Quadratic Convexity and Sums of Squares

Martin Ames Harrison

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Overview

- **Introduction**
  - Motivation
  - Terms and problem statement
  - Tools: matrices and convexity

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- **Main Results**
  - Reformulation and related problems
  - Quadratic maps in general
  - Necessary condition

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  - Quadratic maps in general
  - Necessary condition
- **Back to SOS**
  - Application
  - Conjecture

Introduction - Why SOS?

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$$\inf\{p(x) \mid x \in C\} = \sup\{y \mid (p-y)|_C > 0\}.$$

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SOS approximations are easy to find.

Introduction - Notation

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- $\Sigma_{n,d}$ denotes the SOS *cone*, sums of squares in $\mathbb{R}[x_1, \ldots, x_n]_{2d}$, i.e.

$$g_1^2 + \ldots + g_k^2$$

  for some $g_1, \ldots, g_k \in \mathbb{R}[x_1, \ldots, x_n]_d$ (necessarily).

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- Degree of x^α is $|\alpha| \equiv \sum_j \alpha_j$.
- For $k \in \mathbb{N}$, $\Sigma_{n,d}(k)$ is the set of sums of k (or fewer) squares.

Introduction - Example

The polynomial $p(x, y) = x^4 + x^2y^2 + y^4$ is a sum of 3 squares:

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Will *even fewer* suffice? No! The length of $p(x,y)$ is 2.

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How does one compute $\mathcal{P}(\Sigma_{n,d})$?

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- The convex hull $\operatorname{conv}(S)$ of a set S is the smallest convex set containing S; all convex combinations of elements of S.
- The conic hull $\operatorname{co}(S)$ is the set of all conic combinations of elements of S:

$$\operatorname{co}(S) = \{t_1 x_1 + \ldots + t_k x_k \mid x_i \in S, t_i \geq 0\}.$$

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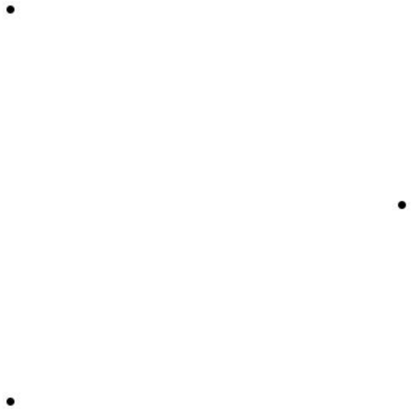

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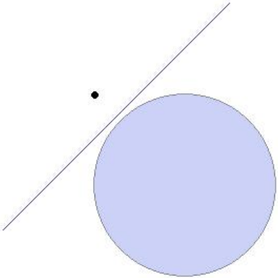

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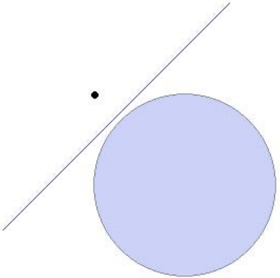

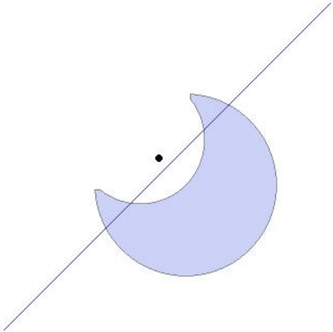

A special property

Let $S \subset \mathbb{R}^m$ be a convex set. If $\mathring{S} = \emptyset$, then S is contained in a proper affine subspace of $\mathbb{R}^m$.

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For $A \in \mathcal{S}_N$, the following are equivalent:

i) A is positive semidefinite.

ii) $A = B^2$ for some $B \in \mathcal{S}_N$.

iii) $x^T A x \geq 0$ for all $x \in \mathbb{R}^N$.

iv) The principal minors of A are nonnegative.

v) $A = LL^T$ for some $L \in M_N(\mathbb{R})$.

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Finally, $K_N^+ \equiv \mathcal{S}_N^+ \cap \{x \in M_N(R) \mid \text{trace}(x) = 1\}$ (example of a spectrahedron).

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- $p = \mathrm{trace}(\mathbf{m}\mathbf{m}^T(\sum_i \mathbf{g_i}\mathbf{g_i}^T))$.
- System of $\binom{n+2d-1}{2d}$ equations $p_\alpha = \mathrm{trace}(A_\alpha \sum_i \mathbf{g_i}\mathbf{g_i})$, where

$$(A_\alpha)_{\beta,\gamma} = \begin{cases} 1, & \text{if } \beta + \gamma = \alpha \\ 0, & \text{otherwise} \end{cases}$$

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- Conversely, $X \in \mathcal{S}_N^+$ satisfying $p_\alpha = \mathrm{trace}(A_\alpha X)$ yields a sum of squares. Write $U^T D U = X$.

$$p = \mathrm{trace}(\mathbf{m}\mathbf{m}^T X) = \mathrm{trace}(\mathbf{m}^T U^T D U \mathbf{m}) = \sum_i \lambda_i (U_i \mathbf{m})^2$$

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- $\ell(p) = \min\{\text{rank}(X) \mid X \in S(p)\}$, where $S(p)$ is the set of all Gram matrices for p.
- $\mathcal{P}(\Sigma_{n,d}) \leq \binom{n+d-1}{d}$

Quadratic Maps

$A = (A_1, \ldots, A_M)$, $M-$tuple of elements of $\mathcal{S}_N$

$A(x) \equiv (x^T A_1 x, \ldots, x^T A_M x)^T \in \mathbb{R}^M$, A is a quadratic map.

- When is $A(\mathbb{R}^N) \equiv \{(x^T A_1 x, \ldots, x^T A_M x) \mid x \in \mathbb{R}^N\}$ convex?

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- Who cares, and what does this have to do with pythagoras numbers?

Quadratic Convexity - motivation

- Relaxation of systems of quadratic equations

$$x^T A_i x = a_i$$

$a = (a_1, \ldots, a_M)$ outside the convex hull of $A(\mathbb{R}^N)$ can be separated by a hyperplane...

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- Joint numerical range of a tuple of hermitian operators - applications in *quantum* physics

$$w(A) = \{(z^* A_1 z, \ldots, z^* A_M z) \mid z \in \mathbb{C}^N \text{ and } \|z\| = 1\}$$

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- The set $\Sigma_{n,d}(k)$ is convex exactly when

$$\Sigma_{n,d} = \Sigma_{n,d}(k),$$

i.e., exactly when $\mathcal{P}(\Sigma_{n,d}) \leq k$.

Quadratic Convexity - first observations

Suppose that $A : \mathbb{R}^N \to \mathbb{R}^M$ *is a quadratic map. Let* $\mathbb{B}$ *denote the unit ball* $\{x \in \mathbb{R}^N \mid \|x\| \leq 1\}$ *in* $\mathbb{R}^N$*.*

Then $A(\mathbb{R}^N)$ *is convex whenever* $A(\mathbb{B})$ *is convex, and* $A(\mathbb{B})$ *is convex whenever* $A(S^{N-1})$ *is convex.*

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The converses are false, but we can reduce the question to one about a *compact* set.

Suppose $A : \mathbb{R}^N \to \mathbb{R}^M$ is a quadratic map and $A(\mathbb{R}^N)$ spans $\mathbb{R}^M$. TFAE

i) $0 \notin A(K_N^+) \equiv \operatorname{conv} A(S^{N-1})$

ii) There is a linear functional ℓ on $\mathbb{R}^M$ such that $\ell \cdot A \equiv \sum_j \ell_j A_j$ is positive definite.

iii) There is an invertible linear operator T on $\mathbb{R}^M$ such that $T \circ A$ is positive definite in each coordinate.

Positive maps - Corollary

- *Suppose that* $A : \mathbb{R}^N \to \mathbb{R}^M$ *is a quadratic map and* $\operatorname{conv} A(S^{N-1})$ *does not contain* 0.

  Then there is an invertible linear operator L *on* $\mathbb{R}^N$ *such that* $A(\mathbb{R}^N)$ *is convex exactly when* $A \circ L(S^{N-1})$ *is convex.*

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- *If* $A(\mathbb{R}^N)$ *is closed and contains no lines, then there is a linear transformation* T *from* $\mathbb{R}^k$ *to* $\mathbb{R}^N$, *for some* $k \leq N$, *such that* $A(\mathbb{R}^N)$ *is convex exactly when* $A \circ T(S^{k-1})$ *is convex.*

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- Recall a property of convex sets: nonempty interior *or* lower dimension
- Sard's Theorem: Critical values have measure zero.
- A quadratic map is polynomial, and so is the derivative.

If $A : \mathbb{R}^N \to \mathbb{R}^M$ *is a quadratic map, then...*

there is a linear operator $T : \mathbb{R}^M \to \mathbb{R}^k$ *satisfying*

i) $(T \circ A)(\mathbb{R}^N)$ *is contained in no proper subspace of* $\mathbb{R}^k$*, and*

ii) $(T \circ A)(\mathbb{R}^N)$ *is convex if and only if* $A(\mathbb{R}^N)$ *is convex.*

Quadratic Convexity - Squashing images

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there is a linear operator $T : \mathbb{R}^M \to \mathbb{R}^k$ *satisfying*

i) $(T \circ A)(S^{N-1})$ *is contained in no proper affine subspace of* $\mathbb{R}^k$*, and*

ii) $(T \circ A)(S^{N-1})$ *is convex if and only if* $A(S^{N-1})$ *is convex.*

SOS revisited

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- Lower bounds on pythagoras numbers, smallest pythagoras number of a nonempty open subset