriction of $\mathcal{L}$ to the fibers over T_i and the fibers over T_j are isomorphic, finishing this part of the problem.

b. There's an obvious map $\operatorname{Pic} X \times \operatorname{Pic} T \to \operatorname{Pic}(X \times T)$ by sending $(\mathcal{L}, \mathcal{M}) \mapsto \psi^*\mathcal{L} \otimes \pi^*\mathcal{M}$. To show surjectivity, suppose $\mathcal{N}$ is a line bundle on $X \times T$: then by (a), $\mathcal{N}_t$ is independent of t, so $\psi^*(\mathcal{N}_t)$ and $\mathcal{N}$ have isomorphic restrictions to each fiber of π, and therefore $\mathcal{N} \otimes \psi^*(\mathcal{N}_t)$ is an invertible sheaf which is trivial along the fibers of π. All we have to do is to show that such an invertible sheaf is a pullback of an invertible sheaf on T. But the final part of exercise III.12.4 does this without any integrality assumptions on T once we know that the pushforward of $\mathcal{N} \otimes \psi^*(\mathcal{N}_t)$ is locally free of rank one on T by theorem III.12.11.

   To show injectivity, suppose $\psi^*\mathcal{L} \otimes \pi^*\mathcal{M} \cong \mathcal{O}_{X \times T}$. Now pull back along $i_t : X \times \{t\} \to X \times T$ for some closed point t: on the one hand, as pullback commutes with tensor products, $i_t \circ \psi = id_X$, and $i_t \circ \psi$ is the closed immersion $\{t\} \to T$, we get $\mathcal{L}$, while on the other hand we get $\mathcal{O}_X$, so $\mathcal{L} \cong \mathcal{O}_X$. The same logic when pulling back along $\{x\} \times T \to X \times T$ for some closed point $x \in X$ gives that $\mathcal{M} \cong \mathcal{O}_T$, and we've shown injectivity.

Chapter IV

Curves

In this chapter, a curve is a complete, nonsingular curve over an algebraically closed field k; a point is a closed point unless one specifically says 'generic point'. Don't forget!

IV.1 Riemann-Roch Theorem

This is one of the most fundamental results in the study of curves. We'll use the material from this section all over the rest of this chapter.

Exercise IV.1.1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Solution. Apply Riemann-Roch to nP to see that

$$l(nP) - l(K - nP) = n + 1 - g.$$

Taking $n > g$, we find that $l(nP) = \dim_k H^0(X, \mathcal{O}_X(nP))$ must be at least two. This shows that the image of $H^0(X, \mathcal{O}_X(nP))$ in $K(X) = \mathcal{L}(nP)_\eta$ under the restriction map is at least two-dimensional and must contain a nonconstant function (η is the generic point of X). Now, the functions in the image of this map are regular except possibly at P, and any nonconstant function must have a pole at P: else the map to $\mathbb{P}^1$ given by this function is neither surjective nor constant, contradicting the fact that the image of a proper scheme is again proper (exercise II.4.4).

Exercise IV.1.2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Solution. We can find f_i which has a pole of some order at P_i and is regular elsewhere by the previous exercise. Now consider $f = \sum f_i$. It is straightforward to verify that this does the job by checking the valuation of f at each point $x \in X$: at any x not among the P_i, the valuation is at least the minimum of the valuations of each f_i at x, and these are all nonnegative, so f is regular at x. At each P_i, f_j for $j \neq i$ is regular, so the valuation of f at P_i is exactly the valuation of f_i at P_i, which is negative and thus f has a pole at each P_i.

Exercise IV.1.3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k which is *not* proper over k. Then X is affine. [*Hint*: Embed X in a (proper) curve $\overline{X}$ over k, and use (Ex. 1.2) to construct a morphism $f : \overline{X} \to \mathbb{P}^1$ such that $f^{-1}(\mathbb{A}^1) = X$.]

Solution. Embed X as an open set in a proper curve $\overline{X}$ by remark II.4.10.2(e). Then the complement of X in $\overline{X}$ is a proper closed set and therefore a finite set of points, say $P_1, \cdots, P_r$. Now use exercise IV.1.2 with those points to obtain a morphism $f : \overline{X} \to \mathbb{P}^1$ with $f^{-1}(\infty) = \{P_1, \cdots, P_r\}$. As this morphism is finite by proposition II.6.8, it is affine, so $f^{-1}(\mathbb{A}^1) = X$ and X is affine.

Exercise IV.1.4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine. [*Hint*: Combine (Ex. 1.3) with (III, Ex. 3.1, Ex. 3.2, Ex. 4.2).]

Solution. By exercises III.3.1 and III.3.2, we may reduce to the case X integral (and still non-proper). Letting $\nu : X' \to X$ be the normalization, a surjective finite map by exercise II.3.8, we can check that X' is not proper over k either: if it were, then X would be proper over k by exercise II.4.4. So X' is affine over k by exercise IV.1.3, and by exercise III.4.2, X must also be affine.

Exercise IV.1.5. For an effective divisor D on a curve X of genus g, show that $\dim |D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Solution. First, observe that we have equality when $D = 0$ or $g = 0$: $h^0(X, \mathcal{O}_X) = 1$ for any curve, while $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg D)) = \deg D + 1$ for any effective divisor on $\mathbb{P}^1$.

Next, we'll reuse some material from the proof of the Riemann-Roch theorem. We have for any divisor D and any point P on a curve X an exact sequence of sheaves

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + P) \to k(P) \to 0.$$

Taking global sections, we get that either $l(D + P) = l(D)$ or $l(D + P) = l(D) + 1$. Now write $D = \sum_{i=1}^{d} P_i$, let $D_j = \sum_{i=1}^{j} P_i$ and consider the sequence $\{l(D_j)\}_{0 \leq j \leq d}$: this is a sequence which starts with 1 and increases by at most 1 each time the index does. So $l(D) = l(D_d) \leq d + 1$, with equality implying $l(D_1) = 2$. But $l(D_1) = 2$ means that there's a nonconstant rational function on X with a single pole at P_1, which implies that X has a degree-one map to $\mathbb{P}^1$, or that $X \cong \mathbb{P}^1$. So $l(D) \leq \deg D + 1$, with equality iff $D = 0$ or $g = 0$.

Exercise IV.1.6. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g + 1$. (Recall that the degree of a finite morphism of curves $f : X \to Y$ is defined as the degree of the field extension $[K(X) : K(Y)]$, (II, §6).)

Solution. Let $D = P_1 + \cdots + P_{g+1}$ be an effective divisor of degree $g + 1$. Then $h^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g + h^1(D) = 2 + h^1(D)$, and therefore we can find a nonconstant rational function $f \in K(X)$ with at most simple poles along a nonempty subset of the P_i and regular elsewhere. I claim that the induced morphism $f : X \to \mathbb{P}^1$ suffices.

To check the claim about degrees, by proposition II.6.8 and exercise III.9.3(a) we have that f is a finite flat morphism, so $f_*\mathcal{O}_X$ is a locally free sheaf, and its rank is equal to the degree of f by looking at the situation at the generic point. Conversely, the rank of $f_*\mathcal{O}_X$ at $\infty \in \mathbb{P}^1$ is equal to the number of points in the preimage counted with multiplicity. By construction of D, this is at most $g + 1$ and we're done.

Exercise IV.1.7. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

a. If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

b. Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geq 2$.

Note. We will see later (Ex. 3.2) that there exist nonhyperelliptic curves. See also (V, Ex. 2.10).

Solution.

a. Plugging in $D = 0$ to Riemann-Roch, we get that $l(0) - l(K) = 1 - g$, so $l(K) = g$. Plugging in $D = K$ to Riemann-Roch, we get that $l(K) - l(0) = \deg K + 1 - g$, or $2g - 2 = \deg K$. Therefore when $g = 2$, $\deg K = 2$ and $|K| = l(K) - 1 = 1$.

   To check that K has no base points, suppose P is a base point: then $l(K - P) = l(K) = 2$, and by Riemann-Roch applied to $D = P$, we have that $l(P) - l(K - P) = \deg P + 1 - 2$, so $l(P) = 1 + 1 - 2 + 2$ and therefore we can find a nonconstant rational function f with a pole only at P. But this means that f defines a degree-one morphism to $\mathbb{P}^1$, giving that $X \cong \mathbb{P}^1$ and contradicting the fact that X is of genus 2. By the constructions of section II.7, we can pick two basis elements for $h^0(K)$ and get a morphism $f : X \to \mathbb{P}^1$ which is nonconstant and thus finite by propostion II.6.8. Per exercise II.6.8, $f^*\mathcal{L}(D) \cong \mathcal{L}(f^*D)$, and by proposition II.6.9 we have that $\deg f \cdot \deg \mathcal{O}(1) = \deg \omega_X$ which implies that $\deg f = 2$ and therefore X is hyperelliptic.

b. Consider the projection of a curve of type $(g + 1, 2)$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$ to the second factor. Restricting to any fiber $\mathbb{P}^1 \times \{pt\}$, we see that the preimage of a point under this projection is two points (possibly counted with multiplicity). Therefore our morphism is nonconstant and thus finite by propostion II.6.8, so we may apply proposition II.6.9 to see that $p_2|_X$ is of degree 2 and we're done.

Exercise IV.1.8. p_a *of a Singular Curve*. Let X be an integral projective scheme of dimension 1 over k, and let $\widetilde{X}$ be its normalization (II, Ex. 3.8). Then there is an exact sequence of sheaves on X,

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \to \sum_{P \in X} \widetilde{\mathcal{O}}_P/\mathcal{O}_P \to 0,$$

where $\widetilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = \text{length}(\widetilde{\mathcal{O}}_P/\mathcal{O}_P)$.

a. Show that $p_a(X) = p_a(\widetilde{X}) + \sum_{P \in X} \delta_P$. [*Hint*: Use (III, Ex. 4.1) and (III, Ex. 5.3).]

b. If $p_a(X) = 0$, show that X is already nonsingular and in fact isomorphic to $\mathbb{P}^1$. This strengthens (1.3.5).

c. (*) If P is a node or an ordinary cusp (I, Ex. 5.6, Ex. 5.14), show that $\delta_P = 1$. [*Hint*: Show first that δ_P depends only on the analytic isomorphism class of the singularity at P. Then compute δ_P for the node an cusp of suitable plane cubic curves. See (V, 3.9.3) for another method.]

Solution. We discussed something like this exact sequence in exercise II.6.9: a key property to remember about the third term is that it's supported on a finite set of closed points because normalization is birational.

a. From exercise III.5.3, we know that $p_a(C) = \dim_k H^1(C, \mathcal{O}_C)$ for any integral scheme of dimension one over an algebraically closed field k, so let's try and get at these quantities via taking cohomology of our exact sequence of sheaves. We have the following long exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, f_*\mathcal{O}_{\widetilde{X}}) \to H^0(\sum_{P \in X} \widetilde{\mathcal{O}}_P/\mathcal{O}_P) \to H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_{\widetilde{X}}) \to 0,$$

where H^1 of the third term is zero by Grothendieck vanishing as it is supported on a finite discrete set. By an application of exercise III.4.1, $H^i(X, f_*\mathcal{O}_{\widetilde{X}}) \cong H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Therefore by our work in the lemmas for exercise II.6.9, we have that the first two terms in our long exact sequence are just k, and we get that

$$0 \to H^0(\sum_{P \in X} \widetilde{\mathcal{O}}_P/\mathcal{O}_P) \to H^1(X, \mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to 0$$

is exact. Finally, since $\delta_P = \dim_k(\widetilde{\mathcal{O}}_P/\mathcal{O}_P)$, cohomology distributes over direct sums, and the alternating sum of dimensions of an exact sequence of k-vector spaces is zero, we obtain the requested equality.

b. From our work in (a), $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$. If this is zero, then both terms in our final short exact sequence must also be zero. This implies that $\widetilde{\mathcal{O}}_P = \mathcal{O}_P$ for all $P \in X$, or that X is already normal. So X is a curve of genus 0, and by example IV.1.3.5, it must be $\mathbb{P}^1$.

c. Let $\widehat{\mathcal{O}}_P$ be the completion of $\mathcal{O}_P$ at its maximal ideal. By our work in the solution of exercise III.10.4, this is a faithfully flat $\mathcal{O}_P$ module, so the sequence

$$0 \to \mathcal{O}_P \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P \to \widetilde{\mathcal{O}}_P \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P \to (\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P \to 0$$

remains exact. Now we have two things to verify: first, that $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ and $(\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P$ have the same dimension as k-vector spaces, and second that normalization commutes with completion so that $\widetilde{\mathcal{O}}_P \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P \cong \widetilde{\widehat{\mathcal{O}}}_P$.

To verify the first statement, it suffices to show that some power of $\mathfrak{m}_P$ acts as zero on $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$: if this is true, the tensor product $(\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P$ is the same as the tensor product $(\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P/\mathfrak{m}_P^N$ for some large enough N, and $\widehat{\mathcal{O}}_P/\mathfrak{m}_P^N \cong \mathcal{O}_P/\mathfrak{m}_P^N$ by theorem II.9.3A(a). But as $M \otimes_R R/I \cong M/IM$, we see that this won't change the underlying k-vector space. As our long exact sequence on homology in (a) shows that $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ injects in to $H^1(X, \mathcal{O}_X)$, which is finite-dimensional over k by theorem III.5.3, $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ is also finite-dimensional and therefore by Nakayama's lemma some power of $\mathfrak{m}_P$ must act as zero on it.

The second statement is a little trickier, and in fact I'm going to just cite it away in lieu of proving it. By some commutative algebra, completion commutes with normalization for (quasi-)excellent rings by Stacks Project tag 0C23. As any field is excellent and any localization of a finite type ring over a (quasi-)excellent ring is again (quasi-)excellent, any local ring of any finite type scheme over a field is excellent (Stacks Project tags 07QW and 07QU).

Therefore from our two statements, $\widetilde{\mathcal{O}}_P \otimes_{\mathcal{O}_P} \widehat{\mathcal{O}}_P \cong \widetilde{\widehat{\mathcal{O}}}_P$, and δ_P is exactly the dimension of $\widetilde{\widehat{\mathcal{O}}}_P/\widehat{\mathcal{O}}_P$ over k, so δ_P depends only on the analytic isomorphism class of the singularity at P. To calculate, we'll use the models $k[[x,y]]/(xy)$ for the node and $k[[x,y]]/(y^2-x^3)$ for the ordinary cusp. The normalization of the former is $k[[x,y]]/(xy) \to k[[x]] \times k[[y]]$ by $f(x,y) \mapsto (f(x,0), f(0,y))$ which has cokernel $0 \times k$. The normalization of the latter is $k[[x,y]]/(y^2 - x^3) \to k[[t]]$ by $x \mapsto t^2$ and $y \mapsto t^3$, which has cokernel spanned by t. In both cases, this cokernel is one-dimensional, giving $\delta_P = 1$.

Exercise IV.1.9. (*) *Riemann-Roch for Singular Curves.* Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.

a. Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e., all $P_i \in X_{reg}$. Then define $\deg D = \sum n_i$. Let $\mathcal{L}(D)$ be the associated invertible sheaf on X, and show that

$$\chi(\mathcal{L}(D)) = \deg D + 1 - p_a.$$

b. Show that any Cartier divisor on X is the difference of two very ample Cartier divisors. (Use (II, Ex. 7.5).)

c. Conclude that every invertible sheave $\mathcal{L}$ on X is isomorphic to $\mathcal{L}(D)$ for some divisor D with support in X_{reg}.

d. Assume furthermore that X is a locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω_X° is an invertible sheaf on X, so we can define the *canonical divisor* K to be a divisor with support in X_{reg} corresponding to ω_X°. Then the formula of (a) becomes

$$l(D) - l(K-D) = \deg D + 1 - p_a.$$

Solution.

a. This is the same proof as the Riemann-Roch theorem: $\chi(\mathcal{O}_X) = 1 - p_a(X)$ by exercise III.5.3, and then the exact sequence argument with $0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to k(P) \to 0$ works exactly the same.

b. Just to clear up any confusion at the beginning, Hartshorne's using the result of corollary II.6.15 without mentioning it much at the start: the Cartier class group is isomorphic to the Picard group for any integral scheme, so it makes sense to ask about whether Cartier divisors are (very) ample or not. From there, let $\mathcal{M}$ be the line bundle corresponding to an arbitrary Cartier divisor D and let $\mathcal{L}$ be any very ample line bundle on X which exists by the projectivity hypothesis. Then $\mathcal{L}$ is also ample by theorem II.7.6, so there exists some n_0 such that for all $n \geq n_0$, $\mathcal{L}^n \otimes \mathcal{M}$ is generated by global sections. By an application of exercise II.7.5(d), we get that $\mathcal{L}^{n+1} \otimes \mathcal{M}$ is very ample. Therefore as $\mathcal{L}^{n+1}$ is very ample, we can write $\mathcal{M} \cong (\mathcal{L}^{n+1} \otimes \mathcal{M}) \otimes (\mathcal{L}^{n+1})^{-1}$ as a difference of two very ample Cartier divisors.

c. By part (b), we only have to prove this for $\mathcal{L}(D)$ the sheaf associated to an effective very ample Cartier divisor. Embedding $X \hookrightarrow \mathbb{P}^N$ by $\mathcal{L}(D)$, we have that D is a hyperplane divisor on $X \subset \mathbb{P}^N$. Since the non-regular points of X are finite, we can just pick a hyperplane in $\mathbb{P}^N$ which misses them to get D' a divisor with support in X_{reg} which is linearly equivalent to D. Then $\mathcal{L}(D') \cong \mathcal{L}(D)$ by proposition II.6.13, and we're done.

d. By proposition II.8.23, X is Cohen-Macaulay, so by theorem III.7.6 we have that

$$H^1(X, \mathcal{L}(D))' \cong \operatorname{Ext}^0(\mathcal{L}(D), \omega_X^\circ).$$

By proposition III.6.7 and III.6.3,

$$\operatorname{Ext}^0(\mathcal{L}(D), \omega_X^\circ) \cong \operatorname{Ext}^0(\mathcal{O}_X, \omega_X^\circ \otimes \mathcal{L}(-D)) \cong H^0(X, \omega_X^\circ \otimes \mathcal{L}(-D)).$$

But the dimension of this last vector space is just $l(K - D)$, so $h^1(X, \mathcal{L}(D)) = l(K - D)$ and as $\chi(\mathcal{L}(D)) = h^0(X, \mathcal{L}(D)) - h^1(X, \mathcal{L}(D))$, we get that the formula from (a) becomes

$$l(D) - l(K - D) = \deg D + 1 - p_a$$

as requested.

Exercise IV.1.10. Let X be an integral projective scheme of dimension 1 over k, which is locally complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{reg}$. Imitate (1.3.7) to show that the map $P \to \mathcal{L}(P - P_0)$ gives a one-to-one correspondence between the points of X_{reg} and the elements of the group $\operatorname{Pic}^\circ X$. This generalizes (II, 6.11.4) and (II, Ex. 6.7).

Solution. By exercise IV.1.9(c), we can write any invertible sheaf $\mathcal{L}$ on X as $\mathcal{L}(D)$ for some Weil divisor D supported in X_{reg}. As X is a locally complete intersection, we may apply exercise IV.1.9(d) to $D = K$ and get $l(K) - l(0) = \deg K + 1 - 1$, or $\deg K = l(K) - 1$. Since $l(K) = \dim H^0(X, \omega_X^\circ)$ which is equal to $\dim H^1(X, \mathcal{O}_X)$ by duality, we get that $l(K) = p_a = 1$ and so $\deg K = 0$.

Now suppose D is a divisor of degree zero. Applying Riemann-Roch again to $D + P_0$, we get $l(D + P_0) - l(K - D - P_0) = \deg(D + P_0) + 1 - 1 = 1$. Since $K - D - P_0$ has negative degree, it has $l(D) = 0$ by an application of lemma IV.1.2. So $l(D + P_0) = 1$, or $\dim |D + P_0| = 0$. This means there's a unique point $P \in X_{reg}$ which is linearly equivalent to $D + P_0$, or that D is linearly equivalent to $P - P_0$ for a unique choice of $P \in X_{reg}$.

IV.2 Hurwitz's Theorem

This is another one of the fundamental tools for working with curves. The statement is pretty straightforward and yet we can calculate a *ton* of neat stuff when we add some elbow grease - most of these problems have some neat conclusions.

Exercise IV.2.1. Use (2.5.3) to show that $\mathbb{P}^n$ is simply connected.

Solution. Just a reminder: we're working over an algebraically closed base field k here (this is important!).

We prove the result by induction on n: the case $n = 1$ is example IV.2.5.3. Now take $n \geq 1$ and assume we've proven the result for all smaller values. Assume $f : X \to \mathbb{P}^n$ is a nontrivial etale covering with X connected. Since f is finite, X is projective over k; as f is smooth, X is smooth over k and therefore regular. This also gives that X is normal and irreducible.

Now we restrict to the preimage of a hyperplane $H \subset \mathbb{P}^n$: since finite etale morphisms are stable under base change, $X \times_{\mathbb{P}^n} H \to H$ is a finite etale cover of $H \cong \mathbb{P}^{n-1}$. I claim that $X \times_{\mathbb{P}^n} H$ is also connected: by exercise III.5.7(d), $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is ample on X, so $X \times \mathbb{P}^n H = f^{-1}(H)$ is the support of an ample divisor on X and by corollary III.7.9 it is connected. By the inductive hypothesis, $X \times_{\mathbb{P}^n} H \to H$ is isomorphic to $id : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$. Therefore $X \to \mathbb{P}^n$ is a finite etale morphism of degree one, or an isomorphism. So $\mathbb{P}^n$ is simply connected.

Exercise IV.2.2. *Classification of Curves of Genus* 2. Fix an algebraically closed field of characteristic $\neq 2$.

a. If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to an automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to an automorphism of $\mathbb{P}^1$.

b. Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x - \alpha_1) \cdots (x - \alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbb{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4).)

c. Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbb{P}^1$, then there exists a unique $\varphi \in \operatorname{Aut} \mathbb{P}^1$ such that $\varphi(P_1) = 0$, $\varphi(P_2) = 1$, $\varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbb{P}^1$, and then normalize by sending the first three to $0, 1, \infty$, respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of k, $\neq 0, 1$.

d. Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of k, $\neq 0, 1$, as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element $\sigma \in \Sigma_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

e. Summing up, conclude that there is a one-to-one correspondence between the sets of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of k, $\neq 0, 1$, module the action of Σ_6 described in (d). In particular, there are many non-isomorphic curves of genus 2. We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}^3_k$ modulo a finite group.

Solution.

a. The first sentence is straight from exercise IV.1.7. As $\operatorname{char} k \neq 2$, our morphism f is separable, so we may apply Hurwitz's formula to see that $2 \cdot 2 - 2 = 2 \cdot (0-2) + \deg R$, or $\deg R = 6$. Since f is a finite flat morphism of degree 2 over an algebraically closed field, the preimage of any closed point is two points counted with multiplicity and therefore $e_p = 2$ when $e_p \neq 1$. This gives that f is ramified at exactly 6 points with ramification index 2 at each. We may see that f is uniquely determined by exercise II.7.2, for instance.

b. Since the field extension $K/k(x)$ is of degree two, we have that f is of degree two. By the logic from part (a), we can test ramification at the finite points by looking at the number of preimages of a closed point under the map $\operatorname{Spec} k[x,z]/(z^2 - \prod(x-\alpha_i)) \to \operatorname{Spec} k[x]$. The fiber at a closed point x_0 is $\operatorname{Spec} k[x]/(x-x_0) \times_{\operatorname{Spec} k[x]} \operatorname{Spec} k[x,z]/(z^2 - \prod(x-\alpha_i))$ or $\operatorname{Spec} k[z]/(z^2 - \prod(x_0 - \alpha_i))$, which is two distinct points exactly when $x_0 \neq \alpha_i$ as we're in characteristic not two. So f is ramified with $e_p = 2$ at the six points $x = \alpha_i$ and nowhere else at the finite points. We can also see by Hurwitz's formula that this implies f is unramified at infinty, too: $\deg R = \sum_{P \in X}(e_P - 1)$ is even, so $e_\infty - 1$ must be even, but $e_\infty \leq 2$ and therefore $e_\infty = 1$. So f is ramified over the six points $x = \alpha_i$ and nowhere else. By an application of Hurwitz's formula, we have $g(X) = 2$.

To show that f is the same map coming from K, it's enough to show that $f^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \omega_X$. As $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a line bundle, by corollary II.6.16 it's $\mathcal{L}(D)$ for some divisor D, and by corollary II.6.9, $\deg D = 2$. Because $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ has a two-dimensional space of global sections, we see that $h^0(\mathcal{L}(D)) = 2$, so an application of Riemann-Roch shows that $l(D) - l(K-D) = 2 + 1 - 2 = 1$, or $l(K-D) = 1$. Combining this with $\deg K - D = 0$, lemma IV.1.2 shows that $K - D \sim 0$, or $K \sim D$ and we're done.

c. Write $P_i = [p_i : q_i]$. Suppose φ is the fractional linear transform given by $[x : y] \mapsto [ax + by : cx + dy]$. To have $\varphi(P_1) = 0$, we must have $[a : b] = [q_1 : -p_1]$. To have $\varphi(P_3) = \infty$, we must have $[c : d] = [q_3 : -p_3]$. So our fractional linear transformation is of the form $[q_1 x - p_1 y : kq_3 x - kp_3 y]$. To have $\varphi(P_2) = 1$, we must have $ap_2 + bq_2 = cp_2 + dq_2$, which uniquely fixes k and totally determines our fractional linear transformation. By exercise I.6.6, this shows that there is a unique automorphism φ and we're done.

d. To check that this defines a group action, we need to verify that given $g, h \in \Sigma_6$ we get the same thing when we apply g, renormalize, apply h, renormalize versus when we apply hg and renormalize. (There actually is some content here!)

Define the *cross-ratio* of four points z_1, z_2, z_3, z_4 in $\mathbb{P}^1$ to be

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

(We'll be thinking of $\mathbb{P}^1$ as $k \cup \infty$ here.) This is the image of z_1 under the fractional linear transformation sending $z_2 \mapsto 0$, $z_3 \mapsto 1$, $z_4 \mapsto \infty$. I claim this quantity is preserved by fractional linear transformations. To prove this, suppose $T(z) = \frac{az+b}{cz+d}$. Then

$$T(x) - T(y) = \frac{(ad - bc)(x - y)}{(cx + d)(cy + d)}$$

by direct calculation, so plugging in to the cross-ratio we get that

$$\frac{(T(z_1) - T(z_2))(T(z_3) - T(z_4))}{(T(z_1) - T(z_4))(T(z_2) - T(z_3))} = \frac{\frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}}{\frac{(ad-bc)(z_1-z_4)}{(cz_1+d)(cz_4+d)} \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

We can use this to verify that the action by Σ_6 really is a group action. Since fractional linear transformations preserve the cross-ratio, for any ordered collection of four elements of $(0, 1, \infty, \beta_1, \beta_2, \beta_3)$, the cross-ratio of those terms after being permuted in both $(hg)(0, 1, \infty, \beta_1, \beta_2, \beta_3)$ and $h(g(0, 1, \infty, \beta_1, \beta_2, \beta_3))$ have the same cross-ratio. But selecting any group which gets sent to $\beta_i', 0, 1, \infty$, we see that β_i' is the same under both actions by our work on the cross-ratio above. So we've shown this is a group action.

e. Yup!

Exercise IV.2.3. *Plane Curves.* Let X be a curve of degree d in $\mathbb{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P (I, Ex. 7.3). Considering $T_P(X)$ as a point of the dual projective plane $(\mathbb{P}^2)^*$, the map $P \to T_P(X)$ gives a morphism of X to its *dual curve* X^* in $(\mathbb{P}^2)^*$ (I, Ex. 7.3). Note that even though X is nonsingular, X^* in general will have singularities. We assume $\operatorname{char} k = 0$ below.

a. Fix a line $L \subset \mathbb{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$, for each point $P \in X$. Show that φ is ramified at P if and only if either (1) $P \in L$, or (2) P is an *inflection point* of X, which means that the intersection multiplicity (I, Ex. 5.4) of $T_P(X)$ with X at P is ≥ 3. Conclude that X has only finitely many inflection points.

b. A line of $\mathbb{P}^2$ is a *multiple tangent* of X if it is tangent to X at more than one point. It is a *bitangent* if it is tangent to X at exactly two points. If L is a multiple tangent of X, tangent to X at the points $P_1, \ldots, P_r$, and if none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an *ordinary r-fold point*, which means a point of multiplicity r with distinct tangent directions (I, Ex. 5.3). Conclude that X has only finitely many multiple tangents.

c. Let $O \in \mathbb{P}^2$ be a point which is not on X, nor on any inflectional or multiple tangent of X. Let L be a line not containing O. Let $\psi : X \to L$ be the morphism defined by projection from O. Show that ψ is ramified at a point $P \in X$ if and only if the line OP is tangent to X at P, and in that case the ramification index is 2. Use Hurwitz's theorem and (I, Ex. 7.2) to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence the degree of the dual curve (sometimes called the *class* of X) is $d(d-1)$.

d. Show that for all but a finite number of points of X, a point O of X lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O.

e. Show that the degree of the morphism φ of (a) is $d(d-1)$. Conclude that if $d \geq 2$, then X has $3d(d-2)$ inflection points, properly counted. (If $T_P(X)$ has intersection multiplicity r with X at P, then P should be counted $r-2$ times as an inflection point. If $r = 3$ we call it an *ordinary inflection point*.) Show that an ordinary inflection point of X corresponds to an ordinary cusp of the dual curve X^*.

f. Now let X be a plane curve of degree $d \geq 2$, and asssume that the dual curve X^* has only nodes and ordinary cusps as singularities (which should be true for sufficiently general X). Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents. [*Hint*: Show that X is the normalization of X^*. Then calculate $p_a(X^*)$ two ways: once as a plane curve of degree $d(d-1)$, and once using (Ex. 1.8).]

g. For example, a plane cubic curve has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one.

h. A plane quartic curve has exactly 28 bitangents. (This holds even if the curve has a tangent with four-fold contact, in which case the dual curve X^* has a tacnode.)

Solution.

a. Suppose $P \in L$. Taking an affine neighborhood and changing coordinates, we may assume $P = (0,0) \in \mathbb{A}^2$, $L = V(y)$, and $X \cap \mathbb{A}^2$ is cut out by a polynomial f with linear term x. Given a point $Q = (Q_x, Q_y) \in X \cap \mathbb{A}^2$, the tangent line is defined by $V(\frac{\partial f}{\partial x}(Q)(x-Q_x)+\frac{\partial f}{\partial y}(Q)(y-Q_y))$. To get $\varphi(Q)$, set $y = 0$ and solve for x: this gives that

$$x = \frac{\frac{\partial f}{\partial x}(Q)Q_x + \frac{\partial f}{\partial y}(Q)Q_y}{\frac{\partial f}{\partial x}(Q)}.$$

Letting t be a coordinate on L and letting x, y be coordinates on X, this gives that our map on local rings $\mathcal{O}_{L,P} \to \mathcal{O}_{X,P}$ is of the form $k[t]_{(t)} \to k[x,y]_{(x,y)}/(f)$ by $t \mapsto \frac{\frac{\partial f}{\partial x}(x,y)x+\frac{\partial f}{\partial y}(x,y)y}{\frac{\partial f}{\partial x}(x,y)}$. By our assumptions on the form of f, we have that $\frac{\partial f}{\partial x}(x,y) = 1 + \cdots$ is a unit while $\frac{\partial f}{\partial y}(x,y)$ is of positive valuation. To check the valuation of x and y in $\mathcal{O}_{X,P}$ is equivalent to computing the intersection multiplicity at the origin of X and $V(x)$ or $V(y)$, respectively. Since $V(y)$ is not tangent to X, we have that this intersection multiplicity is one, while as $V(x)$ is tangent, we have that this intersection multiplicity is at least two. Therefore t maps to something of valuation at least two in $\mathcal{O}_{L,P}$, and so this map is ramified.

If $P \notin L$, then we can pick coordinates so that we have all the same assumptions as in the previous case except that $L = V(z)$ is the line at infinity. The equation of the tangent line is the same, and the intersection of this with L is $[\frac{\partial f}{\partial x}(Q) : \frac{\partial f}{\partial y}(Q) : 0]$. Taking t to be a local coordinate at $[1:0:0]$, we have that our map on local rings $\mathcal{O}_{L,T_PX\cap L} \to \mathcal{O}_{X,P}$ is of the form $k[t]_{(t)} \to k[x,y]_{(x,y)}/(f)$ by $t \mapsto \frac{\frac{\partial f}{\partial y}(x,y)}{\frac{\partial f}{\partial x}(x,y)}$. Now let $f = x + ax^2 + bxy + cy^2 + \cdots$. By our assumptions on f, the denominator is $1 + 2ax + by + \cdots$ and the numerator is $bx + 2cy + \cdots$. The denominator is a unit, while the numerator has valuation at least two iff $c = 0$: by the work in the previous paragraph, the valuation of x is at least two while the valuation of y is exactly one. But $c = 0$ is equivalent to the intersection multiplicity of $V(x)$ and X being at least three (the dimension of $\mathcal{O}_{\mathbb{A}^2,P}/(x,f)$ is equal to the minimum d so that y^d appears in f), so P is an inflection point of X.

So φ is ramified at P iff $P \in L$ or P is an inflection point. By Hurwitz's formula, $\deg R < \infty$, so X must have finitely many inflection points.

b. Calculating the equation of the dual curve is difficult: if X is cut out by the homogeneous equation f, the dual curve is cut out by the ideal $(F, u - \frac{\partial f}{\partial x}, v - \frac{\partial f}{\partial y}, w - \frac{\partial f}{\partial z}) \cap k[u,v,w]$ inside $k[x,y,z,u,v,w]$. So we need a different method to get at the tangent direction, especially along a branch of our curve. We'll use a formal neighborhood and a parametrization by formal power series - our techniques will end up looking rather like what one would see in differential geometry, and that's okay.

Given a map from a smooth curve C to another scheme D, one can take the completion at a point $c \in C$ to get a map $\operatorname{Spec} k[[t]] \to C \to D$. When D is affine space, we can define the embedded tangent space as the common zero locus of the linear forms that pull back to something in the square of the maximal ideal and it's easy to see that we get the same embedded tangent space whether we consider C or the completion of C at c. In particular, when $D = \mathbb{A}^2$, and our morphism $\operatorname{Spec} k[[t]] \to \mathbb{A}^2$ is of the form $t \mapsto (f(t), g(t))$ for power series f, g then the tangent line to our curve at the point $t = t_0$ is given by $\frac{\partial g}{\partial t}(t_0)(x - f(t_0)) - \frac{\partial f}{\partial t}(t_0)(y - f(t_0)) = 0$. We'll use this a few times to figure out what the tangent directions to the map $X \to X^*$ at the point in X^* corresponding to the multiple tangent are.

Suppose $V(x)$ is simply tangent to X at $[0:0:1]$ and $[0:1:0]$. Locally at $[0:0:1]$, X is cut out by $x - uy^2$ for u a unit, and y is a generator of $\mathcal{O}_{X,[0:0:1]}$. This gives us that we can parametrize X in a formal neighborhood by $t \mapsto (ut^2, t)$, which gives tangent line $(x - u(t)t^2) - (u't^2 + 2u(t)t)(y - t) = 0$. Rearranging, this gives us that the parametrization of the dual curve given by $\operatorname{Spec} k[[t]] \to X \to X^*$ is given by

$$t \mapsto [1 : -(u't^2 + 2u(t)t) : u't^3 + ut^2].$$

Taking cooridnates $[A : B : C]$ for $(\mathbb{P}^2)^*$, the tangent line to this parametrization is then given by

$$(u''t^3 + 4u't^2 + 2ut)(B + (u't^2 + 2ut)A) + (u''t^2 + 4u't + 2u)(C - (u't^3 + ut^2)A) = 0$$

which when $t = 0$ gives that the tangent line is $V(C)$. By symmetry, we find that the tangent line to the branch of the dual curve obtained by passing through $[0 : 1 : 0]$ is $V(B)$, and so the tangent directions at $[1 : 0 : 0] \in X^*$ corresponding to different points of intersection between our multiple tangent and X are distinct. Therefore we may conclude that $L \in X^*$ is an ordinary r-fold point and that X has only finitely many multiple tangents.

c. Up to a change of variables, we may assume $O = [0 : 0 : 1]$, $L = V(z)$. Then the map ψ is just $(x, y) \mapsto [x : y]$, and up to a change of coordinates we may assume $P = (0, 1)$. x/y is a coordinate on $\mathbb{P}^1$ at $[0 : 1]$, so our map is ramified iff $\varphi^{-1}(x/y)$ is in the square of the maximal ideal. As $y = 1$ at P, it's invertible there, so $\varphi^{-1}(x/y)$ is in the square of the maximal ideal iff x is, which is the same as saying $V(x)$ has intersection multiplicity at least two with X. From the assumption that O is not on any inflectional tangent, we get that the ramification index is exactly two and no more; from the assumption that O is not on any multiple tangent, we get that each point of ramification of ψ corresponds to a unique line through O tangent to X.

By the results of exercise I.7.2, $g(X) = \frac{1}{2}(d-1)(d-2)$ and $g(L) = 0$, so Hurwitz's theorem gives that $\deg R = (d-1)(d-2) - 2 - d(-2) = d^2 - 3d + 2 - 2 + 2d = d(d-1)$. As R is a sum of distinct points each of which corresponds uniquely to a tangent to X through O by our first paragraph, we have that the number of tangents to X through O is $d(d-1)$.

d. Let O be a point on X not on any inflectional or multiple tangents, and let L be a line not containing O. Let $\psi : X \to L$ be the projection from O. Then ψ is of degree $d-1$, so Hurwitz's formula gives that $(d-1)(d-2)-2 = (d-1)(-2)+\deg R$. After some rearranging, $\deg R = (d+1)(d-2)$, and by the same argument as in part (c), R is a sum of distinct points corresponding uniquely to tangents of X passing through O giving the result.

e. $\varphi^{-1}(P)$ is exactly the set of points $Q \in X$ such that $P \in T_Q X$. If P does not lie on any inflectional or multiple tangents, then by part (c) this set is $d(d-1)$ points, and so $\deg \varphi = d(d-1)$. Applying Hurwitz's theorem, $\deg R = (d-1)(d-2)-2+2d(d-1) = 3d^2-5d$. But this overcounts by d since $L \cap X$ of degree d, so the number of inflection points (properly counted) is $3d^2 - 6d = 3d(d-2)$.

To check that an ordinary inflection point corresponds to an ordinary cusp, we use the same strategy as in (b). We may take $[0 : 0 : 1]$ to be the ordinary inflection point and $V(x)$ to be the tangent line, giving that locally near $[0 : 0 : 1]$, X is cut out by $x - uy^3$ for u a unit and that X is (formally) parametrized by (ut^3, t) near $[0 : 0 : 1]$. The map to the dual curve at the inflection point is therefore given by $t \mapsto [1 : 3ut^2+u't^3 : -(4ut^3+u't^4)] = [1 : t^2(3u+u't) : -t^3(4u+u't)]$, which is a cusp under any reasonable definition (Hartshorne does not supply one).

f. It's clear that $\varphi : X \to X^*$ is finite, as it's a nonconstant map of curves. We also have that it is generically one-to-one on points: the only way to violate injectivity is to have multiple tangents, which happens only finitely often by (b). Since we're in characteristic zero, this implies that φ is in fact birational. As X is smooth, it is normal, so by the universal property of normalization we find that $\varphi : X \to X^*$ is actually the normalization.

As the only singularities of X^* are nodes and ordinary cusps, by exercise IV.1.8 we have that $p_a(X^*) = p_a(X) + I + B$, where I is the number of inflection points and B is the number of bitangents. On the other hand, considering X^* as a plane curve of degree $d(d-1)$, we have that $p_a(X^*) = \binom{d(d-1)-1}{2}$. By (e), $I = 3d(d-2)$, so $B = \binom{d(d-1)-1}{2} - \binom{d-1}{2} - 3d(d-2)$, which after a little light algebra may be confirmed to be $\frac{1}{2}d(d-2)(d-3)(d+3)$.

g. We can see the claim about inflection points by plugging in $d = 3$ to the formula in (e).

The other claim is more interesting. Fix one of the inflection points as the identity of the group law on the plane cubic. All inflection points are 3-torsion points per exercise II.6.6(c). Given two inflection points $P, Q \in X$ and R the third point of intersection of the line $\overline{PQ}$ with X, we have $P + Q + R = 0$ in the group law by exercise II.6.6(a). But this means R must also be 3-torsion, so R is an inflection point too.

h. Plug in $d = 4$ to (f) to get 28 bitangents.

Exercise IV.2.4. *A Funny Curve in Characteristic p.* Let X be the plane quartic curve $x^3y + y^3z + z^3x = 0$ over a field of characteristic 3. Show that X is nonsingular, every point of X is an inflection point, the dual curve X^* is isomorphic to X, but the natural map $X \to X^*$ is purely inseparable.

Solution. The Jacobian is $\begin{pmatrix} z^3 & x^3 & y^3 \end{pmatrix}$ which is of rank one everywhere, so X is nonsingular. That's also the dual map, and it's sort of clear that if $[x_0 : y_0 : z_0] \in X$, then the image under the dual map $[z_0^3 : x_0^3 : y_0^3]$ also satisfies $u^3v + v^3w + w^3u = 0$ and therefore $X^* \cong X$ - we can also verify this from the procedure outlined in the solution to exercise IV.2.3(b). To check that the map is inseparable, we see that the map on function fields $\operatorname{Frac}(k[x, y, z]/(x^3y + y^3z + z^3x))_0$ is given by sending $\frac{x}{z} \mapsto \frac{z^3}{y^3}$ and $\frac{y}{z} \mapsto \frac{x^3}{y^3}$, so the minimal polynomials of the generators of $k(X)$ over $k(X^*)$ are of the form $T^3 - a$ and therefore the extension is purely inseparable.

To check that every point is an inflection point, we expand the dehomogenization of $x^3y + y^3z + z^3x$ near a point $[a : b : 1]$ as a polynomial in $(x - a)$ and $(y - b)$ and inspect the terms to compute the intersection multiplicity. But we can handle the degree one and two terms by the same way we Taylor expand in calculus: take derivatives and evaluate. This gives that in $\mathcal{O}_{\mathbb{P}^2,[a:b:1]}$ we have that $x^3y + y^3 + x = (x - a) + a^3(y - b)$ plus terms of order at least three, and therefore $[a : b : 1]$ is an inflection point. (In fact, one can check that $x^3y + y^3 + x = (x-a) + a^3(y-b) - b(x-a)^3 + (y-b)^3 + (x-a)^3(y-b)$ for any a, b satisfying $a^3b + b^3 + a = 0$ and calculate the intersection multiplicity using that, but guessing this form is a less robust strategy to take to other problems than the Taylor expansion method.)

Exercise IV.2.5. *Automorphisms of a Curve of Genus ≥ 2.* Prove the theorem of Hurwitz [1] that a curve of genus $g \geq 2$ over a field of characteristic 0 has at most $84(g - 1)$ automorphisms. We will see later (Ex. 5.2) or (V, Ex. 1.11) that the group $G = \operatorname{Aut} X$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subset K(X)$ corresponds to a finite morphism of curves $f : X \to Y$ of degree n.

a. If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}(f(P))$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that

$$(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i).$$

b. Since $g \geq 2$, the left hand side of the equation is > 0. Show that if $g(Y) \geq 0$, $s \geq 0$, $r_i \geq 2$, $i = 1, \ldots, s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i) > 0,$$

then the minimum value of this expression is $1/42$. Conclude that $n \geq 84(g-1)$. See (Ex. 5.7) for an example where this maximum is achieved.

Note: It is known that this maximum is achieved for infinitely many values of g (Macbeath [1]). Over a field of characteristic $p > 0$, the same bound holds, proveded $p > g+1$, with one exception, namely the hyperelliptic curve $y^2 = x^p - x$, which has $p = 2g+1$ and $2p(p^2-1)$ automorphism (Roquette [1]). For other bounds on the order of the group of automorphisms in characteristic p, see Singh [1] and Stichtenoth [1].

Solution.

a. This is just the orbit-stabilizer theorem: for any $y \in Y$, $f^{-1}(y)$ forms an orbit for G and so the stabilizers of each point in $f^{-1}(y)$ are conjugate, and in particular of the same order. So the orbit-stabilizer theorem gives us that $f^{-1}(y)$ is of size n/r for some $r|n$, and it's easy to see that if f is ramified at P then f must also be ramified of the same ramification index at gP for any $g \in G$ because the maps $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,f(P)} \cong \mathcal{O}_{Y,f(gP)} \to \mathcal{O}_{X,gP}$ are the same. By finiteness of f, we have that $\dim_k \mathcal{O}_{X_y}(X_y) = n$ is independent of y and equal to $\sum_{P \mapsto y} e_P$, so when $|f^{-1}(y)| = n/r$ we have that $e_P = r$ for all $P \in f^{-1}(y)$.

Hurwitz's theorem then gives that

$$2g-2 = n(2g(Y)-2) + \sum_{P \in X} (e_P - 1)$$

and after dividing by n,

$$(2g-2)/n = 2g(Y) - 2 + \sum_{P \in X} (e_P - 1)/n.$$

As each point P with ramification index $r > 1$ lies in an orbit of n/r points with ramification index r, we have that $\sum_{P \in X} (e_P - 1)/n = \sum_{i=1}^{s} (n/r_i)(r_i - 1)/n = \sum_{i=1}^{s} (1 - 1/r_i)$ and the claim is proven.

b. Let $Q = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i)$. First, note that as $r_i \geq 2$, we have that $(1 - 1/r_i) \geq \frac{1}{2}$ for each i. Therefore if $g(Y) > 0$, then Q is at least $1/2$, so it's at least $1/42$. When $g(Y) = 0$, we're looking for the minimum positive value of $-2 + \sum_{i=1}^{s}(1 - 1/r_i)$. Since $1 - 1/r_i < 1$, s must be at least 3. From the condition on $(1-1/r_i)$, we know that if $s \geq 5$ then $\sum_{i=1}^{s}(1-1/r_i)$ is at least $\frac{5}{2}$ and Q is at least $\frac{1}{2}$, so we can concentrate on $s = 3$ or $s = 4$.

The key observation for the rest of the problem is that for $x_0 < x_1$, $-\frac{1}{x_0} < -\frac{1}{x_1}$. So if $\{r_i'\}_{i\in I}$ can be obtained from $\{r_i\}_{i\in I}$ by successive steps of adding one to one r_i, then $\sum_{i\in I}(1 - \frac{1}{r_i}) < \sum_{i\in I}(1 - \frac{1}{r_i'})$. We'll call this process of adding 1 to an individual r_i among $\{r_i\}_{i\in I}$ a *move*.

When $s = 4$, the minimum positive value of $2 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}$ is $1/6$, achieved when three $r_i = 2$ and one $r_i = 3$: we must have at least one $r_i > 2$, else we get 0; any move must increase $2 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}$ so the minimum value occurs when we make the minimum amount of moves.

When $s = 3$, we're looking for the minimum positive value of $1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3}$. Without loss of generality we may assume $r_1 \leq r_2 \leq r_3$. If $r_1 \geq 3$, then by the same logic as in the previous paragraph we find that the minimum positive value of $1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3}$ is $\frac{1}{12}$. Now set $r_1 = 2$; we must have $r_2 > 2$ else $1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3}$ is negative. If $r_2 = 3$, we're looking to find the minimum positive value of $\frac{1}{6} - \frac{1}{r_3}$ which is $\frac{1}{42}$ and occurs when $r_3 = 7$. If $r_2 = 4$, we're looking to find the minimum positive value of $\frac{1}{4} - \frac{1}{r_3}$ which is $\frac{1}{20}$ when $r_3 = 5$. If $r_2 \geq 5$, then $1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3}$ is at least $\frac{1}{10}$ because $r_3 \geq r_2$, and so we've shown that the minimum value of Q is $\frac{1}{42}$ as requested.

Exercise IV.2.6. *f_* for Divisors.* Let $f : X \to Y$ be a finite morphism of curves of degree n. We define a homomorphism $f_* : \operatorname{Div} X \to \operatorname{Div} Y$ by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$ for any divisor $D = \sum n_i P_i$ on X.

a. For any locally free sheaf $\mathcal{E}$ on Y, of rank r, we define $\det \mathcal{E} = \bigwedge^r \mathcal{E} \in \operatorname{Pic} Y$ (II, Ex. 6.11). In particular, for any invertible sheaf $\mathcal{M}$ on X, $f_*\mathcal{M}$ is locally free of rank n on Y, so we can consider $\det f_*\mathcal{M} \in \operatorname{Pic} Y$. Show that for any divisor D on X,

$$\det(f_*\mathcal{L}(D)) \cong (\det f_*\mathcal{O}_X) \otimes \mathcal{L}(f_*D).$$

Note in particular that $\det(f_*\mathcal{L}(D)) \neq \mathcal{L}(f_*D)$ in general! [*Hint*: First consider an effective divisor D, apply f_* to the exact sequence $0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, and use (II, Ex. 6.11).]

b. Conclude that f_*D depends only on the linear equivalence class of D, so there is an induced homomorphism $f_* : \operatorname{Pic} X \to \operatorname{Pic} Y$. Show that $f_* \circ f^* : \operatorname{Pic} Y \to \operatorname{Pic} Y$ is just multiplication by n.

c. Use duality for a finite flat morphism (III, Ex. 6.10) and (III, Ex. 7.2) to show that

$$\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}.$$

d. Now assume that f is separable, so we have the ramification divisor R. We define the *branch divisor* B to be the divisor f_*R on Y. Show that

$$(\det f_*\mathcal{O}_X)^2 \cong \mathcal{L}(-B).$$

Solution.

a. Let $D = \sum_{i=1}^r n_i P_i$ be an effective divisor and consider the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$. Since f is affine, all higher direct images of quasi-coherent sheaves along f vanish by proposition III.8.5 and theorem III.3.5, so $0 \to f_*\mathcal{O}_X(-D) \to f_*\mathcal{O}_X \to f_*\mathcal{O}_D \to 0$ is also exact. Computing $\det f_*\mathcal{O}_D$ from this resolution, we get $\det f_*\mathcal{O}_D = (\det f_*\mathcal{O}_X) \otimes (\det f_*\mathcal{O}_X(-D))^{-1}$.

On the other hand, $f_*\mathcal{O}_D$ is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_{n_i f(P_i)}$ since f is affine and $f_*\mathcal{O}_D$ is coherent; therefore we can resolve it by

$$0 \to \bigoplus_{i=1}^r \mathcal{O}_Y(-n_i f(P_i)) \to \bigoplus_{i=1}^r \mathcal{O}_Y \to \bigoplus_{i=1}^r \mathcal{O}_{n_i f(P_i)} \to 0.$$

Computing the determinant from this resolution, we get

$$\det f_*\mathcal{O}_D \cong (\det \bigoplus_{i=1}^r \mathcal{O}_Y) \otimes (\det \bigoplus_{i=1}^r \mathcal{O}_Y(-n_i f(P_i)))^{-1}.$$

Since the determinant of a direct sum is the tensor product of the determinants of the factors, we find that $\det f_*\mathcal{O}_D \cong \mathcal{O}_Y(-f_*D)^{-1} \cong \mathcal{O}_Y(f_*D)$. By exercise II.6.11(b), the determinant doesn't depend on the choice of resolution, so $\mathcal{O}_Y(f_*D) \cong (\det f_*\mathcal{O}_X) \otimes (\det f_*\mathcal{O}_X(-D))^{-1}$ for an effective divisor D on X.

Now let $D = D_1 - D_2$ for D_1, D_2 effective divisors on X. By pushing forward the exact sequence

$$0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$$

and taking the determinant of the final term versus twisting by D, pushing forward and taking determinant of the final term, we get that

$$(\det f_*\mathcal{O}_X) \otimes (\det f_*\mathcal{O}_X(-D_1))^{-1} \cong (\det f_*\mathcal{O}_X(D)) \otimes (\det f_*\mathcal{O}_X(-D_2))^{-1}$$

by exercise II.6.11(b). From our earlier work, we have that $(\det f_*\mathcal{O}_X(-D_i))^{-1} \cong \mathcal{O}_Y(f_*D_i) \otimes (\det f_*\mathcal{O}_X)^{-1}$, so substituting this in to our expression we obtain

$$(\det f_*\mathcal{O}_X) \otimes \mathcal{O}_Y(f_*D_1) \otimes (\det f_*\mathcal{O}_X)^{-1} \cong (\det f_*\mathcal{O}_X(D)) \otimes \mathcal{O}_Y(f_*D_2) \otimes (\det f_*\mathcal{O}_X)^{-1},$$

and rearranging we have

$$\det(f_*\mathcal{O}_X(D)) \cong (\det f_*\mathcal{O}_X) \otimes \mathcal{O}_Y(f_*D_1 - f_*D_2) \cong (\det f_*\mathcal{O}_X) \otimes \mathcal{O}_Y(f_*D)$$

as $f_*D_1 - f_*D_2 = f_*D$.

b. As $\mathcal{O}_Y(f_*D) \cong (\det f_*\mathcal{O}_X(D)) \otimes (\det f_*\mathcal{O}_X)^{-1}$ and the RHS only depends on the linear equivalence class of D, the LHS must also only depend on the linear equivalence class of D.

   To check the second claim, it suffices to check for a single point $y \in Y$ by linearity. Since the pullback of y is $\sum n_i P_i$ for $\sum n_i = n$ and $P_i \mapsto y$, we have the result. (To see this, let $\operatorname{Spec} A \subset Y$ be an affine open neighborhood of y where $y = V(a)$ for some $a \in A$. Then f^*y is cut out by f^*a as a function on $\operatorname{Spec} B = f^{-1}(\operatorname{Spec} A) \subset X$. But since f is finite flat, $B \cong A^n$ as A-modules, so $B/(f^*a)B \cong B \otimes_A k(y) \cong A^n \otimes k(y) \cong k(y)^n$. On the other hand, $B/f^*a \cong \prod_{x \mapsto y} \mathcal{O}_{X,x}/f^*a$.)

c. As X and Y are nonsingular curves, Ω_X and Ω_Y are their dualizing sheaves. By exercise III.7.2(a), $f^!\Omega_Y \cong \Omega_X$, and by exercise III.6.10(a), $f^!\Omega_Y \cong \mathcal{H}om_Y(f_*\mathcal{O}_X, \Omega_Y) \cong (f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y$. Per the argument of exercise III.8.4(b) that for a locally free sheaf $\mathcal{E}$ and a line bundle $\mathcal{L}$, $\det(\mathcal{E} \otimes \mathcal{L}) \cong (\det \mathcal{E}) \otimes \mathcal{L}^{\otimes \operatorname{rank} \mathcal{E}}$, taking determinants gives that $\det((f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y) \cong (\det(f_*\mathcal{O}_X)^{-1}) \otimes \Omega_Y^{\otimes n}$, and as determinants commute with dualization we have $\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}$ as requested.

d. By proposition IV.2.3, $\Omega_X \cong f^*\Omega_Y \otimes \mathcal{O}_X(R)$, so after pushing forward along f_* we get that $f_*\Omega_X \cong f_*(f^*\Omega_Y \otimes \mathcal{O}_X(R))$, and applying the projection formula (exercise II.5.1(d)) we get that $f_*\Omega_X \cong \Omega_Y \otimes f_*\mathcal{O}_X(R)$. Taking determinants of both sides and applying the argument of exercise III.8.4(b) again, we have $\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X(R)) \otimes \Omega_Y^{\otimes n}$. Combining this with (a) and (c), we have that

$$(\det f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n} \cong (\det f_*\mathcal{O}_X) \otimes \mathcal{O}_Y(B) \otimes \Omega_Y^{\otimes n}$$

   which rearranges to

$$(\det f_*\mathcal{O}_X)^2 \cong \mathcal{O}_Y(-B)$$

   as desired.

Exercise IV.2.7. *Étale Covers of Degree 2.* Let Y be a curve over a field k of characteristic $\neq 2$. We show that there is a one-to-one correpsondence between finite étale morphisms $f : X \to Y$ of degree 2, and 2-torsion elements of $\operatorname{Pic} Y$, i.e., invertible sheaves on Y with $\mathcal{L}^2 \cong \mathcal{O}_Y$.

a. Given an étale morphism $f : X \to Y$ of degree 2, there is a natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $\mathcal{L}$ be the cokernel. Then $\mathcal{L}$ is an invertible sheaf on Y, $\mathcal{L} \cong \det f_*\mathcal{O}_X$, and so $\mathcal{L}^2 \cong \mathcal{O}_Y$ by (Ex. 2.6). Thus an étale cover of degree 2 determines a 2-torsion element in $\operatorname{Pic} Y$.

b. Conversely, given a 2-torsion element $\mathcal{L}$ in $\operatorname{Pic} Y$, define an $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathcal{L}$ by $\langle a, b\rangle \cdot \langle a', b'\rangle = \langle aa' + \varphi(b \otimes b'), ab' + a'b\rangle$, where φ is an isomorphism of $\mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_Y$. Then take $X = \mathbf{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$ (II, Ex. 5.17). Show that X is an étale cover of Y.

c. Show that these processes are inverse to each other. [*Hint*: Let $\tau : X \to X$ be the involution which interchanges the points of each fibre of f. Use the *trace map* $a \mapsto a + \tau(a)$ from $f_*\mathcal{O}_X \to \mathcal{O}_Y$ to show that the seuqence of $\mathcal{O}_Y$-modules in (a)

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{L} \to 0$$

   is split exact.

Note. This is a special case of the more general fact that for $(n, \operatorname{char} k) = 1$, the étale Galois covers of Y with group $\mathbb{Z}/n\mathbb{Z}$ are classified by the étale cohomology group $H^1_{et}(Y, \mathbb{Z}/n\mathbb{Z})$, which is equal to the group of n-torsion poitns of $\operatorname{Pic} Y$. See Serre [6].

Solution.

a. Since f is finite flat, $f_*\mathcal{O}_X$ is locally free of rank equal to $\deg f = 2$, so $(f_*\mathcal{O}_X)_y$ is a free $\mathcal{O}_{Y,y}$-module of rank two for any $y \in Y$. Letting $\mathcal{L}$ be the cokernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$, we may reduce the exact sequence $0 \to \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y \to \mathcal{L}_y \to 0$ modulo $\mathfrak{m}_y$ to get $k \to k^2 \to \mathcal{L}_y/\mathfrak{m}_y\mathcal{L}_y \to 0$. Since the map $k \to k^2$ may be represented as the pullback of the constant functions on y to the constant functions on $f^{-1}(y) = \{p, q\}$ for two distinct points (this is where we use that f is unramified), the map is injective. So $\mathcal{L}$ is of rank one at every closed point and therefore is a line bundle by exercise II.5.8(c).

   By exercise II.5.16(d), we have that $\mathcal{O}_Y \otimes \mathcal{L} \cong \mathcal{L} \cong \det f_*\mathcal{O}_X$; by exercise IV.2.6 we have $(\det f_*\mathcal{O}_X)^2 \cong \mathcal{O}_Y(-B)$, but since f is etale $R = B = 0$ and $\mathcal{L}^2 \cong \mathcal{O}_Y$.

b. $X \to Y$ is affine, and for any affine open $\operatorname{Spec} A \subset Y$ where $\mathcal{L}$ is free, we have that $X \times_Y \operatorname{Spec} A$ is the spectrum of an A-algebra which is module-isomorphic to A^2. Since such $\operatorname{Spec} A$ cover Y, we have that X is finite flat of degree two. We can also check that $\Omega_{X/Y}$ is zero on this cover by calculating the module of differentials. I claim that $d(a, b) = 0$ for all (a, b): since d distributes over addition, it's enough to see that $d(a, 0) = 0$ and $d(0, b) = 0$. The first follows from the definition of $A \to A \oplus \mathcal{L}(\operatorname{Spec} A)$, while the second follows from the fact that $(0, b)^2 \in A$: as $0 = d(\varphi(b^2), 0) = d(0, b)^2 = 2(0, b)d(0, b)$ and 2 as well as $(0, b)$ are not zero-divisors, we must have $d(0, b) = 0$. So $\Omega_{X/Y} = 0$, and thus by exercise III.10.3, f is etale which implies that X is smooth over k.

c. The map $a \mapsto (a + \tau(a))/2$ from $f_*\mathcal{O}_X \to \mathcal{O}_Y$ is an inverse to the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and therefore splits

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{L} \to 0.$$

   Thus $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}$ and by exercise II.5.17, $X \cong \mathbf{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$. This gives that every etale cover of degree two is of the form $\mathbf{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$ for $\mathcal{L}$ a 2-torsion element of $\operatorname{Pic} Y$, so the work from (a) and (b) is enough to see that these processes are inverse to each other.

IV.3 Embeddings in Projective Space

We're going to change our perspective here - in chapters II and III we mostly considered properties of schemes and morphisms of schemes without reference to an embedding. Now we're going to shift our attention to the question of 'how do curves embed' and find out some neat information.

Exercise IV.3.1. If X is a curve of genus 2, show that a divisor D is very ample $\Leftrightarrow \deg D \geq 5$. This strengthens (3.3.4).

Solution. By corollary IV.3.2(b), $\deg D \geq 5$ implies D is very ample. We may case-work the converse statement. If $\deg D \leq 2$, then by exercise IV.1.5 we have $\dim |D| < \deg D \leq 2$, so the closed immersion coming from D would express D as a closed subvariety of $\mathbb{P}^0$ or $\mathbb{P}^1$, which is clearly nonsense. When $\deg D > 3$, we have $\deg K - D < 0$, so $l(K-D) = 0$ and Riemann-Roch then gives that $l(D) = \deg D - 1$. When $\deg D = 3$, $|D| = 1$ and the logic from earlier applies. When $\deg D = 4$, $|D| = 2$ which embeds X in $\mathbb{P}^2$. But no smooth plane curve can have genus 2 by the degree-genus formula.

Exercise IV.3.2. Let X be a plane curve of degree 4.

a. Show that the effective canonical divisors on X are exactly the divisors $X.L$, where L is a line in $\mathbb{P}^2$.

b. If D is any effective divisor of degree 2 on X, show that $\dim |D| = 0$.

c. Conclude that X is not hyperelliptic (Ex. 1.7).

Solution.

a. By example II.8.20.3, $\omega_X \cong \mathcal{O}_X(1)$, so the effective canonical divisors are just the hyperplane sections.

b. Suppose $D = P_1 + P_2$ is an effective divisor of degree two, and let $D' = Q_1 + Q_2$ be another effective divisor so that $D \sim D'$. Let $L \subset \mathbb{P}^2$ be the line through P_1 and P_2 (in the case where $P_1 = P_2$, take L to be the tangent line to X at P_1). Then $K \sim X.L = P_1 + P_2 + P_3 + P_4 \sim Q_1 + Q_2 + P_3 + P_4$, so Q_1 and Q_2 are on the line determined by P_3 and P_4, which is L. So $\{P_1, P_2\} = \{Q_1, Q_2\}$ and thus $|D| = 0$.

c. A degree-two morphism $X \to \mathbb{P}^1$ comes from a degree-two divisor D on X with $\dim |D| > 0$. But no such divisor exists by (b).

Exercise IV.3.3. If X is a curve of genus ≥ 2 which is a complete intersection (II, Ex. 8.4) in some $\mathbb{P}^n$, show that the canonical divisor K is very ample. Conclude that a curve of genus 2 can never be a complete intersection in any $\mathbb{P}^n$. Cf. (Ex. 5.1).

Solution. Suppose $X \subset \mathbb{P}^n$ can be written as a complete intersection $\bigcap_{i=1}^{n-1} H_i$ where H_i has degree d_i and is cut out by the homogeneous polynomial f_i. Let $r = \sum_{i=1}^{n-1} d_i - n - 1$. I claim that $\omega_X \cong \mathcal{O}_X(r)$, which we'll prove by a combination of proposition III.7.5 which states that $\omega_X \cong \mathcal{E}xt^{n-1}(\mathcal{O}_X, \omega_{\mathbb{P}^n_k})$ and choosing a convenient resolution of $\mathcal{O}_X$. (We'd like to just cite exercise II.8.4(e) for this, but there's an assumption in there that we can choose the hypersurfaces H_i so that each intermediate intersection $\bigcap_{i=1}^{j} H_i$ is nonsingular, and I don't see how to guarantee this.)

First, recall that locally free sheaves are acyclic for sheaf Ext, so they may be used to compute sheaf Ext by the material of III.3. We have the start of a locally free resolution of $\mathcal{O}_X$ from the fact that the homogeneous ideal of X is generated by the f_i: $\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n_k}(-d_i) \to \mathcal{O}_{\mathbb{P}^n_k} \xrightarrow{f} \mathcal{O}_X \to 0$ is exact, where the i^{th} basis vector is sent to f_i. We can continue this to a free resolution known as the *Koszul resolution*: set $K_a = \bigwedge^a \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n_k}(-d_i)\right)$ where the differential from $K_a \to K_{a-1}$ is given by sending $e_1 \wedge \cdots \wedge e_k$ to $\sum_{j=1}^{k} (-1)^{i+1} f(e_i) e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_k$. As the f_i form a regular sequence in $k[x_0, \cdots, x_n]$, $K_\bullet$ forms a free resolution of $\mathcal{O}_X$.

To compute $\mathcal{E}xt^{n-1}(\mathcal{O}_X, \omega_{\mathbb{P}^n_k})$, we can apply $\mathcal{H}om(-, \omega_{\mathbb{P}^n_k})$ to this resolution and inspect the last few terms. Simplifying the wedge products in our Koszul complex a bit, we see that $K_{n-1} \cong \mathcal{O}_{\mathbb{P}^n_k}(-\sum_{i=1}^{n-1} d_i)$ and $K_{n-1} \cong \bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^n_k}(d_j - \sum_{i=1}^{n-1} d_i)$, where the map between them is $1 \mapsto (f_1, \cdots, f_{n-1})$. Applying $\mathcal{H}om(-, \omega_{\mathbb{P}^n_k})$, we see that the tail end of our complex is

$$\bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^n_k}(-n-1-d_j+\sum_{i=1}^{n-1} d_i) \to \mathcal{O}_{\mathbb{P}^n_k}(-n-1+\sum_{i=1}^{n-1} d_i) \to 0$$

where the nontrivial map sends the j^{th} basis vector to f_j. But this is just the complex

$$\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n_k}(-d_i) \to \mathcal{O}_{\mathbb{P}^n_k} \to 0$$

twisted by $\mathcal{O}_{\mathbb{P}^n_k}(-n-1+\sum_{i=1}^{n-1} d_i) \cong \mathcal{O}_{\mathbb{P}^n_k}(r)$, so we find that $\omega_X \cong \mathcal{O}_X(r)$.

Now the problem wraps up quickly. By exercise II.8.4(c), $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(X, \mathcal{O}_X(r)) \cong H^0(X, \omega_X)$ is surjective. As $\dim_k H^0(X, \omega_X)$ is the (geometric) genus of X, we must have that $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ is at least 2, which implies $r > 0$ and therefore ω_X is very ample.

For a curve of genus 2, $|K| = 2g - 2 = 2$ (by Riemann-Roch, see exercise IV.1.7 if you need a reminder), and so it is not very ample by exercise IV.3.1.

Exercise IV.3.4. Let X be the d-uple embedding (I, Ex. 2.12) of $\mathbb{P}^1$ in $\mathbb{P}^d$, for any $d \geq 1$. We call X the *rational normal curve of degree d in $\mathbb{P}^d$*.

a. Show that X is projectively normal, and that its homogeneous ideal can be generated by forms of degree 2.

b. If X is any curve of degree d in $\mathbb{P}^n$, with $d \leq n$, which is not contained in any $\mathbb{P}^{n-1}$, show that in fact $d = n$, $g(X) = 0$, and X differs from the rational normal curve of degree d only by an automorphism of $\mathbb{P}^d$. Cf. (II, 7.8.5).

c. In particular, any curve of degree 2 in any $\mathbb{P}^n$ is a conic in some $\mathbb{P}^2$.

d. A curve of degree 3 in any $\mathbb{P}^n$ must be either a plane cubic curve, or the twisted cubic curve in $\mathbb{P}^3$.

Solution.

a. By exercise I.2.12, the homogeneous coordinate ring of X is $S(X) = k[x,y]^{(d)}$, and we'll show directly that this is integrally closed. From the embedding $S(X) \hookrightarrow k[x,y]$, we get an embedding $\mathrm{Frac}(S(X)) \subset k(x,y)$ and by the fact that the integral closure of $k[x,y]$ in $k(x,y)$, any element of $\mathrm{Frac}(S(X))$ integral over $S(X)$ must be in $k[x,y]$. But by degree considerations such an integral element must be of degree d, which means it must be in $S(X)$.

   Next I claim that $I(X)$ is generated by $z_iz_j - z_kz_l$ where $i+j = k+l$ and z_i is the coordinate corresponding to $x^{d-i}y^i$. It is easy to see by repeated applications of these relations that for any monomial in $I(X) \subset k[z_0, \cdots, z_d]$, we can rewrite it as $z_0^a z_d^b z_i$ for a unique i. Thus if $f \in I(X)_{e+1}$, we may write $f = \sum_{i=0} c_i z_0^a z_d^{e-a} z_i$ and upon evaluating $z_i = x^{d-i}y^i$ we see that the resulting terms are linearly independent because they have different powers of x, and therefore $f \in (z_iz_j - z_kz_l \mid i+j = k+l)$ and we've proven the claim.

b. Consider $X.H$: this is of degree d and $\dim |H| = n$ as $X \not\subset \mathbb{P}^{n-1}$. Pick a point $P \in X$ not in the base locus of $|X.H|$. Then $\deg(X.H - P) = d-1$ with $\dim |X.H - P| = n-1$. Continuing the process, we obtain a divisor D of degree zero with $\dim |D| = n-d$, and by exercise IV.1.5, $n - d = 0$. This also gives $\mathcal{O}_X(X.H) \cong \mathcal{O}_{\mathbb{P}^1}(n)$, and as X is not contained in any hyperplane, the morphism $X \to \mathbb{P}^n$ must be given by a basis of the global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$. Therefore by remark II.7.8.1, X and the rational normal curve are related by an automorphism of $\mathbb{P}^n$.

c. Clearly such a curve cannot be a line in $\mathbb{P}^n$ for reasons of degree. Let D be the intersection of X with a hyperplane. Then $\deg D = 2$ and X is embedded by $\mathcal{O}_X(D)$, so by exercise IV.1.5 we have $l(D) \leq 3$ and the closed immersion induced by $\mathcal{L}(D)$ must factor through a plane.

d. Repeat (c) to get that X cannot be a line and is embedded by $\mathcal{O}_X(D)$ with $l(D) \leq 4$ with equality iff $g(X) = 0$. When $g(X) \neq 0$, $l(D) = 3$ and X is a plane cubic curve; when $g(X) = 0$, $l(D) = 4$ and the closed immersion $X \to \mathbb{P}^n$ factors through a 3-space. By (b), such a curve must be the twisted cubic.

Exercise IV.3.5. Let X be a curve in $\mathbb{P}^3$, which is not contained in any plane.

a. If $O \notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbb{P}^2$, show that $\varphi(X)$ *must* be singular. [*Hint*: Calculate $\dim H^0(X, \mathcal{O}_X(1))$ two ways.]

b. If X has degree d and genus g, conclude that $g < \frac{1}{2}(d-1)(d-2)$. (Use (Ex. 1.8).)

c. Now let $\{X_t\}$ be the flat family of curves induced by the projection (III, 9.8.3) whose fibre over $t = 1$ is X, and whose fibre X_0 over $t = 0$ is a scheme with support $\varphi(X)$. Show that X_0 always has nilpotent elements. Thus the example (III, 9.8.4) is typical.

Solution.

a. If $\varphi(X)$ was nonsingular, then since $X \to \varphi(X)$ is birational, it must be an isomorphism as the birational morphisms of nonsingular projective curves are the isomorphisms. We show that this cannot happen.

   Since X does not lie in any plane in $\mathbb{P}^3$, the map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathcal{O}_X(1))$ must be injective, so $\dim H^0(X, \mathcal{O}_X(1)) \geq 4$. On the other hand, if the map $\varphi : X \to \varphi(X)$ is an isomorphism, then $\mathcal{O}_X(1) \cong \mathcal{O}_{\varphi(X)}(1)$ because they're both isomorphic to $\mathcal{L}_X(D)$ for the divisor given by the intersection of a hyperplane H through O with X. But since a smooth plane curve is a complete intersection, we may apply exercise II.8.4(c) to see that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(X, \mathcal{O}_X(1))$ must be surjective, giving $\dim H^0(X, \mathcal{O}_X(1)) \leq 3$, a contradiction.

b. I claim that $\deg X = \deg \varphi(X)$: consider a hyperplane H through O missing all the finitely many points of $\varphi(X)$ where φ is not an isomorphism. Then the degree of the two divisors $H.X$ and $\varphi(H).\varphi(X)$, and this quantity is exactly $\deg X$ and $\deg \varphi(X)$. Further, X is the normalization of $\varphi(X)$, so by exercise IV.1.8 the (arithmetic) genus g of X is less than the arithmetic genus of $\varphi(X)$, which is $\frac{1}{2}(d-1)(d-2)$ by the degree-genus formula. Therefore $g < \frac{1}{2}(d-1)(d-2)$.

c. If X_0 were reduced, then $X_0 \cong \varphi(X)$ by the uniqueness of the reduced induced subscheme structure. But this gives that $p_a(\varphi(X)) = p_a(X_0) = p_a(X)$, where the second equality comes from theorem III.9.9 which says that the Hilbert polynomial of the fibers in a flat family is an invariant. This contradicts the conclusion of (b), so it cannot be the case that X_0 is reduced.

Exercise IV.3.6. *Curves of Degree 4.*

a. If X is a curve of degree 4 in some $\mathbb{P}^n$, show that either

   (1) $g = 0$, in which case X is either the rational normal quartic in $\mathbb{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbb{P}^3$ (II, 7.8.6), or

   (2) $X \subset \mathbb{P}^2$, in which case $g = 3$, or

   (3) $X \subset \mathbb{P}^3$ and $g = 1$.

b. In the case $g = 1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbb{P}^3$ (I, Ex. 5.11). [*Hint*: Use the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$ to compute $\dim H^0(\mathbb{P}^3, \mathcal{I}_X(2))$, and thus conclude that X is contained in at least two irreducible quadric surfaces.]

Solution.

a. Let D be the hyperplane divisor on X. Since X is of degree four, D is too, and by exercise IV.1.5, we have that $\dim |D| \leq \deg D$ with equality iff $g = 0$. When $g = 0$, X embeds in to

$\mathbb{P}^4$. If X is not contained in any hyperplane, then it is the rational normal quartic by exercise IV.3.4(b). If X is contained in a hyperplane, then it's a rational quartic curve.

If $g \neq 0$, then $\dim |D| \leq 3$. If X embeds in to $\mathbb{P}^3$ and is not contained in a hyperplane, then by exercise IV.3.5(b) we must have that $g < 3$. But D is very ample of degree 4, so by exercise IV.3.1, $g \neq 2$ and thus $g = 1$.

If X embeds in to $\mathbb{P}^2$, then by the degree-genus formula we have that $g = 3$, and we're done.

b. Consider
$$0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$$
and take homology. First, $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{2} = 10$ by the material we developed in section III.5. Second, $h^0(\mathcal{O}_X(2))$ is at most by 8: this space is of dimension $|2H.X|+1$, which is strictly less than $1 + \deg 2H.X = 9$ by exercise IV.1.5 and the fact that our curve is of degree 4. Therefore $h^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 2$ and X is contained in the intersection of two quadric surfaces Q_1 and Q_2 given by the vanishing of two basis elements for $h^0(\mathbb{P}^3, \mathcal{I}_X(2))$.

I claim that X is exactly this intersection. First, Q_1 and Q_2 do not share any irreducible components: if they did, then X would be contained in a plane and not be of genus 1 by (a). Therefore $Q_1 \cap Q_2$ may be written as $\bigcup Z_i$ for Z_i irreducible components all of dimension two. By Bezout's theorem, $\sum \deg Z_i = (\deg Q_1)(\deg Q_2) = 4$, and as $X \subset Q_1 \cap Q_2$ is also of degree four and dimension two, we must have that $X = Q_1 \cap Q_2$.

Exercise IV.3.7. In view of (3.10), one might ask conversely, is every plane curve with nodes a projection of a nonsingular curve in $\mathbb{P}^3$? Show that the curve $xy + x^4 + y^4 = 0$ (assume $\operatorname{char} k \neq 2$) gives a counterexample.

Solution. Let X be our curve $V(xyz^2 + x^4 + y^4) \subset \mathbb{P}^2$ and $X' \subset \mathbb{P}^3$ be the smooth curve which supposedly projects on to X. The projective Jacobian of X is $\begin{pmatrix} yz^2 + 4x^3 & xz^2 + 4y^3 & 2xyz \end{pmatrix}$, which is of rank zero only at $[0 : 0 : 1]$, where the curve has a node. By the degree-genus formula, $p_a(X) = 3$. Applying the formula and calculations from exercise IV.1.8, we see that $p_a(X') = 2$. But this contradicts the conclusions of exercise IV.3.6(a), so no such X' exists.

Exercise IV.3.8. We say that a (singular) integral curve in $\mathbb{P}^n$ is *strange* if there is a point which lies on all the tangent lines at nonsingular points of the curve.

a. There are many singular strange curves, e.g., the curve given parametrically by $x = t$, $y = t^p$, $z = t^{2p}$ over a field of characteristic $p > 0$.

b. Show, however, that if $\operatorname{char} k = 0$, there aren't even any singular strange curves besides $\mathbb{P}^1$.

Solution.

a. I'm going to interpret this as 'show this curve is strange'. The curve in question is $V(y^2 - zw, x^p - yw^{p-1}) \subset \mathbb{P}^3$, which has Jacobian $\begin{pmatrix} 0 & 2y & -w & -z \\ 0 & -w^{p-1} & 0 & -yw^{p-2} \end{pmatrix}$ and this is of rank

< 2 exactly when $w = 0$. Therefore the singular points are at infinity, so it suffices to analyze the behavior of the tangent line at the points in the affine parametrization. At the point (t, t^p, t^{2p}), the tangent direction is $(1, pt^{p-1}, 2pt^{2p-1}) = (1, 0, 0)$, which passes through $(1, 0, 0, 0)$, so this curve is strange.

b. We can mimic some of the strategy of the proof of theorem IV.3.9 here. By repeatedly applying exercise I.4.9 and projecting from a point distinct from A, the point through which the tangent line at every nonsingular point passes, we may consider a curve $X \subset \mathbb{P}^3$ which has tangent line passing through a point $A \in \mathbb{P}^3$ except for finitely many singular points and possibly finitely many points where the projection from our original curve was not an isomorphism.

   Now consider the projection π from A to a plane $\mathbb{P}^2$. If the image of X is a curve, then $X \to \pi(X)$ will be ramified at every point $x \in X$ for which $\overline{AX} = T_x X$ (use the same proof as exercise IV.2.3(c)). But the ramification locus of a morphism in characteristic zero is a proper closed subset, so this set must be finite, which contradicts our setup that $\overline{AX} = T_x X$ for infinitely many $x \in X$. So the image of X must be a point, which means that X is a line.

Exercise IV.3.9. Prove the following lemma of Bertini: if X is a curve of degree d in $\mathbb{P}^3$, not contained in any plane, then for almost all planes $H \subset \mathbb{P}^3$ (meaning a Zariski open subset of the dual projective space $(\mathbb{P}^3)^*$), the intersection $X \cap H$ consists of exactly d distinct points, no three of which are collinear.

Solution. We'll show that the locus of hyperplanes H which intersect X in either less than d distinct points or in three collinear points are proper closed subsets of $(\mathbb{P}^3)^*$.

First, a hyperplane H intersects X in exactly d distinct points iff H does not contain any tangent line of X. We'll show that the set of hyperplanes containing a tangent to X is of dimension at most two in $(\mathbb{P}^3)^*$. Consider the incidence correspondence $I_X \subset \mathbb{P}^3 \times (\mathbb{P}^3)^*$ given by $\{(x, H) \mid x \in X, x \in H, T_x X \subset T_x H\}$. I claim this is closed: clearly $x \in X$ is closed, since $X \subset \mathbb{P}^3$ is closed; $x \in H$ is closed because it comes from evaluating the linear form determining H on x; and $T_x X \subset T_x H$ is closed because it comes from the condition that appending the equation for H to the Jacobian matrix of X does not change the rank, and this is expressible as the vanishing of determinants of certain minors. Next, the fiber of I_X over any point $x \in X$ is of dimension one: the hyperplanes in $(\mathbb{P}^3)^*$ in the fiber are given by the choice of a 2-plane in $T_x\mathbb{P}^3$ containing $T_x X$, and there's a $\mathbb{P}^1$ worth of choices here. Thus I_X is of dimension at most two, so it's projection to $(\mathbb{P}^3)^*$ is a proper closed subset.

Second, a hyperplane contains three collinear points on X iff it contains a multisecant to X. By the logic of the proof of theorem IV.3.10, X has only finitely many multisecants, and each contributes a $\mathbb{P}^1$ worth of points in $(\mathbb{P}^3)^*$ to avoid. Therefore the locus of hyperplanes to avoid is a proper closed subset of $(\mathbb{P}^3)^*$, and we're done.

Exercise IV.3.10. Generalize the statement that "'not every secant is a multisecant"' as follows. If X is a curve in $\mathbb{P}^n$, not contained in any $\mathbb{P}^{n-1}$, and if $\operatorname{char} k = 0$, show that for almost all choices of $n-1$ points $P_1, \ldots, P_{n-1}$ on X, the linear space L^{n-2} spanned by the P_i does not contain any further points of X.

Solution. This is known as the "'general position theorem"' in the literature. We'll use exercise IV.3.9 to prove this. Before we begin we will prove a lemma about the expected size of a fiber of a morphism.

Lemma. *Let $f : X \to Y$ be a dominant morphism of integral schemes so that $K(X)/K(Y)$ is a finite separable extension of degree n. Then there is a dense open set $U \subset Y$ so that $f : f^{-1}(U) \to U$ is finite and for each $y \in U$, $X \times_Y \operatorname{Spec} k(y) \cong \bigcup \operatorname{Spec} F_i$ for fields F_i which are all finite separable extensions of $k(y)$ and $\sum [F_i : k(y)] = n$.*

Proof. By exercise II.3.7, there is an open dense subset $U \subset Y$ so that $f^{-1}(U) \to U$ is finite, so we may assume that $f : X \to Y$ is finite. Restricting our attention to an affine open $\operatorname{Spec} R \subset Y$, we may assume that $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$ for A a finite R-algebra As $K(Y) \to K(X)$ is finite separable, we may apply the theorem of the primitive element to write $K(X) = K(Y)[T]/(p(T))$ with $p(T)$ monic of degree n. Now I claim that there is some $r \in R$ so that $A_r \cong R_r[T]/p(T)$ as R_r-algebras: this fact is true at the generic point and both sides are finitely-generated R-algebras, so by the logic of exercise II.5.7(a), we have the conclusion. This implies that for all $y \in D(r) \subset \operatorname{Spec} R$, we have that the coordinate algebra of X_y is an n-dimensional $k(y)$-algebra.

Next, I claim that f is etale on a dense open subset of X of the form $f^{-1}(U)$ for some $U \subset Y$. First, as $\Omega_{X/Y,\eta_X} = \Omega_{K(X)/K(Y)} = 0$, $\Omega_{X/Y} = 0$ on a dense open subset by exercise II.5.7(a); because f is finite dominant, the image of the set where this is not true must be of positive codimension in and thus a proper closed subset of Y. Similarly, as f is flat at the generic point of Y, exercise III.9.4 gives that f is generically flat and we may apply the same argument about f to see that the image of the set where f is not flat is a proper closed subset of Y. By exercise III.10.3, this gives that f is etale on $f^{-1}(U)$ as claimed.

Finally, by exercise III.10.4, we have our claim about X_y for all y in $U \cap D(r)$. ■

First we claim that we can find an $x \in X$ so that the projection $\pi_x : X \to \mathbb{P}^{r-1}$ is birational on to its image (this is the only place we use $\operatorname{char} k = 0$). By the proof of theorem IV.3.10, there's an open subset of $(x, y) \in X \times X$ so that $\overline{xy}$ is not a multisecant. Therefore we may pick a point $x \in X$ so that for all but finitely many $y \in X$, the line $\overline{xy}$ is not a multisecant, which implies that $\overline{xy} \cap X = \{x, y\}$ set theoretically. This means that π_x is (set-theoretically) one-to-one over each such y, and since π_x satisfies the assumptions of our lemma above, π_x must be birational on to its image.

Now let $U \subset (\mathbb{P}^r)^*$ be the nonempty open set of hyperplanes which do not contain any tangent to X (this is a nonempty open set by the same logic as the proof of exercise IV.3.9). Consider the incidence correspondence $I \subset X \times U$ consisting of pairs (p, H) where $p \in H \cap X$. This is irreducible of dimension r: it's cut out in $X \times U$ by the single equation plugging p in to H. Now consider I_0, the subset of pairs (p, H) so that $p \in H$ and p is a part of a linearly dependent set of points: I claim this is actually a closed subvariety. It's cut out by the condition that $p \in H$ and that there exists an $r \times r$ minor of the matrix of coordinates of the points in $H \cap X$ with determinant zero. Therefore the only way the statement of the exercise can be false is if $I_0 = I$, that is, *for every* point p on a plane H which intersects X transversely, p is part of a set of r points in $H \cap X$ which are linearly dependent.

Finally, pick a point $x \in X$ so that π_x is birational on to its image. Then by the previous paragraph, if the general position theorem fails for X, it must also fail for $\pi_x(X)$: given any H through x, the points $p_2, \cdots, p_r$ which are linearly dependent with x all lie on $\pi_x(H)$ to $\mathbb{P}^{r-1}$, which is a $\mathbb{P}^{r-2}$. But after doing this until we're considering $X \subset \mathbb{P}^3$, we find that we must have a counterexample to what we proved in exercise IV.3.9, and therefore the general position is in fact true.

(Attentive readers will note that we did not ensure that $\pi_x(X)$ was nonsingular - we don't have to. There are two adaptations of the proof of exercise IV.3.9 which are necessary. The first is that we should add the condition H does not pass through any singular point to our construction of the first incidence correspondence. The second is a little more involved: we note that the proof of theorem IV.3.10 which we referenced which contains the statement that X has a secant which is not a multisecant remains valid here, because the proof of proposition IV.3.8 does not require X to be nonsingular and the observation about secants with coplanar tangents from the proof of theorem IV.3.10 is local and does not depend on the smoothness of X.)

Exercise IV.3.11.

a. If X is a nonsingular variety of dimension r in $\mathbb{P}^n$, and if $n > 2r+1$, show that there is a point $O \notin X$, such that the projection from O induces a closed immersion of X into $\mathbb{P}^{n-1}$.

b. If X is the Veronese surface in $\mathbb{P}^5$, which is the 2-uple embedding of $\mathbb{P}^2$ (I, Ex. 2.13), show that each point of every secant line of X lies on infinitely many secant lines. Therefore, the secant variety of X has dimension 4, and so in this case there is a projection which gives a closed immersion of X into $\mathbb{P}^4$ (II, Ex. 7.7). (A theorem of Severi [1] states that the Veronese surface is the only surface in $\mathbb{P}^5$ for which there is a projection giving a closed immersion into $\mathbb{P}^4$. Usually one obtains a finite number of double points with transversal tangent planes.)

Solution.

a. Just as in the proof of proposition IV.3.4, it suffices to project from a point not on the tangent or secant variety of X. Locally, the tangent variety and the secant varieties are images of $X \times \mathbb{P}^r$ and $(X \times X \setminus \Delta) \times \mathbb{P}^1$, so they are of dimension at most $2r$ and $2r+1$, respectively. Therefore their union cannot be all of $\mathbb{P}^n$ and we may take any O in their complement.

b. Suppose $s \in \mathbb{P}^5$ is on the secant line $\overline{pq} \subset \mathbb{P}^5$ for $p, q \in X$. Then the line $\overline{pq} \in \mathbb{P}^2$ gets mapped to a smooth plane conic through $p, q \in \mathbb{P}^5$. But for any smooth plane conic $C \subset \mathbb{P}^2$ and any point $s \in \mathbb{P}^2$ which lies on a secant, s is on infinitely many secants of C: any line through s intersects C in two points counted with multiplicity by Bezout's theorem, and the fact that these are distinct at least once implies they are generically distinct. Therefore the closure of the image of the map $(X \times X \setminus \Delta) \times \mathbb{P}^1$ must have a positive-dimensional fiber over every point, so the image cannot be of dimension 5 by exercise II.3.22(c).

On the other hand, the secant variety contains the tangent variety: any tangent line to any point in X is in the closure of the secant variety by taking the limit of secant lines, just like in calculus. The tangent variety is the image of $X \times \mathbb{P}^2$, and it's not so difficult to see that the embedded tangent plane to any point only intersects X in one point. Therefore the tangent variety is of dimension 4 and the secant variety must also be of dimension 4. The final claim follows from the same logic as (a).

Exercise IV.3.12. For each value of $d = 2, 3, 4, 5$ and r satisfying $0 \leq r \leq \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

Solution. This is some small cases of the more general problem of determining whether for a given degree and collection of singularities, such a curve embedded in some projective variety exists. The problem is completely solved only for planar curves with nodes, where the solution was known classically due to Severi - for any $d \geq 0$ and number of nodes $r \leq \frac{1}{2}(d-1)(d-2)$, there exists an irreducible curve with that many nodes and no other singularities. The idea is that one may smooth out nodes independently of each other, so starting with a projection of the rational normal curve of degree d (which has $\frac{1}{2}(d-1)(d-2)$ nodes by a combination of theorem IV.3.10 and the argument of exercise IV.3.5(b)) we may smooth out any number of nodes. Instead of proving the full result of Severi, which requires a little more deformation theory than we've developed so far, we'll just demonstrate equations for curves of with specified d and r.

Our approach for $r \leq 1$ can be done easily by hand. When $r = 0$, the sum of d^{th} powers works by exercise I.5.5. When $d > 2$ and $r = 1$, the equation $xyz^{d-2} + x^d + y^d$ works: by looking at the final entry of the Jacobian

$$\begin{pmatrix} yz^{d-2} + dx^{d-1} & xz^{d-2} + dy^{d-1} & (d-2)xyz^{d-3} \end{pmatrix}$$

at least one of x, y, or z must be zero. If $x = 0$ we find that the $xz^{d-2} + dy^{d-1}$ entry of the Jacobian becomes dy^{d-1}, forcing $y = 0$ and giving the singular point $[0:0:1]$. $y = 0$ gives the same result by symmetry. If $z = 0$, then the first two entries of the Jacobian become dx^{d-1} and dy^{d-1}, so there are no singular points with $z = 0$. At $[0:0:1]$, the equation of our curve is $xy + x^d + y^d$ which is clearly a node. It's also easy to see that such a variety is irreducible: if it were reducible, it must have at least $d-1$ singular points counted with multiplicity coming from the intersection of two components of positive degree, but here there is exactly one singular point which is a node and counts once.

Our general method for $r > 1$ will be to consider curves of the form $aF + bG$ for $a, b \in k$ where $V(F)$ is d lines crossing with only nodes (up to $\frac{1}{2}(d)(d-1)$ of them) and G vanishes to order at least two at r of the nodes on $V(F)$: generically, we should end up with an irreducible curve with r nodes, so all we have to do is pick a, b which work. To check that our curves are indeed curves and have no other singularities outside of the $r \leq 1$ case, we recommend using a computer algebra system to compute the primary decomposition of (f) and (f, f_x, f_y, f_z) - doing these by hand gets very cumbersome.

d	r	Equation
2	0	$x^2 + y^2 + z^2$
3	0	$x^3 + y^3 + z^3$
3	1	$xyz + x^3 + y^3$
4	0	$x^4 + y^4 + z^4$
4	1	$xyz^2 + x^4 + y^4$
4	2	$y(y-z)(x+y+z)(x-y-z) - (yz+x^2-z^2)^2$
4	3	$y(y-z)(x+y+z)(x-y-z) - (yz-x^2+z^2)^2$
5	0	$x^5 + y^5 + z^5$
5	1	$xyz^3 + x^5 + y^5$
5	2	$y(y-z)(y-2z)(x+y+z)(x-y-z) + (yz+x^2-z^2)^2 z$
5	3	$y(y-z)(y-2z)(x+y+z)(x-y-z) + (yz-x^2+z^2)^2 z$
5	4	$y(y-z)(y-4z)(y-3x+2z)(y+3x+2z) + 4(yz-x^2)^2 z$
5	5	$6x(x-z)(x+z)(y-z)(y+z) + (x^2+y^2-2z^2)^2 z$
5	6	$(y-z)(y-2x)(y+2x)(y-3x+2z)(y+3x+2z) - (yz-x^2)^2 z$

Here's an example of the calculations one would need to make in a computer algebra system to check that the curve given by $f = y(y-z)(x+y+z)(x-y-z) - (yz-x^2+z^2)^2$ for $(d,r) = (4,3)$ is irreducible and singular at the prescribed points using Singular, which has a web interface at `https://www.singular.uni-kl.de/` :

```
> LIB "primdec.lib";
> ring r = 0,(x,y,z),dp;
> poly f = y*(y-z)*(x+y+z)*(x-y-z)-(y*z-x^2+z^2)^2;
> poly fx = 2*x*(-2*x^2+y^2+y*z+2*z^2);
> poly fy = x^2*(2*y+z)-4*y^3-3*y^2*z-z^3;
> poly fz = x^2*(y+4*z)-y^3-3*y*z^2-4*z^3;
> ideal i = f;
> ideal j = f,fx,fy,fz;
> primdecGTZ(i);
[1]:
   [1]:
  _[1]=x4-x2y2+y4-x2yz+y3z-2x2z2+yz3+z4
 [2]:
    _[1]=x4-x2y2+y4-x2yz+y3z-2x2z2+yz3+z4
> primdecGTZ(j);
[1]:
   [1]:
  _[1]=y
_[2]=x-z
 [2]:
    _[1]=y
_[2]=x-z
```

```
[2]:
   [1]:
  _[1]=y+z
_[2]=x
 [2]:
    _[1]=y+z
_[2]=x
[3]:
   [1]:
  _[1]=y
_[2]=x+z
 [2]:
    _[1]=y
_[2]=x+z
[4]:
   [1]:
  _[1]=z3
_[2]=2y3z2+3y2z3-6yz4
_[3]=2y4+18y3z+6y2z2-3yz3
_[4]=33xy2z-5xyz2+46xz3
_[5]=4xy3+13xyz-5xyz2+14xz3
_[6]=2y3+7x2z+3y2z-6yz2
_[7]=2x2y+15x2z+3y2z-12yz2
_[8]=2x3-xy2-xyz-2xz2
 [2]:
    _[1]=x
_[2]=y
_[3]=z
```

What's happening here is that we're 1) loading the library which deals with prime decomposition, 2) declaring a polynomial ring of characteristic zero in three indetereminants `x,y,z` with a specified lexicographical order (don't worry about this final substep if you don't know what it means), 3) inputting our equation for our curve as `f` and its partial derivatives as `fx,fy,fz`, 4) declaring the ideals `i` and `j` as generated by `f` and `f,fx,fy,fz`, respectively, and 5) computing the primary decompositions of these ideals. The results tell us that (f) is prime so it cuts out a curve, and that $V(f, f_x, f_y, f_z)$ is exactly the three points $\{[1:0:1], [0:-1:1], [-1:0:1]\}$.

To check that these are indeed nodes, we compute the partial derivatives of the dehomogenization of f in the chart $z = 1$ and evaluate. We'll check $(1, 0)$ as an example.

- $\frac{\partial f(x,y,1)}{\partial x}(1, 0) = 0$
- $\frac{\partial f(x,y,1)}{\partial y}(1, 0) = 0$
- $\frac{\partial^2 f(x,y,1)}{\partial x^2}(1, 0) = -8$

- $\frac{\partial^2 f(x,y,1)}{\partial x \partial y}(1,0) = 2$
- $\frac{\partial^2 f(x,y,1)}{\partial y^2}(1,0) = 2$

Therefore the $f(x, y, 1)$ has multiplicity two at $(1, 0)$ and the degree-two part expands as $-4(x-1)^2 + 2(x-1)y + y^2 = (-2(x-1) + \frac{1+\sqrt{5}}{2}y)(2(x-1) + \frac{-1+\sqrt{5}}{2}y)$, so it is indeed a node. (We can also verify this by graphing the curve and noticing two distinct tangent directions.)

IV.4 Elliptic Curves

Elliptic curves are a topic that you can write whole books about - for instance, Silverman's *The Arithmetic of Elliptic Curves* is about the same length as Hartshorne and spends the whole book talking about them, instead of a section and scattered examples. If this section sparks your interest, you might want to spend some time with a more comprehensive reference.

Recall that we're omitting $\operatorname{char} k = 2$ in this section.

Exercise IV.4.1. Let X be an elliptic curve over k, with $\operatorname{char} k \neq 2$, let $P \in X$ be a point, and let R be the graded ring $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nP))$. Show that for suitable choice of t, x, y,

$$R \cong k[t, x, y]/(y^2 - x(x - t^2)(x - \lambda t^2)),$$

as a graded ring, where $k[t, x, y]$ is graded by setting $\deg t = 1$, $\deg x = 2$, $\deg y = 3$.

Solution. Let X be an elliptic curve over k, with $\operatorname{char} k \neq 2$, let $P \in X$ be a point, and let R be the graded ring $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nP))$. Show that for suitable choice of t, x, y,

$$R \cong k[t, x, y]/(y^2 - x(x - t^2)(x - \lambda t^2)),$$

as a graded ring, where $k[t, x, y]$ is graded by setting $\deg t = 1$, $\deg x = 2$, $\deg y = 3$.

By proposition IV.4.6, we may assume X is embedded in $\mathbb{P}^2$ as the curve $y^2 = x(x-1)(x-\lambda)$ with $P \mapsto O$, the point at infinity. Consider the graded ring homomorphism $\varphi : k[t, x, y] \to R$ defined by $t \mapsto 1 \in H^0(X, \mathcal{O}_X(P))$ and mapping x, y as in proposition IV.4.6. An arbitrary element of R is a rational function on X which has poles only at P, and therefore may be considered as the restriction of a polynomial function $p(x, y)$ on $\mathbb{A}^2$ to $X \cap \mathbb{A}^2$. But such a polynomial function is in the image of φ because $1, x, y$ are, so φ is surjective.

Next, as in the proof of proposition IV.4.6, $y^2 = x(x-1)(x-\lambda)$ in R, so $(y^2 - x(x-t^2)(x-\lambda t^2)) \in \ker \varphi$, and so φ factors through $k[t, x, y]/(y^2 - x(x - t^2)(x - \lambda t^2))$. Now we count dimensions: by Riemann-Roch, $\dim_k R_i = i$ for all $i > 0$ and $\dim_k R_0 = 1$. To compute the dimension of graded pieces of $k[t, x, y]/(y^2 - x(x-t^2)(x-\lambda t^2))$, we observe that $\dim_k k[t, x, y]_i$ is equal to the coefficient of s^i in the series expansion of $\frac{1}{(1-s)(1-s^2)(1-s^3)}$ because the different monomial basis elements of degree i correspond to solutions to $a+2b+3c = i$. Therefore the dimension of the i^{th} graded piece of $k[t, x, y]/(y^2-x(x-t^2)(x-\lambda t^2))$ is the coefficient of s^i in $\frac{1}{(1-s)(1-s^2)(1-s^3)} - \frac{x^6}{(1-s)(1-s^2)(1-s^3)} = \frac{s^2-s+1}{(1-s)^2}$. But this is just $1 + \frac{s}{(1-s)^2} = 1 + \frac{d}{ds}\frac{1}{1-s}$, which expands as the power series $1 + s + 2s^2 + 3s^3 + \cdots$. So the dimensions of the graded pieces of R and $k[t, x, y]/(y^2 - x(x-t^2)(x-\lambda t^2))$ agree, and therefore our surjective map must also be injective and thus an isomorphism.

Exercise IV.4.2. If D is any divisor of degree ≥ 3 on the elliptic curve X, and if we embed X in $\mathbb{P}^n$ by the complete linear system $|D|$, show that the image of X in $\mathbb{P}^n$ is projectively normal.

Note. It is true more generally that if D is a divisor of degree $\geq 2g+1$ on a curve of genus g, then the embedding of X by $|D|$ is projectively normal (Mumford [4, p. 55]).

Solution. We'll prove the more general version, following Mumford's argument. For any line bundles $\mathcal{L}, \mathcal{M}$ on a scheme X, we can define a map $\alpha : H^0(X, \mathcal{L}) \otimes_{\mathcal{O}_X(X)} H^0(X, \mathcal{M}) \to H^0(X, \mathcal{L} \otimes \mathcal{M})$

by multiplying local sections of $\mathcal{L}$ and $\mathcal{M}$ on open sets where $\mathcal{L}$ and $\mathcal{M}$ are trivial and then gluing. Analyzing this map α will be the main piece of how we solve this problem.

Consider the following commutative diagram, where X is embedded in $\mathbb{P}^n$ by D so that $\mathcal{O}_X(D) = \mathcal{O}_X(1)$:

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r+1)) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \mathcal{O}_X(r)) & \longrightarrow & H^0(X, \mathcal{O}_X(r+1)) \end{array}$$

First, $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(X, \mathcal{O}_X(r))$ is surjective for $r = 0$ (both sides are just k) and $r = 1$ ($|D|$ is a complete linear system). So if we can show that the bottom map is surjective for $r = 1$, then we may conclude that the right map must also be surjective, showing that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) \to H^0(X, \mathcal{O}_X(2))$ is surjective. Proceeding by induction, if we can show that the bottom map is surjective for all $r \geq 1$, then this will show that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(X, \mathcal{O}_X(r))$ is surjective for all $r \geq 0$, which is enough to show that $X \subset \mathbb{P}^n$ is projectively normal by the criteria of exercise II.5.14(d).

We'll prove a slightly more general claim:

Theorem. *Let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on a curve X of genus g with $d = \deg \mathcal{L} \geq 2g + 1$ and $\deg \mathcal{M} \geq 2g$. Then $\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{L} \otimes \mathcal{M})$ is surjective.*

We will need one preliminary result first:

Lemma (Generalized Castelnuovo Lemma). *Suppose $\mathcal{L}$ is an ample, base-point free invertible sheaf on a projective variety X over an algebraically closed field. Let $\mathcal{F}$ be a coherent sheaf on X so that $H^i(\mathcal{F} \otimes \mathcal{L}^{-i}) = 0$ for all $i > 0$. Then $H^i(\mathcal{F} \otimes \mathcal{L}^j) = 0$ when $i + j \geq 0$ and $i > 0$, and $H^0(\mathcal{F} \otimes \mathcal{L}^i) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L}^{i+1})$ is surjective for all $i \geq 0$.*

Proof. We induct on $\dim \operatorname{Supp} \mathcal{F}$. If $\dim = 0$, then $H^i(\mathcal{F} \otimes \mathcal{L}^j) = 0$ for any $i > 0$ by Grothendieck vanishing. To attack the other claim, we can find a global section s of $\mathcal{L}$ which is non-vanishing on all the points of $\operatorname{Supp} \mathcal{F}$: take the map $\varphi_{\mathcal{L}} : X \to \mathbb{P}^l$ which has $\varphi_{\mathcal{L}}^* \mathcal{O}(1) \cong \mathcal{L}$ and pick a hyperplane section which misses all the finitely many points of $\varphi_{\mathcal{L}}(\operatorname{Supp} \mathcal{F})$ (this is where we use the fact that k is at least infinite). Then multiplication by this section is an isomorphism in each stalk of $\mathcal{F}$, so $H^0(\mathcal{F} \otimes \mathcal{L}^i) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L}^{i+1})$ is surjective for any $i > 0$.

Now suppose we've proven the claim for all coherent sheaves $\mathcal{F}'$ with $\dim \operatorname{Supp} \mathcal{F}' < \dim \operatorname{Supp} \mathcal{F}$. For any $x \in X$, we may find a global section s of $\mathcal{L}$ which does not vanish identically on any component of $\operatorname{Supp} \mathcal{F}$: using the same trick as in the previous paragraph, pick a closed point in each component of $\operatorname{Supp} \mathcal{F}$, embed X in to $\mathbb{P}^l$ by $\varphi_{\mathcal{L}}$, and choose a hyperplane section which misses all the images of the finitely many points we chose. Consider the injective map $f \mapsto f \otimes s$ from $\mathcal{F} \otimes \mathcal{L}^{-1} \to \mathcal{F}$, and let $\mathcal{G}$ be the cokernel so that for any i, the sequence

$$0 \to \mathcal{F} \otimes \mathcal{L}^{-i-1} \to \mathcal{F} \otimes \mathcal{L}^{-i} \to \mathcal{G} \otimes \mathcal{L}^{-i} \to 0$$

is exact. Then $\dim \operatorname{Supp} \mathcal{G} < \dim \operatorname{Supp} \mathcal{F}$, and the long exact sequence on homology associated to the above exact sequence gives that

$$H^i(\mathcal{F} \otimes \mathcal{L}^{-i}) \to H^i(\mathcal{G} \otimes \mathcal{L}^{-i}) \to H^{i+1}(\mathcal{F} \otimes \mathcal{L}^{-i-1})$$

is exact. Since the outer two terms are zero by assumption, $\mathcal{G}$ satisfies the assumptions of our lemma and therefore also the conclusions by the inductive hypothesis. On the other hand, the long exact sequence in cohomology associated to $0 \to \mathcal{F} \otimes \mathcal{L}^{-i} \to \mathcal{F} \otimes \mathcal{L}^{-i+1} \to \mathcal{G} \otimes \mathcal{L}^{-i+1} \to 0$ gives that

$$H^i(\mathcal{F} \otimes \mathcal{L}^{-i}) \to H^i(\mathcal{F} \otimes \mathcal{L}^{-i+1}) \to H^i(\mathcal{G} \otimes \mathcal{L}^{-i+1})$$

is exact; the first term vanishes by hypothesis and the third term vanishes by the theorem for $\mathcal{G}$, so the middle term must vanish as well and by induction we've proven the first statement.

To show surjectivity of $H^0(\mathcal{F} \otimes \mathcal{L}^i) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L}^{i+1})$ for all $i \geq 0$, consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{F} \otimes \mathcal{L}^{-1}) \otimes H^0(\mathcal{L}) \arrow[r] \arrow[d, "\alpha"] & H^0(\mathcal{F}) \otimes H^0(\mathcal{L}) \arrow[r] \arrow[d, "\beta"] & H^0(\mathcal{G}) \otimes H^0(\mathcal{L}) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0(\mathcal{F}) \arrow[r] \arrow[ur, dotted, "\delta"'] & H^0(\mathcal{F} \otimes \mathcal{L}) \arrow[r] & H^0(\mathcal{G} \otimes \mathcal{L}) &
\end{tikzcd}$$

By the snake lemma, we get an exact sequence $\operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma$. Our inductive assumption gives that γ is surjective, while the existence of the diagonal map δ shows that the map $\operatorname{coker} \alpha \to \operatorname{coker} \beta$ is zero: any element missed by the image of α is sent by the horizontal map to an element in the image of β. So $\operatorname{coker} \beta$ is zero, or $H^0(\mathcal{F} \otimes \mathcal{L}^i) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L}^{i+1})$ when $i = 0$. Since $\mathcal{F} \otimes \mathcal{L}$ satisfies the same hypotheses, we may replace $\mathcal{F}$ by $\mathcal{F} \otimes \mathcal{L}$ and then conclude our statement for all values of i by induction.

■

Now we prove the main result.

Proof. For the duration of this proof, every cohomology group will be computed on X, so we write $H^i(X, \mathcal{F}) = H^i(\mathcal{F})$.

Let E be an effective divisor of degree $d - g - 1$ to be chosen later. Let $\mathcal{G}$ be the cokernel of the natural map $\mathcal{L}(-E) \to \mathcal{L}$ so that

$$0 \to \mathcal{L}(-E) \to \mathcal{L} \to \mathcal{G} \to 0$$

is exact, where $\mathcal{G}$ is supported on $\operatorname{Supp} E$. If $H^1(\mathcal{L}(-E)) = 0$, then the sequence $0 \to H^0(\mathcal{L}(-E)) \to H^0(\mathcal{L}) \to H^0(\mathcal{G}) \to 0$ is exact, and remains exact after tensoring with $H^0(\mathcal{M})$, so we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{L}(-E)) \otimes H^0(\mathcal{M}) \arrow[r] \arrow[d, "\alpha"] & H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \arrow[r] \arrow[d, "\beta"] & H^0(\mathcal{G}) \otimes H^0(\mathcal{M}) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0(\mathcal{L}(-E) \otimes \mathcal{M}) \arrow[r] & H^0(\mathcal{L} \otimes \mathcal{M}) \arrow[r] & H^0(\mathcal{G} \otimes \mathcal{M}) &
\end{tikzcd}$$

and by the snake lemma, an exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma.$$

I claim that $\operatorname{coker} \gamma = 0$. Since $\mathcal{M}$ is base-point free by corollary IV.3.2, we may find a global section of $\mathcal{M}$ which does not vanish on any point in $\operatorname{Supp} \mathcal{G}$: take the morphism $\phi_{\mathcal{M}}$ to projective space induced by $H^0(\mathcal{M})$ with $\phi_{\mathcal{M}}^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \mathcal{M}$, and then choose any hyperplane section which misses the finitely many points of $\phi_{\mathcal{M}}(\operatorname{Supp} \mathcal{G})$. Therefore multiplication by this section is an isomorphism on all the stalks of $\mathcal{G}$, so γ is surjective.

Next, by Riemann-Roch,

$$h^0(\mathcal{L}(-E)) = \deg(\mathcal{L}(-E)) - (g-1) + h^1(\mathcal{L}(-E)) = d - (d-g-1) - (g-1) + 0 = 2.$$

Our second requirement on E is that $\mathcal{L}(-E)$ should be base-point free, and we want to apply the lemma to see that $H^0(\mathcal{L}(-E)) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L}(-E) \otimes \mathcal{M})$ is surjective. The final requirement is that we'll need $H^1(\mathcal{M} \otimes \mathcal{L}(-E)^{-1}) = H^1(\mathcal{M} \otimes \mathcal{L}^{-1}(E)) = 0$ to deduce that, after which we will have the necessary information to conclude that $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$ is surjective as claimed.

It remains to check that we can find an appropriate E. I claim that it suffices to prove that we can find E so that $h^0(\mathcal{L}(-E-x)) = 1$ for all $x \in X$ in order that $h^1(\mathcal{L}(-E)) = 0$ and $\mathcal{L}(-E)$ is base-point free. As $h^0(\mathcal{L}(-E)) - h^1(\mathcal{L}(-E)) = 2$ by our previous calculation, we must have $h^0(\mathcal{L}(-E)) \geq 2$. Because $h^0(\mathcal{L}(-E)) \leq h^0(\mathcal{L}(-E-x)) + 1$ by propositon IV.3.1, with equality iff $\mathcal{L}(-E)$ is base point free, our new condition implies both that $h^1(\mathcal{L}(-E)) = 0$ and $\mathcal{L}(-E)$ is base-point free. So we need to check that we can find E so that $h^0(\mathcal{L}(-E-x)) = 1$ for all $x \in X$ and $h^1(\mathcal{M} \otimes \mathcal{L}^{-1}(E)) = 0$.

To check that we can find E so that $h^0(\mathcal{L}(-E-x)) = 1$ for all $x \in X$, note that by Riemann-Roch and Serre Duality, $h^0(\mathcal{L}(-E-x)) > 1$ iff $h^1(\mathcal{L}(-E-x)) > 0$ iff $h^0(\omega_X \otimes \mathcal{L}^{-1}(E+x)) > 0$. This last inequality occurs iff there's an effective divisor of degree $2g-2-d+(d-g-1)+1 = g-2$ which is linearly equivalent to $\omega_x \otimes \mathcal{L}^{-1}(E+x)$. But there are only $g-1$ points here that we have control over, so the locus of E which don't satisfy our condition is of dimension at most $g-1$.

Checking that we may choose E so that $h^1(\mathcal{M} \otimes \mathcal{L}^{-1}(E)) = 0$ is similar. Suppose $\deg \mathcal{M} = e+2g$. By Serre duality, $h^1(\mathcal{M} \otimes \mathcal{L}^{-1}(E)) > 0$ iff $h^0(\omega_X \otimes \mathcal{M}^{-1} \otimes \mathcal{L}(-E)) > 0$, which happens iff there's an effective divisor of degree $2g-2-(2g+e)+d-(d-g-1) = g-1-e$ linearly equivalent to $\omega_X \otimes \mathcal{M}^{-1} \otimes \mathcal{L}(-E)$. As in the previous paragraph, there are at most $g-1$ points here that we have control over, so the locus of E which don't satisfy our condition is of dimension at most $g-1$. Since the locus of effective divisors of degree $d-g-1$ is of dimension at least g, we may in fact find an appropriate E, and we're done.

■

Exercise IV.4.3. Let the elliptic curve X be embedded in $\mathbb{P}^2$ so as to have equation $y^2 = x(x-1)(x-\lambda)$. Show that any automorphism of X leaving $P_0 = (0,1,0)$ fixed is induced by an automorphism of $\mathbb{P}^2$ coming from the automorphism of the affine (x,y)-plane given by

$$\begin{cases} x' = ax + b \\ y' = cy. \end{cases}$$

In each of the four cases of (4.7), describe these automorphisms of $\mathbb{P}^2$ explicitly, and hence determine the structure of the group $G = \mathrm{Aut}(X, P_0)$.

Solution. Any automorphism of X leaving P_0 fixed induces an automorphism of $H^0(X, \mathcal{O}_X(nP_0))$ for any $n \in \mathbb{Z}$, and therefore taking $n = 3$ we see that such an automorphism comes from an automorphism of $\mathbb{P}^2$ preserving X and fixing P_0. The automorphisms of $\mathbb{P}^2$ fixing $P_0 = [0 : 1 : 0]$ are linear automorphisms of the form $x \mapsto ax+b$ and $y \mapsto dx+cy+e$. As such a linear automorphism of $\mathbb{P}^2$ will preserve the group structure, we see that such an automorphism must permute the non-identity 2-torsion points $(0, 0)$, $(1, 0)$, and $(\lambda, 0)$. This gives that $d = e = 0$, and therefore our automorphisms must be of the form $x \mapsto ax + b$ and $y \mapsto cy$ as claimed.

To attack the second portion of the problem, we continue with our investigation of the 2-torsion points. Focusing only on the $x \mapsto ax + b$ portion of our automorphisms, we see that we have six candidates to investigate. We start by picking the value of b from $\{0, 1, \lambda\}$, which determines where 0 maps, then choose a based on where we send 1 and check what happens with λ.

- $b = 0$, $a = 1$: This is the identity map.
- $b = 0$, $a = \lambda$: This sends $1 \mapsto \lambda$ and preserves 0, so it must map λ to 1. This occurs iff $\lambda^2 = 1$, which means $\lambda = -1$, giving $j = 1728$.
- $b = 1$, $a = -1$: This sends $0 \mapsto 1$ and $1 \mapsto 0$, so it must preserve λ. This occurs iff $\lambda = 1 - \lambda$, which means $\lambda = \frac{1}{2}$, giving $j = 1728$.
- $b = 1$, $a = \lambda - 1$: This sends $0 \mapsto 1$ and $1 \mapsto \lambda$, so it must map λ to 0. This occurs iff $\lambda^2 - \lambda + 1 = 0$, giving $j = 0$.
- $b = \lambda$, $a = -\lambda$: This sends $0 \mapsto \lambda$ and $1 \mapsto 0$, so it must send λ to 1. This occurs iff $\lambda^2 - \lambda + 1 = 0$, giving $j = 0$.
- $b = \lambda$, $a = 1 - \lambda$: This sends $0 \mapsto \lambda$ and preserves 1, so it must map λ to 0. This occurs iff $-\lambda^2 + \lambda - 1 = 0$, which means $\lambda = 2$, giving $j = 1728$.

Any automorphism on this list can show up with exactly one of $y \mapsto y$ or $y \mapsto -y$, in order to preserve the ideal of our curve, so the automorphism group G is a subgroup of $\mathbb{Z}/2 \times S_3$. This gives us enough information to determine the group structure in all four cases:

- $j \neq 0, 1728$: Only the identity and $y \mapsto -y$ are valid automorphisms, giving $G = \mathbb{Z}/2$.
- $j = 1728$ and $\operatorname{char} k \neq 3$: $\lambda \in \{2, -1, \frac{1}{2}\}$, and our automorphism group is generated by $y \mapsto -y$ and the unique non-identity automorphism from our list above matching the value of λ. This gives $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.
- $j = 0$ and $\operatorname{char} k \neq 3$: In this case, the only valid automorphisms from the list above are the identity and the two cyclic permuations, giving $G = \mathbb{Z}/2 \times \mathbb{Z}/3$.
- $j = 0$ and $\operatorname{char} k = 3$: In this case, all of the above automorphisms are valid, giving $G = \mathbb{Z}/2 \times S_3$, or D_{12}, the dihedral group of 12 elements.

(If you're wondering what the deal is with these special values of j, they correspond to extra symmetry of the set $\{0, 1, \lambda\}$ - over $\mathbb{C}$, for instance, $j = 0$ corresponds to the set forming an equilateral triangle, while $j = 1728$ corresponds to one point being the midpoint of the other two.)

Exercise IV.4.4. Let X be an elliptic curve in $\mathbb{P}^2$ given by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Show that the j-invariant is a rational function of the a_i, with coefficients in $\mathbb{Q}$. In particular, if the a_i are all in some field $k_0 \subset k$, then $j \in k_0$ also. Furthermore, for every $\alpha \in k_0$, there exists an elliptic curve defined over k_0 with j-invariant equal to α.

Solution. Since we're working with $\operatorname{char} k \neq 2$ in this section, we can make the substitution $y \mapsto \frac{1}{2}(y - a_1x - a_3)$ to turn our equation in to

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6,$$

where $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$, and $b_6 = a_3^2 + 4a_6$. Define

$$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,$$

$$c_4 = b_2^2 - 24b_4,$$

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6,$$

and after some tedious algebra one finds that $j = c_4^3/\Delta$. This shows that j is a rational function of the a_i with coefficients in $\mathbb{Q}$.

To check that we can find a curve defined over k_0 with $j = \alpha$, we split in to cases: $j = 0$, $j = 1728$, and all other options. In the first case, $y^2 + y = x^3$ has $j = 0$ and the benefit of being nonsingular in every characteristic; in the second case, $y^2 = x^3 + x$ has $j = 1728$ and is also nonsingular in every characteristic except 2; and finally, $y^2 + xy = x^3 - \frac{36x}{\alpha - 1728} - \frac{1}{\alpha - 1728}$ suffices when $\alpha \neq 0, 1728$.

Exercise IV.4.5. Let X, P_0 be an elliptic curve having an endomorphism $f : X \to X$ of degree 2.

a. If we represent X as a $2 - 1$ covering of $\mathbb{P}^1$ by a morphism $\pi : X \to \mathbb{P}^1$ ramified at P_0, then as in (4.4), show that there is another morphism $\pi' : X \to \mathbb{P}^1$ and a morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$, also of degree 2, such that $\pi \circ f = g \circ \pi'$.

b. For suitable choices of coordinates in the two copies of $\mathbb{P}^1$, show that g can be taken to be the morphism $x \to x^2$.

c. Now show that g is branched over two of the branch points of π, and that g^{-1} of the other two branch points of π consists of the four branch points of π'. Deduce a relation involving the invariant λ of X.

d. Solving the above, show that there are just three values of j corresponding to elliptic cures with endomorphism of degree 2, and find the corresponding values of λ and j. [*Answers*: $j = 2^6 \cdot 3^3$; $j = 2^6 \cdot 5^3$; $j = -3^3 \cdot 5^3$.]

Solution.

a. As $\operatorname{char} k \neq 2$, f is separable, so by Riemann-Hurwitz we find that f is unramified and the preimage of any point is two distinct points. By lemma IV.4.9, f is an endomorphism of (X, P_0) as an abelian group. Combining these observations, we have that $f^{-1}(P_0) = \ker f$ is two distinct points $\{P_0, P_1\}$ which means that P_1 is 2-torsion. We see that the other two 2-torsion points P_2 and P_3 of X must map to the same place under f, since they differ by P_1. Since they can't map to P_0, they have to both map to one of P_1, P_2, or P_3. Up to swapping the labels of P_2 and P_3, we may assume that either $f(P_2) = P_1$ or $f(P_2) = P_2$.

   Now we'll give an explicit equations for π' and g and show that they make the required diagram commute. Our main motivation will be the fact that the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ along the composite $\pi \circ f$ is $\mathcal{O}_X(2P_0 + 2P_1)$, so if g pulls back $\mathcal{O}_{\mathbb{P}^1}(1)$ to $\mathcal{O}_{\mathbb{P}^1}(2)$ (as any degree-two map of $\mathbb{P}^1$ does) then π' should come from the line bundle $\mathcal{O}_X(P_0 + P_1)$. In order to give equations, it will be helpful to think of X as embedded in $\mathbb{P}^2$ with (affine) equation $y^2 = x(x-1)(x-\lambda)$, with $P_0 = [0:1:0]$, $P_1 = [0:0:1]$, $P_2 = [1:0:1]$, and $P_3 = [\lambda : 0 : 1]$.

   Case 1: $f(P_2) = P_1$. Take π to be the map given by $[x, y, z] \mapsto [x : z]$, take g to be the map given by $[x : y] \mapsto [x^2 : y^2]$, and take π' to be the map given by $[x : y : z] \mapsto [dy : x]$ for some $d \in k$ to be chosen later.

   Case 2: $f(P_2) = P_2$. Take π to be the map given by $[x, y, z] \mapsto [x - z : z]$, take g to be the map given by $[x : y] \mapsto [x^2 : y^2]$, and take π' to be the map given by $[x : y : z] \mapsto [dy : x]$ for some $d \in k$ to be chosen later.

   To check that these maps work, we first list how various lines in $\mathbb{P}^2$ intersect X:

   - $V(x)$ has a double intersection with X at P_1 and a single intersection at P_0;
   - $V(y)$ has a single intersection with X at each of P_1, P_2, and P_3;
   - $V(z)$ has a triple intersection with X at P_0;
   - $V(x - z)$ has a double intersection with X at P_2 and a single intersection at P_0.

   This gives us the following information about rational functions on X:

   - $\frac{x}{z}$ has a double zero at P_1, a double pole at P_0, and is regular elsewhere;
   - $\frac{x-z}{z}$ has a double zero at P_2, a double pole at P_0, and is regular elsewhere;
   - $\frac{y}{x}$ has a single zero at P_2 and P_3, a single pole at P_0 and P_1, and is regular elsewhere.

   In either case, this gives that $\pi \circ f$ and $g \circ \pi'$ have the same zeroes and poles on X: they have zeroes of order two at P_2 and P_3, poles of order two at P_0 and P_1, and are regular elsewhere. Therefore their quotient is a rational function with no zeroes or poles on X. But the only such functions are the nonzero constants, so we may choose a value for d so that $\pi \circ f = g \circ \pi'$.

b. We already did this in our construction above, but it is worth pointing out that any separable degree-two map $g : \mathbb{P}^1 \to \mathbb{P}^1$ can be put in to the form $x \mapsto x^2$ up to coordinate changes on the source and the target. By Riemann-Hurwitz, g is ramified at two points, say P_1, P_2 with preimages Q_1, Q_2 respectively. Choose coordinates on both copies of $\mathbb{P}^1$ so that P_1 and Q_1 are both 0 and P_2 and Q_2 are both ∞. Now we can deduce several things about our map g: writing it in coordinates, it's of the form $[X : Y] \mapsto [aX^2 + bXY + cY^2 : dX^2 + eXY + fY^2]$, and $0 \mapsto 0$ plus $\infty \mapsto \infty$ implies $c = d = 0$. The fact that g is ramified above 0 gives that $b = 0$ as well, since $g^{-1}([0 : 1]) = \{[0 : 1], [-b : a]\}$; similar logic looking at ∞ gives that $e = 0$, so our map is really of the form $[aX^2 : fY^2]$ for $a, f \neq 0$. Scaling appropriately, we may arrange that $1 \mapsto 1$ and therefore our map is of the form $x \mapsto x^2$.

c. (A note on terminology: branch points are the images on the target of the points where a morphism is ramified.) As g was chosen to be $x \mapsto x^2$, it is ramified and branched at 0 and ∞. By example IV.4.8.2, the ramification points of the map induced by $|2P_0|$ are the four 2-torsion points of X: P_0, P_1, P_2, and P_3. With either definition of π, their images under π include 0 and ∞, so the first half of the claim is proven.

   To check that g^{-1} of the other two branch points are branch points of π', we note that the preimage under π' of these points can only be one point by comparing with the preimages of these points under $\pi \circ f$. There are precisely four of them, so by Riemann-Hurwitz that's all of the ramified and branch points.

   In order to obtain a relation involving λ, we'll exploit the fact that we have an alternate description of the ramification points of π': since π' is given (up to an automorphism of the target $\mathbb{P}^1$) by projection from the origin P_1 to the line at infinity, the ramification locus consists of the points $Q_i \in X$ so that the line P_1Q_i is tangent to Q_i. To find these, first intersect $y = tx$ with $y^2 = x(x-1)(x-\lambda)$ to see that the two non-zero x coordinates of intersection both solve $t^2x = (x-1)(x-\lambda)$, or $x^2 - (1+\lambda+t^2)x + \lambda = 0$. We're interested in when this has only one solution, which happens exactly when the discriminant vanishes, or $(1+\lambda+t^2)^2 - 4\lambda = 0$. This happens when $t^2 = \pm 2\sqrt{\lambda} - 1 - \lambda = -(1 \pm \sqrt{\lambda})^2$.

   In case 1, when π is $[x : y : z] \mapsto [x : y]$, the branch points of π are 0, 1, λ, and ∞; the branch points 0 and ∞ are also branch points of g, so we should have $d^2(-(1 \pm \sqrt{\lambda})^2)$ be either 1 or λ depending on the $\pm$. Taking the quotient, we find that $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2}$ is equal to either λ or λ^{-1}.

   In case 2, when π is $[x : y : z] \mapsto [x - z : y : z]$, the branch points of π are -1, 0, $\lambda - 1$ and ∞; the branch points 0 and ∞ are also branch points of g, so we should have $d^2(-(\sqrt{\lambda} \pm 1)^2)$ be either -1 or $\lambda - 1$ depending on the $\pm$. Taking the quotient, we find that $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2}$ is equal to either $1 - \lambda$ or $\frac{1}{1-\lambda}$.

d. Let μ be a solution of $\mu^2 = \lambda$.

   We find that our first relation $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2} = \lambda$ becomes $(1+\mu)^2 = \mu^2(1-\mu)^2$, or $(\mu^2+1)(\mu^2 - 2\mu - 1) = 0$. This is solved by $\mu^2 = -1$ and $\mu = 1 \pm \sqrt{2}$, giving $\lambda = -1$ or $\lambda = 3 \pm 2\sqrt{2}$.

We find that our second relation $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2} = \frac{1}{\lambda}$ becomes $\mu^2(1+\mu)^2 = (1-\mu)^2$, or $(\mu^2+1)(\mu^2+2\mu-1) = 0$. This is solved by $\mu^2 = -1$ and $\mu = -1 \pm \sqrt{2}$, giving $\lambda = -1$ or $\lambda = 3 \pm 2\sqrt{2}$.

We find that our third relation $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2} = 1 - \lambda$ becomes $(1+\mu)^2 = (1-\mu)^2(1-\mu^2)$, or $\mu(\mu+1)(\mu^2 - 3\mu + 4) = 0$. This is solved by $\mu = 0$, $\mu = -1$, and $\mu = \frac{1}{2}(3 \pm \sqrt{-7})$, giving $\lambda = 0$ and $\lambda = 1$ which we discard and $\lambda = \frac{1}{2}(1 \pm 3\sqrt{-7})$ which we keep.

We find that our fourth relation $\frac{(1+\sqrt{\lambda})^2}{(1-\sqrt{\lambda})^2} = \frac{1}{1-\lambda}$ becomes $(1+\mu)^2(1-\mu^2) = (1-\mu)^2$, or $\mu(\mu-1)(\mu^2+3\mu+4) = 0$. This is solved by $\mu = 0$, $\mu = 1$, and $\mu = \frac{1}{2}(3 \pm \sqrt{-7})$, giving $\lambda = 0$ and $\lambda = 1$ which we discard and $\lambda = \frac{1}{2}(1 \pm 3\sqrt{-7})$ which we keep.

The value $\lambda = -1$ gives $j = 1728$, the values $\lambda = 3 \pm 2\sqrt{2}$ give $j = 8000$, and the values $\lambda = \frac{1}{2}(1 \pm 3\sqrt{-7})$ give $j = -3375$ matching the answers given.

Exercise IV.4.6.

a. Let X be a curve of genus g embedded birationally in $\mathbb{P}^2$ as a curve of degree d with r nodes. Generalize the method of (Ex. 2.3) to show that X has $6(g-1)+3d$ inflection points. A node does not count as an inflection point. Assume $\operatorname{char} k = 0$.

b. Now let X be a curve of genus g embedded as a curve of degree d in $\mathbb{P}^n$, $n \geq 3$, not contained in any $\mathbb{P}^{n-1}$. For each point $P \in X$, there is a hyperplane H containing P, such that P counts at least n times in the intersection $H \cap X$. This is called an *osculating hyperplane* at P. It generalizes the notion of tangent line for curves in $\mathbb{P}^2$. If P counts at least $n+1$ times in $H \cap X$, we say H is a *hyperosculating hyperplane*, and that P is a *hyperosculation point.* Use Hurwitz's theorem as above, and induction on n, to show that X has $n(n+1)(g-1)+(n+1)d$ hyperosculation points.

c. If X is an elliptic curve, for any $d \geq 3$, embed X as a curve of degree d in $\mathbb{P}^{d-1}$, and conclude that X has exactly d^2 points of order d in its group law.

Solution.

a. Let X' be the embedded curve in $\mathbb{P}^2$, and let $e : X \to X'$ be the embedding. By exercises I.7.2 and IV.1.8, X' has arithmetic genus $\frac{1}{2}(d-1)(d-2)$ and X has genus $\frac{1}{2}(d-1)(d-2) - r$.

We're going to basically run exercise IV.2.3 here, with some slight alterations because X' has nodes. We start with the analogue of part (c): pick a point $O \in \mathbb{P}^2$ not on any inflectional or multiple tangent, where we count each of the individual tangent directions at a node separately.Let $\pi' : X \to \mathbb{P}^1$ be the projection from O, and let $\pi : X \to \mathbb{P}^1$ be the extension of $\pi' \circ e$ to a regular map. Then π is of degree d, so by Riemann-Hurwitz it has ramification divisor of degree $d^2 - d - 2r$. As O did not lie on any multiple or inflectional tangent, all ramification is of degree two and each point of ramification corresponds uniquely to a line through O tangent to X'.

Next, we do the analogues of parts (a) and (e). Let $L \subset \mathbb{P}^2$ be a line not tangent to X' nor passing through any of the nodes of X'. We may then define a rational map $\varphi' : X' \to L$ by sending $x \in X$ to $T_x X \cap L$ and extend the rational map $\varphi' \circ e$ to a regular map $\varphi : X \to \mathbb{P}^1$. Since each point not on a multiple or inflectional tangent is on $d^2 - d - 2r$ tangents by the previous paragraph, this morphism is of degree $d^2 - d - 2r$ and Riemann-Hurwitz says that its ramification divisor is of degree $3d^2 - 5d - 6r$. After removing the ramification coming from $L \cap X'$, we get that X' has $3d^2 - 6d - 6r = 3d^2 - 9d - 6r + 3d = 6(\frac{1}{2}(d-1)(d-2) - r) + 3d = 6(g-1) + 3d$ inflection points (counted correctly).

b. We'll develop some general material first because this stuff is a little difficult to find.

First we show that for any $P \in X$, the set of intersection multiplicities of X at P with an arbitrary hyperplane $H \subset \mathbb{P}^n$ is of cardinality exactly $n+1$ (since H is cut out by some equation h, the intersection multiplicity is just the valuation of h/h_0 in the local ring $\mathcal{O}_{X,P}$ for h_0 some hyperplane missing p). Take any hyperplane H_0 not meeting P and let h_0 be an equation for it: then mapping $h \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ to $h/h_0 \in \mathcal{O}_{X,P}$ is an injective map of vector spaces since X is not contained in any hyperplane. As any two h with different orders of vanishing at p must be linearly independent, we see that the cardinality is at most $n+1$. On the other hand, if the cardinality of our set is less than $n+1$, then any basis we pick for $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ must have two elements h_1, h_2 which give the same intersection multiplicity v at P: this means that h_1/h_0 and h_2/h_0 are both in $\mathfrak{m}^v \setminus \mathfrak{m}^{v+1}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,P}$. But $\mathfrak{m}^v/\mathfrak{m}^{v+1}$ is of dimension one, since $\mathcal{O}_{X,P}$ is a DVR - so there's a linear combination of h_1/h_0 and h_2/h_0 with valuation strictly larger than v. Replacing h_2 by this linear combination, we see that we can repeat this successively to show that we can always find a basis of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ which has at least $n+1$ different intersection multiplicities at p. This tells us that the osculating hyperplane is *unique*, which is a very nice thing to know: writing the equation h for an arbitrary hyperplane in the basis described above, we see that the valuation of h/h_0 is the minimum valuation of h_b/h of among basis vectors h_b appearing with nonzero coefficient.

We call the $n+1$ elements of the above set $a_0 < a_1 < \cdots < a_n$ (this is called the *vanishing sequence*) and we write $\alpha_i = a_i - i$ (this is called the *ramification sequence*) which also has the property that $\alpha_i \leq \alpha_{i+1}$. For any $1 \leq i \leq n$, we can form the $(i-1)$-osculating plane at x, denoted $O_{i-1}(x)$ by taking the intersection of all the hyperplanes through x which intersect X to multiplicity a_i at x. I claim that $O_{i-1}(x)$ varies regularly with x - that is, the $(i-1)$-plane $O_{i-1}(x)$ can be found by looking at the image of x under some map of schemes $X \to \mathbb{G}(i, n+1)$ where $\mathbb{G}(i, n+1)$ is the Grassmanian of i-planes in $\mathbb{A}^n$ (equivalently, projective $(i-1)$-planes in $\mathbb{P}^n$). Let T_i be homogeneous coordinates on $\mathbb{P}^n$ so that $V(T_i)$ intersects $f(X)$ at x with intersection multiplicity a_i. This means we can write the inclusion map $X \to \mathbb{P}^n$ as

$$t \mapsto [1 : t^{1+\alpha_1} g_1(t) : \cdots : t^{n+\alpha_n} g_n(t)]$$

near x, where t is a generator of the maximal ideal of $\mathcal{O}_{X,x}$ and $g_i(0) \neq 0$ are units in $\mathcal{O}_{X,x}$. The coordinates of $O_{i-1}(f, x')$ in $\mathbb{G}(i, n+1)$ for x' near x are then the determinants of the $i \times i$ minors of the matrix which consists of f and all of it's first n derivatives of our parametrization:

$$\begin{pmatrix} 1 & t^{1+\alpha_1}g_1(t) & \cdots & t^{n+\alpha_n}g_n(t) \\ 0 & (1+\alpha_1)t^{\alpha_1}g_1(t)+\cdots & \cdots & (n+\alpha_n)t^{n+\alpha_n-1}g_n(t)+\cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & (\cdots)t^{\alpha_n}g_n(t)+\cdots \end{pmatrix}$$

The minimal valuation with respect to t of the determinant of an $i \times i$ minor is achieved by taking the top left $i \times i$ minor, which has valuation $\alpha_1 + \cdots + \alpha_{i-1}$. The next-lowest valuation is achieved by taking the top i rows, the first $i-1$ columns, and the $(i+1)^{th}$ column, and it is $\alpha_1 + \cdots + \alpha_{i-2} + \alpha_i + 1$. (To check this, note that the lowest valuation among elements in a column occurs at the furthest-down nonzero entry, while the lowest valuation among elements in a row occurs at the furthest-left nonzero entry: by row and column operations starting from the furthest-left column and proceeding to the right, we can transform any square submatrix in to one that only has one nonzero entry per row and column without changing the determinant, and the valuation will just be the sum of the valuations of the nonzero terms remaining. If the nonzero term in the i^{th} column is located in the j^{th} row, it has valuation $\alpha_{i-1} + i - 1 + j - 1$, and summing these across all i we find that the total valuations are exactly as claimed.) So we see that after dividing through by $t^{\alpha_1+\cdots+\alpha_{i-1}}$ our map really does give back the osculating planes at x and the map which picks out the osculating $(i-1)$ plane is ramified at x iff $\alpha_i > \alpha_{i-1}$.

This immediately tells us that there are only finitely many hyperosculating points: in characteristic zero, a map can only be ramified over a proper closed subset and any point with $\alpha_n > 0$ must be ramified under one of the maps to $\mathbb{G}(i, n+1)$ described above. With some cleverness, this will also tell us exactly how many hyperosculating points there are. If we take the determinant of the entire $(n+1) \times (n+1)$ matrix, we find a matrix which vanishes to order $\sum \alpha_i$ at x. If we could assemble this in to a global section of a line bundle on X, then we could just count the degree of this line bundle to give us the number of hyperosculating points.

The key question we need to answer here is that if σ is a section of a line bundle $\mathcal{L}$, what's the line bundle that the section which is locally the derivative of σ with regards to a local coordinate is a section of? We may begin by thinking about how things change when we change the local coordinate. This is the chain rule: if we alter the local coordinate from t to u, then we should change the derivative of our function by a factor of $\frac{\partial u}{\partial t}$. If we want this to be coordinate-invariant, we should tensor by a factor of ω_X: the local sections of this bundle transform the opposite way when we change the local coordinate. Therefore the determinant of the whole $(n+1) \times (n+1)$ matrix above is a global section of $\mathcal{O}_X(1)^{n+1} \otimes \omega_X^{\binom{n+1}{2}}$, which has degree $(n+1)d + \binom{n+1}{2}(2g-2) = n(n+1)(g-1) + (n+1)d$ and we're finished.

c. Embed $X \hookrightarrow \mathbb{P}^{d-1}$ via $\mathcal{O}_X(dP_0)$. Then for any $P \in X$, P is a hyperosculating point of X under this embedding iff there is a hyperplane H so that $H \cap X = dP$. As any hyperplane section of X will be linearly equivalent as a divisor to dP_0, we see that hyperosculating points are those P such that $dP \sim dP_0$, which is exactly saying that they're points of order d in the group law. Plugging in $n = d-1$ and $g = 1$ in to the formula from (b), we find that there are d^2 such points as requested.

Exercise IV.4.7. *The Dual of a Morphism.* Let X and X' be elliptic curves over k, with base points P_0, P_0'.

a. If $f : X \to X'$ is any morphism, use (4.11) to show that $f^* : \operatorname{Pic} X' \to \operatorname{Pic} X$ induces a homomorphism $\widehat{f} : (X', P_0') \to (X, P_0)$. We call this the *dual* of f.

b. If $f : X \to X'$ and $g : X' \to X''$ are two morphisms, then $(g \circ f)^\wedge = \widehat{f} \circ \widehat{g}$.

c. Assume $f(P_0) = P_0'$, and let $n = \deg f$. Show that if $Q \in X$ is any point, and $f(Q) = Q'$, then $\widehat{f}(Q') = n_X(Q)$. (Do the separable and purely inseparable cases separately, then combine.) Conclude that $f \circ \widehat{f} = n_{X'}$ and $\widehat{f} \circ f = n_X$.

d. (*) If $f, g : X \to X'$ are two morphisms preserving the base points P_0, P_0', then $(f+g)^\wedge) = \widehat{f} + \widehat{g}$. [*Hints*: It is enough to show that for any $\mathcal{L} \in \operatorname{Pic} X'$, that $(f+g)^*\mathcal{L} \cong f^*\mathcal{L} \otimes g^*\mathcal{L}$. For any f, let $\Gamma_f : X \to X \times X'$ be the graph morphism. Then it is enough to show (for $\mathcal{L}' = p_2^*\mathcal{L}$) that
$$\Gamma_{f+g}^*(\mathcal{L}') = \Gamma_f^*\mathcal{L}' \otimes \Gamma_g^*\mathcal{L}'.$$
Let $\sigma : X \to X \times X'$ be the section $x \to (x, P_0')$. Define a subgroup of $\operatorname{Pic}(X \times X')$ as follows:
$$\operatorname{Pic}_\sigma = \{\mathcal{L} \in \operatorname{Pic}(X \times X') \mid \mathcal{L} \text{ has degree 0 along each fibre of } p_1, \text{ and } \sigma^*\mathcal{L} = 0 \text{ in } \operatorname{Pic} X\}.$$
Note that this subgroup is isomorphic to the group $\operatorname{Pic}^\circ(X'/X)$ used in the definition of the Jacobian variety. Hence there is a 1-1 correspondence between morphisms $f : X \to X'$ and elements $\mathcal{L}_f \in \operatorname{Pic}_\sigma$ (this defines $\mathcal{L}_f$). Now compute explicitly to show that $\Gamma_g^*(\mathcal{L}_f) = \Gamma_f^*(\mathcal{L}_g)$ for any f, g.
Use the fact that $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$, and the fact that for any $\mathcal{L}$ on X', $p_2^*\mathcal{L} \in \operatorname{Pic}_\sigma^\circ$ to prove the result.]

e. Using (d), show that for any $n \in \mathbb{Z}$, $\widehat{n}_X = n_X$. Conclude that $\deg n_X = n^2$.

f. Show for any f that $\deg \widehat{f} = \deg f$.

Solution. When we specify a morphism by what it does on k-points here, we're really using the equivalence of categories from proposition II.2.6 which requires integral separated schemes of finite type over an algebraically closed field.

a. Since f preserves line bundles of degree zero, we see that f^* restricts to a map between the identity components of $\operatorname{Pic} X'$ and $\operatorname{Pic} X$. By theorem IV.4.11, we can identify X and X' with their Jacobian varieties, so this map between Pic^0s gives a map of their closed points which can be upgraded to a morphism $X' \to X$ sending $P_0' \mapsto P_0$ as the pullback of the structure sheaf is the structure sheaf. Applying lemma IV.4.9, this map is a homomorphism.

b. By functoriality of the pullback, $(g \circ f)^* = g^* \circ f^*$, and as we proceed through the rest of the steps from (a) this equality remains true.

c. We assume f is non-constant here, otherwise everything is fairly obvious.

   Let's do the separable case first. By Riemann-Hurwitz, as $g(X) = g(X') = 1$, f is unramified. Therefore $f^{-1}(Q') = Q + \ker f$ and $f^{-1}(P_0') = \ker f$, so $f^*\mathcal{O}_{X'}(Q' - P_0') \cong \mathcal{O}_X((Q + \ker f) - (\ker f)) \cong \mathcal{O}_X(nQ - nP_0)$ and so $\widehat{f}(Q') = n_X(Q)$.

   If f is purely inseparable, the fiber over any point is a singleton by the work that we did in exercise III.11.4 (fifth paragraph, to be specific). Therefore $f^*\mathcal{O}_{X'}(Q' - P_0') = \mathcal{O}_X(nQ - nP_0)$ and so $\widehat{f}(Q') = n_X(Q)$.

   To combine these in to the total claim, factor f as the composition $f_{insep} \circ f_{sep}$ (for instance, by factoring the field extension $k(X) \subset k(X')$ as the composition of a separable extension and a purely inseparable extension and using the equivalence of categories between field extensions of transcendence degree one and regular projective curves). Then $f^*\mathcal{O}_{X'}(Q' - P_0') \cong \mathcal{O}_X(n_X Q - n_X P_0)$ by applying the above results, so $\widehat{f}(Q') = n_X(Q)$ and we've shown that $\widehat{f} \circ f = n_X$.

   In order to show that $f \circ \widehat{f} = n_{X'}$, consider the composition $f \circ \widehat{f} \circ f$. As $\widehat{f} \circ f = n_X$, the composition is equal to $f \circ n_{X'}$, which is equal to $n_{X'} \circ f$ as f is a homomorphism of abelian groups. Therefore $n_{X'} \circ f = (f \circ \widehat{f}) \circ f$, and I claim that this implies $n_{X'} = f \circ \widehat{f}$. To see this implies equality of $n_{X'}$ and $f \circ \widehat{f}$, consider the pullback of $\Delta_{X'} : X' \to X' \times X'$ along $(n_{X'}, f \circ \widehat{f}) : X' \to X'$. Since X' is separated, this pullback is a closed subscheme of X'. Because it's also the locus where $n_{X'} = f \circ \widehat{f}$, we see that as f is surjective and $n_{X'} \circ f = (f \circ \widehat{f}) \circ f$, it must contain all the points of X'. Thus as X' is reduced, it must actually be all of X' and we have $f \circ \widehat{f} = n_{X'}$.

d. Let's go through the hints in order.

   Since the group operation on the Picard group is tensoring line bundles, $\widehat{f} + \widehat{g}$ applied to some invertible sheaf $\mathcal{L}$ is given by $f^*\mathcal{L} \otimes g^*\mathcal{L}$. Therefore by our construction of the dual morphism, it suffices to show that $(f + g)^*\mathcal{L} \cong f^*\mathcal{L} \otimes g^*\mathcal{L}$ for any invertible sheaf $\mathcal{L}$ of degree zero on X'.

   Next, define $\Gamma_f : X \to X \times X'$ as $id \times f$. Then $p_2 \circ \Gamma_f = f$, so $\Gamma_f^* p_2^* = f^*$, so if we let $\mathcal{L}' = p_2^*\mathcal{L}$, it's enough to show that $\Gamma_{f+g}^*\mathcal{L}' = \Gamma_f^*\mathcal{L}' \otimes \Gamma_g^*\mathcal{L}'$.

   Continuing with the hints, let $\sigma : X \to X \times X'$ be the section $x \to (x, P_0')$. Define a subgroup of $\operatorname{Pic}(X \times X')$ as follows:

$$\operatorname{Pic}_\sigma = \{\mathcal{L} \in \operatorname{Pic}(X \times X') \mid \mathcal{L} \text{ has degree 0 along each fibre of } p_1, \text{ and } \sigma^*\mathcal{L} = 0 \text{ in } \operatorname{Pic} X\}.$$

   To check that this is isomorphic to $\operatorname{Pic}^\circ(X'/X)$ defined in the construction of the Jacobian, we see that the condition $\mathcal{L}$ has degree 0 along each fiber of p_1 is the condition that $\mathcal{L}$ is in $\operatorname{Pic}^\circ(X' \times X)$. The condition that $\sigma^*\mathcal{L} = 0$ uniquely specifies a representative of a class in $\operatorname{Pic}^\circ(X' \times X)$ up to the action of $p_1^* \operatorname{Pic} X$: any class $L \in \operatorname{Pic}^\circ(X'/X)/p_1^* \operatorname{Pic} X$ has a representative $\mathcal{L}$ with $\sigma^*\mathcal{L} = 0 \in \operatorname{Pic} X$ by taking any representative $\mathcal{L}$ and tensoring by $p_1^*\sigma^*\mathcal{L}$. Conversely, if $\mathcal{L}$ and $\mathcal{M} = \mathcal{L} \otimes p_1^*\mathcal{N}$ are two representatives of the same class in $\operatorname{Pic}^\circ(X'/X)/p_1^* \operatorname{Pic} X$ with $\sigma^*\mathcal{L} \cong \sigma^*\mathcal{M} \cong \mathcal{O}_X$, then $\sigma^*\mathcal{L} \cong \sigma^*\mathcal{L} \otimes \sigma^* p_1^*\mathcal{N}$, which implies $\sigma^* p_1^*\mathcal{N} \cong \mathcal{O}_X$. But $p_1 \circ \sigma = id_X$, so $\mathcal{N} \cong \mathcal{O}_X$. Therefore $\operatorname{Pic}_\sigma \cong \operatorname{Pic}^\circ(X'/X)$, so the universal property of the Jacobian says that maps $X \to X'$ are in bijection with elements of $\operatorname{Pic}^\circ(X'/X) \cong \operatorname{Pic}_\sigma$.

Now let's find the representative of $\mathcal{L}_f$ in $\operatorname{Pic}_\sigma$. From the construction of the Jacobian in theorem IV.4.11, the element in $\operatorname{Pic}^\circ(X'/X)$ uniquely corresponding to $f : X \to X'$ is the pullback of $\mathcal{L} = \mathcal{L}(\Delta_{X'}) \otimes p_1^*\mathcal{L}(-P_0')$ along $id_{X'} \times f : X' \times X' \to X' \times X$ (note there's a swapping of orders here when discussing $\operatorname{Pic}_\sigma$ which is really rather confusing - I'll be keeping X' as the first factor here and not doing any switching). To ensure we get the version of this living in $\operatorname{Pic}_\sigma$, we need to adjust by the inverse of the pullback of $\mathcal{L}$ along $\sigma : X \to X' \times X$ by $x \mapsto (P_0', x)$. We note that we may instead adjust $\mathcal{L}$ first so that it's trivial after pullback along $\sigma' : X' \to X' \times X'$ by $x' \mapsto (P_0', x')$: as $\sigma'^*\mathcal{L} \cong \mathcal{L}(-P_0')$, the representative of $\mathcal{L}$ in $\operatorname{Pic}^\circ(X'/X')$ which is trivial along σ' is $\mathcal{L} \otimes p_2^*\mathcal{L}(-P_0') = \mathcal{L}(\Delta_{X'}) \otimes p_1^*\mathcal{L}(-P_0') \otimes p_2^*\mathcal{L}(-P_0')$, so

$$\mathcal{L}_f = (id_{X'} \times f)^*(\mathcal{L}(\Delta_{X'}) \otimes p_1^*\mathcal{L}(-P_0') \otimes p_2^*\mathcal{L}(-P_0')).$$

Writing out the composite pullbacks, one may check that $\Gamma_g^*\mathcal{L}_f \cong (g, f)^*\mathcal{L}(\Delta_{X'}) \otimes f^*\mathcal{L}(-P_0') \otimes g^*\mathcal{L}(-P_0')$ and $\Gamma_f^*\mathcal{L}_g \cong (f, g)^*\mathcal{L}(\Delta_{X'}) \otimes f^*\mathcal{L}(-P_0') \otimes g^*\mathcal{L}(-P_0')$. As $\Delta_{X'}$ is stable under the isomorphism swapping the factors of $X' \times X'$, these two line bundles are the same and $\Gamma_f^*\mathcal{L}_g = \Gamma_g^*\mathcal{L}_f$ for any $f, g : X \to X'$.

To check that $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$, we can quickly observe that they're isomorphic on $\{x\} \times X'$ for any $x \in X$. First, $\mathcal{L}_{f+g}|_{\{x\}\times X} = \mathcal{L}((f+g)(x) - P_0)$, while $\mathcal{L}_f|_{\{x\}\times X} = \mathcal{L}(f(x) - P_0)$ and $\mathcal{L}_g|_{\{x\}\times X} = \mathcal{L}(g(x) - P_0)$, so $\mathcal{L}_f \otimes \mathcal{L}_g$ restricted to any $\{x\} \times X'$ is $\mathcal{L}(f(x) + g(x) - 2P_0)$. Since $f(x) + g(x) = (f(x) + g(x)) + P_0$ as divisors, these two invertible sheaves are isomorphic. Therefore $\mathcal{L}_{f+g}$ and $\mathcal{L}_f \otimes \mathcal{L}_g$ differ by $p_1^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on X by exercise III.12.4. But by the condition that pullback along σ gives $\mathcal{O}_X$, we see that $\mathcal{M} = \mathcal{O}_X$ and so we've verified that $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$ in $\operatorname{Pic}_\sigma$.

Now we can finish the problem: for any line bundle of degree zero on $\mathcal{L}$ on X', the line bundle $\mathcal{L}' = p_2^*\mathcal{L}$ on $X \times X'$ has degree zero along each fiber and as $p_2 \circ \sigma$ is constant, $\sigma^* p_2^*\mathcal{L} = \mathcal{O}_X$. So any line bundle of the form $p_2^*\mathcal{L}$ is the line bundle $\mathcal{L}_h$ associated to some morphism $h : X \to X'$ by our work with $\operatorname{Pic}_\sigma$. Therefore we may take $\mathcal{L}' = \mathcal{L}_h$ in the statement in our third paragraph, so we need to show

$$\Gamma_{f+g}^*\mathcal{L}_h = \Gamma_f^*\mathcal{L}_h \otimes \Gamma_g^*\mathcal{L}_h.$$

Using that $\Gamma_f^*\mathcal{L}_g \cong \Gamma_g^*\mathcal{L}_f$ on both sides, this is equivalent to showing

$$\Gamma_h^*\mathcal{L}_{f+g} = \Gamma_h^*\mathcal{L}_f \otimes \Gamma_h^*\mathcal{L}_g.$$

But as $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$, this is immediate and we've shown that sums and duals commute.

e. The map 1_X is the identity morphism, and it is it's own dual by (c). By (d), we may show that $n_X = 1_X + \cdots + 1_X$ has dual $\widehat{n}_X = \widehat{1}_X + \cdots + \widehat{1}_X = 1_X + \cdots + 1_X = n_X$.

   By (c), $f \circ \widehat{f} = (\deg f)_X$. Setting $f = n_X$ and using the fact that $\widehat{n}_X = n_X$, we see that $n_X \circ n_X = (\deg n_X)_X$, so $\deg n_X = n^2$.

f. Let $n = \deg f$. By (c), $\widehat{f} \circ f = n_X$. Since degrees of finite maps multiply under composition and $\deg n_X = n^2$ by (d), we see that $\deg \widehat{f} = n$ as well.

Exercise IV.4.8. For any curve X, the *algebraic fundamental group* $\pi_1(X)$ is defined as $\varprojlim \operatorname{Gal}(K'/K)$, where K is the function field of X, and K' runs over all Galois extension of K such that the corresponding curve X' is étale over X (III, Ex. 10.3). Thus, for example, $\pi_1(\mathbb{P}^1) = 1$ (2.5.3). Show that for an elliptic curve X,

$$\pi_1(X) = \prod_{l \text{ prime}} \mathbb{Z}_l \times \mathbb{Z}_l \qquad \text{if } \operatorname{char} k = 0;$$

$$\pi_1(X) = \prod_{l \neq p} \mathbb{Z}_l \times \mathbb{Z}_l \qquad \text{if } \operatorname{char} k = p \text{ and } \operatorname{Hasse} X = 0;$$

$$\pi_1(X) = \mathbb{Z}_p \times \prod_{l \neq p} \mathbb{Z}_l \times \mathbb{Z}_l \quad \text{if } \operatorname{char} k = p \text{ and } \operatorname{Hasse} X \neq 0,$$

where $\mathbb{Z}_l = \varprojlim \mathbb{Z}/l^n$ is the l-adic integers.

[*Hints*: Any Galois étale cover X' of an elliptic curve is again an elliptic curve. If the degree of X' over X is relatively prime to p, then X' can be dominated by the cover $n_X : X \to X$ for some integer n with $(n, p) = 1$. The Galois group of the covering n_X is $\mathbb{Z}/n \times \mathbb{Z}/n$. Étale covers of degree divisible by p can occur only if the Hasse invariant of X is not zero.]

Note: More generally, Grothendieck has shown [SGA 1, X, 2.6, p. 272] that the algebraic fundamental group of any curve of genus g is isomorphic to a quotient of the completion, with respect to subgroups of finite index, of the ordinary topological fundamental group of a compact Riemann surface of genus g, i.e., a group with $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ and the relation $(a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) = 1$.

Solution. Any Galois extension K'/K is separable, so $X' \to X$ is separable and by Riemann-Hurwitz, X' has genus 1. Choosing any point over O_X as the identity of X', the map $f : X' \to X$ can be taken to be an isogeny, and then by exercise IV.4.7(c) we may write $n_X = f \circ \widehat{f}$, where $n = \deg f$. By exercise IV.4.7(e), n_X is of degree n^2. If n_X is separable, then the kernel of n_X is of order n^2, which means it must be $(\mathbb{Z}/n)^2$, and so $\operatorname{Aut}(X, n_X) = \operatorname{Gal}(K'/K)$ is also $(\mathbb{Z}/n)^2$ by considering the translation maps by elements of $\ker n_X$.

If $\operatorname{char} k = 0$, then as every map is separable the system of coverings n_X for $n \in \mathbb{Z}_{>0}$ with the ordering $n \leq m$ iff $n \mid m$ forms a cofinal system among the etale Galois covers of X and therefore can be used to calculate the inverse limit. Rewriting $(\mathbb{Z}/n)^2$ as $\prod_{p|n} (\mathbb{Z}/p^{\nu_p(n)})^2$ where $\nu_p(n)$ is the valuation of n with respect to p, it's easy to see that the inverse system breaks up as a product of the inverse systems used to create $\mathbb{Z}_l$ from $\mathbb{Z}/l^a$ and the claim follows.

When $\operatorname{char} k = p > 0$, the endomorphism n_X is automatically separable for all n relatively prime to p, but there might be issues when p divides n. If $f : X' \to X$ has degree $p^r m$ with $(p, m) = 1$, then $f \circ \widehat{f} = m_X \circ p_X^r$, and using the fact that $Fr \circ \widehat{Fr} = \widehat{Fr} \circ Fr = p_X$, we can rewrite p_X^r as $\widehat{Fr}^r \circ Fr^r$. The Frobenius is always inseparable, while by exercise IV.4.15 the dual of the Frobenius is separable iff the Hasse invariant is 1. Therefore the system $n_X : X \to X$ with $(n,p) = 1$ forms a cofinal system when the Hasse invariant is zero, while the system $\widehat{Fr}^r \circ m_X : X^{(p^r)} \to X$ with $(m,p) = 1$ forms a cofinal system when the Hasse invariant is one.

In the Hasse invariant zero case, the same logic from the characteristic zero case shows that our inverse limit breaks in to a product of $\mathbb{Z}_l^2$ for each prime l distinct from p. On the other hand, if the Hasse invariant is nonzero, then the kernel of $\widehat{Fr}$ is a copy of $\mathbb{Z}/p$ and so similar logic shows that our inverse limit breaks in to a product of $\mathbb{Z}_p$ and $\mathbb{Z}_l^2$ for each prime l distinct from p.

Exercise IV.4.9. We say two elliptic curves X, X' are *isogenous* if there is a finite morphism $f : X \to X'$.

a. Show that isogeny is an equivalence relation.

b. For an elliptic curve X, show that the set of elliptic curves X' isogenous to X, up to isomorphism, is countable. [*Hint*: X' is uniquely determined by X and $\ker f$.]

Solution.

a. The identity is an isogeny, so the relation is reflexive. A composition of finite maps is again finite, so the relation is transitive. Symmetry comes from the construction of the dual isogeny in exercise IV.4.7: by part (f), the dual of any isogeny is again an isogeny of the same degree. Being reflexive, symmetric, and transitive, isogeny is an equivalence relation.

b. Factoring $X \to X'$ in to separable and purely inseparable parts, we see that the possible purely inseparable parts are given by repeated applications of the Frobenius per proposition IV.2.5. Therefore it suffices to prove the claim for separable isogenies.

   To show that X' is uniquely determined by X and $\ker f$, let τ_p be the automorphism of X given by translation by p. Define $\Phi = \{\tau_p \mid p \in \ker f\}$, which is a finite subgroup of $\mathrm{Aut}(X)$. Then $k(X)^\Phi$ is a subfield of $k(X)$, the extension $k(X)^\Phi \subset k(X)$ is Galois with Galois group Φ, and by the correspondence between fields of transcendence degree one over k and regular projective curves, $k(X)^\Phi$ is the function field of a unique curve X' and the inclusion $k(X)^\Phi \subset k(X)$ corresponds to a map $\varphi : X \to X'$. As the preimage of any closed point of X' is of size $\ker f$, which is equal to $\deg \varphi$, we have that φ is unramified and therefore by Riemann-Hurwitz X' is in fact an elliptic curve.

   Once we know that X' is uniquely determined by X and $\ker f$, it remains to show that there are only countably many choices of $\ker f$ for any X. Since $\ker f$ is a finite abelian group, it's contained in the n-torision of X for some n. By exercise IV.4.7(e), there are n^2 n-torsion points of X for any n, so there are countably many torsion points on X. As the number of finite subsets of a countable set is again countable, we are finished.

Exercise IV.4.10. If X is an elliptic curve, show that there is an exact sequence

$$0 \to p_1^* \operatorname{Pic} X \oplus p_2^* \operatorname{Pic} X \to \operatorname{Pic}(X \times X) \to R \to 0,$$

where $R = \operatorname{End}(X, P_0)$. In particular, we see that $\operatorname{Pic}(X \times X)$ is bigger than the sum of the Picard groups of the factors. Cf. (III, Ex. 12.6), (V, Ex. 1.6).

Solution. Before we begin, one preliminary about the Jacobian: if C and X are elliptic curves and $\varphi : C \to X$ is a morphism, then there's a unique homomorphism $\widetilde{\varphi} : J_C \to X$ so that $t_{-\varphi(c)} \circ \varphi = \widetilde{\varphi} \circ \alpha_c$ for all $c \in C$ where α_c is the map $\alpha_c(P) = \mathcal{O}_C(P - c)$ and t_x is translation by x. Since $J_C = C$ and α_c is an isomorphism, we may define $\widetilde{\varphi}$ to be the map $t_{-\varphi(c)} \circ \varphi \circ \alpha_c^{-1}$ which is clearly unique. Since this preserves the base point, it's a homomorphism by lemma IV.4.9. To check that this does not depend on c, consider $(\widetilde{\varphi}_c - \widetilde{\varphi}_{c'})\alpha_c(p)$: the first term is just $\widetilde{\varphi}_c \alpha_c(p) = t_{-\varphi(c)}\varphi(p)$, while the second term can be written $\widetilde{\varphi}_{c'}(\alpha_{c'}(p) - \alpha_{c'}(c)) = t_{-\varphi(c')}(\varphi(p)) - t_{-\varphi(c')}(\varphi(c))$, and these cancel.

Given a line bundle $\mathcal{L}$ on $X \times X$ and a point $P \in X$, let $\mathcal{L}_P$ be the pullback of $\mathcal{L}$ along the closed immersion $\{P\} \times X \to X \times X$. Define $\varphi_{\mathcal{L}} : J_X \to J_X$ by sending

$$\mathcal{O}_X(\sum n_i P_i) \mapsto \bigoplus \mathcal{L}_{P_i}^{n_i}.$$

To check that this is well-defined, we note that this is exactly the map $J_X \to J_X$ induced from the map $X \to J_X$ by $P \mapsto \mathcal{L}_P \otimes \mathcal{L}_{P_0}^{-1}$ and apply the preliminary result.

We may also define a map $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(X \times X)$ as the product of the pullbacks associated to the projections. To check that this is injective, consider the composite $\{P\} \times X \to X \times X \overset{p_2}{\to} X$ for any $P \in X$: since this is the identity, it gives the identity map on Picard groups, so if $p_1^*\mathcal{M}_1 \otimes p_2^*\mathcal{M}_2 \cong \mathcal{O}_{X \times X}$, then $\mathcal{M}_1 \cong \mathcal{O}_X$. Similar logic holds for the other factor, and therefore $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(X \times X)$ is injective, and we've proven exactness at the first term of the exact sequence.

To check exactness at the second term, we first show that if $\mathcal{L} \cong \mathcal{O}(\{P\} \times X)$ or $\mathcal{O}(X \times \{P\})$, then $\varphi_{\mathcal{L}} = 0$. In the first case, the restriction of $\mathcal{L}$ to any fiber $\{Q\} \times X$ is trivial because $\mathcal{L}$ is the pullback of $\mathcal{O}_X(P)$ which is locally free and therefore trivial in a neighborhood of Q. In the second case, $\mathcal{L}_Q \cong \mathcal{O}_X(P)$ for any Q, and since we're considering divisors of degree zero, the map $\varphi_{\mathcal{L}} = 0$. Conversely, if $\mathcal{L}$ is such that $\varphi_{\mathcal{L}} = 0$, then we must have that $\mathcal{L}_P \cong \mathcal{L}_{P_0}$ for all $P \in X$. By exercise III.12.4, this means that $\mathcal{L} \cong p_1^*\mathcal{M} \otimes p_2^*\mathcal{L}_{P_0} \otimes$ for some invertible sheaf $\mathcal{M}$ on X, so $\mathcal{L}$ is in the image of $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(X \times X)$ and we've shown exactness at the middle term.

Finally I claim that if $f : J_X \to J_X$ is a homomorphism of Jacobians, then Γ_f, the graph of f is the divisor which gets sent to $f \in \operatorname{Hom}(J_X, J_X)$. One can check rather quickly that $\mathcal{O}_X(P - P_0)$ gets sent to $\mathcal{O}_X(f(P) - f(P_0))$. Therefore we've shown the exact sequence as requrested.

Exercise IV.4.11. Let X be an elliptic curve over $\mathbb{C}$, defined by the elliptic functions with periods $1, \tau$. Let R be the ring of endomorphisms of X.

a. If $f \in R$ is a nonzero endomorphism corresponding to complex multiplication by α, as in (4.18), show that $\deg f = |\alpha|^2$.

b. If $f \in R$ corresponds to $\alpha \in \mathbb{C}$ again, show that the dual $\widehat{f}$ of (Ex. 4.7) corresponds to the complex conjugate $\overline{\alpha}$ of α.

c. If $t \in \mathbb{Q}(\sqrt{-d})$ happens to be integral over $\mathbb{Z}$, show that $R = \mathbb{Z}[\tau]$.

Solution.

a. Let Λ be the lattice $\{n + m\tau \mid n, m \in \mathbb{Z}\}$ and let $\alpha = a + bi$. If multiplication by α gives an endomorphism, then the degree of this endomorphism is just the ratio of the two covolumes of the lattices Λ and $\alpha\Lambda$ because this counts how many points lie over a given point. The covolume of a lattice can be found by taking the determinant of the matrix which has as its columns the coordinates of any two spanning vectors. Since multiplying any two spanning vectors for Λ by α gives a set of spanning vectors of $\alpha\Lambda$, the matrix of spanning vectors transforms by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and therefore the volume changes by a factor of the determinant of this matrix, which is just $a^2 + b^2 = |\alpha|^2$.

b. The dual is the unique morphism so that $\widehat{f} \circ f = n_X$, where $n = \deg f$. As the composition of endomorphisms corresponding to complex numbers α, β is given by the endomorphism corresponding to $\alpha\beta$, we see that $\overline{\alpha}$ is exactly the morphism which composes with α to get $|\alpha|_X$ and therefore must be the dual.

c. By theorem IV.4.19, if $\tau = r + s\sqrt{-d}$ for $r, s \in \mathbb{Q}$, then $R = \{a + b\tau \mid a, b \in \mathbb{Z}$ and $2br, b(r^2 + ds^2) \in \mathbb{Z}\}$. The minimal polynomial of τ over $\mathbb{Q}$ is $T^2 - 2rT + r^2 + s^2 d$, and if τ is integral over $\mathbb{Z}$, all coefficients of this polynomial must actually be in $\mathbb{Z}$. But this implies that $R = \mathbb{Z}[\tau]$ and we're done.

Exercise IV.4.12. Again let X be an elliptic curve over $\mathbb{C}$ determined by the elliptic functions with periods $1, \tau$, and assume that τ lies in the region G of (4.15B).

a. If X has any automorphisms leaving P_0 fixed other than ± 1, show that either $\tau = i$ or $\tau = \omega$, as in (4.20.1) and (4.20.2). This gives another proof of the fact (4.7) that there are only two curves, up to isomorphism, having automorphisms other than ± 1.

b. Now show that there are exactly three values of τ for which X admits an endomorphism of degree 2. Can you match these with the three values of j determined in (Ex. 4.5)? [*Answers*: $\tau = i$; $\tau = \sqrt{-2}$; $\tau = \frac{1}{2}(-1 + \sqrt{-7})$.]

Solution.

a. By exercise IV.4.11(a), an endomorphism corresponding to complex multiplication by α has degree $|\alpha|^2$, so any automorphism given by multiplication by α must have $|\alpha|^2 = 1$. From theorem IV.4.19, we have that the endomorphism ring consists of elements $a + b\tau$ with $a, b \in \mathbb{Z}$ and $2br, b(r^2 + s^2 d) \in \mathbb{Z}$ where $\tau = r + s\sqrt{-d}$ with $r, s \in \mathbb{Q}$. From the assumption that $\tau \in G$, we have $-\frac{1}{2} \leq r < \frac{1}{2}$ and $|\tau| \geq 1$, so we must have $a = 0$ and $b = \pm 1$. Therefore from the requirement that $2br \in \mathbb{Z}$, we have $r = 0$ or $r = -\frac{1}{2}$, which shows us that the only options are $\tau = i$ or $\tau = \omega$.

b. This time we're looking for $\alpha = a + b\tau \in R$ with $|\alpha|^2 = 2$. As the imaginary part of τ is at least $\frac{\sqrt{3}}{2}$, we have $b = \pm 1$, and as the real part of τ is between $-\frac{1}{2}$ and $\frac{1}{2}$, $|a| \leq 1$. Therefore from the requirement that $2br \in \mathbb{Z}$, we have $r = 0$ or $r = -\frac{1}{2}$.

   When $r = 0$ and $a = \pm 1$, the condition that $|a + b\tau|^2 = 2$ gives that $1 + s^2 d = 2$, so $s^2 d = 1$ which is achieved for $\tau \in G$ only by $\tau = i$. When $r = 0$ and $a = 0$, the condition that $|a + b\tau|^2 = 2$ gives that $s^2 d = 2$, which is achieved for $\tau \in G$ only by $\tau = \sqrt{-2}$. When $r = -\frac{1}{2}$, we cannot have $a = -1$ as then $a + b\tau$ has real part $\frac{3}{2}$ and therefore norm more than 2. When $r = -\frac{1}{2}$, then either $a = 0$ or $a = 1$ gives $|a + b\tau|^2 = \frac{1}{4} + s^2 d$, so $s^2 d = \frac{7}{4}$ which is achieved for $\tau \in G$ only by $\tau = \frac{1}{2}(-1 + \sqrt{-7})$.

   To match τ to j, the answers are that $\tau = i$ corresponds to $j = 1728$, $\tau = \sqrt{-2}$ corresponds to $j = 8000$, and $\tau = \frac{1}{2}(-1 + \sqrt{-7})$ corresponds to $j = -3375$. Actually computing j from τ is generally difficult, so we'll use a different approach. By example IV.4.20.1, $\tau = i$ gives $j = 1728$. To match the other two, we note that by our work in exercise IV.4.5 the order-two endomorphism of the curve with $j = 8000$ does not fix any non-identity points in the 2-torsion subgroup while the order-two endomorphism of the curve with $j = -3375$ does fix a non-identity 2-torsion point. The 2-torsion points of the curve with $\tau = \sqrt{-2}$ inside the period parallelogram are $\{0, \frac{1}{2}, \frac{\sqrt{-2}}{2}, \frac{1+\sqrt{-2}}{2}\}$ which after multiplication by $\sqrt{-2}$ become $\{0, \frac{\sqrt{-2}}{2}, -1, \frac{\sqrt{-2}}{2} - 1\}$ respectively, and translating to place all of these in the period parallelogram we see that multiplication by $\sqrt{-2}$ does not fix any non-identity point, giving us our answer.

Exercise IV.4.13. If $p = 13$, there is just one value of j for which the Hasse invariant of the corresponding curve is 0. Find it. [*Answer*: $j = 5 \pmod{13}$.]

Solution. Recall from corollary IV.4.22 that the Hasse invariant of the curve $y^2 = x(x-1)(x-\lambda)$ is zero iff $h = \sum_{i=0}^{k} \binom{k}{i}^2 \lambda^i$ is zero, where $k = \frac{1}{2}(p-1)$. When $p = 13$, we have that

$$h = \lambda^6 + 10\lambda^5 + 4\lambda^4 + 10\lambda^3 + 4\lambda^2 + 10\lambda + 1$$

after reducing modulo 13. To convert the formula for j in terms of λ in to a polynomial, clear denominators, collect terms, and adjust the leading coefficient to arrive at

$$\lambda^6 + 10\lambda^5 + (6-3j)\lambda^4 + (6+6j)\lambda^3 + (6-3j)\lambda^2 + 10\lambda + 1 = 0$$

after reducing modulo 13. Subtracting these two equations, we find $(2-3j)\lambda^4 + (9+6j)\lambda^3 + (2-3j)\lambda^2 = 0$, or

$$(2-3j)(\lambda-1)^2\lambda^2 = 0.$$

By assumption, λ cannot be either 0 or 1. Therefore the only way for this relation to be satisfied is if $2 - 3j = 0$. Hence $j = 5 \pmod{13}$ is the only value of j for which the Hasse invariant of the corresponding curve is 0.

Exercise IV.4.14. The Fermat curve X: $x^3 + y^3 = z^3$ gives a nonsingular curve in characteristic p for every $p \neq 3$. Determine the set $\mathfrak{P} = \{p \neq 3 \mid X_{(p)} \text{ has Hasse invariant } 0\}$, and observe (modulo Dirichlet's theorem) that it is a set of primes of density $\frac{1}{2}$.

Solution. By proposition IV.4.21, we need to determine the coefficient of $(xyz)^{p-1}$ in $(x^3+y^3-z^3)^{p-1}$. If 3 does not divide $p-1$, this coefficient is automatically zero, while if 3 does divide $p-1$ this has coefficient $\pm\frac{(p-1)!}{((p-1)/3)^3}$, which is nonzero modulo p. Therefore $\mathfrak{B} = \{p \mid p = 3k+1, k \in \mathbb{Z}\}$ which has density $1/\varphi(3) = \frac{1}{2}$ by Dirichlet's theorem.

Exercise IV.4.15. Let X be an elliptic curve over a field k of characteristic p. Let $F : X_p \to X$ be the k-linear Frobenius morphism (2.4.1). Use (4.10.7) to show that the dual morphism $\widehat{F} : X \to X_p$ is separable if and only if the Hasse invariant of X is 1. Now use (Ex. 4.7) to show that if the Hasse invariant is 1, then the subgroup of points of order p on X is isomorphic to $\mathbb{Z}/p$; if the Hasse invariant is 0, it is 0.

Solution. Let $X[\varepsilon] = X \times_k \operatorname{Spec} k[\varepsilon]/\varepsilon^2$. From exercise III.4.6, we have an exact sequence $0 \to \varepsilon\mathcal{O}_X \to \mathcal{O}^*_{X[\varepsilon]} \to \mathcal{O}^*_X \to 0$ which after taking cohomology gives us the exact sequence $0 \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic} X[\varepsilon] \to \operatorname{Pic} X \to 0$ of remark IV.4.10.7 (the first map is an injection because $\mathcal{O}^*_{X[\varepsilon]} \to \mathcal{O}^*_X$ is surjective on global sections and the last map is surjective by Grothendieck vanishing). Given an isogeny $f : X \to Y$, pullback gives us an induced map on the cohomologies of these exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1(Y,\mathcal{O}_Y) \arrow[r] \arrow[d] & \operatorname{Pic} Y[\varepsilon] \arrow[r] \arrow[d] & \operatorname{Pic} Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & \operatorname{Pic} X[\varepsilon] \arrow[r] & \operatorname{Pic} X \arrow[r] & 0
\end{tikzcd}$$

Therefore the map on tangent spaces $T_0 J_Y \to T_0 J_X$ is the map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ induced by pullback. Up to pre- and post-composing by translations on J_Y and J_X which are isomorphism and therefore induce isomorphisms on tangent spaces, this shows that the map on tangent spaces associated to $J_Y \to J_X$ is zero iff the map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ induced by f^* is zero. So $\widehat{F}$ is separable iff the Hasse invariant of X is 1.

To make the connection between the Hasse invariant and the p-torsion, consider $p_X = \widehat{F} \circ F$. If $\widehat{F}$ is purely inseparable, then p_X is purely inseparable and every fiber of p_X is a single point, so the subgroup of order-p points is trivial. If $\widehat{F}$ is separable, then every fiber is p points and so p_X has p points in each fiber and the subgroup of order-p points is of size p. Since these situations correspond to Hasse invariant 0 and 1 respectively, we're finished.

Exercise IV.4.16. Again let X be an elliptic curve over k of characteristic p, and suppose X is defined over the field $\mathbb{F}_q$ of $q = p^r$ elements, i.e., $X \subset \mathbb{P}^2$ can be defined by an equation with coefficients in $\mathbb{F}_q$. Assume also that X has a rational point over $\mathbb{F}_q$. Let $F' : X_q \to X$ be the k-linear Frobenius with respect to q.

a. Show that $X_q \cong X$ as schemes over k, and that under this identification, $F' : X \to X$ is obtained by the qth-power map on the coordinates of points of X, embedded in $\mathbb{P}^2$.

b. Show that $1_X - F'$ is a separable morphism and its kernel is just the set $X(\mathbb{F}_q)$ of points of X with coordinates in $\mathbb{F}_q$.

c. Using (Ex. 4.7), show that $F' + \widehat{F}' = a_x$ for some integer a, and that $N = q - a + 1$, where $N = \#X(\mathbb{F}_q)$.

d. Use the fact that $\deg(m + nF') > 0$ for all $m, n \in \mathbb{Z}$ to show that $|a| \leq 2\sqrt{q}$. This is Hasse's proof of the analogue of the Riemann hypothesis for elliptic curves (App. C. Ex. 5.6).

e. Now assume $q = p$, and show that the Hasse invariant of X is 0 if and only if $a \equiv 0 \pmod p$. Conclude for $p \geq 5$ that X has Hasse invariant 0 if and only if $N = p + 1$.

Solution.

a. From the definition in remark IV.2.4.1, X_q is X but with the structure morphism post-composed with $\operatorname{Spec}(x \mapsto x^q)$ on $\mathbb{F}_q$. Since $x \mapsto x^q$ is the identity on $\mathbb{F}_q$, X has the same structure morphism as X over k and is therefore isomorphic to X over k.

   To understand that F' acts as the qth power map, think about the map $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (t^2, t^3)$: on the level of coordinate algebras, this is given by the map $k[x, y] \to k[t]$ by $x \mapsto t^2$, $y \mapsto t^3$. So if the map F' is the q^{th} power map, then it sends $[x : y : z] \mapsto [x^q : y^q : z^q]$.

b. First, by exercise II.8.3, if X and Y are varieties over an algebraically closed field and $x \in X$ and $y \in Y$ are closed points, then $T_{(x,y)}(X \times Y) \cong T_x X \oplus T_y Y$: the cotangent space is $(p_1^* \Omega_{X/k} \oplus p_2^* \Omega_{Y/k}) \otimes k$, where we think of k as the residue field at (x, y). As tensor product distributes over direct sum and commutes with pullback, we see that the cotangent space at (x, y) is the sum of the cotangent spaces at x and y, and then the result follows by taking duals. Next, we observe that given a map of varieties over an algebraically closed field $Z \to X \times Y$ so that $z \mapsto (x, y)$, we have that the tangent map $T_z Z \to T_{(x,y)}(X \times Y) \cong T_x X \oplus T_y Y$ is given by the tangent map $T_z Z \to T_x X$ associated to $Z \to X \times Y \xrightarrow{p_1} X$ and the tangent map $T_z Z \to T_y Y$ associated to $Z \to X \times Y \xrightarrow{p_2} Y$. Finally, we see that given a map $X \times Y \to W$ with $(x, y) \mapsto w$, the tangent map $T_{(x,y)}(X \times Y) \to T_w W$ is the sum of tangent maps $T_x X \to T_w W$ and $T_y Y \to T_w W$ obtained from $X \xrightarrow{(id_X, \{y\})} X \times Y \to W$ and $Y \xrightarrow{(\{x\}, id_Y)} X \times Y \to W$.

   We apply these observations to our problem by writing $1_X - F' : X \to X$ as

$$X \xrightarrow{(1_X, -F')} X \times X \xrightarrow{\mu} X.$$

   The first map gives us the tangent map $T_O X \to T_O X \oplus T_O X$ by $(id, 0)$, while the second map gives us the tangent map $T_O X \oplus T_O X \to T_O X$ by adding tangent vectors (since μ restricts to the identity on $X \times \{O\}$ and $\{O\} \times X$). Therefore since 1_X gives the identity on tangent spaces and F' gives the zero map on tangent spaces, we see that $1_X - F'$ gives a nonzero map on tangent spaces at the identity. By the same argument as in IV.4.15, this implies that the map of tangent spaces associated to $1_X - F'$ is everywhere nonzero, and so $1_X - F'$ is separable.

   To check that the kernel of $1_X - F'$ is those points with coordinates in $\mathbb{F}_q$, note that by (a), F' fixes a point iff it's coordinates are in $X(\mathbb{F}_q)$.

c. By exercise IV.4.7(c), $\widehat{f} \circ f = (\deg f)_X$ for any isogeny $f : X \to X'$. Apply this to the isogeny $1_X - F'$: since duals distribute over sums by (d), $(1_X - F') \circ (1_X - \widehat{F'}) = (\deg 1_X - F')_X$, and as composition distributes over sums of isogenies, we see that $1_X - F' - \widehat{F'} + F \circ \widehat{F} = (\deg 1_X - F')_X$, or $(q+1)_X - (F' + \widehat{F'}) = (\deg 1_X - F')_X$. This shows that $F' + \widehat{F}$ is an integer, and since $1_X - F'$ is a separable isogeny with kernel equal to the $\mathbb{F}_q$-points of X by (b) we have that $N = q - a + 1$ as requested.

d. Using the same tactic as in (c), consider $(m + nF')(\widehat{m + nF'})$. As $\widehat{c} = c$ for any integer c by IV.4.7(e), this is equal to $m^2 + mn(F' + \widehat{F'}) + n^2 q$, or $m^2 + mna + n^2 q$. Setting this equal to zero and solving for m, we find $m = \frac{-na \pm \sqrt{n^2a^2 - 4n^2q}}{2}$, or $m = \frac{-a \pm \sqrt{a^2 - 4q}}{2} n$. If there are two distinct solutions, then there's an open subset in S^1 of the values of $\frac{m}{n}$ where the expression $m^2 + mna + n^2 q$ is negative, which means there must be a rational value of $\frac{m}{n}$ where the expression is negative, and hence integers where the expression is negative. Therefore $a^2 - 4q \leq 0$, which implies $|a| \leq 2\sqrt{q}$.

e. We have that $a = F' + \widehat{F'}$, so $\widehat{F'} = a - F'$. By the same strategy as in (b), we can show that the action on the tangent space at the identity of $a - F'$ is the same as the action on the tangent space at the identity of a. So $\widehat{F'}$ is separable iff a is, which happens exactly when $(a, q) = 1$. Since $\widehat{F'}$ is inseparable iff the Hasse invariant of X is zero by exercise IV.4.15, we see that the Hasse invariant of X is 0 if and only if $a \equiv 0 \pmod{p}$.

   The remaining assertion follows from the fact that $2\sqrt{p} < p$ for $p \geq 5$: if $2\sqrt{p} < p$, then the only value of a with $|a| \leq 2\sqrt{p}$ and $a \equiv 0 \pmod{p}$ is $a = 0$, giving $N = p + 1$.

Exercise IV.4.17. Let X be the curve $y^2 + y = x^3 - x$ of (4.23.8).

a. If $Q = (a, b)$ is a point on the curve, compute the coordinates of the point $P + Q$, where $P = (0, 0)$ as a function of a, b. Use this formula to find the coordinates of nP, $n = 1, 2, \ldots, 10$. [*Check*: $6P = (6, 14)$.]

b. This equation defines a nonsingular curve over $\mathbb{F}_p$ for all $p \neq 37$.

Solution.

a. Let's just quickly refresh our memory on how to do this: if we want to calculate $P + Q$, we take the line ℓ through P and Q, let R be the third point of intersection, and then $-R$ (obtained as the second point of intersection of the vertical line through R) is $P + Q$.

   When $P = Q$, then the line ℓ is just the tangent line at P, which is $x + y = 0$ and intersects X at $R = (-1, 1)$. The other intersection of $x = 1$ with X is $(1, 0)$, so $2P = (1, 0)$.

   When $P \neq Q$ and $a = 0$, the points of intersection of ℓ and X are ∞, P, and $(0, -1)$, with $\infty + P = P$ and $P + (0, 1) = \infty$.

   When $P \neq Q$ and $a \neq 0$, the line ℓ is $y = \frac{b}{a}x$ which plugs in to $y^2 + y = x^3 - x$ to get that the x-coordinates of the intersections of ℓ and X are 0 and the solutions of $a^2x^2 - b^2x - a(b+a) = 0$.

Since we know that $x = a$ is one of them, divide this polynomial by $x - a$ to get that $a^2x^2 - b^2x - a(b+a) = (x-a)(a^2x + a + b)$ using the fact that $a^3 - b^2 = a + b$. So $R = (-\frac{a+b}{a^2}, -\frac{b}{a} \cdot \frac{a+b}{a^2})$ and $P + Q$ are the two intersection points of $x = -\frac{a+b}{a^2}$ with X. Therefore the y-coordinate is the solution of $y^2 + y = (-\frac{a+b}{a^2})^3 + \frac{a+b}{a^2}$ which isn't $-\frac{b}{a} \cdot \frac{a+b}{a^2}$, and by doing polynomial division again we find that $P + Q = (-\frac{a+b}{a^2}, -1 + \frac{ab+b^2}{a^3})$.

This gives the following values of nP:

- $P = (0, 0)$
- $2P = (1, 0)$
- $3P = (-1, -1)$
- $4P = (2, -3)$
- $5P = (\frac{1}{4}, -\frac{5}{8})$
- $6P = (6, 14)$
- $7P = (-\frac{5}{9}, \frac{8}{27})$
- $8P = (\frac{21}{25}, -\frac{69}{125})$
- $9P = (-\frac{20}{49}, -\frac{435}{343})$
- $10P = (\frac{161}{16}, -\frac{2065}{64})$

b. Homogenizing, our curve is cut out by $y^2z + yz^2 - x^3 + xz^2$ which has Jacobian

$$\begin{pmatrix} -3x^2 + z^2 & 2yz + z^2 & y^2 + 2yz + 2xz \end{pmatrix}.$$

In characteristic 2, the singular points must satisfy $x^2 + z^2 = z^2 = y^2 = 0$, and therefore the curve is nonsingular. When the characteristic is not 2, make the change of variables $y \mapsto (y - \frac{1}{2}z)$ so that our equation becomes $y^2z - x^3 + xz^2 - \frac{1}{4}z^3$ which has Jacobian

$$\begin{pmatrix} -3x^2 + z^2 & 2yz & y^2 + 2xz - \frac{3}{4}z^2 \end{pmatrix}.$$

When $z = 0$, the third entry implies $y = 0$ so the only possible singular point at infinity would be $[1 : 0 : 0]$ which is not on our curve.

It remains to check $z = 1$. The second entry tells us that any singular point must have $y = 0$, so we must simultaneously solve the equations $1 - 3x^2 = 0$, $2x - \frac{3}{4} = 0$, and $x^3 - x + \frac{1}{4} = 0$ in order to have a singular point. The second equation tells us $x = \frac{3}{8}$, which when we plug in to the other two equations give $\frac{37}{64}$ and $\frac{-37}{512}$, respectively. As $\frac{37}{64} = \frac{-37}{512} = 0$ iff $p = 37$, we have shown the claim.

Exercise IV.4.18. Let X be the curve $y^2 = x^3 - 7x + 10$. This curve has at least 26 points with integer coordinates. Find them (use a calculator), and verify that they are all contained in the subgroup (maybe equal to all of $X(\mathbb{Q})$?) generated by $P = (1, 2)$ and $Q = (2, 2)$.

Solution. There's not really a great way to do this besides jumping in and calculating a bunch of values of $aP + bQ$. Once you accept this, the best way to compute lots of values is to make a computer do it for you. Here's Sage code which will compute $iP + jQ$ as $-d \leq i, j \leq d$ for integers i, j and print any values it finds which have integer coordinates:

```
E = EllipticCurve([0,0,0,-7,10])
P = E(1,2)
Q = E(2,2)
d = 10
for i in range(-d,d+1):
  for j in range(-d,d+1):
    R = i*P+j*Q
    if denominator(R[0])==denominator(R[1])==1:
      print(i,j,R)
```

You can download Sage yourself or run this on the web at `https://sagecell.sagemath.org/`. The upshot is that we find the following integer points, sorted in increasing order of their x-coordinate:

- $(-3, \mp 2)$: $\pm(P + Q)$
- $(-2, \pm 4)$: $\pm(2P + 3Q)$
- $(-1, \mp 4)$: $\pm 2P$
- $(1, \pm 2)$: $\pm P$
- $(2, \pm 2)$: $\pm Q$
- $(3, \mp 4)$: $\pm(2P + Q)$
- $(5, \pm 10)$: $\pm(3P + Q)$
- $(9, \mp 26)$: $\pm 3P$
- $(13, \pm 46)$: $\pm(P - Q)$
- $(31, \pm 172)$: $\pm(2P + 2Q)$
- $(41, \mp 262)$: $\pm(5P + 2Q)$
- $(67, \pm 548)$: $\pm(6P + Q)$
- $(302, \mp 5248)$: $\pm(4P - Q)$.

It turns out that these are indeed all of the integer points on this curve, and the group $X(\mathbb{Q})$ is in fact generated by P and Q. See *Integer Points on* $y^2 = x^3 - 7x + 10$ by Bremmer and Tzanakis, `https://doi.org/10.2307/2007708` for a reference (note that this paper was published in 1983, six years after Hartshorne's text).

Exercise IV.4.19. Let X, P_0 be an elliptic curve defined over $\mathbb{Q}$, represented as a curve in $\mathbb{P}^2$ defined by an equation with integer coefficients. Then X can be considered as the fibre over the generic point of a scheme $\overline{X}$ over $\operatorname{Spec}\mathbb{Z}$. Let $T \subset \operatorname{Spec}\mathbb{Z}$ be the open subset consisting of all primes $p \neq 2$ such that the fibre $X_{(p)}$ of $\overline{X}$ over p is nonsingular. For any n, show that $n_X : X \to X$ is defined over T, and is a flat morphism. Show that the kernel of n_X is also flat over T. Conclude that for any $p \in T$, the natural map $X(\mathbb{Q}) \to X_{(p)}(\mathbb{F}_p)$ induced on the groups of rational points, maps the n-torsion points of $X(\mathbb{Q})$ *injectively* into the torsion subgroup of $X_{(p)}(\mathbb{F}_p)$ for any $(n,p) = 1$.

By this method one can show easily that the groups $X(\mathbb{Q})$ in (Ex. 4.17) and (Ex. 4.18) are torsion-free.

Solution. Hartshorne is missing a small assumption here: T should consist of all primes $p \neq 2$ such that the fiber $X_{(p)}$ is nonsingular *and of genus one*. (We'll show later on that regular over a perfect field implies smooth, and by a combination of propositions III.9.3 and II.8.10, the geometric genus $\dim_k H^0(X, \Omega_{X/k})$ is preserved under field extensions for smooth curves. Therefore it really does make sense to think of the fibers as elliptic curves. This is one place where it would be nice to have developed technology for varieties over non-algebraically-closed fields more rigorously, but we'll muddle through anyways.)

Our strategy is that we will show that $\overline{X}_T \subset \mathbb{P}^2_T$ can be cut out by an equation of the form $y^2z = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$ for $a_i \in \mathcal{O}_T(T)$, then show that multiplication on X is defined over T so that n_X is defined over T (and the formulas are constant in T). This is both computationally intensive and quite long, so we'll save that for the end and discuss the other portions of the problems first, assuming that we've already proven these claims.

Flatness of n_X over T : To show that n_X is flat over T, we'll need a lemma about upgrading flatness on the fibers of $\overline{X}_T \to T$ to flatness of $\overline{X}_T \to T$.

Lemma. *Let S be a noetherian scheme, and let X and Y be flat locally noetherian S-schemes. Suppose $f : X \to Y$ is an S-morphism, and let $f_s : X_s \to Y_s$ denote the base change of f along the inclusion of a point $s \to S$. If $f_s : X_s \to Y_s$ is flat at $x \in X_s$, then f is flat at $x \in X$. In particular, if f_s is flat for all $s \in S$, then f is flat.*

Proof. The statement that f_s is flat at x means that $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is flat. By the local criteria for flatness mentioned in our solution to exercise III.9.4, it suffices to show that $\operatorname{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}, \mathcal{O}_{X,x}) = 0$.

To do this, choose a free resolution $F_\bullet$ of $\mathcal{O}_{S,s}/\mathfrak{m}_s$ as an $\mathcal{O}_{S,s}$-module. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y}$ is flat, we obtain a free resolution $F_\bullet \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ of $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$ which we may use to compute Tor. But the Tor functors are just the higher homology of $(F_\bullet \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong F_\bullet \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$, which is exact because $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{S,s}$.

■

To apply this lemma, we need to verify that n_X is flat when restricted to each fiber. Our goal is to use proposition II.6.8, which says a nonconstant map of curves is finite flat. This theorem is proven over an algebraically closed field though, so we need to do a little extra work before we can apply it.

Lemma. *Regular implies geometrically regular for schemes of finite type over perfect fields. That is, if $V \to \operatorname{Spec} k$ is a scheme of finite type over a perfect field k and $v \in V$ is a regular point, then all the points above v in $V_{\overline{k}} = V \times_k \overline{k}$ are regular.*

Proof. For $v \in V$, $\mathcal{O}_{V,v}$ is a regular local ring iff $\Omega_{X/k}$ is locally free of rank $\dim_v V$ at v by exercises II.8.1(b) and II.5.7(a). Let $V_{\overline{k}}$ be the base change of X to the algebraic closure. Then $\Omega_{V_{\overline{k}}/\overline{k}}$ is the pullback of $\Omega_{V/k}$ along the projection $V_{\overline{k}} \to V$ by proposition II.8.10, and as the pullback of a locally free sheaf is again locally free of the same rank, we may again apply exercise II.8.1(b) to see that $V_{\overline{k}}$ is regular at any point above v.

■

As $\overline{X}_T \to T$ is a projective morphism, it is of finite type. Because $\overline{X}_T \to T$ can be written $\overline{X}_T \to \mathbb{P}^2_T \to T$ with all fibers hypersurfaces of degree 3, they have the same Hilbert polynomial and by theorem III.9.9, $\overline{X}_T \to T$ is flat. From the material discussed in the second lemma combined with the observation that all residue fields of $\operatorname{Spec} \mathbb{Z}$ are perfect, we see that all the geometric fibers of $\overline{X}_T \to T$ are regular, and therefore $\overline{X}_T \to T$ is smooth.

From the material on faithful flatness we discussed in our solution to exercise III.10.4, it suffices to check flatness of n_X on any fiber $\overline{X}_{(p)}$ of $\overline{X} \to T$ after base change to the algebraic closure. Since $\overline{X}_T \to T$ is smooth, these geometric fibers are smooth curves, and here we may identify the base change of the multiplication-by-n map on the fibers of $\overline{X}_T$ to their algebraic closures to the multiplication-by-n map on the geometric fibers (this relies on calculations of the formula for n_X, which we will show is independent of the base ring), so multiplication by n for $n \neq 0$ is a nonconstant map of curves over an algebraically closed field and therefore by proposition II.6.8 is finite flat. Thus by the first lemma, n_X is flat over T. This immediately implies the kernel $\mathcal{K}$ of n_X is flat over T: it's the base change of $n_X : \overline{X}_T \to \overline{X}_T$ along the map $T \to \overline{X}_T$ sending $t \mapsto (t, [0:0:1])$ which picks out the identity element on each fiber.

Injection on torsion. Let's explain how we define a map from n-torsion points over $\mathbb{Q}$ to n-torsion points over $\mathbb{F}_p$. Given an n-torsion point $[a:b:c]$ in $X(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$, we may assume a, b, and c are coprime integers: clear denominators and then divide through by the greatest common divisor of the resulting integers. Therefore reducing each of a, b, c modulo p will always give a well-defined point in $\overline{X}_{(p)}(\mathbb{F}_p)$, and since the formulas for multiplication by n are defined over T, the reduced version of $[a:b:c]$ modulo p will still be an n-torsion point when $(p) \in T$.

To show that this is an injection where $(n,p) = 1$, we begin by showing that the kernel $\mathcal{K}$ of the multiplication by n map is regular over $T \cap D(n)$. If $\mathcal{K}_{(p)}$ is regular, then it's a disjoint union of spectra of fields, so we can immediately see from the fact that the local ring of any point in $\operatorname{Spec} \mathbb{Z}$ is regular that the local rings of the points in $\mathcal{K}$ over p are regular. It suffices to check that the base change of $\mathcal{K}$ to the algebraic closure of the residue field at a point in $T \cap D(n)$ is regular: since $\mathcal{K} \to T$ is finite, $\mathcal{K}$ and $\mathcal{K}_t$ are both affine and $\mathcal{K}_t$ is the spectrum of a finite $k(t)$-algebra. Therefore $\Omega_{\mathcal{K}_t/\operatorname{Spec} k(t)} \cong \Omega_{X/T}|_{\mathcal{K}_t}$ corresponds to a $k(t)$-module, and it's zero iff the pullback along the base change to the algebraic closure is. But when $(n,p) = 1$, the multiplication-by-n map is separable on an elliptic curve over an algebraically closed field of characteristic p and by Riemann-Hurwitz it is unramified, so the kernel consists of n^2 reduced points which are therefore smooth. So $\mathcal{K}$ is regular on $T \cap D(n)$.

Knowing that $\mathcal{K} \cap D(n)$ is regular means that no two irreducible components of $\mathcal{K} \cap D(n)$ can intersect. If we have two distinct n-torsion points $[a : b : c]$ and $[a' : b' : c']$ which both have gcd 1, then the closures of each of these points viewed in $\mathbb{P}^2_T$ give distinct irreducible components of $\mathcal{K}$, which don't intersect over $T \cap D(n)$. Therefore the reduction map must be injective, and we've shown the claim.

Getting a good equation. Now to the real work of the problem: showing that n_X is defined over T. First we need to do some preparatory work with the equation f of X over $\mathbb{Q}$. f is a homogeneous cubic in three variables with integer coefficients which is satisfied by the specified point $P_0 = [x_0 : y_0 : z_0]$ where $x_0, y_0, z_0 \in \mathbb{Q}$. Up to a scaling action, we may assume the coordinates of P_0 are integers with greatest common divisor 1. By applying the Euclidean algorithm, we can find a sequence of transformations of the form $v \mapsto v + mw$ where $v, w \in \{x, y, z\}$ and $m \in \mathbb{Z}$ which send P_0 to $[0 : 1 : 0]$ and maintain the integer coefficients of f. Therefore we may assume $P_0 = [0 : 1 : 0]$.

The tangent line of $V(f)$ at P_0 is of the form $\alpha x + \beta z = 0$ for some integers α, β not both 0. I claim there exists an integer-valued matrix with determinant one which transforms this in to the line $z = 0$: dividing by the greatest common divisor, we may assume $\gcd(\alpha, \beta) = 1$. Then by Bezout's identity, there exist $\gamma, \delta \in \mathbb{Z}$ such that $\alpha\delta - \beta\gamma = 1$, so the automorphism of $\mathbb{P}^2$ given by

$$\begin{pmatrix} \gamma & 0 & \delta \\ 0 & 1 & 0 \\ \alpha & 0 & \beta \end{pmatrix}$$

has integer entries and sends the line $V(\alpha x + \beta z)$ to $V(z)$ while maintaining the integer entries of f.

Now we'll use these pieces of geometric information to pin down some information about f. Since $[0 : 1 : 0]$ satisfies f, we may conclude that f has no y^3 term. As the tangent line to $[0 : 1 : 0]$ is the projective line cut out by

$$\frac{\partial f}{\partial x}([0:1:0])x + \frac{\partial f}{\partial y}([0:1:0])y + \frac{\partial f}{\partial z}([0:1:0])z = 0,$$

we must have $\frac{\partial f}{\partial x}([0 : 1 : 0]) = \frac{\partial f}{\partial y}([0 : 1 : 0]) = 0$ and $\frac{\partial f}{\partial z}([0 : 1 : 0]) \neq 0$. But the value of $\frac{\partial f}{\partial x}([0 : 1 : 0])$ is the coefficient of xy^2 and the value of $\frac{\partial f}{\partial z}([0 : 1 : 0])$ is the coefficient of y^2z, so we may conclude that f has no xy^2 term and the y^2z term appears with nonzero coefficient. Since P_0 is the identity for the group law on X, the tangent line to P_0 should intersect X at P_0 with multiplicity 3. The intersection multiplicity of X and its tangent line $V(z)$ at $[0 : 1 : 0]$ is the dimension of $\mathbb{Q}[x,z]_{(x,z)}/(z, f(x,1,z)) \cong \mathbb{Q}[x]_{(x)}/(f(x,1,0))$. As $f(x,1,0) = \varepsilon x^2 + \eta x^3$ where ε, η are the coefficients of x^2y and x^3 in f, we see that we must have $\varepsilon = 0$ and $\eta \neq 0$ in order to get intersection multiplicity three at $[0 : 1 : 0]$. So

$$f = c_1y^2z + c_2xyz + c_3yz^2 + c_4x^3 + c_5x^2z + c_6xz^2 + c_7z^3,$$

for $c_i \in \mathbb{Z}$ which we may assume to have greatest common divisor 1 and $c_1, c_4 \neq 0$.

To investigate the properties of $\overline{X}$ over T, we can work over the base ring $R = \mathcal{O}_T(T)$ which is a localization of $\mathbb{Z}$. From here on, we'll switch to considering f as the equation for $\overline{X} \cap \mathbb{P}^2_T \subset \mathbb{P}^2_T$, so $f \in \mathcal{O}_T(T)[x,y,z]$. I claim that c_1 is invertible in R: if p is a prime which divides c_1, then the reduction of f modulo p is not an elliptic curve because the point $[0:1:0]$ is singular and therefore $(p) \notin T$. Multiplying by $1/c_1$, we may assume $c_1 = 1$ in f. Since $(2) \notin T$ by assumption, 2 is invertible in R and we may apply the automorphism $y \mapsto y - \frac{1}{2}(c_2 x + c_3 z)$ to eliminate the xyz and yz^2 terms. Therefore we may assume $c_2 = c_3 = 0$ so our equation is of the form

$$y^2 z + c_4 x^3 + c_5 x^2 z + c_6 x z^2 + c_7 z^3 = 0.$$

Now we look at c_4: if p is a prime dividing c_4, then the reduction of f modulo p is divisible by z and therefore does not cut out an elliptic curve, so $(p) \notin T$. Thus we may assume c_4 is a unit in R. After applying the automorphism $x \mapsto -c_4 x$, $y \mapsto c_4^2 y$, we may multiply through by c_4^{-4} and now x^3 has coefficient -1, so we may take f to be of the form

$$y^2 z - (x^3 + a_2 x^2 z + a_4 x z^2 + a_6)$$

for $a_i \in R$. This is the form we'll use for the rest of the problem.

The smoothness trick. One trick we'll use several times later on is that the ideal $(x^3 + a_2 x^2 z + a_4 x z^2 + a_6, 3x^2 + 2a_2 x + a_4) \subset R[x]$ is the unit ideal. First, the intersection with $R \subset R[x]$ is nonzero: the resultant of the two defining polynomials is an element of R which is in this ideal, and the support of the resultant is exactly the locus where the two defining polynomials have a common solution. But over any $t \in T$, the ideal $(x^3 + a_2 x^2 z + a_4 x z^2 + a_6, 3x^2 + 2a_2 x + a_4) \subset k(t)[x]$ cuts out the singular locus of $V(y^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6) \subset \mathbb{A}^2_{k(t)}$, which is empty by assumption. So the resultant must be a unit in R and thus the ideal is the unit ideal.

Reduction to the multiplication map. In order to show that n_X is defined over T, we'll show that the multiplication map μ is defined over T, which will prove the claim: n_X can be inductively defined as the map $((n-1)_X, id_X) : X \to X \times X$ followed by $\mu : X \times X \to X$. To do this, we'll cover X by two affine open sets, $X' = D(z) \cap X$ and $X'' = D(y) \cap X$, and find explicit equations for the four maps $X' \times X' \to X$, $X' \times X'' \to X$, $X'' \times X' \to X$, and $X'' \times X'' \to X$.

The first chart. First we attack $X' \times X' \to X$. In the ambient space $D(z) \subset \mathbb{P}^2_T$, X' is cut out by the equation $y^2 = x^3 + a_2 x^2 + a_4 x + a_6$. Let (x_1, y_1) and (x_2, y_2) be two points on our curve. Define $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ and $\nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} = y_i - x_i \lambda$ so that $y = \lambda x + \nu$ is the line through (x_1, y_1) and (x_2, y_2). Plugging the equation for the line in to our equation for X', we get $(\lambda x + \nu)^2 = x^3 + a_2 x^2 + a_4 x + a_6$, or

$$0 = x^3 + (a_2 - \lambda^2) x^2 + (a_4 - 2\lambda\nu) x + (a_6 - \nu^2).$$

The roots of this polynomial in x are the three intersection points of the line $y = \lambda x + \nu$ with X'. Their sum is $-(a_2 - \lambda^2) = \lambda^2 - a_2$, and as x_1 and x_2 are roots, the third root must be $\lambda^2 - a_2 - x_1 - x_2$. By plugging this in to the equation for our line, the y-coordinate of the third intersection point is $\lambda(\lambda^2 - a_2 - x_1 - x_2) + \nu$. Then

$$\mu((x_1, y_1), (x_2, y_2)) = [\lambda^2 - a_2 - x_1 - x_2 : -\lambda(\lambda^2 - a_2 - x_1 - x_2) - \nu : 1].$$

Multiplying through by $(x_2 - x_1)^3$ to clear denominators, we get that $\mu((x_1, y_1), (x_2, y_2))$ is given by

$$[(x_2 - x_1)(y_2 - y_1)^2 - (x_2 - x_1)^3 (a_2 + x_1 + x_2) :$$

$$-(y_2-y_1)^3+(x_2-x_1)^2(a_2+x_1+x_2)(y_2-y_1)-(x_2-x_1)^2(y_1x_2-y_2x_1):$$
$$(x_2-x_1)^3],$$

which shows that multiplication is defined everywhere except possibly on common vanishing locus of these functions, which can easily seen to be $V(x_2-x_1, y_2-y_1)$, or the diagonal in $X'\times X'$.

To show that the map is defined on the diagonal, we use the relations $y_i^2 = x_i^3+a_2x_i^2+a_4x_i+a_6$ which hold identically to rewrite our expression for μ. From subtracting these two relations, we find that $y_2^2-y_1^2 = x_2^3-x_1^3+a_2(x_2^2-x_1^2)+a_4(x_2-x_1)$, or $(y_2-y_1)(y_2+y_1) = (x_2-x_1)(x_2^2+x_2x_1+x_1^2+a_2(x_2+x_1)+a_4)$. Letting $\rho = (x_2^2+x_2x_1+x_1^2+a_2(x_2+x_1)+a_4)$, this last expression turns in to $(y_2-y_1)(y_2+y_1) = (x_2-x_1)\rho$. Multiplying through by ρ^3 and rewriting $y_1x_2-y_2x_1 = y_1(x_2-x_1)-x_1(y_2-y_1)$ in the last portion of the second coordinate, our formula for μ becomes

$$[(y_2-y_1)^3(y_2+y_1)\rho^2-(y_2-y_1)^3(y_2+y_1)^3(a_2+x_1+x_2):$$
$$-(y_2-y_1)^3\rho^3+(y_2-y_1)^3(y_2+y_1)^2\rho(a_2+x_1+x_2)-y_1(y_2-y_1)^3(y_2+y_1)^3+x_1(y_2+y_1)^2(y_2-y_1)^3\rho:$$
$$(y_2-y_1)^3(y_2+y_1)^3],$$

which after dividing by $(y_2-y_1)^3$ becomes

$$[(y_2+y_1)\rho^2-(y_2+y_1)^3(a_2+x_1+x_2):$$
$$-\rho^3+(y_2+y_1)^2\rho(a_2+x_1+x_2)-y_1(y_2-y_1)^3(y_2+y_1)^3+x_1(y_2+y_1)^2\rho:$$
$$(y_2+y_1)^3].$$

This expression defines the map except on the common vanishing locus of $V(y_2+y_1,\rho)$, so we've defined μ on $X'\times X'$ except possibly on the locus $V(y_2-y_1, x_2-x_1, y_2+y_1, \rho)$.

I claim that $V(y_2-y_1, x_2-x_1, y_2+y_1,\rho)\subset X'\times X'$ is empty. Indeed, $R[X'\times X']/(y_2-y_1, x_2-x_1, y_2+y_1,\rho)\cong R[X']/(2y_1, 3x_1^2+2a_2x_1+a_4)$ by sending $x_2\mapsto x_1$ and $y_2\mapsto y_1$. But $R[X']/(2y_1, 3x_1^2+2a_2x_1+a_4)\cong R[x_1]/(x_1^3+a_2x_1^2+a_4x_1+a_6, 3x_1^2+2a_2x_1+a_4) = 0$ by the smoothness trick. Therefore we've shown $V(y_2-y_1, x_2-x_1, y_2+y_1,\rho)\subset X'\times X'$ is empty, and $\mu: X'\times X'\to X$ is defined over T.

The second chart. Now we analyze $\mu: X''\times X''\to X$. In the ambient space $D(z)\subset\mathbb{P}^2_T$, X'' is cut out by the equation $z^2 = x^3+a_2x^2z+a_4xz^2+a_6z^3$. Let (x_1,z_1) and (x_2,z_2) be two points on our curve. Define $\lambda = \frac{x_2-x_1}{z_2-z_1}$ and $\nu = \frac{x_1z_1-x_2z_1}{z_2-z_1}$ so that $x=\lambda z+\nu$ is the line through (x_1,z_1) and (x_2,z_2). Plugging $x=\lambda z+\nu$ in to $z = x^3+a_2x^2z+a_4xz^2+a_6z^3$ gives $0 = (\lambda z+\nu)^3+a_2(\lambda z+\nu)^2z+a_4(\lambda z+\nu)z^2+a_6z^3-z$, or

$$z^3(\lambda^3+a_2\lambda^2+a_4\lambda+a_6)+z^2(3\lambda^2\nu+2a_2\lambda\nu+a_4\nu)+z(3\lambda\nu^3+a_2\nu^2-1)+\nu^3.$$

The sum of the roots of this polynomial is $-\frac{3\lambda^2\nu+2a_2\lambda\nu+a_4\nu}{\lambda^3+a_2\lambda^2+a_4\lambda+a_6}$, so the third root is $-\frac{3\lambda^2\nu+2a_2\lambda\nu+a_4\nu}{\lambda^3+a_2\lambda^2+a_4\lambda+a_6}-z_1-z_2$. Multiplying through by $\frac{(z_2-z_1)^3}{(z_2-z_1)^3}$ and letting the new numerator and denominator be α and β respectively, we find

$$\alpha = (x_1z_2-x_2z_1)(3(x_2-x_1)^2+2a_2(x_2-x_1)(z_2-z_1)+a_4(z_2-z_1)^2),$$

$$\beta = (x_2-x_1)^3 + a_2(x_2-x_1)^2(z_2-z_1) + a_4(x_2-x_1)(z_2-z_1)^2 + a_6(z_2-z_1)^3$$

and we can write the multiplication map in coordinates as

$$[\lambda((z_1+z_2)\beta+\alpha) - \nu\beta : \beta : (z_1+z_2)\beta+\alpha].$$

I claim that even though I've written the first coordinate with divisions by (z_2-z_1), the function $\lambda((z_1+z_2)\beta+\alpha)-\nu\beta$ is regular on $X''\times X''$ - after expanding terms and factoring, $(z_2-z_1)\lambda((z_1+z_2)\beta+\alpha)-(z_2-z_1)\nu\beta$ can be written as

$$(z_2-z_1)(x_2-x_1)((z_1+z_2)(a_2(x_2-x_1)^2+a_4(x_2-x_1)(z_2-z_1)+a_6(z_2-z_1)^2)+$$

$$+(x_1z_2-x_2z_1)(2a_2(x_2-x_1)+a_4(z_2-z_1))-(x_2-x_1)^3),$$

so dividing by (z_2-z_1) really does give us a regular function on $X''\times X''$. The expression $[\lambda((z_1+z_2)\beta+\alpha)-\nu\beta : \beta : (z_1+z_2)\beta+\alpha]$ shows that μ is defined except possibly where all the coordinate functions vanish, which I claim is exactly $V(x_2=x_1, z_2=z_1)\subset X''\times X''$. It's clear that if $x_2=x_1$ and $z_2=z_1$, all of these coordinate functions vanish, whereas if all the coordinate functions vanish, then certainly α and β vanish, which I claim implies $x_2=x_1$ and $z_2=z_1$ (that is, $V(\alpha,\beta)\subset V(x_2-x_1, z_2-z_1)$).

If β vanishes and $z_2=z_1$, then we must have $x_2=x_1$, so $V(\alpha,\beta)\cap(V(x_2-x_1,z_2-z_1))^c$ is contained in $D(z_2-z_1)\subset X''\times X''$ and we may assume $z_2\neq z_1$. Now we split in to cases depending on which factor of α vanishes.

Suppose $x_1z_2-x_2z_1=0$. As we cannot have both z_1 and z_2 vanish, without loss of generality we may suppose $z_2\neq 0$ which gives $x_1=x_2\frac{z_1}{z_2}$ and $z_1=z_2\frac{z_1}{z_2}$. Plugging these relations in to β, we see that $\beta=(1-\frac{z_1}{z_2})^3(x_2^3+a_2x_2^2z_2+a_4x_2z_2^2+a_6z_2^3)$, or $\beta=(1-\frac{z_1}{z_2})^3z_2$ from applying our defining equation. As $z_1\neq z_2$, this implies $z_2=0$, contradiction. So $x_1z_2-x_2z_1\neq 0$ and $V(\alpha,\beta)=V(3(x_2-x_1)^2+2a_2(x_2-x_1)(z_2-z_1)+a_4(z_2-z_1)^2,\beta)$ outside $V(x_2-x_1,z_2-z_1)$.

Dividing through by the appropriate power of z_2-z_1, we see that $V(\alpha,\beta)=V(3u^2+2a_2u+a_4, u^3+a_2u^2+a_4u+a_6)$ outside $V(x_2-x_1,z_2-z_1)$ where $u=\frac{x_2-x_1}{z_2-z_1}$. But this is empty: considering the map from $R[U]$ to the coordinate algebra of $V(\alpha,\beta)\cap D(z_2-z_1)$ given by sending $U\mapsto u$, any prime ideal of the coordinate algebra of $V(\alpha,\beta)\cap D(z_2-z_1)$ containing $(3u^2+2a_2u+a_4, u^3+a_2u^2+a_4u+a_6)$ must have preimage a prime ideal of $R[U]$ containing $(3U^2+2a_2U+a_4, U^3+a_2U^2+a_4U+a_6)$. By the smoothness trick, $(3U^2+2a_2U+a_4, U^3+a_2U^2+a_4U+a_6)$ is the unit ideal, so there cannot be any such prime ideals in the coordinate algebra of $V(\alpha,\beta)\cap D(z_2-z_1)$ and therefore $V(\alpha,\beta)\cap D(z_2-z_1)$ is empty. This shows that $V(\alpha,\beta)\subset V(x_2-x_1,z_2-z_1)$.

To pursue the same rewriting strategy as in the first chart, we'll need to use the defining relations $z_i=x_i^3+a_2x_i^2z_i+a_4x_iz_i^2+a_6z_i^3$ to construct a relation of the form $(z_2-z_1)\sigma=(x_2-x_1)\tau$ for polynomials σ,τ. As the equation

$$z_2-z_1=x_2^3-x_1^3+a_2(x_2^2z_2-x_1^2z_1)+a_4(x_2z_2^2-x_1z_1^2)+a_6(z_2^3-z_1^3)$$

reduces to $0=0$ when plugging in $x_2=x_1$ and $z_2=z_1$, it is in the ideal (x_2-x_1,z_2-z_1) and we can find σ and τ from rewriting this equation. We'll choose

$$\sigma=1-a_6(z_2^2+z_2z_1+z_1^2)-a_2x_1^2-a_4(z_1+z_2)x_2,$$

$$\tau = (x_2^2 + x_2x_1 + x_1^2) + a_2(x_1 + x_2)z_2 + a_4z_1^2$$

so $(z_2 - z_1)\sigma = (x_2 - x_1)\tau$. Multiplying the coordinates of our map through by σ^3 and rewriting $x_1z_2 - x_2z_1 = x_1(z_2 - z_1) - z_1(x_2 - x_1)$, our formula for μ becomes

$$[a_2(x_2 - x_1)^3(z_1 + z_2)\sigma^3 + a_4(x_2 - x_1)^3(z_1 + z_2)\sigma^2\tau + a_6(x_2 - x_1)^3(z_1 + z_2)\sigma\tau^2 +$$

$$+(x_1(x_2 - x_1)^2\sigma\tau - z_1(x_2 - x_1)^2\sigma^2)(2a_2(x_2 - x_1)\sigma + a_4(x_2 - x_1)\tau) - (x_2 - x_1)^4\sigma^3 :$$

$$(x_2 - x_1)^3\sigma^3 + a_2(x_2 - x_1)^3\sigma^2\tau + a_4(x_2 - x_1)^3\sigma\tau^2 + a_6(x_2 - x_1)^3\tau^3 :$$

$$((x_2 - x_1)^3\sigma^3 + a_2(x_2 - x_1)^3\sigma^2\tau + a_4(x_2 - x_1)^3\sigma\tau^2 + a_6(x_2 - x_1)^3\tau^3)(z_1 + z_2) +$$

$$+(x_1(x_2 - x_1)\tau - z_1(x_2 - x_1)\sigma)(3(x_2 - x_1)^2\sigma^2 + 2a_2(x_2 - x_1)^2\sigma\tau + a_4(x_2 - x_1)^2\tau^2)]$$

which after dividing by $(x_2 - x_1)^3$ becomes

$$[a_2(z_1 + z_2)\sigma^3 + a_4(z_1 + z_2)\sigma^2\tau + a_6(z_1 + z_2)\sigma\tau^2 + (x_1\sigma\tau - z_1\sigma^2)(2a_2\sigma + a_4\tau) - (x_2 - x_1)\sigma^3 :$$

$$\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3 :$$

$$(\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3)(z_1 + z_2) + (x_1\tau - z_1\sigma)(3\sigma^2 + 2a_2\sigma\tau + a_4\tau^2)].$$

This shows that we've defined $\mu : X'' \times X'' \to X$ except possibly on the locus $V(x_2 - x_1, z_2 - z_1, \sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3, (x_1\tau - z_1\sigma)(3\sigma^2 + 2a_2\sigma\tau + a_4\tau^2))$, which I claim is empty.

Applying the relations $x_2 = x_1$ and $z_2 = z_1$, we get that $\sigma = 1 - 3a_6z_1^2 - a_2x_1^2 - 2a_4x_1z_1$ and $\tau = 3x_1^2 + 2a_2x_1z_1 + a_4z_1^2$ and we are reduced to considering $V(\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3, (x_1\tau - z_1\sigma)(3\sigma^2 + 2a_2\sigma\tau + a_4\tau^2)) \subset X''$. First,

$$x_1\tau - z_1\sigma = 3x_1^3 + 2a_2x_1^2z_1 + a_4x_1z_1^2 + a_2x_1^2z_1 + 2a_4x_1z_1^2 + 3a_6z_1^3 - z_1 = 3(x_1^3 + a_2x_1^2z_1 + a_4x_1z_1^2 + a_6z_1^3) - z_1,$$

and applying $x_1^3 + a_2x_1^2z_1 + a_4x_1z_1^2 + a_6z_1^3 = 0$ we see that $x_1\tau - z_1\sigma = -z_1$. But if $z_1 = 0$, then $x_1 = 0$, giving $\sigma = 1$ and $\tau = 0$, which don't solve $\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3$. So it suffices to consider $V(\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3, 3\sigma^2 + 2a_2\sigma\tau + a_4\tau^2) \subset X''$. By repeating the argument from before involving α and β, we see that $V(\sigma^3 + a_2\sigma^2\tau + a_4\sigma\tau^2 + a_6\tau^3, 3\sigma^2 + 2a_2\sigma\tau + a_4\tau^2) = V(\sigma, \tau)$. But σ and τ are the derivatives of the equation defining X'', which means $V(\sigma, \tau) = \emptyset$ by a variation on the smoothness trick we used before. Therefore we've shown that $\mu : X'' \times X'' \to X$ is defined over T.

The third chart. By symmetry, showing that μ is defined on $X' \times X''$ will show that it's defined on $X'' \times X'$, so it suffices to work with $X' \times X''$. In fact, as we've shown that μ is defined on $X' \times X'$ and $X'' \times X''$, we only need to make sure that $\mu : X' \times X'' \to X$ is defined on a neighborhood of $V(y) \times V(z)$: any point lying in either $D(y) \times X''$ or $X' \times D(z)$ lies in $X'' \times X''$ or $X' \times X'$, respectively. This will focus our calculations.

Let $[x_1 : y_1 : 1]$ and $[x_2 : 1 : z_2]$ be two points on X. The projective line through them is

$$(y_1z_2 - 1)x + (x_2 - x_1z_2)y + (x_1 - x_2y_1)z = 0,$$

which after dehomogenizing with respect to z and solving for y can be written

$$y = \frac{1 - y_1 z_2}{x_2 - x_1 z_2} x + \frac{x_2 y_1 - x_1}{x_2 - x_1 z_2}.$$

As before, define $\lambda = \frac{1-y_1 z_2}{x_2 - x_1 z_2}$ and $\nu = \frac{x_2 y_1 - x_1}{x_2 - x_1 z_2}$ so that our line is $y = \lambda x + \nu$. Plugging this in to our equation $y^2 = x^3 + a_2 x^2 + a_4 x + a_6$, we get

$$0 = x^3 + (a_2 - \lambda^2)x^2 + (a_4 - 2\lambda\nu)x + (a_6 - \nu^2),$$

which has sum of roots $\lambda^2 - a_2$. We already know two of the roots, x_1 and $\frac{x_2}{z_2}$, so the third root must be $\lambda^2 - a_2 - x_1 - \frac{x_2}{z_2}$. This lets us write μ in coordinates as

$$[\lambda^2 - a_2 - x_1 - \frac{x_2}{z_2} : -\lambda(\lambda^2 - a_2 - x_1 - \frac{x_2}{z_2}) - \nu : 1] = [\lambda^2 - (a_2 + x_1 + \frac{x_2}{z_2}) : -\lambda^3 + \lambda(a_2 + x_1 + \frac{x_2}{z_2}) - \nu : 1].$$

Define $A = 1 - y_1 z_2$, $B = x_2 - x_1 z_2$, $C = z_2(a_2 + x_1 + \frac{x_2}{z_2}) = a_2 z_2 + x_1 z_2 + x_2$, and $D = x_2 y_1 - x_1$, we can write $\lambda = \frac{A}{B}$ and $\nu = \frac{D}{B}$, so μ may be written as

$$\left[\frac{A^2}{B^2} - \frac{C}{z_2} : -\frac{A^3}{B^3} + \frac{A}{B}\frac{C}{z_2} - \frac{D}{B} : 1\right].$$

Multiplying through by $B^3 z_2$ to clear denominators, this gives us the expression

$$[A^2 B z_2 - B^3 C : -A^3 z_2 + AB^2 C - B^2 D z_2 : B^3 z_2]$$

for μ.

We immediately have issues when $x_2 = z_2 = 0$: this gives $B = 0$, and all the coordinates of our map vanish. To rectify this, we'll follow the same strategy as in the first two charts. One guiding principle will be that we should be in some sense working in the local ring at $\mathcal{O}_{X,[0:1:0]}$ - here x_2 has valuation 1 and z_2 has valuation 3, and we'll frequently sort terms by valuation in order make sure we handle things correctly. Let's expand the expressions we have for our coordinates.

Expanding the first coordinate, we find

$$A^2 B z_2 - B^3 C = x_2 z_2 - x_2^4 - a_2 x_2^3 z_2 + 2x_1 x_2^3 z_2 - x_1 z_2^2 - 2y_1 x_2 z_2^2 + 3a_2 x_1 x_2^2 z_2^2 + 2x_1 y_1 z_2^3$$
$$-3a_2 x_1^2 x_2 z_2^3 - 2x_1^3 x_2 z_2^3 + y_1^2 x_2 z_2^3 + a_2 x_1^3 z_2^4 + x_1^4 z_2^4 - x_1 y_1^2 z_2^4$$

and we notice that we can apply the rule $z_2 = x_2^3 + a_2 x_2^2 z_2 + a_4 x_2 z_2^2 + a_6 z_2^3$ to rewrite the valuation 4 part at the front: $z_2 - x_2^3 = a_2 x_2^2 z_2 + a_4 x_2 z_2^2 + a_6 z_2^3$, so $x_2 z_2 - x_2^4 = x_2(a_2 x_2^2 z_2 + a_4 x_2 z_2^2 + a_6 z_2^3)$, which gives

$$A^2 B z_2 - B^3 C = 2x_1 x_2^3 z_2 - x_1 z_2^2 - 2y_1 x_2 z_2^2 + a_4 x_2^2 z_2^2 + 3a_2 x_1 x_2^2 z_2^2 + 2x_1 y_1 z_2^3 + a_6 x_2 z_2^3$$
$$-3a_2 x_1^2 x_2 z_2^3 - 2x_1^3 x_2 z_2^3 + y_1^2 x_2 z_2^3 + a_2 x_1^3 z_2^4 + x_1^4 z_2^4 - x_1 y_1^2 z_2^4.$$

Expanding the second coordinate, we get

$$-A^3z_2 + AB^2C - B^2Dz_2 = x_2^3 - z_2 + a_2x_2^2z_2 - 2y_1x_2^3z_2 + 3y_1z_2^2 - 2a_2x_1x_2z_2^2 - 3x_1^2x_2z_2^2 - a_2y_1x_2^2z_2^2$$
$$+3x_1y_1x_2^2z_2^2 + a_2x_1^2z_2^3 + 2x_1^3z_2^3 - 3y_1^2z_2^3 + 2a_2x_1y_1x_2z_2^3 - a_2x_1^2y_1z_2^4 - x_1^3y_1z_2^4 + y_1^3z_2^4$$

and we notice that we can apply the rule $z_2 = x_2^3 + a_2x_2^2z_2 + a_4x_2z_2^2 + a_6z_2^3$ to rewrite the valuation 3 part at the front: $x_2^3 - z_2 = -(a_2x_2^2z_2 + a_4x_2z_2^2 + a_6z_2^3)$, which gives

$$-A^3z_2 + AB^2C - B^2Dz_2 = -2y_1x_2^3z_2 + 3y_1z_2^2 - 2a_2x_1x_2z_2^2 - 3x_1^2x_2z_2^2 - a_4x_2z_2^2 - a_2y_1x_2^2z_2^2$$
$$+3x_1y_1x_2^2z_2^2 + a_2x_1^2z_2^3 + 2x_1^3z_2^3 - 3y_1^2z_2^3 - a_6z_2^3 + 2a_2x_1y_1x_2z_2^3$$
$$-a_2x_1^2y_1z_2^4 - x_1^3y_1z_2^4 + y_1^3z_2^4.$$

Expanding the third coordinate is much quicker:

$$B^3z_2 = x_2^3z_2 - 3x_1x_2^2z_2^2 + 3x_1^2x_2z_2^3 - x_1^3z_2^4$$

which doesn't really require any simplification.

Now define $\sigma = 1 - a_2x_2^2 - a_4x_2z_2 - a_6z_2^2$ so $z_2\sigma = x_2^3$. After multiplying all three coordinates by σ^2, one sees that every term is divisible by x_2^6. Dividing by x_2^6, we find the following expressions for our coordinates:

$$(A^2Bz_2 - B^3C)\cdot\frac{\sigma^2}{x_2^6} = 2x_1\sigma - x_1 - 2y_1x_2 + a_4x_2^2 + 3a_2x_1x_2^2 + 2x_1y_16z_2 + a_6x_2z_2 - 3a_2x_1^2x_2z_2$$
$$-2x_1^3x_2z_2 + y_1^2x_2z_2 + a_2x_1^3z_2^2 + x_1^4z_2^2 - x_1y_1^2z_2^2,$$
$$(-A^3z_2 + AB^2C - B^2Dz_2)\cdot\frac{\sigma^2}{x_2^6} = -2y_1\sigma + 3y_1 - 2a_2x_1x_2 - 3x_1^2x_2 - a_4x_2 - a_2y_1x_2^2 + 3x_1y_1x_2^2$$
$$+a_2x_1^2z_2 + 2x_1^3z_2 - 3y_1^2z_2 - a_6z_2 + 2a_2x_1y_1x_2z_2 - a_2x_1^2y_1z_2^2 - x_1^3y_1z_2^2 + y_1^3z_2^2,$$
$$(B^3z_2)\cdot\frac{\sigma^2}{x_2^6} = \sigma - 3x_1x_2^2 + 3x_1^2x_2z_2 - x_1^3z_2^2.$$

Evaluating at $x_2 = z_2 = 0$, we find that $\sigma = 1$ and our map becomes $[x_1 : y_1 : 1]$. So $\mu : X' \times X'' \to X$ is defined in a neighborhood of $V(y) \times V(z)$, and we've shown that $\mu : X \times X \to X$ is defined over T, proving our claim and finishing the problem.

Exercise IV.4.20. Let X be an elliptic curve over a field k of characteristic $p > 0$, and let $R = \mathrm{End}(X, P_0)$ be its ring of endomorphisms.

a. Let X_p be the curve over k defined by changing the k-structure of X (2.4.1). Show that $j(X_p) = j(X)^{1/p}$. Thus $X \cong X_p$ over k if and only if $j \in \mathbb{F}_p$.

b. Show that p_X in R factors into a product $\pi\widehat{\pi}$ of two elements of degree p if and only if $X \cong X_p$. In this case, the Hasse invariant of X is 0 if and only if π and $\widehat{\pi}$ are associates in R (i.e., differ by a unit). (Use (2.5).)

c. If $\text{Hasse}(X) = 0$ show in any case $j \in \mathbb{F}_{p^2}$.

d. For any $f \in R$, there is an induced map $f^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. This must be multiplication by an element $\lambda_f \in k$. So we obtain a ring homomorphism $\varphi : R \to k$ by sending f to λ_f. Show that any $f \in R$ commutes with the (nonlinear) Frobenius morphism $F : X \to X$, and conclude that if $\text{Hasse}(X) \neq 0$, then the image of φ is $\mathbb{F}_p$. Therefore, R contains a prime ideal $\mathfrak{p}$ with $R/\mathfrak{p} \cong \mathbb{F}_p$.

Solution.

a. By the reasoning of exercise IV.4.16(a), the Frobenius map $X_p \to X$ acts on coordinates by raising them to the p^{th} power. Therefore if X is embedded in $\mathbb{P}^2$ with affine equation $y^2 = x(x-1)(x-\lambda)$, then $X_p \to X$ will be embedded by $y^2 = x(x-1)(x-\lambda^{1/p})$. Plugging $\lambda^{1/p}$ in to the formula for the j-invariant, we see that $j(X_p)$ is a p^{th} root of $j(X)$ because raising everything to the p^{th} power is a homomorphism which gives us back $j(X)$.

b. p_X may be written as the composition of $F : X_p \to X$ and $\widehat{F} : X \to X_p$, so if $\alpha : X_p \to X$ is an isomorphism, then $p_X = (\alpha \circ \widehat{F}) \circ (F \circ \alpha^{-1})$ and both entries in the RHS are of degree p. Further, the Hasse invariant of X being zero is equivalent to $\widehat{F}$ being purely inseparable by exercise IV.4.15, and by proposition IV.2.5 $\widehat{F}$ being purely inseparable means that up to an automorphism, $F = \widehat{F}$ and therefore they are associates in R.

   On the other hand, suppose p_X factors as a product of two elements π, $\widehat{\pi}$ of degree p. At least one of these elements must be purely inseparable, say π. Writing $p_X = \pi \circ \widehat{\pi}$, we see that $\pi : X \to X$ is a purely inseparable morphism of curves of degree p, and so by proposition IV.2.5 we must have $X \cong X_p$.

c. Hasse invariant zero implies that p_X is purely inseparable, so by an application of proposition IV.2.5 we have that $X_{p^2} \cong X$. But this means their j-invariants must be the same, and by the logic of (a) we have $j(X_{p^2})^{p^2} = j(X)$. But the solutions to $x^{p^2} = x$ are just the elements of $\mathbb{F}_{p^2}$, so $j(X) \in \mathbb{F}_{p^2}$.

d. Consider $f \circ F$: the (set-theoretic) kernel of this morphism is the same as the (set-theoretic) kernel of $F \circ f$, so by our proof that a separable isogeny is determined by its kernel from exercise IV.4.9, the maps $f \circ F$ and $F \circ f$ are the same iff they have the same inseparable degree. But that's clearly true, so $f \circ F = F \circ f$.

   Next, if f and F commute, then $F^* f^*$ and $f^* F^*$ are equal as maps $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$, which means that $\lambda_f F^* = F^*(\lambda_f) = \lambda_f^p F^*$. So if $\text{Hass}\,E(X) \neq 0$, then $\lambda_f^p = \lambda_f$ and so $\lambda_f \in \mathbb{F}_p$. Therefore φ has image $\mathbb{F}_p$ and we may take $\mathfrak{p} = \ker \varphi$.

Exercise IV.4.21. Let O be the ring of integers in a quadratic number field $\mathbb{Q}(\sqrt{-d})$. Show that any subring $R \subset O$, $R \neq \mathbb{Z}$, is of the form $R = \mathbb{Z} + f \cdot O$, for a uniquely determined integer $f \geq 1$. This integer f is called the *conductor* of the ring R.

Solution. For any square-free d, the ring of integers of $\mathbb{Q}(\sqrt{d})$ (the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$) is $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d = 1 \pmod 4$ and $\mathbb{Z}[\sqrt{d}]$ otherwise. The minimal polynomial of $a+b\sqrt{d}$ with $a, b \in \mathbb{Q}$, $b \neq 0$ is $x^2 - 2ax + (a^2 - db^2)$, so if $a + b\sqrt{d}$ is monic then $2a$ and $a^2 - db^2$ should be integers. If $2a$ is odd, then $a^2 - db^2$ is an integer iff $2b$ is odd and $d = 1 \pmod 4$. If $2a$ is even, or a is an integer, b must also be an integer as d is square-free. In either case, $O \cong \mathbb{Z}^2$ as $\mathbb{Z}$-modules where the isomorphism is given by taking 1 and $\frac{1+\sqrt{d}}{2}$ or 1 and $\sqrt{d}$ as the generators. (In general, if K is a number field with $[K : \mathbb{Q}] = n$, $O \cong \mathbb{Z}^n$ as $\mathbb{Z}$-modules - consult your favorite algebraic number theory text for a proof.)

Thus any order R in $\mathbb{Q}(\sqrt{-d})$ not equal to $\mathbb{Z}$ is equivalent to a subgroup of $\mathbb{Z}^2$ strictly containing $\mathbb{Z} \times \{0\}$, and therefore must be of the form $\mathbb{Z} \times f\mathbb{Z}$ for some uniquely determined positive integer f. This f is exactly the conductor.

Exercise IV.4.22. (*) If $X \to \mathbb{A}^1_{\mathbb{C}}$ is a family of elliptic curves having a section, show that the family is trivial. [*Hints*: Use the section to fix the group structure on the fibres. Show that the points of order 2 on the fibres form an étale cover of $\mathbb{A}^1_{\mathbb{C}}$, which must be trivial, since $\mathbb{A}^1_{\mathbb{C}}$ is simply connected. This implies that λ can be defined on the family, so it gives a map $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0, 1\}$. Any such map is constant, so λ is constant, so the family is trivial.]

Solution. Heads up: we're not really going to follow the hint here. Hartshorne doesn't really specify enough detail about what he means when he says that $X \to \mathbb{A}^1_{\mathbb{C}}$ is a family. We'll assume that he means a projective family, and we'll name this morphism f. Note that by theorem III.9.9, semicontinuity, and the fact that the fibers over the closed points are all elliptic curves, this implies f is flat.

Our first goal is to show that X is integral and f (and therefore X) is smooth. I claim first that X is reduced. Let $\operatorname{Spec} A \subset X$ be an arbitrary affine open subset. Since $\operatorname{Spec} A \to X \to \mathbb{A}^1_{\mathbb{C}}$ is flat, A is flat over $\mathbb{C}[t]$ and therefore by considering the injection $(t-t_0) \otimes_{\mathbb{C}[t]} A \to A$ for arbitrary $t_0 \in \mathbb{C}$, we see that A has no $(t-t_0)$ torsion for any $t_0 \in \mathbb{C}$. Thus if A has any nontrivial nilpotent elements, they must descend to nontrivial nilpotents of $A/(t - t_0)A$ for some $t_0 \in \mathbb{C}$. But $A/(t - t_0)A$ is the coordinate ring of an affine open subscheme of an elliptic curve over $\mathbb{C}$ and is therefore reduced. Thus A is reduced, and so X is also reduced. Next I claim that X is irreducible. Suppose X has multiple irreducible components $X_1, \cdots, X_n$. Let $X_i^\circ = X \setminus \bigcup_{j \neq i} X_j$ be the open subset of X which consists of points in X_i which are on only one irreducible component. By exercise III.9.1, f is open, so $f(X_i^\circ)$ is a nonempty open subset of $\mathbb{A}^1_{\mathbb{C}}$ for each i and the preimage of any point in $\bigcap f(X_i^\circ)$ has n connected components. But the fiber over any point is an elliptic curve, which is connected. Thus $n = 1$ and X is irreducible, so X is integral. Exercise III.10.2 then gives that for every point of $\mathbb{A}^1_{\mathbb{C}}$, there is an open neighborhoood U so that $f|_{f^{-1}(U)}$ is smooth, and therefore since any open subset containing all the closed points of $\mathbb{A}^1_{\mathbb{C}}$ must be the whole of $\mathbb{A}^1_{\mathbb{C}}$, f is smooth. Hence X is smooth as well.

Now we'll show that X admits an embedding in to $\mathbb{P}^2_{\mathbb{A}^1_{\mathbb{C}}}$ where it's cut out by a Weierstrass equation. We'll use the section $s : \mathbb{A}^1_{\mathbb{C}} \to X$ in order to find the appropriate line bundle. From the existence of a section s, we know that f is surjective. Therefore an application of exercise II.3.22 shows that X is of dimension two. As f is projective, it is separated, so the map s is a closed immersion by an application of exercise II.4.8, and $s(\mathbb{A}^1_{\mathbb{C}})$ is a divisor on the smooth surface X.

Let $\mathcal{L}$ be the invertible sheaf corresponding to this divisor. For any closed point $p \in \mathbb{A}^1_{\mathbb{C}}$, consider $\mathcal{L}|_{X_p} \cong \mathcal{O}_{X_y}(s(p))$: since X_p is an elliptic curve over $\mathbb{C}$, we have that for $n > 0$, $\mathcal{O}_{X_p}(s(p))^n$ has n-dimensional H^0 and 0-dimensional H^1, so by semicontinuity (theorem III.12.8 and corollary III.12.9) we find that for $n > 0$, $f_*\mathcal{L}^n$ is a locally free sheaf of rank n, the map $f_*\mathcal{L}^n \otimes k(p) \to H^0(X_p^n, \mathcal{L}_{X_p})$ is surjective, and all higher direct images of $\mathcal{L}^n$ vanish. Since $\mathbb{A}^1_{\mathbb{C}}$ is affine and $f_*\mathcal{L}^n$ is coherent, we have that it's $\widetilde{M}$ for some finitely generated $\mathbb{C}[t]$-module which is projective of rank n. By the structure theorem for finitely generated modules over a PID, we find that $M \cong (\mathbb{C}[t])^{\oplus n}$.

I claim that for $n > 0$, $\mathcal{L}^n$ is globally generated: if not, the base locus is a closed subscheme of X, and it must intersect X_p for some closed $p \in \mathbb{A}^1_{\mathbb{C}}$. But this is impossible, since $\mathcal{L}^n|_{X_p}$ is globally generated for each p and $f_*\mathcal{L}^n \otimes k(p)$ surjects on to $H^0(X_p, \mathcal{L}^n|_{X_p})$. Next I claim that the map $\varphi : X \to \mathbb{P}^2_{\mathbb{A}^1_{\mathbb{C}}}$ induced by $\mathcal{L}^3$ is a closed immersion. By the following lemma, it is enough to prove that $\varphi_p : X_p \to \mathbb{P}^2_p$, the base change of our $\mathbb{A}^1_{\mathbb{C}}$-map along $p \to \mathbb{A}^1_{\mathbb{C}}$ is a closed immersion for all points $p \in \mathbb{A}^1_{\mathbb{C}}$:

Lemma. *Suppose X and Y are projective S-schemes and $f : X \to Y$ is a projective S-morphism. If $f_s : X_s \to Y_s$ is a closed immersion for all $s \in S$, then f is a closed immersion. Furthermore, if S is of finite type over a field, then it suffices to check this condition for just the closed $s \in S$.*

Proof. Since f_s is topologically the induced morphism on the fibers of X and Y over s, injectivity of f_s for all s implies injectivity of f. By exercise II.4.8, f is projective, so it is closed. Therfore being a continuous closed injective map, f is a homeomorphism on to a closed subset of Y. By exercise III.11.2, f is finite as it is quasi-finite and projective. Therefore $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{Y,f(x)}$-module for any $x \in X$. By the assumption that f_s is a closed immersion for all s, we see that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ becomes surjective after tensoring with $\mathcal{O}_{S,s}/\mathfrak{m}_s$ where $x \mapsto s$. An application of the third isomorphism theorem shows that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is then surjective after tensoring with $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$, so by Nakayama $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective and therefore $X \to Y$ is a closed immersion.

When S is finite type over a field, we alter our strategy slightly. It is enough to show that f restricted to each irreducible component of X equipped with the reduced induced scheme structure is quasi-finite, since X is noetherian and therefore has finitely many irreducible components and these reductions don't change the topological fibers. Replacing Y with the scheme-theoretic image of X, we find ourselves in the situation of exercise II.3.22: by (d), the set of points with fibers of dimension greater than zero form a closed subset, which if they are nonempty must have a closed point since we're working with schemes of finite type over a field. But over every closed point $s \in S$, f_s is a closed immersion and therefore has finite fibers. Thus f is quasi-finite, and we may again apply exercise III.11.2 to find that f is finite. To see that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, let $\mathcal{Q}$ be the cokenrel: since $f_*\mathcal{O}_X$ is coherent, $\mathcal{Q}$ must also be coherent, and therefore the support of $\mathcal{Q}$ must contain a closed point if it is nonempty. But $\mathcal{Q}$ has zero fiber at all closed points by the assumption that f_s is a closed immersion over all closed $s \in S$ and the logic from the first case, so we see that $\mathcal{Q}$ must actually be zero and therefore we've shown f is a closed immersion.

■

When p is a closed point in $\mathbb{A}^1_{\mathbb{C}}$, we see that the map φ_p is the map induced by $(\mathcal{L}^3)_p \cong \mathcal{O}_{X_p}(3s(p))$, which is a closed immersion by proposition IV.3.3. Therefore by an application of the lemma $X \to \mathbb{P}^2_{\mathbb{A}^1_{\mathbb{C}}}$ is a closed immersion. (One may also show this by using fpqc descent plus cohomology and base change - it's a much cleaner argument, but requires more prerequisites. The idea is that base changing the generic fiber to the algebraic closure of $\mathbb{C}(t)$ will give an elliptic curve by smoothness and checking $h^1(\Omega)$, and the induced morphism from the base change of $\mathcal{L}^3$ will be a closed immersion, which implies that the original morphism must have been a closed immersion by descent.)

Choosing $\{1, x, y\}$ as a basis for $f_*\mathcal{L}^3$ as in proposition IV.4.6, we see that the same methods imply we have a $\mathbb{C}[t]$-linear dependence relation among $\{1, x, y, x^2, xy, x^3, y^2\}$. If the coefficient of x^3 or y^2 were to vanish at $t \in \mathbb{A}^1_{\mathbb{C}}$, then we would get a $\mathbb{C}$-linear dependence relation among those same variables viewed as sections of the very ample sheaf $\mathcal{O}_{X_t}(3s(t))$ with the coefficient of x^3 or y^2 zero, which is impossible by the logic of proposition IV.4.6. Therefore the coefficients of x^3 and y^2 are elements in $\mathbb{C}[t]$ which do not vanish anywhere, and hence nonzero constants. Up to a scaling, this means we may assume we have an equation of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ with $a_i \in \mathbb{C}[t]$. Completing the square on the left, we may assume $a_1 = a_3 = 0$, and sending $x \mapsto x - a_2/3$ on the right, we may assume we have an equation of the form $y^2 = x^3 + px + q$ for $p, q \in \mathbb{C}[t]$. Evaluating this at any $t \in \mathbb{A}^1_{\mathbb{C}}$ will produce an equation which the fiber X_t must satisfy in $\mathbb{P}^2_t$, which we see is the equation of the elliptic curve X_t.

Since all of our fibers are elliptic curves, they are smooth. In particular, the derivative of our equation with respect to x and the RHS of our equation cannot be simultaneously zero for any value of t. Eliminating x from $3x^2 + p$ and $x^3 + px + q$ via the resultant, we find that $4p^3 + 27q^2$ vanishes at $t = t_0$ iff $3x^2 + p$ and $x^3 + px + q$ do, so it must be the case that $4p^3 + 27q^2$ is a nonzero constant c independent of t. I claim the only way this can happen is if p and q are constant. Start by considering a map $\mathbb{C}[P, Q]/(4P^3 + 27Q^2 - c) \to \mathbb{C}[t]$ given by sending $P \mapsto p$ and $Q \mapsto q$. This naturally extends to a map $\mathbb{P}^1 \to \operatorname{Proj} \mathbb{C}[P, Q, R]/(4P^3 + 27Q^2R - cR^3)$, and the latter is an elliptic curve. If this map was non-constant, Riemann-Hurwitz would show that the degree of the ramification divisor would be -2, so the map must be constant and p, q are constants. Therefore our family must be trivial.

IV.5 The Canonical Embedding

Hartshorne plays pretty fast and loose with moduli spaces here. It pretty much all works out, but if you're really interested in a rigorous accounting you should consult another source.

Exercise IV.5.1. Show that a hyperelliptic curve can never be a complete intersection in any projective space. Cf. (Ex. 3.3).

Solution. By our work in exercise IV.3.3, if X is the complete intersection of $n-1$ hypersurfaces of degree $d_1, \cdots, d_{n-1}$ in $\mathbb{P}^n$, we have $\omega_X \cong \mathcal{O}_X(r)$, where $r = -n-1+\sum_{i=1}^{n-1} d_i$. Since $l(K) = g$ by Riemann-Roch and we assume that $g \geq 2$, we see that $r > 0$ and therefore ω_X is very ample. But by proposition IV.5.2, $|K|$ is very ample iff X is not hyperelliptic.

Exercise IV.5.2. If X is a curve of genus ≥ 2 over a field of characteristic 0, show that the group $\operatorname{Aut} X$ of automorphisms of X is finite. [*Hint*: If X is hyperelliptic, use the unique g^1_2 and show that $\operatorname{Aut} X$ permutes the ramification points of the 2-fold covering $X \to \mathbb{P}^1$. If X is not hyperelliptic, show that $\operatorname{Aut} X$ permutes the hyperosculation points (Ex. 4.6) of the canonical embedding. Cf. (Ex. 2.5).]

Solution. If X is hyperelliptic, then by proposition IV.5.3, it has a unique g^1_2. This means that given any two degree-two maps $f_1, f_2 : X \to \mathbb{P}^1$, there's an automorphism $\sigma \in \operatorname{Aut} X$ and an automorphism $\tau \in \operatorname{Aut} \mathbb{P}^1$ so that $f_1 \circ \sigma = \tau \circ f_2$: pullback of the unique g^1_2 along σ gives an automorphism of the g^1_2, so the two maps to $\mathbb{P}^1$ given by choosing bases of the global sections differ by an automorphism of $\mathbb{P}^1$ (see II.7 for a reminder if necessary). Therefore we have a homomorphism from $\operatorname{Aut} X$ to the symmetric group on the ramification points, of which there are $2g+2$ by Riemann-Hurwitz. The kernel consists of automorphisms of $k(X)$ fixing $k(\mathbb{P}^1) \subset k(X)$, of which there are at most $[k(X) : k(\mathbb{P}^1)] = 2$ by Galois theory. Therefore $\operatorname{Aut} X$ is an extension of finite groups and therefore finite.

If X is not hyperelliptic, $|K|$ is very ample and therefore embeds X in to $\mathbb{P}^{g-1}$ as a curve of degree $2g-2$. The hyperosculation points of this embedding are exactly the points $P \in X$ with a global section $s \in \Omega_X(X)$ so that s vanishes to order at least g at P. Since Ω_X is preserved by any automorphism, we see that this condition is preserved by automorphisms and so $\operatorname{Aut} X$ acts on the set of hyperosculation points. Since there are $n(n+1)(g-1)+(n+1)d = g^3-g$ hyperosculation points in characteristic zero by exercise IV.4.6, all we need to do is to show the kernel of this map is finite in order to show $\operatorname{Aut} X$ is finite. In fact, we'll show that the kernel is trivial by showing that automorphism of a genus g curve fixing more than $2g+2$ points is trivial. This proves the claim because $g^3-g > 2g+2$ for $g > 2$ (all curves of genus 2 are hyperelliptic anyways, so the only time we'd use the hyperosculation points is when $g > 2$).

To show that a nontrivial automorphism σ of a curve X of genus g has at most $2g+2$ fixed points, I claim that given a nontrivial automorphism σ, we can always find a nonconstant rational function $f \in K(X)$ with at most $g+1$ poles (counted with multiplicity) so that $f - \sigma f$ is not constant. Then $f - \sigma f$ has a zero at every fixed point of σ, and $f - \sigma f$ has at most $2g+2$ poles because each of f and σf have at most $g+1$ poles. As the number of poles and zeroes of any nonconstant rational function on a curve are equal (they're both equal to the degree of the map to $\mathbb{P}^1$ given by f), this proves that σ has at most $2g+2$ fixed points.

First, for any effective divisor of degree $g+1$, we can find a nonconstant function with poles contained in D: by Riemann-Roch, $l(D)-l(K-D)=g+1-g+1=2$, so $l(D)\geq 2$, and therefore $l(D)$ contains a nonconstant rational function. Next, we can always find a degree $g+1$ effective divisor which isn't fixed by σ: just pick some point $x\in X$ not fixed by σ (which we can always find because σ is not the identity) and look at $(g+1)x$. So $(f-\sigma f)_\infty$ is of degree at most $2g+2$ and therefore $(f-\sigma f)_0$ is of degree at most $2g+2$ and thus can contain at most $2g+2$ points, and we've won.

Exercise IV.5.3. *Moduli of Curves of Genus* 4. The hyperelliptic curves of genus 4 form an irreducible family of dimension 7. The nonhyperelliptic ones form an irreducible family of dimension 9. The subset of those having only one g^1_3 is an irreducible family of dimension 8. [*Hint*: Use (5.2.2) to count how many complete intersections $Q\cap F_3$ there are.]

Solution. Two counts have already been covered in the text. By remark IV.5.5.4, $\mathfrak{M}_4$ is an irreducible quasi-projective variety of dimension $3\cdot 4-3=9$. By example IV.5.5.5 (and the techniques of exercise IV.2.2), we can see that the moduli space of hyperelliptic curves of genus 4 form an irreducible subvariety of dimension $2g-1=7$.

One count will require us to do some work. By example IV.5.5.2, X has a unique g^1_3 exactly when it is the intersection of a singular quadric cone and a cubic hypersurface. The space of quadratic equations is $\binom{3+2}{2}=10$ dimensional, and the requirement that our equation cut out something singular is that the 4×4 matrix of partial derivatives is of rank exactly three (this is the same as the usual calculation with the matrix representing our quadratic form, except it also works in characteristic two, which is pretty neat). Rank 3 or less is a codimension-one condition enforced by the determinant, and rank at least three is given by the union of the non-vanishing loci of the 3×3 determinants, so it's an open condition among the $10-1=9$ dimensional space of quadratic equations cutting out singular quadrics. The space of cubic equations is $\binom{3+3}{3}=20$ dimensional, but we need to identify two equations if they differ by a linear multiple of our quadratic equation cutting out the quadric cone. This cuts down the space of cubic equations to $20-4=16$ dimensions. Projectivizing on both factors, we see that the space of choices of a quadric cone and a cubic section on it is of dimension $9-1+16-1=23$. $\operatorname{Aut}\mathbb{P}^3=PGL(4)$, which is $4^2-1=15$ dimensional, acts on this, and so the quotient is of dimension $23-15=8$ as desired.

Exercise IV.5.4. Another way of distinguishing curves of genus g is to ask, what is the least degree of a birational plane model with only nodes as singularities (3.11)? Let X be nonhyperelliptic of genus 4. Then:

a. if X has two g^1_3's, it can be represented as a plane quintic with two nodes, and conversely;

b. if X has one g^1_3, then it can be represented as a plane quintic with a tacnode (I, Ex.5.14d), but the least degree of a plane representation with only nodes in 6.

Solution. Recall from example IV.5.5.2, the canonical embedding of a nonhyperelliptic curve X of genus 4 lands on a unique irreducible quadric surface $Q\subset\mathbb{P}^3$. X has two g^1_3s when Q is nonsingular and one g^1_3 otherwise.

a. Fix $P \in X$ and consider the projection π from P. If $\ell \subset \mathbb{P}^3$ passes through P and intersects X with total multiplicity at least three, then it must intersect Q with total multiplicity at least three, and by Bezout it must lie inside Q. As there are only two lines in Q through any point on Q, the map $X \to \pi(X)$ is an isomorphism except at potentially along the intersection of X with either of those two lines. Therefore $\pi(X)$ is a plane curve of degree 5, and it remains to show that it has two nodes. If we can pick P so that the two lines through P in Q both intersect X in three distinct points, and for either line none of the tangent directions of the points of intersection are coplanar, then projection from P will give that $\pi(X)$ has two nodes.

   The first condition is easy: this is the fiber of the projection from X to either the first or second factor of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ containing no ramification points. As X is a divisor of type $(3,3)$ on Q, these projections are of degree three and therefore they are either separable or purely inseparable. Since $X \not\cong \mathbb{P}^1$, these projections cannot be purely inseparable, else we would contradict proposition IV.2.5. Therefore each projection on to the factors of Q is separable and thus generically unramified. This actually also implies the second condition: the tangent plane at two distinct points (a,b) and (a,c) in $\mathbb{P}^1 \times \mathbb{P}^1$ have common intersection $a \times \mathbb{P}^1$, so as long as the tangent lines are not the line $\{a\} \times \mathbb{P}^1$, they're not coplanar. Since we chose P so that X intersects the two lines in Q through P in three distinct points, none of these points has tangent line equal to either of these lines. Therefore $\pi(X)$ has at most nodes as singularities and at most two nodes. By the degree-genus formula and exercise IV.1.8, we see that a plane quintic with at most nodes as singularities has genus $6-r$ where r is the number of nodes, and therefore $\pi(X)$ is actually a plane quintic with two nodes.

   Conversely, if X can be represented as a plane quintic X' with two nodes, the projection from each node is a degree-three map to $\mathbb{P}^1$. To show that these are distinct, pick a line ℓ through the first node but not the second which meets X' in three distinct non-node points x_1, x_2, x_3. If the two projections were the same g^1_3, then the other projection should also send x_1, x_2, x_3 to the same point. But the line through these three points does not meet the other node, and we're done.

b. If X has one g^1_3, then it's the complete intersection of a quadric cone Q and a cubic hypersurface C which does not containing the cone point of the quadric. Consider the projection π from a point $P \in X$. If a line through P intersects X with multiplicity at least three, then it must intersect Q with multiplicity and least three and therefore be contained in Q by Bezout. Since a quadric cone contains only one line through any non-cone-point, this means that $X \to \pi(X)$ is an isomorphism away from the points on the line $L \subset Q$ through P. Therefore $\pi(X)$ is a plane quintic with at most one singular point, and it remains to verify the condition about the tacnode.

   The projection map from the cone point to $\mathbb{P}^2$ sends X to a smooth conic in $\mathbb{P}^2$, or a $\mathbb{P}^1$. This map is of degree three, and by the same logic as (a) it is generically unramified. Therefore we may choose P so that ℓ, the line in Q through P contains three points of X. Call these other two points P_1, P_2. By the same argument as (a), the tangent lines to P_1 and P_2 are not ℓ. On the other hand, they are coplanar this time: the tangent planes to any smooth point on the same line in Q agree. Therefore the projection from P identifies P_1 and P_2 as well as their tangent lines.

Now it remains to check that this gives a tacnode per the definition in exercise I.5.14(d). We'll have to assume char $k \neq 2$ here for that result to hold. Since π is birational and P_1 and P_2 are the only points of X lying over $\pi(P_1) \in \pi(X)$, we have that $\mathcal{O}_{\pi(X),\pi(P_1)} = \mathcal{O}_{X,P_1} \cap \mathcal{O}_{X,P_2}$ in $k(X)$ by theorem II.4.11A. Next, we observe that if H is a line in $\mathbb{P}^2$ cut out by a linear form l, then $\pi^{-1}(H)$ is cut out by l as well, so when we compute the intersection multiplicity of $\pi(X)$ with H at $\pi(P_1)$, we get the same thing as if we sum the intersection multiplicities of X with $V(H) \subset \mathbb{P}^3$. Noting that P_1 and P_2 have coplanar tangent lines, any plane containing ℓ but missing this common tangent direction will give an intersection multiplicity of two. On the other hand, the plane which contains these common tangent lines can only intersect X with multiplicity two at each of P, P_1, and P_2: since X is not contained in a plane, the plane intersects X with total multiplicity six and at least multiplicity two at each of those three points. Therefore $i(H, \pi(X); \pi(P_1)) = 2$ for all but one line H, and that exceptional line has multiplicity 4.

This gives us that $\mu_{\pi(P_1)}(\pi(X)) = 2$, where μ means multiplicity as in exercises I.5.3 and I.5.4, so $\pi(P_1)$ is a double point of $\pi(X)$ and by exercise I.5.14(d) it has analytic local ring isomorphic to $k[[x,y]]/(y^2 - x^r)$ for some r. I claim $r = 4$: expanding the equation for X in $\mathcal{O}_{\mathbb{P}^2,\pi(P_1)}$ we get an equation of the form $f_2 + f_3 + f_4 + f_5 + \cdots$ where each f_i is homogeneous of degree i from the fact that $\pi(P_1)$ is a double point, and f_2 is a square by the fact that $i(H, \pi(X); \pi(P_1)) = 2$ for all but one line. Taking x, y as coordinates on $\mathbb{P}^2$ at $\pi(P_1)$ and applying a linear change of coordinates so that $f_2 = y^2$, we see that $i(V(y), \pi(X); \pi(P_1)) = 4$, so the lowest r so that x^r appears in our local equation of f is 4, and following the proof of exercise I.5.14(d) we see that this must be the same r as is found there, so $\pi(P_1)$ is a tacnode.

Now let us check that the minimum degree of a plane representation X' of X with only nodes is 6. By exercise IV.1.8(a), the arithmetic genus of X' must be at least 4, so by the degree-genus formula X' has degree at least 5. If X' has degree 5 and only nodes, then by exercise IV.1.8 parts (a) and (c), it must have two nodes. Repeating the argument from part (a) of this exercise, each node gives a distinct g^1_3, which cannot happen. On the other hand, by theorem IV.3.10, we can find a point $O \notin X \subset \mathbb{P}^3$ (where X is embedded via the canonical embedding) so that the projection from O gives a birational model of X with at most nodes as singularities, and this model is degree 6 as the canonical embedding is degree 6.

Exercise IV.5.5. *Curves of Genus* 5. Assume X is not hyperelliptic.

a. The curves of genus 5 whose canonical model in $\mathbb{P}^4$ is a complete intersection $F_2.F_2.F_2$ form a family of dimension 12.

b. X has a g^1_3 if and only if it can be represented as a plane quintic with one node. These form an irreducible family of dimension 11. [*Hint*: If $D \in g^1_3$, use $K - D$ to map $X \to \mathbb{P}^2$.]

c. (*) In that case, the conics through the node cut out the canonical system (not counting the fixed points at the node). Mapping $\mathbb{P}^2 \to \mathbb{P}^4$ by this linear system of conics, show that the canonical curve X is contained in a cubic surface $V \subset \mathbb{P}^4$, with V isomorphic to $\mathbb{P}^2$

with one point blown up (II, Ex. 7.7). Furthermore, V is the union of all the trisecants of X corresponding to the g_3^1 (5.5.3), so V is contained in the intersection of all the quadric hypersurfaces containing X. Thus V and the g_3^1 are unique.

Note. Conversely, if X does not have a g_3^1, then its canonical embedding is a complete intersection, as in (a). More generally, a classical theorem of Enriques and Petri shows that for any nonhyperelliptic curve of genus $g \geq 3$, the canonical model is projectively normal, and it is an intersection of quadric hypersurfaces unless X has a g_3^1 or $g = 6$ and X has a g_5^2. See Saint-Donat [1].

Solution.

a. First, $\dim \Gamma(\mathbb{P}^4, \mathcal{O}(2)) = \binom{4+2}{2} = 15$ and $\dim \Gamma(X, \mathcal{O}_X(2K)) = 4g - 4 + 1 - g = 3g - 3 = 12$. If we have a complete intersection of three quadrics, the span of their defining equations must be three-dimensional inside $\Gamma(\mathbb{P}^4, \mathcal{O}(2))$. Conversely, by repeated applications of Bertini's theorem, a generic three-dimensional subspace of $\Gamma(\mathbb{P}^4, \mathcal{O}(2))$ gives a smooth complete intersection of three quadrics. So it suffices to compute the dimension of $G(3, 15)$, the Grassmannian of 3-planes in 15-space. But this is $3(15 - 3) = 36$, and after quotienting out by the action of $\operatorname{Aut} \mathbb{P}^4 = PGL(4 + 1)$, we drop the dimension by $(4 + 1)^2 - 1 = 24$, and therefore get a moduli space of dimension 12 as requested.

b. This problem is false as stated: one must allow cusps as well. To be precise, the statement we'll prove is that a non-hyperelliptic curve X of genus 5 has a g_3^1 iff it can be represented as a plane quintic with a double point, which must be a node or a cusp.

   If X can be represented as a plane quintic with a double point A, then a line ℓ through A intersects the curve in a divisor of the form $2A + P + Q + R$ by Bezout, and therefore projection from the node to $\mathbb{P}^1$ which sends $P + Q + R$ to $\ell \cap \mathbb{P}^1$ is a map of degree 3 and gives a g_3^1.

   Conversely, suppose X has a g_3^1. Let D be an effective divisor consisting of three distinct points in this g_3^1. Since g_3^1 is a map to $\mathbb{P}^1$, $\dim |D| \geq 1$, while by Clifford's theorem (IV.5.4) $\dim |D| \leq \frac{1}{2} \deg D = \frac{3}{2}$, so $\dim |D| = 1$ and $l(D) = 2$. (Some sources assume the phrase 'has a g_d^r' to mean the existence of a divisor D with $\deg D = d$ and $\dim |D| = r$, but Hartshorne doesn't exactly say this, so we prove it here for this case anyways.) By Riemann-Roch, $l(D) - l(K - D) = \deg D + 1 - g = 3 + 1 - 5 = -1$, so $l(K - D) = 3$ and D is contained in a line. Now consider $D + P$ for arbitrary P: since $l(D) = 2$, $l(D + P) \geq 2$, and by Clifford's theorem, $l(D + P) < 3$, so $l(D + P) = 2$ and by Riemann-Roch $l(K - D - P) = 2$. Therefore $K - D$ is base-point free by proposition IV.3.1, and so $K - D$ gives a degree 5 map from X to $\mathbb{P}^2$ which we can see as projection from the line spanned by D to some disjoint $\mathbb{P}^2$. Since the map induced by $K - D$ is of degree 5, the image is either of degree 5 and the map is birational, or the image is of degree 1 and the map is 5-to-1. I claim $\varphi_{K-D}(X)$ is not of degree 1 in $\mathbb{P}^2$: if it were, then X is contained in the linear span of the line formed by D in the canonical embedding and the line we project to. But this implies that X is contained in a hyperplane in the canonical map, contradicting the fact that the canonical map is nondegenerate. (We haven't mentioned this before, but it's not so hard to see - degeneracy would mean there's some nonzero global section of $\Omega_{X/k}$ which vanishes everywhere, which from the fact that $\Omega_{X/k}$ is a line bundle would imply that there are elements of $\mathcal{O}_X(U)$ for small enough U which evaluate to zero everywhere but are nonzero, contradicting the fact that X is reduced.)

Therefore the image of X under φ_{K-D} is a plane quintic X', and by exercise IV.1.8, X' must have a unique singular point with $\delta = 1$. I claim this is a double point: by projecting from this point, if it's a triple point, X is hyperelliptic, while if it's a quadruple point, X is rational. Applying exercise IV.1.8(c), δ depends only on the analytic isomorphism type, so write $k[[x,y]]/(f)$, where $f = f_2 + f_3 + \cdots$ for the analytic local ring. Assuming characteristic not 2, we can apply our work in exercise I.5.14(d) to see that the analytic local ring is isomorphic to $k[[x,y]]/(y^2 - x^r)$, and the normalization of this is $k[[x,y]]/(y^2 - x^r) \to k[[t]]$ by $y \mapsto t^r$ and $x \mapsto t^2$ when r odd and $k[[x,y]]/(y^2 - x^r) \mapsto k[[t]] \times k[[t]]$ by $y \mapsto (t^{r/2}, -t^{r/2})$ and $x \mapsto (t,t)$ when r is even. The cokernels of these maps are the spans of $\{t, t^3, \cdots, t^{r-2}\}$ and $\{(0,1), (0,t), \cdots, (0, t^{r/2-1})\}$, so the only way to get $\delta = 1$ is either $r = 2$ or $r = 3$. (The work in characteristic two is similar, but one has to contend with cusps locally looking like $y^2 + x^2y + x^3$ and so on. The key observation is that if $y^2 + yv(x) + w(x)$ is the analytic equation of our cusp, then the order of v is at least two, the order of w is at least three, $\frac{y}{x}$ belongs to the integral closure always, and $\frac{y}{x^2}$ belongs to the integral closure whenever the order of w is four or more. Therefore w must be of order 3, and we have a cusp.) This allows for a cusp or a node.

To show that allowing cusps is necessary, we'll show that any curve of genus 5 or more cannot be trigonal in two different ways and exhibit a plane curve with a cusp and not a node. First, we need to talk about the *base-point free pencil* trick: suppose $\mathcal{L}$ and $\mathcal{M}$ are line bundles on a curve C with $s_1, s_2 \in \Gamma(C, \mathcal{L})$ so that the vanishing locii of s_1 and s_2 are disjoint. Then

$$0 \to \mathcal{M} \otimes \mathcal{L}^{-1} \to \mathcal{M} \oplus \mathcal{M} \to \mathcal{M} \otimes \mathcal{L} \to 0$$

is exact, where the first map is (locally) given by $t \mapsto (ts_2, -ts_1)$ and the second map is (locally) given by $(t,t') \mapsto ts_1 + t's_2$. Then $h^0(\mathcal{M} \otimes \mathcal{L}) \geq 2h^0(\mathcal{M}) - h^0(\mathcal{M} \otimes \mathcal{L}^{-1})$ from taking global sections. We'll first use this to show that any curve of genus at least 3 or more cannot be both hyperelliptic and trigonal: setting $\mathcal{M}$ to be the line bundle associated to a g^1_2 and $\mathcal{L}$ to be the line bundle associated to a g^1_3, we see that $\mathcal{M} \otimes \mathcal{L}^{-1}$ has no sections and therefore C admits a divisor D with $\deg D = 5$ and $l(D) \geq 4$. By Riemann-Roch, $l(K - D) = l(D) - \deg D - 1 + g =\geq g - 2$, so D is special and by Clifford's theorem we must have $l(D) - 1 \leq \frac{5}{2}$, a contradiction. Now we can show that any curve of genus 5 cannot be trigonal in two different ways by setting $\mathcal{L}$ and $\mathcal{M}$ to be two different g^1_3s: then $\mathcal{M} \otimes \mathcal{L}^{-1}$ cannot have global sections, as it is of degree zero but not isomorphic to $\mathcal{O}_C$. Therefore $h^0(\mathcal{M} \otimes \mathcal{L}) \geq 2h^0(\mathcal{M})$, or there exists a divisor D on X with $\deg D = 6$ and $l(D) = 4$. By Riemann-Roch, $l(K - D) = 4 - 6 - 1 + 5 = 2$, so D is special and since $D \neq K$, we may apply Clifford's theorem to see that $l(D) - 1 < \frac{6}{2}$, a contradiction. So no curve of genus ≥ 5 can be trigonal in two different ways.

I claim that this implies that any two realizations of a fixed genus 5 curve as a plane quintic with a singularity are projectively equivalent. Suppose the realization is the map associated to the divisor D: then $\deg D = 5$ and $l(D) \geq 3$. By Riemann-Roch, $l(D) - l(K-D) = \deg D + 1 - g = 1$, so D is special and by Clifford's theorem, $l(D) = 3$. This means that $K - D$ is a g^1_3, and by our work above it is *the* g^1_3. So all realizations of a plane quintic with a singularity are projectively equivalent. Since singularity type is preserved under projective equivalence, it remains to produce a cuspidal plane cubic. The plane quintic with affine equation $y^2 - x^3 + x^5 + y^5$ in characteristic zero will do: it is singular at $(0,0)$ and nowhere else.

To count the dimension of this family, we'll compute the dimension of the space of plane quintics with a fixed double point singularity with $\delta = 1$. First, the vector space of plane quintics is of dimension $\binom{2+5}{2} = 21$, and assuming our singular point is $[0:0:1]$, we must have that the z^5, xz^4, and yz^4 terms all vanish. We should also enforce that the coefficients of the terms x^2z^3, xyz^3, and y^2z^3 aren't all zero, plus that the Jacobian doesn't vanish anywhere else on our curve (we might have to worry about some stuff related to cusps, but the existence of a cusp is a condition that happens when $b^2 - 4ac = 0$, so all of that stuff occurs in the closure of this cell of the moduli space and it won't affect the dimension). These are all open, so they don't drop the dimension. Therefore all that's left to do is homogenize and take the quotient by the action of $\operatorname{Aut}\mathbb{P}^2$ fixing $[0:0:1]$. This drops the dimension by $1+6=7$, so we end up with a $21-3-1-6=11$ dimensional space as requested.

c. From part (b), the realization as a plane quintic with a singularity is given by the complete linear system associated to $K-D$ where D is a g^1_3, and D can be recovered as the projection from the singular point. (We'll pretend we don't already know that there's a unique g^1_3 for now.) Therefore the hyperplane divisor on the realization is $K-D$, and the conic divisor on the realization is $2(K-D)$. Considering the conics through the singular point gives $2K-2D-P-Q$, where P and Q are the points living over the singularity ($P \neq Q$ when the singular point is a node, while $P = Q$ when it's a cusp). Now note that $\deg 2K-2D-P-Q = 8$ and $l(2K-2D-P-Q) \geq 5$ since the latter is the dimension of the space of conics through a point in $\mathbb{P}^2$. Applying Riemann-Roch, we find $l(2K-2D-P-Q) - l(K-(2K-2D-P-Q)) = 8+1-5 = 4$, so $l(K-(2K-2D-P-Q)) \geq 1$, but that's a divisor of degree zero, so we must actually have $2K-2D-P-Q = K$. So we've shown the first claim.

The second claim follows from exercise II.7.7: mapping $\mathbb{P}^2 \to \mathbb{P}^4$ by the linear system of conics through a fixed point, we find that after blowing up the point in question, the image is a cubic surface V. Since our singularity is a node or a cusp, one blowup is enough to resolve the singularity, and therefore the strict transform of the singular quintic in the blown-up $\mathbb{P}^2$ is our curve X. As $X \to \widetilde{\mathbb{P}}^2 \to \mathbb{P}^4$ are all closed immersions and the composite is given by the map φ_K associated to the canonical divisor, $\varphi_K(X)$ is contained in V. Next, by exercise II.7.7, lines through the singular point are taken to disjoint lines in $\mathbb{P}^4$ contained in V. Because D can be recovered as the projection from the singular point, the lines through the singular point form a 1-parameter family of disjoint trisecants to the canonical curve. Since the intersection of any trisecant with any quadric hypersurface containing X is three points counted with multiplicity by definition, Bezout implies that any trisecant to the canonical curve is contained in any quadric hypersurface containing the canonical curve.

To show V is unique, we show that there is can only be one non-degenerate cubic surface contained in the intersection of three quadric hypersurfaces (recall from (a) that $X \subset \mathbb{P}^4$ is contained in three quadrics). Let our three quadrics be Q_1, Q_2, Q_3. Since we're looking for a non-degenerate cubic surface, each Q_i must be irreducible. As Q_1 and Q_2 are independent and irreducible, $Q_1 \cap Q_2$ is of pure dimension 2 and degree 4, so it's a union of irreducible surfaces of degrees $1+1+1+1$, $1+1+2$, $2+2$, $1+3$, or 4. Among those possibilities, the only way to get a surface of degree 3 after intersecting with Q_3 is the $1+3$ case, where one component of $Q_1 \cap Q_2$ is already a surface of degree 3 and this cubic surface is already contained in Q_3. Therefore V is unique, and if we can show that there's only one family of lines on V we'll have that the g^1_3 is unique too.

Consider a curve in $\widetilde{\mathbb{P}}^2$ which does not meet the exceptional divisor, or equivalently a curve in $\mathbb{P}^2$ missing the blowup point P. Then intersecting the image of this curve in V under the embedding $(\mathbb{P}^2 \setminus P) \to \mathbb{P}^4$ with a hyperplane of $\mathbb{P}^4$ not containing this curve will give the degree of the embedded curve. But hyperplanes in $\mathbb{P}^4$ pull back to conics in $\mathbb{P}^2$ by exercise II.7.7, and the fact that our curve misses the blowup point means that the intersection in $\mathbb{P}^2 \setminus P$ is of degree at least two. Thus the image of our curve can't be a line, and the only lines in V are the strict transforms of lines in $\mathbb{P}^2$ through the blowup point P, so the g^1_3 is unique.

Exercise IV.5.6. Show that a nonsingular plane curve of degree 5 has no g^1_3. Show that there are nonhyperelliptic curves of genus 6 which cannot be represented as a nonsingular plane quintic curve.

Solution. Let's prove something slightly more ambitious: no smooth plane curve of degree $d > 2$ has a g^1_e for any $e < d-1$ and e either coprime to the characteristic or e a prime (where we use the 'strict' definition of a g^1_e as a divisor D with $\deg D = e$ and $|D| = 1$). We'll need a more geometric interpretation of Riemann-Roch for this: starting from $l(D) - l(K-D) = \deg D + 1 - g$, we can rewrite this as $l(D) - 1 = \deg D - 1 - g + l(K-D) + 1$. It's clear we can interpret $l(D) - 1$ as $\dim |D|$, and we can also interpret $-g + 1 + l(K-D)$ as $-(g - (l(K-D) - 1))$, the negative of the dimension of the space of hyperplanes in $\mathbb{P}^{g-1}$ which vanish on the image of D under the canonical embedding. So Riemann-Roch also says that $\dim |D| = \deg D - 1 - \dim \overline{\varphi_K(D)}$, where the last term is the dimension of the image of D under the canonical embedding (suitably construed when D contains points of the form nP). This says that $|D|$ is equal to the difference between the 'expected' dimension of the span of $\deg D$ points and the dimension of the actual span of $\deg D$ points under φ_K, the canonical embedding. In particular, if we have a g^1_e, this says that there are e (distinct) points on the canonical curve which lie in an $(e-2)$-plane: our conditions on e imply that the map induced by the g^1_e is separable, and therefore generically unramified. We'll show that if $e < d-1$ this cannot happen by finding a section of the canonical bundle which vanishes on any $e-1$ of these points but not the e^{th}.

For a smooth plane curve X of degree d, the canonical bundle is $\mathcal{O}_X(d-3)$ by exercise II.8.4(e). It's enough to find a section of this sheaf which satisfies our vanishing/nonvanishing conditions: the restriction map on global sections is injective because the kernel is $\Gamma(\mathcal{I}_X(d-3)) \cong \Gamma(\mathcal{O}_{\mathbb{P}^2}(-d+d-3)) \cong \Gamma(\mathcal{O}_{\mathbb{P}^2}(-3)) = 0$. If $e < d-1$, then $e - 1 \leq d-3$, and it suffices to solve the problem for $d-3$ points: for each point, pick a linear form which vanishes on it but not on our final point, and multiply all of them together to find a degree $d-3$ polynomial which satisfies our conditions. This proves our claim.

To demonstrate that there is a nonhyperelliptic genus 6 curve which cannot be represented as a nonsingular plane quintic, we have two options: we can either note that by remark IV.5.5.4, the moduli space of genus 6 curves is 15-dimensional while the space of plane quintics up to projective equivalence is $\binom{2+5}{2}-1-8=12$-dimensional, or we can demonstrate a genus 6 curve with a g^1_3 (since such curve cannot be hyperelliptic by the argument involving the base-point free pencil trick from our solution to exercise IV.5.5). For an explicit example, consider a smooth divisor of type $(3,4)$ on $\mathbb{P}^1\times\mathbb{P}^1$. By the results of exercise II.5.6, this has genus $(3-1)(4-1)=6$, and the projections on to the factors give maps to $\mathbb{P}^1$ of degree 3 and 4, respectively.

Exercise IV.5.7.

a. Any automorphism of a curve of genus 3 is induced by an automorphism of $\mathbb{P}^2$ via the canonical embedding.

b. (*) Assume $\operatorname{char} k\neq 3$. If X is the curve given by

$$x^3y+y^3z+z^3x=0,$$

the group $\operatorname{Aut} X$ is the simple group of order 168, whose order is the maximum $84(g-1)$ allowed by (Ex. 2.5). See Burnside [1, §232] or Klein [1].

c. (*) Most curves of genus 3 have no automorphisms except the identity. [*Hint*: For each n, count the dimension of the family of curves with an automorphism T of order n. For example, if $n=2$, then for suitable choice of coordinates, T can be written as $x\to -x$, $y\to y$, $z\to z$. Then there is an 8-dimensional family of curves fixed by T; changing coordinates there is a 4-dimensional family of such T, so the curves having an automorphism of degree 2 form a family of dimensional 12 inside the 14-dimensional family of all plane curves of degree 4.]

Note: More generally it is true (at least over $\mathbb{C}$) that for any $g\geq 3$, a "'sufficiently general"' curve of genus g has no automorphisms except the identity–see Baily [1].

Solution.

a. Per the results of proposition IV.2.3 and the observation that an automorphism must be unramified, we see that the pullback of the canonical bundle along an automorphism of curves must be the canonical bundle. Therefore an automorphism of a curve induces an automorphism of the global sections of the canonical bundle, and therefore an automorphism of $\mathbb{P}^{g-1}$, viewed as the projectivization of the global sections of the canonical bundle. So any automorphism of our curve of genus 3 comes from an automorphism of $\mathbb{P}^{g-1}=\mathbb{P}^2$ via the canonical embedding.

b. This problem is quite difficult. Part of the issue is that the discovery of the Klein quartic actually went the other way: Klein started with the simple group of order 168 and found an irreducible 3-dimensional representation, which leads to the equation of the Klein quartic as a particular invariant of the representation. Much more detail is available in the MSRI volue *The Eightfold Way: The Beauty of Klein's Quartic Curve*, edited by Silvio Levy. (Noam Elkies' article there is a particular highlight, plus it contains an English translation of Klein's original paper.) I'm not sure that Hartshorne is really expecting you to do it all yourself, especially given that he provided two references in the problem statement, but we'll make a valiant effort.

Let's talk a little bit about characteristic issues before we begin. First, note that in characteristic 7, the curve K defined by $V(X^3Y + Y^3Z + Z^3X)$ is not smooth: the point $[2:4:1]$ is a singular point. So we'll ignore the characteristic 7 case, and we'll also skip the characteristic two case because Hurwitz's theorem on the maximal order of the automorphism group doesn't necessarily hold there (see the remark at the end of the statement of exercise IV.2.5), even though the curve is smooth in characteristic two.

Now for our strategy. As there's one simple group of order 168 and it has presentation

$$\langle x, y \mid x^2 = y^3 = (zy)^7 = [x, y]^4 = 1\rangle,$$

we'll find two nontrivial automorphisms $\widetilde{x}$ and $\widetilde{y}$ of the Klein quartic which obey those relations. By the first isomorphism theorem we get a surjective homomorphism on to the subgroup of automorphisms of the Klein quartic generated by $\widetilde{x}$ and $\widetilde{y}$ from the simple group of order 168, which must be an isomorphism since that group is simple and our automorphisms are nontrivial. From the Hurwitz bound, this implies that the automorphisms of the Klein quartic are actually the group generated by $\widetilde{x}$ and $\widetilde{y}$.

We begin by finding some automorphisms of the Klein quartic. One automorphism is very direct to find: the cyclic permutation of the variables $X \mapsto Z$, $Y \mapsto X$, $Z \mapsto Y$ fixes the equation defining the Klein quartic. We'll call this automorphism β and note it is represented by the permutation matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We can also find another type of automorphism without going to too much trouble. Consider an automorphism given by sending $X \mapsto \lambda_1 X$, $Y \mapsto \lambda_2 Y$, $Z \mapsto Z$ where λ_1 and λ_2 are roots of unity. Making these substitutions in the defining equation of the Klein quartic, we find

$$X^3Y + Y^3Z + Z^3X \mapsto \lambda_1^3\lambda_2 X^3Y + \lambda_2^3 YZ + \lambda_1 Z^3X,$$

so writing $\lambda_1 = \zeta_n^a$ and $\lambda_2 = \zeta_n^b$ for ζ_n a primitive n^{th} root of unity we find that $3a+b = 3b = a$ (mod n). Thus the only nontrivial automorphisms of this form happen when $n = 7$ and $a = 3b$. Declaring ζ to be a primitive 7^{th} root of unity for the rest of the problem, taking $a = 6$, $b = 2$, and then scaling so that the resulting matrix has determinant 1, we find an automorphism which we call γ given by the matrix

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^4 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.$$

By direct computation, one finds that $\beta^3 = 1$, $\gamma^7 = 1$, and $\beta\gamma\beta^{-1} = \gamma^2$, so these two automorphisms generate a non-abelian subgroup of order 21. We need more automorphisms.

By part (a), any automorphism of the Klein quartic is induced by a linear automorphism of $\mathbb{P}^2$, and therefore by the same logic as in problem IV.5.2 such an automorphism must permute the inflection points of K. Since the coordinate points $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$ are inflection points and are taken to the columns of our matrix, we can search for inflection points of K and attempt to construct a linear automorphism by setting the columns of our matrix as these inflection points. In order to find inflection points of K, we'll need a lemma.

Lemma. *Let $F \in k[X,Y,Z]$ be a homogeneous polynomial of degree d with no linear factors. Suppose k is an algebraically closed field where $d-1$ is invertible. Let $Hess_F$ be the Hessian matrix of second-order partial derivatives of F, and let H be its determinant. If $p \in V(F)$ is a regular point with projective tangent line L, then $i(V(F), V(H); p) = i(V(F), L; p) - 2$.*

This implies that if $V(F)$ is a smooth plane curve of degree d, then we can find the inflection points of $V(F)$ as the intersection of $V(F)$ and $V(H)$ (assuming $d-1$ is invertible in k).

Proof. From caluculus, if J is the Jacobian, the Hessian transforms as $J^t Hess_F J$ under a change of variables, so under a linear change of coordinates on $\mathbb{P}^2$ the Hessian determinant is multiplied by a nonzero constant. Therefore we may assume up to a linear change of coordinates that $p = [0:0:1]$ and the tangent line is $V(Y)$.

Now we manipulate $Hess_F$ to get something that is easier to calculate with. Starting from

$$Hess_F = \begin{pmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial Y \partial X} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial Z \partial X} & \frac{\partial^2 F}{\partial Z \partial Y} & \frac{\partial^2 F}{\partial Z^2} \end{pmatrix}$$

multiply the final line by Z and add X times the first line and Y times the second line to it. This transforms our determinant by a factor of Z, and by an application of Euler's lemma, which states that $\sum X_i \frac{\partial P}{\partial X_i} = nP$ for a homogeneous polynomial P of degree n, the final line has entries $(d-1)\frac{\partial F}{\partial X}$, $(d-1)\frac{\partial F}{\partial Y}$, and $(d-1)\frac{\partial F}{\partial Z}$. Doing the same process but with columns instead of rows gives us the following matrix with determinant HZ^2:

$$\begin{pmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & (d-1)\frac{\partial F}{\partial X} \\ \frac{\partial^2 F}{\partial Y \partial X} & \frac{\partial^2 F}{\partial Y^2} & (d-1)\frac{\partial F}{\partial Y} \\ (d-1)\frac{\partial F}{\partial X} & (d-1)\frac{\partial F}{\partial Y} & d(d-1)^2 f \end{pmatrix}.$$

This has determinant

$$(d-1)\frac{\partial F}{\partial X}\left(\frac{\partial^2 F}{\partial X\partial Y}\frac{\partial F}{\partial Y}-\frac{\partial^2 F}{\partial Y^2}\frac{\partial F}{\partial X}\right)+(d-1)\frac{\partial F}{\partial Y}\left(\frac{\partial^2 F}{\partial X\partial Y}\frac{\partial F}{\partial Y}-\frac{\partial^2 F}{\partial X^2}\frac{\partial F}{\partial Y}\right)+F(\cdots)$$

which may be rewritten $HZ^2=(d-1)G+F(\cdots)$.

Now we calculate the intersection multiplicity at p with this G: since $V(Z)$ misses p, the two intersection multiplicities $i(V(F),V(H);p)$ and $i(V(F),V(HZ^2);p)$ are equal, and therefore by definition the intersection multiplicity is the dimension of $\mathcal{O}_{\mathbb{P}^2,p}/(f,(d-1)g+f)$ where f,g are dehomogenizations of F,G respectively. From our coordinate change earlier, f can be written as $y(1+\cdots)+x^e(c+\cdots)$ where $c\in k^\times$ and $e=i(V(F),L;p)$. Now we examine g: from our description of HZ^2, we find that

$$G=\left(\frac{\partial F}{\partial Y}\right)^2\frac{\partial^2 F}{\partial X^2}+\left(\frac{\partial F}{\partial X}\right)^2\frac{\partial^2 F}{\partial Y^2}-2\frac{\partial F}{\partial X}\frac{\partial F}{\partial Y}\frac{\partial^2 F}{\partial X\partial Y},$$

and so term-by-term analysis shows that the first term contributes a term of the form x^{e-2} while all other terms are either divisible by y or of degree $e-1$ or higher. Therefore if we move to the power series ring $k[[x,y]]/(f,g)$ to compute the intersection multiplicity (which does not change the result since the local ring $\mathcal{O}_p/(f,g)$ is already complete) we see that $k[[x,y]]/(f,g)\cong k[[x,y]]/(y,x^{e-2})$ and we've proven the claim.

■

With the lemma in hand, our first step is to find the inflection points via intersecting the Hessian with our curve.. The Hessian of $X^3Y+Y^3Z+Z^3X$ is $-54(X^5Z+Y^5X+Z^5Y-5X^2Y^2Z^2)$, so outside of characteristic two and three the inflection points are the simultaneous solutions of $X^3Y+Y^3Z+Z^3X$ and $X^5Z+Y^5X+Z^5Y-5X^2Y^2Z^2$. Dehomogenizing with respect to Z and taking the resultant of these two equations with respect to y, we find that the x-coordinates of the inflection points in the chart $D(Z)$ are the solutions of $x(x^{21}+57x^{14}-289x^7-1)$. If $\operatorname{char} k\neq 7$, then the second factor is separable and has distinct roots, so adding in $[1:0:0]$ and $[0:1:0]$ this shows that there are exactly 24 inflection points of the Klein quartic and all inflection points are simple inflection points, i.e. $i(X,T_pX;p)=3$.

Factoring $x^{21}+57x^{14}-289x^7-1$ over the integers (with the help of a computer algebra system), the factor of smallest degree is x^3+x^2-2x-1. The solutions to this polynomial are $\zeta+\zeta^{-1}$ as ζ varies among the primitive 7^{th} root of unity: $(\zeta+\zeta^{-1})^2=\zeta^2+2+\zeta^{-2}$ and $(\zeta+\zeta^{-1})^3=\zeta^3+3\zeta+3\zeta^{-1}+\zeta^{-3}$, so $(\zeta+\zeta^{-1})^3+(\zeta+\zeta^{-1})^2-(\zeta+\zeta^{-1})-1=\zeta^{-3}\frac{\zeta^7-1}{\zeta-1}=0$. Plugging this in to the dehomogenizations of the defining equations for the Klein Quartic and its Hessian and solving for y, we find that $y=-(1+\zeta^2+\zeta^{-2})$ and so $[\zeta+\zeta^{-1}:-(1+\zeta^2+\zeta^{-2}):1]$ is an inflection point for each primitive 7^{th} root of unity ζ.

This description of these inflection points isn't very symmetric compared to the coordinate points, though. One may observe that scaling each of these inflection points by $\zeta-\zeta^{-1}$, we get something much more symmetric: our point $[\zeta+\zeta^{-1}:-(1+\zeta^2+\zeta^{-2}):1]$ transforms in to

$[\zeta^2-\zeta^{-2}:\zeta^4-\zeta^{-4}:\zeta-\zeta^{-1}]$. Fixing one particular choice of ζ, this gives us the three points $[\zeta^2-\zeta^{-2}:\zeta^4-\zeta^{-4}:\zeta-\zeta^{-1}]$, $[\zeta^4-\zeta^{-4}:\zeta-\zeta^{-1}:\zeta^2-\zeta^{-2}]$, and $[\zeta-\zeta^{-1}:\zeta^2-\zeta^{-2}:\zeta^4-\zeta^{-4}]$. Trying different ways to organize these in to a 3×3 matrix, we find that the matrix

$$\begin{pmatrix} \zeta-\zeta^{-1} & \zeta^2-\zeta^{-2} & \zeta^4-\zeta^{-4} \\ \zeta^2-\zeta^{-2} & \zeta^4-\zeta^{-4} & \zeta-\zeta^{-1} \\ \zeta^4-\zeta^{-4} & \zeta-\zeta^{-1} & \zeta^2-\zeta^{-2} \end{pmatrix}$$

squares to $-7I$, and after a slightly interminable amount of algebra, we find that the automorphism described by this matrix multiplies the equation for the Klein quartic by -49 and therefore it stabilizes the quartic. We let α denote this automorphism of order two, and we'll represent α by $\frac{1}{\sqrt{-7}}$ times the matrix above during our subsequent computations. Direct computation shows that $\alpha\beta\alpha=\beta^{-1}$, but $\alpha\gamma\alpha$ is not immediately recognizable as some product of α, β, and γ.

Searching for products of α, β, and γ to use as x and y is difficult and time consuming, so we ask a computer to do it for us. Specifically, we use the GAP system for discrete algebra, available at `https://www.gap-system.org/` (all code in this problem was executed in version 4.11.1). Here is the code which takes the group $\langle \alpha,\beta,\gamma \mid \alpha^2=\beta^3=\gamma^7=(\alpha\beta)^2=\beta\gamma\beta^{-1}\gamma^{-2}\rangle$ and tests whether it has quotients which are isomorphic to the simple group of order 168:

```
gap> f:=FreeGroup("a","b","c");
<free group on the generators [ a, b, c ]>
gap> rels:=ParseRelators(f,"aba=B,bcB=c2,a2,b3,c7");
[ (a*b)^2, b*c*b^-1*c^-2, a^2, b^3, c^7 ]
gap> g:=f/rels;
<fp group on the generators [ a, b, c ]>
gap> q:=GQuotients(g,PSL(3,2));
[ [ a, b, c ] -> [ (4,6)(5,7), (2,6,4)(3,7,5), (1,2,4,3,6,7,5) ],
  [ a, b, c ] -> [ (2,4)(3,5), (2,6,4)(3,7,5), (1,2,4,3,6,7,5) ],
  [ a, b, c ] -> [ (2,6)(3,7), (2,6,4)(3,7,5), (1,2,4,3,6,7,5) ] ]
```

The code shows that there are three different ways to take a quotient of $\langle \alpha,\beta,\gamma \mid \alpha^2=\beta^3=\gamma^7=(\alpha\beta)^2=\beta\gamma\beta^{-1}\gamma^{-2}\rangle$ and get the simple group of order 168.

To attempt to try and find generators x and y, we look for an explicit isomorphism. We first ask GAP to put our elements in normal form (note that this is not guaranteed to work, but we get lucky), then define the simple group of order 168 (and sanity check that our presentation is correctly dealt with), and finally set up the isomorphism between our presentation and the internal presentation of the simple group of order 168 in GAP:

```
gap> SetReducedMultiplication(g);
gap> f2:=FreeGroup("x","y");
<free group on the generators [ x, y ]>
```

```
gap> psl:=f2/ParseRelators(f2,"x2,y3,(xy)7,[x,y]4");
<fp group on the generators [ x, y ]>
gap> Size(psl);
168
gap> isop:=IsomorphismPermGroup(psl);
[ x, y ] -> [ (1,2)(4,5), (2,3,4)(5,6,7) ]
```

Now we can ask for explicit representations of x and y based on α, β, and γ for each of the three quotients we found:

```
gap> isoA:=IsomorphismGroups(Image(isop),Image(q[1]));
[ (1,2)(4,5), (2,3,7,5)(4,6) ] -> [ (1,4)(3,6), (2,3)(4,6,5,7) ]
gap> isoA:=isop*isoA;
[ x, y ] -> [ (1,4)(3,6), (1,2,3)(4,5,7) ]
gap> List(GeneratorsOfGroup(psl),
> x->PreImagesRepresentative(q[1],ImagesRepresentative(isoA,x)));
[ (b*c)^2*(a*c)^2*b*a, a*b*a*c*a ]
gap> isoB:=IsomorphismGroups(Image(isop),Image(q[2]));
[ (1,2)(4,5), (2,3,7,5)(4,6) ] -> [ (1,4)(3,6), (2,3)(4,6,5,7) ]
gap> isoB:=isop*isoB;
[ x, y ] -> [ (1,4)(3,6), (1,2,3)(4,5,7) ]
gap> List(GeneratorsOfGroup(psl),
> x->PreImagesRepresentative(q[2],ImagesRepresentative(isoB,x)));
[ b*c*b*a*c*a*b*a*c*b, b*c*b*(a*c*a*b*a*c)^2*a*b*a ]
gap> isoC:=IsomorphismGroups(Image(isop),Image(q[3]));
[ (1,2)(4,5), (2,3,7,5)(4,6) ] -> [ (1,4)(3,6), (2,3)(4,6,5,7) ]
gap> isoC:=isop*isoC;
[ x, y ] -> [ (1,4)(3,6), (1,2,3)(4,5,7) ]
gap> List(GeneratorsOfGroup(psl),
> x->PreImagesRepresentative(q[3],ImagesRepresentative(isoC,x)));
[ c*a*c*b*(c*a)^2*(b*a*c)^2*b, a*b*a*c*b*a ]
```

This gives us three options:

(i) $x = (\beta\gamma)^2(\alpha\gamma)^2\beta\alpha$ and $y = \alpha\beta\alpha\gamma\alpha$,

(ii) $x = \beta\gamma\beta\alpha\gamma\alpha\beta\alpha\gamma\beta$ and $y = \beta\gamma\beta(\alpha\gamma\alpha\beta\alpha\gamma)^2\alpha\beta\alpha$,

(iii) $x = \gamma\alpha\gamma\beta(\gamma\alpha)^2(\beta\alpha\gamma)^2\beta$ and $y = \alpha\beta\alpha\gamma\beta\alpha$.

To test which of these are valid, we'll use the following SageMath code, executed in the online cell at their website `https://sagecell.sagemath.org/`:

```
R = QQbar
M = MatrixSpace(R,3,3)
z = R.zeta(7)

A = M([(z-z^6)/sqrt(-7),(z^2-z^5)/sqrt(-7),(z^4-z^3)/sqrt(-7),
       (z^2-z^5)/sqrt(-7),(z^4-z^3)/sqrt(-7),(z-z^6)/sqrt(-7),
       (z^4-z^3)/sqrt(-7),(z-z^6)/sqrt(-7),(z^2-z^5)/sqrt(-7)])
B = M([0,0,1, 1,0,0, 0,1,0])
G = M([z,0,0, 0,z^4,0, 0,0,z^2])

#GAP option 1
#X=(B*G)^2*(A*G)^2*B*A
#Y=A*B*A*G*A

#GAP option 2
#X=B*G*B*A*G*A*B*A*G*B
#Y=B*G*B*(A*G*A*B*A*G)^2*A*B*A

#GAP option 3
X=G*A*G*B*(G*A)^2*(B*A*G)^2*B
Y=A*B*A*G*B*A

print("x:",X)
print("y:",Y)
print("x^2:",X^2)
print("y^3:",Y^3)
print("(xy)^7:",(X*Y)^7)
print("[x,y]^4]:",(X*Y*X*Y*Y)^4)
```

(Sage is built on Python, so the octothorpe is the comment symbol - use this to try the different presentations if you wish.) We find that using the third presentation, all our relations are satisfied (all matrix products are a scalar multiple of the identity) and X and Y act nontrivially on $\mathbb{P}^2$. Thus we really have found a subgroup of $\operatorname{Aut} K$ isomorphic to the simple group of order 168, and by the Hurwitz bound this implies that $\operatorname{Aut} K$ is in fact the simple group of order 168 in characteristic not 2, 3, or 7.

c. An automorphism T of $\mathbb{P}^2$ of order n can be written up to a change of basis as

$$\begin{pmatrix} \zeta^a & 0 & 0 \\ 0 & \zeta^b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ζ is a primitive n^{th} root of unity, $\gcd(a,b,n) = 1$, and $0 \leq b < a < n$. This acts on homogeneous polynomials of degree 4 as follows, where we specify them by the exponents of x and y:

	x^0	x^1	x^2	x^3	x^4
y^0	1	ζ^a	ζ^{2a}	ζ^{3a}	ζ^{4a}
y^1	ζ^b	ζ^{a+b}	ζ^{2a+b}	ζ^{3a+b}	
y^2	ζ^{2b}	ζ^{a+2b}	ζ^{2a+2b}		
y^3	ζ^{3b}	ζ^{a+3b}			
y^4	ζ^{4b}				

A curve stable under our automorphism is exactly given by a polynomial consisting of monomial terms that all transform the same way under T, so to find their dimension before coordinate changes it's enough to count the number of matching terms in the above table. Let s be the additive order of a modulo n. If $s \geq 4$, then the first row contains at most two entries which are the same, and all the entries in any other fixed row are distinct, giving a maximum of 5 matching entries and therefore a 4-dimensional locus of curves before taking coordinate changes in to account. If $s = 3$ and $b \neq 0$, then the additive order of b modulo n is at least s. If it's strictly greater, then we can use the logic of the $s \geq 4$ case applied to the columns, while if it's the same we must have $n = 3$, $a = 2$, and $b = 1$, which we can write out and find a maximum of 5 matching entries. Therefore in each of these cases, after projectivizing we find a locus of curves of dimension at most 4 and applying the $PGL(3)$ action we find that they fill out a space of dimension at most $4+8 = 12$ in the space of all plane curves of degree 4.

When $s = 2$ or $s = 3$ and $b = 0$, we can again fill out the table and see that we have at most 9 or 7 matching entries, respectively, leading to locii of dimension 8 or 6, respectively. This time, the $PGL(3)$ action has a 4-dimensional stabilizer given by matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

so after applying the $PGL(3)$ action we find that they fill out spaces of dimension at most $8+8-4 = 12$ and $6+8-4 = 10$ in the in the space of all plane curves of degree 4. Since there are only finitely many n to consider, the locus of genus 3 curves admitting an automorphism forms an at most 12-dimensional locus inside the 14-dimensional space of plane curves of degree 4.

IV.6 Classfication of Curves in $\mathbb{P}^3$

Opening remarks!

Exercise IV.6.1. A rational curve of degree 4 in $\mathbb{P}^3$ is contained in a unique quadric surface Q, and Q is necessarily nonsingular.

Solution. Let X be our curve, and consider the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$. Twisting by 2 and taking global sections, we get an exact sequence

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to \cdots .$$

As $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{2} = 10$ and $\dim H^0(\mathcal{O}_X(2)) = \dim H^0(\mathcal{O}_{\mathbb{P}^1}(8)) = \binom{1+8}{8} = 9$, we must have that $\dim H^0(\mathcal{I}_X(2)) \geq 1$ and therefore X is contained in at least one quadric surface.

First, X cannot be contained in any singular quadric surface: if X is contained in a plane, then by the degree-genus formula we would have $g = 3$; if X is contained in a quadric cone (the unique irreducible singular quadric surface) then X must have genus 1 by remark IV.6.4.1(d). Now suppose X was contained in two distinct nonsingular quadric surfaces: then it must be contained in their intersection, and since that intersection is one-dimensional and has degree 4, X must actually equal that intersection. But such a curve has genus 1 by remark IV.6.4.1(c). So X lies on a unique quadric Q and Q must be nonsingular.

Exercise IV.6.2. A rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

Solution. Let X be our curve, and consider the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$. Twisting by 3 and taking global sections, we get an exact sequence

$$0 \to H^0(\mathcal{I}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3)) \to \cdots .$$

As $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(3)) = \binom{3+3}{3} = 20$ and $\dim H^0(\mathcal{O}_X(3)) = \dim H^0(\mathcal{O}_{\mathbb{P}^1}(15)) = \binom{1+15}{15} = 16$, we must have that $\dim H^0(\mathcal{I}_X(3)) \geq 1$ and therefore X is contained in at least one cubic surface.

The same procedure but twisting by 2 instead of 3 shows that $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{2} = 10$ and $\dim H^0(\mathcal{O}_X(2)) = \dim H^0(\mathcal{O}_{\mathbb{P}^1}(10)) = \binom{1+10}{10} = 11$, so cohomology doesn't force an inclusion in a quadric surface. One very low-tech way of finding such an example is to just play around. Consider the curve X given by the map $[s : t] \mapsto [s^5 : s^4t : s^3t^2 + st^4 : t^5]$. Plugging this in to a quadratic $\sum_{0\leq i\leq j\leq 3} c_{ij}x_ix_j$, we obtain the equation

$$c_{00}s^{10} + c_{01}s^9t + c_{02}(s^8t^2 + s^6t^4) + c_{03}t^{10} + c_{11}s^8t^2 + c_{12}(s^7t^3 + s^5t^5) + c_{13}s^4t^6+$$

$$+c_{22}(s^6t^4 + 2s^4t^6 + s^2t^8) + c_{23}(s^3t^7 + st^9) + c_33t^{10}.$$

In order for this to be zero, we find that all the c_{ij} must actually be zero: by examining s^6t^4, s^7t^3, s^2t^8, and st^9 we find that any c_{ij} with i or j equal to 2 must vanish, and then the remaining c_{ij} are each associated to a unique term s^at^b and must also be zero.

To check that this is a closed immersion, we first note that on $D(x_0)$ we can recover t/s as $\frac{s^4t}{s^5}$, so our map is a closed immersion over all of $D(s) \subset \mathbb{P}^1$. The coordinate patch $D(x_3)$ requires a bit more work: the algebra generated by the ratios of the sections here is $k[x^5, x^4, x^3+x]$ where $x = \frac{s}{t}$. But as $x = x^3+x-(x^3+x)^3+x^5x^4+3x^4(x^3+x)$ we have $k[x^5,x^4,x^3+x] = k[x]$ and we see that this is a closed immersion over all of $D(t) \subset \mathbb{P}^1$.

Exercise IV.6.3. A curve of degree 5 and genus 2 in $\mathbb{P}^3$ is contained in a unique quadric surface Q. Show that for any abstract curve X of genus 2, there exists embeddings of degree 5 in $\mathbb{P}^3$ for which Q is nonsingular, and there exist other embeddings of degree 5 for which Q is singular.

Solution. Let X be our curve, and consider the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$. Twisting by 2 and taking global sections, we get

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to \cdots .$$

As $\mathcal{O}_X(2) = \mathcal{O}_X(D)$ for a divisor D of degree 10, by Riemann-Roch we have $l(D)-l(K-D) = 10+1-2$, and as $\deg K = 2g-2 = 2$, we have $\deg K - D < 0$ so $l(K-D) = 0$ and $l(D) = 9$. Thus as $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ is $\binom{3+2}{3} = 10$ dimensional, $\dim H^0(\mathcal{I}_X(2)) > 0$, so X must be contained in a quadric surface. If $\dim H^0(\mathcal{I}_X(2)) > 1$, then X is contained in two distinct quadric surfaces and therefore either Z is contained in a plane or X is contained in a dimension-one subvariety of degree 4. The former cannot happen by the degree-genus formula, and the latter cannot happen because X is of degree 5. Thus C is contained in a unique quadric surface.

Now let X be an arbitrary genus 2 curve. By corollary IV.3.2(b), any divisor D on X of degree 5 will be very ample; by Riemann-Roch, any such divisor has $l(D) = 5+1-2 = 4$, so the complete linear system $|D|$ embeds X in to $\mathbb{P}^3$ as a curve of degree 5 and genus 2. From our work above, Z is contained in a unique irreducible quadric surface Q which must be either a quadric cone or the smooth quadric surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

I claim a degree 5 divisor of the form $D = 2K + P$ embeds X on a cone in $\mathbb{P}^3$. We may view $l(D-(K+P))$ as the codimension of the linear span of K and P under the embedding induced by D, and since $D-(K+P) \sim K$, we get that $l(D-(K+P)) = 2$, or that K and P span a line in $\mathbb{P}^3$. Therefore X under this embedding has a family of trisecants which all meet at P: all the effective divisors $P_1+P_2+P \sim K+P$. Since any trisecant must be contained in any quadric surface containing X by Bezout, we see that this family of trisecants lies on Q. This implies that Q cannot be $\mathbb{P}^1 \times \mathbb{P}^1$: there's no family of lines on $\mathbb{P}^1 \times \mathbb{P}^1$ which all meet at a single point. Therefore Q must be a cone.

Now I claim that any degree 5 divisor D not linearly equivalent to $2K+P$ for any P will embed X on a smooth quadric surface in $\mathbb{P}^3$. First, I claim that X has trisecants under this embedding, and that the trisecants are exactly the effective divisors linearly equivalent to $D-K$. If E is the divisor of three points which are collinear under the embedding induced by D, then $l(D-E) = 2$, so $D-E$ is a divisor of degree 2 with $\dim |D-E| = 1$, and by Clifford's theorem, $D-E \sim K$. Conversely, $l(D-K) = 2$, so $|D-K|$ is nonempty and any effective divisor E linearly equivalent to $D-K$ gives a trisecant under the embedding induced by D. Every such trisecant must be contained in Q, and I claim that no two trisecants intersect, which will show that we are on the nonsingular quadric surface. To begin, two trisecants cannot intersect outside of X: if they did, the linear span of the two trisecants would be of dimension at most two, so $l(D-E-E')$ should be positive. But $D-E-E'$ is of negative degree. Alternatively, two trisecants cannot intersect on X: given a trisecant to X and a hyperplane containing it, the other two points of intersection form a canonical divisor by our discussion above; therefore an intersection of two trisecants on X would give a hyperplane section linearly equivalent to $2K+P$. But D is linearly equivalent to a hyperplane section of X embedded by D, so this would imply $D \sim 2K+P$ in contradiction to our assumptions.

Such divisors always exist by dimension reasons: an effective divisor of degree 5 corresponds to a point of X^5 up to reordering, so the parameter space of such divisors is 5-dimensional, while an effective canonical divisor corresponds to the preimage of a point under the canonical map $\varphi_K : X \to \mathbb{P}^1$, and so the space of effective divisors linearly equivalent to $2K+P$ for some $P \in X$ is three-dimensional.

Exercise IV.6.4. There is no curve of degree 9 and genus 11 in $\mathbb{P}^3$. [*Hint*: Show that it would have to lie on a quadric surface, then use (6.4.1).]

Solution. Let X be our curve. First we show that X must lie on a quadric surface. Consider the exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$. Twisting by 2 and taking global sections, we get

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to \cdots .$$

As $\mathcal{O}_X(2) = \mathcal{O}_X(D)$ for an effective divisor D of degree 18, by Riemann-Roch we have that $l(D) - l(K-D) = 18 + 1 - 11 = 8$. Since $l(D) = \dim H^0(\mathcal{O}_X(2))$, it suffices to show that $l(K-D) < 2$ in order to show that $H^0(\mathcal{I}_X(2)) \neq 0$.

By Clifford's theorem (theorem IV.5.4), $l(K-D) - 1 = \dim |K-D| \leq \frac{1}{2}\deg(K-D) = 1$ with equality iff $D = 0$ or $D = K$ or X hyperelliptic and D a multiple of the unique g^1_2. Since strict equality will give $l(K-D) < 2$ and $K-D$ is not equal to 0 or K, all we need to do is to show that it's not equal to g^1_2. As any g^1_2 is generated by global sections and D is very ample, if $K-D$ was equal to g^1_2, we would have that $K = (K-D) + D = g^1_2 + D$ is very ample by exercise II.7.5(d). But by proposition IV.5.2, K is very ample iff X is not hyperelliptic, so $l(K-D) < 2$ and X lies on a quadric surface.

If X lies on a nonsingular quadric surface, then by remark IV.6.4.1(c), we would need to solve $9 = a+b$ and $11 = ab - a - b + 1$ for integers a, b. Plugging in $b = 9-a$ to the second equation, we see that we're looking for integer solutions to $a^2 - 9a + 19 = 0$, which the quadratic formula shows does not exist. If X lies on a quadric cone, then from remark IV.6.4.1(d), we would need to have $d = 2a+1$ and $g = a^2 - a$. To make $d = 9$, take $a = 4$ giving $g = 12$, contradiction. Else, X must lie in a plane, which contradicts the degree-genus formula as a plane curve of degree 9 has genus 28.

Exercise IV.6.5. If X is a complete intersection of surfaces of degrees a, b in $\mathbb{P}^3$, then X does not lie on any surface of degree $< \min(a, b)$.

Solution. If X is a complete intersection of surfaces of degrees a, b, then the ideal sheaf $\mathcal{I}$ is the sheaf associated to the graded module (f_a, f_b) where f_a and f_b are two (coprime, irreducible) polynomials cutting out the surfaces of degree a and b respectively. It's clear that this module has no elements of degree $< \min(a, b)$, so there are no homogeneous polynomials of degree $< \min(a, b)$ vanishing on X. (I know this feels slightly handwavy, but everything's ok by the work we did in exercise II.8.4: in particular, by (a), we have that $\mathcal{I}_X = \mathcal{I}_{H_1} + \mathcal{I}_{H_2}$ as subsheaves of $\mathcal{O}_{\mathbb{P}^3}$. So global sections of the sum are sums of global sections of $\mathcal{I}_{H_1} \cong \mathcal{O}_{\mathbb{P}^3}(a)$ and $\mathcal{I}_{H_2} \cong \mathcal{O}_{\mathbb{P}^3}(b)$, and we can use the fact that $\Gamma_*(\mathcal{O}_{\mathbb{P}^3}(r)) = S(r)$ from earlier in section II.5.)

Exercise IV.6.6. Let X be a projectively normal curve in $\mathbb{P}^3$, not contained in any plane. If $d = 6$, then $g = 3$ or 4. If $d = 7$, then $g = 5$ or 6. Cf. (II, Ex. 8.4) and (III, Ex. 5.6).

Solution. By exercise II.8.4(c), if X is projectively normal, then $H^0(\mathcal{O}_{\mathbb{P}^3}(r)) \to H^0(\mathcal{O}_X(r))$ is surjective for all r. Let D be the hyperplane divisor on X, so that $\mathcal{O}_X(1) = \mathcal{O}_X(D)$. Then the condition that X does not lie in any plane says $H^0(\mathcal{I}_X(1)) = 0$, so $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_X(D))$ is an isomorphism and $l(D) = \binom{3+1}{1} = 4$.

When $d = 6$, proposition IV.6.3 says that if D is special then $g = 4$. If D is nonspecial, then $l(K - D) = 0$ and by Riemann-Roch, $l(D) = \deg D + 1 - g$, so $g = 6 + 1 - 4 = 3$. Therefore if $d = 6$, $g = 3$ or 4.

When $d = 7$, theorem IV.6.4 says that $g \leq \frac{1}{4}(d^2 - 1) - d + 1 = 6$. Proposition IV.6.3 states that if D is special then $g \geq \frac{7}{2} + 1$, so $g \geq 5$. If D is nonspecial, then $l(K - D) = 0$ and by Riemann-Roch, $l(D) = \deg D + 1 - g$, so $g = 7 + 1 - 4 = 4$. But then $2D$ is also nonspecial, so $l(2D) = 2 \deg D + 1 - g = 14 + 1 - 4 = 11$ which along with the fact that $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{3} = 10$ contradicts projective normality. Therefore if $d = 7$, $g = 5$ or 6.

Exercise IV.6.7. The line, the conic, the twisted cubic and the elliptic quartic curve in $\mathbb{P}^3$ have no multisecants. Every other curve in $\mathbb{P}^3$ has infinitely many multisecants. [*Hint*: Consider a projection from a point of the curve to $\mathbb{P}^2$.]

Solution. We begin with a little discussion of the term multisecant. According to page 310, a multisecant is a line meeting our curve in at least three distinct points. This is somewhat annoying for this problem: it would be much easier if we just assumed there were no lines meeting our curve with total multiplicity three. In fact, at present, I'm not sure how to show that Hartshorne's definition implies the one we would prefer and I don't see how to solve the problem without this better definition, so we're just going to assume that's what was meant.

If our curve X lies in a plane, then it has multisecants iff it is of degree ≥ 3: for any plane curve of degree < 3, any line intersects it at most twice by Bezout, while for a plane curve of degree ≥ 3 any line in that plane is a multisecant by Bezout.

If our curve X does not lie in a plane, pick an arbitrary point P and project from P to $\mathbb{P}^2$. Since X has no multisecants, the projection map is one-to-one: if two points Q and Q' mapped to the same place, then P, Q, and Q' are collinear and form a multisecant. The projection map also separates tangent vectors: if it didn't separate tangent vectors at some point $Q \neq P$, the tangent line to Q must pass through P, so the tangent line at Q meets the curve in multiplicity at least three. To check that it separates tangent vectors at P, we can choose coordinates on $\mathbb{P}^3$ so that $P = [1 : 0 : 0 : 0]$ and X is locally parametrized by $[1 : u_1 t : u_2 t^a : u_3 t^b]$ where t is a uniformizer of $\mathcal{O}_{X,P}$, u_i are units in $\mathcal{O}_{X,P}$, and $1 < a \leq b$. The condition that $T_p X$ meets X to multiplicity at most two is that $a \leq 2$ (the tangent line is $V(x_1, x_2)$, so the intersection multiplicity is $\dim \mathcal{O}_{X,P}/(u_2t^a, u_3t^b) = \min(a,b))$, and therefore $a = 2$. The projection of P is therefore locally parametrized by $[u_1t : u_2t^a : u_3t^b] = [u_1 : u_2t : u_3t^{b-1}]$ and so the projection of X is isomorphic to X.

Since the projection is of degree $d-1$, we must have $g(X) = \frac{1}{2}(d-2)(d-3)$. On the other hand, by Castelnuovo's bound, we must have $g(X) \leq \frac{1}{4}d^2 - d + 1$ or $g(X) \leq \frac{1}{4}(d^2-1) - d + 1$ if d is even or odd, respectively. After solving these inequalities, the only possibilities for d are 2, 3, or 4. As a curve of degree two is contained in a plane, we only need to check what happens with a non-degenerate curve of degree 3 or 4. When $d = 3$, the genus must be 0, which gives the rational normal curve of degree 3; when $d = 4$, the genus must be 1, which gives the elliptic quartic.

When $d > 4$, every point on a nondegenerate curve must lie on a multisecant: else by the work above the projection from a point not on a multisecant would embed our curve of genus g as a plane curve of degree $d-1$, contradicting Castelnuovo's bound. As every multisecant intersects our curve in finitely many points, this implies that there are infinitely many multisecants to any curve in $\mathbb{P}^3$ of degree $d > 4$.

Exercise IV.6.8. A curve X of genus g has nonspecial divisor D of degree d such that $|D|$ has no base points if and only if $d \geq g+1$.

Solution. Here's one thing to get nit-picky about: according to Hartshorne's definition of a base point on page 158, the trivial linear system $k = \Gamma(X, \mathcal{O}_X)$ is base-point free for any curve X. When $g = 0$, $l(K) = 0$, so $D = 0$ is nonspecial and base-point free, and the statement is wrong. One way to fix this is to add the assumption that $D \neq 0$. If we do, then everything works out because if $|D|$ is base-point free and $D \neq 0$, then $l(D) > 1$: if $l(D) = 1$ and $|D|$ is base-point free, this means $H^0(D) \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$ is surjective by lemma II.7.8, and if $h^0(D) = 1$ then this is a surjective map of line bundles and therefore an isomorphism by exercise II.7.1.

Suppose D is nonspecial and has no base points. By an application of Riemann-Roch, we have that $l(D) = d+1-g$, and since $l(D) > 1$ by the first paragraph, we see that $d - g \geq 1$, or $d \geq g+1$ as requested.

Now suppose we're searching for a non-special divisor D of degree $d \geq g+1$ such that $|D|$ is base-point free. We'd like to mimic the proof of proposition IV.6.1, except for the fact that the proof kind of stinks - it's very rough to figure out what's going on at the end. Thankfully, this proof has been improved on in MathOverflow question 409898, and we mimic that proof instead. (For better or worse, this will rely on some materials that have not been completely covered yet. We will avoid constructing the Picard scheme here, though it does exist, it's a variety in our situation, and you may read about it elsewhere: EGA, FGA explained, etc. We will also avoid talking about the details of constructing the quotient of a projective variety by a finite group in defining $\mathrm{Div}^d(X)$, which again you may read about elsewhere.)

By proposition IV.3.1(a), D is base point free iff $\dim |D-P| = \dim |D| - 1$ for all $P \in X$. Restating in terms of $l(D)$, this last condition is the same as $l(D-P) = l(D) - 1$ for all $P \in X$. If this condition holds, then D is nonspecial exactly when $D - P$ is nonspecial: Riemann-Roch on D tells us that $l(K-D) = l(D) + g - d - 1$, while Riemann-Roch on $D-P$ tells us that $l(K-D+P) = l(D-P) + g - (d-1) - 1 = l(D-P) + g - d$, so $l(D-P) - 1 = l(D)$ gives that $l(K-D) = l(K-D+P)$. Up to replacing D by a linearly equivalent divisor D', we may assume $D' - P$ is effective.

Let $\mathrm{Pic}^d(X)$ be the scheme which parametrizes line bundles of degree d on X. Let $\mathrm{Div}^d(X) = X^d/S_d$ be the scheme parametrizing all effective divisors of degree d on X. Our goal is to show that the set S of divisors $D \in \mathrm{Div}^d(X)$ which are linearly equivalent to a divisor of the form $E+P$ for E an effective special divisor of degree $d-1$ and P a point is of dimension at most g.

Let $D_u \subset X \times \mathrm{Div}^d(X)$ be the universal effective divisor of degree d, so that $\mathcal{O}_{X\times \mathrm{Div}^d(X)}(D_u)$ induces a map

$$\varphi_d : \mathrm{Div}^d(X) \to \mathrm{Pic}^d(X)$$

by sending $D \mapsto \mathcal{O}_X(D)$. Therefore the fibers of φ_d are (up to a finite quotient) $|D|$.

Now write $\mathrm{SpDiv}^d \subset \mathrm{Div}^d(X)$ for the closed subscheme of all special effective divisors. By Riemann-Roch, $l(D) = \deg d + 1 - g + l(K-D)$, so if D is special, then

$$\dim \varphi_d^{-1}(\varphi_d(D)) \geq d - g + 1.$$

Since $\dim \mathrm{SpDiv}^d \leq g-1$ as in Hartshorne's proof of proposition IV.6.1, we have that

$$\dim \varphi_d(\mathrm{SpDiv}^d) \leq (g-1) - (d-g+1) = 2g-2-d.$$

Next, let $\alpha : X \times \mathrm{Div}^{d-1}(X) \to \mathrm{Div}^d(X)$ by $(P, D) \mapsto P + D$ and let $\beta : X \times \mathrm{Pic}^{d-1}(X) \to \mathrm{Pic}^d(X)$ by $(P, \mathcal{L}) \mapsto \mathcal{L}(P)$. We obtain the following commutative diagram:

$$\begin{tikzcd} X \times \mathrm{SpDiv}^{d-1} \arrow[r] & X \times \mathrm{Div}^{d-1}(X) \arrow[r, "\alpha"] \arrow[d, "id_X \times \varphi_{d-1}"'] & \mathrm{Div}^d(X) \arrow[d, "\varphi_d"] \\ & X \times \mathrm{Pic}^{d-1}(X) \arrow[r, "\beta"] & \mathrm{Pic}^d(X) \end{tikzcd}$$

Let $T = \beta(id_X \times \varphi_{d-1}(X \times \mathrm{SpDiv}^{d-1})) \subset \mathrm{Pic}^d(X)$, which has dimension at most $2g-2-(d-1)+1 = 2g-d$. As the dimension of the general fiber of φ_d is $d-g$, with equality for nonspecial $\mathcal{L} \in \mathrm{Pic}^d(X)$, the dimension of $\varphi_d^{-1}(T)$ is at most the maximum of the dimension of the special divisors in $\mathrm{Div}^d(X)$ and the dimension of T plus the dimension of the fiber, or $\max(g-1, d-g+2g-d) = g$. Therefore $\dim \varphi_d^{-1}(T) \leq g$, and this is exactly the subscheme which parametrizes effective divisors $D \subset X$ which are linearly equivalent to $E + P$, the sum of a special effective divisor $E \subset X$ and a point $P \in X$. So we've shown that if $d \geq g+1$, we can find a nonspecial base-point free divisor of degree d on X.

Exercise IV.6.9. (*) Let X be an irreducible nonsingular curve in $\mathbb{P}^3$. Then for each $m \gg 0$, there is a nonsingular surface F of degree m containing X. [*Hint*: Let $\pi : \widetilde{P} \to \mathbb{P}^3$ be the blowing-up of X and let $Y = \pi^{-1}(X)$. Apply Bertini's theorem to the projective embedding of $\widetilde{\mathbb{P}}$ corresponding to $\mathcal{I}_X \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(m)$.]

Solution. We'll proceed a little differently from the hint: we'll use the method of the proof of Bertini's theorem to show the result directly.

First, we define the incidence correspondence

$$B_d = \{(f, p) \mid p \in V(f), V(f) \text{ singular at } p\} \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^3}(d))) \times \mathbb{P}^3.$$

This is a closed subscheme cut out by the vanishing locus of $f(p)$ and $\frac{\partial f}{\partial x_i}(p)$ for all i. Next, the injection $\mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^3}(d)$ induces an injection on their global sections, which gives a linear closed immersion $\mathbb{P}(\Gamma(\mathcal{I}_X(d))) \to \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^3}(d)))$. We let $B'_d \subset \mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times \mathbb{P}^3$ be the intersection of B_d and $\mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times \mathbb{P}^3$. Our goal is to show that for all $d \gg 0$, $\dim B'_d < \dim \mathbb{P}(\Gamma(\mathcal{I}_X(d)))$: this implies that the image of B'_d in $\mathbb{P}(\Gamma(\mathcal{I}_X(d)))$ under the projection is a proper closed subscheme, and any closed point outside $\pi(B'_d)$ represents a smooth surface in $\mathbb{P}^3$ containing X.

We will need one short observation about the dimension of a scheme of finite type over a field before we move on to the meat of the problem: if Y is a scheme of finite type over a field and $U \subset Y$ is an open set, then $\dim Y \leq \max(\dim U, \dim U^c)$. The proof begins by considering an irreducible subscheme $Z \subset Y$. Either $Z \cap U \neq \emptyset$ and $\dim Z = \dim Z \cap U \leq \dim U$, or $Z \subset U^c$ and $\dim Z \leq \dim U^c$. As the dimension of Y is the dimension of its largest irreducible component, we have shown the statement. This lets us split our analysis of B'_d in to two cases: $B'_d \cap \mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times (\mathbb{P}^3 \setminus X)$ and $B'_d \cap \mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times X$.

Now suppose $\mathcal{I}_X(d)$ is globally generated, which is true for all $d \gg 0$ by theorem II.5.17. Let $p \in \mathbb{P}^3 \setminus X$. By global generation,

$$\mathcal{I}_X(d) \to (\mathcal{I}_X(d))_p \cong \mathcal{O}_{\mathbb{P}^3,p} \to \mathcal{O}_{\mathbb{P}^3,p}/\mathfrak{m}_p^2$$

is surjective, so the projectivization of the kernel of this map is exactly the fiber of B'_d over p. As $\dim_k \mathcal{O}_{\mathbb{P}^3,p}/\mathfrak{m}_p^2 = 4$, the fiber of B'_d over p is of codimension 4 in $\mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times \mathbb{P}^3$. Therefore the dimension of $B'_d \cap \mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times (\mathbb{P}^3 \setminus X)$ is at most $\dim \mathbb{P}(\Gamma(\mathcal{I}_X(d))) - 1$.

On the other hand, let $p \in X$. By global generation, the image of the map

$$\mathcal{I}_X(d) \to (\mathcal{I}_X(d))_p = \mathcal{I}_{X,p} \subset \mathcal{O}_{\mathbb{P}^3,p}$$

contains a generating set for $\mathcal{I}_{X,p}$. Now consider the map $\mathcal{I}_{X,p} \to k^4$ given by mapping $f \to \{\frac{\partial f}{\partial x_i}(p)\}$. By the projective Jacobian criteria (exercise I.5.8) and the fact that X is smooth at p, the kernel of this map is two-dimensional. Thus the fiber of B'_d over p is of codimension 2 in $\mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times \mathbb{P}^3$, and the dimension of $B'_d \cap \mathbb{P}(\Gamma(\mathcal{I}_X(d))) \times X$ is at most $\dim \mathbb{P}(\Gamma(\mathcal{I}_X(d))) - 1$.

This completes the problem: when $\mathcal{I}_X(d)$ is globally generated, our observation about dimension shows that $\dim B'_d < \dim \mathbb{P}(\Gamma(\mathcal{I}_X(d)))$ and so any point in $\mathbb{P}(\Gamma(\mathcal{I}_X(d)))$ outside the projection of B'_d represents a smooth surface of degree d containing X.

Chapter V

Surfaces

Coming soon!

Appendix A

Intersection Theory

Coming soon!

Appendix B

Transcendental Methods

Coming soon!

Appendix C

The Weil Conjectures

We'll use series expansions like

$$\log(\frac{1}{1-t}) = \sum_{r=1}^{\infty} \frac{t^r}{r}$$

without too much discussion in these solutions. We'll also typically work formally and ignore convergence issues.

Exercise C.5.1. Let X be a disjoint union of locally closed subschemes X_i. Then show that

$$Z(X,t) = \prod Z(X_i,t).$$

Solution. Let $N_{Y,r} = \#\{Y(\mathbb{F}_{q^r})\}$. If the X_i partition X, then $X_i(\mathbb{F}_{q^r})$ partition $X(\mathbb{F}_{q^r})$, so $N_{X,r} = \sum N_{X_i,r}$. Therefore $Z(X,t) = \exp(\sum_{r=1}^{\infty} N_{X,r}\frac{t^r}{r})$, or $\exp(\sum_{r=1}^{\infty}(\sum_i N_{X_i,r})\frac{t^r}{r})$, which is equal to $\prod_i \exp(\sum_{r=1}^{\infty} N_{X_i,r}\frac{t^r}{r}) = \prod_i Z(X_i,t)$.

Exercise C.5.2. Let $X = \mathbb{P}^n_k$, where $k = \mathbb{F}_q$, and show from the definition of the zeta function that

$$Z(\mathbb{P}^n,t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^nt)}.$$

Verify the Weil conjectures for $\mathbb{P}^n$.

Solution. We may write $\mathbb{P}^n = \bigsqcup_{i=0}^n \mathbb{A}^i$. As $\mathbb{A}^i(\mathbb{F}_{q^r}) = (q^r)^i$, we have

$$Z(\mathbb{A}^i,t) = \exp(\sum_{r=1}^{\infty}(q^it)^r/r) = \exp(-\log(1-q^it)) = \frac{1}{(1-q^it)}.$$

Applying the previous exercise, we see that

$$Z(\mathbb{P}^n,t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^nt)}$$

which proves the rationality of the zeta function and the analogue of the Riemann hypothesis for $\mathbb{P}^n$. The Betti numbers are also straightforward: $B_i = 1$ for i even and $B_i = 0$ for i odd, just as the Betti numbers for $\mathbb{C}P^n$, which is $(\mathbb{P}^n_{\mathbb{Z}})_h$.

To check the functional equation, we first need to determine E, the self-intersection number of $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. First, $A(\mathbb{P}^n \times \mathbb{P}^n) \cong A(\mathbb{P}^n) \otimes A(\mathbb{P}^n)$: by property A11 of the Chow ring in section A.2 applied to $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$, we find that $A(\mathbb{P}^n \times \mathbb{P}^n) \cong A(\mathbb{P}(\mathcal{E}))$ is a free $A(\mathbb{P}^n)$-module generated by $1, h, \cdots, h^n$ where h is the hyperplane divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, which is exactly our claim. Let s, t be the classes of hyperplanes on the first and second $\mathbb{P}^n$, respectively. Writing $\Delta = \sum_{i+j=n} a_{ij} s^i t^j$, we see that $\Delta^2 = \sum a_{ij} a_{i'j'} s^n t^n$ where $i + i' = j + j' = n$ because $s^{n+1} = t^{n+1} = 0$. On the other hand, we may compute the intersection product of $\mathbb{P}^i, \mathbb{P}^j \subset \mathbb{P}^n$ by taking $\Delta.(\mathbb{P}^i \times \mathbb{P}^j)$, and by the structure of the Chow ring for projective space in example A.2.0.1, we have that this is 1 iff $i + j = n$ and 0 else. This implies that $a_{ij} = 1$ for all $i + j = n$, and therefore $\Delta^2 = n + 1$. Taking $Z(\frac{1}{q^n t})$, we get

$$\frac{1}{(1 - \frac{1}{q^n t})(1 - \frac{1}{q^{n-1} t}) \cdots (1 - \frac{1}{t})} = \frac{(q^n t)(q^{n-1} t) \cdots (t)}{(q^n t - 1)(q^{n-1} t - 1) \cdots (t - 1)}$$

which is equal to $(-1)^{n+1} q^{n(n+1)/2} t^{n+1} Z(t)$, and the functional equation is satisfied.

Exercise C.5.3. Let X be a scheme of finite type over $\mathbb{F}_q$, and let $\mathbb{A}^1$ be the affine line. Show that

$$Z(X \times \mathbb{A}^1, t) = Z(X, qt).$$

Solution. First we reduce to the case where X is affine by stratifying X by locally closed affine subschemes. For each of the finitely-many irreducible components of X, pick an affine open neighborhood contained entirely inside that irreducible component. Then the complement of these neighborhoods are a closed subscheme of X of strictly smaller dimension, so it again has finitely many irreducible components. Now repeat until we've exhausted X. By problem A.5.1, it suffices to prove the claim for each of these pieces, so we may assume X is affine.

Now consider X as a closed subset of some $\mathbb{A}^n$. Then $N_{X,r}$ is the number of points of $\overline{X} \subset \overline{\mathbb{A}^n}$ with coordinates in $\mathbb{F}_{q^r}$. As any point in $\overline{X}$ with coordinates in $\mathbb{F}_{q^r}$ determines q^r points of $\overline{X \times \mathbb{A}^1}$ with coordinates in $\mathbb{F}_{q^r}$ by plugging in any element of $\mathbb{F}_{q^r}$ in to the final coordinate of $\overline{X \times \mathbb{A}^1} \subset \overline{\mathbb{A}^{n+1}}$, we see that $N_{X \times \mathbb{A}^1, r} = q^r N_{X,r}$. So $\exp(\sum_{r=1}^\infty N_{X \times \mathbb{A}^1, r} t^r / r) = \exp(\sum_{r=1}^\infty N_{X,r} (qt)^r / r)$ and we have the desired identity.

Exercise C.5.4. The *Riemann zeta function* is defined as

$$\zeta(s) = \prod \frac{1}{1 - p^{-s}},$$

for $s \in \mathbb{C}$, the product being taken over all prime integers p. If we regard this function as being associated with the scheme $\operatorname{Spec} \mathbb{Z}$, it is natural to define, for any scheme X of finite type over $\operatorname{Spec} \mathbb{Z}$,

$$\zeta_X(s) = \prod (1 - N(x)^{-s})^{-1}$$

where the product is taken over all closed points $x \in X$, and $N(x)$ denotes the number of elements in the residue field $k(x)$. Show that if X is of finite type over $\mathbb{F}_q$, then this function is connected to $Z(X,t)$ by the formula

$$\zeta_X(s) = Z(X, q^{-s}).$$

[*Hint*: take $d\log$ of both sides, replace q^{-s} by t, and compare.]

Solution. Let's follow the hint, starting with $\zeta_X(s)$:

$$\frac{\partial}{\partial s}\log\prod(1-N(x)^{-s})^{-1} = \frac{\partial}{\partial s}\sum -\log(1-N(x)^{-s}) = \sum\frac{\log(N(x))}{1-N(x)^s} = \sum\frac{-N(x)^{-s}\log(N(x))}{1-N(x)^{-s}}$$

and letting $\deg x = [k(x):\mathbb{F}_q]$ so that $N(x) = q^{\deg x}$ we get that this is equal to

$$\sum -\frac{q^{-s\deg x}(\deg x)\log(q)}{1-q^{-s\deg x}}$$

where the sums and product are over all closed points of x. For $Z(X, q^{-s})$, we have

$$\frac{\partial}{\partial s}\log(\exp(\sum_{r=1}^{\infty} N_r\frac{q^{-rs}}{r})) = \sum_{r=1}^{\infty} -N_r\log(q)q^{-rs}.$$

Our task is therefore to prove that

$$\sum_{x\in X \text{ closed}}\frac{q^{-s\deg x}(\deg x)}{1-q^{-s\deg x}} = \sum_{r=1}^{\infty} N_r q^{-rs}.$$

For a fixed closed $x \in X$, we have that the summand on the left hand side is equal to

$$(\deg x)(q^{-s\deg x} + q^{-2s\deg x} + q^{-3s\deg x} + \cdots).$$

Now we investigate the contribution of a fixed closed $x \in X$ to N_r. Since X is finite type over $\mathbb{F}_q$, $k(x)$ is a finite extension of $\mathbb{F}_q$ by Zariski's lemma, and as $\mathbb{F}_q$ is perfect, $k(x)$ is a separable extension. By the theorem of the primitive element, we may write $k(x) = \mathbb{F}_q[\alpha]/(\rho(\alpha))$ for some separable polynomial ρ with $\deg\rho = \deg x$. Then the number of points living over x in $X\times_{\mathbb{F}_q}\mathbb{F}_{q^r}$ is the number of $\mathbb{F}_{q^r}$ points of $\operatorname{Spec} k(x)\times_X X\times_{\mathbb{F}_q}\mathbb{F}_{q^r}$, equivalently of $\operatorname{Spec} k(x)\times_{\mathbb{F}_q}\mathbb{F}_{q^r} \cong \operatorname{Spec}\mathbb{F}_{q^r}[\alpha]/(\rho(\alpha))$. This has $\mathbb{F}_{q^r}$ points exactly when $\rho(\alpha)$ factors, which happens exactly when ρ splits completely, which happens exactly when $\deg x$ divides r, in which case we get $\deg x$ points. So x contributes $\deg x$ points to N_r when $\deg x$ divides r and contributes nothing to N_r otherwise. This shows that the coefficients of q^{-rs} on both sides of our hoped-for equality are equal, and therefore we're done.

Exercise C.5.5. Let X be a curve of genus g over k. Assuming the statements (1.1) to (1.4) of the Weil conjectures, show that $N_1, N_2, \cdots, N_g$ determine N_r for all $r \geq 1$.

Solution. By the Weil conjectures, $\dim H^1(X, \mathbb{Q}_l) = B_1 = \deg P_1(t) = 2g$, so $Z(X,t) = \frac{P_1(t)}{(1-t)(1-qt)}$ where $P_1 = \prod_{j=1}^{2g}(1-\alpha_j t)$ for algebraic integers α_j. Rewriting this Zeta function in exponential form, we have

$$\exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty}\left(\frac{t^r}{r} + \frac{q^r t^r}{r} - \sum_{j=1}^{2g}\frac{\alpha_j^r t^r}{r}\right)\right)$$

and therefore $N_r = 1 + q^r - \sum_{j=1}^{2g}\alpha_j^r$ for all r.

Knowing N_r for $1 \le r \le g$ tells us half of the information we need to determine $\sum_{j=1}^{2g}\alpha_j^r$ for all r; we'll get the other half from the functional equation. By the portion of the Weil conjectures dealing with the Betti numbers, $E = \sum(-1)^i B_i$, so on a curve of genus g we get $E = 2-2g$. Therefore the functional equation gives us that

$$Z(\frac{1}{qt}) = \pm q^{1-g} t^{2-2g} Z(t)$$

and the LHS expands as

$$\frac{\prod_{i=j}^{2g}(1-\frac{\alpha_j}{qt})}{(1-\frac{1}{qt})(1-\frac{1}{t})} = \frac{t^{2-2g}q^{1-g}\prod_{j=1}^{2g}(\sqrt{q}t - \frac{\alpha_j}{\sqrt{q}})}{(1-t)(1-qt)}$$

so $P_1(t) = \prod_{j=1}^{2g}(1-\alpha_j t) = \prod_{j=1}^{2g}(\sqrt{q}t - \frac{\alpha_j}{\sqrt{q}})$. Using this new expression for $P_1(t)$ and rewriting this Zeta function in exponential form, we find that

$$Z(X,t) = \exp\left(\sum_{r=1}^{\infty}\left(1 + q^r + \sum_{j=1}^{2g}\frac{q^r}{\alpha_j^r}\right)\frac{t^r}{r}\right)$$

and therefore $N_r = 1 + q^r + \sum_{j=1}^{2g}\frac{q^r}{\alpha_j^r}$ and so we also know $\sum_{j=1}^{2g}\alpha_j^{-r}$ for all $1 \le r \le g$.

To show that knowing $\sum_{j=1}^{2g}\alpha_j^r$ for $-g \le r \le g$ suffices to determine $\sum_{j=1}^{2g}\alpha_j^r$ for all r, we'll show that the rational functions $p_r = \sum_{j=1}^{2g} x_j^r$ for $-g \le r \le g$ generate $\mathbb{Q}(x_1, \cdots, x_{2g})^{\Sigma_{2g}}$, the field of rational symmetric functions on $2g$ variables. It suffices to show that the elementary symmetric polynomials $e_i(x_j)$ for $0 < i \le 2g$ are in the field generated by the p_r, since by the fundamental theorem of symmetric polynomials these generate $\mathbb{Q}(x_1, \cdots, x_{2g})^{\Sigma_{2g}}$. By Newton's identities, p_r for $0 < r \le g$ give us $e_i(x_j)$ for $0 < i \le g$, so it remains to construct the other half of the $e_i(x_j)$.

Let's investigate p_r for $r < 0$. We may write

$$p_r = \frac{e_{2g-1}(x_j^r)}{e_{2g}(x_j^r)} = \frac{e_{2g-1}(x_j^r)}{e_{2g}(x_j)^r},$$

and the numerator $e_{2g-1}(x_j^r)$ may be written as a sum of products of the form $\prod_{i=1}^{N} e_{r_i}(x_j)$ where $\sum r_i = 2g-1$. I claim that in fact this sum of products is of the form

$$\sum_{a_1 \le \cdots \le a_r, a_1 + \cdots + a_r = r(2g-1)} c_{a_1, \cdots, a_r} \prod_{i=1}^{r} e_{a_i}(x_j)$$

where each c is nonzero. The proof is by Gauss' algorithm to express a symmetric polynomial in terms of elementary symmetric polynomials - by examining leading terms, it's not difficult to see that $e_{2g-1}(x_j^r) = e_{2g-1}(x_j)^r + \cdots$ and then fill in the remaining $\cdots$ as products after computing multinomial coefficients, but this is a little messy and I'm choosing to skip over the details.

The upshot is that each p_r for $-g \leq r < 0$ may be expressed as a polynomial in $\frac{e_{2g-1}}{e_{2g}}, \cdots, \frac{e_{2g-r+1}}{e_{2g}}$ plus a term of the form $c\frac{e_{2g-r}}{e_{2g}}$ for some nonzero rational c. In particular, this gives us all we need to prove our claim about the field of symmetric functions: the generating set $\{p_r\}_{-g\leq r<1}$ reduces the the generating set $\{\frac{e_s}{e_{2g}}\}_{g\leq s\leq 2g-1}$ by some elementary algebra, and given that we know e_g, this lets us recover e_{2g} and then every e_r for $g < r \leq 2g$, and we're done.

Exercise C.5.6. Use (IV, Ex. 4.16) to prove the Weil conjectures for elliptic curves. First note that for any r,

$$N_r = q^r - (f^r + \hat{f}^r) + 1$$

where $f = F'$. Then calculate $Z(t)$ formally and conclude that

$$Z(t) = \frac{(1-ft)(1-\hat{f}t)}{(1-t)(1-qt)}$$

and hence

$$Z(t) = \frac{1-at+qt^2}{(1-t)(1-qt)},$$

where $f + \hat{f} = a_X$. This proves rationality immediately. Verify the functional equation. Finally, if we write

$$1 - at + qt^2 = (1-\alpha t)(1-\beta t),$$

show that $|a| \leq 2\sqrt{q}$ if and only if $|\alpha| = |\beta| = \sqrt{q}$. Thus the analogue of the Riemann hypothesis is just (IV, Ex. 4.16d).

Solution. By exercise IV.4.16(d), $N_r = q^r - (f^r + \hat{f}^r) + 1$, so by the definition of the Zeta function we have

$$Z(X,t) = \exp(\sum_{r=1}^{\infty}(q^r - (f^r + \hat{f}^r) + 1)\frac{t^r}{r}) = \frac{(1-ft)(1-\hat{f}t)}{(1-qt)(1-t)}.$$

Combining this with the fact that $f\hat{f} = q$ by exercise IV.4.7(c) and the fact that $f + \hat{f}$ is an integer a_X by exercise IV.4.16(c), we have that $Z(X,t) = \frac{1-a_X t+qt^2}{(1-qt)(1-t)}$, which verifies rationality.

The functional equation follows from a little algebra:

$$Z(X, \frac{1}{qt}) = \frac{1 - a\frac{1}{qt} + q\frac{1}{q^2t^2}}{(1-\frac{1}{qt})(1-q\frac{qt}{)}} = \frac{1 - \frac{a}{qt} + \frac{1}{qt^2}}{(1-\frac{1}{qt})(1-\frac{1}{t})}$$

and multiplying by $\frac{qt^2}{qt^2}$, we find this is just

$$\frac{qt^2 - at + 1}{(qt-1)(t-1)}$$

or $\pm Z(X,t)$.

See parts (b) and (c) of exercise C.5.7 for the final claim.

Exercise C.5.7. Use (V, Ex. 1.10) to prove the analogue of the Riemann hypothesis (1.3) for any curve C of genus g defined over $\mathbb{F}_q$. Write $N_r = 1 - a_r + q^r$. Then according to (V, Ex. 1.10),

$$|a_r| \leq 2g\sqrt{q^r}.$$

On the other hand, by (4.2) the zeta function of C can be written

$$Z(t) = \frac{P_1(t)}{(1-t)(1-qt)}$$

where

$$P_1(t) = \prod_{i=1}^{2g}(1 - \alpha_i t)$$

is a polynomial of degree $2g = \dim H^1(C, \mathbb{Q}_l)$.

a. Using the definition of the zeta function and taking logs, show that

$$a_r = \sum_{i=1}^{2g}(\alpha_i)^r$$

for each r.

b. Next show that

$$|a_r| \leq 2g\sqrt{q^r} \text{ for all } r \quad \Leftrightarrow \quad |\alpha_i| \leq \sqrt{q} \text{ for all } i.$$

[*Hint*: One direction is easy. For the other, use the power series expansion

$$\sum_{i=1}^{2g} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{r=1}^{\infty} a_r t^r$$

for suitable $t \in \mathbb{C}$.]

c. Finally, use the functional equation (4.4) to show that $|\alpha_i| \leq \sqrt{q}$ for all i implies that $|\alpha_i| = \sqrt{q}$ for all i.

Solution.

a. By definition, $\log(Z(X,t)) = \sum_{r=1}^{\infty} N_r \frac{t^r}{r}$. On the other hand, from the expression $Z(t) = \frac{P_1(t)}{(1-t)(1-qt)}$, we have $\log(Z(t)) = \sum_{r=1}^{\infty}(1 + q^r - \sum_{i=1}^{2g} \alpha_i^r)\frac{t^r}{r}$, so by matching coefficients we have $N_r = 1 + q^r - a_r = 1 + q^r - \sum_{i=1}^{2g} \alpha_i^r$, so $a_r = \sum_{i=1}^{2g} \alpha_i^r$ as requested.

b. If $|\alpha_i| \leq \sqrt{q}$ for all i, then

$$|a_r| \leq \sum_{i=1}^{2g} |\alpha_i|^r \leq \sum_{i=1}^{2g} \sqrt{q^r} = 2g\sqrt{q^r}.$$

Conversely, writing

$$\sum_{i=1}^{2g} \frac{\alpha_i t}{1-\alpha_i t} = \sum_{r=1}^{\infty} a_r t^r$$

as per the hint, if $|a_r| \leq 2g\sqrt{q^r}$, then the RHS is holomorphic for $|t| < q^{-1/2}$. But if $|\alpha_i| > \sqrt{q}$ for some i, then the LHS has a pole at $t = \frac{1}{\alpha_i}$ which is inside $|t| < q^{-1/2}$.

c. By the same sort of technique with the functional equation as in exercise C.5.5, we have that $P_1(t) = \prod_{j=1}^{2g}(1-\alpha_j t) = \prod_{j=1}^{2g}(\sqrt{q}t - \frac{\alpha_j}{\sqrt{q}})$. Therefore the roots of these two polynomials must be equal. The LHS has roots $\alpha_1^{-1}, \cdots, \alpha_{2g}^{-1}$ and the RHS has roots $\frac{\alpha_1}{q}, \cdots, \frac{\alpha_{2g}}{q}$. Therefore for any i there is a j so that $|\alpha_i\alpha_j| = q$, and since $|\alpha_i|$ and $|\alpha_j|$ are both at most $\sqrt{q}$ by (b), we see that they must be exactly $\sqrt{q}$ in order for this relation to hold.