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CSCI 426: Simulation

Homework 6

7.1.3 If X is a continuous random variable with pdf $f(\cdot)$, mean μ and variance σ^2 prove that

$$\int_x x^2 f(x)\, dx = \mu^2 + \sigma^2$$

where the integration is over all possible values of X.

Proof: We know that the formula for the variance is $\sigma^2 = E[(x-\mu)^2]$ which, when reduced, gives us $E[x^2 - 2\mu x + \mu^2] = E[x^2] - 2\mu^2 + \mu^2 = E[x^2] - \mu^2$. Since our given integral is $E[x^2]$, we have from the previous formula that $E[x^2] = E[(x-\mu)^2] - (\mu^2 - 2\mu^2) = \sigma^2 + \mu^2$

7.2.4 A continuous random variable X is $Weibull(a, b)$ if the real-valued parameters a and b are positive, the possible values of X are $x > 0$, and the cdf is:

$$F(x) = 1 - \exp(-(bx)^a)$$

What are the pdf and idf?

Solution: To find the pdf $f(x)$, we simply take the derivative, which gives us:

$$f(x) = \frac{a(bx)^a \cdot \exp(-(bx)^a)}{x}$$

To find the idf, we take the inverse function of $F(x)$, replacing the term $F(x)$ with u, where $u \in (0, 1)$, which gives us:

$$u = 1 - \exp(-(bx)^a)$$
$$1 - u = \exp(-b^a x^a)$$
$$\ln(1-u) = -b^a x^a$$
$$\frac{-\ln(1-u)}{b^a} = x^a$$
$$\sqrt[a]{\frac{-\ln(1-u)}{b^a}} = x$$
$$\frac{\sqrt[a]{-\ln(1-u)}}{b} = x = F^{-1}(u)$$

We know this will be real-valued for all $u \in (0, 1)$ because $\ln(x)$ is negative for all $x \in (0, 1)$.

7.3.5 (a) Derive the equations for the mean and standard deviation of a $Triangular(a, b, c)$ random variable.

Solution: To get the mean, we evaluate $\int_a^c xf(x)\ dx$, which, since our function is defined piecewise, gives us $\mu = \int_a^b xf(x)\ dx + \int_b^c xf(x)\ dx$, which is

$$\mu = \int_a^b x\frac{2(x-a)}{(b-a)(c-a)}\ dx + \int_b^c x\frac{2(b-x)}{(b-a)(b-c)}\ dx$$

Our final value will be:

$$\mu = \left(\frac{2}{(b-a)(c-a)}\right)\left(\frac{b^3}{3} - \frac{a^3}{3} - \frac{b^2a}{2} + \frac{a^3}{2}\right) + \left(\frac{2}{(b-a)(b-c)}\right)\left(\frac{c^2b}{2} - \frac{b^3}{2} - \frac{c^3}{3} + \frac{b^3}{3}\right)$$

which reduces to

$$\mu = \frac{a+b+c}{3}$$

To get the standard deviation, we do it all over again, except with $\int_a^c x^2 f(x)\ dx$, and take the square root to obtain a final value of:

$$\sigma = \sqrt{\left(\frac{2}{(b-a)(c-a)}\right)\left(\frac{b^4}{4} - \frac{a^4}{4} - \frac{b^3a}{3} + \frac{a^4}{3}\right) + \left(\frac{2}{(b-a)(b-c)}\right)\left(\frac{c^3b}{3} - \frac{b^4}{3} - \frac{c^4}{4} + \frac{b^4}{4}\right)}$$

which reduces to

$$\sigma = \frac{\sqrt{(a-b)^2 + (a-c)^2 + (b-c)^2}}{6}$$

(b) Similarly, derive the results for the cdf and idf.

Solution: To get the cdf, we integrate $f(x)$ piecewise from a to x, then take the integral from x to b subtracted from 1 to obtain our cdf $F(x)$. We will get, for values x such that $a < x \le c$,

$$\frac{2}{(b-a)(c-a)}\int_a^x x - a\ dx = \frac{2}{(b-a)(c-a)}\frac{x^2 - 2ax - a^2 + 2a^2}{2} = \frac{(x-a)^2}{(b-a)(c-a)}$$

And for values from c to b, we will get

$$1 - \frac{2}{(b-a)(b-c)}\int_x^b b - x\ dx = 1 - \frac{2}{(b-a)(b-c)}\frac{2b^2 - b^2 - 2bx + x^2}{2} = 1 - \frac{(b-x)^2}{(b-a)(b-c)}$$

Our final piecewise function will therefore be:

$$F(x) = \begin{cases} 0 & x \le a \\ \frac{(x-a)^2}{(b-a)(c-a)} & a < x \le c \\ 1 - \frac{(b-x)^2}{(b-a)(b-c)} & c < x < b \\ 1 & x \ge b \end{cases}$$

To get the idf, we take the inverse of $F(x)$, replacing the term $F(x)$ with u, where $u \in (0,1)$. Our derivation for values of x between a and c will be:

$$u = \frac{(x-a)^2}{(b-a)(c-a)}$$

$$u(b-a)(c-a) = (x-a)^2$$
$$\sqrt{u(b-a)(c-a) = x-a}$$
$$x = \sqrt{u(b-a)(c-a)} + a$$

This function will be defined for values of x between 0 and $\frac{(c-a^2)}{(b-a)(c-a)} = \frac{c-a}{b-a}$
To get the second piece of the idf, we take the inverse of the function for values of x between c and b. We get:

$$u = 1 - \frac{(b-x)^2}{(b-a)(b-c)}$$

$$(1-u)(b-a)(b-c) = (b-x)^2$$
$$\sqrt{(1-u)(b-a)(b-c) = b-x}$$
$$x = b - \sqrt{(1-u)(b-a)(b-c)}$$

This function will be defined from values of x between $\frac{c-a}{b-a}$ and 1, giving us an idf of:

$$F^{-1}(u) = \begin{cases} \sqrt{u(b-a)(c-a)} + a & 0 < u \leq \frac{c-a}{b-a} \\ b - \sqrt{(1-u)(b-a)(b-c)} & \frac{c-a}{b-a} < u < 1 \end{cases}$$