Mathematical Epidemiology

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Problem Sheet 2

for **Monday, 6 November 2023, 12:00pm**.

Please upload your solutions to Stud.IP. Solutions will be discussed in the following problem class. Please note in your solutions if you have difficulties with something or would like the next problem class to go through a particular topic.

1. **Estimating R_0 for rubella** A serological survey of freshman at Yale University found that 25% were susceptible to rubella at the beginning of the year and 9.65% were susceptible at the end of the year. Assume an SIR epidemic model and that an epidemic has taken place over the year.

   (a) Find the implicit equation for the final epidemic size.

   (b) Determine the value of R_0 by numerically solving the equation in (a).

2. **SIRS** Some diseases confer only temporary immunity, so that there is a flow of individuals from the immune class R back to the susceptible class S. Such diseases are known as *SIRS* diseases. Like *SIS* diseases, they may die out or remain endemic. For this question, ignore demographic processes. Assume a density-dependent incidence.

   (a) Write down a model for such a disease. Let β be the transmission parameter, γ the recovery rate and ρ the rate with which individual lose immunity and become susceptible again.

   (b) Draw a transfer diagram.

   (c) Show that the total population size is constant.

   (d) Find R_0 for this disease and interpret it.

   (e) Rewrite the model in terms of the fractions of the population being susceptible, infected and immune. Arrive at the following form:

   $$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}T} &= \gamma[-R_0\,u\,v + \varphi(1-u-v)]\,,\\ \frac{\mathrm{d}v}{\mathrm{d}T} &= \gamma[R_0\,u-1]\,v\,.\end{aligned}$$

   How does φ need to be chosen?

   (f) Determine the equilibria and their stability. What is the condition for the disease to remain endemic in the population?

3. **Flatten the curve** [10] The following table gives the number of COVID-19 cases reported in Germany, beginning with the day when the cases exceeded 1 per million inhabitant (to avoid random effects) and lasting for three weeks (until March 26th, 2020).

Days	0	1	2	3	4	5	6	7	8	9	10
Cases	138	284	163	55	237	157	271	802	693	733	1043
	11	12	13	14	15	16	17	18	19	20	
	1174	1144	1042	5940	4049	3276	3311	4438	2342	4954	

Consider the SIR model

$$\frac{\mathrm{d}S}{\mathrm{d}T} = -\beta SI, \quad \frac{\mathrm{d}I}{\mathrm{d}T} = \beta SI - \gamma I, \quad \frac{\mathrm{d}R}{\mathrm{d}T} = \gamma I,$$

and assume that the total population size is $N = S + I + R = 80$ million (in Python `N=80e6`), the relative recovery rate $\gamma = 0.125$ per day, and the transmission coefficient $\beta = 3.86$`e-9`.

(a) Solve the SIR model numerically for the initial condition $S(0) = N, I(0) = I_0, R(0) = 0$, where $I_0 = 138$ as in the data, and plot $S(T)$, $I(T)$, and $R(T)$ over 100 days. [4]

(b) Suppose the health system in Germany has a capacity of 30,000 intensive care units with ventilation. Suppose that measures are taken on day 20, which become effective immediately. That is, assume a different value of β from day 20 onward. Use your code from (a) to solve the SIR model numerically for the initial condition $I(20) = 4954$ and to plot $I(T)$ over the time period $[20, 200]$. By trying different values of β, explore by how many percent the transmission coefficient needs to be reduced such that the number of cases will not exceed the health care capacity at any time over the next 180 days? You may assume that 1.32% of cases will need ventilation.[1] Produce a diagram that shows three solutions of $I(T)$ with β values at this critical value, slightly above this critical value, and at a levels such that $I(T)$ declines. [5]

(c) Use the criterion $R_0 = 1$ to find the percentage by which β needs to be reduced for the number of cases to decline. [1]

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[1] Report No. 9, Imperial College COVID-19 Response Team