Problem Sheet 6

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1 Operations Over Context-Free Languages

1.1 Proof that Context-Free Languages are closed over the Union Operation

If two languages (L_1, L_2) are context-free, there exist corresponding context-free grammars:

$$G_1 = (V_1, \Sigma_1, R_1, S_1)$$
$$G_2 = (V_2, \Sigma_2, R_2, S_2)$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates $L_1 \cup L_2$ where:

$$V = V_1 \cup V_2 \cup \{S\}$$
$$\Sigma = \Sigma_1 \cup \Sigma_2$$
$$R = R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}$$

Where S is a new start variable

Clearly G generates $L_1 \cup L_2$.

$$S \implies S_1 \implies ...(\text{any string from } L(G_1))$$
$$S \implies S_2 \implies ...(\text{any string from } L(G_2))$$

This completes the proof for closure of Context-Free Languages under the union operation.

1.2 Proof that Context-Free Languages are closed over the Concatenation Operation

If two languages (L_1, L_2) are context-free, there exist corresponding context-free grammars:

$$G_1 = (V_1, \Sigma_1, R_1, S_1)$$
$$G_2 = (V_2, \Sigma_2, R_2, S_2)$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates $L_1 \circ L_2$ where:

$$V = V_1 \cup V_2 \cup \{S\}$$
$$\Sigma = \Sigma_1 \cup \Sigma_2$$
$$R = R_1 \cup R_2 \cup \{S \to S_1 S_2\}$$

Where S is a new start variable

Clearly G generates $L_1 \circ L_2$.

$$S \implies S_1 S_2 \implies (\text{any string from } L(G_1)) \circ (\text{any string from } L(G_2))$$

This completes the proof for closure of Context-Free Languages under the concatenation operation.

1.3 Proof that Context-Free Languages are closed over the Kleene Star Operation

If a language L is context-free, there exists a corresponding context-free grammars:

$$G_1 = (V_1, \Sigma_1, R_1, S_1)$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates L^* where:

$$\begin{aligned}
&V = V_1 \cup \{S\} \\
&\Sigma = \Sigma_1 \\
&R = R_1 \cup \{S \to S_1 S, S \to \varepsilon\}
\end{aligned}$$

Where S is a new start variable

Clearly G generates L^*.

$$\begin{aligned}
S &\implies \varepsilon \\
S &\implies S_1 S \implies \text{(any string from } L(G_1)) \circ \text{(any string from } L(G))
\end{aligned}$$

This completes the proof for closure of Context-Free Languages under the Kleene Star operation.

2 Pumping Lemma for Context Free Languages

Prove the following language is not context-free:

$$L = \{a^p \in \{a\}^* \mid p \text{ is a prime number}\}$$

Assume L is context-free, then the context-free pumping lemma applies, with a pumping length of n.

Then consider a^p where $p \geq n$ is a prime number.

Then by the lemma, $a^p = uvxyz$ with the usual conditions:

Let $s = |uxz|$, and $r = |vy|$. Notice that $p = s + r$

Then $|uv^ixy^iz| = s + ir$, and by the context-free pumping lemma, $a^{s+ir} \in L$, i.e. $s + ir$ is prime.

Choose a string $w = a^k$ where k is the next prime after the pumping length n plus 2. i.e $|w| = n + 2$.

Therefore, given that $|vxy| \leq n$, $|uz| \geq 2$. Therefore, $|uxz| \geq 2$ which is equivalent to $s \geq 2$.

Choose $i = s$, therefore $k = s + ir \implies k = s + sr \implies k = s(r + 1)$

Given that $s \geq 2$, k is not prime, and therefore $w \notin L$

3 Prove that Context-Free Languages are not closed over the Intersection Operation

Given two languages:

$$L_1 = \{a^k b^k c^j \mid k, j \geq 0\}$$
$$L_2 = \{a^j b^k c^k \mid k, j \geq 0\}$$

We construct grammars that accept the languages to prove they are context free.

We construct a grammar $G_1 = (V_1, \Sigma_1, R_1, S_1)$ that generates L_1 where:

$V_1 = \{S_1, E_1, F_1\}$
$\Sigma = \{a, b, c\}$
$R = \{$
$S_1 \rightarrow E_1 F_1$
$E_1 \rightarrow a E_1 b \mid \varepsilon$
$F_1 \rightarrow c F_1 \mid \varepsilon$
$\}$

Where S_1 is the start variable

We construct a grammar $G_2 = (V_2, \Sigma_2, R_2, S_2)$ that generates L_2 where:

$V_2 = \{S_2, E_2, F_2\}$
$\Sigma = \{a, b, c\}$
$R = \{$
$S_2 \rightarrow F_2 E_2$
$E_2 \rightarrow b E_2 c \mid \varepsilon$
$F_2 \rightarrow a F_2 \mid \varepsilon$
$\}$

Where S_2 is the start variable

The language $L = L_1 \cap L_2$, is:

$$L = \{a^k b^k c^k \mid k \geq 0\}$$

L can be proven to not be a context free language by the pumping lemma.

Assume L is context free, then the pumping lemma applies, which some pumping length n.

Consider $w = a^n b^n c^n$, the length $|w| = 3n$ which is greater than n, so we can apply the pumping lemma.

$$w = uvxyz$$

Since $|vxy| \leq n$, one of the following must be true:

vy does not contain any c symbols

or

vy does not contain any a symbols

Consider yv^2xy^2z. The number of a, b, c symbols cannot be equal.

The lemma says $yv^2xy^2z \in L$. This is a contradiction, so L must not be context-free.

Therefore the context free languages are not closed under the pumping lemma.