To prove : The Laplacian operator is rotationally invariant

Show that for any image f :

$$f_{xx} + f_{yy} = f_{uu} + f_{vv}$$

where, $u = xcos\theta - ysin\theta$ and $v = xsin\theta + ycos\theta$

L.H.S.

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = cos\theta$, $\frac{\partial v}{\partial x} = sin\theta$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}cos\theta + \frac{\partial f}{\partial v}sin\theta$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial u}cos\theta + \frac{\partial f}{\partial v}sin\theta) = \frac{\partial^2 f}{\partial x \partial u}cos\theta + \frac{\partial^2 f}{\partial x \partial v}sin\theta$

We know that partial derivatives can be exchanged, thus $\frac{\partial^2 f}{\partial x \partial u} = \frac{\partial^2 f}{\partial u \partial x}$

$\frac{\partial^2 f}{\partial x \partial u}cos\theta + \frac{\partial^2 f}{\partial x \partial v}sin\theta = \frac{\partial^2 f}{\partial u \partial x}cos\theta + \frac{\partial^2 f}{\partial v \partial x}sin\theta = cos\theta\frac{\partial}{\partial u}(\frac{\partial f}{\partial x}) + sin\theta\frac{\partial}{\partial v}(\frac{\partial f}{\partial x})\ldots$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}cos\theta + \frac{\partial f}{\partial v}sin\theta \ldots = cos\theta\frac{\partial}{\partial u}(\frac{\partial f}{\partial u}cos\theta + \frac{\partial f}{\partial v}sin\theta) + sin\theta\frac{\partial}{\partial v}(\frac{\partial f}{\partial u}cos\theta + \frac{\partial f}{\partial v}sin\theta)$

$= \frac{\partial^2 f}{\partial u^2}cos^2\theta + \frac{\partial^2 f}{\partial u \partial v}sin\theta cos\theta + \frac{\partial^2 f}{\partial v \partial u}cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2}sin^2\theta$

$= \frac{\partial^2 f}{\partial u^2}cos^2\theta + 2\frac{\partial^2 f}{\partial u \partial v}cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2}sin^2\theta$

$\rightarrow f_{xx} = \frac{\partial^2 f}{\partial u^2}cos^2\theta + 2\frac{\partial^2 f}{\partial u \partial v}cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2}sin^2\theta$

Similarly for f_{yy}

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -sin\theta$, $\frac{\partial v}{\partial y} = cos\theta$

$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial u}sin\theta + \frac{\partial f}{\partial v}cos\theta$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(-\frac{\partial f}{\partial u}sin\theta + \frac{\partial f}{\partial v}cos\theta) = -\frac{\partial^2 f}{\partial y \partial u}sin\theta + \frac{\partial^2 f}{\partial y \partial v}cos\theta$

$= -\frac{\partial^2 f}{\partial u \partial y}sin\theta + \frac{\partial^2 f}{\partial v \partial y}cos\theta = -sin\theta\frac{\partial}{\partial u}(\frac{\partial f}{\partial y}) + cos\theta\frac{\partial}{\partial v}(\frac{\partial f}{\partial y})$

$= -sin\theta\frac{\partial}{\partial u}(-\frac{\partial f}{\partial u}sin\theta + \frac{\partial f}{\partial v}cos\theta) + cos\theta\frac{\partial}{\partial v}(-\frac{\partial f}{\partial u}sin\theta + \frac{\partial f}{\partial v}cos\theta)$

$= \frac{\partial^2 f}{\partial u^2} sin^2\theta - \frac{\partial^2 f}{\partial u \partial v} sin\theta cos\theta - \frac{\partial^2 f}{\partial v \partial u} cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2} cos^2\theta$

$= \frac{\partial^2 f}{\partial u^2} sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v} cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2} cos^2\theta$

$\rightarrow f_{yy} = \frac{\partial^2 f}{\partial u^2} sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v} cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2} cos^2\theta$

$f_{xx} + f_{yy} = \frac{\partial^2 f}{\partial u^2} cos^2\theta + \ 2\frac{\partial^2 f}{\partial u \partial v} cos\theta sin\theta \ + \frac{\partial^2 f}{\partial v^2} sin^2\theta + \frac{\partial^2 f}{\partial u^2} sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v} cos\theta sin\theta + \frac{\partial^2 f}{\partial v^2} cos^2\theta$

$= \frac{\partial^2 f}{\partial u^2}(cos^2\theta + sin^2\theta) + \frac{\partial^2 f}{\partial v^2}(cos^2\theta + sin^2\theta) \ldots \ cos^2\theta + sin^2\theta = 1 \ldots = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = f_{uu} + f_{vv}$

R.H.S. $\rightarrow f_{uu} + f_{vv}$

Therefore, L.H.S. = R.H.S.

Hence Proved