$X_1, \ldots, X_n \sim f(x \mid \theta_1, \ldots, \theta_n)$

maximum likelihood

$$L(\theta_1, \ldots, \theta_n \mid X_1, \ldots, X_n) = \prod_{i=1}^{n} f(x_1, \ldots, x_n \mid \theta_1, \ldots, \theta_n)$$

$$\ell(\theta_1, \ldots, \theta_n \mid X_1, \ldots, X_n) = \sum \log f(x_1, \ldots, x_n \mid \theta_1, \ldots, \theta_n)$$

$\Rightarrow$ $\hat{\theta}_1^{MLE}, \ldots, \hat{\theta}_n^{MLE}$ the values of the parameters that maximize this function

$\Rightarrow$ method of moments

$\underline{X} \sim f(x \mid \theta_1, \ldots, \theta_r)$

$E[X^k] = g_k(\theta_1, \ldots, \theta_r)$

find $\hat{\theta}_1^{MM}, \ldots, \hat{\theta}_k^{MM}$

$$g_k(\hat{\theta}_1^{MM}, \ldots, \hat{\theta}_1^{MM}) = \frac{1}{n} \sum_{i=1}^{n} x_i^k$$

$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$E[X] = \mu$

$E[X^2] = \mu^2 + \sigma^2$

μ, σ s/t $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\mu^2 + \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

↳ point estimates

↳ come up with the best for which member of the famil the data came from

↳ noise from the sample, what are a reasonable set of vals for our parameter(s)?

$(1-\alpha) \cdot 100\%$ confidence interval is a statistic (L, U)

s/t $P(\theta \in (L, U)) = 1 - \alpha$

Confidence Intervals with Normally distributed

Con...

data

$Z \sim N(0,1)$

z_α is the value

$$P(Z \geq z_\alpha) = \alpha$$

$X_1,\dots,X_n \sim N(\mu, \sigma)$ μ unknown, σ known

I want to estimate μ

→ point estimate: $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Y = sum of n independent normal random variables with expectation $\mu_1,\dots,\mu_n$, standard dev $\sigma_1,\dots,\sigma_n$

$$Y \sim N\left(\sum_{i=1}^{n}\mu_i, \sqrt{\sigma_1^2+\dots+\sigma_n^2}\right)$$

$$E[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\cdot n\mu = \mu$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\Big(\underbrace{\sum_{i=1}^{n}\mathrm{Var}(X_i)}_{= n\sigma^2}\Big)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

standard

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1-\alpha$$

$$\Longleftrightarrow$$

$$P\left(-z_{\alpha/2}\cdot \sigma/\sqrt{n} < \bar{X} - \mu < z_{\alpha/2}\cdot \sigma/\sqrt{n}\right) = 1-\alpha$$

$$\Longleftrightarrow$$

$$P\left(\mu \in \underline{\underline{\bar{X} \pm z_{\alpha/2}\cdot \sigma/\sqrt{n}}}\right) = 1-\alpha$$

$$\left(\bar{X} - \frac{\sigma}{\sqrt{n}}\cdot z_{\alpha/2},\ \bar{X} + \frac{\sigma}{\sqrt{n}}\cdot z_{\alpha/2}\right)$$

↳ $(1-\alpha)\%$ confidence

one-sided confidence intervals interval for μ

$$P\left(\frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}} < Z_\alpha\right) = 1-\alpha.$$

$$P\left(\mu > \bar{X} - \frac{Z_\alpha \sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$\left(x - \frac{Z_\alpha \sigma}{\sqrt{n}}, \infty\right)$$ "upper"

$$\left(-\infty, x + \frac{Z_\alpha \sigma}{\sqrt{n}}\right)$$

We're not going to know the variance!

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\Longrightarrow$ normal distributions

$\bar{X}$ & S^2 are independent

$$\chi^2_n \text{ d.f.} = \sum_{i=1}^{n} Z_i^2 \qquad Z_i \overset{iid}{\sim} N(0,1)$$

$\bar{X} \sim N(\mu, \sigma^2)$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{degrees of freedom}$$

T distribution with n. d.f.

$Z \sim N(0,1)$

$Y \sim \chi^2_n$ $\qquad T_n = \dfrac{Z}{\sqrt{Y/n}}$

$$\Longrightarrow \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T_{n-1}$$

$t_{\alpha,n}$ is the value s/t P(T distn with n d.f $> t_{\alpha,n-1}$) $= \alpha$

$$P\left(-t_{\alpha/2,n-1} < \sqrt{n}\,\frac{(\bar{X}-\mu)}{S} < t_{\alpha/2,n-1}\right) = 1-\alpha$$

$$\hookrightarrow P\left(\mu \in \bar{X} \pm \frac{t_{\alpha/2,n-1} \cdot S}{\sqrt{n}}\right) = 1-\alpha$$

we can derive one-sided confidence intervals in the same way as we did for the normal distn

→ data are normally distributed,
and we're trying to estimate μ

$X_1, \dots, X_n \sim \exp(\lambda)$ . . 99% confidence interval for λ

Bootstrap

Def a bootstrap sample is when re-sample the data with replacement $X_1, \dots, X_n$

Def a discrete uniform random variable
$\text{dunif}(\{x_1, \dots, x_n\})$
↳ $P(x_i) = 1/n$

$p(i) \sim \text{dunif}(\{1, \dots, n\})$
$Y_i = X_{p(i)}$ for $i = 1, \dots, n$

Ex $X_1 = 1$, $X_2 = 0.7$, $X_3 = 1.9$
→ generate 3 $\text{dunif}(\{1,2,3\})$

$p(1) = 1 \quad p(2) = 1 \quad p(3) = 2$

$Y_1 = X_{p(1)} = X_1 = \underline{1}$

$Y_2 = X_{p(2)} = 1$

$Y_3 = X_{p(3)} = 0.7$

More general version of this, would be draw m bootstrap

$p_{ij} \sim \text{Unif}(\{1, \dots, n\})$

$Y_{ij} = X_{p_{ij}}$ $\quad i = 1, \dots, m$, $\quad j = 1, \dots, n$

m bootstrap samples all of length n

θ estimators $\phi_n()$

$\hat{\theta}^i_{bs} = \phi_n(Y_{i1}, \dots, Y_{in})$.

$\Longrightarrow X_1, \dots, X_5 \sim \exp(\lambda)$ $\quad \hat{\lambda}_{MLE} = \frac{n}{\sum X_i}$

m different bootstrap samples

$\hat{\lambda}^i_{BS} = \frac{n}{\sum_{j=1}^{n} Y_{ij}}$

Sample percentile $100p$ percentile in sample point

$X_1, \dots, X_n$

$X_{(1)}, \dots, X_{(n)}$

np is integer: average of $X_{(np)}$ & $X_{(np)+1}$

np is not an int: $X_{\lceil np \rceil}$

100 data point $\Longrightarrow$ 50th percentile

is the avg of 50th and 51st

sorted points

50.5 percentile 51th data point

$Y_{11}, \dots, Y_{1n} \longrightarrow \hat{\lambda}^{1}_{(BS)} = \frac{}{\sum_{j=1}^{n} Y_{1j}}$

$\vdots$

$Y_{m1}, \dots, Y_{mn} \longrightarrow \hat{\lambda}^{m}_{(BS)} = \frac{}{\sum_{j=1}^{n} Y_{mj}}$

$(1-\alpha)\%$ confidence

$\longrightarrow$ bootstrap confidence

$\left(\alpha/2 \text{ percentile } \hat{\lambda}_{BS},\ 1-\alpha/2 \text{ percentile } \hat{\lambda}_{BS}\right)$