Robert Johns
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CSCI 678: Statistical Analysis of Simulation Models

Homework 10

1. For the MA(2) process:

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}$$

we have

$$\begin{aligned} E[X_t] &= E[\beta_0 Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}] \\ &= \beta_0 E[Z_t] + \beta_1 E[Z_{t-1}] + \beta_2 E[Z_{t-2}] \\ &= 0 \end{aligned}$$

because the expected value of each noise term is 0. Also, because the noise terms are iid, we have that

$$\begin{aligned} V[X_t] &= V[\beta_0 Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}] \\ &= \beta_0^2 V[Z_t] + \beta_1^2 V[Z_{t-1}] + \beta_2^2 V[Z_{t-2}] \\ &= (\beta_0^2 + \beta_1^2 + \beta_2^2)\sigma_Z^2 \end{aligned}$$

Since $E[X_t] = E[X_{t+1}] = 0$ we see that

$$\mathrm{Cov}(X_t, X_{t+1}) = E[(\beta_0 Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2})(\beta_0 Z_{t+1} + \beta_1 Z_t + \beta_2 Z_{t-1})]$$

Because $E[Z_s Z_t]$ is σ_Z^2 if $s = t$ and 0 otherwise, we see that

$$\gamma(1) = \beta_0\beta_1\sigma_Z^2 + \beta_1\beta_2\sigma_Z^2 = \beta_1(\beta_0 + \beta_2)\sigma_Z^2$$

Similarly, for $\gamma(2)$, we have

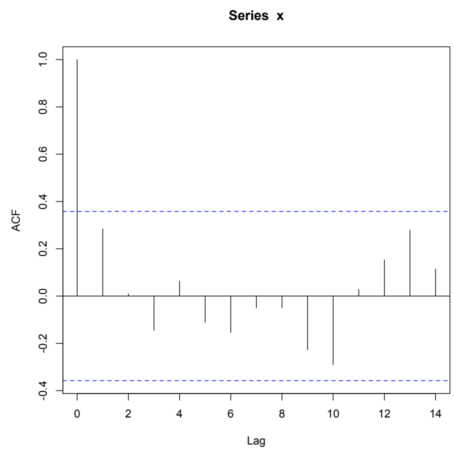
$$\mathrm{Cov}(X_t, X_{t+2}) = E[(\beta_0 Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2})(\beta_0 Z_{t+2} + \beta_1 Z_{t+1} + \beta_2 Z_t)] = \beta_0\beta_2\sigma_Z^2$$

A pattern emerges: we see see that $\gamma(k) = 0$ for $|k| > 2$, so the autocorrelation function for an MA(2) process is

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{\beta_1(\beta_0+\beta_2)}{\beta_0^2+\beta_1^2+\beta_2^2} & k = 1, -1 \\ \frac{\beta_0\beta_2}{\beta_0^2+\beta_1^2+\beta_2^2} & k = 2, -2 \\ 0 & |k| > 2 \end{cases}$$

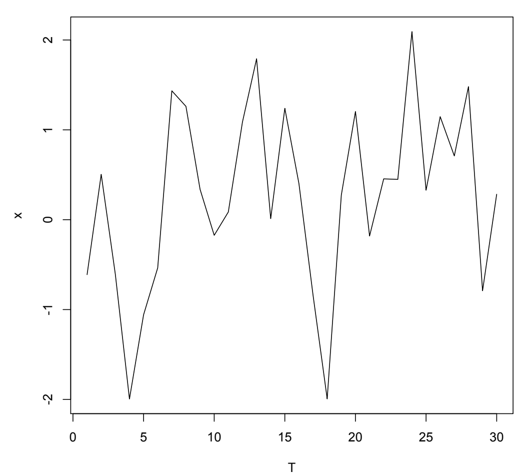
2. (a) We can alternately express this model as $X_t = \Theta(B)Z_t$, where $\Theta(B) = 1 + 2B + 10B^2$ and B is the backshift operator. From the notes, we know that the model is invertible if the roots of $\Theta(B)$ lie outside the complex unit circle. Applying the quadratic formula, we see that the roots of $\Theta(B)$ are $-1 + 6i$ and $-1 - 6i$, so the model in invertible.

(b)



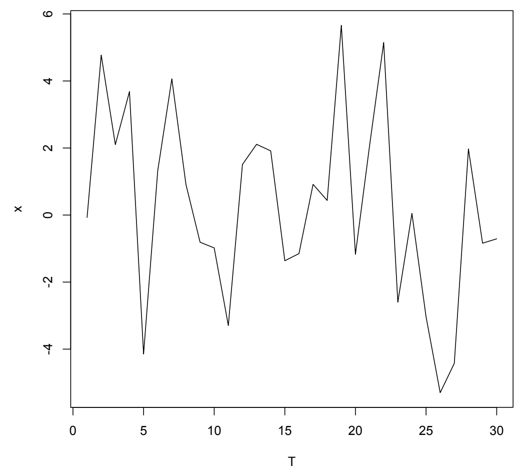
(c) The program asm10a.r was used to create the function. The code and plot of the results are below:

```
realma <- function(iq, parms, xmu, sig, n) {
realma <- function(iq, parms, xmu, sig, n) {
       x = c(1:n)
       z = rnorm(iq, xmu, sig)
       T = c(1:n)
       t = 0
       while (t < n) {
             t = t + 1
     z = c(rnorm(1,xmu,sig), z)
     for (i in 1:iq){
       x[t] = parms[i]*z[i]}
}
plot(T,x, type = "l")
return(x)
}
```

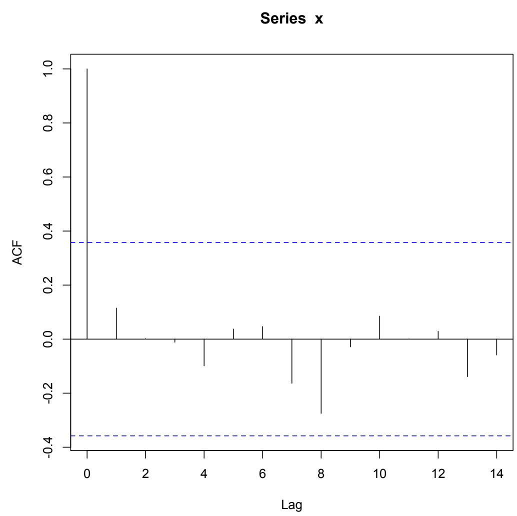


3. (a) Using a backshift operator, we can express this model as $X_t = \Theta(B)Z_t$, where $\Theta(B) = 3 - 2B$ and B is the backshift operator. From the notes, we know that the model will be invertible if $|-\frac{3}{2}| > 1$. Since $|-\frac{3}{2}| = \frac{3}{2} > 1$, we see that the model is invertible.

(b)



(c) The same program `asm10a.r` was used to produce the following plot:



4. Applying the result from question 1, we have that

$$\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{\beta_1(\beta_0 + \beta_2)}{\beta_0^2 + \beta_1^2 + \beta_2^2} = \frac{0.7(1 - 0.2)}{1 + 0.49 + 0.04} = 0.366013 \\
\rho(2) &= \frac{\beta_0\beta_2}{\beta_0^2 + \beta_1^2 + \beta_2^2} = \frac{-0.2}{1 + 0.49 + 0.04} = -0.13071895
\end{aligned}$$

As the model is MA(2), the $\rho(k) = 0$ for all $k > 2$.

5. The model under consideration is $X_t = \sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}$. Because the expected value of the noise terms is 0, we see that $E[X_t] = 0, V[X_t] = \sigma_Z^2$. So then:

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+k}) &= E\left[\left(\frac{Z_t}{m+1} + \frac{Z_{t-1}}{m+1} + \ldots + \frac{Z_{t-m}}{m+1}\right)\left(\frac{Z_{t+k}}{m+1} + \frac{Z_{t+k-1}}{m+1} + \ldots + \frac{Z_{t+k-m}}{m+1}\right)\right] \\
&= \frac{1}{m+1} E[(Z + t + Z_{t-1} + \ldots + Z_{t-m})(Z_{t+k} + Z_{t+k-1} + \ldots + Z_{t+k-m})]
\end{aligned}$$

Because $E[Z_x Z_y] = 0$, all but $m+1-k$ terms will be 0, so $\mathrm{Cov}(X_t, X_{t+k}) = \frac{m+1-k}{m+1}\sigma_Z^2$. So our acf is:

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{m+1-k}{m+1} & k = 1, \ldots, m \\ 0 & k > m \end{cases}$$

6. We see that the variance of the infinite order MA process $X_t = Z_t + C(Z_{t-1} + Z_{t-1} + \ldots)$ is:

$$\begin{aligned}
V[X_t] &= V[Z_t + C(Z_{t-1} + Z_{t-1} + \ldots)] \\
&= V[Z_t] + C^2(V[Z_{t-1}] + V[Z_{t-1}] + \ldots) \\
&= \sigma_Z^2 + C^2 \sum_{i=1}^{\infty} \sigma_Z^2
\end{aligned}$$

So, as the variance is infinite, the process is non-stationary.

For the series of differences $Y_t = X_t - X_{t-1} = (Z_t + c\sum_{i=1}^{\infty} Z_{t-i}) - (Z_{t-1} + c\sum_{i=2}^{\infty} Z_{t-i}) = Z_t + (C-1)Z_{t-1}$, we have that $E[Y_t] = 0$ and $V[Y_t] = (C^2 - 2C + 2)\sigma_Z^2$, which are both constant, so the series is stationary. The autocorrelation function for Y_t is:

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{C-1}{C^2-2C+2} & k = 1 \\ 0 & k > 1 \end{cases}$$

7. Our model is $X_t = \mu + Z_t + \beta Z_{t-1}$. We see that $E[X_t] = \mu$ and $V[X_t] = (1+\beta^2)\sigma_Z^2$. We have that

$$\mathrm{Cov}(X_t, X_{t+1}) = E[(X_t - \mu) * (X_{t+1} - \mu)] - E[X_t - \mu] * E[X_{t+1} - \mu]$$

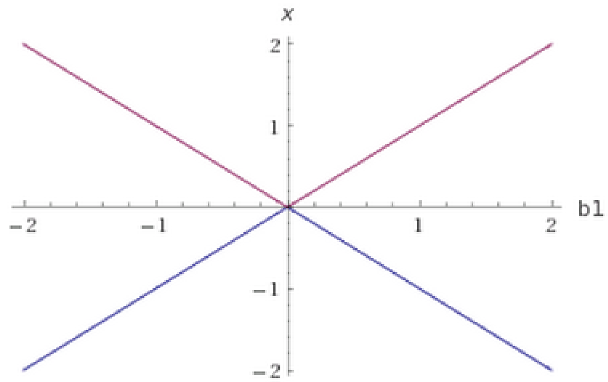
Since the individual expectations are each 0, the covariance is the expectation of the product. Also

$$\begin{aligned}
E[(X_t - \mu)(X_{t+1} - \mu)] &= E[(\mu + Z_t + \beta Z_{t-1} - \mu)(\mu + Z_{t+1} + \beta Z_t - \mu)] \\
&= E[(Z_t + \beta Z_{t-1})(Z_{t+1} + \beta Z_t)]
\end{aligned}$$

which does not depend on μ. The autocorrelation function is

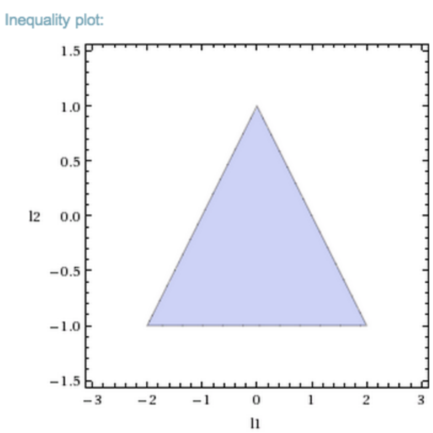
$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{\beta}{1+\beta^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

8. (a) In the following picture, $b1 = \beta_0, x = \beta_1$:



(b) On the edges of the invertibility region, $|\beta_0| = |\beta_1| = \beta$, so $\rho(1) = \frac{\beta_0\beta_1}{\beta_0^2+\beta_1^2} = \pm\frac{\beta^2}{2\beta^2} = \pm\frac{1}{2}$. So we have that, $-0.5 < \rho(1) < 0.5$.

9. In the following picture (obtained from WolframAlpha), $l1 = \lambda_1, l2 = \lambda_2$



The level curves are $\rho(1) = \frac{\lambda_1}{1-\lambda_2} = .5$ and $\rho(2) = \frac{\lambda_1^2}{1-\lambda_2} + \lambda_2$.

10. We can rewrite the process as $(1 - B - cB^2)X_t = Z_t$, where B is the backshift operator. If the roots of $1 - B - cB^2 = 0$ lie outside the unit circle, the process is stationary. From the quadratic formula, the roots are $\frac{1+\sqrt{1+4c}}{2}$ and $\frac{1-\sqrt{1+4c}}{2}$. In polar form, radius is

$$r = \sqrt{\left(\frac{-1+\sqrt{1+4c}}{2c}\right)\left(\frac{-1-\sqrt{1+4c}}{2c}\right)} = \sqrt{\frac{1-1-4c}{4c^2}} = \sqrt{\frac{-1}{c}}$$

So then to be invertible, we must have that $r > 1$, so $\sqrt{\frac{-1}{c}} > 1$, which is true if $-1 < c < 0$.

Since $B = 1$ is a root of the equation and lies on the unit circle, we see that the process is not stationary.

11. Two plots, one of the time series and one of the autocorrelation function, follow.