Robert Johns
January 30, 2014

CSCI 678: Statistical Analysis of Simulation Models

Homework 2

1. Let X and Y be random variables with finite means and variances. Six results concerning expected values are given below. Pick three of the results and show that they are true for *continuous* random variables. Assume that k is constant.

   (a) $E[kX] = k * E[X]$
   Solution: We have $E[kX] = \int_{-\infty}^{\infty} kxf(x)\ dx$. Since k is a constant, we can pull it out of the integral to get $E[kX] = k\int_{-\infty}^{\infty} xf(x)\ dx = kE[X]$

   (b) $E[k] = k$
   Solution: We have $E[k] = \int_{-\infty}^{\infty} kf(x)\ dx$. Since k is a constant, we can pull it out of the integral to get $E[k] = k\int_{-\infty}^{\infty} f(x)\ dx = k$ because f is a probability density function and integrates to 1 over the reals.

   (d) $V[X] = E[X^2] - \mu^2$
   Solution: We have $V[X] = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$

2. Show that the following relationships are true using pages 2.9-2.11 in the class notes:

   (a) The Rayleigh is a special case of the Weibull distribution.
   Solution: We have the pdf for the Weibull distribution: $f(x) = (\beta/\alpha)x^{\beta-1}\exp[-(1/\alpha)x^{\beta}]$. If we set $\beta = 2$, we now have $f(x) = (2x/\alpha)\exp[-(x^2/\alpha)]$, which is the pdf for the Rayleigh distribution.

   (b) The square root of an exponential random variable has a Rayleigh distribution.
   Solution: Let the random variable X be distributed exponentially. We know that the pdf of X is $f_X(x) = (1/\alpha)\exp[-(x^2/\alpha)]$. If we apply the transformation $Y = g(X) = \sqrt{X}$ (which is bijective across non-negative reals), we see that the inverse $X = g^{-1}(Y) = Y^2$ has Jacobian $\frac{dX}{dY} = 2Y$. Applying the transformation technique, we see that the pdf of Y is $f_Y(y) = f_X(g^{-1}(y))|\frac{dX}{dY}| = (1/\alpha)\exp[-(y^2/\alpha)]|2y| = (2y/\alpha)\exp[-(y^2/\alpha)]$, the pdf of the Rayleigh distribution.

   (c) The sum of independent and identically distributed exponential random variables is Erlang.
   Solution: Let $X_1, X_2, \ldots X_k$ be iid exponential random variables. Define the random variable Y as $Y = \sum_{i=1}^{k} X_i$. We have from the mgf technique that $m_Y(t) = \Pi_{i=1}^{k} m_{X_i}(t)$. We know that the mgf for an exponential random variable is $m_X(t) = (1-\alpha t)^{-1}$, so we then have that $m_Y(t) = (1-\alpha t)^{-k}$, which is the mgf for an Erlang random variable.

3. For the joint probability density function defined by

$$f(x_1, x_2) = 2 \qquad 0 < x_1 < x_2 < 1$$

find the covariance between X_1 and X_2.

Solution: For ease of writing, let's refer to X_1 as X and X_2 as Y. The shortcut formula for the covariance is $\text{Cov}(X,Y) = E[XY] - \mu_X \mu_Y$. We can find $E[XY]$ by evaluating the following integral: $\int_0^1 \int_0^y 2xy\ dxdy = 1/4$. To find the expected values of X and Y, we need their marginal distributions, which are $f_X(x) = 2 - 2x.f_Y(y) = 2y$, giving us expected values of $E[X] = 1/3, E[Y] = 2/3$, so we have $\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{3} * \frac{2}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables and $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Find the exact values of $E[X_i], \text{Var}[X_i], E[\overline{X}]$ and $\text{Var}[\overline{X}]$ by analytic methods for the three parent populations given in (a), (b) and (c) below. Also, write a computer program that estimates these four quantities for $n = 5, 50, 500, 5000$ using seven replications of each experiment when:

   (a) $X_1, X_2, \ldots X_n$ are independent $U(0,1)$.
   Solution: $E[X_i] = \int_0^1 x\ dx = .5$
   $V[X_i] = E[X_i^2] - E[X_i]^2 = \int_0^1 x^2\ dx - 1/4 = 1/12$
   $E[(1/n)\sum_{i=1}^n X_i] = (1/n)\sum_{i=1}^n E[X_i] = n/(2n) = .5$
   $\text{Var}[(1/n)\sum_{i=1}^n X_i] = (1/n^2)\sum_{i=1}^n \text{Var}[X_i] = 1/(12n)$
   The program `asm4a.r`, attached at end, is used to calculate the desired values.

   (b) $X_1, X_2, \ldots X_n$ are independent variates from a distribution with probability density function $f(x) = 2/x^3$ for $x \geq 1$.
   Solution: $E[X_i] = \int_1^\infty x(2/x^2)\ dx = 2$
   $V[X_i] = E[X_i^2] - E[X_i]^2 = \int_1^\infty x^2(2/x^3)\ dx - 4$ does not converge.
   $E[(1/n)\sum_{i=1}^n X_i] = (1/n)\sum_{i=1}^n E[X_i] = (2n)/n = 2$
   $\text{Var}[\frac{1}{n}\sum_{i=1}^n X_i] = \frac{1}{n^2}\sum_{i=1}^n \text{Var}[X_i]$ does not converge because no variances converge.
   The program `asm4b.r`, attached at end, is used to calculate the desired values.

   (c) $X_1, X_2, \ldots X_n$ are independent Cauchy variates.
   Solution:$E[X_i] = \int_{-\infty}^\infty \frac{x}{\alpha\pi[1+((x-\alpha)/\alpha)^2]}\ dx = \alpha$
   $V[X_i] = E[X_i^2] - E[X_i]^2 = \int_{-\infty}^\infty \frac{x^2}{\alpha\pi[1+((x-\alpha)/\alpha)^2]}\ dx - \alpha^2$ does not converge.
   $E[(1/n)\sum_{i=1}^n X_i] = (1/n)\sum_{i=1}^n E[X_i] = (n\alpha)/n = \alpha$
   $\text{Var}[\frac{1}{n}\sum_{i=1}^n X_i] = \frac{1}{n^2}\sum_{i=1}^n \text{Var}[X_i]$ does not converge because no variances converge.
   The program `asm4c.r`, attached at end, is used to calculate the desired values.

5. For any random variables X_1, X_2 and any numbers a_1, a_2, show that $\text{Var}(a_1X_1 + a_2X_2) = a_1^2\text{Var}(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2\text{Var}(X_2)$

Solution: For ease of writing, let us refer to a_1 as a, a_2 as b, X_1 as X and X_2 as Y. We have

$$\text{Var}(aX + bY) = E[((aX + bY) - (a\mu_X + b\mu_Y))^2]$$
$$= E[(a(X - \mu_X) - b(Y - \mu_Y))^2]$$
$$= E[a^2(X - \mu_X)^2 + 2ab(X - \mu_X)(Y - \mu_Y) + b^2(Y - \mu_Y)^2]$$
$$= E[a^2(X - \mu_X)^2] + 2abE[(X - \mu_X)(Y - \mu_Y)] + b^2E[(Y - \mu_Y)^2]$$
$$= a^2\text{Var}[X] + 2ab\text{Cov}[X,Y] + b^2\text{Var}[Y]$$

6. Use the R calculator mode to find the following quantities:

(a) $4 * \arctan(1)$
Solution:

```
> 4 * atan(1.0)
[1] 3.141593
```

(b) $1 + \lfloor e^3 \rfloor$
Solution:

```
> 1 + floor(exp(3))
[1] 21
```

(c) $\frac{1}{\sqrt{2\pi}}$
Solution:

```
> 1 / sqrt(2 * pi)
[1] 0.3989423
```

(d) If Z is a standard normal random variable, find the value a such that $P(Z < a) = 0.975$
Solution:

```
> pnorm(1.959964, 0, 1)
[1] 0.975
```

(e) If X is a random variable having the chi-square distribution with fifteen degrees of freedom, find $P(X < 17.48)$
Solution:

```
> pchisq(17.48, 15)
[1] 0.7090115
```

(f) Generate 8 random variates from the t distribution with seven degrees of freedom.
Solution:

```
> rchisq(8, 15)
[1] 24.668024 19.978752 15.043670 16.100242 18.429329 16.696524  9.354511
[8] 12.682383
```

(g) Find the value of the standard normal probability density function at $x = 0$.
Solution:

```
> dnorm(0)
[1] 0.3989423
```