Universal Connectivity as Mathematical Necessity: A Zero-Axiom Constructive Proof in Coq

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Abstract

We present a machine-verified constructive proof that any relational structure can be extended to one exhibiting universal connectivity, requiring **zero axioms** beyond the core logic of the Coq proof assistant. The construction introduces a distinguished element—"the Whole"—to which every element relates, while provably preserving all existing relational structure. This result provides rigorous mathematical foundations for relational ontology, demonstrating that universal connectivity is not a metaphysical assumption but a mathematical affordance always available through canonical completion. The formalization includes categorical infrastructure (morphisms, isomorphisms, composition with unit and associativity laws), iterated completion for fractal relational structures, and a comprehensive axiom audit verifying the zero-axiom claim through computational tests. The complete Coq development is publicly available and serves as a verified foundation for the Universal Connectivity Framework (UCF/GUTT).

Keywords: formal verification, relational ontology, universal connectivity, constructive mathematics, Coq, category theory, serial relations

1 Introduction

The question of whether reality is fundamentally composed of discrete entities or constitutive relations has occupied philosophers from Leibniz to contemporary process metaphysicians. Relational ontologies—frameworks asserting that relations are ontologically primary rather than derivative—face a persistent challenge: how can one rigorously formalize the claim that "everything is connected" without either (a) assuming it axiomatically, or (b) rendering it vacuously true?

This paper presents a resolution: a constructive proof, fully formalized in the Coq proof assistant, that *any* relational structure can be extended to one exhibiting universal connectivity. The extension:

- Introduces a distinguished element Whole ("the Whole")
- Ensures every element relates to Whole
- Preserves all existing relational structure exactly
- Requires **zero axioms** beyond Coq's core calculus

The significance is threefold. First, universal connectivity is revealed as a *mathematical necessity*—not a metaphysical postulate but a constructive theorem. Second, the zero-axiom property means the result rests on no hidden assumptions; it is derivable from pure type theory. Third, the formalization provides machine-verified foundations for subsequent theoretical development in the Universal Connectivity Framework (UCF/GUTT).

1.1 Contributions

1. A **constructive proof** that every relation can be completed to a serial (universally connected) relation via the option type construction (Section 3).
2. A **layered record hierarchy** capturing progressively stronger extension properties: basic extensions, pointed extensions, fresh-pointed extensions, and serial extensions (Section 4).
3. **Categorical infrastructure**: morphisms and isomorphisms between extensions, with verified identity, composition, unit laws, and associativity (Section 5).
4. An **iterated completion** construction for nested relational structures with a fractal connectivity theorem (Section 6).
5. A **machine-checkable axiom audit** demonstrating zero axiom dependencies through computational tests and exhaustive `Print Assumptions` verification (Section 7).

The complete Coq development is available at:

`https://github.com/relationalexistence/UCF-GUTT/tree/main/Library/`

2 Background and Motivation

2.1 Relational Ontology

Traditional ontology takes *substances* or *entities* as primitive, with relations defined derivatively as obtaining between pre-existing things. Relational ontology inverts this priority: relations are fundamental, and what we call "entities" are stable patterns within relational structures.

This view has roots in Leibniz's monadology, Whitehead's process philosophy, and contemporary structural realism in philosophy of science. However, these frameworks typically treat universal connectivity as an assumption—either explicitly axiomatized or implicitly presupposed.

The present work demonstrates that universal connectivity need not be assumed. Given *any* relational structure—however sparse, disconnected, or pathological—one can *construct* an extension that is universally connected while preserving the original structure exactly.

2.2 Seriality in Relational Logic

In the logic of relations, a relation R on a set A is **serial** if every element has at least one successor:

$$\forall x \in A.\, \exists y \in A.\, R(x, y)$$

Seriality is weaker than totality (which requires $R(x, y) \vee R(y, x)$ for all x, y) and weaker than reflexivity (which requires $R(x, x)$ for all x). Yet seriality captures a crucial property: no element is a "dead end."

A relation is **pointed-serial** if there exists a distinguished element w such that every element relates to w:

$$\exists w \in A.\, \forall x \in A.\, R(x, w)$$

Pointed-seriality is strictly stronger than seriality: it provides a *uniform witness* for the existential, enabling constructive reasoning.

Our main construction produces pointed-serial extensions: not merely "every element has some successor" but "every element relates to *this specific element*"—the Whole.

2.3 Why Zero Axioms Matters

In formal mathematics, an **axiom** is a proposition accepted without proof. While axioms are sometimes necessary (e.g., the axiom of choice for certain classical results), they represent assumptions that could, in principle, be false or lead to inconsistency.

A **zero-axiom proof** derives its conclusions from pure logic and definitions alone. In Coq, this means:

- No `Axiom` or `Parameter` declarations
- No use of `admit` (unproven goals)
- No dependencies on axioms in imported libraries

The significance is epistemic: a zero-axiom theorem cannot be false unless the underlying logic itself is inconsistent. Since Coq's Calculus of Inductive Constructions is widely trusted (with decades of scrutiny and consistency proofs relative to set theory), zero-axiom results carry maximal certainty.

For a claim as foundational as "universal connectivity is always achievable," this level of rigor is essential.

3 The Whole-Completion Construction

3.1 Core Definitions

Let U be any type (representing the universe of discourse). The **Whole-completion** of U is defined using the option type:

$$U^+ \triangleq U \uplus \{\mathsf{Whole}\} \cong \mathsf{option}\, U$$

where Whole is represented by None and elements of U are embedded via Some. We write U^+ to denote the extended carrier.

Definition 3.1 (Carrier, Injection, Point)**.**

$$\begin{aligned} \mathsf{carrier}(U) &\triangleq \mathsf{option}\, U \\ \mathsf{inject}(u) &\triangleq \mathsf{Some}(u) \\ \mathsf{Whole} &\triangleq \mathsf{None} \end{aligned}$$

Given a relation $R : U \to U \to \mathsf{Prop}$, we define its **lifted relation** R' on U^+:

Definition 3.2 (Lifted Relation)**.**

$$R'(x, y) \triangleq \begin{cases} R(a, b) & \textit{if } x = \mathsf{inject}(a) \textit{ and } y = \mathsf{inject}(b) \\ \top & \textit{if } y = \mathsf{Whole} \\ \bot & \textit{if } x = \mathsf{Whole} \textit{ and } y = \mathsf{inject}(b) \end{cases}$$

In Coq, this is implemented by pattern matching:

```
Definition lift_rel (R : U →U →Prop) (x y : carrier) : Prop :=
  match x, y with
  | Some a, Some b ⇒R a b
  | _,      None   ⇒True
  | None,   Some _ ⇒False
  end.
```

3.2 Core Properties

The construction satisfies four essential properties:

Theorem 3.3 (Conservativity)**.** *For all $a, b : U$ and $R : U \to U \to \mathsf{Prop}$:*

$$R'(\mathsf{inject}(a), \mathsf{inject}(b)) \iff R(a, b)$$

Proof. By definition of lift, when both arguments are injected elements, R' reduces to R. □

Theorem 3.4 (Pointed Seriality). *For all $R : U \to U \to \mathsf{Prop}$ and $x : U^+$:*

$$R'(x, \mathsf{Whole})$$

Proof. By case analysis on x. If $x = \mathsf{inject}(a)$, then $R'(\mathsf{inject}(a), \mathsf{None}) = \top$ by the second case of Definition 3.2. If $x = \mathsf{Whole}$, then $R'(\mathsf{None}, \mathsf{None}) = \top$ similarly. □

Theorem 3.5 (Freshness). *For all $u : U$:*

$$\mathsf{inject}(u) \neq \mathsf{Whole}$$

Proof. $\mathsf{Some}(u) \neq \mathsf{None}$ by the disjointness of constructors in inductive types. □

Theorem 3.6 (Terminality). *For all $R : U \to U \to \mathsf{Prop}$ and $u : U$:*

$$\neg R'(\mathsf{Whole}, \mathsf{inject}(u))$$

Proof. By definition, $R'(\mathsf{None}, \mathsf{Some}(u)) = \bot$. □

3.3 Concrete Example: Completing the Natural Numbers

To ground the abstract construction, consider the natural numbers $\mathbb{N}$ with the strict order $<$. This relation is *not* serial on $\mathbb{N}$: there is no y such that $0 < y$ when we also require $y <$ something (the chain must continue forever).

Applying Whole-completion:

- $\mathsf{carrier}(\mathbb{N}) = \mathsf{option}\,\mathbb{N} = \{\mathsf{Some}(0), \mathsf{Some}(1), \mathsf{Some}(2), \ldots, \mathsf{None}\}$
- $\mathsf{Whole} = \mathsf{None}$ (the new universal sink)
- $<' (\mathsf{Some}(m), \mathsf{Some}(n)) \iff m < n$ (preserved)
- $<' (x, \mathsf{None}) = \top$ for all x (everything relates to Whole)
- $<' (\mathsf{None}, \mathsf{Some}(n)) = \bot$ (Whole is terminal)

Now every element has a successor: natural numbers can continue their $<$-chains as before, but can also "exit" to Whole. The number 0, previously a dead-end for predecessor search, now relates to Whole. The completion is minimal—we added exactly one element—and conservative—the original ordering on $\mathbb{N}$ is unchanged.

In Coq:

```
Example five_has_successor :
  ∃ y, WholeCompletion.lift_rel lt (Some 5) y.
Proof.
  ∃ None. (* Whole *)
  apply WholeCompletion.serial.
Qed.
```

Corollary 3.7 (Weak Seriality). *For all R, the relation R' is serial:*

$$\forall x : U^+. \exists y : U^+. R'(x, y)$$

Proof. Take $y = \mathsf{Whole}$ and apply Theorem 3.4. □

Corollary 3.8 (No Dead Ends). *For all R:*

$$\neg \exists x : U^+. \forall y : U^+. \neg R'(x, y)$$

Proof. Suppose such an x exists. Then in particular $\neg R'(x, \mathsf{Whole})$, contradicting Theorem 3.4. □

Theorem 3.9 (Injectivity of Embedding). *For all $a, b : U$:*

$$\mathsf{inject}(a) = \mathsf{inject}(b) \implies a = b$$

Proof. Some is injective by properties of inductive types. □

Theorem 3.10 (Point Self-Loop). *For all R:*

$$R'(\mathsf{Whole}, \mathsf{Whole})$$

Proof. $R'(\mathsf{None}, \mathsf{None}) = \top$ by the second case of Definition 3.2. □

Theorem 3.11 (Unique Universal Sink). *If $w : U^+$ satisfies $\forall R. \forall x. R'(x, w)$, then $w = \mathsf{Whole}$.*

Proof. Suppose $w = \mathsf{inject}(u)$ for some u. Take R to be the empty relation. Then $R'(\mathsf{Whole}, \mathsf{inject}(u)) = \bot$, contradicting the assumption. Hence $w = \mathsf{Whole}$. □

4 The Record Hierarchy

To capture extensions at varying levels of structure, we define a hierarchy of record types:

Definition 4.1 (Universe Extension). *A **universe extension** of U consists of:*
- *A carrier type* $\mathsf{carrier}$
- *An injection* $\mathsf{inject} : U \to \mathsf{carrier}$
- *A lift operation* $\mathsf{lift} : (U \to U \to \mathsf{Prop}) \to (\mathsf{carrier} \to \mathsf{carrier} \to \mathsf{Prop})$
- *A conservativity proof:* $\mathsf{lift}(R)(\mathsf{inject}(a), \mathsf{inject}(b)) \iff R(a, b)$

Definition 4.2 (Pointed Universe Extension). *A **pointed universe extension** adds:*
- *A distinguished point* $\mathsf{point} : \mathsf{carrier}$

Definition 4.3 (Fresh-Pointed Universe Extension). *A **fresh-pointed universe extension** adds:*
- *Freshness:* $\forall u. \mathsf{inject}(u) \neq \mathsf{point}$

Definition 4.4 (Pointed-Serial Extension). *A **pointed-serial extension** adds:*
- *Serial point:* $\forall R. \forall x. \mathsf{lift}(R)(x, \mathsf{point})$

The Whole-completion construction provides canonical inhabitants:

Theorem 4.5. *For any type U, the option-type construction yields a pointed-serial extension.*

Proof. Conservativity is Theorem 3.3, freshness is Theorem 3.5, and serial point is Theorem 3.4. □

5 Categorical Structure

Universe extensions form a category with rich structure.

5.1 Morphisms

Definition 5.1 (Extension Homomorphism). *A **homomorphism** $f : E_1 \to E_2$ between extensions of U consists of:*
- *A map* $f : \mathsf{carrier}(E_1) \to \mathsf{carrier}(E_2)$
- *Injection commutes:* $f(\mathsf{inject}_{E_1}(u)) = \mathsf{inject}_{E_2}(u)$
- *Lift preserves:* $\mathsf{lift}_{E_1}(R)(x, y) \implies \mathsf{lift}_{E_2}(R)(f(x), f(y))$

Definition 5.2 (Extension Isomorphism). *An **isomorphism** consists of homomorphisms $f : E_1 \to E_2$ and $g : E_2 \to E_1$ with $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$.*

5.2 Identity and Composition

Definition 5.3 (Identity Extension). *The **identity extension** of U has* $\mathsf{carrier} = U$, $\mathsf{inject} = \mathrm{id}$, $\mathsf{lift} = \mathrm{id}$.

Definition 5.4 (Extension Composition). *Given E_1 over U and E_2 over* $\mathsf{carrier}(E_1)$, *their composition $E_1 \gg E_2$ has:*

$$\begin{aligned}\mathsf{carrier}(E_1 \gg E_2) &= \mathsf{carrier}(E_2)\\ \mathsf{inject}_{E_1 \gg E_2}(u) &= \mathsf{inject}_{E_2}(\mathsf{inject}_{E_1}(u))\\ \mathsf{lift}_{E_1 \gg E_2}(R) &= \mathsf{lift}_{E_2}(\mathsf{lift}_{E_1}(R))\end{aligned}$$

5.3 Category Laws

Theorem 5.5 (Left Unit). $\mathrm{id}_U \gg E \cong E$

Theorem 5.6 (Right Unit). $E \gg \mathrm{id}_{\mathsf{carrier}(E)} \cong E$

Theorem 5.7 (Associativity). $(E_1 \gg E_2) \gg E_3 \cong E_1 \gg (E_2 \gg E_3)$

All three isomorphisms are witnessed by identity maps on the underlying carriers, verified in Coq.

6 Iterated Completion and Fractal Connectivity

For nested relational structures, we define n-fold Whole-completion:

Definition 6.1 (Iterated Carrier).

$$\begin{aligned}\mathsf{iter\_carrier}(0, U) &= U\\ \mathsf{iter\_carrier}(n+1, U) &= \mathsf{option}(\mathsf{iter\_carrier}(n, U))\end{aligned}$$

At depth n, the structure has n distinct "Whole" elements at different levels.

Theorem 6.2 (Fractal Connectivity). *For any n, U, R, and element $u : U$, the embedded element relates to the Whole at every level $\ell \leq n$.*

This captures the self-similar nature of nested relational tensors: connectivity is preserved at all scales.

Corollary 6.3 (Inner and Outer Wholes). *In a double completion ($n = 2$):*

- *Elements relate to both the inner Whole (*$\mathsf{Some}(\mathsf{None})$*) and outer Whole (*$\mathsf{None}$*)*
- *The inner Whole relates to the outer Whole*
- *The outer Whole does not relate to the inner Whole (terminality)*

This asymmetry—outer envelops inner but not vice versa—captures the hierarchical structure of nested relational systems.

7 Verification Methodology

The zero-axiom claim is verified through two complementary methods:

7.1 Computational Tests

Axioms in Coq have no computational content. Therefore, tests using `reflexivity` (which requires definitional equality) would *fail* if the definitions were axioms:

```
Definition test_lift_rel_some_some :
  lift_rel (fun x y ⇒x < y) (Some 1) (Some 2) = (1 < 2).
Proof. reflexivity. Qed.
```

The audit file includes 10+ such computational tests, all succeeding.

7.2 Print Assumptions Audit

Coq's `Print Assumptions` command lists all axioms a definition depends on. The audit file invokes this on 150+ definitions, lemmas, and theorems. Every invocation returns:

```
Closed under the global context.
```

This exhaustive check confirms no axiom dependencies anywhere in the library.

7.3 Verification Instructions

To independently verify:

```
git clone https://github.com/relationalexistence/UCF-GUTT.git
cd UCF-GUTT/Library
coqc -w +all Top__Extensions__Base.v
coqc -w +all Top__Extensions__WholeCompletion.v
coqc -w +all Top__Extensions__Composition.v
coqc -w +all Top__Extensions__Prelude.v
coqc -w +all Top__Extensions__Extras.v
coqc -w +all Top__Extensions__axiomaudit.v
```

Successful compilation with zero errors and warnings certifies the claims.

8 Philosophical Implications

8.1 Universal Connectivity as Affordance

The construction reveals universal connectivity not as a property the universe *has* but as a structure one can always *construct*. This shifts the metaphysical question from "Is everything connected?" to "Can any relational structure be completed to universal connectivity?"—and the answer is a constructive *yes*.

8.2 The Ontological Status of the Whole

The Whole is:

- **Emergent**: It arises from the completion, not posited a priori
- **Unique**: It is the only element to which everything relates (Theorem 3.11)
- **Terminal**: It relates only to itself among embedded elements (Theorem 3.6)
- **Fresh**: It is genuinely new, not collapsing into any existing element (Theorem 3.5)

These properties distinguish the Whole from both arbitrary additions and trivial identifications.

8.3 Relations as Primary

The construction requires no assumptions about what U is—it works for any type. The relational structure R is similarly unconstrained. This universality supports the relational ontologist's claim: relations can be fundamental because relational completion is always available, regardless of the underlying entities.

8.4 Constructive vs. Classical

The entire development is constructive, using no excluded middle or choice axioms. This has computational significance: the witnesses in existential statements are extractable as programs. When we prove "every element has a successor," we provide the successor explicitly (Whole).

9 Related Work

Serial modal logics. The seriality condition appears in deontic and doxastic logics, ensuring every world has an accessible successor. Our construction provides a canonical way to *force* seriality.

Compactifications in topology. One-point compactification adds a point at infinity to make non-compact spaces compact. Our construction is analogous: adding Whole to make non-serial relations serial.

Monads in category theory. The option type is the "maybe monad." Our lift operation makes this monad compatible with relational structure.

Domain theory. Adding a bottom element to partial orders is standard in denotational semantics. Our Whole is a *top* element relationally (everything maps to it) rather than a bottom.

Process philosophy. Whitehead's "extensive connection" aims at universal relatedness. We provide formal machinery achieving this.

10 Conclusion

We have presented a machine-verified, zero-axiom proof that any relational structure can be extended to one with universal connectivity. The Whole-completion construction is:

- **Canonical**: Using the option type, a standard construction in type theory
- **Conservative**: Preserving all existing relational structure exactly
- **Minimal**: Adding only what is necessary (one element, the Whole)
- **Universal**: Working for any type and any relation
- **Constructive**: Providing explicit witnesses, not mere existence claims
- **Verified**: Machine-checked with exhaustive axiom auditing

The layered record hierarchy, categorical infrastructure, and iterated completion for fractal structures provide a comprehensive foundation for relational mathematics.

For the Universal Connectivity Framework, this library establishes that its core postulate—universal connectivity—is not an assumption but a theorem. The philosophical implications extend beyond any particular framework: universal connectivity is a mathematical affordance, always available, requiring nothing beyond pure constructive logic.

10.1 Limitations

Several limitations of the present work merit acknowledgment:

- **Binary relations only.** The construction addresses binary relations $R : U \to U \to \mathsf{Prop}$. Extension to n-ary relations and higher-order relational structures remains future work.

- **Constructive bias.** The development is entirely constructive (no excluded middle or choice). While this strengthens the results computationally, some classical mathematicians may prefer formulations using classical logic. We note that all results remain valid classically, as constructive proofs are a fortiori classical proofs.
- **Abstract framework.** The library provides foundational infrastructure but does not yet include extensive concrete applications. The `Extras.v` file contains examples (natural number ordering, divisibility, equivalence relations), but domain-specific applications (e.g., social network completion, knowledge graph extension) would benefit practitioners.
- **Pre-print status.** This work has not yet undergone formal peer review. Independent verification is possible via the public repository and the axiom audit methodology described in Section 7.

10.2 Future Work

- **n-ary relations**: Extending the construction to relations of arbitrary arity, with corresponding n-ary seriality conditions.
- **Limits and colimits** in the category of extensions, establishing completeness properties.
- **Higher-order structure**: Systematic treatment of extensions of extensions and their coherence conditions.
- **Connection to nested relational tensors**: Formal bridge to UCF/GUTT physics applications, particularly quantum mechanical and relativistic recovery theorems.
- **Computational extraction**: Using Coq's extraction mechanism to generate certified OCaml or Haskell code.
- **Classical variants**: Parallel development using classical axioms for comparison, and investigation of which results genuinely require constructivity.

Acknowledgments

This work represents over sixty years of theoretical development in relational ontology. The author thanks the Coq development team for creating the proof assistant that made machine verification possible.

Note on Publication

This paper is intended for submission to venues such as the *Journal of Formalized Reasoning* (JFR), *Logical Methods in Computer Science* (LMCS), or philosophy of science journals such as *Synthese*. A preprint is available on arXiv. The author welcomes correspondence regarding applications, extensions, or corrections.

Data Availability

The complete Coq development is available at:

`https://github.com/relationalexistence/UCF-GUTT/tree/main/Library/`

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A Complete Coq Definition of Lift

```
Module WholeCompletion.
  Section WithU.
    Variable U : Type.

    Definition carrier : Type := option U.
    Definition inject (u : U) : carrier := Some u.
    Definition point : carrier := None.

    Definition lift_rel (R : U →U →Prop) (x y : carrier) : Prop :=
      match x, y with
      | Some a, Some b ⇒R a b
      | _,      None   ⇒True
      | None,   Some _ ⇒False
      end.

    Lemma serial : ∀(R : U →U →Prop) (x : carrier),
      lift_rel R x point.
    Proof.
      intros R x. unfold lift_rel, point. destruct x; exact I.
    Qed.

    Lemma lift_conservative : ∀(R : U →U →Prop) (a b : U),
      lift_rel R (inject a) (inject b) ↔R a b.
    Proof.
      intros R a b. unfold lift_rel, inject. tauto.
    Qed.

  End WithU.
End WholeCompletion.
```

B Axiom Audit Excerpt

```
(* Computational test: would FAIL if lift_rel were an axiom *)
Definition test_lift_rel_some_some :
  WholeCompletion.lift_rel (fun x y : nat ⇒x < y) (Some 1) (Some 2)
  = (1 < 2).
Proof. reflexivity. Qed.

(* Print Assumptions audit *)
Print Assumptions WholeCompletion.serial.
```

```
(* Output: Closed under the global context. *)

Print Assumptions WholeCompletion.lift_conservative.
(* Output: Closed under the global context. *)

Print Assumptions SerialComposition.fractal_connectivity.
(* Output: Closed under the global context. *)
```