A new constitutive equation for a solid material

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ABSTRACT: The present author (Nakaza), introduced a new elastic theory in 2005. According to this theory, the internal stress of an isotropic elastic material consists of an elastic stress governed by Hooke's law and an internal pressure governed by the state equation. In conventional theory, the internal stress consists of the two elastic stresses corresponding to the two elastic moduli and thus the force causes the Poisson effect cannot be explained, even if we observe the occurrence of the lateral strains under a uniaxial loading. According to Nakaza's elastic theory, Hooke's law has only one elastic modulus and in this sense, the new theory is consistent with the assertion of Navier who used only one elastic constant to derive his fundamental elastic theory. The new elastic theory may change the stress evaluation from the existing theory.

1 INTRODUCTION

Even if we observe deformations of a material (i.e., a distribution of strain), we cannot immediately judge whether the distribution corresponds to the distribution of the internal stress. A strain distribution is converted into a stress distribution based on Hooke's law, which is characterized as having two elastic moduli. For example, when we observe the deformation of the material under longitudinal (vertical) uniaxial loading, not only longitudinal strain, but also lateral strains are generally observed in the material. In such a case, non-zero internal stress exists only in the longitudinal direction in the material, i.e., the stresses in the lateral directions are evaluated to be zero.

In this case, the distortions in the transverse directions are caused by the Poisson effect. By relying on the traditional elastic theory, the stress that causes these cannot be explained. Then, by further loading the material, the material shows cracks or splitting in the longitudinal direction, leading to failure. At this time, the fracture surface is in the longitudinal direction. Why did the material crack run vertically? The conventional theory cannot answer this simple question because two elastic moduli are used in the traditional theory

In the present paper, a new elastic theory is introduced, and its characteristics are discussed. According to the new elastic theory, the simple questions stated above are clearly resolved. Furthermore, it is shown that the distribution of internal stresses received from outer loading directly corresponds to the measured deformations (strain distribution). Finally, a new failure criterion is suggested.

2 OUTLINES OF THE NEW ELASTICITY THEORY

Here, we will derive the new elastic theory of Nakaza (2005) as a development of the conventional isotropic elastic theory. First, the linear relationship between stress and strain in the conventional elasticity theory is as follows (Fung, 1994) :

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} + \beta dT\delta_{ij} \tag{1}$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain tensor, λ and μ are the moduli of elasticity, dT is the temperature change, β is the coefficient of linear thermal expansion, and δ_{ij} is the Kronecker delta.

Equation (1) shows the constitutive equation (general Hooke's law) introduced in the traditional elastic theory, considering the thermal expansion. Nakaza (2005, 2009, 2010) recognized the existence of pressure in the internal stress of the elastic body. The relationship among pressure change, material density change and temperature change is generally given as follows, according to the thermodynamic equation of state:

$$p = R\rho T\left(\frac{d\rho}{\rho} + \frac{dT}{T}\right) \tag{2}$$

where p is the inner pressure change, ρ is the density, $d\rho$ is the density change, T is the temperature, dT is the temperature change, and R is the material constant.

Therefore, by introducing Equation (2) to Equation (1), we have the following relationship:

$$\sigma_{ij} = -p\delta_{ij} + 2E\varepsilon_{ij} \tag{3}$$

where, E is the modulus characterizing the elasticity, which may be a function of a state defined by inner pressure and temperature, and is uni-modular for an elastic isotropic material. From Equation (1) and (3), the relationship $E = \mu$ is clearly identified.

Note that when we consider the plasticity and the viscosity of a material, we may introduce the plasticity strain, the rate of strain and viscosity coefficient to the fundamental relationship, i.e., Eq. (3).

Equation (3) is the linear relationship between the stress and strain defined by Nakaza (2005). The first term on the right-hand side of Equation (3) represents the change in the internal pressure governed by the state equation shown by Equation (2) and the second term represents the elastic stress governed by Hooke's law. In this way, the elastic modulus is defined as only one constant for Hooke's law of an isotropic elastic material, which conforms to Navier's theorem claiming that there should be only one elastic modulus (Timoshenko, 1988).

The internal pressure change shown in Equation (3) causes isotropic expansion or contraction of a material. Therefore, we move the term of internal pressure to the left-hand side in order to obtain the following equation:

$$\sigma_{ij} + p\delta_{ij} = 2E\varepsilon_{ij}. \tag{4}$$

Alternatively, this equation can be written as follows:

$$\tau_{ij} = 2E\varepsilon_{ij} \tag{5}$$

where τ_{ij} is termed *elastic stress*.

The physical meaning of Equation (4) is interpreted such that the left-hand side represents the stresses as the action that causes deformations of an elastic material, and the right-hand side represents the appearance showing the resistance of the material by elastic springs counter to the actions of the stresses. Equation (3) will be further transformed into another type of equation through the discussion in the following sections.

3 PHYSICAL INTERPRETATION OF THE POISSON EFFECT

Although conventional elastic theory recognizes the occurrence of strain due to the Poisson effect, it cannot explain what stress causes it. On the other hand, the theory of Nakaza shown in Equation (3) is explained below. Here, the case of uniaxial compression of an elastic bar in the vertical direction is assumed. At this time, the relationship between stress and strain in the longitudinal direction is given as follows:

$$\sigma_3 = -p + 2E\varepsilon_3 \tag{6}$$

where the suffix 3 indicates the direction of the bar axis.

For the lateral directions, the following relations are given:

$$0 = -p + 2E\varepsilon_1 \tag{7}$$

And

$$0 = -p + 2E\varepsilon_2. \tag{8}$$

From these, we have

$$p = 2E\varepsilon_1 \tag{9}$$

and

$$p = 2E\varepsilon_2. \tag{10}$$

In these equations, $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(\sigma_1, \sigma_2, \sigma_3)$ are the principal strain and stress.

From Equations (9) and (10), it is concluded that the Poisson effect is an isotropic deformation caused by the internal pressure change. According to Hooke's law, the change in internal pressure causes isotropic deformation of the material.

In Equation (6), the external load is supported by a change in internal pressure and elastic stress. Since the deformation caused by the pressure change is isotropic, the following relationship can be given:

$$p = 2E\varepsilon_p \tag{11}$$

where ε_p is the isotropic strain caused by the internal pressure change. This can be measured as the strain due to the Poisson effect being observed. From this equation, we have the pressure change in a material.

Substituting Equation (11) in to Equation (6), we have:

$$\sigma_3 = 2E(\varepsilon_3 - \varepsilon_p). \tag{12}$$

The stress, indicated by the left-hand side of Equation (12), due to the outer loading on a material is supported by the elastic stress formed by the strain observed after subtraction of the isotropic strain. The strain $(\varepsilon_3 - \varepsilon_p)$ can be written in ε'_3, which is termed the *effective strain* corresponding to the stresses purely caused by outer loads.

In traditional theory, for a uniaxial test of an elastic bar, Poisson's ratio has been defined as

$$\varepsilon_1 = \varepsilon_2 = -\nu\varepsilon_3 \tag{13}$$

where ν is the Poisson's ratio.

In Equation (13), the Poisson effect is not properly defined as satisfying the isotropic condition for an isotropic elastic material. Since in the direction of axial loading, any strain caused by the Poisson effect has not explicitly been introduced.

From Equations (3) and (12), considering a constant temperature condition, we have:

$$\varepsilon_p = -\,\theta\,\varepsilon_{kk} \tag{14}$$

$$\theta = \nu/(1-2\nu). \tag{15}$$

Instead of using Poisson's definition for Poisson's ratio, as in Equation (13), we here introduce the *modified Poisson's ratio*, θ, properly defined as satisfying the isotropic condition of the Poisson effect.

From Equation (11) and (14), for the state change for a constant temperature, obtaining the relative volume change ε_{kk}, we have:

$$p = 2E\,(-\theta\,\varepsilon_{kk}). \tag{16}$$

4 EFFECTS OF TEMPERATURE

For temperature changes, under no outer loading on a material, the new theory explains that temperature change causes an inner pressure change as shown by Equation (2), so that the pressure change causes strain as shown in Equation (3). From the constitutive equation (Equation (3) and the state Equation (2)), we have

$$p = 2E\varepsilon_{ij} \tag{17}$$

which shows the thermal expansions of an elastic material due to temperature changes. According to Hooke's law, these thermal expansions are dynamically caused by inner pressure changes. When we introduce the inner pressure changes due to the temperature changes, Equation (17) shows that the modulus between the pressure change and the strain can be given by the elasticity, E.

5 FAILURE CRITERIA FOR ISOTROPIC MATERIALS

There are many types of failure criteria, and the criteria of Tresca and von Mises are well known as basic failure criteria for isotropic materials. These are mathematically related to invariants of the stress and/or strain tensor (Chen & Saleeb, 1982).

In the traditional theory, the stress tensor is split into two parts, the pure hydrostatic pressure and the pure shear stress, as follows:

$$\sigma_{ij} = -\bar{p}\delta_{ij} + 2G\big(\varepsilon_{ij} - 1/3\varepsilon_{kk}\delta_{ij}\big). \tag{18}$$

where G is the shear modulus, $\bar{p}$ is the mean stress (pure hydrostatic stress) and the second term is the deviatoric stress (stress deviator) tensor.

Denoting the deviatoric stress tensor by σ'_{ij}, we have:

$$\sigma_{ij} = -\bar{p}\delta_{ij} + \sigma'_{ij} \tag{19}$$

or

$$\sigma_{ij} = -\bar{p}\delta_{ij} + 2G\varepsilon'_{ij}. \tag{20}$$

Here ε'_{ij} is the deviatoric strain tensor and is given as:

$$\varepsilon'_{ij} = \big(\varepsilon_{ij} - 1/3\varepsilon_{kk}\delta_{ij}\big). \tag{21}$$

We can now compare Equation (20) with Equation (3) in order to determine the differences between these equations.

Tresca's failure criteria are related to the maximum shear stress at a point in a material as follows:

$$max\left[\frac{1}{2}(\sigma_1-\sigma_2),\ \frac{1}{2}(\sigma_2-\sigma_3),\ \frac{1}{2}(\sigma_3-\sigma_1)\right] = k \tag{22}$$

where σ_1, σ_2, and σ_3 are the principal stresses, and k is the failure (yield) strength.

On the other hand, von Mises criteria are related to the second invariant of deviatoric stress tensor as follows:

$$J_2 - k^2 = 0 \tag{23}$$

where J_2 is the second invariant of the deviatoric stress tensor.

There are also Drucker-Prager failure criteria relating the two invariants to the above fundamental failure criterions:

$$\alpha I_1 + \sqrt{J_2} - k = 0 \tag{24}$$

where α is a material constant, and I_1 is the first invariant of the stress tensor.

The failure criteria introduced here, and many other failure criteria are thus associated with the invariants of the stress tensor or strain tensor. This is due to the mathematical property whereby the invariants do not depend on how coordinate axes are set. However, there is no obvious physical reason to associate these invariants with failure criteria.

The failure criteria discussed above indicate that at any point in an isotropic elastic body, when the stress or strain reaches a certain level, cracking or splitting occurs. There is a problem, however, with the setting of the failure surface. For example, in the case of uniaxial pure compression for which the loading state is very simple, the fracture surface cannot be set even if we see that the stress state reaches a certain failure criterion. Therefore, according to the conventional theory, dual criteria (Wu, 1974) are recommended based on both the stress tensor for checking the failure point and the strain tensor for setting the fracture surface.

In contrast to these conventional failure criteria, Nakaza's new theory presents a physically clear and simple method.

First, Equation (3) gives the following equation for the work done by the stress and the inner pressure:

$$\int \left(\sigma_{ij} + p\delta_{ij}\right) d\varepsilon_{ij} = E\varepsilon_{ij}^2. \tag{25}$$

The right-hand side of Equation (25) shows the *strain energy* as the work done by the internal pressure p and the stress σ_{ij}.

As shown in Equation (4), it is physically explained that the isotropic elastic material is deformed under the influence of the stresses due to the actions of both the external force and the internal pressure change. The material withstands the actions of the stresses with the reaction of the elastic stress produced by a mechanical function of an elastic spring as an elastic body. That is, as a failure criterion, it is sufficient to investigate the magnitude of the elastic stress and/or the strain energy represented by the right-hand side of Equation (4) and the strain energy term of Equation (25).

Instead, in term of the traditional theory of elasticity, the strain energy is given as follows:

$$\int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} K\varepsilon_{kk}^2 + G\varepsilon'^2_{ij} \tag{26}$$

or

$$\int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} \lambda\varepsilon_{kk}^2 + \mu\varepsilon_{ij}^2 \tag{27}$$

where K and G are the bulk and the shear modulus, respectively, ε_{kk} is the volumetric strain, and ε'_{ij} is the deviatoric tensor of the strain tensor. Comparing Equations (26) and (27) with Equation (25), we can completely explain the physical differences between these equations.

According to Equation (25), for example, if we consider the strain energy, the failure criterion is given as follows:

$$E\varepsilon_{ij}^2 = k^2 \tag{28}$$

where k is the failure (yield) strength, which may generally be a function of state defined in terms of the inner pressure and temperature.

The failure criteria shown in Equation (28) are similar to the von Mises failure criteria shown in the Equation (23), though the von Mises failure criteria focus on the deviatoric stress tensor of the stress tensor. In the new theory, the criteria, for example, is physically based on the strain energy that shows the material deformation enduring both stresses due to external force and internal pressure. Of course, creating a failure criterion based on the maximum principal stress or the maximum shear stress of the elastic stress is straightforward, shown as follows:

$$E\varepsilon_1 = k \tag{29}$$

or

$$max\left[\frac{1}{2}E(\varepsilon_1 - \varepsilon_2), \frac{1}{2}E(\varepsilon_2 - \varepsilon_3), \frac{1}{2}E(\varepsilon_3 - \varepsilon_1)\right] = k \tag{30}$$

In the new elastic theory, there is no necessity to introduce the deviatoric stress or strain tensor as used in the conventional elastic theory. Furthermore, in the new failure criteria, in Equation (28), (29), and (30), the distributions of the stress and the strain directly correspond to each other such that the new theory has the advantage that the failure surface based on the stress completely coincides with the failure surface based on the strain.

Therefore, according to the new theory, there is no need to include a dual failure criteria system. The problem that arose in the case of uniaxial compression discussed earlier is also completely resolved by the new theory.

Figure 1 shows the strengths of two types of concrete specimens under combined loading of pure compression, pure tension, pure shear, compression and shear, and tensile and shear. One of the concrete specimens has normal strength (compression strength, $\sigma_{cr} = 28\,MPa$) and the other has high strength (compression strength, $\sigma_{cr} = 43\,MPa$). Following Equation (5), the elastic stress is evaluated. In Figure 1, both concrete specimens are shown in the graph.

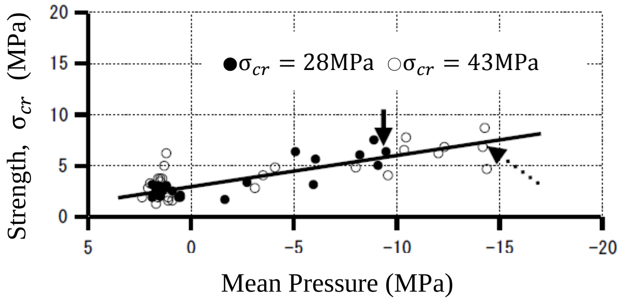

Figure 1. Strengths σ_{cr} of two types of concrete specimens under combined loading of pure compression, pure tension, pure shear, compression and shear, and tension and shear. The solid line indicates Equation (29) for which k is a function of pressure and temperature changes. The two arrows indicate the pure compression strengths for two types of concrete specimens. All experimental data are from Okajima (1970).

The result based on the new theory, represented by the solid line, creates a failure criterion in wide range from pure tension to pure compression. The *x*-axis of Figure 1, although indicated as the pure hydrostatic (mean) pressure $\bar{p}$, can be easily rewritten as the pressure p, as follows:

$$p = \frac{3\nu}{1+\nu}\,\bar{p}. \qquad (31)$$

Let us note that the failure criterion, k, may not be a constant or a linear function, but rather must be a nonlinear function of pressure and temperature.

6 CONCLUSION

Based on the principles of physics, Nakaza presented a new elastic theory and described the differences between the new elastic theory and the conventional elastic theory. In the present paper, based on the new theory, the physical mechanism of the Poisson effect and the dynamics of the thermal expansion of an elastic body were explained. The theory of failure criteria for isotropic elastic materials was then explained. The failure criteria based on the elastic stress was applied to two types of concrete specimens that have different compressive strengths, and the effectiveness of the new theory was demonstrated.

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