

Problem Sheet 6

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1 Operations Over Context-Free Languages

1.1 Proof that Context-Free Languages are closed over the Union Operation

If two languages (L_1, L_2) are context-free, there exist corresponding context-free grammars:

$$\begin{aligned}G_1 &= (V_1, \Sigma_1, R_1, S_1) \\G_2 &= (V_2, \Sigma_2, R_2, S_2)\end{aligned}$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates $L_1 \cup L_2$ where:

$$\begin{aligned}V &= V_1 \cup V_2 \cup \{S\} \\ \Sigma &= \Sigma_1 \cup \Sigma_2 \\ R &= R_1 \cup R_2 \cup \{S \rightarrow S_1, S \rightarrow S_2\}\end{aligned}$$

Where S is a new start variable

Clearly G generates $L_1 \cup L_2$.

$$\begin{aligned}S &\Rightarrow S_1 \Rightarrow \dots (\text{any string from } L(G_1)) \\ S &\Rightarrow S_2 \Rightarrow \dots (\text{any string from } L(G_2))\end{aligned}$$

This completes the proof for closure of Context-Free Languages under the union operation.

1.2 Proof that Context-Free Languages are closed over the Concatenation Operation

If two languages (L_1, L_2) are context-free, there exist corresponding context-free grammars:

$$\begin{aligned}G_1 &= (V_1, \Sigma_1, R_1, S_1) \\G_2 &= (V_2, \Sigma_2, R_2, S_2)\end{aligned}$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates $L_1 \circ L_2$ where:

$$\begin{aligned}V &= V_1 \cup V_2 \cup \{S\} \\ \Sigma &= \Sigma_1 \cup \Sigma_2 \\ R &= R_1 \cup R_2 \cup \{S \rightarrow S_1 S_2\}\end{aligned}$$

Where S is a new start variable

Clearly G generates $L_1 \circ L_2$.

$$S \Rightarrow S_1 S_2 \Rightarrow (\text{any string from } L(G_1)) \circ (\text{any string from } L(G_2))$$

This completes the proof for closure of Context-Free Languages under the concatenation operation.

1.3 Proof that Context-Free Languages are closed over the Kleene Star Operation

If a language L is context-free, there exists a corresponding context-free grammars:

$$G_1 = (V_1, \Sigma_1, R_1, S_1)$$

Now, we construct a grammar $G = (V, \Sigma, R, S)$ that generates L^* where:

$$V = V_1 \cup \{S\}$$

$$\Sigma = \Sigma_1$$

$$R = R_1 \cup \{S \rightarrow S_1 S, S \rightarrow \varepsilon\}$$

Where S is a new start variable

Clearly G generates L^* .

$$S \implies \varepsilon$$

$$S \implies S_1 S \implies (\text{any string from } L(G_1)) \circ (\text{any string from } L(G))$$

This completes the proof for closure of Context-Free Languages under the Kleene Star operation.