Hypothesis testing

↳ can we exclude certain probability distributions from being the distribution the data

Ex Say I'm pharma testing a statin

$X_1, \ldots, X_n$ = change in cholesterol levels after 60 days in trial

$\sim N(\mu, \sigma)$

→ postulate a null hypothesis

$H_0: \mu \geq 0$

$H_A: \mu < 0$

Construct a test statistic

Two kinds of error:

Type I error: false positive,
reject H_0, when H_0 is actually

Type II: false negative,
fail to reject H_0, when H_0 is
not true

it's easy to control the "level" of Type I
error

station example $\underline{\underline{\sigma = 1}}$

$n = 9$

under H_0: $\mu = 0$

$C = -1$

reject H_0 if $\bar{X} < -1$

$\bar{X} \sim N(0, \frac{1}{3})$

$P(\bar{X} < -1)$

$= P\left(\frac{\bar{X}}{1/3} < \frac{-1}{1/3}\right) = P(Z < -3)$

scipy.stats.norm.cdf(-10)

$= 7.6 \times 10^{-24}$ → really small

(-3) ⟹ .0013

...

...

⟹ specify the level (admissible amount of type I)

$\alpha = .05$

Say the level α has been specified (it's $\alpha = .05$)

$X_1, \dots, X_n \sim N(\mu, \sigma)$ σ known

$H_0: \mu = k$ test at level

$H_A: \mu \neq k$

$H_0: \bar{X} \sim N\left(k, \frac{\sigma}{\sqrt{n}}\right)$

$$\frac{\bar{X} - k}{\sigma / \sqrt{n}} \sim N(0, 1)$$

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - k}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha$$

$$P \begin{cases} \bar{X} > k + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \\ \text{or} \\ \bar{X} < k - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \end{cases} = \alpha$$

σ^2 unknown sample size n

$$P\left(-t_{\alpha/2, n-1} < \frac{\bar{X} - k}{s/\sqrt{n}} < t_{\alpha/2, n-1}\right)$$

$$\bar{X} > k + t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}}$$

$$\bar{X} < k - t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}}$$

Power of a test

↳ suppose I've specified my null hypothesis and the collection of alternatives

↳ I've chosen α

↳ calculated the critical region / rejection region cutoffs

power of a test (given a specific alternative hypothesis)

is the probability of rejecting H_0 if the specific alternative is true

$1 - P(\text{type II error})$

$X_1, \ldots, X_9 \sim N(\mu, \sigma)$ $\quad$ σ known to be 1

$H_0: \mu \leq 0$ $\quad$ $\alpha = .05$

$H_A: \mu > 0$

reject H_0 if

$$\bar{X} > z_{.05} / \sqrt{9} \approx \frac{1.65}{3} = .55$$

sps $\mu = .6$ what is the power of the test

$$P(\bar{X} > .55) \quad \text{if } \mu = .6$$

under $H_A: \mu = .6$

$$\bar{X} \sim N(.6, \tfrac{1}{3})$$

```
1 - scipy.stats.norm.cdf(.55, loc=.6,
                          scale=1/3)
```

$$P(\bar{X} > .55)$$

$$= P\left(\frac{\bar{X} - .6}{1/3} > \frac{.55 - .6}{1/3} \right)$$

$$P(Z > -.15)$$

```
1 - scipy.stats.norm.cdf(-.15)
```

$$\approx .56$$

why do a power calculation?

Last concept: p-value

$X_1, \dots, X_{100} \sim N(\mu, 1)$

$\bar{X} \sim N(\mu, 1/100)$

under $H_0: \mu = 0$

$H_A: \mu \neq 0$

cutoff $|\bar{X}| > .196$

$\bar{X} = -.197$ → p-value = .049

→ p-value essentially quantifies how unlikely our outcome was under H_0

→ p-value is the smallest level α under which the null hypothesis would have been rejected.

Ex $X_1, \dots, X_9 \sim N(\mu, 1)$

$H_0: \mu = 0$

$H_A: \mu \neq 0$

⇒ $\pm z_{\alpha/2} \cdot \frac{1}{3}$

$\bar{X} = c$

α s/t $\quad |c| = z_{\alpha/2} / 3$

$c = 1$

$P(|\bar{X}| > |c|)$

$= 2P(\bar{X} > |c|)$

$= .0026$

Hypothesis testing is equivalent to calculating a confidence interval

① $H_0: \mu = \mu_0$, and I want to perform a level α test.

$X_1, \dots, X_n \Rightarrow$ calculate a $(1-\alpha)100\%$ CI for μ

$\Rightarrow$ reject H_0 if $\mu_0 \notin$ confidence region at level α

$H_0: \mu \leq \mu_0$, I use a one-sided upper confidence interval

$H_0: \mu \geq \mu_0$, I use a one-sided lower confidence interval

(2) Can also calculate the p-value by finding the smallest α s/t $(1-\alpha)100\%$ CI contains

μ_0

(3) Can also calculate CIs by bootstrap

$(1-\alpha)100\%$ CI for μ

$$\bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

hypothesis test under $H_0: \mu = k$

critical region $|\bar{X} - k| > z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

From office hrs:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \text{np.mean}(X)$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2} \qquad \text{np.std}(X)$$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$ np.std(x, ddof=1)

$$CI: \bar{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}$$

$t_{\alpha/2,n-1}$ = scipy.stats.t.ppf(1-α/2, n-1)

$$f(x|\alpha) = \begin{cases} C(\alpha)\,x^{\alpha-1}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

$$\underbrace{\int_0^1 C(\alpha)\,x^{\alpha-1}\,dx = 1}$$

$$\int_0^1 x^{\alpha-1}\,dx = \frac{1}{\alpha}$$

$$C(\alpha) = \alpha$$

$$f(x|\alpha) = \alpha\,x^{\alpha-1}$$

$$\int_{}^{1}$$

$$E[X] = \int_0^1 x \cdot \alpha x^{\alpha-1} \, dx$$

$$= \int_0^1 \alpha x^{\alpha} \, dx$$

X k-th moment of X

$E[X^k]$

$\theta_1, \ldots, \theta_r$ so for $r = 1$ or 2

$$E[X^k] = g_k(\theta_1, \ldots, \theta_r)$$

m_k = k-th sample moment

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

r equations in r unknowns

$k = 1, \ldots, r$

$$m_k = g_k(\theta_1, \ldots, \theta_r)$$

$X_1, \ldots, X_n \sim \mathrm{Bern}(p)$

$$E[X_i] = p$$

$$m_1 = p$$

$$X_1, \ldots, X_n \sim Gamma(\alpha, \lambda) \quad r = 2$$

$$E[X] = \frac{\alpha}{\lambda} \qquad g_1(\alpha, \lambda) = \frac{\alpha}{\lambda}$$

$$E[X^2] = \frac{\alpha^2 + \alpha}{\lambda^2} \qquad g_2(\alpha, \lambda) = \frac{\alpha^2 + \alpha}{\lambda^2}$$

$$m_1 = \frac{\alpha}{\lambda} \implies \underline{\alpha = m_1 \lambda}$$

$$m_2 = \frac{\alpha^2 + \alpha}{\lambda^2}$$

$$m_2 = \frac{m_1^2 \lambda^2 + m_1 \lambda}{\lambda^2} = m_1^2 + \frac{m_1}{\lambda}$$

$$m_2 = m_1^2 + \frac{m_1}{\lambda} \qquad \underline{\underline{\hat{\lambda}_{MM}}} = \frac{m_1}{m_2 - m_1^2}$$

$$\hat{\alpha}_{MM} = \frac{m_1^2}{m_2 - m_1^2}$$

$$f(x|\alpha) = \begin{cases} \alpha \cdot x^{\alpha-1}, & 0 < x < 1 \\ 0, & o/w \end{cases}$$

$$E[X] = \int_0^1 x \cdot \alpha \cdot x^{\alpha-1}\, dx$$

$$= \alpha \int_0^1 x^{\alpha}\, dx = \frac{\alpha}{\alpha+1}$$

$$m_1 = \frac{\alpha}{\alpha+1}$$