$$\mathscr{F}(f'(x)) = jw\mathscr{F}(f(x))$$

Yes, we can use the fourier transform to compute the gradient at every pixel. Assuming a 1D case for simplicity, take 1D DFT of our image, multiply it pixel by pixel with (jw), and then take the 1D IDFT back into the spatial domain.