random variable maps a sample space to a real #

↳ random number that is based on the outcome of experiment

↳ discrete random variable can take either a finite or countable # of values

countable list → 1-1 correspondence between items in the list and the positive integers

$\underline{\underline{X}}$ $\{ \underline{\underline{x_1, x_2, \ldots}} \} : r$

↳ generate a random number X

$$p(x) = P(\underline{X} = x)$$

$$p(3) = P(\underline{X} = 3)$$

$$1 \geq p(x) \geq 0$$

$$x \notin r \Rightarrow p(x) = 0$$

$$x \in r \Rightarrow p(x) > 0$$

$$\sum_{x_i \in r} p(x_i) = 1$$

$\underline{Ex}$ $\underline{X} \to \{1, 2, 3\}$

$p(1) = .5$

$p(2) = .4$

$F(0) = 0 \Rightarrow$ there are no values in r ≤ 0

$\underline{\underline{F(1)}} = .5 = \sum_{x_i \in \{1,2,3\}} p(x_i) = p(1)$

$p(3) = .1$

$F(2) = p(1) + p(2)$ $\quad x_i \le 1$

$= .9$

$F(3) = 1.0$ $\quad F(2.5) = .9$

cdf cumulative distribution function

$X \to r$

$$F(x) = P(X \le x)$$

discrete with $r = \{x_1, \dots\}$

pmf p

$$F(x) = \sum_{\substack{x_i \le x \\ x_i \in r}} p(x_i)$$

Continuous random variable

can take any value in some range

$[a,b]$ or even some union of these ranges

pdf is the analogue of the pmf

for a continuous random variable

$$P(X = x) = 0$$

$$P(X \in (c,d)) = \int_c^d \underline{\underline{f(x)}}\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \cancel{\ge} \; f(x) \ge 0$$

$$P(X \in (x-\Delta x, x+\Delta x)) \approx 2\Delta x f(x)$$

$$= \int_{x-\Delta x}^{x+\Delta x} f(x)\,dx$$

a rectangle with height $f(x)$ and width $2\Delta x$

→ cdf in this case

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$$

$$\frac{d}{dx} F(x) = f(x)$$

Ex

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{everywhere else} \end{cases}$$

what is the cdf of this

$$F(-1) = \int_{-\infty}^{-1} 0 \cdot dx = 0$$

$$F(3) = 1 = \underbrace{\int_{-\infty}^{0} 0 \cdot dx}_{=0} + \int_{0}^{1} 2x \, dx + \underbrace{\int_{1}^{3} 0 \cdot dx}_{=0}$$

$$= x^2 \Big|_0^1 = 1$$

$$F(x) = \begin{cases} 0 & , \ x < 0 \\ x^2 & , \ x \in [0,1] \\ 1 & , \ x > 1 \end{cases}$$

$$P(a < X \leq b) = F(b) - F(a)$$

$$\{X \le b\} = \{\underline{X} \le a\} \cup \{a < \underline{X} \le b\}$$

$$\underbrace{P(\underline{X} \le b)}_{F(b)} = \underbrace{P(\underline{X} \le a)}_{F(a)} + P(a < \underline{X} \le b)$$

Expectation

↳ the expectation of random variable $\underline{X}$ is the probability weighted avg value of $\underline{X}$

$x_1, \ldots, x_n \qquad w_1, \ldots, w_n \qquad \sum w_i = 1$

$$\sum_{i=1}^{n} w_i x_i$$

discrete random variable:

$\underline{X} \to r = \{x_1, \ldots\}$

pmf p

$$E[\underline{X}] = \sum_{x \in r} p(x_i) \cdot x_i$$

$X \to \{1, 2, 3\}$

$p(1) = .5$, $p(2) = .4$, $p(3) = .1$

$$E[X] = (.5)(1) + (.4)(2) + (.1)(3)$$

$$= 1.6$$

continuous random $\to$ density f

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$\underline{Ex}$ $f(x) = \begin{cases} 2x, & x \in (0,1) \\ 0, & o/w \end{cases}$

$$E[X] = \int_0^1 2x \cdot x\, dx$$

$$\cancel{+ \int_{-\infty}^{0} 0 \cdot x\, dx + \int_1^{\infty} 0 \cdot x\, dx}$$

$$= \int_0^1 2x^2\, dx$$

$$= \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

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property of expectation

↳ random variable

↳ $g(\cdot)$ → want to calculate the expectation of $g(x)$

$$E[g(X)] = \sum_{x_i} g(x_i) \cdot p(x_i) \quad \text{(discrete)}$$

$$= \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \quad \text{(cont)}$$

<u>Variance of a random variable</u>

$\mu = E[X]$

$$Var(X) = E[(X - \mu)^2]$$

= "the average squared distance from the average value" identity $Var(X) = E[X^2] - E[X]^2$

standard deviation of X is

$\sqrt{Var(X)}$

Let's suppose we have two random variables X and Y → discrete

and let's they are both defined on the sample space

discrete $p(x, y) = P(X = x \cap Y = y)$

<u>Ex</u> roll two four-sided dice

X_1 = dice 1 outcome

X_2 = dice 2 outcome

$X = \min(X_1, X_2) \rightarrow \{1, 2, 3, 4\}$

$Y = \max(X_1, X_2) \longrightarrow \{1,2,3,4\}$

$p(1,1) = P(X=1 \text{ and } Y=1)$

$= P(\text{both } X_1 \text{ and } X_2 \text{ are } 1)$

$= 1/16$

$p(2,1) = 0$

$p(1,2) = P(\{1,2\} \cup \{2,1\}) = 1/8$

$p(1,3) = P(\{1,3\} \cup \{3,1\}) = 1/8$

$p(1,4) = 1/8$

→ joint distribution for continuous random variables

$[a,b], [c,d] \in \mathbb{R}$ $\qquad f(x,y)$

$$P(X \in [a,b] \cap Y \in [c,d])$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Def $g(x,y)$ X and Y r.v.s with joint pmf p or density f

$$E[g(x,y)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx\, dy$$

$$= \sum_{y_i}\sum_{x_i} p(x_i,y_i) \cdot g(x_i,y_i)$$

$\Longrightarrow$ $Cov(X,Y)$ $X, Y \sim f$

$E[X] = \mu_X$ $E[Y] = \mu_Y$

$$= E[(X - \mu_X)(Y - \mu_Y)]$$

= "probability weighted average of the product of signed difference of X and Y from their means"

$$= E[XY] - E[X]E[Y]$$

$$Cov(X,X) = Var(X)$$