(3) a) $P = A^T A$

$Q = A A^T$

$A = (m \times n)$ size Matrix

$[n \geq m]$

P is $n \times n$ Matrix

Q is $m \times m$ Matrix

y has dimensions $n \times 1$

z has dimensions $m \times 1$

$$y^T P y = y^T A^T A y = (Ay)^T A y$$

Ay is $(m \times 1)$ matrix

Let $Ay = [a_1 \quad a_2 \quad \dots \quad a_n]$

$$(Ay)^T Ay = a_1^2 + a_2^2 \dots + a_n^2$$

$\therefore y^T P y \geq 0$.

Similarly

$$z^T Q z = z^T A A^T z = (A^T z)^T A^T z$$

$A^T z$ is $n \times 1$ matrix

Let $A^T z = [b_1 \quad b_2 \quad \dots \quad b_n]$

$$(A^T z)^T A^T z = b_1^2 + b_2^2 + \dots b_n^2$$

$\therefore z^T Q z \geq 0$

* Hence Proved.

$P = A^TA$

Let eigen vector of A^TA be x

$$A^TA\,x = \lambda x \quad \text{---}(1)$$

x is a $n\times 1$ vector

Multiply (1) by x^T

$$x^TA^TAx = x^T\lambda x = \lambda x^Tx$$

$(Ax)^TAx \geq 0$

$\lambda x^Tx = \lambda \sum x_i^2$

So for $\lambda \sum x_i^2 \geq 0$, λ has to be greater than zero

$\therefore \lambda \geq 0$

Similarly same proof for $Q = AA^T$

b) If, $Pu = \lambda u$

Show ⇒ $QAu = \lambda Au$

$$A^TAu = \lambda u$$

Multiply by A

$AA^TAu = A\lambda u$

$QAu = \lambda Au$

Au is eigenvector of Q with eigenvalue λ

⇒ u is a $(n\times 1)$ vector

If, $Qv = \mu v$

Show ⇒ $PA^Tv = \mu A^Tv$

$$AA^Tv = \mu v$$

Multiply by A^T

$A^TAA^Tv = A^T\mu v$

$PA^Tv = \mu A^Tv$

A^Tv is eigenvector of P with eigen value μ

⇒ v is a $(m\times 1)$ vector

* Hence Proved

c) $Qv_i = \lambda_i v_i$

$AA^T v_i = \lambda_i v_i$

$$u_i \triangleq \frac{A^T v_i}{\|A^T v_i\|_2}$$

To prove, $Au_i = r_i v_i$, where r_i is real and non-negative

$$Au_i = \frac{AA^T v_i}{\|A^T v_i\|_2} = \frac{\lambda_i v_i}{\|A^T v_i\|_2} = r_i v_i$$

$$r_i = \frac{\lambda_i}{\|A^T v_i\|_2} \Rightarrow \left[\, \|A^T v_i\|_2 \geq 0 \,\right]$$

We have proved that eigen vector of Q are non-negative in the first part, therefore
$\lambda_i \geq 0$

So, $r_i \geq 0$

* Hence Proved.

d) $U \Sigma V^T$

$$= [v_1 | v_2 | v_3 \ldots | v_m] \begin{bmatrix} \frac{\lambda_1}{\|A^T v_1\|} & 0 & 0 & 0 \ldots 0 \\ 0 & \frac{\lambda_2}{\|A^T v_2\|} & \ldots & \\ 0 & 0 & 0 & \frac{\lambda_n}{\|A^T v_n\|} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

$$= [v_1 | v_2 | \ldots | v_m] \begin{bmatrix} \frac{\lambda_1 u_1}{\|A^T v_1\|} \\ \vdots \\ \frac{\lambda_n u_n}{\|A^T v_n\|} \end{bmatrix}$$

$$\begin{bmatrix} u_i^T u_j = 0 \text{ for } i \neq j \\ v_i^T v_j = 0 \text{ for } i \neq j \end{bmatrix} \text{given}$$

Therefore,

$$U \Sigma V^T = A$$

* Hence Proved.