Problem 1

(a) $f(x|\alpha) = \begin{cases} C(\alpha)\, x^{\alpha-1}, & 0 < x < 1 \\ 0, & o/w \end{cases}$

$$\int_0^1 x^{\alpha-1}\, dx = \frac{1}{\alpha} x^{\alpha} \Big|_0^1 = \frac{1}{\alpha}$$

$$C(\alpha) = \alpha$$

(b) $E[\overline{X}] = \alpha \int_0^1 x \cdot x^{\alpha-1}\, dx$

$$= \alpha \int_0^1 x^{\alpha}\, dx = \frac{\alpha}{\alpha+1}$$

$$E[\overline{X}^2] = \alpha \int_0^1 x^{\alpha+1}\, dx = \frac{\alpha}{\alpha+2}$$

$$Var(X) = \frac{\alpha}{\alpha+2} - \left(\frac{\alpha}{\alpha+1}\right)^2$$

$$= \frac{\alpha(\alpha+1)^2 - \alpha^2(\alpha+2)}{(\alpha+2)(\alpha+1)^2}$$

$$= \frac{\alpha^3 + 2\alpha^2 + \alpha - \alpha^3 - 2\alpha^2}{(\alpha+2)(\alpha+1)^2}$$

$$= \frac{\alpha}{(\alpha+2)(\alpha+1)^2}$$

(c) $L(\alpha \mid x_1, \dots, x_n)$

$$= \alpha^n \prod_{i=1}^{n} x_i^{\alpha - 1}$$

$\ell(\alpha \mid x_1, \dots, x_n)$

$$= n \log(\alpha) + (\alpha - 1)\sum_{i=1}^{n} \log(x_i)$$

(d) $$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(x_i)$$

$$\hat{\alpha}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \log(x_i)}$$

MOM:

$$\bar{x} = \frac{\alpha}{\alpha+1}$$

$$(\alpha+1)\bar{x} = \alpha$$

$$\alpha(\bar{x}-1) = -\bar{x}$$

$$\hat{\alpha}_{MOM} = \frac{\bar{x}}{1-\bar{x}}$$

Problem 2 See Jupyter Notebook

Problem 3

(a) $P(X=i) = P(X=i \mid \lambda_1)P(\lambda_1)$

$$+ P(X=i \mid \lambda_2)(1-P(\lambda_1))$$

$$= e^{-\lambda_1}\lambda_1^i \qquad e^{-\lambda_2}\lambda_2^i$$

$$= \frac{\quad}{i!} p + \frac{\quad}{i!} (1-p)$$

$$= \frac{1}{i!}\left(e^{-\lambda_1}\lambda_1^{i} p + e^{-\lambda_2}\lambda_2^{i}(1-p)\right)$$

(b) $L(\lambda_1, \lambda_2, p \mid X_1, \ldots, X_n)$

$$= \prod_{i=1}^{n} \frac{1}{x_i!}\left(e^{-\lambda_1}\lambda_1^{x_i} p + e^{-\lambda_2}\lambda_2^{x_i}(1-p)\right)$$

$\ell(\lambda_1, \lambda_2, p \mid X_1, \ldots, X_n)$

$$= -\sum_{i=1}^{n} \log(x_i!) + \sum_{i=1}^{n} \log\left(e^{-\lambda_1}\lambda_1^{x_i} p + e^{-\lambda_2}\lambda_2^{x_i}(1-p)\right)$$

c, d, e → see