Task 1

(a) $\frac{dI}{dt} \overset{!}{=} 0 \Leftrightarrow 0 = \beta(\alpha)\, S^* I^* - (d+\alpha+\theta) I^*$

$$\boxed{I_1^* = 0} \text{ oder } \beta(\alpha)\, S^* - (d+\alpha+\theta) = 0$$

$$\Leftrightarrow \boxed{S_1^* = \frac{d+\alpha+\theta}{\beta(\alpha)}}$$

$$\frac{dS}{dt} \overset{!}{=} 0 \Leftrightarrow 0 = bS^* + bI^* + \theta I^* - \beta(\alpha)\, S^* I^*$$

$$\xRightarrow{I_1^*=0} \boxed{0 = S_2^*} \Rightarrow \boxed{E_1 = (0,0)}$$

$$\xRightarrow{S_1^*} I_2^* \left(b + \theta - \beta(\alpha) S_1^* \right) + b S_1^* = 0$$

$$I_2^* = \frac{-b S_1^*}{b + \theta - \beta(\alpha) S_1^*}$$

$$= \frac{-b \frac{(d+\alpha+\theta)}{\beta(\alpha)}}{(b+\theta) - \cancel{\beta(\alpha)} \frac{(d+\alpha+\theta)}{\cancel{\beta(\alpha)}}}$$

$$= \frac{-b\,(d+\alpha+\theta)}{\beta(\alpha)\left[(b\cancel{+\theta}) - d - \alpha \cancel{-\theta})\right]}$$

$$I_2^* = \frac{+b\,(d+\alpha+\theta)}{\beta(\alpha)\,(d+\alpha-b)}$$

$$\Rightarrow E_2 = \left(\frac{d+\alpha+\theta}{\beta(\alpha)} ; \frac{b\,(d+\alpha+\theta)}{\beta(\alpha)\,(d+\alpha-b)} \right.$$

$$\frac{\partial\left(\frac{dS}{dt}\right)}{\partial S} = b - \beta(\alpha) I \qquad \frac{\partial\left(\frac{dS}{dt}\right)}{\partial I} = b + \theta - \beta(\alpha) S$$

$$\frac{\partial\left(\frac{dI}{dt}\right)}{\partial S} = \cancel{b+\theta}\ \beta(\alpha) I \qquad \frac{\partial\left(\frac{dI}{dt}\right)}{\partial I} = \beta(\alpha) S - (d+\alpha+\theta)$$

$$J\Big|_{E_1} = \begin{pmatrix} b & b+\theta \\ 0 & -(d+\alpha+\theta) \end{pmatrix} \qquad b > 0 \Rightarrow E_1 \text{ ist instabil}$$

$$J\Big|_{E_2} = \begin{pmatrix} b - \cancel{\beta(\alpha)} \frac{b \cdot (d+\alpha+\theta)}{\cancel{\beta(\alpha)}(d+\alpha-b)} & b+\theta - \left[\cancel{\beta(\alpha)} \frac{(d+\alpha+\theta)}{\cancel{\beta(\alpha)}}\right] \\ \cancel{\beta(\alpha)} \frac{b \cdot (d+\alpha+\theta)}{\cancel{\beta(\alpha)}(d+\alpha-b)} & \cancel{\beta(\alpha)} \frac{\cancel{(d+\alpha+\theta)}}{\cancel{\beta(\alpha)}} - \cancel{(d+\alpha+\theta)} \end{pmatrix}$$

$$= \begin{pmatrix} b - \frac{b \cdot (d+\alpha+\theta)}{d+(\alpha-b)} & (b \cancel{+\theta} - d - \alpha \cancel{-\theta}) \\ \frac{b \cdot (d+\alpha+\theta)}{(d+\alpha-b)} & 0 \end{pmatrix}$$

$$Tr\left(J\big|_{E_2}\right) = \underbrace{b}_{>0} \underbrace{\left(1 - \frac{(d+\alpha+\theta)}{d+\alpha-b}\right)}_{< 0} = b \cdot \left(\frac{-(b+\theta)}{(d+\alpha)-b}\right) < 0$$

$$\Rightarrow \quad 1 - \frac{d+\alpha+\theta}{d+\alpha-b} < 0$$

$$d+\alpha-b < d+\alpha+\theta$$

$$-b < \theta$$

$$0 < \underbrace{\theta+b}_{>0} \quad \checkmark$$

$$\det(J|_{E_2}) = -\left[b-d-\alpha \cdot \frac{b\cdot(d+\alpha+\theta)}{(d+\alpha-b)} \right]$$

$$= \cancel{(d+\alpha-b)} \cdot \frac{b\cdot(d+\alpha+\theta)}{\cancel{(b+\alpha-b)}}$$

$$= b(d+\alpha+\theta) > 0 \quad \checkmark$$

(b)

the invasion fitness $f(\alpha,\alpha')$ is the per-capita growth-rate of a mutant α' shortly after the mutation take place. therefor the function $f(\alpha,\alpha')$ depents on the strain of each parameter α and α'. α gets us the environment Equilibrium and α' gets us the growth rate.

(c)

$$\frac{dI'}{dt} = I' \underbrace{\left(\beta(\alpha')\, S^*(\alpha) - (d+\alpha'+\theta)\right)}_{f(\alpha;\alpha')}$$

$$f(\alpha,\alpha') = \beta(\alpha') \frac{(d+\alpha+\theta)}{\beta(\alpha)} - (d+\alpha+\theta)$$

$$= \beta(\alpha') \left[\frac{1}{R_0(\alpha)} - \frac{1}{R_0(\alpha')} \right]$$

$$S(\alpha) = \frac{\partial f}{\partial \alpha'}\Big|_{\alpha=\alpha'}$$

$$= \underbrace{\frac{\partial \beta(\alpha')}{\partial \alpha'}\left[\frac{1}{R_0(\alpha)} - \frac{1}{R_0(\alpha')}\right]}_{=0}\Bigg|_{\alpha=\alpha'} + \beta(\alpha') \cdot \frac{\frac{\partial R_0(\alpha')}{\partial \alpha'}}{R_0(\alpha')^2}\Bigg|_{\alpha=\alpha'}$$

$$= \frac{\beta(\alpha')}{R_0(\alpha')^2} \quad \frac{\partial R_0(\alpha')}{\partial \alpha'}$$

$$= \frac{\beta(\alpha')}{\frac{\beta(\alpha')^2}{(d+\alpha'+\theta)^2}} \quad \frac{\partial \frac{\beta(\alpha')}{(d+\alpha+\theta)}}{\partial \alpha'}$$

$$= \frac{\cancel{(d+\alpha'+\theta)}^2}{\beta(\alpha')} \cdot \frac{\frac{\partial(\beta(\alpha'))}{\partial \alpha'} \cdot (d+\alpha'+\theta) - \beta(\alpha') \cdot 1}{\cancel{(d+\alpha'+\theta)^2}}$$

$$= \frac{(d+\alpha'+\theta)}{\beta(\alpha')} \quad \frac{\partial \beta(\alpha')}{\partial \alpha'} \quad - 1$$

$$= \frac{(d+\alpha'+\theta)}{\beta(\alpha')} \cdot \frac{(\cancel{\alpha'}+c) - \cancel{\alpha'}}{(\alpha'+c)^2} \quad - 1$$

$$= \frac{(d+\alpha'+\theta)}{\frac{\alpha'}{\cancel{(\alpha'+c)}}} \cdot \frac{c}{(\alpha'+c)^{\cancel{2}}} \quad - 1$$

$$= \frac{c \cdot (a+\alpha'+\theta)}{\alpha' \cdot (\alpha'+c)} \quad - 1$$

d)

$$s(\alpha^*) = \frac{(d + \sqrt{c(d+\theta)} + \theta)c}{c(d+\theta) + \sqrt{c(d+\theta)} \cdot c} - 1 = \frac{cd + c\sqrt{c(d+\theta)} + c\theta}{cd + c\theta + c\sqrt{c(d+\theta)}} - 1$$

$$= 0$$

$$\frac{\partial s(\alpha')}{\partial \alpha'} = \frac{\partial}{\partial \alpha'}\left[\frac{(d + \alpha' + \theta)c}{\alpha'(\alpha'+c)} - 1\right]$$

$$= \frac{c(\alpha'(\alpha'+c)) - (d+\alpha'+\theta)c \cdot (2\alpha'+c)}{(\alpha'(\alpha'+c))^2}$$

$$= \frac{c\alpha'^2 + c^2\alpha' - [((d+\theta)c + \alpha'c) \cdot (2\alpha'+c)]}{(\alpha'(\alpha'+c))^2}$$

$$= \frac{\boxed{c(\alpha')^2} + \boxed{c^2\alpha'} - 2\alpha'(d+\theta)c - c^2(d+\theta) - \boxed{2c(\alpha')^2} - \boxed{\alpha' c^2}}{(\alpha'(\alpha'+c))^2}$$

$$= \frac{-2\alpha'(d+\theta)c - c^2(d+\theta) - c(\alpha')^2}{(\alpha'(\alpha'+c))^2}$$

$$\alpha^* = \sqrt{c(d+\theta)}$$

$$\frac{\partial(s(\alpha'))}{\partial \alpha'} < 0$$

$$0 > \frac{-2\alpha'(d+\theta)c - c^2(d+\theta) - c(\alpha')^2}{(\alpha'(\alpha'+c))^2}$$

$$0 > -2\sqrt{c(d+\theta)}(d+\theta)c - c^2(d+\theta) - c\, c(d+\theta)$$

$$0 > -2\sqrt{c(d+\theta)}(d+\theta)c - 2c^2(d+\theta)$$

$$0 > -2c(d+\theta)\left(\sqrt{c(d+\theta)} + c\right)$$