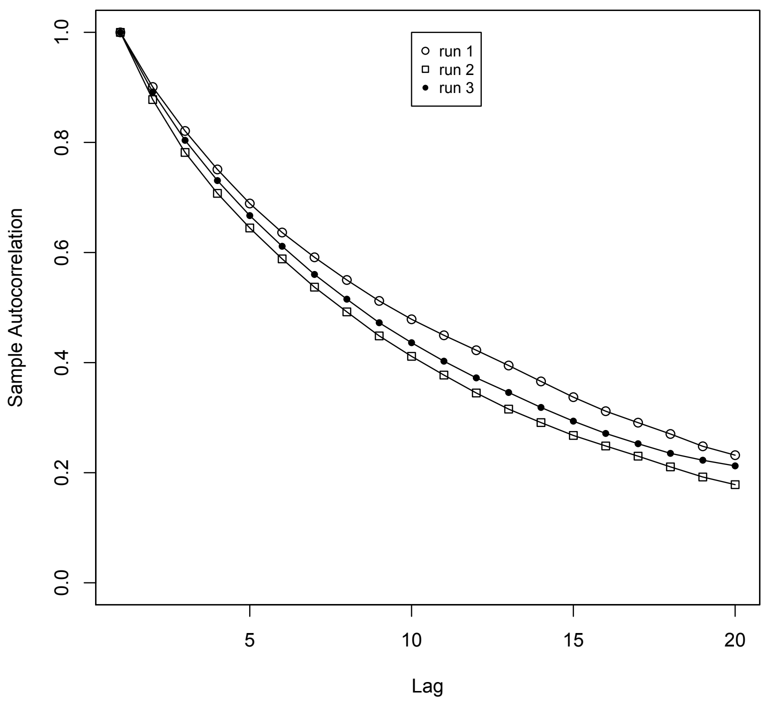
Robert Johns
February 6, 2014

CSCI 678: Statistical Analysis of Simulation Models

Homework 3

1. Modify the M/G/1 queueing simulation program from Assignment 1 to print out the sample autocorrelations for the wait times for lags 1 to 20 using the formulas given on page 2.29. Make 3 replications using different sets of random variables and plot three correlograms on one set of axes.

   Solution: See attached code `ams3a.c`



2. For the exponential distribution: $f(x) = \frac{1}{\alpha}e^{-x/\alpha}, \;\; x \geq 0$

   (a) Find the mgf.
   Solution: $M_X(t) = E[e^{tx}] = \int_0^\infty \frac{1}{\alpha}e^{x(t-1/\alpha)}\, dx = (1-\alpha t)^{-1}$

   (b) Generate the first four raw moments. using the result from part (a)
   Solution: From the formula $E[X^r] = \frac{d^r m_X(t)}{dx^r}\Big|_{t=0}$, we get

   $E(X) = m'_X(0) = \frac{\beta}{(\beta * t-1)^2}\Big|_{t=0} = \beta$

   $E(X^2) = m''_X(0) = \frac{-2\beta}{(\beta * t-1)^3}\Big|_{t=0} = -2\beta$

   $E(X^3) = m'''_X(0) = \frac{6\beta^3}{(\beta * t-1)^4}\Big|_{t=0} = 6\beta^3$

   $E(X^4) = m^{(4)}_X(0) = \frac{24\beta^4}{(\beta * t-1)^5}\Big|_{t=0} = 24\beta^4$

(c) What is the mode of the exponential distribution?

Solution: It will be the maximum value of the pdf of the exponential distribution. Examining the distribution graphically, we see that this happens at $x = 0$.

(d) What is the median of the exponential distribution?

Solution: First, we need the cdf. To do this, we evaluate $\int_0^m f(w)\,dw = 1 - e^{-m/\beta}$. Then we set the equation equal to 1/2 and solve. We get $1/2 = 1 - e^{-m/\beta}$, or $e^{-m/\beta} = 1/2$, or $-m/\beta = \ln(1/2)$, or $m = -\beta \ln(1/2)$.

(e) Give an expression for determining a fractile of an exponential R.V.

Solution: We see, following our process from part (d), that the only effect that the specific fractile had on our answer was the term inside the natural logarithm. An appropriate expression for the pth fractile, therefore, would be $x_p = -\beta \ln(p)$.

3. State the following complex numbers in each of the three standard forms:

(a) $3 + 4i$

Solution: $a + bi : 3 + 4i$

Polar: $5e^{.6435i}$

Sines, Cosines: $5(\cos(.6435) + i \sin(.6435))$

(b) $10e^{(\pi i)/4}$

Solution: $a + bi : 5\sqrt{2} + 5\sqrt{2}i$

Polar: $10e^{(\pi i)/4}$

Sines, Cosines: $10(\cos(\pi/4) + i \sin(\pi/4))$

(c) $4(\cos(2\pi/3) + i \sin(2\pi/3))$

Solution: $a + bi : -2 + 2\sqrt{3}i$

Polar: $4e^{(2\pi i)/3}$

Sines, Cosines: $4(\cos(2\pi/3) + i \sin(2\pi/3))$

4. Consider independent random variables Y_1, Y_2 and Y_3. Using only expectations, variances and covariances, derive (and simplify) expressions for:

(a) $\mathrm{Cov}(Y_1 + Y_3, Y_2 + Y_3)$

Solution: We have the shortcut formula for covariance: $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$, which gives us

$$\mathrm{Cov}(Y_1 + Y_3, Y_2 + Y_3) = E[(Y_1 + Y_3)(Y_2 + Y_3)] - E(Y_1 + Y_3)E(Y_2 + Y_3)$$

$$= E(Y_1Y_2 + Y_1Y_3 + Y_2Y_3 + Y_3^2) - [E(Y_1) + E(Y_3)][E(Y_2) + E(Y_3)]$$

$$= E(Y_1Y_2) + E(Y_1Y_3) + E(Y_2Y_3) + E(Y_3^2) - [E(Y_1)E(Y_2) + E(Y_1)E(Y_3) + E(Y_2)E(Y_3) + E(Y_3)^2]$$

$$= [E(Y_1Y_2) - E(Y_1)E(Y_2)] + [E(Y_1Y_3) - E(Y_1)E(Y_3)] + [E(Y_2Y_3) - E(Y_2)E(Y_3)] + [E(Y_3^2) - E(Y_3)^2]$$

$$= \mathrm{Cov}(Y_1, Y_2) + \mathrm{Cov}(Y_1, Y_3) + \mathrm{Cov}(Y_2, Y_3) + \mathrm{Var}(Y_3)$$

$$= \mathrm{Var}(Y_3)$$

(b) ρ for $(Y_1 + Y_3, Y_2 + Y_3)$

Solution: The formula for ρ of two random variables X and Y is $\frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$. So we have

$$\begin{aligned}
\rho &= \text{Var}(Y_3)/(\sqrt{\text{Var}(Y_1 + Y_3)\text{Var}(Y_2 + Y_3)}) \\
&= \text{Var}(Y_3)/\sqrt{[\text{Var}(Y_1) + \text{Var}(Y_3)][\text{Var}(Y_2) + \text{Var}(Y_3)]} \\
&= \text{Var}(Y_3)/\sqrt{\text{Var}(Y_1)\text{Var}(Y_2) + \text{Var}(Y_1)\text{Var}(Y_3) + \text{Var}(Y_2)\text{Var}(Y_3) + \text{Var}(Y_3)^2}
\end{aligned}$$

5. Find the characteristic function for the $U(0,1)$ distribution. Show that $\phi_x(0) = 1$.

   Solution: We have $\phi_X(t) = E[e^{itx}] = \int_0^1 e^{itx}\,dx$. Performing a u-substitution, where $u = itx$ and $du = it\,dx$, we have $\int_0^1 e^{itx}\,dx = \frac{-i}{t}\int_0^1 e^u\,du = \frac{-i}{t}e^{itx}\Big|_0^1 = \frac{-i}{t}(e^{it} - 1) = \frac{i}{t}(1 - e^{it}) = \frac{i}{t}(1 - \cos(it) - i\sin(it)) = \frac{i - i\cos(it) + \sin(it)}{t} = \phi_X(t)$

   We see that $\phi_X(0) = E[e^{i*0*x}] = E[e^0] = E[1] = 1$

6. Suppose that X and Y are jointly discrete random variables with

$$f(x,y) = \begin{cases} \frac{2}{n(n+1)} & x = 1, 2, \ldots n; y = 1, 2\ldots, x \\ 0 & \text{otherwise} \end{cases}$$

   Find $f_X(x), f_Y(y)$ and determine whether X and Y are independent.

   Solution: Firstly, a neat party trick tells us that X and Y cannot be independent, because they are not defined on a product space; the space they are defined on is the discrete representation of a triangle. Now, for the marginals. A cursory example of the geometry gives us the functions:

$$f_X(x) = \begin{cases} \frac{2x}{n(n+1)} & x \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2(n-y+1)}{n(n+1)} & y \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

7. For a single run of the M/G/1 queue simulation from question 1, determine the sample minimum, maximum and the quartiles of the customer wait times by writing the wait times to a file and reading those vales into R. Submit code in C and R, and the values for the minimum, maximum and quartiles.

   Solution: See attached code `asm3b.c`. The short program `asm3c.r` is written below. The results are also printed blow.

Results	*Program*
`"min:" 1.000149`	`data <- scan("data.txt")`
`"max:" 24.56443`	`print("min:")`
`"quartiles" 25% 50% 75%`	`print(min(data))`
`1.831689 3.040863 5.097327`	`print("max:")`
	`print(max(data))`
	`print("quartiles")`
	`print(quantile(data, c(.25, .5, .75 )))`